CS/ECE/ME 532

Homework 1: Matrices, vectors, and norms

due: Friday September 16, 2016

1. **Matrix multiplication.** The local factory makes widgets and gizmos. Making one widget requires 3 lbs of materials, 4 parts, and 1 hour of labor. Making one gizmo requires 2 lbs of materials, 3 parts, and 2 hours of labor.

   a) Write the information above in a matrix. What do the rows and columns represent?

   b) Suppose materials cost \$1/lb, parts cost \$10 each, and labor costs \$100/hr. Write this information in a vector. Write out a matrix-vector multiplication that calculates the total cost of making widgets and gizmos.

   c) Suppose the factory receives an order for 3 widgets and 4 gizmos. Again using matrix multiplication, find the total material, parts, and labor required to fill the order.

   d) Calculate the total cost for the order (using, you guessed it, matrix multiplication)

2. **Linear dynamical systems.** Linear dynamical systems are a popular way of modeling mechanical and electrical systems. In general, the model takes the form:

$$\begin{aligned} \boldsymbol{x}(t+1) &= A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= C\boldsymbol{x}(t) + D\boldsymbol{u}(t) \end{aligned} \qquad \text{for } t = 0, 1, \dots, N$$

   For example, in an engine model, the inputs $\boldsymbol{u}(0), \boldsymbol{u}(1), \dots, \boldsymbol{u}(N) \in \mathbb{R}^m$ could represent the throttle, fuel, and air injected at each timestep and the outputs $\boldsymbol{y}(0), \boldsymbol{y}(1), \dots, \boldsymbol{y}(N) \in \mathbb{R}^k$, could represent the engine RPM and torque at each timestep. The matrices A, B, C, D characterize the complicated dependence of the outputs on the inputs and $\boldsymbol{x}(0), \boldsymbol{x}(1), \dots, \boldsymbol{x}(N) \in \mathbb{R}^n$ are internal *state* variables. Note that n might be quite large, even if m and k are small! Find a matrix G that satisfies:

$$\begin{bmatrix} \boldsymbol{y}(0) \\ \vdots \\ \boldsymbol{y}(N) \end{bmatrix} = G \begin{bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{u}(0) \\ \vdots \\ \boldsymbol{u}(N) \end{bmatrix} \qquad \text{where } G \in \mathbb{R}^{k(N+1)\times(n+m(N+1))}$$

   Note that G should only depend on A, B, C, D; it should not contain any $\boldsymbol{u}$'s, $\boldsymbol{y}$'s, or $\boldsymbol{x}$'s.

3. **Norm nonnegativity.** Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a function that satisfies the following three properties:

   - $f(\mathbf{0}) = 0$ (zero property)
   - $f(a\boldsymbol{x}) = |a| f(\boldsymbol{x})$ for all $a \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^n$ (absolute homogeneity)
   - $f(\boldsymbol{x} + \boldsymbol{y}) \le f(\boldsymbol{x}) + f(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ (triangle inequality)

   Use the properties above to prove that $f(\boldsymbol{x}) \ge 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

4. **Norm additivity.** Suppose $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on $\mathbb{R}^n$.

   a) Prove that $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_a + \|\boldsymbol{x}\|_b$ is also a norm on $\mathbb{R}^n$.

   b) Sketch the norm ball in $\mathbb{R}^2$ for the norm $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 + \|\boldsymbol{x}\|_\infty$.