CS/ECE/ME 532
Homework 2: subspaces and linear equations

due: Friday September 23, 2016

1. **Rank of a product.** Suppose that $C = AB$ where $A \in \mathbb{R}^{m\times k}$ and $B \in \mathbb{R}^{k\times n}$. Prove the following inequality: $\text{rank}(C) \le \min(m, n, k)$. *Hint:* think about how we proved that $\text{rank}(xy^\mathsf{T}) = 1$ in class.

2. **Subspace properties.** Suppose $S, T \subseteq \mathbb{R}^n$ are subspaces.

   a) Prove that the sum $S + T$ is a subspace. Here, $S + T = \{s + t \mid s \in S \text{ and } t \in T\}$, i.e. the set of vectors that can be written as the sum of a vector from S and a vector from T.

   b) Prove that the intersection $S \cap T$ is a subspace. Here, $S \cap T = \{x \mid x \in S \text{ and } x \in T\}$, i.e. the set of vectors belonging to both S and T.

3. **Mostly zeros.** Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   a) Find a basis for $\text{range}(A)$ and find a basis for $\text{null}(A)$.

   b) Find a vector b such that $Ax = b$ has no solutions, or explain why no such b can exist. Repeat the question for the case of exactly one solution, and the case of infinitely many solutions.

4. **Linear equations.** This problem concerns linear equations and their solutions.

   a) Find all solutions to the following system of equations.

$$\begin{aligned} x + 3y + 6z &= 1 \\ 2x + 7y + 15z &= -1 \end{aligned}$$

   b) Find all solutions to the following equation.

$$x + 4y + 10z = 2$$

   c) Find all (x, y, z) that simultaneously satisfy the equations of parts (a) and (b).

   d) Sketch the set of solutions to parts (a), (b), and (c) in 3D on the same axes.

5. **Existence of solutions.** We saw that $Ax = b$ will have at least one solution if $b \in \text{range}(A)$. However, this property can be difficult to check! An alternate way is to compare $\text{rank}(A)$ and $\text{rank}(\begin{bmatrix} A & b \end{bmatrix})$. If they are the same, then $Ax = b$ has at least one solution. If they are different, then $Ax = b$ has no solutions. Explain why this works. *Note:* $\begin{bmatrix} A & b \end{bmatrix} \in \mathbb{R}^{m\times(n+1)}$ is the matrix formed by including b as an extra column of A.