Foreword. The authors consider the adventure combat model only, and discard the expedition combat model as trivial and uninteresting.

1 Notation

For a unit, u say, let $\mathsf{He}(u)$ and $\mathsf{Red}(u)$ denote its current hit points and damage reduction; let Acc_u, Spl_u, $\overline{\mathsf{Dm}}_u$, and $\underline{\mathsf{Dm}}_u$ denote its accuracy, splash chance, maximum, and minimum damage. We will refer to the damage before reduction is taken into account as *pure* damage. Furthermore, define a *unit group*, or simply *group*, as a collection of a certain number of units, $\{u_1, \cdots, u_n\}$, sharing the *same unit type*. An *army* is an ordered collection $(v_1, \cdots, v_n)$ of units of possibly different types. If G is a group, $\mathcal{A}$ is an army, v is a unit, h is an integer, we write

$$\binom{G}{\mathcal{A}}; \quad \binom{G}{v}; \quad \binom{G}{h}$$

to denote the (combat) pairing of G versus $\mathcal{A}$; G vs. a singleton group $\{v\}$; or G vs. a unit with h hit points, respectively. If G_1, G_2 are groups, and $\mathcal{A}_1, \mathcal{A}_2$ are armies, we write

$$\binom{G_1}{\mathcal{A}_1} \bigsqcup \binom{G_2}{\mathcal{A}_2}; \quad \binom{G_1}{\mathcal{A}_1} \bigcup \binom{G_2}{\mathcal{A}_2}$$

to denote the pairing $(G_1 \cup G_2)$ vs. $(\mathcal{A}_1 \cup \mathcal{A}_2)$ when the corresponding sub-pairings can or cannot (resp.) be done independently.

2 Assumptions

The standing assumptions throughout this work are as follows:

1. Units attack sequentially, one at a time.
2. Hit points are always non-negative integer-valued.
3. Damage is always non-negative integer-valued.

Example 1. *Three units, u_1, u_2, and u_2, attack v_1. Suppose v_1 has 21 hit points, and damage reduction is 0.4. The first two attackers deal pure damage of 13 each, the last attacker deals pure damage of 9. The inflicted damage from u_1 and u_2 will be $\lfloor 13 \cdot (1 - 0.4) \rfloor = 7$ each. The inflicted damage from u_3 will be $\lfloor 9 \cdot (1 - 0.4) \rfloor = 5$, respectively. Thus the total damage will be $(7 + 7 + 5) = 19$, and v_1 will survive with $(21 - 19) = 2$ hit point remaining.*

Note that if one dropped the assumption that units attacked sequentially, the inflicted damage would be

$$\lfloor (13 + 13 + 9) \cdot (1 - 0.4) \rfloor = 21$$

instead, and v_1 would be killed.

3 Combat Mechanics

In what follows, we consider a group $G = \{u_1, u_2, \cdots, u_n\}$ attacking an army $\mathcal{A} = (v_1, v_2, \cdots, v_m)$. The enemy units are attacked in that order: v_1 first, while it is alive, then v_2, etc. Let d_i denote the pure damage dealt by u_i (if any), $1 \leqslant i \leqslant n$, and $h_j = \mathsf{He}(v_j)$, $r_j = \mathsf{Red}(v_j)$, $1 \leqslant j \leqslant m$. *Each* unit in the attacking group, $u_i, 1 \leqslant i \leqslant n$, can deal *potential* damage

$$\begin{aligned} &\overline{\mathsf{Dm}}_u \quad \text{with probability } \mathsf{Acc}_u, \\ &\underline{\mathsf{Dm}}_u \quad \text{with probability } 1 - \mathsf{Acc}_u. \end{aligned}$$

The potential damage dealt by i-th unit can be written as $\overline{\mathsf{Dm}}_u \cdot B_i + \underline{\mathsf{Dm}}_u \cdot (1 - B_i)$, where B_i, $1 \leqslant i \leqslant n$ are i.i.d. Bernoulli random variables with probability of success Acc_u. The *actual* damage d_i might be different; to that end there are two distinct attacking patterns: splash damage, and truncated damage.

Consider the following example: u_1 attacks v_1 with 100 damage, and v_1 has only 20 hit points left; in either case v_1 will be killed. If this was a *splash damage* attack, the remaining 80 damage will be inflicted onto v_2. If this was a *truncated damage* attack, the remaining 80 damage will be lost (discarded).

When attacking, each unit will deal splash damage with probability Spl_u; with probability $(1 - \mathsf{Spl}_u)$ it will be truncated damage. We will now consider two cases: (i) all units in the group deal splash damage; and (ii) the general case.

Uniform splash damage. This is the simplest case: the excess damage is always spread over consecutive units sharing the same damage reduction. We will start with an example emphasizing the importance of uniform damage reduction.

Example 2. *Two defending units: first with* 2 *hit points and damage reduction of* 0.5*, second with* 6 *hit points and no damage reduction, are attacked by a group with* $\overline{\mathsf{Dm}} = 5$*,* $\underline{\mathsf{Dm}} = 3$*, and* $\mathsf{Acc} = 1$*. To demonstrate that the order of attack matters, consider two scenarios: the attacker deals (i) high, low, low; and (ii) low, low, high .*

In the first case the defender's hit points will be $(2, 6) \to (0, 6) \to (0, 3) \to (0, 0)$.

In the second case the defender's hit points will be $(2, 6) \to (1, 6) \to (0, 6) \to (0, 1)$.

In the pure splash damage scenario with uniform damage reduction r, the actual damage dealt is the same as reduced potential damage, and it suffices to consider the total reduced damage dealt by the attacking group versus the total hit points of the defending army:

$$d = \sum_{i=1}^{n} d_i = \left\lfloor (1-r)\,\overline{\mathsf{Dm}}_u \right\rfloor \cdot \left(\sum_{i=1}^{n} B_i \right) + \left\lfloor (1-r)\,\underline{\mathsf{Dm}}_u \right\rfloor \cdot \left(n - \sum_{i=1}^{n} B_i \right) \qquad \text{vs.} \qquad h = \sum_{j=1}^{m} h_i .$$

Put otherwise, $d = \left\lfloor (1-r)\,\overline{\mathsf{Dm}}_u \right\rfloor \cdot X + \left\lfloor (1-r)\,\underline{\mathsf{Dm}}_u \right\rfloor \cdot (n - X)$, where X is a binomial random variable with parameters n and Acc_u, since B_i are i.i.d.: $X \sim \mathrm{Bin}(n, \mathsf{Acc}_u)$.

Mixture damage. This is the general case: the excess damage is lost with probability Spl_u. It implies that the actual damage dealt by two consecutive attacking units need not be independent.

On the other hand, if the defending unit has h hit points, it will take at least $\lceil h / \overline{\mathsf{Dm}}_u \rceil$ units to eliminate it. The course of attack is outlined in the following algorithm.

Algorithm 1 (conservative)**.** *Consider a group* $G = \{u_1, u_2, \cdots, u_n\}$ *attacking an army* $\mathcal{A} = (v_1, v_2, \cdots, v_m)$*. Set* $i = 1$ *and* $j = 1$*, the (one-based) index of the attacking and defending units, respectively. Let* $h = \mathsf{He}(v_1)$ *and* $r = \mathsf{Red}(v_1)$ *be the health and damage reduction of the first defending unit.*

1. *Let* $\overline{\mathsf{Dm}}_u^* = \left\lfloor (1-r)\,\overline{\mathsf{Dm}}_u \right\rfloor$ *and* $\underline{\mathsf{Dm}}_u^* = \left\lfloor (1-r)\,\underline{\mathsf{Dm}}_u \right\rfloor$ *be the effective high and low damage, and* $h^* = \lceil h/(1-r) \rceil$ *be the pure damage required to kill the current defending unit.*
2. *Let* $a = \left\lceil h / \overline{\mathsf{Dm}}_u^* \right\rceil$ *be the least number of units required to kill the current defending unit.*
3. *If* a *is greater than* $(n - i + 1)$*, the remaining units in the attacking group, update* $a \leftarrow (n - i + 1)$.
4. *Generate* $X \sim \mathrm{Bin}\,(a, \mathsf{Acc}_u)$*, the number of units (out of the first* a*) dealing high damage.*
5. *Let* $d = \overline{\mathsf{Dm}}_u \cdot X + \underline{\mathsf{Dm}}_u \cdot (a - X)$ *and* $d^* = \overline{\mathsf{Dm}}_u^* \cdot X + \underline{\mathsf{Dm}}_u^* \cdot (a - X)$ *be the pure and effective damage, respectively.*
6. *Update* $i \leftarrow i + a$ *(point to the next attacking unit).*
7. *Update* $h \leftarrow h - d^*$ *(damage the current defending unit), and* $d \leftarrow d - h^*$ *(potential overshoot pure damage).*
8. *If* $h < 0$ *(the current defending unit has been killed and there is still leftover damage), generate* $Y \sim \mathrm{Bin}\,(1, \mathsf{Spl}_u)$*, the chance of splash damage (note that it is not necessary if* $h = 0$*). Otherwise set* $Y = 0$.
9. *If* $Y = 0$ *(no splash), update* $d \leftarrow 0$ *(no overshoot damage).*
10. *While* $h \leqslant 0$*:*
    (a) *update* $j \leftarrow j + 1$ *(proceed to the next defending unit);*
    (b) *if* $j = m + 1$*, terminate: the entire army* $\mathcal{A}$ *has been eliminated;*
    (c) *update* $h \leftarrow \mathsf{He}(v_j)$*,* $r \leftarrow \mathsf{Red}(v_j)$*;*
    (d) *update* $d^* \leftarrow \lfloor d \cdot (1-r) \rfloor$ *(effective damage) and* $h^* \leftarrow \lceil h/(1-r) \rceil$ *(pure damage required to kill the next defending unit);*
    (e) *update* $h \leftarrow h - d^*$ *(damage the next defending unit), and* $d \leftarrow d - h^*$ *(overshoot pure damage).*
11. *If* $i = n + 1$*, terminate: the entire group* G *has attacked, and exactly* $(j - 1)$ *units from* $\mathcal{A}$ *have been killed; the* j*-th unit in* $\mathcal{A}$ *has only* h *hit points left.*
12. *Otherwise repeat steps 1–11.*

4 Special Abilities and Battle Structure

In this section we will discuss all battle or unit characteristics, modifiers, and abilities, that affect the normal course of the battle.

The key assumptions of the battle mechanics (as implemented by yours humbly) are as follows: damage can be fractional (or irrational even), hit points are always integer-valued. To allow for that, damage is always rounded *up*. For example, when a group of units is hit with 31.416 damage, the outcome would be exactly the same as if it was hit with 32 damage.

Before the battle starts, there are two stages. First, each army's bonuses are applied. Next, each army's penalties are applied.

1. Pre-battle bonus stage.
   (a) Explosive ammunition: every friendly unit in *ranged* category gains *attack weakest target* ability. Note that this will be ignored if the enemy army has *intercept*.
   (b) Accuracy bonus (additive): a given category of friendly unit's accuracy is increased by this amount. If the unmodified accuracy is a and the accuracy bonus is x, the adjusted accuracy would be $(a + x)$, clipped to $[0.0, 1.0]$.
   (c) Splash bonus (additive): a given category of friendly unit's splash chance is increased by this amount. If the unmodified splash chance is s and the accuracy bonus is x, the adjusted splash chance would be $(s + x)$, clipped to $[0.0, 1.0]$.
   (d) Minimum/maximum damage bonus (additive): a given category of friendly unit's minimum/maximum damage is increased by this amount. If the unmodified minimum/maximum damage is d and the damage bonus is x, the adjusted damage would be $(d + x)$.
2. Pre-battle penalty stage.
   (a) Boss health reduction (multiplicative): every enemy unit with *boss* special ability will have a health penalty. If a boss unit has h hit points and the boss health reduction is x, its actual hit points would become $\lceil h \cdot (1 - x) \rceil$.
   (b) Dazzle: every enemy unit's accuracy is set to 0.0.
   (c) Intercept: every enemy unit is stripped of *attack weakest target* ability.

Some modifiers are applied during the battle. Suppose the current round is $n \geqslant 1$.

1. Direct damage reduction (multiplicative): every enemy unit's damage is reduced by this amount. If a unit has d damage and the direct damage reduction is x, the actual damage would become $\lceil d \cdot (1 - x) \rceil$.
2. Tower damage reduction (multiplicative). If a friendly unit with *tower bonus* special ability were to be hit with d and the tower damage reduction is x, the actual inflicted damage would become $\lceil d \cdot (1 - x) \rceil$. Ignored if the attacking enemy unit has *ignore tower bonus* ability.
3. Frenzy bonus (multiplicative): every friendly unit's damage is multiplied by this amount. If the unmodified damage is d and the frenzy bonus is x, the adjusted damage would be $\left\lceil d \cdot (1 + x)^{(n-1)} \right\rceil$.

5 Timing

Consider a battle between two armies. Let T_f, T_d, T, and S be random variables

$$\begin{aligned} T_f &= \{\text{fighting time in rounds}\}, \quad T_d = \{\text{destruction time in rounds}\}, \\ T &= \{\text{total time in rounds}\} = T_f + T_d, \\ S &= \{\text{loser's camp and surviving unit types}\}. \end{aligned}$$

Note that $T_d \perp\!\!\!\perp T_f \,|\, S$, and write $T_d^s \triangleq (T_d \,|\, S = s)$.

The result of fighting phase is the pair (T_f, S). Given s, the observed value of S, the result of destruction phase is T_d^s. Ultimately, we are interested in distributions of T_f, T_d, and T.

It is worth noting that destruction time, T_s, can be written as a stopping time. Let m denote the number of surviving unit types, and write $s = (c, \{u_i\}_{1 \leqslant i \leqslant m})$. Let h denote the number of hit points of the loser's camp, and p_i, $\overline{d_i}$, and $\underline{d_i}$ denote the accuracy, maximum, and minimum destruction damage of u_i. Let

$$D_t^s = \sum_{i=1}^{m} \left(\overline{d_i} B_i^t + \underline{d_i} (1 - B_i^t) \right), \quad t \geqslant 1,$$

be the damage per round inflicted by the survivors, where $\{B_i^t\}_{1\leqslant i\leqslant m, t\geqslant 1}$ are independent Bernoulli random variables, $B_i^t \sim \mathrm{Bernoulli}(p_i)$. Then

$$T_t^s = \inf\left\{t \geqslant 0\,:\, \sum_{k=1}^{t} D_k^s \geqslant h\right\}.$$

6 Monte-Carlo simulations

Let n and n_s denote the number of fighting phase simulations and destruction phase simulations, respectively. Then after simulating the fighting phase one can approximate

$$\begin{aligned}
&\mathbf{P}(T_f = r, S = s) \approx \left(\sum_{i=1}^{n} \mathbb{1}\{(t_{f,i}, s_i) = (r,s)\}\right)\Big/ n, \quad \mathbf{P}(S = s) \approx \left(\sum_{i=1}^{n} \mathbb{1}\{(t_{f,i}, s_i) = (\cdot,s)\}\right)\Big/ n,\\
&\mathbf{P}(T_f = t) \approx \left(\sum_{i=1}^{n} \mathbb{1}\{(t_{f,i}, s_i) = (t,\cdot)\}\right)\Big/ n. \qquad (6.1)
\end{aligned}$$

After simulating the destruction phase for every observed value of s, one gets

$$\begin{aligned}
\mathbf{P}(T_d^s = t) &= \mathbf{P}(T_d = t \,|\, S = s) \approx \left(\sum_{i=1}^{n_s} \mathbb{1}\{t_{d,i}^s = t\}\right)\Big/ n_s,\\
\mathbf{P}(T_d = t) &= \sum_{s\in\mathcal{S}} \mathbf{P}(T_d = t \,|\, S = s)\, \mathbf{P}(S = s) = \sum_{s\in\mathcal{S}} \mathbf{P}(T_d^s = t)\, \mathbf{P}(S = s), \qquad (6.2)\\
\mathbf{P}(T = t) &= \sum_{r=0}^{t} \mathbf{P}(T_f = r, T_d = t - r) = \sum_{s\in\mathcal{S}} \sum_{r=0}^{t} \mathbf{P}(T_f = r, T_d = t-r \,|\, S = s)\, \mathbf{P}(S = s)\\
&= \sum_{s\in\mathcal{S}} \sum_{r=0}^{t} \mathbf{P}(T_f = r \,|\, S = s)\, \mathbf{P}(T_d = t-r \,|\, S = s)\, \mathbf{P}(S = s)\\
&= \sum_{s\in\mathcal{S}} \sum_{r=0}^{t} \mathbf{P}(T_f = r, S = s)\, \mathbf{P}(T_d^s = t - r) \qquad (6.3)
\end{aligned}$$