Analysis 1, Tutorium 11

29. 1. 2021

Aufgabe 1 (Partialbruchzerlegung)**.** Finde Stammfunktionen von

$$f\colon]-1,3[\to \mathbb{R}, \quad t \mapsto \frac{1}{(t-3)(t+1)},$$
$$g\colon]0,1[\to \mathbb{R}, \quad x \mapsto \frac{x-3}{x^3-5x^2+8x-4}.$$

Partialbruchzerlegung:

FTA: $q \in \mathbb{C}[z]$ Polynom, $n = \deg q$

$$\Rightarrow \quad q = \alpha \prod_{i=1}^{s} (z-\alpha_i)^{e_i}$$

$\alpha_i \in \mathbb{C}$ paarweise verschieden

$\frac{p}{q}$, $p, q \in \mathbb{C}[z]$ $\quad \deg p < \deg q$

$$= \sum_{i=1}^{s} \left[\frac{A_{i,e_i}}{(z-\alpha_i)^{e_i}} + \frac{A_{i,e_i-1}}{(z-\alpha_i)^{e_i-1}} + \dots + \frac{A_{i,1}}{(z-\alpha_i)} \right]$$

$A_{i,j} \in \mathbb{C}[z]$

$$f(t) = \frac{1}{(t-3)^1(t+1)^1} = \frac{A}{(t-3)^1} + \frac{B}{(t+1)^1}$$

$$\Updownarrow \qquad A, B \in \mathbb{C}[t]$$
$$\deg A, \deg B \overset{?}{\leq} 1$$

$$1 = (t+1)A + (t-3)B$$
$$= t(A+B) + (A-3B)$$

$$\Rightarrow \quad A+B=0, \qquad A-3B=1$$

Lineare Gleichung

$$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}^{-1} = \frac{1}{-4}\begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{-4}\begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{-4}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$f(t) = \frac{1/4}{t-3} + \frac{-1/4}{t+1}, \qquad -1 < t < 3$$

hat die Stammfkt $F(t) = \frac{1}{4}\log(\underline{3-t}) + $

$$- \frac{1}{4}\log(t+1)$$

$$= \frac{1}{4}\log\left(\frac{3-t}{t+1}\right)$$

$$\uparrow$$

$$\log(ab) = \log a + \log b$$

$$\log : \mathbb{R}^+ \xrightarrow{\cong} \mathbb{R}$$

$$\mathbb{R}^+ = \mathbb{R}_{>0}$$

$$h(t) = \frac{1}{t-\alpha}$$

$$\frac{d}{dt}\log(t-\alpha) \qquad\qquad \frac{d}{dt}\log(\alpha-t)$$

$$\| \qquad\qquad\qquad \|$$

$$\frac{1}{t-\alpha}\underbrace{\frac{d}{dt}(t-\alpha)}_{=1} \qquad\qquad \frac{1}{\alpha-t}\underbrace{\frac{d}{dt}(\alpha-t)}_{=-1}$$

$$\|$$

$$\frac{1}{t-\alpha}$$

$$\log|t-\alpha|$$

$$g(x) = \frac{x-3}{x^3-5x^2+8x-4}$$

$\rightsquigarrow$ 1. Schritt: Faktorisiere den Nenner x^3-5x^2+8x-4

(Gauß-Lemma: rationale Nullstellen teilen den konstanten Koeff.)

$$x^3-5x^2+8x-4 = (x-1)\underbrace{(x^2-4x+4)}_{=(x-2)^2}$$

$$\begin{array}{l}
\quad x^3-5x^2+8x-4 \\
-(x^3-x^2) \\
\hline
\quad -4x^2+8x-4 \\
\quad -(-4x^2+4x) \\
\hline
\qquad 4x-4 \\
\qquad -(4x-4) \\
\hline
\qquad\quad 0
\end{array}$$

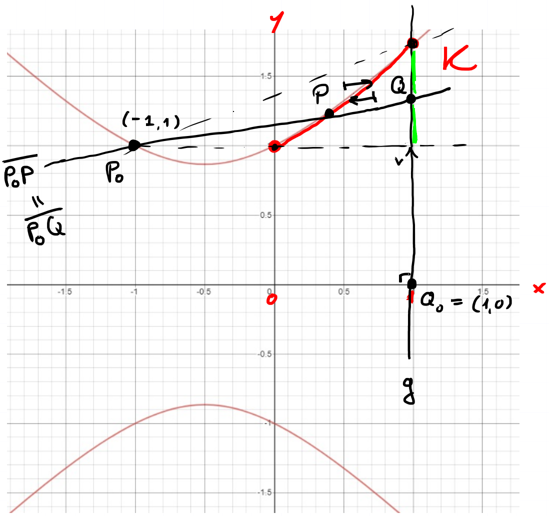
$$g(x) = \frac{A}{(x-2)^2} + \frac{B}{x-2} + \frac{C}{x-1}$$

Aufgabe 2 (Euler-Substitution)**.** Berechne das Integral

$$\int_0^1 \frac{dx}{\sqrt{1+x+x^2}}.$$

$$\underbrace{K:\ y^2 = 1+x+x^2}$$

$$\Longleftrightarrow \quad K = \{(x,y) \mid y^2 = 1+x+x^2\}$$



$P_0 \in K$ $\qquad 0 \le x \le 1$

Gerade $g = Q_0 + \mathbb{R} v = \{Q_0 + tv \mid t \in \mathbb{R}\} = \{(1,t) \mid t \in \mathbb{R}\}$

$$K: \; y^2 = 1 + x + x^2$$

$$g := \{(1,t) \mid t \in \mathbb{R}\}$$

$\underline{P \mapsto Q}$: $\quad P = \left(x, \sqrt{1+x+x^2}\right)$

$$\overline{PP_0} = P_0 + (P - P_0)\mathbb{R}$$

$P_0 = (-1, 1)$ $\quad = \left\{\left(-1 + \lambda(x+1),\; 1 + \lambda\left(\sqrt{1+x+x^2} - 1\right)\right) \,\middle|\, \lambda \in \mathbb{R}\right\}$

$$1 \overset{!}{=} -1 + \lambda(x+1)$$

$$\lambda = \frac{2}{x+1}$$

$$\{Q\} = g \cap \overline{PP_0} = \{(1,t) \mid t \in \mathbb{R}\}$$

$$\Rightarrow Q = \Big(1, \underbrace{1 + \frac{2}{x+1}\left(\sqrt{1+x+x^2} - 1\right)}_{=:t}\Big)$$

$\underline{\underset{\in g}{Q} \mapsto P}$: $\quad \{P\} = \overline{QP_0} \cap K$

$Q = (1,t) \in g$

$$= \Big\{\Big(\underbrace{-1 + \lambda(1+1)}_{=x},\; 1 + \lambda(t-1)\Big) \,\Big|\, \lambda \in \mathbb{R}\Big\} \cap K$$

$$(1+\lambda(t-1))^2 = \cancel{1} + \cancel{(-1} + \cancel{2\lambda}) + (-1+2\lambda)^2$$

$$4\lambda^2 - \cancel{4\lambda} + \cancel{1}$$

$$\parallel$$

$$\lambda^2(t-1)^2 + 2\lambda(t-\cancel{1}) + \cancel{1}$$

$$\Leftrightarrow \quad \lambda^2(t-1)^2 + 2\lambda t = 4\lambda^2 \qquad \Big| : \lambda \neq 0$$

$$\lambda = \frac{2t}{4-(t-1)^2} \qquad \uparrow \quad \lambda = 0 \rightsquigarrow P_0$$

$$x = -1 + 2\lambda = \dots = -\frac{(t+3)(t-1)}{(t-3)(t+1)}$$

$$\sqrt{1+x+x^2} = y = 1+\lambda(t-1)$$

$$= \dots = -\frac{t^2+3}{(t-3)(t+1)}$$

$$d(x(t)) = x'(t)\,dt = \dots = 4\frac{t^2+3}{(t-3)^2(t+1)^2}\,dt$$

$$\parallel$$

$$dx$$

$$\int_0^1 \frac{dx}{y(x)} \underset{\substack{t(0)=1 \\ t(1)=\sqrt{3}}}{\overset{}{=}} \int_1^{\sqrt{3}} \frac{4\frac{\cancel{t^2+3}}{(t-3)^{\cancel{2}}(t+1)^{\cancel{2}}}}{-\frac{\cancel{t^2+3}}{\cancel{(t-3)(t+1)}}}\,dt$$

$$= -4\int_1^{\sqrt{3}} \frac{1}{(t-3)(t+1)}\,dt$$

$$\parallel$$

$$\| $$

$$-4 \left[\frac{1}{4} \log\left(\frac{3-t}{t+1} \right) \right]_{t=1}^{\sqrt{3}}$$

$$\|$$

$$-1 \left(\log\left(\frac{3-\sqrt{3}}{\sqrt{3}+1} \right) - \underbrace{\log(1)}_{=0} \right)$$

$$= \log\left(\frac{\sqrt{3}+1}{3-\sqrt{3}} \right) = \log\left(\frac{(\sqrt{3}+1)(3+\sqrt{3})}{6} \right) = \log\left(1 + \frac{2}{\sqrt{3}}\right)$$

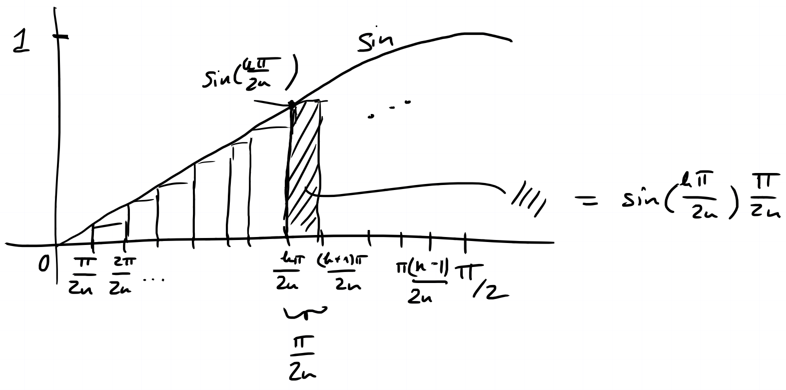
Nachtrag: $\int_0^{\pi/2} \sin x \; dx$

$$\sum_{k=0}^{n-1} \sin\left(\frac{\pi k}{2n} \right) \underset{\uparrow}{=} \operatorname{Im}\left(\frac{1-i}{1-e^{i\frac{\pi}{2n}}} \right)$$

letztes Mal

$$\underset{\uparrow}{=} \frac{1}{2i} \left(\frac{1-i}{1-e^{i\pi/2n}} - \frac{1+i}{1-e^{-i\pi/2n}} \right)$$

$\operatorname{Im}(z) = \frac{z-\bar{z}}{2i}$



$$\int_0^{\pi/2} \sin x \, dx = \lim_{n\to\infty} \frac{\pi}{2n} \sum_{k=0}^{n-1} \sin\left(\frac{k\pi}{2n}\right)$$

$$= \lim_{n\to\infty} \frac{\pi}{2n} \frac{1}{2i} \left(\frac{1-i}{1-e^{i\pi/2n}} - \frac{1+i}{1-e^{-i\pi/2n}} \right)$$

$$= \lim_{x\to 0} \frac{-1}{2} \left((1-i) \underbrace{\frac{x}{1-e^{x}}}_{\to -1} + (1+i) \underbrace{\frac{-x}{1-e^{-x}}}_{\to -1} \right)$$

" $x_n = \frac{i\pi}{2n}$ "

$$\overset{\uparrow}{=} + \frac{1}{2} (1-i+1+i) = 1 .$$

$$e^x = 1 + x + O(x^2) \quad , \quad x \to 0$$

$x \in \mathbb{C}$

$$\Downarrow$$

$$e^x - 1 = x + O(x^2)$$

$$\Downarrow$$

$$\frac{e^x - 1}{x} = 1 + O(x)$$

$$\|$$

$$\frac{\sum_{i=0}^{\infty} \frac{x^i}{i!} - 1}{x} = \frac{\sum_{i=1}^{\infty} \frac{x^i}{i!}}{x} = \sum_{i=1}^{\infty} \frac{x^{i-1}}{i!}$$

$$\|$$

$$1 + O(x)$$

$z \mapsto \frac{1}{z}$ stetig

$\Longrightarrow$ f stetig $\Rightarrow$ $\frac{1}{f}$ stetig

$(\Leftarrow) \left[f(x_n) \to f(x) \iff \frac{1}{f(x_n)} \to \frac{1}{f(x)} \right]$