CS/ECE/ME 532 - Fall 2017 - Rebecca Willett

Homework 1: Vectors and Matrices

Due Wednesday, September 20th, 2017

Roumen Guha

1. Matrix multiplication.

Settlers of Catan is the <u>favorite game of the Green Bay Packers</u>
<u>(https://www.washingtonpost.com/news/early-lead/wp/2015/01/16/the-green-bay-packers-are-</u>
<u>obsessed-with-settlers-of-catan/?utm_term=.0c5c6419f03d)</u>. In this game, participants build roads,
settlements, and cities by using resources such as wood, bricks, wheat, sheep, and ore. The
number of resources required for each building project are reflected in Figure 1.

Figure 1: Building costs in Settlers of Catan

**a) Write the information above in a matrix. What do the rows represent? What do the
columns represent?**

One possible representation would be:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The rows represent the requirements for Road, Settlement, City and Development Card
respectively. Each row vector contains the requirements for that construction.

And the columns represent Wood, Bricks, Wheat, Sheep and Ore respectively. Each column vector represents the quantities of a resource used in different constructions.

b) Departing from the game somewhat, suppose resources cost $1 for each unit of wood, $2 for brick, $3 for sheep, $5 for wheat, and $8 for ore. Write this information in a vector. Write out a matrix-vector multiplication that calculates the total cost of buying roads, settlements, and cities.

$$c = \begin{bmatrix} 1 & 2 & 3 & 5 & 8 \end{bmatrix}^T$$

Therefore, $t = Ac$

```
In [5]:  A = [1 1 0 0 0
             1 1 1 1 0
             0 0 2 0 3
             0 0 1 1 1]

         c = [1
              2
              3
              5
              8]

         t = A*c
```

```
Out[5]:  4-element Array{Int64,1}:
          3
         11
         30
         16
```

c) Suppose you want to crush Corey Linsley by building a city, two settlements, and six road lengths connecting them. Again using matrix multiplication, and the total resources required to fill the order.

$$g = \begin{bmatrix} 6 & 2 & 1 & 0 \end{bmatrix}^T$$

Therefore, $r = g^T A$

```
In [7]:  g = [6
              2
              1
              0]

         r = transpose(g)*A
```

```
Out[7]:  1×5 Array{Int64,2}:
          8  8  4  2  3
```

d) Calculate the total cost for the order (using, you guessed it, matrix multiplication)

The total cost of the order is $g^T A c$

```
In [8]: total = transpose(g) * A * c
```

```
Out[8]: 1-element Array{Int64,1}:
         70
```

2.

Let $X = [x_1 x_2 \ldots x_n] \in \mathbb{R}^{p \times n}$, where $x_i \in \mathbb{R}^p$ is the ith column of X. Consider the matrix

$$C = \frac{XX^T}{n}$$

a) Express C as a sum of rank-1 matrices (i.e., columns of X times rows of X^T).

$$C = \frac{\sum_{i=1}^{n} x_i x_i^T}{n}$$

b) Assuming $x_1, x_2, \ldots, x_n$ are linearly independent, what is the rank of C?

If all columns $x_1, x_2, \ldots, x_n$ are linearly independent, meaning n linearly independent column vectors, that means that the rank of C is $\leq min(p, n)$.

3.

Let

$$X = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

a) What is the rank of X?

```
In [10]: X = [1 0 1 0 1
              0 1 0 1 1
              0 0 0 0 0
              0 0 1 1 1]

         rank(X)
```

```
Out[10]: 3
```

b) What is the rank of XX^T?

```
In [11]:  rank(X*transpose(X))
```

Out[11]: 3

c) Find a set of linearly independent columns in X.

One maximal linearly independent set of vectors is $\{x_1, x_2, x_3\}$

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$