3. Heat-Pipe Design [10 pts]

A heated fluid at temperature T (degrees above ambient temperature) flows in a pipe with fixed length and circular cross section with radius r. A layer of insulation, with thickness w, surrounds the pipe to reduce heat loss through the pipe walls (w is much smaller than r). The design variables in this problem are T, r, and w.

The energy cost due to heat loss is roughly equal to $\alpha_1 Tr/w$. The cost of the pipe, which has a fixed wall thickness, is approximately proportional to the total material, i.e., it is given by $\alpha_2 r$. The cost of the insulation is also approximately proportional to the total insulation material, i.e., roughly $\alpha_3 rw$. The total cost is the sum of these three costs.

The heat flow down the pipe is entirely due to the flow of the fluid, which has a fixed velocity, i.e., it is given by $\alpha_4 Tr^2$. The constants α_i with $i \in \{1, 2, 3, 4\}$ are **all positive**, as are the variables T, r, and w.

Now the problem: maximize the total heat flow down the pipe, subject to an upper limit C_{max} on total cost, and the constraints

$$T_{\min} \leq T \leq T_{\max}, \qquad r_{\min} \leq r \leq r_{\max} \qquad w_{\min} \leq w \leq w_{\max}, \qquad w \leq 0.1r$$

a) Express this problem as a geometric program, and convert it into a convex optimization problem.

① $\max\limits_{T, r, w \geq 0} \quad (\alpha_4 \cdot T \cdot r^2)$

$$\text{s.t.} \quad (\alpha_1 T \cdot r/w) + (\alpha_2 \cdot r) + (\alpha_3 \cdot r \cdot w) \leq C_{max}$$

$$T_{min} \leq T \leq T_{max}$$

$$r_{min} \leq r \leq r_{max}$$

$$w_{min} \leq w \leq w_{max}$$

$$w \leq 0.1 r$$

② $\min\limits_{T, r, w \geq 0} \quad (\alpha_4^{-1} T^{-1} r^{-2})$

$$\text{s.t.} \quad \frac{T - T_{min}}{T_{max} - T_{min}} \leq 1 \qquad \frac{(\alpha_1 T r/w) + (\alpha_2 \cdot r) + (\alpha_3 \cdot r \cdot w)}{C_{max}} \leq 1$$

$$\frac{r - r_{min}}{r_{max} - r_{min}} \leq 1 \qquad 10 \left(\frac{w}{r}\right) \leq 1$$

$$\frac{w - w_{min}}{w_{max} - w_{min}} \leq 1$$

③ Define

$$x := \ln(T) \iff T = \exp(x)$$

$$y := \ln(r) \iff r = \exp(y)$$

$$z := \ln(w) \iff w = \exp(z)$$

$\min\limits_{x, y, z} \quad -\ln(\alpha_4) - x - 2y$

$$x \leq \ln(T_{max}) \qquad e^{z-y} \leq 0.1$$

$$y \leq \ln(r_{max}) \qquad \frac{e^{\ln(\alpha_1) + x + y - z} + e^{\ln(\alpha_2) + y} + e^{\ln(\alpha_3) + y + z}}{e^{\ln(C_{max})}} \leq 1$$

$$z \leq \ln(w_{max})$$

$$\ln\left(e^{\ln(\alpha_1) + x + y - z} + e^{\ln(\alpha_2) + y} + e^{\ln(\alpha_3) + y + z}\right) \leq \ln(C_{max}$$

$$\downarrow$$

$$e^{\ln(\alpha_1) + x + y - z} + e^{\ln(\alpha_2) + y} + e^{\ln(\alpha_3) + y + z - C_{max}} \leq 0$$

b) Consider a simple instance of this problem, where $C_{max} = 500$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$. Also assume for simplicity that **each variable has a lower bound of zero** and **no upper bound**. Solve this problem using JuMP. Use the Mosek solver and the command **@NLconstraint(...)** to specify nonlinear constraints such as log-sum-exp functions. Note: Mosek can solve general convex optimization problems! What is the optimal T, r, and w?

In [1]:
```
using JuMP, Mosek

a = ones(4)
C_max = 500

m = Model(solver=MosekSolver(LOG=0))

@variable(m, x >= 0)
@variable(m, y >= 0)
@variable(m, z >= 0)

@NLexpression(m, T, exp(x))
@NLexpression(m, r, exp(y))
@NLexpression(m, w, exp(z))

@NLexpression(m, heat_loss_cost, a[1]*T*r/w)
@NLexpression(m, pipe_cost, a[2]*r)
@NLexpression(m, insulation_cost, a[3]*r*w)
@NLexpression(m, total_cost, heat_loss_cost + pipe_cost + insulation_cost)

@NLconstraint(m, exp(log(a[1]) + x + y - z) + exp(log(a[2]) + y) + exp(log(a[3]) + y + z) - C_max <= 0)
@NLconstraint(m, exp(z - y) <= 0.1)

@NLobjective(m, Min, log(a[4]) - x - 2y)

println(solve(m))

println("We achieved a maximum heat flow of ", getvalue(T) * getvalue(r)^2)
println("T = ", getvalue(T))
println("r =  ", getvalue(r))
println("w = ", getvalue(w))
println("Our total cost is: ", getvalue(total_cost))
```

```
Optimal
We achieved a maximum heat flow of 51305.90644653668
T = 23.840238958644314
r =  46.39042810824172
w = 4.639042747423223
Our total cost is: 500.0000000182839
```

In [2]:

```julia
# attempting something I saw on Piazza (@196)
# inverted objective, then solve for minimization
# NOT MY FINAL ANSWER

using JuMP, Mosek

m1 = Model(solver=MosekSolver(LOG=0))

@variable(m1, T >= 0)
@variable(m1, r >= 0)
@variable(m1, w >= 0)

@NLexpression(m1, x, log(T))
@NLexpression(m1, y, log(r))
@NLexpression(m1, z, log(w))

@NLexpression(m1, heat_loss_cost, a[1]*T*r/w)
@NLexpression(m1, pipe_cost, a[2]*r)
@NLexpression(m1, insulation_cost, a[3]*r*w)
@NLexpression(m1, total_cost, heat_loss_cost + pipe_cost + insulation_cost)

@NLconstraint(m1, a[1]*T*r/w + a[2]*r + a[3]*r*w <= C_max)
@NLconstraint(m1, w <= 0.1*r)

@NLobjective(m1, Min, (a[4]*T*r^2)^-1)

println(solve(m1))

println("We achieved a maximum heat flow of ", getvalue(T) * getvalue(r)^2)
println("T = ", getvalue(T))
println("r =  ", getvalue(r))
println("w = ", getvalue(w))
println("Our total cost is: ", getvalue(total_cost))
```

```
Optimal
We achieved a maximum heat flow of 48667.67592771966
T = 20.24756171792293
r =  49.02684407505225
w = 4.3118627352713395
Our total cost is: 490.64317317489616
```