Homework 4: Duality

Due date: 11:00pm on Sunday February 25, 2018
See the course website for instructions and submission details.

1. **The chess problem.** A small joinery makes two different sizes of boxwood chess sets. The small set requires 3 hours of machining on a lathe, and the large set requires 2 hours. There are four lathes with skilled operators who each work a 40 hour week, so we have 160 lathe-hours per week. The small chess set requires 1 kg of boxwood, and the large set requires 4 kg. Unfortunately, boxwood is scarce and only 200 kg per week can be obtained. When sold, each of the large chess sets yields a profit of \$8, and one of the small chess set has a profit of \$5. The problem is to decide how many sets of each kind should be made each week so as to maximize profit.

   a) Write out the primal LP. Plot the feasible set and solve the LP graphically. Be sure to label the axes and indicate units. Label the optimal point and find the optimal objective.

   b) Repeat all the same steps as in part **a)** but for the dual LP this time. Verify that the optimal dual objective is the same as the optimal objective of part **a)**.

2. **Stigler's supplement.** Consider Stigler's diet problem from Homework 2. To help further lower the cost of your diet, a friend offers to sell you calcium supplements. Each calcium pill contains 500 mg of calcium.

   a) What is the most you would be willing to pay per pill? **Hint:** use duality!

   b) Suppose you can buy calcium pills at a cost \$0.01 each. What is your new optimal diet? How much money does it save compared to the original optimal diet that didn't have access to the calcium supplement?

3. **Dual interpretation.** Suppose $t \in [0, 2\pi]$ is a parameter. Consider the following LP:

$$\begin{aligned} \underset{p,q,r,s}{\text{minimize}} \quad & p + q + r + s \\ \text{subject to:} \quad & p - r = \cos(t) \\ & q - s = \sin(t) \\ & p, q, r, s \geq 0 \end{aligned}$$

   a) Plot the optimal objective of this LP as a function of t. Can you explain what you see?
   Hint: you can do this by looping over values of t, and solving a separate LP for each different value of t. To interpret what you're seeing, you may want to separately consider the cases where $\cos(t)$ and $\sin(t)$ are positive or negative (four cases).

   b) Write out the dual LP. Interpret and solve the dual LP graphically. Does your solution agree with the solution found in part **a)**?