Derivatives of the log likelihood *min_dist_to_sides*

March 3, 2025

Loss Function: distance to tabletop top side

In our 3D table detection problem, the table is modeled by a 9-dimensional parameter vector

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \\ \alpha \\ \beta \\ \gamma \\ \mathrm{w} \\ \mathrm{d} \\ \mathrm{h} \end{bmatrix}, \tag{1}$$

where x, y, z define the position of the table in the room frame, α, β, γ define the rotation of the table in the room frame, and w, d, h define the width, depth, and height of the table respectively. We want to compute the *segment* distance of a point, given in the tabletop frame, to a specific side of the tabletop. The *segment* distance depends on the relative position of the point wrt the side: if the point projects onto the side, the distance is the perpendicular projection; if the point projects outside the table, the distance is the minimum of the distances to the two closest sides. More formally, the differentiable version of the *segment* distance is given by,

$$d = (w_{in}d_{in} + w_{left}d_{left} + w_{right}d_{right}), \tag{2}$$

where the d_x are the three possible cases and the w_x are weights that depend on the result of the projection. To discard points inside the tabletop, we apply a softplus gate (ReLU) to the signed distance to the side that makes zero all negative values.

1. Tabletop to top side Transformation

Let the length of the current side be

$$\mathrm{s} = \frac{\mathrm{d}}{2}. \tag{3}$$

We define the transformation from the top of the table to the current side as a Pose3 object:

$$T_{t2s} = \text{Pose3}\Big(\mathbf{R} = \mathbf{I},\ \mathbf{t} = \begin{bmatrix} 0 \\ \text{s} \\ 0 \end{bmatrix}\Big), \tag{4}$$

2. Transforming the Input Point

The $SE(3)$ transformation from the tabletop frame to the top side frame is given by,

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \text{d} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

Let $p \in \mathbb{R}^3$ be the point (in the room frame) whose distance we wish to compute. We transform p into the coordinate system of the corresponding side:

$$\text{ps} = T^{-1}(p) = \begin{bmatrix} ps_x \\ ps_y \\ ps_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\text{d} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y - d \\ p_z \\ 1 \end{bmatrix} \tag{6}$$

We define T_{t2s} as the object *gtsam::Pose3(R, t)* and use its method *Pose3::transformTo(p,* $H_{ps}(3x6)$*,* $H_{pt}(3x3)$*)* to convert the point p to the side frame. The method provides the Jacobians with respect to the pose parameters and the input point p.

$$T_{t2s} = \text{Pose3}\Big(\mathbf{R} = \mathbf{I},\ \mathbf{t} = \begin{bmatrix} 0 \\ \text{s} \\ 0 \end{bmatrix}\Big), \tag{7}$$

The $3x3$ Jacobian wrt to the input point is given by:

$$H_{\text{pts}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

The $3x6$ Jacobian wrt to the table parameters is given by:

$$H_{\text{table}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

The -1 element corresponds to the partial derivative of the transformed point's (y)-coordinate with respect to the table's d parameter. This reflects the negative translation along the y-axis in the transformation matrix.
Note: the other three sides will have a different transformation matrix.

2. ReLU on the y coordinate

We now compute a ReLU function to check if the point is inside the table. As the side's frames are defined to have the y-axis pointing outwards and the x-axis pointing right along the side, we use the y-coordinate for all sides. If the y is positive then the point is outside the tabletop:

$$\mathrm{g} = \mathrm{softplus}\Big(ps_y\Big), \tag{10}$$

where $\mathrm{softplus}(x) = \ln\big(1 + \exp(x)\big)$ is a smooth approximation of the ReLU function. It is zero if ps_y is negative (inside the table) and increases smoothly to 1 as ps_y increases.
Its derivative is the logistic function:

$$\frac{d}{dx}\,\mathrm{softplus}(x) = \frac{1}{1 + \exp(-x)}. \tag{11}$$

The g gate will be multiplied by the distance to the side to ensure that the distance is zero if the point is inside the table.

3. Segment distance

We now compute the segment distance (SD) from the point to the side. Let's define the side as a segment AB with extremes in the top-left corner A and the top-right corner B of the side. In the side frame, the segment is defined by the points:

$$A = \begin{bmatrix} -\mathrm{s} \\ 0 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} \mathrm{s} \\ 0 \\ 0 \end{bmatrix}. \tag{12}$$

as the X-axis is aligned with the segment. The distance from the point to the segment is given by the projection of the point onto the segment:

$$t = \frac{ps_x}{(B_x - A_x)} + 0.5. \tag{13}$$

The constant 0.5 is added to the projection to center it in the segment. Now we check if the projection is inside the segment using a sigmoid function. The sigmoid smooths the "saturation" of the projection on $[0, 1]$. We use two variables s and $s1$:

$$s0 = \mathrm{logistic}(\beta t) \quad s1 = \mathrm{logistic}(\beta(t-1)) \tag{14}$$

$s0$ goes from 0 to 1 close to $t = 0$ $s1$ goes from 0 to 1 close to $t = 1$
and the logistic function is defined as:

$$\mathrm{logistic}(x) = \frac{1}{1 + \exp(-x)}. \tag{15}$$

We compute now the soft weights for each region:

$$\begin{aligned} w_{in} &= s0(1 - s1) \\ w_{left} &= (1 - s0)s1 \\ w_{right} &= s1 \end{aligned}$$

And the squared distance to each segment is given by:

$$\begin{aligned} d_{in} &= {ps_y}^2 \\ d_{left} &= |ps - A|^2 \\ d_{right} &= |ps - B|^2 \end{aligned}$$

The combined distance is given by:

$$d = w_{in}d_{in} + w_{left}d_{left} + w_{right}d_{right} \tag{16}$$

We now apply the previously defined gate to d to obtain the final distance:

$$fd = g * d \tag{17}$$

4. Jacobian with respect to table parameters v

We need to compute the Jacobian $\frac{\partial fd}{\partial v(7)}$. The derivative with respect to the point **p** will be addressed separately.
The relevant quantities are:

$$ps = t2s1.transformTo(\mathbf{p}) \tag{18}$$
$$h_{pose}(:,4) = \text{y-component of h}_{pose} \tag{19}$$
$$t2s1 = \text{gtsam::Pose3(gtsam::Rot3::Identity()}, \text{gtsam::Point3}(0.0, v(7)/2, 0.0)) \tag{20}$$
$$A = (-v(7)/2, 0, 0) \tag{21}$$
$$B = (v(7)/2, 0, 0) \tag{22}$$
$$t_1 = \frac{ps_x}{B_x - A_x} + 0.5 \tag{23}$$
$$s_0 = \text{logistic}(\beta t_1) \tag{24}$$
$$s_1 = \text{logistic}(\beta(t_1 - 1)) \tag{25}$$
$$w_{in} = s_0(1 - s_1) \tag{26}$$
$$w_{left} = (1 - s_0) \tag{27}$$
$$w_{right} = s_1 \tag{28}$$
$$d_{in} = ps_y^2 \tag{29}$$
$$d_{left} = ||ps - A||^2 \tag{30}$$
$$d_{right} = ||ps - B||^2 \tag{31}$$
$$d = w_{in}d_{in} + w_{left}d_{left} + w_{right}d_{right} \tag{32}$$
$$g = \text{softplus}(ps_y) \tag{33}$$
$$fd = g \cdot d \tag{34}$$

Applying the chain rule:

$$\frac{\partial fd}{\partial v(7)} = g \cdot \frac{\partial d}{\partial v(7)} + d \cdot \frac{\partial g}{\partial v(7)} \tag{35}$$

The second term can be computed as:

$$\frac{\partial g}{\partial v(7)} = \frac{\partial softplus(ps_y)}{\partial ps_y} \frac{\partial ps_y}{\partial v(7)} = \text{logistic}(ps_y)[h_{pose}(:,4)]_y \tag{36}$$

The Jacobian $\frac{\partial dist}{\partial v(7)}$ can be computed as the sum of the contributions from the three terms:

$$\frac{\partial d}{\partial v(7)} = \underbrace{\frac{\partial(w_{in}d_{in})}{\partial v(7)}}_{K} + \underbrace{\frac{\partial(w_{left}d_{left})}{\partial v(7)}}_{L} + \underbrace{\frac{\partial(w_{right}d_{right})}{\partial v(7)}}_{M} \tag{37}$$

Term K: $\frac{\partial(w_{in}d_{in})}{\partial v(7)}$

Applying the product rule:

$$\frac{\partial(w_{in}d_{in})}{\partial v(7)} = w_{in}\frac{\partial d_{in}}{\partial v(7)} + d_{in}\frac{\partial w_{in}}{\partial v(7)} \tag{38}$$

Derivative $\frac{\partial d_{in}}{\partial v(7)}$:
Recall $d_{in} = ps_y^2$. The dependency on $v(7)$ is through ps.

$$\frac{\partial d_{in}}{\partial v(7)} = \frac{\partial d_{in}}{\partial ps}\frac{\partial ps}{\partial v(7)} \tag{39}$$

$$\frac{\partial d_{in}}{\partial ps} = (0, 2ps_y, 0)^T \tag{40}$$

Let $\mathbf{h}_{pose,v7}$ be the column of H_{pose} (the Jacobian from 'transformTo') corresponding to the derivative with respect to $v(7)$.

$$\frac{\partial ps}{\partial v(7)} = \mathbf{h}_{pose,v7} \tag{41}$$

Thus,

$$\frac{\partial d_{in}}{\partial v(7)} = (0, 2ps_y, 0)^T\mathbf{h}_{pose,v7} = 2ps_y[\mathbf{h}_{pose,v7}]_y \tag{42}$$

Derivative $\frac{\partial w_{in}}{\partial v(7)}$:
w_{in} depends on $v(7)$ through s_0, s_1, t_1, ps_x, A_x, and B_x.

$$\frac{\partial w_{in}}{\partial v(7)} = \frac{\partial w_{in}}{\partial s_0}\frac{\partial s_0}{\partial t_1}\frac{\partial t_1}{\partial v(7)} + \frac{\partial w_{in}}{\partial s_1}\frac{\partial s_1}{\partial t_1}\frac{\partial t_1}{\partial v(7)} \tag{43}$$

and,

$$\frac{\partial t_1}{\partial v(7)} = \frac{\partial t_1}{\partial ps_x}\frac{\partial ps_x}{\partial v(7)} + \frac{\partial t_1}{\partial A_x}\frac{\partial A_x}{\partial v(7)} + \frac{\partial t_1}{\partial B_x}\frac{\partial B_x}{\partial v(7)} \tag{44}$$

Computing the derivatives:

$$\frac{\partial w_{in}}{\partial s_0} = 1 - s_1 \tag{45}$$

$$\frac{\partial w_{in}}{\partial s_1} = -s_0 \tag{46}$$

$$\frac{\partial s_0}{\partial t_1} = \beta \cdot \text{logistic}(\beta t_1)(1 - \text{logistic}(\beta t_1)) \tag{47}$$

$$\frac{\partial s_1}{\partial t_1} = \beta \cdot \text{logistic}(\beta(t_1 - 1))(1 - \text{logistic}(\beta(t_1 - 1))) \tag{48}$$

$$\frac{\partial t_1}{\partial ps_x} = \frac{1}{B_x - A_x} = \frac{1}{v(7)} \tag{49}$$

$$\frac{\partial ps_x}{\partial v(7)} = [\mathbf{h}_{pose,v7}]_x \tag{50}$$

$$\frac{\partial t_1}{\partial A_x} = -\frac{ps_x}{(B_x - A_x)^2} = -\frac{ps_x}{v(7)^2} \tag{51}$$

$$\frac{\partial t_1}{\partial B_x} = \frac{ps_x}{(B_x - A_x)^2} = \frac{ps_x}{v(7)^2} \tag{52}$$

$$\frac{\partial A_x}{\partial v(7)} = -1/2 \tag{53}$$

$$\frac{\partial B_x}{\partial v(7)} = 1/2 \tag{54}$$

Thus,

$$\frac{\partial t_1}{\partial v(7)} = \frac{1}{v(7)}[\mathbf{h}_{pose,v7}]_x - \frac{ps_x}{v(7)^2}(-1/2) + \frac{ps_x}{v(7)^2}(1/2) = \frac{1}{v(7)}[\mathbf{h}_{pose,v7}]_x + \frac{ps_x}{v(7)^2} \tag{55}$$

and,

$$\begin{aligned}\frac{\partial w_{in}}{\partial v(7)} = [&(1 - s_1)\beta \cdot \text{logistic}(\beta t_1)(1 - \text{logistic}(\beta t_1)) \\ &-s_0\beta \cdot \text{logistic}(\beta(t_1 - 1))(1 - \text{logistic}(\beta(t_1 - 1)))] \\ &\left(\frac{1}{v(7)}[\mathbf{h}_{pose,v7}]_x + \frac{ps_x}{v(7)^2}\right)\end{aligned} \tag{56}$$

Combining for Term K,

$$
\begin{aligned}
\frac{\partial(w_{in}d_{in})}{\partial v(7)} = w_{in} 2 ps_y [\mathbf{h}_{pose,v7}]_y \\
&+ d_{in} \left[(1-s_1)\beta \cdot \text{logistic}(\beta t_1)(1-\text{logistic}(\beta t_1)) \right. \\
&\left. - s_0 \beta \cdot \text{logistic}(\beta(t_1-1))(1-\text{logistic}(\beta(t_1-1)))\right] \\
&\left(\frac{1}{v(7)}[\mathbf{h}_{pose,v7}]_x + \frac{ps_x}{v(7)^2}\right)
\end{aligned} \tag{57}
$$

Term L: $\frac{\partial(w_{left}d_{left})}{\partial v(7)}$

Applying the product rule:

$$
\frac{\partial(w_{left}d_{left})}{\partial v(7)} = w_{left}\frac{\partial d_{left}}{\partial v(7)} + d_{left}\frac{\partial w_{left}}{\partial v(7)} \tag{58}
$$

Derivative $\frac{\partial d_{left}}{\partial v(7)}$: Recall $d_{left} = ||ps - A||^2$.

$$
\frac{\partial d_{left}}{\partial v(7)} = \frac{\partial d_{left}}{\partial ps}\frac{\partial ps}{\partial v(7)} + \frac{\partial d_{left}}{\partial A}\frac{\partial A}{\partial v(7)} \tag{59}
$$

$$
\frac{\partial d_{left}}{\partial ps} = 2(ps - A)^T \tag{60}
$$

$$
\frac{\partial ps}{\partial v(7)} = \mathbf{h}_{pose,v7} \tag{61}
$$

$$
\frac{\partial d_{left}}{\partial A} = -2(ps - A)^T \tag{62}
$$

$$
\frac{\partial A}{\partial v(7)} = (-1/2, 0, 0)^T \tag{63}
$$

Therefore:

$$
\frac{\partial d_{left}}{\partial v(7)} = 2(ps - A)^T\mathbf{h}_{pose,v7} - 2(ps - A)^T(-1/2, 0, 0)^T = 2(ps - A)^T\mathbf{h}_{pose,v7} + (ps_x - A_x) \tag{64}
$$

Derivative $\frac{\partial w_{left}}{\partial v(7)}$:
Recall $w_{left} = (1 - s_0)$.

$$
\frac{\partial w_{left}}{\partial v(7)} = \frac{\partial w_{left}}{\partial s_0}\frac{\partial s_0}{\partial t_1}\frac{\partial t_1}{\partial v(7)} \tag{65}
$$

$$\frac{\partial w_{left}}{\partial s_0} = -1 \tag{66}$$

$$\frac{\partial s_0}{\partial t_1} = \beta \cdot \text{logistic}(\beta t_1)(1 - \text{logistic}(\beta t_1)) \tag{67}$$

$$\frac{\partial t_1}{\partial v(7)} = \frac{\partial t_1}{\partial ps_x}\frac{\partial ps_x}{\partial v(7)} + \frac{\partial t_1}{\partial A_x}\frac{\partial A_x}{\partial v(7)} + \frac{\partial t_1}{\partial B_x}\frac{\partial B_x}{\partial v(7)} \tag{68}$$

Using values calculated for Term K:

$$\frac{\partial t_1}{\partial v(7)} = \frac{1}{v(7)}[\mathbf{h}_{pose,v7}]_x + \frac{ps_x}{v(7)^2} \tag{69}$$

$$\frac{\partial w_{left}}{\partial v(7)} = -\beta \cdot \text{logistic}(\beta t_1)(1 - \text{logistic}(\beta t_1)) \left(\frac{1}{v(7)}[\mathbf{h}_{pose,v7}]_x + \frac{ps_x}{v(7)^2}\right) \tag{70}$$

Combining for Term L:

$$\begin{aligned}\frac{\partial(w_{left}d_{left})}{\partial v(7)} = w_{left} &\left(2(ps - A)^T \mathbf{h}_{pose,v7} + (ps_x - A_x)\right) \\ &- d_{left}\beta \cdot \text{logistic}(\beta t_1)(1 - \text{logistic}(\beta t_1)) \left(\frac{1}{v(7)}[\mathbf{h}_{pose,v7}]_x + \frac{ps_x}{v(7)^2}\right)\end{aligned} \tag{71}$$

Term M: $\frac{\partial(w_{right}d_{right})}{\partial v(7)}$

Applying the product rule:

$$\frac{\partial(w_{right}d_{right})}{\partial v(7)} = w_{right}\frac{\partial d_{right}}{\partial v(7)} + d_{right}\frac{\partial w_{right}}{\partial v(7)} \tag{72}$$

Derivative $\frac{\partial d_{right}}{\partial v(7)}$: Recall $d_{right} = ||ps - B||^2$.

$$\frac{\partial d_{right}}{\partial v(7)} = \frac{\partial d_{right}}{\partial ps}\frac{\partial ps}{\partial v(7)} + \frac{\partial d_{right}}{\partial B}\frac{\partial B}{\partial v(7)} \tag{73}$$

$$\frac{\partial d_{right}}{\partial ps} = 2(ps - B)^T \tag{74}$$

$$\frac{\partial ps}{\partial v(7)} = \mathbf{h}_{pose,v7} \tag{75}$$

$$\frac{\partial d_{right}}{\partial B} = -2(ps - B)^T \tag{76}$$

$$\frac{\partial B}{\partial v(7)} = (1/2, 0, 0)^T \tag{77}$$

Therefore:

$$\frac{\partial d_{right}}{\partial v(7)} = 2(ps-B)^T \mathbf{h}_{pose,v7} - 2(ps-B)^T(1/2,0,0)^T = 2(ps-B)^T \mathbf{h}_{pose,v7} - (ps_x - B_x) \quad (78)$$

Derivative $\frac{\partial w_{right}}{\partial v(7)}$: Recall $w_{right} = s_1$.

$$\frac{\partial w_{right}}{\partial v(7)} = \frac{\partial w_{right}}{\partial s_1}\frac{\partial s_1}{\partial t_1}\frac{\partial t_1}{\partial v(7)} \quad (79)$$

$$\frac{\partial w_{right}}{\partial s_1} = 1 \quad (80)$$

$$\frac{\partial s_1}{\partial t_1} = \beta \cdot \text{logistic}(\beta(t_1-1))(1-\text{logistic}(\beta(t_1-1))) \quad (81)$$

Using the value of $\frac{\partial t_1}{\partial v(7)}$ calculated before:

$$\frac{\partial w_{right}}{\partial v(7)} = \beta \cdot \text{logistic}(\beta(t_1-1))(1-\text{logistic}(\beta(t_1-1)))\left(\frac{1}{v(7)}[\mathbf{h}_{pose,v7}]_x + \frac{ps_x}{v(7)^2}\right) \quad (82)$$

Combining for Term M:

$$\begin{aligned}\frac{\partial(w_{right}d_{right})}{\partial v(7)} &= w_{right}\left(2(ps-B)^T\mathbf{h}_{pose,v7} - (ps_x - B_x)\right) \\ &+ d_{right}\beta \cdot \text{logistic}(\beta(t_1-1))(1-\text{logistic}(\beta(t_1-1)))\left(\frac{1}{v(7)}[\mathbf{h}_{pose,v7}]_x + \frac{ps_x}{v(7)^2}\right) \quad (83)\end{aligned}$$

4. Jacobian with respect to table parameters: final step

The aggregated K + L + M derivative gives,

$$\frac{\partial d}{\partial v(7)} = K + L + M \quad (84)$$

and the final Jacobian can now be computed as a $1x6$ left-block matrix inside the $1x9$ matrix that the loss function returns:

$$\frac{\partial fd}{\partial v(7)} = g \cdot \underbrace{\frac{\partial d}{\partial v(7)}}_{1x6} + d \cdot \underbrace{\frac{\partial g}{\partial v(7)}}_{1x6} \quad (85)$$

Jacobian with Respect to the Input Point p ($\mathbf{H}_2$)

We now need to compute the Jacobian $\frac{\partial fd}{\partial \mathbf{p}}$. Recall that:

$$fd = g * (w_{in} d_{in} + w_{left} d_{left} + w_{right} d_{right}) \tag{86}$$

The point $\mathbf{p}$ affects the fd function only through the transformed point ps. Therefore, we can apply the chain rule:

$$\frac{\partial fd}{\partial p} = \frac{\partial fd}{\partial ps} \frac{\partial ps}{\partial p} \tag{87}$$

Let's break this down into two parts:

Derivative $\frac{\partial ps}{\partial p}$

We already know that $\frac{\partial ps}{\partial \mathbf{p}} = H_{point}$.

Derivative $\frac{\partial fd}{\partial ps}$

Applying the product rule:

$$\frac{\partial fd}{\partial ps} = \frac{\partial (g \cdot d)}{\partial ps} = g \cdot \frac{\partial d}{\partial ps} + d \cdot \frac{\partial g}{\partial ps} \tag{88}$$

Derivative $\frac{\partial d}{\partial ps}$: This is the derivative of the 'softmax(d)' function with respect to the transformed point 'ps'. We need to consider the contributions of all three terms (d_{in}, d_{left}, and d_{right}), as well as the weights.

$$\frac{\partial dist}{\partial ps} = \frac{\partial (w_{in} d_{in})}{\partial ps} + \frac{\partial (w_{left} d_{left})}{\partial ps} + \frac{\partial (w_{right} d_{right})}{\partial ps} \tag{89}$$

Applying the product rule to each term:

$$\frac{\partial (w_{in} d_{in})}{\partial ps} = w_{in} \frac{\partial d_{in}}{\partial ps} + d_{in} \frac{\partial w_{in}}{\partial ps} \tag{90}$$

$$\frac{\partial (w_{left} d_{left})}{\partial ps} = w_{left} \frac{\partial d_{left}}{\partial ps} + d_{left} \frac{\partial w_{left}}{\partial ps} \tag{91}$$

$$\frac{\partial (w_{right} d_{right})}{\partial ps} = w_{right} \frac{\partial d_{right}}{\partial ps} + d_{right} \frac{\partial w_{right}}{\partial ps} \tag{92}$$

Now we need to compute the individual derivatives:
The derivative $\frac{\partial d_{in}}{\partial ps}$: Recall $d_{in} = ps_y^2$.

$$\frac{\partial d_{in}}{\partial ps} = \frac{\partial (ps_y^2)}{\partial ps} = (0, 2ps_y, 0) \tag{93}$$

The derivative $\frac{\partial d_{left}}{\partial ps}$: Recall $d_{left} = ||ps - A||^2$.

$$\frac{\partial d_{left}}{\partial ps} = 2(ps - A)^T \tag{94}$$

The derivative $\frac{\partial d_{right}}{\partial ps}$: Recall $d_{right} = ||ps - B||^2$.

$$\frac{\partial d_{right}}{\partial ps} = 2(ps - B)^T \tag{95}$$

The derivative $\frac{\partial w_{in}}{\partial ps}$: w_{in} depends on ps through t_1.

$$\frac{\partial w_{in}}{\partial ps} = \frac{\partial w_{in}}{\partial s_0}\frac{\partial s_0}{\partial t_1}\frac{\partial t_1}{\partial ps} + \frac{\partial w_{in}}{\partial s_1}\frac{\partial s_1}{\partial t_1}\frac{\partial t_1}{\partial ps} \tag{96}$$

$$\frac{\partial t_1}{\partial ps} = \frac{\partial t_1}{\partial ps_x}\frac{\partial ps_x}{\partial ps} = \frac{1}{B_x - A_x}(1,0,0) = \frac{1}{v(7)}(1,0,0) \tag{97}$$

We already have $\frac{\partial w_{in}}{\partial s_0}$, $\frac{\partial w_{in}}{\partial s_1}$, $\frac{\partial s_0}{\partial t_1}$, and $\frac{\partial s_1}{\partial t_1}$ from the previous derivation with respect to $v(7)$. Substituting:

$$\begin{aligned}\frac{\partial w_{in}}{\partial ps} =& [(1 - s_1)\beta \cdot \text{logistic}(\beta t_1)(1 - \text{logistic}(\beta t_1)) \\ & -s_0\beta \cdot \text{logistic}(\beta(t_1 - 1))(1 - \text{logistic}(\beta(t_1 - 1)))] \\ & \frac{1}{v(7)}(1,0,0)\end{aligned} \tag{98}$$

The derivative $\frac{\partial w_{left}}{\partial ps}$: w_{left} also depends on ps through t_1.

$$\frac{\partial w_{left}}{\partial ps} = \frac{\partial w_{left}}{\partial s_0}\frac{\partial s_0}{\partial t_1}\frac{\partial t_1}{\partial ps} \tag{99}$$

Using the previously calculated values:

$$\frac{\partial w_{left}}{\partial ps} = -\beta \cdot \text{logistic}(\beta t_1)(1 - \text{logistic}(\beta t_1))\frac{1}{v(7)}(1,0,0) \tag{100}$$

The derivative $\frac{\partial w_{right}}{\partial ps}$: w_{right} also depends on ps through t_1.

$$\frac{\partial w_{right}}{\partial ps} = \frac{\partial w_{right}}{\partial s_1}\frac{\partial s_1}{\partial t_1}\frac{\partial t_1}{\partial ps} \tag{101}$$

Using the previously calculated values:

$$\frac{\partial w_{right}}{\partial ps} = \beta \cdot \text{logistic}(\beta(t_1 - 1))(1 - \text{logistic}(\beta(t_1 - 1)))\frac{1}{v(7)}(1,0,0) \tag{102}$$

Combining Everything:

$$
\begin{aligned}
\frac{\partial d}{\partial ps} = w_{in}(0, 2ps_y, 0) + d_{in}\,[&(1-s_1)\beta \cdot \text{logistic}(\beta t_1)(1-\text{logistic}(\beta t_1)) \\
&-s_0\beta \cdot \text{logistic}(\beta(t_1-1))(1-\text{logistic}(\beta(t_1-1)))]\,\frac{1}{v(7)}(1,0,0) \\
&+ w_{left}2(ps-A)^T + d_{left}\left[-\beta \cdot \text{logistic}(\beta t_1)(1-\text{logistic}(\beta t_1))\right. \\
&\left.\frac{1}{v(7)}(1,0,0)\right] \\
&+ w_{right}2(ps-B)^T + d_{right}\left[\beta \cdot \text{logistic}(\beta(t_1-1))(1-\text{logistic}(\beta(t_1-1)))\right. \\
&\left.\frac{1}{v(7)}(1,0,0)\right] \qquad (103)
\end{aligned}
$$

where $\frac{\partial d}{\partial ps}$ is given by the long expression above.

Derivative $\frac{\partial g}{\partial ps}$

$$\frac{\partial g}{\partial ps} = [0, \text{logistic}(ps_y), 0] \qquad (104)$$

Jacobian with Respect to the Input Point p ($\mathbf{H}_2$): final step

The final Jacobian $H2$ is a 1x3 matrix:

$$H2 = \frac{\partial fd}{\partial p} = \frac{\partial fd}{\partial ps}\frac{\partial ps}{\partial p} = g\,(\underbrace{\frac{\partial d}{\partial ps}}_{1x3} + d\,\underbrace{\frac{\partial g}{\partial ps}}_{1x3})\,\underbrace{H_{point}}_{3x3} \qquad (105)$$