Unit : 01 Predicate Calculus

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* Sentence : A collection of words making a complete grammatical structure with meaning and sense is known as sentence.

* Proposition (Statement) : A declarative sentence which can be either true or false is known as a proposition.
→ It is also known as statement.

For example :

1). New Delhi is a capital of India.
2). The square of 4 is 16.
3). Bring me coffee.
4). Where are you going?
5). This statement is false.

→ 1, 2 : is proposition as sentence 1 and 2 are true.
→ 3 : we can NOT decide whether it is true or false as it is not a declarative sentence [Command.].
→ 4 : is a question. So, it is not a proposition
→ 5 : has both true as well as false value. So, 5^{th} sentence is NOT a proposition.

→ Propositions are denoted by p, q, r OR P, Q, R and also known as propositional variables.

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* Truth value :-
The truth or falsity of a statement is known as its truth value. (True or false)

* TYPES OF STATEMENTS / PROPOSITIONS :-

* Compound statement :
A proposition which is combination of two or more propositional variables, is known as compound statement. It is also known as molecular or composite statement.

'and' & 'or'

* Atomic statement : (Primary / Premitive)
A proposition consisting of only a single propositional variable or a single propositional constant is known as atomic statement. [which is not breakable.]
It is also known as primary or premitive statement.

* Truth table :
A truth table is a table that shows the truth value of a compound proposition for all possible value.

* Connectives [Basic logical operations] :
The words or phrases which are used to form a proposition are known as connectives.

There are five basic connectives :-

1) Conjuction : AND
2) Disjunction : OR
3) Negation : NOT
4) Conditional : If... then
5) Bi-conditional : If and only if

1. Conjuction (AND) :-

→ If p and q are two statements then conjuction of p and q is the compound statement of the form "p and q" and it is denoted by $p \wedge q$, which is true when both p and q are true.

→ Truth table :

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2. Disjunction (OR) :-

→ If p and q are two statements then disjunction of p and q is the compound statement of the form "p or q" and it is denoted by $p \vee q$, which is false when both p and q are false.

→ Truth table :

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

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3. Negation (NOT) :-

→ If p is any proposition, the negation of p is denoted by '$\sim p$' or '$\neg p$', is a proposition which is false when p is true and true when p is false. It is also known as unary operator.

→ Truth table :

p	$\sim p$
T	F
F	T

(Implication)

4. Conditional (If... then) :-

→ If p and q are two statement then conditional statement of p and q is the compound statement of the form "if p then q" and it is denoted by $p \to q$, which is false when p is true and q is false.

→ Truth table :

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

* Remark :-

$$p \to q$$

Hypothesis or antecedent — Conclusion Consequent

→ The connectives if ... then can be also read as follows :

p implies q.

p is sufficient for q.

p only if q.

q is necessary for p.

q if p.

q follows from p.

q is consequence of p.

* $p \to q \equiv \sim p \vee q$.

5. Bi-conditional (If and only if) :-

→ If p and q are two statements then bi-conditional statement of p and q is the compound statement of the form "if p only if q" and it is denoted by $p \leftrightarrow q$, which is true when p and q both are true or false simultaneously.

→ Truth table :

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

* Converse, Contra positive and Inverse of an implication :-

→ There are some related implication.

→ When $p \rightarrow q$ is an implication,

Converse : $q \rightarrow p$

Contra positive : $\sim q \rightarrow \sim p$

Inverse : $\sim p \rightarrow \sim q$

* Remark :-

→ A conditional proposition $(p \rightarrow q)$ and its converse $(q \rightarrow p)$ or inverse $(\sim p \rightarrow \sim q)$ are not logically equivalent.

On the other hand, a conditional proposition $(p \rightarrow q)$ and it's contra positive $(\sim q \rightarrow \sim p)$ are logically equivalent. [Can be checked using the truth table.]

→ The importance of the contra positive derives from the fact that mathematical theorems in the form $p \rightarrow q$ can sometimes be proved easily when restated in the form $\sim q \rightarrow \sim p$.

Ex :- Prove that if x^2 is divisible by 4, then x is even.

→ Here, let p : x^2 is divisible by 4

and q : x is even.

→ The implication is of the form $p \to q$. So, the contra positive is $\sim q \to \sim p$. which states in words that: If x is odd, then x^2 is not divisible by 4.

→ Since x is odd; let $x = 2k+1$; $k \in N$. where k = positive integer number.

Now, $x^2 = (2k+1)^2$

$= 4k^2 + 4k + 1$

$\therefore x^2 = 4\left[k^2 + k + \frac{1}{4}\right]$; $k \in N$.

Here, $k^2 + k$ is an integer but $1/4$ is not an integer.

Therefore x^2 is not divisible by 4.

Ex:- State converse, contra positive and inverse of the following statements.

1) If it rain then the crop will grow.

→ p : it rain, q : the crop will grow.

→ Converse $(q \to p)$: If the crop grow, then there has been rain.

→ Contra positive $(\sim q \to \sim p)$:

If the crop does not grow, then there has been no rain.

→ Inverse $(\sim p \to \sim q)$: If it does not rain, then the crop will not grow.

2) If a triangle is not isosceles then it is not equilateral.

→ p : A triangle is not isosceles.
q : it is not equilateral.

→ Converse $(q \rightarrow p)$:
If a triangle is not equilateral then it is not isosceles.

→ Contra positive $(\sim q \rightarrow \sim p)$:
If a triangle is equilateral then it is isosceles.

→ Inverse $(\sim p \rightarrow \sim q)$:
If a triangle is isosceles then it is equilateral.

Ex:- If a triangle is isosceles, then two of its sides are equal.

→ p : A triangle is isosceles
q : two of its sides are equal.

→ Converse $(q \rightarrow p)$:
If the two sides of a triangle are equal then it is an isosceles.

→ Contra positive $(\sim q \rightarrow \sim p)$:
If the two sides of a triangle are not equal then it is not an isosceles.

→ Inverse $(\sim p \rightarrow \sim q)$:
If a triangle is not isosceles then two of its sides are not equal.

Ex:- If there is not unemployment in India, then the Indian's won't go to the USA for employment.

→ p : There is not unemployment in India.
q : The Indian's won't go to the USA for employment.

→ Converse $(q \rightarrow p)$:
If the Indians won't go to the USA for employment then there will not be unemployment in India.

→ Contra positive $(\sim q \rightarrow \sim p)$:
If the Indian's will go to the USA for employment then there will be unemployment in India.

→ Inverse $(\sim p \rightarrow \sim q)$:
If there is unemployment in India, then the Indian's will go to the USA for employment.

Ex:- If p represents "This book is good" and q represents "This book is cheap", then write the following sentences in symbolic form.

1) $P \wedge Q$: This book is good and cheap.
2) $P \vee Q$: This book is either good or cheap.
3) $\sim P \wedge Q$: This book is not good and cheap.
4) $\sim Q \wedge P$: This book is costly and/but good.
5) $\sim P \wedge \sim Q$: This book is neither good nor cheap.

Ex:- If P : It is raining. Q : I have the time. R : I will go to a movie. Write the sentences in English corresponding to the following propositional form $(\sim P \wedge Q) \Leftrightarrow R$, $(Q \rightarrow R) \wedge (R \rightarrow Q)$, $\sim Q \vee R$ and $R \rightarrow (\sim P \wedge Q)$.

→ P : It is raining
$\sim P$: It is not raining.

→ Q : I have the time.
$\sim Q$: I have no time.

→ R : I will go to the movie.
$\sim R$: I will not go to the movie.

1) $(\sim P \wedge Q) \leftrightarrow R$
→ I will go to the movie if and only if it is not raining and I will have time

2) $(Q \rightarrow R) \wedge (R \rightarrow Q)$ or $(R \Leftrightarrow Q)$
→ I will go to the movies if and only if I have time.

3). $\sim Q \vee R$

→ I have no time or I will go to a movie.

4) $R \rightarrow (\sim p \wedge q)$

→ I will go to a movie only if it is not raining and I have the time.

Ex:- Construct a truth table for each compound proposition.

1. $p \wedge (\sim q \vee q)$

p	q	$\sim q$	$(\sim q \vee q)$	$p \wedge (\sim q \vee q)$
T	T	F	T	T
T	F	T	T	T
F	T	F	T	F
F	F	T	T	F

2. $(p \vee q) \wedge (p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \vee q$	$p \rightarrow q$	$(p \vee q) \wedge (p \rightarrow q)$	$q \rightarrow p$	$(p \vee q) \wedge (p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T	T
T	F	T	F	F	T	F
F	T	T	T	T	F	F
F	F	F	T	F	T	F

~~3. $(P \vee Q) \rightarrow (P \vee R) \rightarrow (Q \vee R)$.~~

~~P~~	~~Q~~	~~R~~	~~$(P \vee Q)$~~	~~$(P \vee R)$~~
~~T~~	~~T~~		~~T~~	
~~T~~	~~F~~		~~T~~	
~~F~~	~~T~~		~~T~~	
~~F~~	~~F~~		~~F~~	

Extra Practice Examples :-

1. $(P \vee Q) \rightarrow (P \vee R) \rightarrow (Q \vee R)$

P	Q	R	$(P \vee Q)$	$(P \vee R)$	$(Q \vee R)$	$(P \vee Q) \rightarrow (P \vee R)$	$(P \vee Q) \rightarrow (P \vee R) \rightarrow (Q \vee R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
F	T	T	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	F	F	T	F
T	F	F	T	T	F	F	T
F	T	F	T	F	T	T	T

2. $(\sim P \leftrightarrow \sim Q) \leftrightarrow Q \leftrightarrow R$

P	Q	R	$\sim P$	$\sim Q$	$(\sim P \leftrightarrow \sim Q)$	$(\sim P \leftrightarrow \sim Q) \leftrightarrow Q$	$(\sim P \leftrightarrow \sim Q) \leftrightarrow Q \leftrightarrow R$
T	T	T	F	F	T	T	T
T	T	F	F	F	T	T	F
T	F	T	F	T	F	T	T
T	F	F	F	T	F	T	F
F	T	T	T	F	F	F	F
F	F	T	T	T	T	F	F
F	T	F	T	F	F	F	T
F	F	F	T	T	T	F	T

3. $(p \wedge q) \vee (\sim p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$(\sim p \wedge q)$	$(p \wedge \sim q)$	$(\sim p \wedge \sim q)$	Answer
T	T	F	F	T	F	F	F	T
T	F	F	T	F	F	T	F	T
F	T	T	F	F	T	F	F	T
F	F	T	T	F	F	F	T	T

4. $p \wedge (q \vee r)$

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	F	T	T	F
F	T	F	T	F
F	T	T	T	F
F	F	F	F	F

5. $\sim (p \vee q) \vee (\sim p \wedge \sim q)$

p	q	$p \vee q$	$\sim (p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	Answer
T	T	T	F	F	F	F	F
T	F	T	F	F	T	F	F
F	T	T	F	T	F	F	F
F	F	F	T	T	T	T	T

* ALGEBRA OF PROPOSITION :-

1. Idempotent law :-

$p \vee p \equiv p$
$p \wedge p \equiv p$

2. Associative law :-

$p \vee (q \vee r) \equiv (p \vee q) \vee r$
$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

3. Commutative law :-

$p \vee q \equiv q \vee p$
$p \wedge q \equiv q \wedge p$

4. Distributive law :-

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

5. De Morgan's law :-

* $\sim (p \vee q) \equiv \sim p \wedge \sim q$
* $\sim (p \wedge q) \equiv \sim p \vee \sim q$

6. Involution law :-

$\sim (\sim p) = p$

7. Complement law :- $p \to q \equiv \sim p \vee q$

$\sim F \equiv T$

$\sim T \equiv F$

$p \vee \sim p \equiv T$

$p \wedge \sim p \equiv F$

8. Identity law :-

$p \vee F \equiv p$

$p \vee T \equiv T$

$p \wedge T \equiv p$

$p \wedge F \equiv F$

* SOME DERIVED CONNECTIVES :-

1. NAND :-

NAND is the negation of conjuction of two statements. Assume p and q are any two statements then NAND of p and q is a proposition which is false when both p and q are true otherwise false. It is denoted by $p \uparrow q$.

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

$p \uparrow q \equiv \sim (p \wedge q)$

2. NOR :-

NOR is negation of disjunction of two statements. Assume p and q be two proposition. NOR of p and q is a proposition which is true when both p and q are false. otherwise true.

It is denoted by $p \downarrow q$.

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

$p \downarrow q \equiv \sim(p \vee q)$

3. XOR (Exclusive OR) :-

It is a proposition that is true when exactly one of p and q is true but not both and otherwise it is false.

It is denoted by $p \oplus q$.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

XOR : p or q but not both

$\Rightarrow$ p or q and not both

$\Rightarrow (p \vee q) \wedge \sim(p \wedge q)$

* $p \uparrow q \equiv \sim(p \wedge q)$
* $p \downarrow q \equiv \sim(p \vee q)$
* $p \oplus q \equiv (p \vee q) \wedge \sim(p \wedge q)$

* LOGICAL EQUIVALENCE :-

→ If two propositions P and Q have the same truth values in every possible case, the propositions are logically equivalent.
It is denoted by $p \equiv q$.

→ To test whether two proposition P and Q are logically equivalent.

Method : 1 : Using truth table.
Method : 2 : Using Algebra of proposition
[Properties of proposition]

Method 1 : Using Truth Table

Step 1 : Construct the truth table for compound statement P and Q.

Step 2 : Check each combination of truth values of the propositional variables to see whether the value of P is same as the truth value of Q.
If in each row the truth ~~table~~ value of P is the same as the truth value of Q, then P and Q are logically equivalent.

→ Number of combinations : 2^n ; $n = 3$ ($\because$ p, q, r)

Ex:- Use truth tables to prove

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r).$$

p	q	r	$q \wedge r$	$\boxed{p \vee (q \wedge r)}$	$p \vee q$	$p \vee r$	$\boxed{(p \vee q) \wedge (p \vee r)}$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	~~F~~ T	T	T
F	T	T	T	T	~~F~~ T	T	T
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F
F	T	F	F	F	T	~~T~~ F	~~T~~ F
T	F	F	F	T	T	~~F~~ T	~~F~~ T

→ Since the entries of $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ is same,

$\therefore$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

Ex:- Use truth tables to prove

$$\sim (p \wedge q) \equiv (\sim p \vee \sim q)$$

p	q	$p \wedge q$	$\boxed{\sim (p \wedge q)}$	$\sim p$	$\sim q$	$\boxed{\sim p \vee \sim q}$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	F	F	T	T	T	T
F	T	F	T	T	F	T

→ Since the entries of $\sim (p \wedge q)$ column and $\sim p \vee \sim q$ column are same.

Therefore, $\sim (p \wedge q) \equiv \sim p \vee \sim q$.

$p \to q \equiv \sim p \vee q$

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Method 2 : Using Algebra of Proposition (8 - laws)
(well known identities)

Ex:- Show that $(p \wedge q) \vee (p \wedge \sim q) \equiv p$.

→ $(p \wedge q) \vee (p \wedge \sim q) \equiv p \wedge (q \vee \sim q)$ ($\because$ distributive law)
$\equiv p \wedge T$ ($\because$ Complement law).
$\equiv p$ ($\because$ Identity law).

Ex:- Show that $(P \to q) \wedge (R \to q) \equiv (P \vee R) \to q$

→ $(P \to q) \wedge (R \to q) \equiv (\sim P \vee q) \wedge (\sim R \vee q)$
($\because$ Equivalent form of $p \to q \equiv \sim p \vee q$)
$\equiv (q \vee \sim P) \wedge (q \vee \sim R)$
($\because$ Commutative law)
$\equiv q \vee (\sim P \wedge \sim R)$
($\because$ Distributive law)
$\equiv q \vee (\sim (P \vee R))$
($\because$ De Morgan's law).
$\equiv (\sim (P \vee R)) \vee q$
($\because$ commutative law).
$\equiv (P \vee R) \to q$
($\because$ Equivalent form of $p \to q \equiv \sim p \vee q$)

→ Hence proved.

Ex:- Show that $(\sim P \wedge (\sim q \wedge R)) \vee (q \wedge R) \vee (P \wedge R) \equiv R$.

→ L.H.S. $= (\sim P \wedge (\sim q \wedge R)) \vee (q \wedge R) \vee (P \wedge R)$
$\equiv [(\sim P \wedge \sim q) \wedge R] \vee (q \wedge R) \vee (P \wedge R)$
($\because$ Associative law).

$\equiv [\sim(P \vee Q) \wedge R] \vee (Q \wedge R) \vee (P \wedge R)$

(∵ De' Morgan's law)

$\equiv [\sim(P \vee Q) \wedge R] \vee [(Q \vee P) \wedge R]$ (∵ Distributive)

$\equiv [\sim(P \vee Q) \wedge R] \vee [(P \vee Q) \wedge R]$

(∵ Commutative law)

$\equiv [\sim(P \vee Q) \vee (P \vee Q)] \wedge R$

(∵ Distributive law)

$\equiv T \wedge R$ (∵ Complement law)

$\equiv R$ (∵ Identity law)

* Practice Example :-

Ex :- Show that the following statements are logically equivalent using truth table.

1. $p \leftrightarrow q \equiv [(p \rightarrow q) \wedge (q \rightarrow p)]$

p	q	$p \rightarrow q$	$q \rightarrow p$	$[(p \rightarrow q) \wedge (q \wedge p)]$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

→ The values of the $p \leftrightarrow q$ column and $[(p \rightarrow q) \wedge (q \wedge p)]$ column are same.

→ Therefore, we can say that,

$p \leftrightarrow q \equiv [(p \rightarrow q) \wedge (q \rightarrow p)]$.

2. $p \leftrightarrow q \equiv [(p \vee q) \rightarrow (p \wedge q)]$

p	q	$p \leftrightarrow q$	$p \vee q$	$p \wedge q$	$[(p \vee q) \rightarrow (p \wedge q)]$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	F	F
F	F	T	F	F	T

→ The values of the column $p \leftrightarrow q$ and $[(p \vee q) \rightarrow (p \wedge q)]$ are same.
Therefore, $p \leftrightarrow q \equiv [(p \vee q) \rightarrow (p \wedge q)]$.

3. $(p \vee q) \rightarrow r \equiv [(p \rightarrow r) \wedge (q \rightarrow r)]$

p	q	r	$p \rightarrow r$	$q \rightarrow r$	$p \vee q$	$(p \vee q) \rightarrow r$	$[(p \rightarrow r) \wedge (q \rightarrow r)]$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	T	T	T	T	T
F	T	T	T	T	T	T	T
F	F	T	T	T	F	T	T
F	T	F	T	F	T	F	F
F	F	F	T	T	F	T	T
T	F	F	F	T	T	F	F.

→ The values of the columns $(p \vee q) \rightarrow r$ and $[(p \rightarrow r) \wedge (q \rightarrow r)]$ are same.

→ Therefore we can say that,
$(p \vee q) \rightarrow r \equiv [(p \rightarrow r) \wedge (q \rightarrow r)]$

Ex:- Show that the following statements are logically equivalent without using truth table.

4. $p \leftrightarrow q \equiv (p \vee q) \rightarrow (p \wedge q)$

→ L.H.S. $= p \leftrightarrow q$

$= (p \rightarrow q) \wedge (q \rightarrow p)$

$= (\sim p \vee q) \wedge (\sim q \vee p)$

$(\because p \rightarrow q = \sim p \vee q)$

$= [(\sim p \vee q) \wedge \sim q] \vee [(\sim p \vee q) \wedge p]$

$(\because$ Distributive law $)$

$= [\sim q \wedge (\sim p \vee q)] \vee [p \wedge (\sim p \vee q)]$

$(\because$ Commutative law $]$

$= (\sim q \wedge \sim p) \vee (\sim q \wedge q) \vee$

$(p \wedge \sim p) \vee (p \wedge q)$

$(\because$ Distributive law $)$

$= \sim (q \vee p) \vee F \vee F \vee (p \wedge q)$

$(\because$ Complement law $)($De'morgan$)$

$= \sim (p \vee q) \vee F \vee (p \wedge q)$

$(\because$ Commutative law $)$

$= \sim (p \vee q) \vee (p \wedge q)$

$(\because$ Identity law $)$

$= (p \vee q) \rightarrow (p \wedge q)$

$(\because$ we know that

$p \rightarrow q = \sim p \vee q)$

* TAUTOLOGY :-

- A statement which is always true is known as tautology.
- It is denoted by t or T (sometimes).
- It is also known as a logical truth.

Ex :-

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

(always true)

$\therefore p \vee \sim p$ is tautology.

- There are two methods to check whether the given proposition is a tautology.
  1) Using truth table method.
  2) Using algebra of proposition / identities.

Ex :- Show : $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$ is Tautology.

- Let $\alpha = (P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$

P	Q	R	$(Q \to R)$	$P \to (Q \to R)$	$P \to Q$	$P \to R$	$(P \to Q) \to (P \to R)$	α
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	T	T	T
T	F	F	T	T	F	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	F	T	T	T	T	T	T
F	F	T	T	T	T	T	T	T

→ Since the truth table values is true for all possible values of the propositional variables which can be seen in the last column of the table, the given proposition is a Tautology.

Ex:- Using identities, prove that the following statement is a Tautology :

$$q \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$$

→ $q \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$

$\equiv [(q \vee p) \wedge (q \vee \sim q)] \vee (\sim p \wedge \sim q)$

(∵ Distributive law)

$\equiv [(p \vee q) \wedge T] \vee \sim (p \wedge q)$

(∵ Complement law and Commututive, De' morgan's law)

$\equiv (p \wedge q) \vee \sim (p \wedge q)$

(∵ Identity law)

$\equiv T$

(∵ Complement law)

→ Hence we can say that the given statement is a Tautology.

* CONTRADICTION :-

→ A statement which is always false is known as contradiction or fallacy.

→ It is denoted by c or F.

→ It is also known as Contradiction.

Ex:-

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

$\therefore p \wedge \sim p$ is a contradiction (always false)

Ex:- Verify that the proposition $p \wedge (q \wedge \sim p)$ is a contradiction.

→

p	q	$\sim p$	$q \wedge \sim p$	$p \wedge (q \wedge \sim p)$
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F
T	T	F	F	F

→ Since the truth table values is false for all possible values of the propositional variables which can be seen in the last column of the table, the given proposition is a contradiction.

Ex:- Show that $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction using the truth table and identities.

→

p	q	$p \vee q$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

→ Since the truth table values of the last column for the all possible values is false, the given statement is contradiction.

→ $(p \vee q) \wedge (\sim p \wedge \sim q)$.

$= (p \vee q) \wedge \sim (p \vee q)$
(∵ De'morgan's law)

$= F$ (∵ Complement law)

→ Hence, we can say that the given statement is a contradiction.

* CONTINGENCY :-

→ A statement which is neither tautology nor contradiction (fallacy) is known as contingency.

Ex:-

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

$p \vee q$ → which has both the values T and F.

* FUNCTIONALLY COMPLETE SET OF CONNECTIVES :-

$\{\wedge, \vee, \sim, \rightarrow, \leftrightarrow\}$

→ Any set of connective in which every formula can be expressed in terms of an equivalence formula containing the connectives from the set is called functionally complete set of connectives.

*** Remark :-**

A minimal functionally complete set of connectives does not contain a connective which can be expressed in terms of other connectives.

$\{\vee, \sim, \rightarrow, \leftrightarrow\}$ can not be generated. ⇒ Not functionally complete set of connectives.

singleton set of connectives (first three) → $\{\wedge\}$, $\{\vee\}$, $\{\sim\}$, $\{\wedge, \vee\}$ are <u>not</u> functionally complete set of connectives.

∵ we can not generate others except it, in $\{\wedge, \vee, \sim, \rightarrow, \leftrightarrow\}$.

Ex:- Prove that $\{\wedge, \sim\}$ and $\{\vee, \sim\}$ is a minimal functionally complete set of connectives.

→ We have five basic connectives. $\{\wedge, \vee, \rightarrow, \leftrightarrow, \sim\}$.

→ To eliminate the conditional, one uses the following logical equivalence.

$$p \rightarrow q \equiv \sim p \vee q \quad \text{—— (1)}$$

→ There are two ways to express the biconditional as given by,

*

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q) \quad \text{—} (2)$$

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p) \quad \text{—} (3).$$

/* $\equiv (\sim p \vee q) \wedge (\sim q \wedge p)$. $= (p \wedge q) \vee (\sim p \wedge \sim q)$ */

Using (1) in (3), we get,

$$p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p).$$

Thus all the conditional and biconditional can be replaced by the three connectives $\wedge$, $\vee$ and $\sim$.

<u>Ex</u>:- Write an equivalent formula for $P \wedge (Q \leftrightarrow R)$ which contains neither the biconditional nor the conditional.

→ $P \wedge (Q \leftrightarrow R) \equiv P \wedge (Q \rightarrow R) \wedge (R \rightarrow Q)$

$\equiv P \wedge (\sim Q \vee R) \wedge (\sim R \vee Q)$

<u>Ex</u>:- Write an equivalent formula for $P \wedge (Q \leftrightarrow R) \vee (R \leftrightarrow P)$ which contains neither the biconditional nor the conditional.

→ $P \wedge (Q \leftrightarrow R) \vee (R \leftrightarrow P)$

$\equiv P \wedge (Q \rightarrow R) \wedge (R \rightarrow Q) \vee (R \rightarrow P) \wedge (P \rightarrow R)$

$\equiv P \wedge (\sim Q \vee R) \wedge (\sim R \vee Q) \vee$

$(\sim R \vee P) \wedge (\sim P \vee R)$

* NORMAL FORM :

→ By comparing the truth tables, one determines whether two logical expression P and Q are equivalent.

→ But the process is very tedius when the number of variables increases.

→ A better method is to transform the expression P and Q to some standard forms of expression P' and Q' such that a simple comparison of P' and Q' shows whether two logical expression P' and Q' are equivalent or not.

→ The standard forms are called normal forms or canonical forms.

* Elementary SUM :

→ In logical expression, a sum of the variables and their negation is called elementary sum.

→ Ex :- p, $\sim p$, $p \vee q$, $\sim p \vee \sim q$, $\sim p \vee q$, $p \vee \sim q$.

* Elementary PRODUCT :

→ In logical expression, a product of the variables and their negation is called an elementary product.

→ **Ex:-** p, $\sim p$, $p \wedge q$, $\sim p \wedge \sim q$, $\sim p \wedge q$, $p \wedge \sim q$.

three variables ($p \wedge q \wedge r$, $p \wedge q \wedge \sim r$)

1. Disjunctive Normal Form (DNF) :-

→ A logical expression is said to be in disjunctive normal form if it is <u>the sum of elementary products.</u>

* Procedure to obtain a DNF of a given logical expression :

1) Remove conditional ($\rightarrow$) and bi-conditional ($\leftrightarrow$) using an equivalent expression which contains negation, disjunction and conjunction only.

2) Eliminate negation before the sum and products using De Morgan's Law.

3) Apply distributive law until a sum of elementary product is obtained.

* Obtain the DNF / disjunctive normal form of the given examples :-

1) $p \wedge (p \rightarrow q)$

$\Rightarrow p \wedge (p \rightarrow q) \equiv p \wedge (\sim p \vee q)$

$\equiv (p \wedge \sim p) \vee (p \wedge q)$

($\because$ distributive law)

which is the required DNF.

2) $(p \wedge \sim(q \wedge r)) \vee (p \rightarrow q)$

$\equiv (p \wedge \sim(q \wedge r)) \vee (\sim p \vee q)$

logically equivalent statement

$\equiv (p \wedge (\sim q \vee \sim r)) \vee (\sim p \vee q)$

De Morgan's law.

$\equiv [(p \wedge \sim q) \vee (p \wedge \sim r)] \vee (\sim p \vee q)$

Distributive law.

$\equiv (p \wedge \sim q) \vee (p \wedge \sim r) \vee \sim p \vee q.$

which is the required DNF.

3) $p \vee (\sim p \rightarrow (q \vee (q \rightarrow \sim r)))$

$\equiv p \vee (\sim p \rightarrow (q \vee (\sim q \vee \sim r)))$

logically equivalent.

$\equiv p \vee (p \vee (q \vee (\sim q \vee \sim r)))$

$\equiv p \vee p \vee q \vee \sim q \vee \sim r$

Associatives law.

$\equiv p \vee q \vee \sim q \vee \sim r$

which is the required DNF.

Remark: Disjunctive normal form (DNF) of logical expression is not unique.

2. Conjunctive Normal Form (CNF) :-

→ A logical expression is said to be in disjunctive form if it is the product of elementary sum.

* Procedure to obtain CNF of a given logical expression :

1) Remove conditional (→) and bi-conditional (↔) using an equivalent expression which contains negation, disjunction and ~~conjunct~~ conjunction only.

2) Eliminate negation before the sum and products using De Morgan's law.

3) Apply distributive law until the product of elementary sum is obtained.

* Obtain the CNF / conjunctive normal form of the given examples :

1) $p \wedge (p \rightarrow q)$

→ $p \wedge (p \rightarrow q) \equiv p \wedge (\sim p \vee q)$

(Removed conditional by logical connectives.)

2) $[q \vee (p \wedge r)] \wedge \sim[(p \vee r) \wedge q]$

$\equiv [q \vee (p \wedge r)] \wedge [\sim(p \vee r) \vee \sim q]$

De Morgan's law.

$\equiv [q \vee (p \wedge r)] \wedge [(\sim p \wedge \sim r) \vee \sim q]$

De Morgan's law.

$\equiv [(q \vee p) \wedge (q \vee r)] \wedge [(\sim q \vee \sim p) \wedge (\sim q \vee \sim r)]$

Distributive law.

$\equiv (q \vee p) \wedge (q \vee r) \wedge (\sim q \vee \sim p) \wedge (\sim q \vee \sim r)$

Which is required CNF.

Remark: Conjunctive normal form (CNF) of logical expression is not unique.

* Points to remember for DNF and CNF.

1. $p \rightarrow q \equiv \sim p \vee q$

2. $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$

3. De Morgan's law :

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$
$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

4. Distributive law :

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

V ✱ PRINCIPAL DISJUNCTIVE NORMAL FORM :-
(PDNF)

→ Minterms : Let p and q be two statement variables then $p \wedge q$, $\sim p \wedge q$, $p \wedge \sim q$, $\sim p \wedge \sim q$ are called minterms of p and q.

Single propositional variable will not be here.

→ The number of minterms in n variables is 2^n.
For example, the minterm for three variables p, q and r are
① $p \wedge q \wedge r$, ② $\sim p \wedge q \wedge r$, ③ $p \wedge \sim q \wedge r$, ④ $p \wedge q \wedge \sim r$,
⑤ $\sim p \wedge \sim q \wedge r$, ⑥ $p \wedge \sim q \wedge \sim r$, ⑦ $\sim p \wedge q \wedge \sim r$, ⑧ $\sim p \wedge \sim q \wedge \sim r$.

→ The truth table for the minterms of p and q :

p	q	$p \wedge q$	$\sim p \wedge q$	$p \wedge \sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F
T	F	F	F	T	F
F	T	F	T	F	F
F	F	F	F	F	T

→ The truth table for the minterms of p, q and r:

p	q	r	~p	~q	~r	1	2	3	4	5	6	7	8
T	T	T	F	F	F	T	F	F	F	F	F	F	F
T	T	F	F	F	T	F	F	F	T	F	F	F	F
T	F	T	F	T	F	F	F	T	F	F	F	F	F
F	T	T	T	F	F	F	T	F	F	F	F	F	F
T	F	F	F	T	T	F	F	F	F	F	T	F	F
F	F	T	T	T	F	F	F	F	F	T	F	F	F
F	T	F	T	F	T	F	F	F	F	F	F	T	F
F	F	F	T	T	T	F	F	F	F	F	F	F	T

* Remarks :

1) From the truth table it is clear that no minterms are equivalent.

MCQ 2) Each minterms has truth value T for exactly one combination of the truth values of the variables p and q.

* PDNF :

→ PDNF of a given formula can be defined as an equivalent formula consisting of disjunctives of minterms. [elementary product which contains the entire variables] only.

→ This is also known as sum of products of canonical form.

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→ There are two ways to obtain the principal disjunctive normal form :

i) Using truth table
ii) Using algebra of identities/proposition.

* Method : 1 : Using truth table.

Step 1 : Construct a truth table for the given compound statement.

Step 2 : For every truth value T of the given proposition, select the minterms which has also true value T for the same combination of truth value of the statement variable.

Step 3 : The disjunctive of minterms selected in step 2 is the required principal disjunctive normal form.

Remark : In PDNF, the repeated minterms are mentioned only once.

Ex:- Obtain the (PDNF) principal disjunctive normal form using the truth table of $p \rightarrow q$.

→ Truth table :

p	q	$p \rightarrow q$	Minterms
T	T	T	$p \wedge q$
T	F	F	–
F	T	T	$\sim p \wedge q$
F	F	T	$\sim p \wedge \sim q$

$p \rightarrow T$

$\sim p \rightarrow F$

/* → The column containing $p \to q$ has truth value T for three combinations of the truth values of p and q.

→ T in the first row of $p \to q$ corresponds to the minterms $p \wedge q$.
Similarly T in the third and fourth row of $p \to q$ corresponds to the minterms $\sim p \wedge q$ and $\sim p \wedge \sim q$ respectively. */

→ Thus, the PDNF of $p \to q$ is,

$$(p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$$

* Method : 2 : Without using truth table.

Step 1 : Obtain the disjunctive normal form.
Step 2 : Drop elementary products which are contradictions. (cancel it out)
Step 3 : If p_i and $\sim p_i$ are missing in an elementary product, replace it by α.
$(\alpha \wedge p_i) \vee (\alpha \wedge \sim p_i)$.
Step 4 : Repeat step 3 until all elementary products are reduced to sum of minterms. Identical minterms appearing in the disjunction are deleted.

* Points to be remembered for PDNF without using truth table :

→ After finding DNF, we apply identity

law, complement law and distributive law.

1) Identity law for DNF : $q \equiv q \wedge T$

2) Complement law for DNF : $\sim q \vee q \equiv T$

3) Distributive law.

Ex :- Obtain the (PDNF) principal disjunctive normal form of the following without using the truth table.

1) $p \to q$

→ $p \to q \equiv \sim p \vee q$ (DNF)

→ $\sim p \equiv (\sim p \wedge T)$ (Identity law)

$\equiv [\sim p \wedge (\sim q \vee q)]$ (Complement law)

$\equiv [(\sim p \wedge \sim q) \vee (\sim p \wedge q)]$

(Distributive law)

→ $q \equiv q \wedge T$ (Identity law)

$\equiv (q \wedge (\sim p \vee p))$ (Complement law)

$\equiv [(q \wedge \sim p) \vee (q \wedge p)]$ (Distributive law)

→ Thus, the required PDNF is,

$p \to q \equiv [(\sim p \wedge \sim q) \vee (\sim p \wedge q)] \vee [(q \wedge \sim p) \vee (q \wedge p)]$

$\equiv [(\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (q \wedge p)]$

2) $q \vee (p \vee \sim q)$

→ $q \vee (p \vee \sim q) \equiv q \vee p \vee \sim q$ (DNF) law

$\equiv (q \wedge T) \vee (p \wedge T) \vee (\sim q \wedge T)$ (Identity law)

$\equiv [q \wedge (p \vee \sim p)] \vee [p \wedge (q \vee \sim q)]$
$\vee [\sim q \wedge (p \vee \sim p)]$
(∵ Complement law)

$\equiv (q \wedge p) \vee (q \wedge \sim p) \vee (p \wedge q) \vee (p \wedge \sim q)$
$\vee (\sim q \wedge p) \vee (\sim q \wedge \sim p)$

$\equiv (q \wedge p) \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$

which is required PDNF.

3). $(\sim p \wedge q \wedge \sim r) \vee (q \wedge r)$

→ In the given formula, $(\sim p \wedge q \wedge \sim r)$ is already a minterm, so we need to convert only $(q \wedge r)$ into minterms.

→ $(q \wedge r) \equiv (q \wedge r)$ (DNF)
$\equiv (q \wedge r) \wedge T$ (Identity law)
$\equiv (q \wedge r) \wedge (p \vee \sim p)$ (Complement law)
$\equiv [(q \wedge r) \wedge p] \vee [(q \wedge r) \wedge \sim p]$
(Distributive law)
$\equiv (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$

∴ The required PDNF is,

$(\sim p \wedge q \wedge \sim r) \vee (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$

4) $p \leftrightarrow q$

→ $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$

$\equiv [(\sim p \vee q) \wedge \sim q] \vee [(\sim p \vee q) \wedge p]$
($\because$ Distributive law)

Contradiction $\equiv (\sim p \wedge \sim q) \vee (\underline{q \wedge \sim q}) \vee$
$(\underline{\sim p \wedge p}) \vee (q \wedge p)$
($\because$ Distributive law)

$\equiv (\sim p \wedge \sim q) \vee (q \wedge p)$
which is required PDNF.

5) $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$ (DNF)
we need minterms

$\rightarrow (p \wedge q) \equiv (p \wedge q) \wedge T$ (Identity law)
$\equiv (p \wedge q) \wedge (\sim r \vee r)$ (Complement law)
$\equiv (p \wedge q \wedge \sim r) \vee (p \wedge q \wedge r)$
(Distributive law)

$\rightarrow (\sim p \wedge r) \equiv (\sim p \wedge r) \wedge T$ (Identity law)
$\equiv (\sim p \wedge r) \wedge (q \vee \sim q)$ (Comp. law)
$\equiv (\sim p \wedge r \wedge q) \vee (\sim p \wedge r \wedge \sim q)$
($\because$ Distributive law)
$\equiv (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$
(Commutative law)

$\rightarrow (q \wedge r) \equiv (q \wedge r) \wedge T$ (Identity law)
$\equiv (q \wedge r) \wedge (p \vee \sim p)$ (Comple. law)
$\equiv (q \wedge r \wedge p) \vee (q \wedge r \wedge \sim p)$
(Distri. law)
$\equiv (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$
(Commutative law).

→ The required PDNF is,

$(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

$\equiv (p \wedge q \wedge \sim r) \vee (p \wedge q \wedge r) \vee$
$(\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r) \vee$
$(p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$

$\equiv (p \wedge q \wedge \sim r) \vee (p \wedge q \wedge r) \vee$
$(\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$

6) $p \vee (\sim p \rightarrow (\sim q \rightarrow r))$

→ $p \vee (\sim p \rightarrow (\sim q \rightarrow r))$

$\equiv p \vee [\sim p \rightarrow (q \vee r)]$ (logically equivalent form)
$\equiv p \vee [p \vee (q \vee r)]$ ($\because$ logically equivalent form)
$\equiv p \vee p \vee q \vee r$
$\equiv p \vee q \vee r$ ($\because$ Idempotent law)
which is required PDNF.

→ $p \equiv p \wedge T$
$\equiv p \wedge (q \vee \sim q)$ ($\because$ Identity law)
$\equiv (p \wedge q) \vee (p \wedge \sim q)$ (Distri. law)
$\equiv [(p \wedge q) \vee (p \wedge \sim q)] \wedge T$ (Identity law)

$\equiv [(p \wedge q) \vee (p \wedge \sim q)] \wedge (r \vee \sim r)$
($\because$ Comple. law)

$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee$
$(p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r)$
(Distri. law)

$\rightarrow q \equiv q \wedge T$ (Identity law)

$\equiv q \wedge (p \vee \sim p)$ (Complement law)

$\equiv (q \wedge p) \vee (q \wedge \sim p)$ (Distributive law)

$\equiv [(p \wedge q) \vee (\sim p \wedge q)] \wedge T$

(Identity law)

$\equiv [(p \wedge q) \vee (\sim p \wedge q)] \wedge (r \vee \sim r)$

(Complement law)

$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee$
$(\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$

(Distributive law)

$\rightarrow r \equiv r \wedge T$ (Identity law)

$\equiv r \wedge (q \vee \sim q)$ (Complement law)

$\equiv (r \wedge q) \vee (r \wedge \sim q)$ (Distri. law).

$\equiv [(r \wedge q) \vee (r \wedge \sim q)] \wedge T$

(Identity law)

$\equiv [(r \wedge q) \vee (r \wedge \sim q)] \wedge (p \vee \sim p)$

(Complement law)

$\equiv (r \wedge q \wedge p) \vee (r \wedge q \wedge \sim p) \vee$
$(r \wedge \sim q \wedge p) \vee (r \wedge \sim q \wedge \sim p)$

(Distributive law)

$\equiv (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r) \vee$
$(p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$

(Commutative law).

$\therefore$ The required PDNF is,

$\rightarrow p \vee q \vee r \equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee$
$(p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee$
$\cancel{(p \wedge q \wedge r)} \vee \cancel{(p \wedge q \wedge \sim r)} \vee$

$(\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee$
$(p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r) \vee$
$(p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$.

$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee$
$(p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee$
$(\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee$
$(\sim p \wedge \sim q \wedge \sim r)$

which is required PDNF.

* Advantages of obtaining principal disjunctive normal form (PDNF) are as below :

1) The PDNF of a given formula is always unique.

2) Two formulas are equivalent if and only if their principal disjunctive normal form (PDNF) coincide / same.

3) If the given compound proposition is a tautology, then its PDNF will contain all the possible minterms of its components.

Ex: $p \rightarrow (p \vee q)$ is a tautology. and its PDNF will contain all the possible minterms of its components.

proof: $p \rightarrow (p \vee q) \equiv \sim p \vee (p \vee q)$
$\equiv \sim p \vee p \vee q$ (DNF)

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* PCNF :

→ PCNF of a given formula is defined as an equivalent formula consists of conjuctive of max terms only.

→ It is also called the product of sums of a canonical form.

→ There are two ways ~~to~~ to obtain the principal conjunctive normal form :

i) Using truth table.
ii) Using algebra of proposition / without using truth table.

* Method : 1 : Using truth table.

Step 1 : Construct a truth table for the given compound statement.

Step 2 : For every truth value F of the given proposition, select the maxterms which has also true value F for the same combination of truth value of the statement variable.

Step 3 : The conjunctive of maxterms selected in step 2 is the required principal conjunctive normal form.

Remark : In PCNF, the repeated max terms are mentioned only once.

* PDNF : F eliminate
* PCNF : T eliminate

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Ex:- Obtain the (PCNF) principal conjunctive normal form using the truth table of $p \wedge q$.

→ Truth table :

p	q	$p \wedge q$	Maxterms
T	T	T	—
T	F	F	$\sim p \vee q$
F	T	F	$p \vee \sim q$
F	F	F	$p \vee q$

☆ ↓ $\sim p \rightarrow T$, $p \rightarrow F$

∴ The required PCNF is,
$(\sim p \vee q) \wedge (p \vee \sim q) \wedge (p \vee q)$

* Method : 2 : without using truth table.

→ As per PDNF with some changes.

* Points to be remembered for PCNF without using truth table :

→ After finding CNF, we apply identity law, complement law and distributive law.
1) Identity law for CNF : $q \equiv q \vee F$.
2) Complement law for CNF : $\sim q \wedge q \equiv F$
3) Distributive law.

Ex:- Obtain the (PCNF) principal conjunctive normal form of the following without using the truth table.

1) $p \wedge q$

$\Rightarrow p \wedge q \equiv (p \vee F) \wedge (q \vee F)$ (Identity law)

$\equiv (p \vee (q \wedge \sim q)) \wedge (q \vee (p \wedge \sim p))$ (Complement law)

$\equiv (p \vee q) \wedge (p \vee \sim q) \wedge (q \vee p) \wedge (q \vee \sim p)$ (Distri. law)

$\equiv (p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q)$

which is required PCNF.

2) $(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$

$\Rightarrow (\sim p \rightarrow r) \wedge (p \leftrightarrow q)$

$\equiv (p \vee r) \wedge (p \rightarrow q) \wedge (q \rightarrow p)$

$\equiv (p \vee r) \wedge (\sim p \vee q) \wedge (\sim q \vee p)$

(∵ logically equivalent form)

$\equiv (p \vee r \vee F) \wedge (\sim p \vee q \vee F) \wedge (\sim q \vee p \vee F)$

(Identity law)

$\equiv [(p \vee r) \vee (q \wedge \sim q)] \wedge [(\sim p \vee q) \vee (r \wedge \sim r)] \wedge [(\sim q \vee p) \vee (r \wedge \sim r)]$

(Complement law)

$\equiv [(p \vee r) \vee q] \wedge [p \vee r \vee \sim q] \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (\sim q \vee p \vee r) \wedge (\sim q \vee p \vee \sim r)$

(Distributive law)

$\equiv (p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (\sim p \vee q \vee r) \wedge$
$(\sim p \vee q \vee \sim r) \wedge (\sim p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r)$

(Commutative law)

$\Rightarrow$ which is required PCNF.

* Advantages of obtaining principal conjunctive normal form (PCNF) are as below:

1) The PCNF of a given formula is always unique.

2) Every compound proposition, which is not a tautology, has an equivalent principal conjunctive normal form.

3) If the given compound proposition is a contradiction, then its principal conjunctive normal form will contain all possible maxterms of it's components.

Ex :- $q \wedge \sim (p \rightarrow q)$ is a contradiction and its PCNF will contain all the possible maxterms of its components.

proof :- $q \wedge \sim (p \rightarrow q) \equiv q \wedge \sim (\sim p \vee q)$

$\equiv q \wedge (p \wedge \sim q)$

$\equiv p \wedge q \wedge \sim q$

which is of CNF.

~~p p~~

$\cancel{(p \vee F) \vee F \equiv [p \vee (q \wedge \sim q)] \vee (r \wedge \sim r)}$

$\equiv$

$$\rightarrow \quad p \equiv (p \vee F) \vee F \quad (\because \text{Identity law})$$
$$\equiv [p \vee (q \wedge \sim q)] \vee (r \wedge \sim r) \quad (\text{Complement})$$
$$\equiv [(p \vee q) \wedge (p \vee \sim q)] \vee (r \wedge \sim r)$$
$$(\text{Distri. law})$$

$$\cancel{\equiv (p \vee q) \wedge (r \wedge \sim r)}$$
$$\equiv (p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge$$
$$(p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r)$$

$$\rightarrow \quad q \equiv (q \vee F) \vee F \quad (\text{Identity law})$$
$$\equiv [q \vee (p \wedge \sim p)] \vee (r \wedge \sim r) \quad (\text{Complement})$$
$$\equiv [(q \vee p) \wedge (q \vee \sim p)] \vee (r \wedge \sim r)$$
$$(\text{Distri. law})$$
$$\equiv (p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge$$
$$(\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$$
$$(\text{Distri law}).$$

$$\rightarrow \quad \sim q \equiv (\sim q \vee F) \vee F \quad (\text{Identity law})$$
$$\equiv [\sim q \vee (p \wedge \sim p)] \vee (r \wedge \sim r) \quad (\text{Compl. law})$$
$$\equiv [(\sim q \vee p) \wedge (\sim q \vee \sim p)] \vee (r \wedge \sim r)$$
$$(\text{Distri. law})$$
$$\equiv (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge$$
$$(\sim p \vee \sim q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$$

$$\therefore \quad p \wedge q \wedge \sim q \equiv (p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge$$
$$(p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge$$
$$(p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge$$
$$(\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge$$
$$(p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge$$
$$(\sim p \vee \sim q \vee r) \wedge (\sim p \vee \sim q \vee \sim r).$$

$\equiv (p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (p \vee \sim q \vee r)$
$\wedge (p \vee \sim q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge$
$(\sim p \vee q \vee \sim r) \wedge (\sim p \vee \sim q \vee r) \wedge$
$(\sim p \vee \sim q \vee \sim r)$

which is required PCNF.

* Practice Examples :-

** Obtain the PDNF and PCNF of the following

1) $(\sim p \vee \sim q) \rightarrow (p \leftrightarrow \sim q)$

2) $q \wedge (p \vee \sim q)$

3) $p \vee (\sim p \rightarrow (q \vee (\sim q \rightarrow r)))$

4) $(p \rightarrow (q \wedge r)) \wedge (\sim p \rightarrow (\sim q \wedge \sim r))$

5) $p \rightarrow ((p \rightarrow q) \wedge \sim(\sim q \vee \sim p))$

* LOGIC IN PROOF :-

→ Theorem :
A theorem is a proposition that can be proved to be true.

→ Premises :
It is a proposition on the basis of which we would be able to draw a conclusion.
One can think of premise as an evidence or an assumption.

Therefore we initially assume that something is true and on the basis of that we draw the conclusion.

→ Conclusion :
It is a proposition that is reached from the given set of premises.
One can think of it as the result of the assumptions that we made in an argument.

→ Argument :
An argument is a sequence of a statement that ends with the conclusion.
All statements except last one are known as premises or hypothesis. and the last statement is known as conclusion.

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→ Proof :
An argument which establishes the truth of the theorem is known as proof.

→ Valid Argument :
An argument is said to be valid, if all premises are true, conclusion must be true in the same possibility.

→ Validity of the argument can be checked using two methods :

1) Truth table technique.
2) Without using truth table technique / using the rule of inference.

* Method : 1 :→ Truth - Table Technique :

Step 1 : Construct a truth-table for premises and the conclusion.

Step 2 : Find the rows in which all the premises are true. This rows are called critical rows.

Step 3 : Check conclusion of all critical rows.
A] If in each ~~row~~ critical row, conclusion is true then the argument is valid.
B] If there is a row in which conclusion is false then the argument form is invalid.

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Ex :- Determine whether the following argument forms are valid or invalid.

$p \to q \vee \sim r$

$q \to p \wedge r$

$\therefore p \to r$

→ Let, $P_1 = p \to q \vee \sim r$ and $C = p \to r$

$P_2 = q \to p \wedge r$.

p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	P_1	P_2	C
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	F
T	F	T	F	F	T	F	T	T
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	T
F	T	F	T	T	F	T	F	T
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

→ One critical row contains false value.

$\therefore$ the conclusion is invalid.

* Practice Examples :

→ Determine whether the conclusion C follows logically from the premises H_1 and H_2 using truth table technique.

1) $H_1 : P \to Q$, $H_2 : P$, $C : Q$
2) $H_1 : \sim P$, $H_2 : P \leftrightarrow Q$, $C : \sim(P \wedge Q)$
3) $H_1 : P \to Q$, $H_2 : \sim(P \wedge Q)$, $C : \sim P$

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* Method : 2 :→ Rules of Inference
(without using truth table)

→ The rules of inference are criteria for determining the validity of an argument.

→ Any conclusion which is arrived by following rules of inference is called a valid conclusion and the argument is called a valid argument.

Rule of inference	Implication form	Name
1) p $\therefore p \vee q$	$p \rightarrow (p \vee q)$	Addition
2) $p \wedge q$ $\therefore p$	$(p \wedge q) \rightarrow p$	Simplification
3) p q $\therefore p \wedge q$	$[(p) \wedge (q)] \rightarrow p \wedge q$	Conjunction
4) $p \rightarrow q$ p $\therefore q$	$[(p \rightarrow q) \wedge p] \rightarrow q$	Modus Ponens
5) $p \rightarrow q$ $\sim q$ $\therefore \sim p$	$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$	Modus Tollens

Rule of inference	Implication form	Name
6) $p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical Syllogism
7) $p \vee q$ $\sim p$ $\therefore q$	$[(p \vee q) \wedge \sim p] \to q$	Disjunction Syllogism
8) $(p \to q) \wedge (r \to s)$ $p \vee r$ $\therefore q \vee s$	$[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$	Constructive Dilemma.
9) $(p \to q) \wedge (r \to s)$ $\sim q \vee \sim s$ $\therefore \sim p \vee \sim r$	$[(p \to q) \wedge (r \to s) \wedge (\sim q \vee \sim s)] \to (\sim p \vee \sim r)$	Destructive Dilemma.

* Procedure :

→ To obtain a valid argument, we have number of premises given with the conclusion.

→ We will make a table in which on the left side formula is given and on the right, we indicate ~~the premises and the rules of infe~~ whether the proposition is premise or a conclusion.

→ If it is conclusion, we indicate the premises and the rules of inference or logical identities used for deriving the conclusion.

Ex:- Check the validity of following arguments with the rules of inference.

P_1 : $p \to q$ $\quad$ C : r

P_2 : $q \to r$

P_3 : p

1) $p \to q$	premise 1
2) $q \to r$	premise 2
3) $p \to r$	line 1 & 2, Hypothetical Syllogism
4) p	premise 3.
5) r	line 3 & 4, Modus Ponens.

∴ Given argument is valid.

Ex:- Can we conclude S from the following premises?

(i) $P \to Q$: P_1 $\quad$ and $\quad$ C : S.

(ii) $P \to R$: P_2

(iii) $\sim (Q \wedge R)$: P_3

(iv) $S \vee P$: P_4

1) $P \to Q$	premise 1.
2) $P \to R$	premise 2.
3) $(P \to Q) \wedge (P \to R)$	Line 1, 2, Conjunction Rule.
4) $\sim (Q \wedge R)$	premise 3.
5) $\sim Q \vee \sim R$	Line 4 & De' morgan's law.
6) $\sim P \vee \sim P$	Line 3, 5 & Destructive Dilemma
7) $\sim P$	Line 6 and Idempotent law.

8) $S \vee P$ premise 4.

9) S Line 7, 8 and disjunctive syllogism

→ Thus, S is the conclusion from the given premises.

∴ Given argument is valid.

Ex:- Derive S from the following premises using valid argument.

1) $P \to Q$ C : S.

2) $Q \to \sim R$

3) R

4) $P \vee S$.

→ 1) $P \to Q$ premise 1.

2) $Q \to \sim R$ premise 2.

3) $P \to \sim R$ line 1, 2, Hypothetical Syllogism.

4) R premise 3.

5) $\sim(\sim R)$ line 4, double negation

6) $\sim P$ line 3, 5, Modus tollens

7). $P \vee S$ premise 4.

8). S line 6, 7, Disjunction syllogism.

→ Thus, S is the conclusion from the given premises.

Ex:- Check the validity of the following argument :

→ If Ram has completed B.E. or MBA,

then he is assured of a good job. If Ram is assured of a good job, then he is happy. Ram is not happy. So, Ram has not completed MBA.

Solⁿ :→ Let p : Ram has completed B.E.
q : Ram has completed MBA.
r : Ram is assured of a good job.
s : Ram is happy.

→ The given premises are ...
1) $(p \vee q) \rightarrow r$ $C : \sim q$.
2) $r \rightarrow s$
3) $\sim s$.

→
1)	$(p \vee q) \rightarrow r$	premise 1
2)	$r \rightarrow s$	premise 2.
3)	$(p \vee q) \rightarrow s$	line 1, 2, Hypothetical syllogism.
4)	$\sim s$	premise 3.
5).	$\sim(p \vee q)$	line 3, 4, Modus Tollens
6)	$\sim p \wedge \sim q$	line 5, De' Morgan's law.
7)	$\sim q$.	line 6, Simplification rule

→ Thus, $\sim q$ is the conclusion from the given premises.

Ex :- Check the validity of the following arguments :

If milk is black then every cow is white. If every cow is white then it has four legs. If every cow has four legs then every buffalo is white and brisk. The milk is black. Therefore, the buffalo is white.

Sol^n : → p : The milk is black.
q : Every cow is white.
r : Every cow has four legs.
s : Every buffalo is white.
t : Every buffalo is brisk.

→ The given premises are ...

1). $p \to q$ C : s
2) $q \to r$
3) $r \to (s \wedge t)$
4) p

→ 1) $p \to q$ premise 1
2) $q \to r$ premise 2.
3) $p \to r$ line 1, 2 Hypothetical syllogism
4) $r \to (s \wedge t)$ premise 3.
5) $p \to (s \wedge t)$ line 3, 4, Hypothetical syllogism
6). p premise 4.
7). $s \wedge t$ line 5, 6, Modus Pollens
8) s line 7, Simplification.

∴ The argument is valid. — 20DCS103
Rushik. Rathod.