2. Quadratic from Positivity [10 pts]

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You're presented with the constraint:

$$2x^2 + 2y^2 + 9z^2 + 8xy - 6xz - 6yz \leq 1 \quad \textbf{(1)}$$

a) It turns out the above constraint is not convex. In other words, the set of (x, y, z) satisfying the constraint (1) is not an ellipsoid. Explain why this is the case.

In [4]:
```
# Symmetric equivalent matrix
V = [2  4 -3
     4  2 -3
    -3 -3  9]

val, vec = eig(V)

println(val)
println(vec)
```

```
[-2.0,3.0,12.0]
[0.707107 -0.57735 -0.408248; -0.707107 -0.57735 -0.408248; 0.0 -0.57735 0.816497]
```

By eigenvalue decomposition and some manipulation, we get:

$$2x^2 + 2y^2 + 9z^2 + 8xy - 6xz - 6yz = -2p^2 + 3q^2 + 12r^2$$

Which means it can be written as:

$$-2p^2 + 3q^2 + 12r^2 \leq 1$$

which a nonconvex quadratic constraint, simply because of the negative eigenvalue, which makes this matrix **non-PSD**.

b) Show that the following QCQP is unbounded:

$$\begin{array}{ll} \text{maximize} & x^2 + y^2 + z^2 \\ \text{subject to} & (1) \end{array}$$

Hint: this is not a convex QCQP because as seen above, (1) is not convex. Moreover, the objective is not convex because it involves *maximizing* a positive definite quadratic. So do not attempt to solve this using JuMP! Instead, show how to construct a vector (x, y, z) of arbitrarily large magnitude that satisfies (1).

$$2x^2 + 2y^2 + 9z^2 + 8xy - 6xz - 6yz \leq 1 \quad \textbf{(1)}$$

Try $(x, y, z) = (c, -c, 0)$

$$2c^2 + 2c^2 + 0 + -8c^2 - 0 - 0 \leq 1$$

$$-4c^2 \leq 1$$

$$\because c^2 \textbf{ must be positive when } c \in \mathbb{R}$$

$$\therefore -4c^2 \leq 0$$

and $-4c^2 < 1$, which means that the constraint **is always satisfied** when (x, y, z) is of the form $(c, -c, 0)$.

Our objective, $x^2 + y^2 + z^2$, when $(x, y, z) = (c, -c, 0)$, becomes $2c^2$ which can be made arbitrarily large by choosing larger values of c. Because we have shown that any vector of the form $(c, -c, 0)$ will have this property of being valid under the constraint (1), the solution is shown to be unbounded.