Analyzing Non-linear Ordinary Differential Equations

MATH 26600

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1 Introduction

1.1 Linearizing a system of ODE's

Let

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x,y) \tag{1}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = G(x,y) \tag{2}$$

be a system of first-order differential equations. The steady-state solutions of this system are the solutions for which $x(t)$ and $y(t)$ are invariant. That is to say,

$$F(x,y) = 0 \tag{3}$$
$$G(x,y) = 0 \tag{4}$$

We can analyzing this system around these equilibrium points by making linear approximations of the function around these points. Using first order Taylor series expansion for F and G we get

$$F(x,y) \approx F(x_0,y_0) + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0) \tag{5}$$
$$G(x,y) \approx G(x_0,y_0) + G_x(x_0,y_0)(x-x_0) + G_y(x_0,y_0)(y-y_0) \tag{6}$$

Where x_0 and y_0 are the equilibrium points for the system. The system can be re-written in matrix-vector notation as

$$\begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \tag{7}$$

Now, let $x - x_0$ be u and $y - y_0$ be v, and $\vec{v}$ be the vector $\langle u, v \rangle$

$$\therefore \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = J\vec{v} \tag{8}$$

Where J is the Jacobian matrix,

$$J = \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix}$$

Now, Eq. 8 is an eigenvalue problem. The local stability and the behaviour of the system around the equilibrium points can be infered from the eigenvalues of J.

Example 1. [1, p. 488] Consider the system for $(x, y \geq 0)$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(10 - x - y)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(30 - 2x - y)$$

The Jacobian of the system is

$$J = \begin{bmatrix} 10 - 2x - y & -x \\ -2y & 30 - 2x - 2y \end{bmatrix}$$

The system has equilibrium points at $(0,0)$, $(10,0)$, $(0,30)$. Analyzing the system at $(0,0)$,

$$J|_{(0,0)} = \begin{bmatrix} 10 & 0 \\ 0 & 30 \end{bmatrix}$$

Since J is a diagonal matrix, the eigenvalues of J are the elements along its diagonal. That is, $\lambda_{1,2} = \{10, 30\}$. Since both the eigenvalues are real and positive, the point $(0,0)$ is a nodal-source. Similarly, analyzing the system at $(10,0)$,

$$J|_{(10,0)} = \begin{bmatrix} -10 & -10 \\ 0 & 10 \end{bmatrix}$$

The eigenvalues of J are $\lambda_{1,2} = \{-10, 10\}$. Since both the eigenvalues are real and nonzero, and $\lambda_1 < 0$, $\lambda_2 > 0$, the point $(10,0)$ is a saddle point.

2 Hamiltonian Systems

3 Dissipative Systems

References

[1] Paul Blanchard, Robert L. Devaney, and Glen R. Hall. *Differential equations.* Boston, MA : Brooks/Cole, Cengage Learning, 2012., 2012.