CS/ECE/ME 532
Homework 1: Vectors and Matrices

1. **Matrix multiplication.** *Settlers of Catan* is the favorite game of the Green Bay Packers[1]. In this game, participants build roads, settlements, and cities by using resources such as wood, bricks, wheat, sheep, and ore. The number of resources required for each building project are reflected in Figure 1.

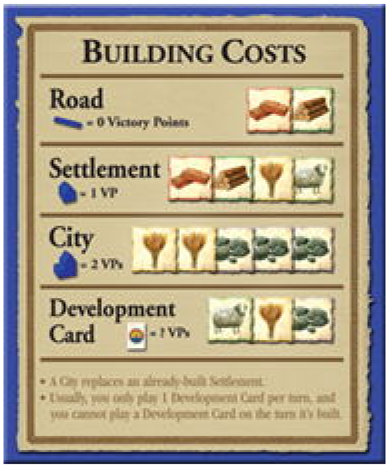

Figure 1: Building costs in Settlers of Catan

a) Write the information above in a matrix. What do the rows represent? What do the columns represent?

SOLUTION: One possible representation is:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 3 \end{bmatrix}$$

The rows represent the requirements for different building projects (roads, settlements, and cities). Each is a row vector that contains the number of bricks, wood, wheat, sheep, and ore required. The columns represent the number of bricks (first column), wood, (second column) and the other resources (third - fifth column). Each column is a vector where the first component corresponds to roads and the second component corresponds to settlements and the their component corresponds to cities.

b) Departing from the game somewhat, suppose resources cost \$1 for each unit of wood, \$2 for brick, \$3 for sheep, \$5 for wheat, and \$8 for ore. Write this information in a vector. Write out a matrix-vector multiplication that calculates the total cost of buying roads, settlements, and cities.

[1] `https://www.washingtonpost.com/news/early-lead/wp/2015/01/16/the-green-bay-packers-are-obsessed-with-settlers-`
`?utm_term=.0c5c6419f03d`

SOLUTION: Write the material costs in a column vector: $c = \begin{bmatrix}2 & 1 & 5 & 3 & 8\end{bmatrix}^T$. The matrix multiplication that calculates total cost is:

$$t = Ac = \begin{bmatrix}1 & 1 & 0 & 0 & 0\\ 1 & 1 & 1 & 1 & 0\\ 0 & 0 & 2 & 0 & 3\end{bmatrix}\begin{bmatrix}2\\1\\5\\3\\8\end{bmatrix} = \begin{bmatrix}3\\11\\34\end{bmatrix}$$

So the total cost of making roads, settlements, and cities are \$3, \$11, and \$34, respectively.

c) Suppose you want to crush Corey Linsley by building a city, two settlements, and six road lengths connecting them. Again using matrix multiplication, find the total resources required to fill the order.

SOLUTION: Write the order in a column vector: $g = \begin{bmatrix}6 & 2 & 1\end{bmatrix}^T$. The matrix multiplication that calculates the requirements is:

$$r = g^T A = \begin{bmatrix}6 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0 & 0 & 0\\ 1 & 1 & 1 & 1 & 0\\ 0 & 0 & 2 & 0 & 3\end{bmatrix} = \begin{bmatrix}8 & 8 & 4 & 2 & 3\end{bmatrix}$$

So to crush Corey, we would require 8 bricks, 8 wood, 4 wheat, 2 sheep, and 3 ore.

d) Calculate the total cost for the order (using, you guessed it, matrix multiplication)

SOLUTION: The total cost is $g^T Ac = 74$

e) Get up and running with either Matlab or Python. In your language of choice, write a script that computes the matrix multiplications in the previous parts of this problem.

SOLUTION: We will show the code for part c; the other are similar. g = [3;4]; A = [3 4 1; 2 3 2]; r = g'*A

2. Let $\boldsymbol{X} = [\boldsymbol{x}_1\ \boldsymbol{x}_2\ \cdots\ \boldsymbol{x}_n] \in \mathbb{R}^{p\times n}$, where $\boldsymbol{x}_i \in \mathbb{R}^p$ is the ith column of $\boldsymbol{X}$. Consider the matrix

$$\boldsymbol{C} = \frac{\boldsymbol{X}\boldsymbol{X}^T}{n}.$$

a) Express $\boldsymbol{C}$ as a sum of rank-1 matrices (i.e., columns of $\boldsymbol{X}$ times rows of $\boldsymbol{X}^T$).

SOLUTION: Split X into its columns and carry out the multiplication:

$$\tfrac{1}{n}XX^T = \tfrac{1}{n}\begin{bmatrix}x_1 & \dots & x_n\end{bmatrix}\begin{bmatrix}x_1^T\\ \vdots\\ x_n^T\end{bmatrix} = \tfrac{1}{n}\sum_{k=1}^{n} x_k x_k^T$$

each $x_k x_k^T$ is an $p \times p$ rank-1 matrix.

b) Assuming $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are linearly independent, what is the rank of $\boldsymbol{C}$?

SOLUTION: Simple approach: Let $v_i = (i^{\text{th}} \text{ row of } X)^T$. Then

$$XX^T = \begin{bmatrix} Xv_1 & Xv_2 & \cdots & Xv_p \end{bmatrix}$$

so each column of XX^T is a weight sum of the columns of X. Thus $\text{rank}(C) = \text{rank}(X) = n$.

Alternative approach: We will prove something more general, that

$$\text{rank}(X) = \text{rank}(X^TX) = \text{rank}(XX^T) = n$$

(i) $\text{rank}(X) = n$. This is true because the columns of X are independent.

(ii) $\text{rank}(X^TX) = n$. Since $X^TX \in \mathbb{R}^{n\times n}$, We can prove the result by showing that X^TX has independent columns. If a linear combination of the columns is zero, then $X^TXw = 0$ for some vector w. This implies that $0 = w^TX^TXw = \|Xw\|^2$. If a norm is zero, its argument must be zero. So $Xw = 0$. Using the fact that X has independent columns, we conclude that $w = 0$. Therefore, X^TX has independent columns, which means $\text{rank}(X^TX) = n$.

(iii) $\text{rank}(XA) \leq \text{rank}(X)$ for any matrix A. This is because each column of XA is a linear combination of columns of X. So the span of the columns of XA is a subset (possibly equal) of the span of the columns of X.

(iv) $\text{rank}(AX) \leq \text{rank}(X)$. Same argument as in (ii) applied to rows instead of columns.

(v) $\text{rank}(X^TXX^TX) = n$. To see why this is true, notice that from (ii), X^TX is full-rank and therefore invertible. Since $X^TXX^TX(X^TX)^{-1}(X^TX)^{-1} = I$, we conclude that X^TXX^TX is invertible as well, and therefore has rank n.

(vi) $\text{rank}(XX^T) = n$. Using (iii)–(iv), we have:

$$n = \text{rank}(X^TXX^TX) \leq \text{rank}(XX^T) \leq \text{rank}(X) = n$$

Therefore $\text{rank}(XX^T) = n$ and we are done.

Note: there is a much easier way to solve this problem using the SVD—we will learn it later on in the course!

3. Let

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

a) What is the rank of $\boldsymbol{X}$?

b) What is the rank of $\boldsymbol{X}\boldsymbol{X}^T$?

c) Find a set of linearly independent columns in $\boldsymbol{X}$.

SOLUTION: The rank of X is 3. The first three columns are clearly linearly independent, and the last two columns are linear combinations of the first three. Namely:

$$x_4 = x_3 + x_2 - x_1 \quad \text{and} \quad x_5 = x_3 + x_2$$

Note that we can deduce a rank bound simply by examining the rows. Since there are only three nonzero rows, the rank can't be any larger than 3. As for the matrix XX^T, it is equal to:

$$XX^T = \begin{bmatrix} 3 & 1 & 0 & 2 \\ 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 3 \end{bmatrix}$$

Again, this matrix has rank at most 3 because of the zero column. The rank of XX^T is actually exactly 3, because the remaining three columns are linearly independent.