Applied Mathematical Modelling 40 (2016) 2837–2844

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Solving engineering models using hyperbolic matrix functions☆

Emilio Defez[a,*], Jorge Sastre[b], Javier Ibáñez[c], Jesús Peinado[c]

[a] Instituto de Matemática Multidisciplinar, Universitat Politècnica de València, Camino de Vera s/n, Valencia, 46022 Spain
[b] Instituto de Telecomunicaciones y Aplicaciones Multimedia, Universitat Politècnica de València, Camino de Vera s/n, Valencia, 46022 Spain
[c] Instituto de Instrumentación para Imagen Molecular, Universitat Politècnica de València, Camino de Vera s/n, Valencia, 46022 Spain

ARTICLE INFO

Article history:
Received 19 December 2013
Revised 31 July 2015
Accepted 23 September 2015
Available online 22 October 2015

Keywords:
Hermite matrix polynomial
Hyperbolic matrix functions
Series expansion

ABSTRACT

In this paper a method for computing hyperbolic matrix functions based on Hermite matrix polynomial expansions is outlined. Hermite series truncation together with Paterson–Stockmeyer method allow to compute the hyperbolic matrix cosine efficiently. A theoretical estimate for the optimal value of its parameters is obtained. An efficient and highly-accurate Hermite algorithm and a MATLAB implementation have been developed. The MATLAB implementation has been compared with the MATLAB function `funm` on matrices of different dimensions, obtaining lower execution time and higher accuracy in most cases. To do this we used an NVIDIA Tesla K20 GPGPU card, the CUDA environment and MATLAB. With this implementation we get much better performance for large scale problems.

1. Introduction

Coupled partial differential problems are frequently found in many different fields of technology and science: elastic and inelastic contact problems of solids, biochemistry, magnetohydrodynamic flows, cardiology, diffusion problems, etc., see for example [1–4] and references therein. In particular, coupled hyperbolic systems appear in microwave heating processes [5] and optics [6] for instance.

In [7], an exact solution of coupled hyperbolic systems of the form

$$\left.\begin{aligned} u_{tt}(x,t) - Au_{xx}(x,t) &= 0, 0 < x < 1\ ,\ t > 0,\\ u(0,t) + B_1 u_x(0,t) &= 0,\ t > 0,\\ A_2 u(1,t) + B_2 u_x(1,t) &= 0,\ t > 0,\\ u(x,0) &= f(x),\ 0 \le x \le 1,\\ u_t(x,0) &= g(x),\ 0 \le x \le 1, \end{aligned}\right\} \tag{1.1}$$

where A, B_1, A_2, B_2 are $r \times r$ complex matrices, and u, f, g are r-vector valued functions, was constructed in terms of a series which used hyperbolic cosine and sine of a matrix, respectively defined by

$$\cosh(Ay) = \frac{e^{Ay} + e^{-Ay}}{2}, \sinh(Ay) = \frac{e^{Ay} - e^{-Ay}}{2}, A \in \mathbb{C}^{r\times r}. \tag{1.2}$$

☆ This work has been supported by Spanish Ministerio de Educación TIN2014-59294-P.

* Corresponding author. Tel.: +34963877663.
E-mail addresses: edefez@imm.upv.es (E. Defez), jorsasma@iteam.upv.es (J. Sastre), jjibanez@dsic.upv.es (J. Ibáñez), jpeinado@dsic.upv.es (J. Peinado).

http://dx.doi.org/10.1016/j.apm.2015.09.050
S0307-904X(15)00584-3/

Functions of a matrix are used in many areas of science and arise in numerous applications in engineering [8]. In particular, matrix exponential e^A and matrix functions sine and cosine have been those that have received the most attention, see [9–14] for example, not only for its computational difficulties but also for its importance in solving differential systems of first and second order.

For the numerical solution of problem (1.1), analytic-numerical approximations are most suitably obtained by truncation of the exact series solution given in [7]. Thus, we need to compute approximations of both matrix functions hyperbolic cosine and sine with good accuracy and efficiency. It is well known that this computation can be reduced to the computation of the cosine of a matrix, due to the identities

$$\cosh(A) = \cos(iA), \quad \sinh(A) = i\cos\left(A - \frac{i\pi}{2}I\right),$$

but this approach has the disadvantage, however, to require complex arithmetic even though the matrix A is real, which is usually the case in applications. Furthermore, it is possible to reduce the computation of $\sinh(A)$ to the computation of $\cosh(A)$ by the relation

$$\sinh(A) = -i\cosh\left(-A - \frac{\pi}{2}iI\right), \tag{1.3}$$

but obviously, formula (1.3) also requires complex arithmetic although matrix A is real, which contributes substantially to the computational overhead. Direct calculation through the exponential matrix using (1.2) is also costly. A method based on Hermite series of the hyperbolic matrix cosine was recently presented in the *17th European Conference on Mathematics for Industry 2012* (Lund, Sweden) and also published in [15]. In this paper a bound of the error was given, however, an algorithm for its calculation was not discussed. In the *Mathematical Modelling in Engineering & Human Behaviour 2013* Conference(Valencia, Spain) [16] this same idea was applied to compute the sine and hyperbolic cosine of a matrix, finding different bounds, more accurate than those obtained in [15]. An algorithm based on the recurrence relation of three terms was developed, which allowed the simultaneous calculation of both functions. The proceedings of that conference were published as a book chapter [17] and the summaries were published in [16,18].

In this work, we propose an algorithm to compute both matrix functions, $\sinh(A)$ and $\cosh(A)$, simultaneously, avoiding complex arithmetic whenever possible, following a similar approach to that used in [14] for computing approximations of the matrix cosine. The proposed method uses Hermite matrix polynomial expansions of both matrix functions in order to provide a very accurate and competitive method for computing them. For this purpose, firstly we have to determinate error bounds of the approximation error of both functions using a series of Hermite matrix polynomials, bounds different from and more accurate than the bounds presented in [15,17]. Then, we have developed an algorithm based on Paterson–Stockmeyer method [19], instead of the algorithm based on the recurrence relation of three terms, in which the corresponding scaling of the matrix and the order of the approximation have been determined from the bound expression. To analyze the accuracy of the proposed method, implementations have been developed in MATLAB . We have also performed numerous tests to compare this algorithm with other state-of-the-art algorithms (`funm` MATLAB function). Finally, high performance implementations have been developed on large dimension matrix problems by using GPGPUs cards.

This work is organized as follows. Sections 2 and 3 summarize previous results of Hermite matrix polynomials and include a Hermite series expansion of the matrix hyperbolic cosine and sine with the respectively error bounds. An algorithm for the proposed method is given in Section 4. Section 5 deals with several numerical tests in order to investigate the accuracy of the proposed method and the method behavior for large scale problems. Both of them were done with MATLAB. When evaluating the performance of the method for large scale problems, we also did a GPGPU (General Purpose Graphics Processing Unit) implementation. The idea is to compare the CPU vs GPGPU implementations.

In this paper, $\lceil x \rceil$ rounds to the nearest integer greater than or equal to x. The matrices I_r and $\theta_{r\times r}$ in $\mathbb{C}^{r\times r}$ denote the matrix identity and the null matrix of order r, respectively. We will use subordinate matrix norms $\|A\|$, $A \in \mathbb{C}^{r\times r}$, and $\|A\|_1$ denotes the usual 1-norm. If $\mathcal{A}(k,n)$ is a matrix in $\mathbb{C}^{r\times r}$ for $n \geq 0$, $k \geq 0$, then [20]:

$$\sum_{n\geq 0}\sum_{k\geq 0}\mathcal{A}(k,n) = \sum_{n\geq 0}\sum_{k=0}^{n}\mathcal{A}(k,n-k). \tag{1.4}$$

2. Hermite matrix polynomial series expansions of hyperbolic matrix cosine: error bound

The following properties of Hermite matrix polynomials which have been established in [13,20,21] will be used in this paper. From (3.4) of [21, p. 25] the nth Hermite matrix polynomial is defined by

$$H_n\left(x, \frac{1}{2}A^2\right) = n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(xA)^{n-2k}}{k!\,(n-2k)!}, \tag{2.1}$$

for an arbitrary matrix A in $\mathbb{C}^{r\times r}$. From [13], we obtain

$$\cosh(Ay) = e^{\frac{1}{\lambda^2}}\sum_{n\geq 0}\frac{1}{\lambda^{2n}(2n)!}H_{2n}\left(y\lambda, \frac{1}{2}A^2\right), \ \lambda \in \mathbb{R}^+. \tag{2.2}$$

Denoting by $CH_N(\lambda, A^2)$ the Nth partial sum of series (2.2) for $y = 1$, one gets the approximation

$$CH_N(\lambda, A^2) = e^{\frac{1}{\lambda^2}} \sum_{n=0}^{N} \frac{1}{\lambda^{2n}(2n)!} H_{2n}\left(\lambda, \frac{1}{2}A^2\right) \approx \cosh(A),\ \lambda \in \mathbb{R}^+. \tag{2.3}$$

As in [22, pp. 1913], a bound for Hermite matrix polynomials $\left\|H_{2n}\left(x, \frac{1}{2}A^2\right)\right\|$ based on $||A^2||$ is obtained, see [23], using the Taylor series for the hyperbolic cosine $\cosh(y) = \sum_{n\geq 0} y^{2n}/(2n)!$. Taking norms in (2.1), one gets

$$\left\|H_{2n}\left(x, \frac{1}{2}A^2\right)\right\| \leq (2n)! \sum_{k=0}^{n} \frac{\left(\left\|A^2\right\|^{\frac{1}{2}}\right)^{2(n-k)}}{k!(2(n-k))!} |x|^{2(n-k)}. \tag{2.4}$$

By using (1.4), it follows that

$$\begin{aligned} e\cosh\left(|x|\left\|A^2\right\|^{\frac{1}{2}}\right) &= \sum_{n\geq 0} \frac{\left(|x|\left\|A^2\right\|^{\frac{1}{2}}\right)^{2n}}{(2n)!} \sum_{k\geq 0} \frac{1}{k!} = \sum_{n\geq 0}\sum_{k\geq 0} \frac{\left(|x|\left\|A^2\right\|^{\frac{1}{2}}\right)^{2n}}{(2n)!k!} \\ &= \sum_{n\geq 0}\sum_{k=0}^{n} \frac{\left(\left\|A^2\right\|^{\frac{1}{2}}\right)^{2(n-k)}}{k!(2(n-k))!} |x|^{2(n-k)}, \end{aligned} \tag{2.5}$$

and as a consequence

$$\sum_{k=0}^{n} \frac{\left(\left\|A^2\right\|^{\frac{1}{2}}\right)^{2(n-k)}}{k!(2(n-k))!} |x|^{2(n-k)} \leq e\cosh\left(|x|\left\|A^2\right\|^{\frac{1}{2}}\right). \tag{2.6}$$

Multiplying by $(2n)!$ in (2.6) and using (2.4), we have the result:

$$\left\|H_{2n}\left(x, \frac{1}{2}A^2\right)\right\| \leq (2n)!\, e\cosh\left(x\left\|A^2\right\|^{\frac{1}{2}}\right),\ \forall x \in \mathbb{R},\ n \geq 0,\ \forall A \in \mathbb{C}^{r\times r}. \tag{2.7}$$

Taking into account approximation (2.3) and bound (2.7), it follows that, for $\lambda \neq 0$:

$$\begin{aligned} \left\|\cosh(A) - CH_N(\lambda, A^2)\right\| &\leq e^{\frac{1}{\lambda^2}} \sum_{k\geq N+1} \frac{1}{\lambda^{2k}(2k)!} \left\|H_{2k}\left(\lambda, \frac{1}{2}A^2\right)\right\| \\ &\leq e^{\frac{1}{\lambda^2}+1} \cosh\left(\lambda\left\|A^2\right\|^{\frac{1}{2}}\right) \sum_{k\geq N+1} \frac{1}{\lambda^{2k}} \\ &= e^{\frac{1}{\lambda^2}+1} \cosh\left(\lambda\left\|A^2\right\|^{\frac{1}{2}}\right)\left[\sum_{k\geq 0} \frac{1}{\lambda^{2k}} - \sum_{k=0}^{N} \frac{1}{\lambda^{2k}}\right]. \end{aligned} \tag{2.8}$$

Simplifying (2.8), we obtain finally the bound:

$$\left\|\cosh(A) - CH_N(\lambda, A^2)\right\| \leq \frac{e^{\frac{1}{\lambda^2}+1} \cosh\left(\lambda\left\|A^2\right\|^{\frac{1}{2}}\right)}{(\lambda^2 - 1)\lambda^{2N}}. \tag{2.9}$$

3. Hermite matrix polynomial series expansions of hyperbolic matrix sine: error bound

Analogously, one gets a similar expression for

$$\sinh(Ay) = e^{\frac{1}{\lambda^2}} \sum_{n\geq 0} \frac{1}{\lambda^{2n+1}(2n+1)!} H_{2n+1}\left(y\lambda, \frac{1}{2}A^2\right),\ \lambda \in \mathbb{R}^+. \tag{3.1}$$

Denoting by $SH_N(A, \lambda)$ the Nth partial sum of series (3.1) for $y = 1$, one gets

$$SH_N(\lambda, A) = e^{\frac{1}{\lambda^2}} \sum_{n=0}^{N} \frac{1}{\lambda^{2n+1}(2n+1)!} H_{2n+1}\left(\lambda, \frac{1}{2}A^2\right) \approx \sinh(A),\ A \in \mathbb{C}^{r\times r}. \tag{3.2}$$

Now we can obtain a bound for Hermite matrix polynomials $\left\|H_{2n+1}\left(x,\frac{1}{2}A^2\right)\right\|$ based on $\|A^2\|$, see [23], using the Taylor series for the hyperbolic sine $\sinh(y)=\sum_{n\geq 0} y^{2n+1}/(2n+1)!$. Taking norms in (2.1), one gets

$$\left\|H_{2n+1}\left(x,\frac{1}{2}A^2\right)\right\| \leq (2n+1)!|x|\|A\|\sum_{k=0}^{n}\frac{\left(\left\|A^2\right\|^{\frac{1}{2}}\right)^{2(n-k)}}{k!(2(n-k)+1)!}|x|^{2(n-k)}. \tag{3.3}$$

By using (1.4), we obtain for $x\neq 0$:

$$\begin{aligned}\frac{e\sinh\left(|x|\left\|A^2\right\|^{\frac{1}{2}}\right)}{|x|\|A^2\|^{\frac{1}{2}}} &= \sum_{n\geq 0}\frac{\left(|x|\left\|A^2\right\|^{\frac{1}{2}}\right)^{2n}}{(2n+1)!}\sum_{k\geq 0}\frac{1}{k!} = \sum_{n\geq 0}\sum_{k\geq 0}\frac{\left(|x|\left\|A^2\right\|^{\frac{1}{2}}\right)^{2n}}{(2n+1)!k!}\\ &= \sum_{n\geq 0}\sum_{k=0}^{n}\frac{\left(\left\|A^2\right\|^{\frac{1}{2}}\right)^{2(n-k)}}{k!(2(n-k)+1)!}|x|^{2(n-k)},\end{aligned} \tag{3.4}$$

and

$$\sum_{k=0}^{n}\frac{\left(\left\|A^2\right\|^{\frac{1}{2}}\right)^{2(n-k)}}{k!(2(n-k)+1)!}|x|^{2(n-k)} \leq \frac{e\sinh\left(|x|\left\|A^2\right\|^{\frac{1}{2}}\right)}{|x|\|A^2\|^{\frac{1}{2}}}. \tag{3.5}$$

Multiplying by $(2n+1)!|x|\|A\|$ in (3.5) and using (3.3), we have the result:

$$\left\|H_{2n+1}\left(x,\frac{1}{2}A^2\right)\right\| \leq (2n+1)!\,\frac{\|A\|}{\|A^2\|^{\frac{1}{2}}}e\sinh\left(|x|\left\|A^2\right\|^{\frac{1}{2}}\right),\ \forall x\in\mathbb{R}\sim\{0\},\ n\geq 0,\ \forall A\in\mathbb{C}^{r\times r}. \tag{3.6}$$

Since $\sinh(0)=0$ and $H_{2n+1}\left(0,\frac{1}{2}A^2\right)=\theta_{r\times r}$, the bound (3.6) holds also when $x=0$.
Taking into account (3.2) and the bound (3.6), it follows that, for $\lambda\neq 0$:

$$\begin{aligned}\left\|\sinh(A)-SH_N(\lambda,A^2)\right\| &\leq e^{\frac{1}{\lambda^2}}\sum_{k\geq N+1}\frac{1}{\lambda^{2k+1}(2k+1)!}\left\|H_{2k+1}\left(\lambda,\frac{1}{2}A^2\right)\right\|\\ &\leq e^{\frac{1}{\lambda^2}+1}\frac{\|A\|}{\|A^2\|^{\frac{1}{2}}}\sinh\left(|\lambda|\left\|A^2\right\|^{\frac{1}{2}}\right)\sum_{k\geq N+1}\frac{1}{\lambda^{2k+1}}\\ &= e^{\frac{1}{\lambda^2}+1}\frac{\|A\|}{\lambda\|A^2\|^{\frac{1}{2}}}\sinh\left(|\lambda|\left\|A^2\right\|^{\frac{1}{2}}\right)\left[\sum_{k\geq 0}\frac{1}{\lambda^{2k}}-\sum_{k=0}^{N}\frac{1}{\lambda^{2k}}\right].\end{aligned} \tag{3.7}$$

Simplifying (3.7), we have the following bound:

$$\left\|\sinh(A)-SH_N(\lambda,A^2)\right\| \leq e^{\frac{1}{\lambda^2}+1}\frac{\|A\|}{\|A^2\|^{\frac{1}{2}}}\frac{\sinh\left(|\lambda|\left\|A^2\right\|^{\frac{1}{2}}\right)}{(\lambda^2-1)\lambda^{2N+1}}. \tag{3.8}$$

4. Hermite algorithm

In this section we describe an algorithm based on Hermite series for computing the matrix hyperbolic cosine.

Preprocessing and postprocessing can be used to reduce the norm of matrix A. An available technique for that purpose is based on balancing [8, p. 299]. This technique attempts to balance the norms of the kth row and kth column, for each k, by means of a diagonal similarity transformation defined by a non singular matrix V. If

$$\breve{A}=V^{-1}AV$$

is the obtained matrix in the preprocessing, then the postprocessing is as follows:

$$B=V\cosh(\tilde{B})V^{-1}.$$

For the evaluation of $CH_N(\lambda,\tilde{A}^2)$ the Horner and Paterson–Stockmeyer's method can be applied [19,24], obtaining previously with MATLAB the symbolic expression of $CH_N(\lambda,\tilde{A}^2)$ as a polynomial of matrix $\tilde{A}^2$ with degree N and coefficients depending on λ, for each considered value of N. The formula $\cosh(2A)=2\cosh^2(A)-I_r$ is used to recover $\cosh(\breve{A})$ from the matrix $\tilde{B}$ obtained in Step 4.

The optimal values of N for computing $CH_N(\lambda,\tilde{A}^2)$ by means of the Paterson–Stockmeyer method belong to $S_N=\{1,2,4,6,9,12,16,20,\ldots\}$[24, p. 6454]. If N_k is the order of Hermite approximation, where k is the position of N in S_N, then

Algorithm 1 (`herm`) Given a matrix $A \in \mathbb{C}^{r\times r}$ and the order N of Hermite matrix approximation of the hyperbolic cosine function, this algorithm computes $B \cong \cosh(A)$.

1: Preprocessing of matrix A: $\check{A}$.
2: Compute optimal values of λ and s.
3: SCALING PHASE: $\tilde{A} = \check{A}/2^s$
4: Compute $\tilde{B} = CH_N(\lambda, \tilde{A}^2)$, where CH_N is the Hermite approximation of the hyperbolic cosine function.
5: **for** $i = 1 : s$ **do**
6: $\quad \tilde{B} = 2\tilde{B}^2 - I$
7: **end for**
8: Postprocessing of matrix $\tilde{B}$: B.

Table 1
Tables of $\lambda_{\min}$ and Θ_N. (a) Optimal values of $\lambda_{\min}$. (b) Values of Θ_N from (4.2).

(a) $\lambda_{\min}$		(b) Θ_N	
N=1	28614.37029738510	N=1	$1.398832216450000\times10^{-4}$
N=2	1304.997141358828	N=2	$4.597769511080000\times10^{-3}$
N=4	110.4336118931741	N=4	$9.055511153551000\times10^{-2}$
N=6	38.32012920933002	N=6	$3.653432599794136\times10^{-1}$
N=9	17.32558067391524	N=9	1.154363749580479
N=12	11.29953801535487	N=12	2.300989971177028
N=16	8.081170359288837	N=16	4.207370311219608
N=20	6.566785645725286	N=20	6.395990872756508

the number of matrix product evaluations in Algorithm 1 is $k + s$. For computing $\cosh(A)$ with IEEE double precision arithmetic, we take the error in (2.9) to be lower than or equal to the unit roundoff in double precision floating-point $u = 2^{-53}$, i.e.,

$$\frac{e^{\frac{1}{\lambda^2}+1}\cosh\left(\lambda\left\|(\check{A}/2^s)^2\right\|^{\frac{1}{2}}\right)}{(\lambda^2-1)\lambda^{2N}} \leq u.$$

Then

$$s \geq \frac{1}{2}\log_2\left(\left\|\check{A}^2\right\|\right) + \log_2\left(\frac{\lambda}{\operatorname{arc\,cosh}\left(\frac{u(\lambda^2-1)\lambda^{2N}}{e^{1+1/\lambda^2}}\right)}\right). \tag{4.1}$$

We choose the parameter $\lambda = \lambda_{\min}$, where $\lambda_{\min}$ is the value such that the right-hand side of (4.1) reaches the minimum value. If we define

$$\Theta_N = \frac{1}{\lambda_{\min}}\operatorname{arc\,cosh}\left(\frac{u(\lambda_{\min}^2-1)\lambda_{\min}^{2N}}{e^{1+1/\lambda_{\min}^2}}\right), \tag{4.2}$$

then, using (4.1), it follows that

$$s = \left\lceil \log_2\left(\frac{\sqrt{||\check{A}^2||_1}}{\Theta_N}\right)\right\rceil. \tag{4.3}$$

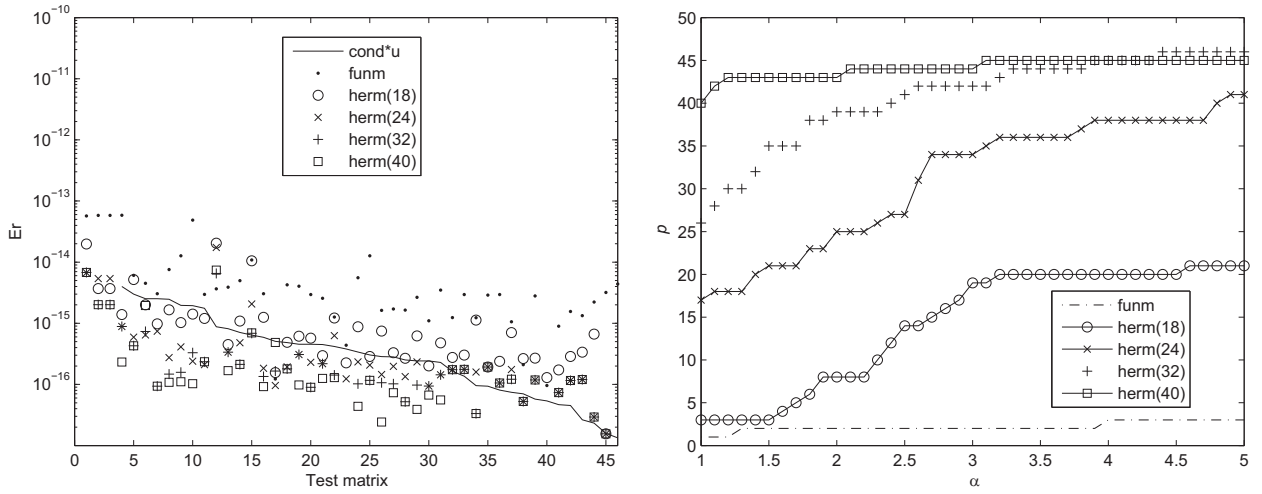
We choose N in {1, 2, 4, 6, 9, 12, 16, 20}, because the corresponding values of $\lambda_{\min}$ are negative for $N = 25, 30$. The values of $\lambda_{\min}$ are listed in Table 1. These values have been computed by using the symbolic functions `diff` and `solve` from the Symbolic Math Toolbox 5 of MATLAB. The values Θ_N of (4.2) are listed in Table 1.

There is no analogue of Algorithm 1 for computing the hyperbolic sine, because the corresponding formula $\sinh(2A) = 2\sinh(A)\cosh(A)$ requires to compute hyperbolic cosine. However, a similar algorithm to Algorithm 12.8 from [8, p. 298], which computes the sine and cosine of a matrix, can be developed to compute both the hyperbolic sine and hyperbolic cosine.

5. Numerical tests

5.1. MATLAB implementation

In this section a MATLAB implementation of Algorithm 1 has been compared with MATLAB function `funm`. We used an Apple Macintosh iMac (mid 2011) with a quadcore i5-2400S 2.5 GHz processor and 12 Gb of RAM. All the tests were carried out using MATLAB R2012a and OS X 10.6.8.



(a) Normwise relative errors (b) Performance profile

Fig. 1. Comparative for matrices of the Toolbox package.

5.1.1. First case study

In the first case study we have used 49 matrices from the Matrix Computation Toolbox [25] of dimension equal to 8. The "*exact*" value of $\cosh(A)$ has been calculated by using [32/32] diagonal Padé method with scaling and squaring with 64-digit precision in an iterative way: different increasing scalings starting from that provided in [26] for `expm` were used, until the norm of the relative difference between the approximations converted to IEEE double precision arithmetic was zero. The [32/32] diagonal Padé approximation was evaluated with a matrix power aggregation similar to that proposed in [26, p. 1183].

Fig. 1 a shows the normwise relative errors. This figure shows the relative errors of `funm` and `herm(m)`, where $m = 2N$ is the order of Hermite approximation, sorted in decreasing order, and a solid line that represents the unit roundoff multiplied by the relative condition number of the hyperbolic matrix cosine at X[8, p. 56]. For a method to perform in a backward and forward stable manner, its error should lie not far above this line on the graph [26, p. 1188]. Fig. 1a shows that all functions perform in a numerically stable way on tests.

Performance profiles [27] are presented in Fig. 1b. This figure shows the performances of the compared functions, where α varies between 1 and 5 in steps equal to 0.1, and p is the probability that the considered function has a relative error lower or equal than α-times the smallest error over all the methods. As shown in this figure, `herm` with maximum order $m = 40$ is the most accurate function, and it was achieved with very similar cost to `herm` with maximum orders m= 18, 24, 32 or 40 , i.e. N= 9, 12, 16 or 20. Hence, we consider $m = 40$ as the best choice of maximum order for `herm`.

5.1.2. Second case study

In the second case study we have used 100 diagonalizable matrices of dimension equal to 512. These matrices have been generated as $A = QDQ$, where $Q = H/\sqrt{512}$, with H a Hadamard matrix of dimension 512. Diagonal matrices D have been randomly generated, with 2-norms varying between 1 and 100. The hyperbolic cosine of A has been computed as $\cosh(A) = Q\cosh(D)Q$, using 64 digits of precision.

For all matrices, the relative error of the Hermite implementation was lower than the relative error of `funm` function. The total time for all the 100 executions for our implementation is 5.77 s and for `funm` 21.32 s.

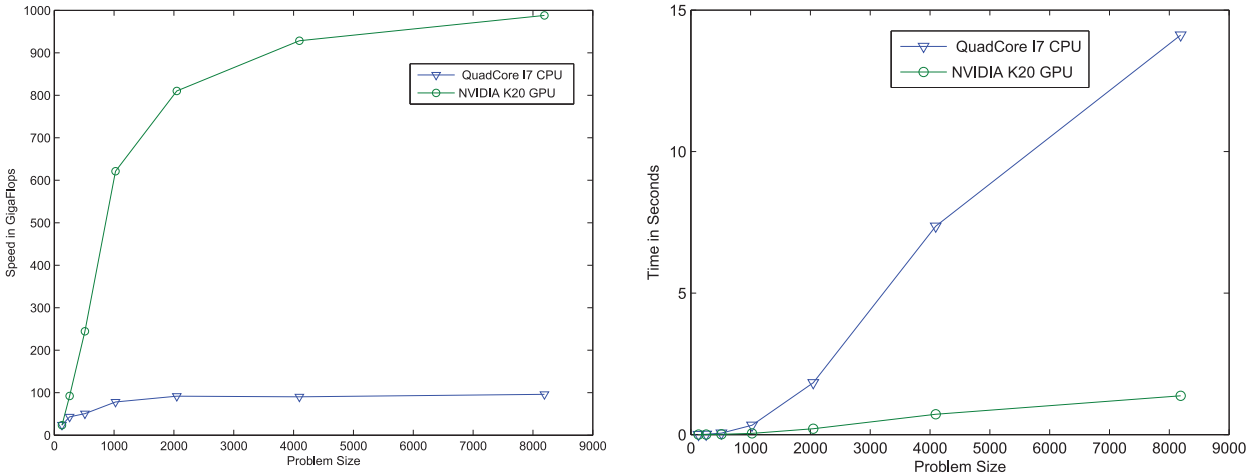
5.2. GPGPU implementation

Sometimes it is interesting to get the solution for large scale problems. In this case it is interesting to use techniques to improve the speed of the resolution of the problem as much as possible. One of these techniques consists of using one or several GPGPU (General Purpose Graphics Processing Unit) cards. We developed a parallel version of the former algorithm using an NVIDIA GPGPU card (from now on GPU) with the unified architecture and the CUDA [28,29] environment. To do this, a special version of basic linear algebra subprograms (BLAS) for GPUs, called CUBLAS [30] was used. Moreover, we developed a MATLAB © toolkit for GPGPUS [31,32] to use our implementation from MATLAB in such a way [33] that if the user has a graphic card, the performance of the implementation is improved. If the user does not have such a card, the algorithm can also be run using the machine CPU. Experimental results on a NVIDIA K20 (Kepler) latest (2013) [34] generation card was used. We compared the results with the code from MATLAB R2012b, using an INTEL QuadCore i7-3820 (3.6 GHz) CPU. All the results are shown in flops and seconds.

In our algorithm the most important computational cost is due to the number of square matrix products [35]. Using this, the computational cost of our problem in flops, can be approximated as the number of products times the cost of a product.

Table 2
Table comparing the number of Gflops and time (in seconds) in CPU vs GPU.

Size	Speedup	GflopsCPU	GflopsGPU	T.CPU	T.GPU
128	0.97	24.14	23.53	0.02781	0.002867
256	2.13	43.18	92.33	0.09168	0.004304
512	4.81	50.74	244.58	0.05063	0.010526
1024	7.93	78.25	621.20	0.33054	0.041680
2048	8.81	91.92	810.18	1.83277	0.208033
4096	10.26	90.41	928.48	7.36381	0.717720
8192	10.28	96.09	987.88	14.1168	1.373200



(a) Performance in GFLOPS (b) Time in seconds

Fig. 2. GPU (K20) vs CPU (i7 Quadcore) in GFLOPS (a) and seconds (b) when increasing the problem size.

In this test, we have used matrix dimensions equal to 2^n, $n = 7, 8, \ldots, 13$, comparing the GPU vs the CPU performance in flops and seconds. All the matrices were generated randomly changing the matrix 1-norm, from 1 to 100 (in steps of 10). Each test was repeated 3 times (Table 2). From Table 1 and Fig. 2a, we can see that GPU is much faster than CPU, inasmuch the GPU is a state of the art NVIDIA K20 while the CPU is an INTEL i7 CPU, not an INTEL XEON CPU. Anyway, our parallel implementation performance is good because it reaches almost the peak of the GPU card (1Tflop) [34]. Speedup refers to the number of times the parallel algorithm (using the GPU) is faster than the sequential algorithm (using the CPU).

6. Conclusions

An algorithm for computing the hyperbolic matrix cosine, based on a Hermite matrix polynomial expansion, and a new bound of the absolute error are presented. The numerical experiments show that the MATLAB implementation developed has lower execution times and higher accuracy than the MATLAB function `funm`. Also, we have developed a parallel algorithm, which uses a state of the art GPGPU, an NVIDIA K20 card, to see if the implementation based on the above algorithm could efficiently compute the hyperbolic matrix cosine. The results have been good because in all cases, the parallel version works much faster than the serial version. Even, we almost get the theoretical maximum performance for the K20 when running our implementation (1 Teraflop/s). This can be interesting for solving large scale problems.

References

[1] M. Sezgin, Magnetohydrodynamics flows in a rectangular duct, Int. J. Numer. Methods Fluids 7 (7) (1987) 697–718.
[2] A. King, C. Chou, Mathematical modeling simulation and experimental testing of biochemical systems crash response, J. Biomech. 9 (1976) 301–317.
[3] L. Jódar, E. Navarro, J. Martín, Exact and analytic-numerical solutions of strongly coupled mixed diffusion problems, Proc. Edinburgh Math. Soc. 43 (2000) 269–293.
[4] V. Soler, E. Defez, M.V. Ferrer, J. Camacho, On exact series solution of strongly coupled mixed parabolic problems, Abstr. Appl. Anal. (2013) Article ID 524514, 9 pages, doi:10.1155/2013/524514.
[5] D. Pozar, Microwave Engineering, Addison-Wesley, New York, 1991.
[6] P. Das, Optical Signal Processing, Springer, New York, 1991.
[7] L. Jódar, E. Navarro, A. Posso, M. Casabán, Constructive solution of strongly coupled continuous hyperbolic mixed problems, Appl. Numer. Math. 47 (3) (2003) 477–492.
[8] N.J. Higham, Functions of Matrices: Theory and Computation, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2008.
[9] J. Sastre, J. Ibáñez, P. Ruiz, E. Defez, Efficient computation of the matrix cosine, Appl. Math. Comput. 219 (2013) 7575–7585.
[10] G.I. Hargreaves, N.J. Higham, Efficient algorithms for the matrix cosine and sine, Numer. Algorithms 40 (2005) 383–400.

[11] A.H. Al-Mohy, N.J. Higham, A new scaling and squaring algorithm for the matrix exponential, SIAM J. Matrix Anal. Appl. 31 (3) (2009) 970–989.
[12] J. Sastre, J.J. Ibáñez, E. Defez, P.A. Ruiz, Accurate matrix exponential computation to solve coupled differential models in engineering, Math. Comput. Model. 54 (2011) 1835–1840.
[13] E. Defez, J. Sastre, J.J. Ibáñez, P.A. Ruiz, Computing matrix functions solving coupled differential models, Math. Comput. Model. 50 (5–6) (2009) 831–839.
[14] E. Defez, J. Sastre, J.J. Ibáñez, P.A. Ruiz, Computing matrix functions arising in engineering models with orthogonal matrix polynomials, Math. Comput. Model. 57 (7-8) (2013) 1738–1743.
[15] M. Fontes, M. Günther, N. Marheineke (Eds.), Progress in Industrial Mathematics at ECMI 2012, Mathematics in Industry, vol. 19, Springer-Verlag, Berlin/Heidelberg, 2014.
[16] E. Defez, J. Sastre, J.J. Ibáñez, J. Peinado, M. Tung, A method to approximate the hyperbolic sine of a matrix, Int. J. Complex Syst. Sci. 4 (1) (2014) 41–45.
[17] J.C.C. López, L. Jódar, R.J. Villanueva (Eds.), Mathematical Modelling Social Sciences and Engineering, Nova Science Publishers, Inc., New York, 2014.
[18] L. Jódar, L. Acedo, J.C. Cortés, M. Ehrhardt (Eds.), Modelling for Engineering, & Human Behavior 2013, Instituto Universitario de Matemática Multidisciplinar, UPV, Valencia, 2013.
[19] M.S. Paterson, L.J. Stockmeyer, On the number of nonscalar multiplications necessary to evaluate polynomials, SIAM J. Comput. 2 (1) (1973) 60–66.
[20] E. Defez, L. Jódar, Some applications of Hermite matrix polynomials series expansions, J. Comput. Appl. Math. 99 (1998) 105–117.
[21] L. Jódar, R. Company, Hermite matrix polynomials and second order matrix differential equations, J. Approximation Theory Appl. 12 (2) (1996) 20–30.
[22] E. Defez, M.M. Tung, J. Sastre, Improvement on the bound of hermite matrix polynomials, Linear Algebra Appl. 434 (2011) 1910–1919.
[23] G.I. Hargreaves, N.J. Higham, Efficient algorithms for the matrix cosine and sine, Numer. Algorithms 40 (2005) 383–400.
[24] J. Sastre, J.J. Ibáñez, E. Defez, P.A. Ruiz, Efficient orthogonal matrix polynomial based method for computing matrix exponential, Appl. Math. Comput. 217 (2011) 6451–6463.
[25] N.J. Higham, The test matrix toolbox for MATLAB: Numerical Analysis Report No. 237, Manchester, England, 1993.
[26] N.J. Higham, The scaling and squaring method for the matrix exponential revisited, SIAM J. Matrix Anal. Appl. 26 (4) (2005) 1179–1193.
[27] E.D. Dolan, J.J. Moré, Benchmarking optimization software with performance profiles, Math. Program. 91 (2002) 201–213.
[28] NVIDIA CUDA Compute Unified Device Architecture, NVIDIA, 2009.
[29] CUDA C Programming Guide, NVIDIA Corporation, 5.0 ed., 2012.
[30] CUDA. CUBLAS Library, NVIDIA, 2013.
[31] NVIDIA, Accelerating MATLAB with CUDA using MEX Files, 2007.
[32] MathWorks, Application Guidelines. Accelerating MATLAB Code using GPUS, 2010.
[33] J. Peinado, J. Ibáñez, V. Hernández, E. Arias, Speeding up solving of differential matrix Riccati equations using GPGPU computing and MATLAB, Concurrency Comput.: Pract. Exp. 24 (12) (2012) 1334–1348.
[34] NVIDIA, Tesla GPU High Performance Computing for Servers, 2013.
[35] G. Golub, C. Van Loan, Matrix Computations, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, 1996.