

# MODELLING FOR ENGINEERING AND HUMAN BEHAVIOUR 2018

Instituto Universitario de Matemática Multidisciplinar

**MATHEMATICAL  
MODELLING IN  
ENGINEERING  
& HUMAN  
BEHAVIOUR 2018**

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**L. Jódar, J. C. Cortés and L. Acedo( Editors )**

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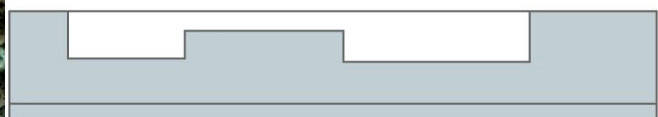
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Instituto de Matemática Multidisciplinar



**UNIVERSITAT  
POLITÈCNICA  
DE VALÈNCIA**

**Ciudad Politécnica de la Innovación**



# **MODELLING FOR ENGINEERING, & HUMAN BEHAVIOUR 2018**

Instituto Universitario de Matemática Multidisciplinar

Universitat Politècnica de València

Valencia 46022, SPAIN

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# Some new Hermite matrix polynomials series expansions and their applications in hyperbolic matrix sine and cosine approximation \*

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## 1 Introduction and notation

Hermite matrix polynomial  $H_n(x, A)$  has the generating function, see [1]:

$$e^{xt\sqrt{2A}} = e^{t^2} \sum_{n \geq 0} \frac{H_n(x, A)}{n!} t^n, \quad (1)$$

from following expressions for the matrix hyperbolic sine and cosine are derived:

$$\left. \begin{aligned} \cosh(xt\sqrt{2A}) &= e^{t^2} \sum_{n \geq 0} \frac{H_{2n}(x, A)}{(2n)!} t^{2n} \\ \sinh(xt\sqrt{2A}) &= e^{t^2} \sum_{n \geq 0} \frac{H_{2n+1}(x, A)}{(2n+1)!} t^{2n+1} \end{aligned} \right\}, \quad x \in \mathbb{R}, |t| < \infty. \quad (2)$$

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Recently we have shown the following formulas which are a generalization of formulas (2):

$$\left. \begin{aligned} \sum_{n \geq 0} \frac{H_{2n+1}(x, A)}{(2n)!} t^{2n} &= e^{-t^2} \left[ H_1(x, A) \cosh \left( xt\sqrt{2A} \right) - 2t \sinh \left( xt\sqrt{2A} \right) \right], \\ \sum_{n \geq 0} \frac{H_{2n+2}(x, A)}{(2n+1)!} t^{2n+1} &= e^{-t^2} \left[ H_1(x, A) \sinh \left( xt\sqrt{2A} \right) - 2t \cosh \left( xt\sqrt{2A} \right) \right], \\ \sum_{n \geq 0} \frac{H_{2n+3}(x, A)}{(2n+1)!} t^{2n+1} &= e^{-t^2} \left[ (H_2(x, A) + 4t^2 I) \sinh \left( xt\sqrt{2A} \right) - 4t H_1(x, A) \cosh \left( xt\sqrt{2A} \right) \right]. \end{aligned} \right\} \quad (3)$$

We will use formulas (3) to obtain a new expansion of the hyperbolic matrix sine and cosine in Hermite matrix polynomials series.

Throughout this paper, we denote by  $\mathbb{C}^{r \times r}$  the set of all the complex square matrices of size  $r$ . We denote by  $\Theta$  and  $I$ , respectively, the zero and the identity matrix in  $\mathbb{C}^{r \times r}$ . If  $A \in \mathbb{C}^{r \times r}$ , we denote by  $\sigma(A)$  the set of all the eigenvalues of  $A$ . For a real number  $x$ ,  $\lfloor x \rfloor$  denotes the lowest integer not less than  $x$  and  $\lceil x \rceil$  denotes the highest integer not exceeding  $x$ .

We recall that for a positive stable matrix  $A \in \mathbb{C}^{r \times r}$  the  $n$ -th Hermite matrix polynomial is defined in [1] by:

$$H_n(x, A) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k \left( \sqrt{2A} \right)^{n-2k}}{k!(n-2k)!} x^{n-2k}, \quad (4)$$

which satisfies the three-term matrix recurrence:

$$\left. \begin{aligned} H_m(x, A) &= x\sqrt{2A}H_{m-1}(x, A) - 2(m-1)H_{m-2}(x, A), \quad m \geq 1, \\ H_{-1}(x, A) &= \Theta, \quad H_0(x, A) = I. \end{aligned} \right\} \quad (5)$$

## 2 Some new Hermite matrix series expansions for the hyperbolic matrix cosine and sine

Let  $A \in \mathbb{C}^{r \times r}$  be a positive stable matrix, then the matrix polynomial  $H_1(x, A) = \sqrt{2A}x$  is invertible if  $x \neq 0$ . Substituting  $\sinh \left( xt\sqrt{2A} \right)$  given in (2) into the first expression of (3) we obtain the following new rational expression for the hyperbolic matrix cosine in terms of Hermite matrix polynomials:

$$\begin{aligned} \cosh \left( xt\sqrt{2A} \right) &= e^{t^2} \left( \sum_{n \geq 0} \frac{H_{2n+1}(x, A)}{(2n)!} \left( 1 + \frac{2t^2}{2n+1} \right) t^{2n} \right) [H_1(x, A)]^{-1}, \\ x \in \mathbb{R} \sim \{0\}, |t| < +\infty. \end{aligned} \quad (6)$$

Substituting  $\sinh \left( xt\sqrt{2A} \right)$  given in (2) into the second expression of (3) and using the three-term matrix recurrence (5) we obtain the expression of  $\cosh \left( xt\sqrt{2A} \right)$  given in (2).

On the other hand, replacing the expression of  $\sin \left( xt\sqrt{2A} \right)$  given in (2) into the third expression of (3), we obtain another new rational expression for the hyperbolic matrix cosine in terms of Hermite matrix polynomials:

$$\begin{aligned} \cosh \left( xt\sqrt{2A} \right) &= \\ &= \frac{-e^{t^2}}{4} \left[ \sum_{n \geq 0} \frac{H_{2n+3}(x, A)}{(2n+1)!} t^{2n} - (H_2(x, A) + 4t^2 I) \star \left( \sum_{n \geq 0} \frac{H_{2n+1}(x, A)}{(2n+1)!} t^{2n+1} \right) \right] [H_1(x, A)]^{-1}, \\ x \in \mathbb{R} \sim \{0\}, |t| < +\infty. \end{aligned} \quad (7)$$

Comparing (7) with (6), we observe that it always has a matrix product more when evaluating (7), the matrix product remarked by symbol “ $\star$ ” in (7). Due to the importance of reducing the number of matrix products, see [2–4] for more details, we will focus mainly on the expansion (6).

From (4), it follows that, for  $x \neq 0$ :

$$\begin{aligned} H_{2n+1}(x, A) [H_1(x, A)]^{-1} &= \frac{(2n+1)!}{x} \sum_{k=0}^n \frac{(-1)^k x^{2(n-k)+1} (2A)^{n-k}}{k!(2(n-k)+1)!} \\ &= \tilde{H}_{2n+1}(x, A), \end{aligned} \quad (8)$$

where

$$\tilde{H}_n(x, A) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k \left( \sqrt{2A} \right)^{n-2k-1}}{k!(n-2k)!} x^{n-2k}, \quad (9)$$

so the right side of (8) is still defined in the case where the matrix  $A$  is singular. In this way, we can re-write the relation (6) in terms of the matrix polynomial  $\tilde{H}_{2n+1}(x, A)$ :

$$\cosh(xt\sqrt{2A}) = e^{t^2} \left( \sum_{n \geq 0} \frac{\tilde{H}_{2n+1}(x, A)}{(2n)!} \left(1 + \frac{2t^2}{2n+1}\right) t^{2n} \right), \quad (10)$$

$x \in \mathbb{R}, |t| < +\infty.$

Replacing the matrix  $A$  by matrix  $A^2/2$  in (10) we can avoid the square roots of matrices, and taking  $x = \lambda, \lambda \neq 0, t = 1/\lambda$ , we finally obtain

$$\cosh(A) = e^{\frac{1}{\lambda^2}} \left( \sum_{n \geq 0} \frac{\tilde{H}_{2n+1}(\lambda, \frac{1}{2}A^2)}{(2n)! \lambda^{2n+1}} \left(1 + \frac{2}{(2n+1)\lambda^2}\right) \right), 0 < \lambda < +\infty. \quad (11)$$

### 3 Numerical approximations

Truncating the given series (11) until order  $m$ , we obtain the approximation  $CH_m(\lambda, A) \approx \cosh(A)$  defined by

$$CH_m(\lambda, A) = e^{\frac{1}{\lambda^2}} \left( \sum_{n=0}^m \frac{\tilde{H}_{2n+1}(\lambda, \frac{1}{2}A^2)}{(2n)! \lambda^{2n+1}} \left(1 + \frac{2}{(2n+1)\lambda^2}\right) \right), 0 < \lambda < +\infty. \quad (12)$$

Working analogously to the proof of the formula (3.6) of [5] one gets, for  $x \neq 0$  the following bound:

$$\left\| \tilde{H}_{2n+1} \left( x, \frac{1}{2}A^2 \right) \right\|_2 \leq (2n+1)! \frac{e \sinh \left( |x| \|A^2\|_2^{1/2} \right)}{|x| \|A^2\|_2^{1/2}}. \quad (13)$$

Then we can obtain the following expression for the approximation error:

$$\begin{aligned} \|\cosh(A) - CH_m(\lambda, A)\|_2 &\leq e^{\frac{1}{\lambda^2}} \sum_{n \geq m+1} \frac{\left\| \tilde{H}_{2n+1}(\lambda, \frac{1}{2}A^2) \right\|_2}{(2n)! \lambda^{2n+1}} \left(1 + \frac{2}{(2n+1)\lambda^2}\right) \\ &\leq \frac{e^{1+\frac{1}{\lambda^2}} \sinh \left( \lambda \|A^2\|_2^{1/2} \right)}{\lambda^2 \|A^2\|_2^{1/2}} \sum_{n \geq m+1} \frac{2n+1}{\lambda^{2n}} \left(1 + \frac{2}{(2n+1)\lambda^2}\right). \end{aligned} \quad (14)$$

Taking  $\lambda > 1$  it follows that  $\frac{2}{(2n+1)\lambda^2} < 1$ , and one gets

$$\begin{aligned} \sum_{n \geq m+1} \frac{2n+1}{\lambda^{2n}} \left(1 + \frac{2}{(2n+1)\lambda^2}\right) &\leq 2 \sum_{n \geq m+1} \frac{2n+1}{\lambda^{2n}} \\ &= \frac{4 + (4m+6)(\lambda^2 - 1)}{\lambda^{2m} (\lambda^2 - 1)^2}, \end{aligned}$$

$m$	$z_m$	$\lambda_m$
2	0.0020000000061361199	909.39256098888882
4	0.079956209874370632	99.997970988888895
6	0.34561400005673254	39.999499988888893
9	1.1120032200657	17.997896988889799
12	2.2373014291079998	11.882978988901458
16	4.1086396680000004	7.9999999964157498

Table 1: Values of  $z_m$  and  $\lambda_m$  for  $\cosh(A)$ .

	$m_1 = 2$	$m_2 = 4$	$m_3 = 6$	$m_4 = 9$	$m_5 = 12$	$m_6 = 16$
$\bar{m}_k$	1	2	3	5	7	11
$\tilde{m}_k$	1	2	4	10	13	17
$f_{m_k}(\max)$	0	0	$1.9 \cdot 10^{-17}$	$6.0 \cdot 10^{-19}$	$1.4 \cdot 10^{-26}$	$1.3 \cdot 10^{-35}$

Table 2: Values  $\bar{m}_k$ ,  $\tilde{m}_k$ , and  $f_{\max}$ .

thus from (14) we finally obtain:

$$\|\cosh(A) - CH_m(\lambda, A)\|_2 \leq \frac{e^{1+\frac{1}{\lambda^2}} \sinh\left(\lambda \|A^2\|_2^{1/2}\right) (4 + (4m+6)(\lambda^2 - 1))}{\|A^2\|_2^{1/2} \lambda^{2m+2} (\lambda^2 - 1)^2}. \quad (15)$$

From this expression (15) we derived the optimal values  $(\lambda_m; z_m)$  such that

$$z_m = \max \left\{ z = \|A^2\|_2; \frac{e^{1+\frac{1}{\lambda^2}} \sinh\left(\lambda z^{1/2}\right) (4 + (4m+6)(\lambda^2 - 1))}{z^{1/2} \lambda^{2m+2} (\lambda^2 - 1)^2} < u \right\}$$

where  $u$  is the unit roundoff in IEEE double precision arithmetic,  $u = 2^{-53}$ . The optimal values of  $m$ ,  $z$  and  $\lambda$  have been obtained with MATLAB. The results are given in the Table 1.

If  $\cosh(A)$  is calculated from the Taylor series, then the absolute forward error of the Hermite approximation of  $\cosh(A)$ , denoted by  $E_f$ , can be computed as

$$E_f = \|\cosh(A) - P_{m_k}(B)\| = \left\| \sum_{i \geq \bar{m}_k} f_{m_k, i} B^i \right\| \cong \left\| \sum_{i \geq \tilde{m}_k} f_{m_k, i} B^i \right\|,$$

where the values of  $\bar{m}_k$  and  $\tilde{m}_k$  for each  $m_k \in \{2, 4, 6, 9, 12, 16\}$  appear in the Table 2.

Scaling factor  $s$  and the order of Hermite approximation  $m_k$  are obtained by the following:

**Theorem 3.1** ([6]) Let  $h_l(x) = \sum_{i \geq l} p_i x^i$  be a power series with radius of convergence  $w$ ,  $\tilde{h}_l(x) = \sum_{i \geq l} |p_i| x^i$ ,  $B \in \mathbb{C}^{n \times n}$  with  $\rho(B) < w$ ,  $l \in \mathbb{N}$  and  $t \in \mathbb{N}$  with  $1 \leq t \leq l$ . If  $t_0$  is the multiple of  $t$  such that  $l \leq t_0 \leq l + t - 1$  and

$$\beta_t = \max\{d_j^{1/j} : j = t, l, l+1, \dots, t_0-1, t_0+1, t_0+2, \dots, l+t-1\},$$

where  $d_j$  is an upper bound for  $\|B^j\|$ ,  $d_j \geq \|B^j\|$ , then

$$\|h_l(B)\| \leq \tilde{h}_l(\beta_t).$$

We have empirically verified that by neglecting the coefficients whose absolute value is lower than  $u$ , the efficiency results are far superior to the state-of-the-art algorithms, with also excellent accuracy.

## 4 Numerical experiments

The MATLAB's implementation `coshmtayher` is a modification of the MATLAB's code `coshher` given in [5], replacing the original Hermite approximation `coshher` by the new Hermite matrix polynomial obtained from (11). In this section, we compare the new MATLAB function developed in this paper, `coshmtayher`, with the functions `coshher` and `funmcosh`:

- **`coshmtayher`**. New code based on the new developments of Hermites matrix polynomials (11).
- **`coshher`**. Code based on the Hermite series for the hyperbolic matrix cosine [5].
- **`funmcosh`**. MATLAB function `funm` for compute matrix functions, i. e. the hyperbolic matrix cosine.

The tests have been develop using MATLAB (R2017b), runing on an Apple Macintosh iMac 27" (iMac retina 5K 27" late 2015) with a quadcore INTEL i7-6700K 4 Ghz processor and 16 Gb of RAM.

The following sets of matrices have been used:

- a) One hundred diagonalizable matrices of size  $128 \times 128$ . Table 3 show the percentage of cases in which the relative errors of `coshmtayher` (new Hermite code) are lower, greater or equal than the relative errors of `coshher` (Hermite code) and `funmcosh` (funm code). Table 4 shows the matrix products of each method. Graphics with the Normwise relative errors, see [7, p. 253] and Performance Profile, see [7, p. 254], are given in Figure 1.
- b) One hundred non diagonalizables matrices of size  $128 \times 128$  with multiple eigenvalues randomly generated. Table 5 shows the percentage of cases in

which the relative errors of `coshmtayher` are lower, greater or equal than the relative errors of `coshher` and `funmcosh`. Table 6 shows the matrix products of each method. Graphics of the Normwise relative errors and the Performance Profile are given in Figure 2.

- c) Ten matrices from the Eigtool MATLAB [8] package with size  $128 \times 128$ , and thirty matrices from the function `matrix` of the Matrix Computation Toolbox [9] with dimensions lower or equal than 128. These matrices have been chosen because they have more varied and significant characteristics. Table 7 shows the percentage of cases in which the relative errors of `coshmtayher` are lower, greater or equal than the relative errors of `coshher` and `funmcosh`. Table 8 shows the matrix products of each method. Graphics of the Normwise relative errors and the Performance Profile are given Figure 3.

$E(\text{coshmtayher}) < E(\text{coshher})$	47.50%
$E(\text{coshmtayher}) > E(\text{coshher})$	50.00%
$E(\text{coshmtayher}) = E(\text{coshher})$	3.00%
$E(\text{coshmtayher}) < E(\text{funmcosh})$	100.00%
$E(\text{coshmtayher}) > E(\text{funmcosh})$	0.00%
$E(\text{coshmtayher}) = E(\text{funmcosh})$	0.00%

Table 3: Comparative between the methods

<i>cosmtayher</i>	<i>coshher</i>	<i>funmcosh</i>
671	973	1500

Table 4: Matrix products

$E(\text{coshmtayher}) < E(\text{coshher})$	52.50%
$E(\text{coshmtayher}) > E(\text{coshher})$	47.00%
$E(\text{coshmtayher}) = E(\text{coshher})$	1.00%
$E(\text{coshmtayher}) < E(\text{funmcosh})$	100.00%
$E(\text{coshmtayher}) > E(\text{funmcosh})$	0.00%
$E(\text{coshmtayher}) = E(\text{funmcosh})$	0.00%

Table 5: Comparative between the methods



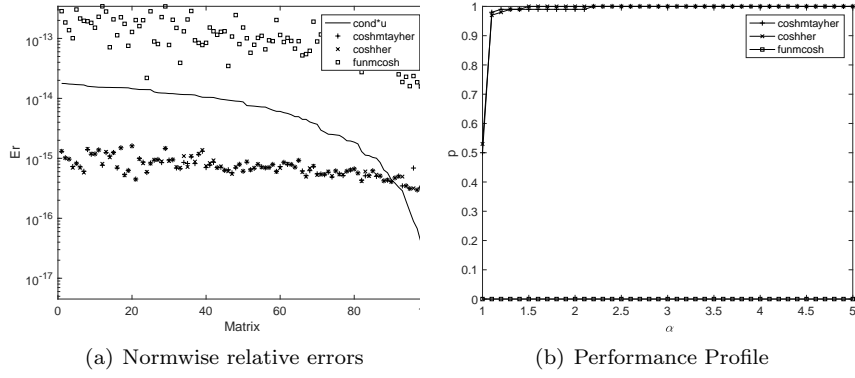


Figure 1: Diagonalizable matrices

<i>cosmtayher</i>	<i>coshher</i>	<i>funmcosh</i>
685	989	1500

Table 6: Matrix products

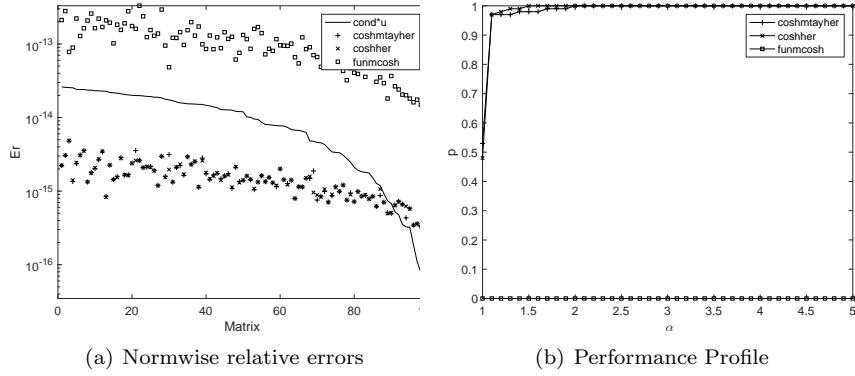


Figure 2: Non diagonalizable matrices

$E(\coshmtayher) < E(\coshher)$	57.50%
$E(\coshmtayher) > E(\coshher)$	30.00%
$E(\coshmtayher) = E(\coshher)$	12.50%
$E(\coshmtayher) < E(funmcosh)$	97.50%
$E(\coshmtayher) > E(funmcosh)$	2.50%
$E(\coshmtayher) = E(funmcosh)$	0.00%

Table 7: Comparative between the methods

<i>cosmtayher</i>	<i>coshher</i>	<i>funmcosh</i>
191	315	600

Table 8: Matrix products

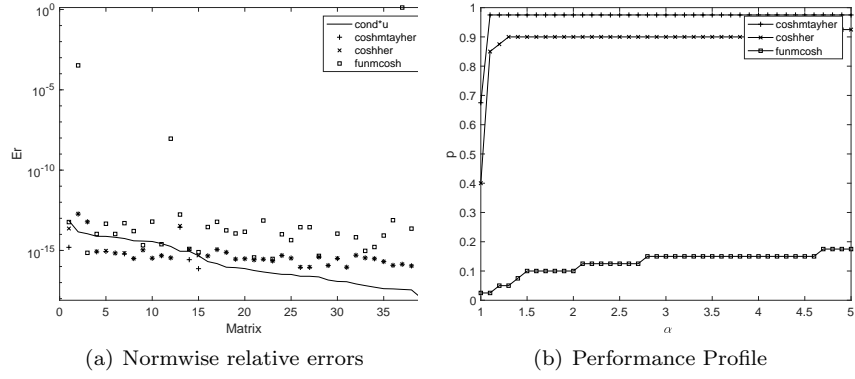


Figure 3: Matrices from the Eigtool and the Matrix Computation Toolbox packages

## 5 Conclusions

The more accurate are the implementations based on the Hermite series: the initial MATLAB implementation (*coshher*) and the proposed MATLAB implementation based on (11) (*cosmtayher*). Also, the new implementation (*cosmtayher*) have considerably lower computational costs than the other functions.

## References

- [1] J. Jódar, R. Company, Hermite matrix polynomials and second order matrix differential equations, *Approximation Theory and its Applications* 12 (2) (1996) 20–30.
- [2] J. Sastre, J. Ibáñez, E. Defez, P. Ruiz, New scaling-squaring Taylor algorithms for computing the matrix exponential, *SIAM Journal on Scientific Computing* 37 (1) (2015) A439–A455.
- [3] P. Alonso, J. Peinado, J. Ibáñez, J. Sastre, E. Defez, Computing matrix trigonometric functions with gpus through matlab, *The Journal of Supercomputing* (2018) 1–14.

- [4] J. Sastre, Efficient evaluation of matrix polynomials, *Linear Algebra and its Applications* 539 (2018) 229–250.
- [5] E. Defez, J. Sastre, J. Ibáñez, J. Peinado, Solving engineering models using hyperbolic matrix functions, *Applied Mathematical Modelling* 40 (4) (2016) 2837–2844.
- [6] J. Sastre, J. Ibáñez, P. Ruiz, E. Defez, Efficient computation of the matrix cosine, *Applied Mathematics and Computation* 219 (14) (2013) 7575–7585.
- [7] N. J. Higham, *Functions of Matrices: Theory and Computation*, SIAM, Philadelphia, PA, USA, 2008.
- [8] T. Wright, Eigtool, version 2.1, URL: [web.comlab.ox.ac.uk/pseudospectra/eigtool](http://web.comlab.ox.ac.uk/pseudospectra/eigtool).
- [9] N. J. Higham, *The test matrix toolbox for MATLAB (Version 3.0)*, University of Manchester Manchester, 1995.