2. Hyperbolic program [10 pts]

In this problem, we start with a problem that doesn't appear to be convex and show that it is in fact convex by converting it into an SOCP.

a) Recall from class that for any $w \in \mathbb{R}^n$ and $x, y \in \mathbb{R}$, the following constraints are equivalent:

$$w^\mathsf{T} w \leq xy, \quad x \geq 0, \quad y \geq 0 \qquad \Longleftrightarrow \qquad \left\| \begin{bmatrix} 2w \\ x - y \end{bmatrix} \right\| \leq x + y$$

Suppose we have an optimization problem with variables $t \geq 0$ and $x \in \mathbb{R}^n$. Express the constraint: $t(a^T x + b) \geq 1$ as a second-order cone constraint. Specifically, write the constraint in the form $\|Ax + b\| \leq c^T x + d$. What are A, b, c, d, x?

2(a)

Note: $w^T w \leq xy$, $x \geq 0$, $y \geq 0$ $\Leftrightarrow$ $\left\| \begin{bmatrix} 2w \\ x - y \end{bmatrix} \right\| \leq x + y$

Problem: $t \geq 0$, $x \in \mathbb{R}^n$

Express $t(a^T x + b) \geq 1$ as an SOCC

i.e. $\|Ax + b\| \leq c^T x + d$

$t(a^T x + b) \geq 1$

let $y = a^T x + b$ $\Rightarrow$ $\left\| \begin{bmatrix} 0 & 0 \\ a^T & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} \right\| \leq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore t(a^T x + b) \geq 1$

$\equiv (a^T x + b) \leq -1/t$ $\Rightarrow$ $\left\| \underbrace{\begin{bmatrix} 0 & 0 \\ a^T & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ b \end{bmatrix}}_{b} \right\| \leq \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{c^T} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ -1/t \end{bmatrix}}_{d}$

b) Consider the following hyperbolic optimization problem **(note the nonlinear objective)**:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & \sum_{i=1}^{p} 1/(a_i^{\mathsf{T}} x + b_i) & \\ \text{subject to} & a_i^{\mathsf{T}} x + b_i > 0, & i = 1, \ldots, p \\ & c_j^{\mathsf{T}} x + d_j \geq 0, & j = 1, \ldots, q \end{array}$$

Write this optimization problem as an SOCP. *Hint: the first part of this problem is **very** relevant!*

(b) $\min\limits_{x} \sum_{i=1}^{p} (a_i^T x + b_i)^{-1}$

s.t. $a_i^T x + b_i > 0$, for $i = 1, \dots, p$

$c_j^T x + d_j \geq 0$, for $j = 1, \dots, q$

SOLUTION: $\min\limits_{x, w_i} \sum_{i=1}^{p} (w_i)$

$0 < \dfrac{1}{a_i^T x + b_i} \leq w_i$ for $i = 1, \dots, p \Rightarrow \underbrace{1 \leq w_i (a_i^T x + b_i)}_{\text{of the same form as part (a)}}$ for $i = 1, \dots,$

$a_i^T x + b_i > 0$, for $i = 1, \dots, p$

$c_j^T x + d_j \geq 0$, for $j = 1, \dots, q$

$x \geq 0$