CS/ECE/ME 532
Homework 2: subspaces and linear equations

due: Friday September 23, 2016

1. Rank of a product. Suppose that $C = AB$ where $A \in \mathbb{R}^{m\times k}$ and $B \in \mathbb{R}^{k\times n}$. Prove the following inequality: $\operatorname{rank}(C) \le \min(m, n, k)$. *Hint:* think about how we proved that $\operatorname{rank}(xy^\mathsf{T}) = 1$ in class.

SOLUTION: Since $C \in \mathbb{R}^{m\times n}$, we already know that $\operatorname{rank}(C) \le \min(m, n)$. It remains to show that $\operatorname{rank}(C) \le k$. The rough idea is that each column of C is a linear combination of columns of A, and so the span of the columns of C can be no larger than the span of the columns of A. This limits the rank of C to be at most the rank of A, which is at most $\min(m, k)$.

One way to write this mathematically is to look at the range of C:

$$\begin{aligned}\operatorname{range}(C) &= \{Cx \mid x \in \mathbb{R}^n\}\\ &= \{ABx \mid x \in \mathbb{R}^n\}\\ &= \{Ay \mid y \in \operatorname{range}(B)\}\\ &\subseteq \left\{Ay \;\middle|\; y \in \mathbb{R}^k\right\}\\ &= \operatorname{range}(A)\end{aligned}$$

Since the rank of a matrix is the same as the dimension of its range, we have:

$$\operatorname{range}(C) \subseteq \operatorname{range}(A) \quad\implies\quad \operatorname{rank}(C) \le \operatorname{rank}(A)$$

And $\operatorname{rank}(A) \le \min(m, k)$, which completes the proof.

2. Subspace properties. Suppose $S, T \subseteq \mathbb{R}^n$ are subspaces.

a) Prove that the sum $S + T$ is a subspace. Here, $S + T = \{s + t \mid s \in S \text{ and } t \in T\}$, i.e. the set of vectors that can be written as the sum of a vector from S and a vector from T.

b) Prove that the intersection $S \cap T$ is a subspace. Here, $S \cap T = \{x \mid x \in S \text{ and } x \in T\}$, i.e. the set of vectors belonging to both S and T.

SOLUTION: For each case, we'll prove the three subspace properties separately (zero is in the set, closure under addition, and closure under scalar multiplication).

a)
- Since $0 \in S$ and $0 \in T$ (both are subspaces), and $0 = 0 + 0$, then $0 \in S + T$.
- If $x, y \in S + T$, we can write $x = s_1 + t_1$ and $y = s_2 + t_2$ for some $s_i \in S$ and $t_i \in T$. Now $x + y = (s_1 + t_1) + (s_2 + t_2) = (s_1 + s_2) + (t_1 + t_2)$. Now $s_1 + s_2 \in S$ and $t_1 + t_2 \in T$, so we have expressed $x + y$ as a sum of an element from S and an element from T, i.e. $x + y \in S + T$.
- If $x \in S + T$, we can write $\alpha x = \alpha(s + t) = \alpha s + \alpha t$ for some $s \in S$ and $t \in T$. So we have expressed αx as a sum of an element from S and an element from T, i.e. $\alpha x \in S + T$.

b)
- Since $0 \in S$ and $0 \in T$ (both are subspaces), then $0 \in S \cap T$.
- If $x, y \in S \cap T$, then $x, y \in S$, which means $x + y \in S$ and $\alpha x \in S$. Similarly, we have $x, y \in T$, so $x + y \in T$ and $\alpha x \in T$. It follows that $x + y \in S \cap T$ and $\alpha x \in S \cap T$, as required.

3. Mostly zeros. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Find a basis for range(A) and find a basis for null(A).

b) Find a vector b such that $Ax = b$ has no solutions, or explain why no such b can exist. Repeat the question for the case of exactly one solution, and the case of infinitely many solutions.

SOLUTION:

a) The zero columns don't contribute to the range, so we are left with

$$\text{range}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

We see the range has dimension 2, which means the nullspace must have dimension 3 (so they sum to 5, the total number of columns). One way to find a basis for the nullspace is to add 1's at each zero column, since this is sure to create a zero vector. This yields:

$$\text{null}(A) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

b)
- **No solutions**: In this case, we must find $b \notin \text{range}(A)$. Since we already found a basis for range(A) in part a, we can see for example that if we pick $b = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^\mathsf{T}$, there will be no solutions to $Ax = b$.
- **Exactly one solution**: This is impossible, since $\text{null}(A) \neq \{0\}$. If x is any solution to $Ax = b$, then $x + w$ will also be a solution for any $w \in \text{null}(A)$. The solution can therefore never be unique.
- **Infinitely many solutions**: We already saw that any b that yields one solution will yield infinitely many. So any such b will do. In other words, we can pick any $b \in \text{range}(A)$, and there will be infinitely many solutions to $Ax = b$. For example, $b = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^\mathsf{T}$.

4. Linear equations. This problem concerns linear equations and their solutions.

a) Find all solutions to the following system of equations.

$$\begin{aligned} x + 3y + 6z &= 1 \\ 2x + 7y + 15z &= -1 \end{aligned}$$

b) Find all solutions to the following equation.

$$x + 4y + 10z = 2$$

c) Find all (x, y, z) that simultaneously satisfy the equations of parts (a) and (b).

d) Sketch the set of solutions to parts (a), (b), and (c) in 3D on the same axes.

SOLUTION:

a) We can perform Gaussian elimination to quickly get a particular solution:

$$\begin{aligned} x + 3y + 6z &= 1 \\ 2x + 7y + 15z &= -1 \end{aligned}$$

Perform $r_2 - 2r_1 \mapsto r_2$ and obtain:

$$\begin{aligned} x + 3y + 6z &= 1 \\ y + 3z &= -3 \end{aligned}$$

Perform $r_1 - 3r_2 \mapsto r_1$ and obtain:

$$\begin{aligned} x - 3z &= 10 \\ y + 3z &= -3 \end{aligned}$$

Each choice of z yields a particular solution. For example, $z = 0$ yields $x = 10$, $y = -3$. To find the nullspace, we should make the left-hand side zero, which can be accomplished for example with $x = 3$, $y = -3$, $z = 1$. The general solution is therefore:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \qquad \text{for any } \alpha_1 \in \mathbb{R}$$

This makes sense, because we expect the nullspace to have dimension 1.

b) For the equation $x+4y+10z = 2$, the equation is already in reduced form! A particular solution is for example $(2, 0, 0)$. The nullspace has dimension 2, and two independent ways of getting zero are $(4, -1, 0)$, and $(10, 0, -1)$. So the general solution is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 10 \\ 0 \\ -1 \end{bmatrix} \qquad \text{for any } \alpha_2, \alpha_3 \in \mathbb{R}$$

c) To simultaneously solve all three equations, the most straightforward way is to write them as a single system of equations:

$$\begin{bmatrix} 1 & 3 & 6 \\ 2 & 7 & 15 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

From here, we can perform Gaussian elimination, or invert the 3×3 matrix. The solution is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 7 & 15 \\ 1 & 4 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 & -6 & 3 \\ -5 & 4 & -3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 22 \\ -15 \\ 4 \end{bmatrix}$$

Another way to solve this problem is to set the two solutions we found from parts a and b equal to one another. In other words:

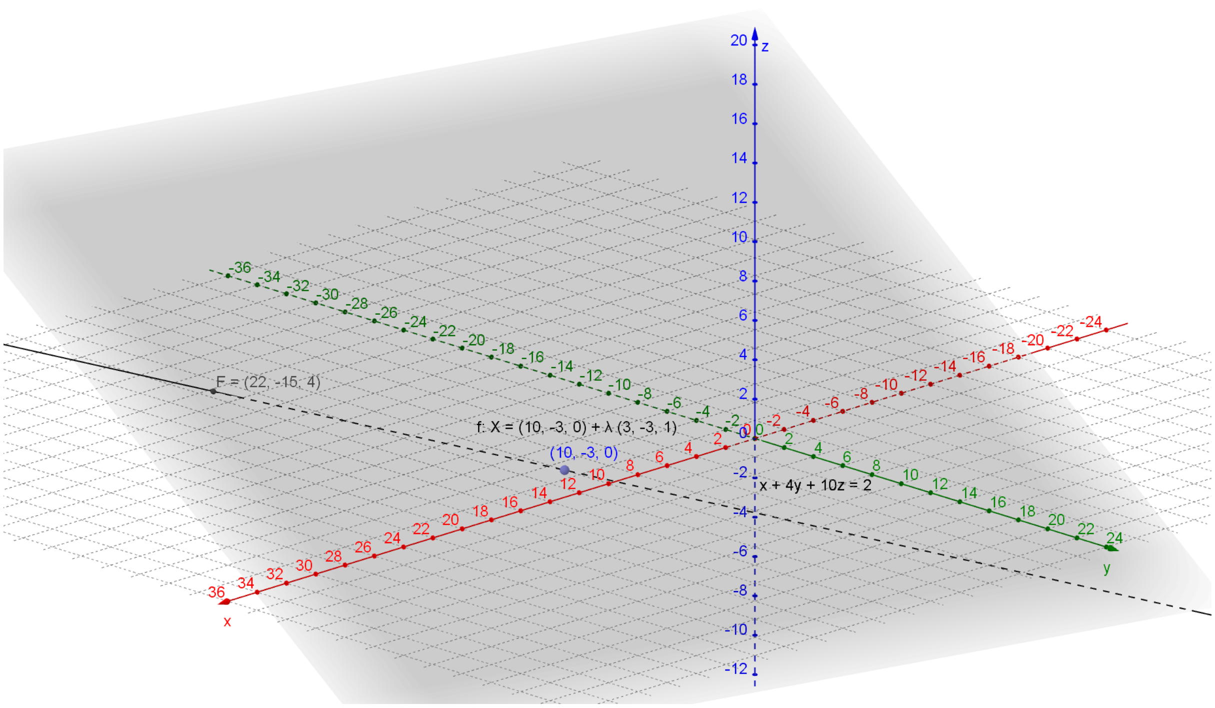
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 10 \\ 0 \\ -1 \end{bmatrix} \tag{1}$$

After simplification, this becomes:

$$\begin{bmatrix} 8 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 & 4 & 10 \\ 3 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

This system is slightly easier to solve, and its unique solution is $\alpha_1 = 4$, $\alpha_2 = 15$, $\alpha_3 = -4$. We can then substitute these α_k values back into either representation (1) and obtain $x = 22$, $y = -15$, $z = 4$, which is the same solution we found using the other method.

d) See below for a sketch of the line, plane, and intersection. This wasn't an easy sketch to draw clearly, so we'll accept anything that looks remotely close to being right.



5. Existence of solutions. We saw that $Ax = b$ will have at least one solution if $b \in \text{range}(A)$. However, this property can be difficult to check! An alternate way is to compare $\text{rank}(A)$ and $\text{rank}(\begin{bmatrix} A & b \end{bmatrix})$. If they are the same, then $Ax = b$ has at least one solution. If they are different, then $Ax = b$ has no solutions. Explain why this works. *Note:* $\begin{bmatrix} A & b \end{bmatrix} \in \mathbb{R}^{m\times(n+1)}$ is the matrix formed by including b as an extra column of A.

SOLUTION: Recall that the rank of a matrix is the dimension of its range. Informally, if adding an additional column to A makes the range larger, that means the new column *was not* a linear combination of the existing columns. Likewise, if the new column does not change the range, it must have been a linear combination of the columns of A.

We can prove this rigorously by comparing the range of A to the range of $\begin{bmatrix} A & b \end{bmatrix}$. By adding the extra column b, the range becomes strictly larger. In other words, $\text{range}(A) \subseteq \text{range}\begin{bmatrix} A & b \end{bmatrix}$. To prove this, it's enough to show that every $w \in \text{range}(A)$ also satisfies $w \in \text{range}\begin{bmatrix} A & b \end{bmatrix}$. Since $w \in \text{range}(A)$, there exists some $x \in \mathbb{R}^n$ such that $w = Ax$. But we can also write:

$$w = Ax = \begin{bmatrix} A & b \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix}$$

therefore $w \in \text{range}\begin{bmatrix} A & b \end{bmatrix}$ as well. So we've shown that $\text{range}(A) \subseteq \text{range}\begin{bmatrix} A & b \end{bmatrix}$.

- If we have equality, $\text{range}(A) = \text{range}\begin{bmatrix} A & b \end{bmatrix}$. Since the range is the span of columns, this implies $b \in \text{range}(A)$. Note that in this case, we will have $\text{rank}(A) = \text{rank}\begin{bmatrix} A & b \end{bmatrix}$ because the rank is the dimension of the range and both ranges are equal.
- If we have strict inequality $\text{range}(A) \subset \text{range}\begin{bmatrix} A & b \end{bmatrix}$. This means there exists some vector in the right set that doesn't belong in the left set. This implies $b \notin \text{range}(A)$. In this case, we will have $\text{rank}(A) < \text{rank}\begin{bmatrix} A & b \end{bmatrix}$ (in fact, $\text{rank}\begin{bmatrix} A & b \end{bmatrix} = \text{rank}(A) + 1$).