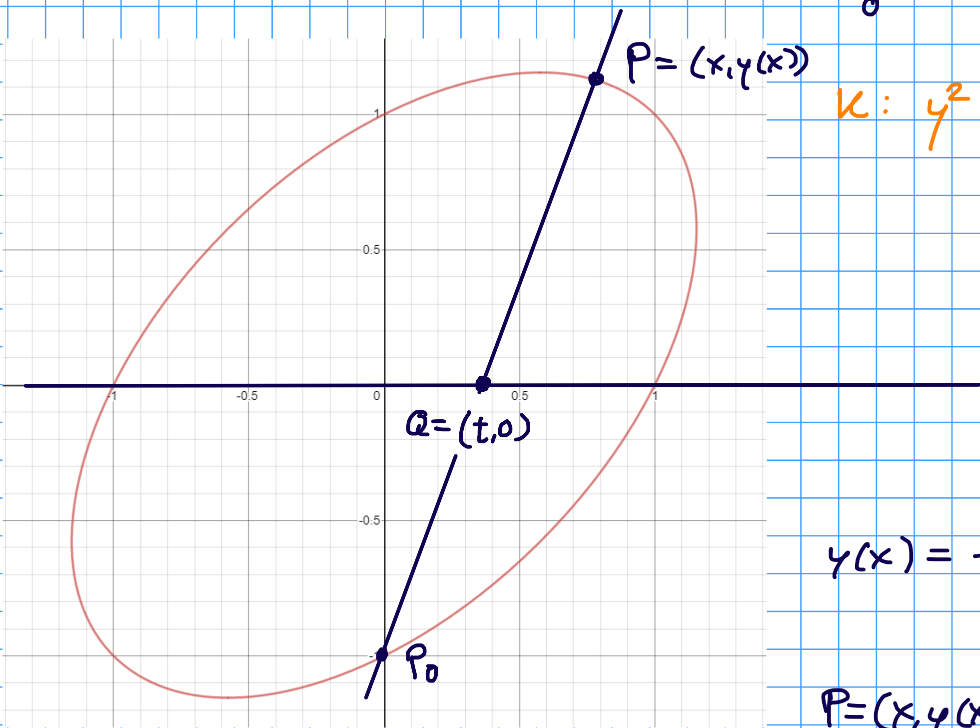
Berechnung des Integrals $\displaystyle\int_0^1 \frac{2\,dx}{x+\sqrt{4-3x^2}}$

$K:\ y^2 - xy + x^2 - 1 = 0$



$\ell = \mathbb{R}(1,0)$

$$y(x) = \frac{x+\sqrt{4-3x^2}}{2} \geq 0, \quad x \in [0,1].$$

$P = (x, y(x)) \in K$, Mitternachtsformel.

$P_0 = (0,-1) \in K$.

Gerade durch P & P_0

$$= P_0 + \mathbb{R}(P - P_0) = \{(0,-1) + \lambda(x, y(x)+1) \mid \lambda \in \mathbb{R}\}$$

schneidet die reelle Achse für $1 = \lambda(y(x)+1)$

$$\iff \lambda = \frac{1}{y(x)+1},$$

Führe also die Substitution

$$t = \lambda x = \frac{x}{y(x)+1}$$ durch.

Eine kurze Rechnung liefert die Umkehrabbildung

$$x(t) = -\frac{t^2-2t}{t^2-t+1},$$

und damit $$dx = x'(t)\,dt = -\frac{t^2+2t-2}{(t^2-t+1)^2}\,dt$$

und $$y(x(t)) = \frac{x(t)}{t} - 1 = -\frac{t-2}{t^2-t+1} - 1 = -\frac{t^2-1}{t^2-t+1}.$$

Insgesamt: $$\int_0^1 \frac{dx}{y(x)} = \int_0^{1/2} \frac{t^2+2t-2}{(t^2-1)(t^2-t+1)}\,dt$$

$$\underset{\substack{\uparrow\\ \text{Partialbruchzerlegung}}}{=} \int_0^{1/2} \left(\frac{2-t}{t^2-t+1} + \frac{1/2}{t-1} + \frac{1/2}{t+1} \right) dt$$

$$= \cancel{-\frac{1}{2}\log\left(\frac{3}{4}\right)} + \boxed{\sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right)} + \cancel{\frac{1}{2}\left(\log\left(\frac{3}{2}\right) + \log\left(\frac{1}{4}\right)\right)}$$

$$\int_0^{1/2} \frac{1}{t \pm 1}\,dt = \Big[\log|t \pm 1|\Big]_0^{1/2} = \begin{cases} \log\left(3/2\right) \\ \log\left(1/2\right) \end{cases}$$

$$\int_0^{1/2} \frac{2-t}{t^2-t+1}\,dt = -\int_0^{1/2} \frac{t-\frac{1}{2}-\frac{3}{2}}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}\,dt =$$

$$\underset{\substack{\uparrow\\ \text{S.187f. im Skript}}}{=} -\left[\frac{1}{2}\log\left(\left(t-\tfrac{1}{2}\right)^2+\tfrac{3}{4}\right)\right]_0^{1/2} + \frac{3}{2}\left[\sqrt{\frac{4}{3}}\arctan\left(\frac{t-\frac{1}{2}}{\sqrt{3/4}}\right)\right]_0^{1/2}$$

$$= -\frac{1}{2}\log\left(\frac{3}{4}\right) + \sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right)$$

Endergebnis: $$\int_0^1 \frac{dx}{y(x)} = \sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{2\sqrt{3}} \approx 0{,}9069$$