280823 - Mooring Systems

Lesson 3 - Line Modelling - Finite Element Analysis

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Outline

1. Finite Element Analysis
   - Strong Form
   - Weak Form

Conservation of Mass:

$$
\begin{aligned}
\frac{\mathrm{D}M}{\mathrm{D}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int \rho \,\mathrm{d}V = 0 \\
\frac{\mathrm{D}\rho}{\mathrm{D}t} &+ \rho \nabla \cdot \dot{x} = 0 \\
\frac{\partial \rho}{\partial t} &+ \nabla \cdot (\rho \dot{x}) = 0
\end{aligned} \tag{1}
$$

Finite Element Analysis
Strong Form
Weak Form

Material Derivative

$$
\begin{aligned}
&\gamma(x(t), t) \\
&\frac{\mathrm{D}\gamma}{\mathrm{D}t} = \frac{\partial \gamma}{\partial t} + \dot{x}\nabla(\gamma)
\end{aligned}
\tag{2}
$$

Reynolds transport theorem

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\Omega}\gamma\,\mathrm{d}\Omega = \int_{\Omega}\frac{\partial\gamma}{\partial t}\,\mathrm{d}\Omega + \int_{\partial\Omega}(\dot{x}\cdot n)\gamma\,\mathrm{d}\Gamma \tag{3}$$

Finite Element Analysis
Strong Form
Weak Form

Divergence theorem

$$\int_{\Omega} \nabla \cdot \gamma \, \mathrm{d}\Omega = \int_{\partial\Omega} (\gamma \cdot n) \, \mathrm{d}\Gamma \tag{4}$$

Conservation of Linear Momentum

$$\sum F = \sum_i m_i a_i = \sum F_{\text{body}} + \sum F_{\text{boundary}} = 0$$
$$\frac{\mathrm{D} M \dot{x}}{\mathrm{D} t} = \frac{\mathrm{d}}{\mathrm{d} t} \int (\rho \dot{x}) \, \mathrm{d} V = 0 \tag{5}$$
$$\rho \frac{\mathrm{D} \dot{x}}{\mathrm{D} t} = \nabla \cdot \sigma + \rho b$$

Conservation of Angular Momentum

$$\sigma = \sigma^T \tag{6}$$

Conservation of Linear Momentum

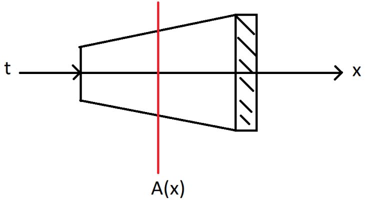

Figure 1: Schematic of a rod.

$$\sum F = t + R + b = 0 \tag{7}$$

Conservation of Linear Momentum

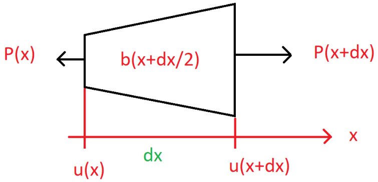

Figure 2: Differential element

1. Equilibrium of forces.

$$
\begin{aligned}
&-f(x) + b(x + \frac{dx}{2})dx + f(x + dx) = 0 \\
&\frac{f(x + dx) - f(x)}{dx} + b(x + \frac{dx}{2}) = 0 \\
&\lim(dx) \to 0 \\
&\mathrm{d}f(x) + b(x) = 0
\end{aligned}
\tag{8}
$$

Finite Element Analysis
Strong Form
Weak Form

2 Constitutive Law.

$$\sigma(x) = \frac{f(x)}{A(x)} \rightarrow f(x) = \sigma(x) \cdot A(x) \tag{9}$$

Finite Element Analysis
Strong Form
Weak Form

3 Notation of Strain and Stress-Strain relationship:

$$
\begin{aligned}
&\varepsilon = \frac{l(f)}{l_0} \\
&\varepsilon \approx \frac{u(x+dx)-u(x)}{dx} \\
&\lim(dx) \rightarrow 0 \\
&\varepsilon := \frac{\mathrm{d}u}{\mathrm{d}x} \\
&\sigma := E \cdot \varepsilon
\end{aligned}
\tag{10}
$$

Equilibrium Equation:

$$\frac{d}{dx}(AE\frac{\mathrm{d}u}{\mathrm{d}t}) + b = 0 \qquad \forall x \in \Omega := [0, l_0] \tag{11}$$

Boundary Conditions:

$$\begin{cases} \sigma(x=0) = E\frac{\mathrm{d}u}{\mathrm{d}t}|_{x=0} = \frac{p(x=0)}{A(x=0)} = -t \\ u(x=l_0) = \bar{u} = 0 \end{cases} \tag{12}$$

Strong Form

$$
\begin{cases}
\dfrac{d}{dx}\left(AE\dfrac{\mathrm{d}u}{\mathrm{d}t}\right) + b = 0 & ,\forall x \in \Omega \\[2ex]
E\dfrac{\mathrm{d}u}{\mathrm{d}t} = -t & ,\forall x \in \partial\Omega_N \\[2ex]
u = \bar{u} & ,\forall x \in \partial\Omega_D
\end{cases}
\tag{13}
$$

Weak Form

We constrain the function $u(x)$ and a test function $v(x)$ to reside in the space :

$$u, v \quad \in \mathcal{L}_2 := \left[||f(x)||_{\mathcal{L}2} \equiv \left(\int_\Omega |f|^2 \right)^{\frac{1}{2}} < \infty \right] \tag{14}$$

Assume integrals $\int_\gamma$ are respect $\mathrm{d}\gamma$:

$$\begin{cases} \int_\Omega v \cdot (\nabla \cdot (AE\nabla(u)) + b) = 0 & , \forall v \quad \in \quad \Omega \\ v \cdot A(E\nabla(u) + t) = 0 & , \forall x \quad \in \quad \partial\Omega_N \\ v = 0 & , \forall v \quad \in \quad \partial\Omega_D \end{cases} \tag{15}$$

Weak Form

Expanding, using integral by parts and divergence theorem:

$$
\begin{cases}
\int_\Omega v\cdot\nabla\cdot(AE\nabla(u)) + \int_\Omega v\cdot b = 0 & ,\forall v \in \Omega \\
\\
-\int_\Omega \nabla(v)\cdot(AE\nabla(u)) \\
+\int_\Omega \nabla\cdot(v\cdot(AE\nabla(u))) + \int_\Omega v\cdot b = 0 & ,\forall v \in \Omega \\
\\
-\int_\Omega \nabla(v)\cdot(AE\nabla(u)) \\
+\int_{\partial\Omega} (n\cdot(v\cdot(AE\nabla(u)))) + \int_\Omega v\cdot b = 0 & ,\forall v \in \Omega
\end{cases}
\tag{16}
$$

Weak Form

Expanding and using integral by parts:

$$
\begin{cases}
-\int_\Omega \nabla(v)\cdot(AE\nabla(u)) \\
+\Big[(n\cdot(v\cdot(AE\nabla(u))))\Big]\Big|_{x=0}^{x=l_0} + \int_\Omega v\cdot b = 0 & ,\forall v \in \Omega \\
\\
-\int_\Omega \nabla(v)\cdot(AE\nabla(u)) \\
-v\cdot(A\sigma)\big|_{x=0} + \int_\Omega v\cdot b = 0 & ,\forall v \in \Omega
\end{cases}
\tag{17}
$$

Weak Form

Dirichlet (prescribed) boundary conditions on $v(x = l_0) = 0$

$$-\int_{\Omega} \nabla(v) \cdot (AE\nabla(u)) = v \cdot t + \int_{\Omega} v \cdot b, \forall v \in \Omega \qquad (18)$$

THE END

References

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Continuum Mechanics for Engineers. Theory and Problems.
Authors, first edition, 2016.
URL: https://www.researchgate.net/publication/308650155_Continuum_Mechanics_for_Engineers_Theory_and_Problems_First_edition_September_2016, doi:10.13140/RG.2.2.25821.20961.