CS/ECE/ME 532
Homework 5: matrix norms and the SVD

due: Friday October 21, 2016

1. Induced norms. In class, we defined the induced 2-norm of a matrix $A \in \mathbb{R}^{m \times n}$ as follows.

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

Prove that the induced 2-norm is indeed a norm.

2. Image of a circle. Plot the image of a unit circle in $\mathbb{R}^2$ when each point is multiplied by $A = \left(\begin{smallmatrix} 3 & -2 \\ -1 & 5 \end{smallmatrix}\right)$. Also overlay the scaled left singular vectors $\sigma_1 u_1$ and $\sigma_2 u_2$ on your plot and verify that they line up with the axes of the ellipse.

3. Dimension reduction. Load the file `sdata.csv` which contains a 1000×3 matrix of data. Each row of the matrix is a point (x_i, y_i, z_i) in $\mathbb{R}^3$. We will approximate this data set as an affine one-dimensional space (a line that doesn't pass through the origin).

a) Find the line that best approximates the data in the sense of minimizing the sum of the squares of the projections of all points onto the line. Plot the line and the data on the same axes and verify that the line approximates the points. *Hint:* before finding the line, shift every point so that the data has zero mean. You can make 3D scatter plots in Matlab by using `plot3`.

b) Instead of using three numbers (x_i, y_i, z_i) to describe each data point, we can now use a single number w_i, which is the position along the line of the projected data point. Give a formula that converts (x, y, z) to w and the reverse formula, which converts w to a point (x, y, z).

c) Convert the data set to w_i coordinates, and plot a histogram of the $\{w_i\}$ to see how the points are distributed. Use 20 equally spaced bins for the histogram.

4. Image compression. In this problem, we'll use low-rank approximations to compress an image.

a) Load the file `bucky.csv` which contains a matrix $A \in \mathbb{R}^{600 \times 400}$ of grayscale values scaled to lie between 0 and 1. Plot the image. In Matlab, you can do this via the commands:

```
A = csvread('bucky.csv');
figure; imagesc(A,[0 1])
colormap gray; axis image; axis off
```

b) Plot the singular values of A. What do you notice?

c) Approximate A as a rank r matrix by only keeping the r largest singular values and making the rest zero. Try this for $r \in \{10, 20, 50, 100\}$ and plot the corresponding compressed images.

d) Compare the space required to store the full A matrix with the space required to store the rank r approximation of A; how many times smaller is the storage requirement for $r \in \{10, 20, 50, 100\}$? You may assume that storage space requirements are proportional to the number of numbers that must be stored. e.g. a 10×10 matrix contains 100 numbers.