CS/ECE/ME 532
Homework 2: Norms and Least Squares

1. Define the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}$ as

$$\Phi(\boldsymbol{x}) \;=\; \max_{1 \le i \le n} i|x_i|.$$

Is $\Phi(\boldsymbol{x})$ a norm?

2. Equivalence of norms. For each case below find positive constants a and b (possibly different in each case) so that for every $\boldsymbol{x} \in \mathbb{R}^n$

(i) $a\|\boldsymbol{x}\|_1 \;\le\; \|\boldsymbol{x}\|_2 \;\le\; b\|\boldsymbol{x}\|_1$ (HINT: Use Cauchy-Schwarz inequality, which states that for any vectors u, v, we have: $u^T v \le \|u\|_2\|v\|_2$ and a vector of all ones.)

(ii) $a\|\boldsymbol{x}\|_1 \;\le\; \|\boldsymbol{x}\|_\infty \;\le\; b\|\boldsymbol{x}\|_1$

3. Let

$$\boldsymbol{A} \;=\; \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

a) What is the rank of $\boldsymbol{A}$?

b) Suppose that $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$. Derive an explicit formula for $\boldsymbol{x}$ in terms of $\boldsymbol{y}$.

4. Answer the following questions. Make sure to explain your reasoning.

a) Are the columns of the following matrix linearly independent?

$$\boldsymbol{A} \;=\; \begin{bmatrix} +0.92 & +0.92 \\ -0.92 & +0.92 \\ +0.92 & -0.92 \\ -0.92 & -0.92 \end{bmatrix}$$

b) Are the columns of the following matrix linearly independent?

$$\boldsymbol{A} \;=\; \begin{bmatrix} +1 & +1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & -1 \end{bmatrix}$$

c) Are the columns of the following matrix linearly independent?

$$\boldsymbol{A} \;=\; \begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 8 \end{bmatrix}$$

d) What is the rank of the following matrix?

$$\boldsymbol{A} = \begin{bmatrix} +5 & +2 \\ -5 & +2 \\ +5 & -2 \end{bmatrix}$$

e) Suppose the matrix in part c is used in the least squares optimization $\min_{\boldsymbol{x}} \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2$. Does a unique solution exist?

5. Consider the following matrix and vector:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

a) Find the solution $\widehat{\boldsymbol{x}}$ to $\min_{\boldsymbol{x}} \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2$.

b) Make a sketch of the geometry of this particular problem in $\mathbb{R}^3$, showing the columns of $\boldsymbol{A}$, the plane they span, the target vector $\boldsymbol{b}$, the residual vector and the solution $\widehat{\boldsymbol{b}} = \boldsymbol{A}\widehat{\boldsymbol{x}}$.

6. Polynomial fitting. Suppose we observe pairs of points (a_i, b_i), $i = 1, \ldots, m$. Imagine these points are measurements from a scientific experiment. The variables a_i are the experimental conditions and the b_i correspond to the measured response in each condition. Suppose we wish to fit a degree $d < m$ polynomial to these data. In other words, we want to find the coefficients of a degree d polynomial p so that $p(a_i) \approx b_i$ for $i = 1, 2, \ldots, m$. We will set this up as a least-squares problem.

a) Suppose p is a degree d polynomial. Write the general expression for $p(a_i) = b$.

b) Express the $i = 1, \ldots, m$ equations as a system in matrix form $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. Specifically, what is the form/structure of $\boldsymbol{A}$ in terms of the given a_i.

c) Write a Matlab or Python script to find the least-squares fit to the $m = 30$ data points in
`polydata.mat`. Plot the points and the polynomial fits for $d = 1, 2, 3$.

7. Recall the cereal calorie prediction problem discussed in class. The data matrix for this problem is

$$\boldsymbol{A} = \begin{bmatrix} 25 & 0 & 1 \\ 20 & 1 & 2 \\ 40 & 1 & 6 \end{bmatrix}.$$

Each row contains the grams/serving of carbohydrates, fat, and protein, and each row corresponds to a different cereal (*Frosted Flakes, Grape-Nuts, Teenage Mutant Ninja Turtles*).

The total calories for each cereal are

$$\boldsymbol{b} = \begin{bmatrix} 110 \\ 110 \\ 210 \end{bmatrix}.$$

a) Write a small program (e.g., in Matlab or Python) that solves the system of equations $\boldsymbol{Ax} = \boldsymbol{b}$. Recall the solution $\boldsymbol{x}$ gives the calories/gram of carbohydrate, fat, or protein. What is the solution?

b) The solution may not agree with the known calories/gram, which are 4 for carbs, 9 for fat and 4 for protein. We suspect this may be due to rounding the grams to integers, especially the fat grams. Assuming the true value for calories/gram is

$$\boldsymbol{x}^\star = \begin{bmatrix} 4 \\ 9 \\ 4 \end{bmatrix},$$

and that the total calories, grams of carbs, and grams of protein are correctly reported above, determine the "correct" grams of fat in each cereal.

c) Now suppose that we predict total calories using a more refined breakdown of carbohydrates, into total carbohydrates, complex carbohydrates and sugars (simple carbs). So now we will have 5 features to predict calories (the three carb features + fat and protein). So let's suppose we measure the grams of these features in 5 different cereals to obtain this data matrix

$$\boldsymbol{A} = \begin{bmatrix} 25 & 15 & 10 & 0 & 1 \\ 20 & 12 & 8 & 1 & 2 \\ 40 & 30 & 10 & 1 & 6 \\ 30 & 15 & 15 & 0 & 3 \\ 35 & 20 & 15 & 2 & 4 \end{bmatrix},$$

and the total calories in each cereal

$$\boldsymbol{b} = \begin{bmatrix} 104 \\ 97 \\ 193 \\ 132 \\ 174 \end{bmatrix}.$$

Can you solve $\boldsymbol{Ax} = \boldsymbol{b}$? Carefully examine the situation in this case. Is there a solution that agrees with the true calories/gram?