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Computing Hyperbolic Matrix Functions Using Orthogonal Matrix Polynomials

Emilio Defez, Jorge Sastre, Javier Ibáñez, and Pedro A. Ruiz

Abstract Hyperbolic matrix functions play a fundamental role in the exact solution of coupled partial differential systems of hyperbolic type. For the numerical solution of these problems, analytic-numerical approximations are most suitable obtained by using the hyperbolic matrix functions $\sinh(A)$ and $\cosh(A)$. It is well known that the computation of both functions can be reduced to the cosine of a matrix $\cos(A)$, which can be effectively calculated, with the disadvantage, however, to require complex arithmetic even though the matrix A is real. In this work we focus on approximate calculation of the hyperbolic matrix cosine $\cosh(A)$ using the truncation of a Hermite matrix polynomials series for $\cosh(A)$. The proposed approximation allows the efficient computation of this matrix function. An illustrative example is given.

E. Defez (✉)
Instituto de Matemática Multidisciplinar, Universitat Politècnica de València, Camino de Vera s/n, 46022 Valencia, Spain
e-mail: edefez@imm.upv.es

J. Sastre
Instituto de Telecomunicaciones y Aplicaciones Multimedia, Universitat Politècnica de València, Camino de Vera s/n, 46022 Valencia, Spain
e-mail: jorsasma@iteam.upv.es

J. Ibáñez • P.A. Ruiz
Instituto de Instrumentación para Imagen Molecular, Universitat Politècnica de València, Camino de Vera s/n, 46022 Valencia, Spain
e-mail: jjibanez@dsic.upv.es; pruiz@dsic.upv.es

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1 Introduction

Coupled partial differential systems are frequent in many different situations [1–5] and many other fields. Coupled hyperbolic systems appear in microwave heating processes [6] and optics [7] for instance. The exact solution of a class of this problems, see [8], is given in terms of matrix functions, in particular, of hyperbolic sine and cosine of a matrix, sinh(A), cosh(A), defined respectively by

$$\cosh(Ay) = \frac{e^{Ay} + e^{-Ay}}{2}, \quad \sinh(Ay) = \frac{e^{Ay} - e^{-Ay}}{2}. \tag{1}$$

For the numerical solution of these problems, analytic-numerical approximations are most suitable obtained by using the hyperbolic matrix functions sinh(A) and cosh(A), see [8]. It is well known that the computation of both functions can be reduced to the cosine of a matrix, because $\sinh(A) = i\cos(A - \frac{i\pi}{2}I)$ and $\cosh(A) = cos(iA)$. Thus, the matrix cosine can be effectively calculated, [9, 10], with the disadvantage, however, to require complex arithmetic even though the matrix A is real, which contributes substantially to the computational overhead. Direct calculation through exponential matrix using (1) is costly. In this paper, we apply Hermite matrix polynomials to approximate sinh(A) and cosh(A), providing sharper bounds for Hermite matrix polynomials and the approximation error. Throughout this paper, $[x]$ and $Re(z)$ denote the integer part of the real number x and the real part of a complex number z. For a matrix $A \in C^{r\times r}$, $\| A \|_2$ and $\sigma(A)$ denote the two-norm and the spectrum (the set of all the eigenvalues) of the matrix A, respectively, and I_r denotes the identity matrix of order r.

2 Hermite Matrix Polynomial Series Expansions of Matrix Hyperbolic Cosine

For the sake of clarity in the presentation of the following results we recall some properties of Hermite matrix polynomials which have been established in [9, 11, 12]. From (3.4) of [11], for an arbitrary matrix A in $C^{r\times r}$, the nth Hermite matrix polynomial satisfies

$$H_n\left(x, \frac{1}{2}A^2\right) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (xA)^{n-2k}}{k!(n-2k)!}, \tag{2}$$

and from its generating function in (3.1) and (3.2) [11] one gets

$$e^{txA-t^2I} = \sum_{n\geq 0} H_n\left(x, \frac{1}{2}A^2\right) t^n/n!, \quad x, t \in C, |t| < \infty, \tag{3}$$

Taking $y = tx$ and $\theta = 1/t$ in (3) it follows that

$$e^{Ay} = e^{\frac{1}{\theta^2}} \sum_{n\geq 0} \frac{1}{\theta^n n!} H_n\left(\theta y, \frac{1}{2}A^2\right), \quad (\theta, y) \in C^2, \quad A \in C^{r\times r}. \tag{4}$$

It is important to pay attention to the fact that the matrix A which defines the Hermite matrix polynomial sequence must be *positive definite*, see [12], i.e. $Re(z) > 0$ for all $z \in \sigma(A)$. This positive stable condition was imposed on the matrix A to guarantee the existence of $\sqrt{A}$ and some integral properties of Hermite polynomials, see [11], but it is not necessary to guarantee the expansion (4). Now, we will look for the Hermite matrix polynomials series expansion of the matrix hyperbolic cosine $\cos(Ax)$. To obtain it, given an arbitrary matrix $A \in C^{r\times r}$, by (1) using (4) and taking into account that, from [11], it follows that

$$H_n(-x, A) = (-1)^n H_n(x, A),$$

one gets the locking for expression:

$$\cosh(Ay) = e^{-\frac{1}{\lambda^2}} \sum_{n\geq 0} \frac{1}{\lambda^{2n}(2n)!} H_{2n}\left(y\lambda, \frac{1}{2}A^2\right). \tag{5}$$

Denoting by $CH_N(\lambda, A^2)$ the Nth partial sum of series (5) for $y = 1$, one gets the approximation

$$CH_N(\lambda, A^2) = e^{-\frac{1}{\lambda^2}} \sum_{n=0}^{N} \frac{1}{\lambda^{2n}(2n)!} H_{2n}\left(\lambda, \frac{1}{2}A^2\right) \approx \cosh(A), \ \lambda \in C. \tag{6}$$

From [10] we have the following bound $\left\|H_{2n}\left(x, \frac{1}{2}A^2\right)\right\|$ for Hermite matrix polynomials based on $||A^2||$:

$$\left\|H_{2n}\left(x, \frac{1}{2}A^2\right)\right\| \leq (2n)!\, e \cosh\left(x \left\|A^2\right\|^{\frac{1}{2}}\right), \ \forall x \in R, \ n \geq 0, \ \forall A \in C^{r\times r}. \tag{7}$$

Taking into account approximation (6) and bound (7), it follows that

$$\left\|\cosh(A) - CH_N(\lambda, A^2)\right\| \leq \frac{e^{1-\frac{1}{\lambda^2}} \cosh\left(\lambda \left\|A^2\right\|^{\frac{1}{2}}\right)}{(\lambda^2 - 1)\lambda^{2N}}. \tag{8}$$

A similar approximate expression (6) and error bound (8) can be found for $\sinh(A)$.

3 Example

Let A be the non-diagonalizable matrix defined by

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Using the minimal theorem the exact value of $\cosh(A)$ is

$$\cosh(A) = \begin{pmatrix} 7.389056098931 & -3.62686040784702 & 3.62686040784702 \\ 5.8459754641154 & -2.0837797730318 & 3.62686040784702 \\ 2.21911505626839 & -2.21911505626839 & 3.76219569108363 \end{pmatrix}.$$

Using (8), if $\lambda > 1$, for an admissible error $\varepsilon > 0$, we need choose a positive integer N so that the next inequality holds:

$$N \geq \frac{\log\left(\frac{e^{\left(1-\frac{1}{\lambda^2}\right)}\cosh\left(\lambda\left\|A^2\right\|^{\frac{1}{2}}\right)}{(\lambda^2-1)\,\epsilon}\right)}{2\log\lambda} \tag{9}$$

For example, if $\lambda = 1.8$ and $\varepsilon = 10^{-5}$ we need $N = 15$ to provide the required accuracy:

$$CH_{15}(1.8, A^2) = \begin{pmatrix} 7.3890560989307 & -3.62686040784702 & 3.62686040784702 \\ 5.8459754641154 & -2.08377977303177 & 3.62686040784702 \\ 2.21911505626839 & -2.21911505626839 & 3.76219569108363 \end{pmatrix},$$

and

$$\left\|\cosh(A) - CH_{15}(1.8, A^2)\right\|_2 = 1.85095 \times 10^{-15}.$$

In practice, the number of terms required to obtain a prefixed accuracy uses to be smaller than the one provided by (9). So for instance, taking the same $\lambda = 1.8$ and $N = 6$ one gets:

$$CH_6(1.8, A^2) = \begin{pmatrix} 7.3890548171477 & -3.6268592817884 & 3.6268592817884 \\ 5.84597418233707 & -2.08377864697777 & 3.6268592817884 \\ 2.21911490054867 & -2.21911490054867 & 3.76219553535930 \end{pmatrix},$$

and

$$\left\|\cosh(A) - CH_6(1.8, A^2)\right\|_2 = 2.90352 \times 10^{-6}.$$

The choice of parameter λ can still be refined. For example, taking $\lambda = 5$ and $N = 9$ one gets

$$\left\|\cosh(A) - CH_9(5, A^2)\right\|_2 = 3.07199 \times 10^{-14}.$$

Similar results are being obtained for $\sinh(A)$.

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