PLAN & MEMO

DATE 11-12-2022

PLACE

MEMBER

# Partial Derivative

**Ex: 1**

Find $f_x(1,3)$ and $f_y(1,3)$ for the function $f(x,y) = 2x^3y^2 + 2y + 4x$.

**Solution**

$$f(x,y) = 2x^3y^2 + 2y + 4x$$

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial x}(2x^3y^2 + 2y + 4x).$$

$$\frac{\partial f(x,y)}{\partial x} = 6x^2y^2 + 4.$$

1,3

$$\frac{\partial f(1,3)}{\partial x} = 6(1)^2(3)^2 + 4$$

$$\frac{\partial f(1,3)}{\partial x} = 58$$

Ans

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}(2x^3y^2 + 2y + 4x)$$

$$\frac{\partial f(x,y)}{\partial y} = 4x^3y + 2$$

$$\frac{\partial f(1,3)}{\partial y} = 4(1)^3(3) + 2$$

$$\frac{\partial f(1,3)}{\partial y} = 14$$

Ans.

DUNLOP

Ex. 2
Find $f_x(x,y)$ and $f_y(x,y)$ for $f(x,y) = 2x^3y^2 + 2y + 4x$, and use those partial derivatives to compute $f_x(1,3)$ and $f_y(1,3)$.

Already Done!

Ex. 3
If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$

Solution

$$f(x,y) = x^3 + x^2y^3 - 2y^2$$

$$f_x(x,y) = 3x^2 + 2xy^3$$

$$f_x(2,1) = 3(2)^2 + 2(2)(1)^3$$

$$f_x(2,1) = 12 + 4$$

$$f_x(2,1) = 16 \text{ Ans}$$

$$f_y(x,y) = 3x^2y^2 - 4y$$

$$f_y(2,1) = 3(2)^2(1)^2 - 4(1)$$

$$f_y(2,1) = 12 - 4$$

$$f_y(2,1) = 8 \text{ Ans}$$

Ex. 1

Find $\partial f/\partial y$ as a function of $f(x,y) = y\sin xy$

solution

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y\sin xy)$$

$\because U \cdot V = v\, du/dv + u\, \frac{dv}{du}$

$$\frac{\partial f}{\partial y} = \sin xy \cdot 1 + y\cos xy\,(x)$$

$$\frac{\partial f}{\partial y} = \sin xy + xy\cos xy.$$

Ans

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y\sin xy)$$

$$\frac{\partial f}{\partial x} = y^2\cos xy$$

Ans

Ex. 2

Find $f_x$ and $f_y$ as function if

$$f(x,y) = \frac{2y}{y+\cos x}.$$

Solution

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right)$$

$$\because \frac{U}{V} = \frac{v \cdot du - u \cdot dv}{v^2}$$

$$f_x = \frac{y+\cos x \cdot (0) - 2y\,(0-\sin x)}{(y+\cos x)^2}$$

$$f_x = \frac{2y\sin x}{(y+\cos x)^2}$$ Ans

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right)$$

$$\because \frac{u}{v} = \frac{v.du - u.dv}{v^2}$$

$$f_y = \frac{(y+\cos x)2 - 2y(1)}{(y+\cos x)^2}$$

$$f_y = \frac{2\cancel{y} + 2\cos x - \cancel{2y}}{(y+\cos x)^2}$$

$$f_y = \frac{2\cos x}{(y+\cos x)^2}$$ Ans

Ex: 8

Find $\frac{\partial z}{\partial x}$ if the equation $yz - \ln z = x+y$ defines $z$ as a function of the two variables $x$ and $y$ and the partial derivative exists.

Solution

$$\frac{\partial}{\partial x}(yz - \ln z) = \frac{\partial}{\partial x}(x+y)$$

$$\frac{\partial z}{\partial x}\cdot y - \frac{1}{z}\cdot\frac{\partial z}{\partial x} = \cancel{\frac{\partial x}{\partial x}}\cdot 1 + 0$$

$$\frac{\partial z}{\partial x}\left(y - \frac{1}{z}\right) = 1$$

$$\frac{\partial z}{\partial x}\left(\frac{yz-1}{z}\right) = 1$$

$$\frac{\partial z}{\partial x} = \frac{1}{\frac{yz-1}{z}}$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz-1} \quad \text{Ans}$$

Ex: 8

If $f(x,y) = \sin\left(\frac{x}{1+y}\right)$. Calculate $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$.

solution

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right)\cdot\frac{\partial}{\partial x}\left(\frac{x}{1+y}\right).$$

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right)\left(\frac{1}{1+y}\right).$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) \Rightarrow x \cdot (1+y)^{-1}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \left(-x(1+y)^{-2}(0+1)\right)$$

$$\frac{\partial f}{\partial y} = -\cos\left(\frac{x}{1+y}\right)\left(\frac{x}{(1+y)^2}\right)$$

Ex: 02

$D = \sqrt{x^2 + y^2}$ $\quad x = 3 \quad , \quad y = 4.$

$D^2 = x^2 + y^2.$

$$\frac{\partial D}{\partial x} = \frac{\partial}{\partial x}(D^2) = \frac{\partial}{\partial x}(x^2 + y^2)$$

$$\Rightarrow \cancel{2}D \frac{\partial D}{\partial x} = \cancel{2}x$$

$$\Rightarrow \frac{\partial D}{\partial x} = \frac{x}{D}$$

$$\Rightarrow \frac{\partial D}{\partial x} = \frac{3}{5}$$

$D^2 = x^2 + y^2$
$D^2 = 3^2 + 4^2$
$D^2 = 5^2$
$D = 5.$

Thus, D is increasing at a rate of 3/5 unit per unit increase in x at point (3,4)

Ex: 4

The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$.

Solution

The slope is the value of partial derivative $\partial z/\partial y$ at $(1, 2)$.

$$z = x^2 + y^2$$

$$\frac{\partial}{\partial y}(z) = \frac{\partial}{\partial y}(x^2 + y^2)$$

$$\frac{\partial z}{\partial y} = 0 + \frac{\partial y^2}{\cancel{\partial y}} \cdot 2y$$

$$\frac{\partial z}{\partial y} = 2y \qquad y = 2$$

$$\frac{\partial z}{\partial y} = 2(2)$$

$$\frac{\partial z}{\partial y} = 4.$$

Estimating partial derivatives from tabular data

Ex
$W(T, v)$ $(T, v) = (25, 10)$.

Estimate the partial derivative of W wrt to v at $(T, v) = (25, 10)$.

Solution

$\because f_x(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$

$\frac{\partial w}{\partial v} = \lim_{\Delta v \to 0} \frac{w(25, 10 + \Delta v) - f(25, 10)}{\Delta v}$

$\frac{\partial w}{\partial v} = \lim_{\Delta v \to 0} \frac{w(25, 10 + \Delta v) - 15}{\Delta v}$

with $\Delta v = 5$

$\frac{\partial w}{\partial v} = \lim_{\Delta v \to 0} \frac{w(25, \overset{10+5}{15}) - 15}{5}$

$\frac{\partial w}{\partial v} = \lim_{\Delta v \to 0} \frac{13 - 15}{5}$ applying limit

$\frac{\partial w}{\partial v} = -\frac{2}{5}$ °F/mi/h

with $\Delta v = -5$

$\frac{\partial w}{\partial v} = \lim_{\Delta v \to 0} \frac{w(25, \overset{10-5}{5}) - 15}{-5}$

$\frac{\partial w}{\partial v} = \lim_{\Delta v \to 0} \frac{19 - 15}{-5}$ applying limit

$\frac{\partial w}{\partial v} = \frac{+4}{-5} = -\frac{4}{5}$ °F/mi/h.

Taking average $\frac{-3}{5} = -0.6°F/min$ for both approximations as our estimation for P.D.

## Implicit Partial Derivatives.

Ex: 1

Find the slope of the sphere $x^2+y^2+z^2=1$ in the $y$-direction at the points $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$.

Solution

$$\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right) = \frac{\partial}{\partial y}(1).$$

$$0 + 2y\frac{\partial y}{\partial y} + 2z\frac{\partial z}{\partial y} = 0.$$

$$\cancel{2}z \cdot \frac{\partial z}{\partial y} = -\cancel{2}y$$

$$\frac{\partial z}{\partial y} = -\frac{y}{z}$$

for upper hemisphere

$$\frac{\partial z}{\partial y}\Big|_{\left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)} = -\frac{1/\cancel{3}}{2/\cancel{3}} = -\frac{1}{2}$$

for lower hemisphere

$$\frac{\partial z}{\partial y}\Big|_{\left(\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right)} = \cancel{-}+\frac{1/\cancel{3}}{\cancel{-}2/\cancel{3}} = \frac{1}{2}.$$

Ex: 02

Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z$ is defined implicitly as a function of $x$ & $y$ by the eq. $x^3 + y^3 + z^3 + 6xyz = 1$

Solution

$$\frac{\partial z}{\partial x} \Rightarrow \frac{\partial}{\partial x}\left(x^3+y^3+z^3+6xyz\right) = \frac{\partial}{\partial x}(1)$$

$$\Rightarrow 3x^2\frac{\partial x}{\partial x} + 0 + 3z^2\frac{\partial z}{\partial x} + 6yz\frac{\partial x}{\partial x} + 6xz\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow 3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x}\left(3z^2 + 6xy\right) = -3x^2 - 6yz$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-3(x^2+2yz)}{3z^2+6xy}$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\cancel{3}(x^2+2yz)}{\cancel{3}(z^2+2xy)}$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{x^2+2yz}{z^2+2xy}.$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^3+y^3+z^3+6xyz\right) = \frac{\partial}{\partial y}(1)$$

$$\frac{\partial z}{\partial y} : 0 + 3y^2\frac{\partial y}{\partial y} + 3z^2\frac{\partial z}{\partial y} + 6xz\frac{\partial y}{\partial y} + 6xy\frac{\partial z}{\partial y} = 0$$

PLAN & MEMO

DATE
PLACE
MEMBER

$$\Rightarrow 3y^2 + 3z^2\left(\frac{\partial z}{\partial y}\right) + 6xz + 6xy\left(\frac{\partial z}{\partial y}\right) = 0$$

$$\Rightarrow \frac{\partial z}{\partial y}\left(3z^2 + 6xy\right) = -3y^2 - 6xz$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-3\left(y^2 + 2xz\right)}{3z^2 + 6xy}$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-\cancel{3}\left(y^2 + 2xz\right)}{\cancel{3}\left(z^2 + 2xy\right)}$$

$$\Rightarrow \frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy} \quad \text{Ans}$$

Ex: 1

$$f(x,y) = \begin{cases} \dfrac{-xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

(a) Show that $f_x(x,y)$ and $f_y(x,y)$ exist at all points.

Solution

$$f_x(x,y) = \frac{(x^2+y^2)(-y) - (-xy)(2x+0)}{(x^2+y^2)^2}$$

$$f_x(x,y) = \frac{-x^2y - y^3 + 2x^2y}{(x^2+y^2)^2}$$

$$f_x(x,y) = \frac{x^2y - y^3}{(x^2+y^2)^2}.$$

$$f_y(x,y) = \frac{(x^2+y^2)(-x) - (-xy)(0+2y)}{(x^2+y^2)^2}$$

DUNLOP

$$f_y(x,y) = \frac{-x^3 - xy^2 + 2xy^2}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{-x^3 + 2xy^2}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{2xy^2 - x^3}{(x^2+y^2)^2}$$

$$\because \quad f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(\overset{0+\Delta x = \Delta x}{x+\Delta x}, y) - f(x,y)}{\Delta x}$$

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x}.$$

Since $f(0,0) = 0$

& when $y=0 \Rightarrow f(\Delta x, 0) = 0.$

Hence

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x}$$

$$f_x(0,0) = 0.$$

$$\because \quad f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}.$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y}.$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y}$$

$$f_y(0,0) = 0$$

at $(0,0)$, values of both P.Ds are 0.

PLAN & MEMO

DATE
PLACE
MEMBER

(b) Explain why f is not continuous at (0,0)

$$f(x,y) = -\frac{xy}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{-xy}{x^2+y^2}$$

applying limit will give undefined since function is undefined with limits, so it is not continuous at (0,0)

P.D with more than 3 vars.
For a function (x,y,z) of three variables, there three partial derivatives.

$f_x(x,y,z)$  $f_y(x,y,z)$  $f_z(x,y,z)$.

$w = f(x,y,z)$  $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$

Ex. 1 if $f(x,y,z) = x^3y^2z^4 + 2xy + z$.

$f_x(x,y,z) = 3x^2y^2z^4 + 2y$

$f_y(x,y,z) = 2x^3yz^4 + 2x$.

$f_z(x,y,z) = 4x^3y^2z^3 + 1$.

Ex. 02

If $f(\rho, \theta, \phi) = \rho^2 \cos\phi \sin\theta$.

$$f_\rho(\rho, \theta, \phi) = 2\rho \cos\phi \sin\theta$$

$$f_\theta(\rho, \theta, \phi) = +\rho^2 \cos\phi \cos\theta$$

$$f_\phi(\rho, \theta, \phi = -\rho^2 \sin\phi \sin\theta.$$

Laplace Equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Show ~~that~~ the function $\{u(x,y) = e^x \sin y$ is a ~~function~~ solution of Laplace's equation

$$u(x,y) = e^x \sin y.$$

$$\Rightarrow u_x = e^x \sin y.$$

$$\Rightarrow u_{xx} = e^x \sin y.$$

$$\Rightarrow u_y = +e^x \cos y.$$

$$\Rightarrow u_{yy} = -e^x \sin y$$

PLAN & MEMO

DATE

PLACE

MEMBER

acc to Laplace's equation.

$$e^x \sin y + (-e^x \sin y) = 0$$

$$e^x \sin y - e^x \sin y = 0$$

$$0 = 0$$

Hence, $u(x,y)$ ~~follows~~ satisfies Laplace's equation.

Wave Equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Ex. 1

Verify that the function $u(x,t) = \sin(x-at)$ satisfies the wave equation.

$$u(x,y) = \sin(x-at).$$

$$\Rightarrow u_t = \cos(x-at).(0-a).$$

$$u_t = -a\cos(x-at).$$

$$\Rightarrow u_{tt} = -a(-\sin(x-at).(0-a))$$

$$u_{tt} = -a^2 \sin(x-at).$$

$\Rightarrow$ ~~$u_{xx}$~~ =

$u(x,y) = \sin(x - at)$

$\Rightarrow u_x = \cos(x - at)(1 - 0)$

$u_x = \cos(x - at).$

$\Rightarrow u_{xx} = -\sin(x - at)(1 - 0)$

$u_{xx} = -\sin(x - at).$

$\because$ $u_{tt} = a^2 u_{xx}.$

$-a^2 \sin(x - at) = -a^2 . \sin(x - at).$

Hence, it satisfies wave equation.