Unit : 01 First order and First degree Ordinary Differential Equations

1. Form the differential equation satisfied by $x^2+y^2=c$, where c is an arbitrary constant.

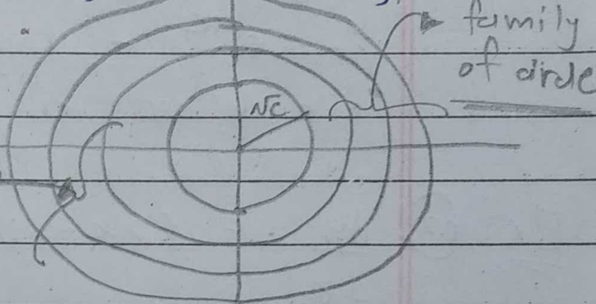
→ Given eqⁿ is $x^2+y^2=c$. —① , c = constant

Differentiating ① w.r.t. x, we get,

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\therefore \boxed{x + y \cdot \frac{dy}{dx} = 0}$$

represents



3 Form the d.e. satisfied by $y = a\sin x + b\cos x$, where a and b are arbitrary constants.

→ Given eqⁿ is $y = a\sin x + b\cos x$ —①

a, b = constants

Differentiating ① w.r.t. x, we get,

$$\frac{dy}{dx} = a\cos x - b\sin x$$

$$\therefore \frac{d^2y}{dx^2} = -a\sin x - b\cos x = -(a\sin x + b\cos x)$$

$$\therefore \boxed{\frac{d^2y}{dx^2} = -y} \quad (\because \text{eq}^n \text{ ①}).$$

* Variable Separable Method :-

Ex: ① Solve the d.e. $e^{2x}\cos y \cdot dx - e^{x} \cdot \sin y \cdot dy = 0$.

→ Here, Given eqⁿ is $e^{2x} \cdot \cos y \cdot dx - e^{x} \cdot \sin y \cdot dy = 0$.

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$\therefore\ e^{x} \cdot dx - \cdot \tan y \cdot dy = 0$ (dividing ① by $e^{x} \cdot \cos y$).

→ Integrating both the sides,

$$\int e^{x} \cdot dx - \int \tan y \cdot dy = c$$

$\therefore\ e^{x} - \log \sec y = c$, c = arbitrary constant.

which is a general sol^n.

Ex: 02 $\dfrac{dy}{dx} = (2x-1)(y+1)$ ——— ①. | $y(0)=0$.

$$\therefore\ \frac{1}{y+1} \cdot dy = (2x-1) \cdot dx$$

$\therefore$ Integrating both the sides,

$$\int \frac{1}{y+1} \cdot dy = \int (2x-1) \cdot dx + c$$

$$\therefore\ \log|y+1| = x^2 - x + c$$

$\therefore$ ~~$\log|y+1| - x^2 + x = c$~~ , c = arbitrary constant.

$$\therefore\ y+1 = e^{x^2-x+c}$$

$$\therefore\ 1+y = e^{x^2-x} \cdot e^{c}$$

$\therefore\ y = -1 + e^{x^2-x} \cdot \underbrace{c_1}$ ②, $c_1 = e^{c}$

which is a general sol^n of ①.

→ We are given $y(0)=0$.

$x=0 \Rightarrow y=0$.

$\therefore$ at $x=0,\ y=0$, eq^n ② becomes,

$0 = -1 + c_1$

$\therefore \quad c_1 = 1.$

$\therefore$ ② becomes, $\boxed{y = e^{x^2 - x} - 1}$

which is a particular solⁿ of ①.

*. Linear Differential Equations :-
(Leibnitz Differential Equations)

Ex:-01 Solve the differential equation $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$

$\rightarrow$ We are given $(x+1)\cdot\frac{dy}{dx} - y = e^{3x}(x+1)^2$

$\therefore \frac{dy}{dx} - \frac{1}{(x+1)}\cdot y = (x+1)\cdot e^{3x}$ | dividing by $(x+1)$ both the sides

Which is a Linear differential eqⁿ.

$\therefore$ Here, $p(x) = \frac{-1}{(x+1)}$ and $Q(x) = e^{3x}(x+1)$

Now, $\text{I.F.} = e^{\int p(x)\cdot dx}$

$= e^{-\int \frac{1}{x+1}\cdot dx}$

$= e^{-\log|x+1|}$

$= e^{\log_e (x+1)^{-1}}$

$\therefore (\text{I.F}) = (x+1)^{-1}.$

→ Thus the sol^n is,

$$y = \frac{1}{(I.F)}\left[\int g(x) \times (I.F).dx + c\right]$$

$$= \frac{1}{(x+1)^{-1}}\left[\int e^{3x}(x+1)(x+1)^{-1}.dx + c\right]$$

$$= (x+1)\left[\int e^{3x}.dx + c\right]$$

$$= (x+1)\left[\frac{e^{3x}}{3} + c\right]$$, c = arbitrary constant.

Ex:-02 (5) Solve the d.e $\frac{dy}{dx} + \frac{2y}{x} = x^3$.

→ Here, we are given $\frac{dy}{dx} + \frac{2}{x}\cdot y = x^3$.

where, $p(x) = \frac{2}{x}$ and $g(x) = x^3$.

which is a linear differential eq^n (must write)

$$\therefore (I.F.) = e^{\int p(x).dx}$$

$$= e^{2\int \frac{1}{x}.dx}$$

$$= e^{2.\log x} = x^2.$$

→ Thus the sol^n is,

$$y = \frac{1}{(I.F)}\left[\int g(x).(I.F.) dx + c\right]$$

$$= \frac{1}{x^2}\left[\int x^3.x^2.dx + c\right]$$

$$= \frac{1}{x^2}\left[\int x^5.dx + c\right]$$

$$\therefore \quad y = \frac{1}{x^2}\left[\frac{x^6}{6} + c\right]$$

$$\therefore \quad \boxed{y \cdot x^2 = \frac{x^6}{6} + c ,} \quad c = \text{arbitrary constant.}$$

Ex:-03 (6) Solve the d.e. $\frac{dx}{dy} + \frac{x}{y \cdot \log y} = \frac{1}{y}$

→ We are given $\frac{dx}{dy} + \frac{1}{y \cdot \log y} \cdot x = \frac{1}{y}$ which is a linear d.e.

where; $p(y) = \frac{1}{y \cdot \log y}$ & $g(y) = \frac{1}{y}$.

$$\therefore \quad (\text{I.F.}) = e^{\int p(y) \cdot dy}$$

$$= e^{\int \frac{1}{y \cdot \log y} \cdot dy}$$

$$= e^{\int \frac{1/y}{\log y} \cdot dy} \qquad \int \frac{f'(x)}{f(x)} \cdot dx = \log(f(x))$$

$$= e^{\log(\log y)}$$

Extra: $\int f'(x) \cdot [f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1}$

$$\therefore \quad (\text{I.F.}) = \log y.$$

→ Thus, the sol^n is,

$$x = \frac{1}{(\text{I.F.})}\left[\int g(y) \cdot (\text{I.F.}) \cdot dy + c\right]$$

$$\therefore \quad x = \frac{1}{\log y}\left[\int \frac{1}{y} \cdot \log y \cdot dy + c\right].$$

$$\therefore \quad x \cdot \log y = \frac{(\log y)^2}{2} + c$$

$$\cancel{\therefore \quad x = \frac{\log y}{2} + c}$$

$$\therefore \quad \boxed{x \cdot \log y = \frac{(\log y)^2}{2} + c}$$

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* Bernoulli's Differential Equation :-

Ex:-01 Solve the d.e. $\frac{dy}{dx} + \frac{y}{x} = x^2y^2$.

→ We are given $\frac{dy}{dx} + \frac{y}{x} = x^2y^2$ —①. d.f.

which is of the form of Bernoulli's eq^n with $n=2$

$\therefore \frac{1}{y^2} \cdot \frac{dy}{dx} + \frac{y}{x \cdot y^2} = x^2$ | dividing by y^2 both the sides

$\therefore y^{-2} \cdot \frac{dy}{dx} + \frac{1}{xy} = x^2$

Let $\frac{1}{y} = v$.

$\therefore -\frac{1}{y^2} \cdot \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore -y^{-2} \cdot \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore -\frac{dv}{dx} + \frac{1}{x} \cdot v = x^2$

$\therefore \frac{dv}{dx} - \frac{1}{x} \cdot v = -x^2$.

which is linear d. eq^n with $p(x) = -\frac{1}{x}$ and $g(x) = -x^2$.

$\therefore (I.F.) = e^{\int p(x) \cdot dx}$

$= e^{-\int \frac{1}{x} \cdot dx}$

$= e^{-\log x}$

$= e^{\log (x)^{-1}}$

$= x^{-1}$ OR

$= \frac{1}{x}$

$\therefore$ the general sol^n is given by,

$$v \cdot \cancel{\ } = \frac{1}{(I.F)} \int g(x) \cdot (I.F)\, dx + c$$

$$\therefore\ v \cdot \frac{1}{x} = \left[-\int x^2 \cdot \frac{1}{x} \cdot dx + c \right] (x^{-1})$$

$$\therefore\ v \cdot {}^1\!/_x = \left[-\int x \cdot dx + c \right] (x^{-1})$$

$$\therefore\ v \cdot \frac{1}{x} = \left[-\frac{x^2}{2} + c \right] (x^{-1}).$$

$$\therefore\ \boxed{\frac{1}{y} = -\frac{x^2}{2} + c} \qquad \left(\because v = \frac{1}{y} \right)$$

which is the general sol^n of eq^n (1).

Ex:-02 (7) Solve the d.e. $\frac{dy}{dx} + 3 \cdot \frac{y}{x} = 2x^4 y^4$.

$\rightarrow$ Here, we are given $\frac{dy}{dx} + 3 \cdot \frac{y}{x} = 2x^4 \cdot y^4$ ——— (1)

which is Bernoulli's d.eq^n with $n = 4$.

$$\therefore\ \frac{1}{y^4} \frac{dy}{dx} + 3 \cdot \frac{1}{x \cdot y^3} = 2 \cdot x^4 \qquad \text{dividing by } y^4 \text{ both the sides}$$

Now, let $\frac{1}{y^3} = v$.

$$\therefore\ \therefore\ -\frac{1}{y^4} \cdot \frac{dy}{dx} = \frac{dv}{dx} \left(\frac{1}{3} \right)$$

$$\therefore\ \frac{1}{y^4} \cdot \frac{dy}{dx} = -\frac{dv}{dx} \left(\frac{1}{3} \right)$$

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$$\therefore\ -\frac{dv}{dx} \left(\frac{1}{3} \right) + \frac{3}{x} \cdot v = 2 \cdot x^4.$$

$\therefore \dfrac{dv}{dx} - \dfrac{9}{x} \cdot v = -6x^4.$ | Multiplying by 3

which is linear d. eq^n. where $p(x) = -\dfrac{9}{x}$

and $q(x) = -6x^4.$

$$\therefore (I.F.) = e^{\int p(x).dx}$$

$$= e^{-9 \int 1/x \, . dx}$$

$$= e^{-9 . \log x}$$

$$= e^{\log (x)^{-9}}$$

$$= x^{-9}.$$

$\therefore$ the general sol^n of (1) is given by,

$$v = \frac{1}{(I.F.)}\left[\int q(x). (I.F.)\, dx + c\right]$$

$$\therefore v = \frac{1}{x^{-9}}\left[\int (-6x^4).(x^{-9}).dx + c\right]$$

$$\therefore v = \frac{1}{x^{-9}}\left[\int (-6).\, x^{-5} dx + c\right]$$

$$\therefore v = \frac{1}{x^{-9}}\left[-6 . \frac{x^{-4}}{(-4)} + c\right]$$

$$\therefore v = \frac{1}{x^{-9}}\left[\frac{3x^{-4}}{2} + c\right]$$

$$\therefore y^{-3} . x^{-9} = \frac{3}{2} . x^{-4} + c$$

which is the general sol^n of eq^n (1).

Ex:-03 (8). Solve the d.e. $\frac{dy}{dx} = 2y \cdot \tan x + y^2 \cdot \tan^2 x$.

if $x = 0$, then $y = 1$.

→ Here, we are given $\frac{dy}{dx} = 2y \cdot \tan x + y^2 \cdot \tan^2 x$

$\therefore \frac{dy}{dx} - 2y \cdot \tan x = y^2 \cdot \tan^2 x$

which is Bernoulli's d. eq^n. with $n = 2$.

$\therefore \frac{1}{y^2} \cdot \frac{dy}{dx} - \frac{2}{y} \cdot \tan x = \tan^2 x$ | dividing by y^2 both the sides.

$\therefore y^{-2} \cdot \frac{dy}{dx} - \frac{2}{y} \cdot \tan x = \tan^2 x$

Now, let $\frac{1}{y} = v$. always let only positive

$\therefore -\frac{1}{y^2} \cdot \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore y^{-2} \cdot \frac{dy}{dx} = -\frac{dv}{dx}$.

$\therefore$ substituting values, we have,

$\therefore -\frac{dv}{dx} - 2 \cdot \tan x \cdot v = \tan^2 x$.

$\therefore \frac{dv}{dx} + 2 \cdot \tan x \cdot v = -\tan^2 x$

which is linear d. eq^n where $p(x) = 2 \cdot \tan x$.
and $q(x) = -\tan^2 x$.

$\therefore (I.F.) = e^{\int p(x) \cdot dx}$.

$$\therefore (I.F.) = e^{\int 2 \tan x \, dx}$$

$$\therefore (I.F.) = e^{2 \cdot \log |\sec|} = \sec^2 x$$

$\therefore$ the general sol^n is given by,

$$v = \frac{1}{(I.F.)}\left[\int g(x).(I.F.)dx + c\right]$$

$$\therefore v = \frac{1}{\sec^2 x}\left[-\int \tan^2 x \cdot \sec^2 x \cdot dx + c\right]$$

$$\therefore v = \frac{1}{\sec^2 x}\left[-\frac{\tan^3 x}{3} + c\right]$$

$$\therefore \frac{\sec^2 x}{y} = -\frac{\tan^3 x}{3} + c \quad \text{——— (2)}$$

which is the general sol^n of eq^n (1).

now, $x = 0 \Rightarrow y = 1$.

substituting these values into eq.^{n} (2),

$$\frac{\sec^2(0)}{(1)} = -\frac{\tan^3(0)}{3} + c$$

$$\therefore \boxed{1 = c}$$

$\rightarrow$ from eq. (2), $\frac{\sec^2 x}{y} = -\frac{\tan^3 x}{3} + 1$.

which is the particular sol^n of eq^n (1).

* Exact and Non-exact differential eq^n :-

Ex:-01 Solve the differential eq^n $(y^2 \cdot e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0$.

→ Here, we are given,

$$(y^2 \cdot e^{xy^2} + 4x^3)\,dx + (2xy \cdot e^{xy^2} - 3y^2)\,dy = 0$$

which is of the type $M(x,y)\,dx + N(x,y)\,dy = 0$.

where,

$M = y^2 \cdot e^{xy^2} + 4x^3$.

$\therefore \frac{\partial M}{\partial y} = 2y \cdot e^{xy^2} + y^2 \cdot e^{xy^2} \cdot 2yx$

$= 2y \cdot e^{xy^2} + e^{xy^2} \cdot 2y^3 x$

$N = 2xy \cdot e^{xy^2} - 3y^2$.

$\therefore \frac{\partial N}{\partial x} = 2xy \cdot e^{xy^2} \cdot y^2 + e^{xy^2} \cdot 2y$

$= 2y \cdot e^{xy^2} + e^{xy^2} \cdot 2y^3 x$

$\therefore$ give d. eq^n is exact.

$\therefore$ It's general sol^n is given by,

$$\int_{y\ \text{const}} M \cdot dx + \int \left(\begin{array}{c} \text{terms of } N \\ \text{containing } y \\ \text{only} \end{array} \right) \cdot dy = c$$

$$\therefore \int_{y\cdot\text{const}} (y^2 \cdot e^{xy^2} + 4x^3) \cdot dx + \int (-3y^2) \cdot dy = c$$

$$\therefore \frac{y^2 \cdot e^{xy^2}}{y^2} + \frac{4x^4}{4} - \frac{3 \cdot y^3}{3} = c$$

$$\therefore e^{xy^2} + x^4 - y^3 = c$$

Ex :- 02 (9) $(y^2 + \tan x - 5x^2)\,dx + (2xy - 9)\,dy = 0$

→ Here, we are given,

$(y^2 + \tan x - 5x^2)\,dx + (2xy - 9)\cdot dy = 0$

which is of the type $M(x,y)\cdot dx + N(x,y)\cdot dy = 0$

where,

$M = y^2 + \tan x - 5x^2$ $\qquad$ $N = 2xy - 9$

$\therefore \dfrac{\partial M}{\partial y} = 2y$ $\qquad$ $\therefore \dfrac{\partial N}{\partial x} = 2y$

∴ Given d. eq^n is exact d. eq^n.

∴ It's general sol^n is given by,

$$\int_{y\text{-const}} M(x,y)\,dx + \int \left(\begin{matrix}\text{terms of } N\\ \text{containing only}\\ y\end{matrix}\right) dy = C$$

$$\therefore \int_{y\text{ const}} (y^2 + \tan x - 5x^2)\cdot dx + \int (-9)\cdot dy = C$$

$$\therefore 0 + \log|\sec x| - \frac{5x^3}{3} - 9y = C$$

$$\therefore \log\sec x - \frac{5}{3}x^3 - 9y = C$$

Ex:-03 (10) Solve the eq^n $(x^2+y^2+3)dx + (2xy + \sin^2 y)dy = 0$.

→ Here, we are given, $(x^2+y^2+3)dx + (2xy+\sin^2 y)\cdot dy = 0$

which is of the type $M(x,y)\,dx + N(x,y)\cdot dy = 0$.

where,

$M = x^2+y^2+3$ $\quad\quad N = 2xy + \sin^2 y$

$\therefore \dfrac{\partial M}{\partial y} = 2y$ $\quad\quad \dfrac{\partial N}{\partial x} = 2y$.

$\therefore$ the given eq^n is exact.

$\therefore$ It's general sol^n is given by,

$$\int_{y\ \text{const}} M(x,y)\cdot dx + \int \left\{ \underset{\text{terms of } N \text{ containing only } y}{} \cdot dy \right\} = C$$

$$\cancel{\therefore \int_{y\,\text{const}} (x^2+y^2+3)\,dx + \int \frac{\sin^2 y}{x}\,dy = C}$$

$$\cancel{\therefore \frac{x^3}{3} + 3x + y^2 x + \frac{\sin^3 y}{y} = C}$$

$$\therefore \int_{y\ \text{const}} (x^2+y^2+3)\cdot dx + \int \sin^2 y \cdot dy = C$$

$$\therefore \int_{y\ \text{const}} (x^2+y^2+3)\,dx + \int \left(\frac{1}{2} - \frac{\cos 2y}{2}\right)\cdot dy = C$$

$$\therefore \frac{x^3}{3} + x\cdot y^2 + 3x + \frac{y}{2} - \frac{\sin 2y}{4} = C$$

Non exact

Ex:- 01 Solve the differential equation
$x^2y \cdot dx - (x^3 + xy^2) \cdot dy = 0$

→ Here, we are given, $x^2y \cdot dx - (x^3+xy^2) \cdot dy = 0$ —— (1)
where,

$M = x^2y$ $\qquad$ $N = -(x^3 + xy^2)$

$\therefore \dfrac{\partial M}{\partial y} = x^2$ $\qquad$ $\dfrac{\partial N}{\partial x} = -3x^2 - y^2$

→ minus consider કરવાનો

$\therefore \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

$\therefore$ eq^n (1) is not exact.

→ Observe that eq^n (1) is homogeneous d. eq^n.

$$\therefore (I.F.) = \frac{1}{x \cdot M + y \cdot N}$$

$$= \frac{1}{x^3y + (-x^3y - xy^3)}$$

$$= \frac{1}{x^3y - x^3y - xy^3} = \boxed{\frac{-1}{xy^3}}$$

→ Now, multiplying (1) with I.F. we get,

$$\left(\frac{-1}{xy^3}\right) \cdot x^2y\, dx - \left(\frac{-1}{xy^3}\right) \cdot x^3 dy - \left(\frac{-1}{xy^3}\right) \cdot xy^2 dy = 0$$

$$\therefore -\frac{x}{y^2} \cdot dx + \frac{x^2}{y^3} \cdot dy + \frac{1}{y} \cdot dy = 0$$

$$\therefore -\frac{x}{y^2} \cdot dx + \left(\frac{x^2}{y^3} + \frac{1}{y}\right) dy = 0 \quad \text{—— (2)}$$

→ Comparing eq. (2) with

$$M^* dx + N^* dy = 0.$$

we get, $M^* = -\dfrac{x}{y^2}$ $N^* = \dfrac{x^2}{y^3} + \dfrac{1}{y}$

$\therefore \dfrac{\partial M^*}{\partial y} = \dfrac{2x}{y^3}$. $\dfrac{\partial N^*}{\partial x} = \dfrac{2x}{y^3}$.

$\therefore$ eq^n (2) is exact.

$\therefore$ the general sol^n of (1) is given by,

$$\int_{y\ \text{const}} M^*\, dx + \int \left(\begin{matrix}\text{terms of } N^* \\ \text{containing } y \\ \text{only}\end{matrix}\right) \cdot dy = C$$

$$\therefore \int_{y\ \text{const}} \left(-\frac{x}{y^2}\right) \cdot dx + \int \frac{1}{y} \cdot dy = C$$

$$\therefore \boxed{-\frac{x^2}{2y^2} + \log y + c = C}$$

Ans.

Ex:-02 Solve the d.e. $(x^2y^2 + 2)y \cdot dx + (2 - x^2y^2) \cdot x\, dy = 0$.

$\to$ Here, we are given,

$(x^2y^2 + 2)y \cdot dx + (2 - x^2y^2) \cdot x\, dy = 0$ — (1)

where,

$M = (x^2y^2 + 2)y = x^2y^3 + 2y$

$\therefore \dfrac{\partial M}{\partial y} = 3y^2x^2 + 2$

and $N = (2 - x^2y^2) \cdot x = 2x - x^3y^2$

$\therefore \dfrac{\partial N}{\partial x} = 2 - 3x^2y^2$.

$\therefore \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ $\therefore$ eq^n (1) is not exact. d.e.

→ Since, (1) is of the form
$f_1(xy) \cdot y \cdot dx + f_2(xy) \cdot x \cdot dy = 0$.

$$\therefore (I.F.) = \frac{1}{x \cdot M - y \cdot N}$$

$$= \frac{1}{x[x^2y^3+2y] - y[2x - x^3y^2]}$$

$$= \frac{1}{x^3y^3 + \cancel{2xy} - \cancel{2xy} + x^3y^3}$$

$$\therefore (I.F.) = \frac{1}{2x^3y^3}$$

→ Now, multiplying (1) by (I.F.) we get,

$$\frac{1}{2x^3y^3}(x^2y^3+2y) \cdot dx + \frac{1}{2x^3y^3}(2x - x^3y^2)dy = 0$$

$$\therefore \left(\frac{1}{2x} + \frac{2}{2x^3y^2}\right)dx + \left(\frac{2}{2y^3x^2} - \frac{1}{2y}\right)dy = 0$$

$$\therefore \frac{1}{2}\left\{\left(\frac{1}{x} + \frac{2}{x^3y^2}\right) \cdot dx + \left(\frac{2}{y^3x^2} - \frac{1}{y}\right)dy\right\} = 0$$

$$\therefore \left(\frac{1}{x} + \frac{2}{x^3y^2}\right) \cdot dx + \left(\frac{2}{x^2y^3} - \frac{1}{y}\right) \cdot dy = 0 \quad \text{--- (2)}$$

→ Comparing (2) with $M^* dx + N^* dy = 0$;
we get,

$M^* = \frac{1}{x} + \frac{2}{x^3y^2}$	$N^* = \frac{2}{x^2y^3} - \frac{1}{y}$
$\therefore \frac{\partial M^*}{\partial y} = -4 \cdot \frac{1}{x^3y^3}$	$\therefore \frac{\partial N^*}{\partial x} = -4 \frac{1}{x^3y^3}$

∴ eqⁿ (2) is exact d.eⁿ

∴ the general solⁿ is given by,

$$\int_{y\ \text{const}} M^* \, dx + \int \left(\begin{array}{c} \text{terms of } N^* \\ \text{containing only} \\ y \end{array} \right) dy = C$$

$$\therefore \int_{y\ \text{const}} \left(\frac{1}{x} + \frac{2}{x^3 y^2} \right) \cdot dx + \int \left(-\frac{1}{y} \right) \cdot dy = C$$

$$\therefore \log x - \frac{2}{2x^2 y^2} - \log y = C$$

$$\therefore \boxed{\log\left(\frac{x}{y}\right) - \frac{1}{x^2 y^2} = C}$$ Ans

Ex:-03 Solve the d.e $(xy^3 + y) \cdot dx + (2x^2y^2 + 2x + 2y^4) \, dy = 0$.

→ Here, we are given

$$(xy^3 + y) \, dx + (2x^2y^2 + 2x + 2y^4) \cdot dy = 0 \quad \text{(1)}$$

where,

$M = xy^3 + y$ $\qquad$ $N = 2x^2y^2 + 2x + 2y^4$

$\therefore \frac{\partial M}{\partial y} = 3xy^2 + 1$ $\qquad$ $\therefore \frac{\partial N}{\partial x} = \cancel{4x^2y + 8y^3}$

$= 4xy^2 + 2 + 0$

$= 4xy^2 + 2$

∴ (1) is not exact as $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{(xy^2+1)y}\left(4xy^2+2-3xy^2-1\right)$$

$$= \frac{1}{(xy^2+1)y}(xy^2+1)$$

$$= \frac{1}{y} = g(y).$$

$\therefore$ $(I.F.) = e^{\int g(y) \cdot dy}$

$= e^{\int 1/y \cdot dy} = e^{\log y} = y.$

$\Rightarrow$ Now, multiplying ① with i.f., we get,

$$(xy^4+y^2)\,dx + (2x^2y^3+2xy+2y^5)\,dy = 0 \quad \text{②}$$

Comparing ② with $M^* dx + N^* dy = 0$,

we get, $M^* = xy^4 + y^2$ and

$\therefore \frac{\partial M^*}{\partial y} = 4xy^3 + 2y.$

$N^* = 2x^2y^3 + 2xy + 2y^5$

$\therefore \frac{\partial N^*}{\partial x} = 4xy^3 + 2y$

$\therefore$ ② is exact d. eq^n.

$\therefore$ the general eq^n is given by,

$$\int_{y\ \text{const}} M^*\,dx + \int \left(\substack{\text{terms of } N^* \\ \text{containing only} \\ y}\right) \cdot dy = C$$

$\therefore \int_{y\ const.} (xy^4 + y^2)\,dx + \int 2y^5\,dy = c$

$\therefore \dfrac{x^2y^4}{2} + 0 + \dfrac{2y^6}{6} = C$

$\therefore \boxed{\dfrac{x^2y^4}{2} + \dfrac{y^6}{3} = C}$ Ans

Ex: 04 (14) Solve the d. eq^n.

$(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$

→ Here, we are given.

$(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\cdot dy = 0. \quad \text{—— (1)}$

where $M = y^4 + 2y$ and $N = xy^3 + 2y^4 - 4x$

$\therefore \dfrac{\partial M}{\partial y} = 4y^3 + 2.$ $\therefore \dfrac{\partial N}{\partial x} = y^3 + 0 - 4$

$= y^3 - 4$

$\therefore \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ $\therefore$ the eq^n is Not exact.

→ Now, ~~$\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$~~

~~$= \dfrac{1}{xy^3 + 2y^4 - 4x}\left(4y^3 + 2 - y^3 + 4\right)$~~

~~$= \dfrac{3y^3 + 6}{xy^3 + 2y^4 - 4x}$~~

$=$

Note :- always choose that eq^n which has easy Multiplication and division or $\boxed{\text{minimum variables}}$

→ Now, $\frac{1}{M}\left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}\right)$

$$= \frac{1}{y(y^3+2)}\left(y^3 - 4 - 4y^3 - 2\right)$$

$$= \frac{1}{y(y^3+2)}\left(-3y^3 - 6\right)$$

$$= -\frac{3}{y} = f(x).$$

$$\therefore \text{ I.F.} = e^{\int f(x)\cdot dx}$$

$$= e^{-\int 3/y \cdot dx}$$

$$= e^{\log y^{-3}} = \boxed{\frac{1}{y^3}}$$

→ multiplying I.F. with eq^n 1 we get,

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0 \quad \text{—(2)}$$

Comparing (2) with $M^* dx + N^* dy = 0$.

we get, $M^* = y + \frac{2}{y^2}$ | $N^* = x + 2y - \frac{4x}{y^3}$

$\therefore \frac{\partial M^*}{\partial y} = 1 - \frac{4}{y^3}$. | $\therefore \frac{\partial N^*}{\partial x} = 1 - \frac{4}{y^3}$.

$$\therefore \frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$$

$\therefore$ the eq^n (2) is exact eq^n.

$\therefore$ the general solⁿ is given by,

$$\int_{y\ const} M^* \, dx + \int \left(\begin{array}{c} \text{terms of } N^* \\ \text{containing only} \\ y \end{array} \right) \cdot dy = C$$

$$\therefore \int_{y\ const.} \left(y + \frac{2}{y^2} \right) \cdot dx + \int 2y \cdot dy = c$$

$$\therefore xy + \frac{2x}{y^2} + \frac{2y^2}{2} = c$$

$$\therefore \boxed{xy + \frac{2 \cdot x}{y^2} + y^2 = c} \quad \text{Ans}$$

Ex: 05 (12) Solve the d. eqⁿ $(1+xy)y \cdot dx + (1-xy)x\,dy = 0.$

$\rightarrow$ Here, we are given $(1+xy)y \cdot dx + (1-xy) \cdot x\,dy = 0$. — (i)

where, $M = (1+xy)y$
$\therefore M = y + xy^2$
$\therefore \frac{\partial M}{\partial y} = 1 + 2xy$

$N = x - x^2 y$
$\therefore \frac{\partial N}{\partial x} = 1 - 2xy$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ the (1) is not exact. but it is of the form

$$f_1(xy) \cdot y \cdot dx + f_2(xy)\,x \cdot dy = 0.$$

$$\therefore \text{I.F.} = \frac{1}{M \cdot x - N \cdot y}$$

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$\therefore$ I.F. $= \dfrac{1}{xy + x^2y^2 - xy + x^2y^2}$

$\therefore$ I.F. $= \dfrac{1}{2x^2y^2}$

→ Now, multiplying I.f. with eqⁿ (1), we get,

$$\frac{1}{2x^2y^2}(y + xy^2) \cdot dx + \frac{1}{2x^2y^2}(x - x^2y) = 0.$$

$$\therefore \left(\frac{1}{2x^2y} + \frac{1}{2x}\right) \cdot dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right) dy = 0 \quad (2)$$

→ Comparing (2) with $M^* dx + N^* dy = 0$
we get,

$$M^* = \frac{1}{2x^2y} + \frac{1}{2x} \qquad N^* = \frac{1}{2xy^2} - \frac{1}{2y}$$

$$\therefore \frac{\partial M^*}{\partial y} = -\frac{1}{2x^2y^2} \qquad \frac{\partial N^*}{\partial x} = -\frac{1}{2x^2y^2}$$

$$\therefore \frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$$

$\therefore$ the eqⁿ (2) is exact.

→ $\therefore$ the g. solⁿ of eqⁿ (2) is given by,

$$\int_{y\,\text{const}} M^* dx + \int \left(\begin{array}{c}\text{terms of } N^* \\ \text{containing only} \\ y\end{array}\right) dy = C$$

$$\therefore \int_{y\ const.} \left(\frac{1}{2x^2 y} + \frac{1}{2x} \right) . dx + \int . \left(\frac{-1}{2y} \right) . dy = C.$$

$$\therefore -\frac{1}{2xy} + \frac{1}{2} \log x - \frac{1}{2} \log y = C$$

$$\therefore \left\{ -\frac{1}{2xy} + \frac{1}{2}(\log x - \log y) \right\} = C$$

$$\therefore \boxed{-\frac{1}{2xy} + \frac{1}{2} \log\left(\frac{x}{y}\right) = C} \quad \text{Ans.}$$

Ex:-06 (13) Solve the d. eq^n.

$$(x^2 + y^2 + 3)\, dx - 2xy \cdot dy = 0$$

→ Here, we are given

$$(x^2 + y^2 + 3)\, dx - 2xy\, dy = 0 \quad \text{——— (1)}$$

let $M = x^2 + y^2 + 3$ | $N = -2xy$

$\therefore \frac{\partial M}{\partial y} = 2y$ | $\therefore \frac{\partial N}{\partial x} = -2y$.

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ (1) is not exact eq^n.

→ Now, $\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$

$$= \frac{1}{-2xy} \left(2y + 2y \right)$$

$$= \frac{4y}{-2xy} = \boxed{\frac{-2}{x}} = f(x).$$

$\therefore$ I.F. $= e^{\int f(x)\cdot dx}$

$= e^{\int (-2/x)\cdot dx}$

$= e^{-2\log|x|}$ $= x^{-2} = \boxed{1/x^2}$

$\Rightarrow$ Now, multiplying (1) with I.F., we get,

$$\left(1 + \frac{y^2}{x^2} + \frac{3}{x^2}\right)\cdot dx - \frac{2y}{x}\cdot dy = 0 \quad \text{--- (2)}$$

$\Rightarrow$ Comparing (2) with $M^* dx + N^* dy = 0$
we get,

$M^* = 1 + \frac{y^2}{x^2} + \frac{3}{x^2}$.	$N^* = -\frac{2y}{x}$
$\therefore \frac{\partial M^*}{\partial y} = \frac{2y}{x^2}$.	$\therefore \frac{\partial N^*}{\partial x} = \frac{2y}{x^2}$

$$\therefore \frac{\partial N^*}{\partial y} = \frac{\partial N^*}{\partial x}$$

$\therefore$ eq. (2) is the exact eq^n.

$\Rightarrow$ $\therefore$ the g. sol^n is given by,

$$\int_{y\ const} M^* dx + \int \left(\begin{matrix}\text{terms of } N^* \\ \text{containing } y \text{ only}\end{matrix}\right) dy = C$$

$$\therefore \int \left(1 + \frac{y^2}{x^2} + \frac{3}{x^2}\right) dx + \int 0 \cdot dy = C$$

$$\therefore x + \left(-\frac{y^2}{x}\right) - \frac{3}{x} = C$$

$\therefore \boxed{x - \frac{y^2}{x} - \frac{3}{x} = c}$ — Ans

Ex:-07 (11) $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

→ We are given, $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$ — (1)

$\therefore$ where $M = x^2y - 2xy^2$ | $N = -x^3 + 3x^2y$

$\therefore \frac{\partial M}{\partial y} = x^2 - 4xy$ | $\therefore \frac{\partial N}{\partial x} = -3x^2 + 6xy$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ (1) is not exact eq^n.

→ Observe that eq^n (1) is homogeneous d. eq^n.

$$\therefore I.F. = \frac{1}{M.x + N.dy}$$

$$= \frac{1}{\cancel{x^3y} - 2x^2y^2 \cancel{- x^3y} + 3x^2y^2}$$

$$\therefore I.F. = \frac{1}{x^2y^2}$$

→ Now, multiplying eq. (1) with I.F., we get,

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \quad \text{— (2)}$$

→ Comparing it with $M^*\,dx + N^*\,dy = 0$, we get,

$M^* = \frac{1}{y} - \frac{2}{x}$ $\quad \frac{\partial M^*}{\partial y} = -\frac{1}{y^2}$

and $N^* = \frac{-x^2}{y^2} + \frac{3}{y}$.

$\therefore \frac{\partial N^*}{\partial x} = -\frac{1}{y^2}$

$\therefore \frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$

$\therefore$ eq^n (2) is 'exact d. eq^n.

$\therefore$ the general sol^n is given by,

$$\int_{y\ \text{const}} M^* \, dx + \int \left(\begin{matrix} \text{terms of } N^* \\ \text{only containing} \\ y \end{matrix} \right) dy = C$$

$$\therefore \int_{y\ \text{const}} \left(\frac{1}{y} - \frac{2}{x} \right) dx + \int \frac{3}{y} \, dy = C$$

$$\therefore \boxed{\frac{x}{y} - 2 \log x + 3 \log y = C}$$ Ans:-