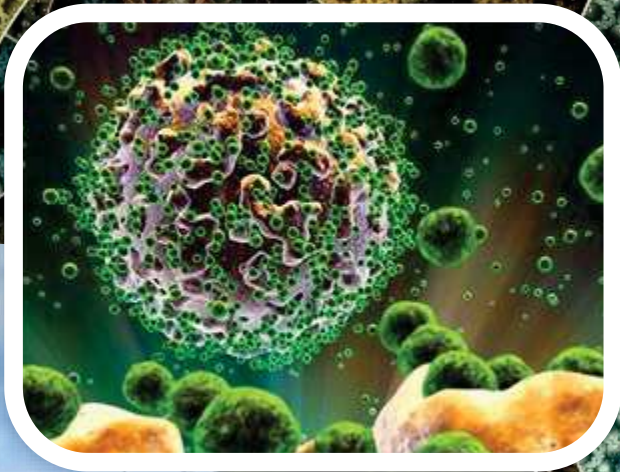
MODELLING FOR ENGINEERING AND HUMAN BEHAVIOUR 2018

Instituto Universitario de Matemática Multidisciplinar



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UNIVERSITAT POLITÈCNICA DE VALÈNCIA

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CONTENTS

Some new Hermite matrix polynomials series expansions and their applications in hyperbolic matrix sine and cosine approximation *

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1 Introduction and notation

Hermite matrix polynomial $H_n(x, A)$ has the generating function, see [1]:

$$e^{xt\sqrt{2A}} = e^{t^2} \sum_{n \geq 0} \frac{H_n(x, A)}{n!} t^n, \tag{1}$$

from following expressions for the matrix hyperbolic sine and cosine are derived:

$$\left.\begin{array}{rcl} \cosh\left(xt\sqrt{2A}\right) & = & e^{t^2} \displaystyle\sum_{n \geq 0} \frac{H_{2n}(x, A)}{(2n)!} t^{2n} \\ \sinh\left(xt\sqrt{2A}\right) & = & e^{t^2} \displaystyle\sum_{n \geq 0} \frac{H_{2n+1}(x, A)}{(2n+1)!} t^{2n+1} \end{array}\right\}, \quad x \in \mathbb{R}, |t| < \infty. \tag{2}$$

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Recently we have shown the following formulas which are a generalization of formulas (2):

$$\left.\begin{aligned}\sum_{n\geq 0}\frac{H_{2n+1}(x,A)}{(2n)!}t^{2n} &= e^{-t^2}\left[H_1(x,A)\cosh\left(xt\sqrt{2A}\right)-2t\sinh\left(xt\sqrt{2A}\right)\right],\\ \sum_{n\geq 0}\frac{H_{2n+2}(x,A)}{(2n+1)!}t^{2n+1} &= e^{-t^2}\left[H_1(x,A)\sinh\left(xt\sqrt{2A}\right)-2t\cosh\left(xt\sqrt{2A}\right)\right],\\ \sum_{n\geq 0}\frac{H_{2n+3}(x,A)}{(2n+1)!}t^{2n+1} &= e^{-t^2}\left[\left(H_2(x,A)+4t^2 I\right)\sinh\left(xt\sqrt{2A}\right)-4tH_1(x,A)\cosh\left(xt\sqrt{2A}\right)\right].\end{aligned}\right\} \tag{3}$$

We will use formulas (3) to obtain a new expansion of the hyperbolic matrix sine and cosine in Hermite matrix polynomials series.

Throughout this paper, we denote by $\mathbb{C}^{r\times r}$ the set of all the complex square matrices of size r. We denote by Θ and I, respectively, the zero and the identity matrix in $\mathbb{C}^{r\times r}$. If $A\in\mathbb{C}^{r\times r}$, we denote by $\sigma(A)$ the set of all the eigenvalues of A. For a real number x, $\lfloor x\rfloor$ denotes the lowest integer not less than x and $\lceil x\rceil$ denotes the highest integer not exceeding x.

We recall that for a positive stable matrix $A\in\mathbb{C}^{r\times r}$ the $n-$th Hermite matrix polynomial is defined in [1] by:

$$H_n(x,A)=n!\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^k\left(\sqrt{2A}\right)^{n-2k}}{k!(n-2k)!}x^{n-2k}, \tag{4}$$

which satisfies the three-term matrix recurrence:

$$\left.\begin{aligned}&H_m(x,A)=x\sqrt{2A}H_{m-1}(x,A)-2(m-1)H_{m-2}(x,A)\ ,\ m\geq 1,\\ &H_{-1}(x,A)=\Theta\ ,\ H_0(x,A)=I\ .\end{aligned}\right\} \tag{5}$$

2 Some new Hermite matrix series expansions for the hyperbolic matrix cosine and sine

Let $A\in\mathbb{C}^{r\times r}$ be a positive stable matrix, then the matrix polynomial $H_1(x,A)=\sqrt{2A}x$ is invertible if $x\neq 0$. Substituting $\sinh\left(xt\sqrt{2A}\right)$ given in (2) into the first expression of (3) we obtain the following new rational expression for the hyperbolic matrix cosine in terms of Hermite matrix polynomials:

$$\cosh\left(xt\sqrt{2A}\right) = e^{t^2}\left(\sum_{n\geq 0}\frac{H_{2n+1}(x,A)}{(2n)!}\left(1+\frac{2t^2}{2n+1}\right)t^{2n}\right)[H_1(x,A)]^{-1}, \quad x\in\mathbb{R}\sim\{0\}, |t|<+\infty. \tag{6}$$

Substituting $\sinh\left(xt\sqrt{2A}\right)$ given in (2) into the second expression of (3) and using the three-term matrix recurrence (5) we obtain the expression of $\cosh\left(xt\sqrt{2A}\right)$ given in (2).

On the other hand, replacing the expression of $\sin\left(xt\sqrt{2A}\right)$ given in (2) into the third expression of (3), we obtain another new rational expression for the hyperbolic matrix cosine in terms of Hermite matrix polynomials:

$$\cosh\left(xt\sqrt{2A}\right) =$$

$$= \frac{-e^{t^2}}{4}\left[\sum_{n\geq 0}\frac{H_{2n+3}(x,A)}{(2n+1)!}t^{2n} - \left(H_2(x,A)+4t^2I\right)\star\left(\sum_{n\geq 0}\frac{H_{2n+1}(x,A)}{(2n+1)!}t^{2n+1}\right)\right][H_1(x,A)]^{-1}, \quad x\in\mathbb{R}\sim\{0\}, |t|<+\infty. \tag{7}$$

Comparing (7) with (6), we observe that it always has a matrix product more when evaluating (7), the matrix product remarked by symbol "$\star$" in (7). Due to the importance of reducing the number of matrix products, see [2–4] for more details, we will focus mainly on the expansion (6).

From (4), it follows that, for $x\neq 0$:

$$\begin{aligned} H_{2n+1}(x,A)[H_1(x,A)]^{-1} &= \frac{(2n+1)!}{x}\sum_{k=0}^{n}\frac{(-1)^k x^{2(n-k)+1}(2A)^{n-k}}{k!(2(n-k)+1)!} \\ &= \widetilde{H}_{2n+1}(x,A), \end{aligned} \tag{8}$$

where

$$\widetilde{H}_n(x,A) = n!\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{(-1)^k\left(\sqrt{2A}\right)^{n-2k-1}}{k!(n-2k)!}x^{n-2k}, \tag{9}$$

so the right side of (8) is still defined in the case where the matrix A is singular. In this way, we can re-write the relation (6) in terms of the matrix polynomial $\widetilde{H}_{2n+1}(x,A)$:

$$\cosh\left(xt\sqrt{2A}\right) = e^{t^2}\left(\sum_{n\geq 0}\frac{\widetilde{H}_{2n+1}(x,A)}{(2n)!}\left(1+\frac{2t^2}{2n+1}\right)t^{2n}\right), \quad x\in\mathbb{R}, |t|<+\infty. \tag{10}$$

Replacing the matrix A by matrix $A^2/2$ in (10) we can avoid the square roots of matrices, and taking $x=\lambda, \lambda\neq 0, t=1/\lambda$, we finally obtain

$$\cosh(A) = e^{\frac{1}{\lambda^2}}\left(\sum_{n\geq 0}\frac{\widetilde{H}_{2n+1}\left(\lambda,\frac{1}{2}A^2\right)}{(2n)!\lambda^{2n+1}}\left(1+\frac{2}{(2n+1)\lambda^2}\right)\right), 0<\lambda<+\infty. \tag{11}$$

3 Numerical approximations

Truncating the given series (11) until order m, we obtain the approximation $CH_m(\lambda,A)\approx\cosh(A)$ defined by

$$CH_m(\lambda,A) = e^{\frac{1}{\lambda^2}}\left(\sum_{n=0}^{m}\frac{\widetilde{H}_{2n+1}\left(\lambda,\frac{1}{2}A^2\right)}{(2n)!\lambda^{2n+1}}\left(1+\frac{2}{(2n+1)\lambda^2}\right)\right), 0<\lambda<+\infty. \tag{12}$$

Working analogously to the proof of the formula (3.6) of [5] one gets, for $x\neq 0$ the following bound:

$$\left\|\widetilde{H}_{2n+1}\left(x,\frac{1}{2}A^2\right)\right\|_2 \leq (2n+1)!\frac{e\sinh\left(|x|\left\|A^2\right\|_2^{1/2}\right)}{|x|\left\|A^2\right\|_2^{1/2}}. \tag{13}$$

Then we can obtain the following expression for the approximation error:

$$\begin{aligned}\left\|\cosh(A)-CH_m(\lambda,A)\right\|_2 &\leq e^{\frac{1}{\lambda^2}}\sum_{n\geq m+1}\frac{\left\|\widetilde{H}_{2n+1}\left(\lambda,\frac{1}{2}A^2\right)\right\|_2}{(2n)!\lambda^{2n+1}}\left(1+\frac{2}{(2n+1)\lambda^2}\right) \qquad (14)\\ &\leq \frac{e^{1+\frac{1}{\lambda^2}}\sinh\left(\lambda\left\|A^2\right\|_2^{1/2}\right)}{\lambda^2\left\|A^2\right\|_2^{1/2}}\sum_{n\geq m+1}\frac{2n+1}{\lambda^{2n}}\left(1+\frac{2}{(2n+1)\lambda^2}\right).\end{aligned}$$

Taking $\lambda>1$ it follows that $\dfrac{2}{(2n+1)\lambda^2}<1$, and one gets

$$\begin{aligned}\sum_{n\geq m+1}\frac{2n+1}{\lambda^{2n}}\left(1+\frac{2}{(2n+1)\lambda^2}\right) &\leq 2\sum_{n\geq m+1}\frac{2n+1}{\lambda^{2n}}\\ &= \frac{4+(4m+6)(\lambda^2-1)}{\lambda^{2m}\left(\lambda^2-1\right)^2},\end{aligned}$$

m	z_m	λ_m
2	0.0020000000061361199	909.39256098888882
4	0.079956209874370632	99.997970988888895
6	0.34561400005673254	39.999499988888893
9	1.1120032200657	17.997896988889799
12	2.2373014291079998	11.882978988901458
16	4.1086396680000004	7.9999999964157498

Table 1: Values of z_m and λ_m for $\cosh(A)$.

	$m_1 = 2$	$m_2 = 4$	$m_3 = 6$	$m_4 = 9$	$m_5 = 12$	$m_6 = 16$
$\bar{m}_k$	1	2	3	5	7	11
$\tilde{m}_k$	1	2	4	10	13	17
$f_{m_k}(\max)$	0	0	$1.9 \cdot 10^{-17}$	$6.0 \cdot 10^{-19}$	$1.4 \cdot 10^{-26}$	$1.3 \cdot 10^{-35}$

Table 2: Values $\bar{m}_k$, $\tilde{m}_k$, and $f_{\max}$.

thus from (14) we finally obtain:

$$\left\|\cosh(A) - CH_m(\lambda, A)\right\|_2 \leq \frac{e^{1+\frac{1}{\lambda^2}} \sinh\left(\lambda \left\|A^2\right\|_2^{1/2}\right)\left(4 + (4m+6)(\lambda^2 - 1)\right)}{\left\|A^2\right\|_2^{1/2} \lambda^{2m+2} \left(\lambda^2 - 1\right)^2}. \tag{15}$$

From this expression (15) we derived the optimal values $(\lambda_m; z_m)$ such that

$$z_m = \max\left\{ z = \left\|A^2\right\|_2 ; \frac{e^{1+\frac{1}{\lambda^2}} \sinh\left(\lambda z^{1/2}\right)\left(4 + (4m+6)(\lambda^2 - 1)\right)}{z^{1/2}\lambda^{2m+2}\left(\lambda^2 - 1\right)^2} < u \right\}$$

where u is the unit roundoff in IEEE double precision arithmetic, $u = 2^{-53}$. The optimal values of m, z and λ have been obtained with MATLAB. The results are given in the Table 1.

If $\cosh(A)$ is calculated from the Taylor series, then the absolute forward error of the Hermite approximation of $\cosh(A)$, denoted by E_f, can be computed as

$$E_f = \left\|\cosh(A) - P_{m_k}(B)\right\| = \left\| \sum_{i \geqslant \bar{m}_k} f_{m_k,i} B^i \right\| \cong \left\| \sum_{i \geqslant \tilde{m}_k} f_{m_k,i} B^i \right\|,$$

where the values of $\bar{m}_k$ and $\tilde{m}_k$ for each $m_k \in \{2, 4, 6, 9, 12, 16\}$ appear in the Table 2.

Scaling factor s and the order of Hermite approximation m_k are obtained by the following:

Theorem 3.1 ([6]) *Let* $h_l(x) = \sum\limits_{i\geq l} p_i x^i$ *be a power series with radius of convergence* w, $\tilde{h}_l(x) = \sum\limits_{i\geq l} |p_i| x^i$, $B \in \mathbb{C}^{n\times n}$ *with* $\rho(B) < w$, $l \in \mathbb{N}$ *and* $t \in \mathbb{N}$ *with* $1 \leqslant t \leqslant l$. *If* t_0 *is the multiple of* t *such that* $l \leqslant t_0 \leqslant l + t - 1$ *and*

$$\beta_t = \max\{d_j^{1/j} : j = t, l, l+1, \ldots, t_0 - 1, t_0 + 1, t_0 + 2, \ldots, l + t - 1\},$$

where d_j *is an upper bound for* $||B^j||$, $d_j \geqslant ||B^j||$, *then*

$$||h_l(B)|| \leqslant \tilde{h}_l\left(\beta_t\right).$$

We have empirically verified that by neglecting the coefficients whose absolute value is lower than u, the efficiency results are far superior to the state-of-the-art algorithms, with also excellent accuracy.

4 Numerical experiments

The MATLAB's implementation `coshmtayher` is a modification of the MATLAB's code `coshher` given in [5], replacing the original Hermite approximation `coshher` by the new Hermite matrix polynomial obtained from (11). In this section, we compare the new MATLAB function developed in this paper, `coshmtayher`, with the functions `coshher` and `funmcosh`:

- `coshmtayher`. New code based on the new developments of Hermites matrix polynomials (11).
- `coshher`. Code based on the Hermite series for the hyperbolic matrix cosine [5].
- `funmcosh`. MATLAB function *funm* for compute matrix functions, i. e. the hyperbolic matrix cosine.

The tests have been develop using MATLAB (R2017b), runing on an Apple Macintosh iMac 27" (iMac retina 5K 27" late 2015) with a quadcore INTEL i7-6700K 4 Ghz processor and 16 Gb of RAM.
The following sets of matrices have been used:

a) One hundred diagonalizable matrices of size 128×128. Table 3 show the percentage of cases in which the relative errors of `coshmtayher` (new Hermite code) are lower, greater or equal than the relative errors of `coshher`(Hermite code) and `funmcosh` (funm code). Table 4 shows the matrix products of each method. Graphics with the Normwise relative errors, see [7, p. 253] and Performance Profile, see [7, p. 254], are given in Figure 1.

b) One hundred non diagonalizables matrices of size 128×128 with multiple eigenvalues randomly generated. Table 5 shows the percentage of cases in which the relative errors of `coshmtayher` are lower, greater or equal than the relative errors of `coshher` and `funmcosh`. Table 6 shows the matrix products of each method. Graphics of the Normwise relative errors and the Performance Profile are given in Figure 2.

c) Ten matrices from the Eigtool MATLAB [8] package with size 128×128, and thirty matrices from the function `matrix` of the Matrix Computation Toolbox [9] with dimensions lower or equal than 128. These matrices have been chosen because they have more varied and significant characteristics. Table 7 shows the percentage of cases in which the relative errors of `coshmtayher` are lower, greater or equal than the relative errors of `coshher` and `funmcosh`. Table 8 shows the matrix products of each method. Graphics of the Normwise relative errors and the Performance Profile are given Figure 3.

$E(coshmtayher) < E(coshher)$	47.50%
$E(coshmtayher) > E(coshher)$	50.00%
$E(coshmtayher) = E(coshher)$	3.00%
$E(coshmtayher) < E(funmcosh)$	100.00%
$E(coshmtayher) > E(funmcosh)$	0.00%
$E(coshmtayher) = E(funmcosh)$	0.00%

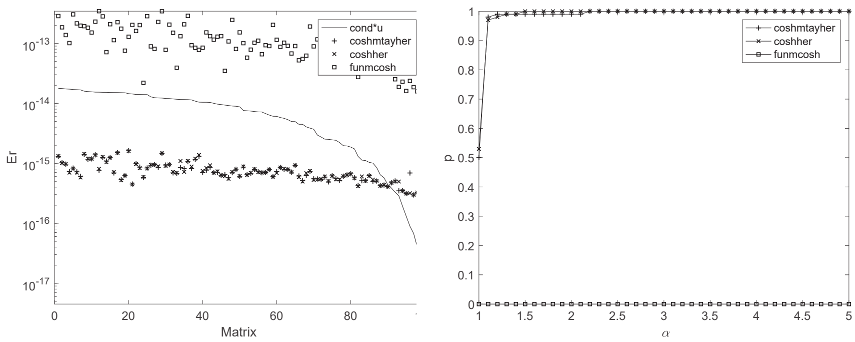
Table 3: Comparative between the methods

cosmtayher	*coshher*	*funmcosh*
671	973	1500

Table 4: Matrix products

$E(coshmtayher) < E(coshher)$	52.50%
$E(coshmtayher) > E(coshher)$	47.00%
$E(coshmtayher) = E(coshher)$	1.00%
$E(coshmtayher) < E(funmcosh)$	100.00%
$E(coshmtayher) > E(funmcosh)$	0.00%
$E(coshmtayher) = E(funmcosh)$	0.00%

Table 5: Comparative between the methods

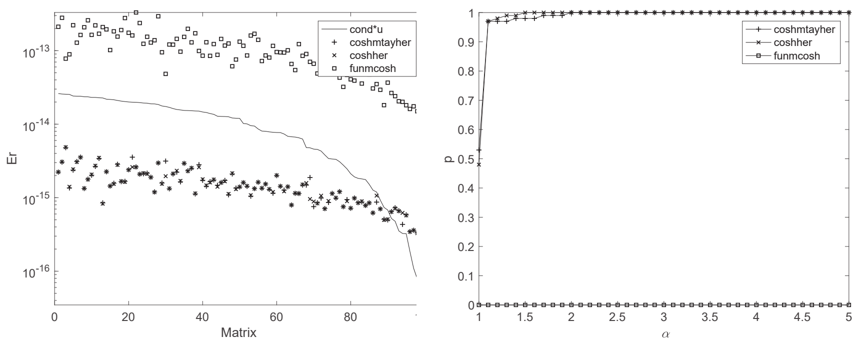


(a) Normwise relative errors (b) Performance Profile

Figure 1: Diagonalizable matrices

cosmtayher	*coshher*	*funmcosh*
685	989	1500

Table 6: Matrix products



(a) Normwise relative errors (b) Performance Profile

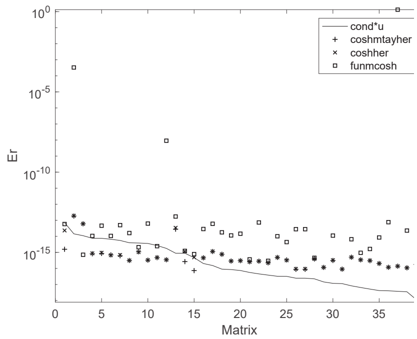
Figure 2: Non diagonalizable matrices

$E(coshmtayher) < E(coshher)$	57.50%
$E(coshmtayher) > E(coshher)$	30.00%
$E(coshmtayher) = E(coshher)$	12.50%
$E(coshmtayher) < E(funmcosh)$	97.50%
$E(coshmtayher) > E(funmcosh)$	2.50%
$E(coshmtayher) = E(funmcosh)$	0.00%

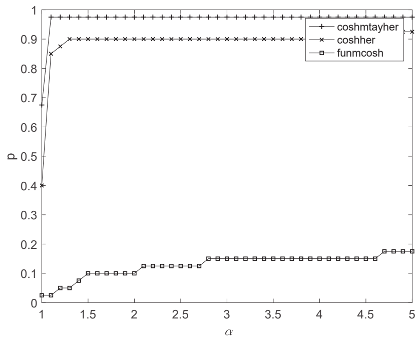
Table 7: Comparative between the methods

cosmtayher	*coshher*	*funmcosh*
191	315	600

Table 8: Matrix products



(a) Normwise relative errors



(b) Performance Profile

Figure 3: Matrices from the Eigtool and the Matrix Computation Toolbox packages

5 Conclusions

The more accurate are the implementations based on the Hermite series: the initial MATLAB implementation (`coshher`) and the proposed MATLAB implementation based on (11) (`coshmtayher`). Also, the new implementation (`coshmtayher`) have considerably lower computational costs than the other functions.

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