Double Integral Over Rectangular Region & Non-Rectangular Region

Ex: Evaluate $\int_1^3 \int_2^4 (40-2xy)\, dy\, dx$.

Solu $\int_1^3 \left| 40y - \frac{\cancel{2}xy^2}{\cancel{2}} \right|_2^4 dx$

$\int_1^3 \left[\{40(4) - x(4)^2\} - \{40(2) - x(2)^2\} \right] dx$

$\int_1^3 \{160 - 16x - 80 + 4x\} dx$

$\int_1^3 (80 - 12x)\, dx$

$\left| 80x - \frac{\cancel{12}^{6} x^2}{\cancel{2}} \right|_1^3$

$\left[\{80(3) - 6(3)^2\} - \{80(1) - 6(1)^2\} \right]$

$240 - 54 - 80 + 6$

$= 112$ Ans

$\int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy$

$\int_2^4 \left| \left(40x - \cancel{2}x^2 y / \cancel{2}\right) \right|_1^3 dy$

$\int_2^4 \left[\{40(3) - (3)^2 y\} - \{40(1) - (1)^2 y\} \right]$

$\int_2^4 (120 - 9y - 40 + y)\, dy$

$\int_2^4 (80 - 8y)\, dy$

$\left| 80y - \cancel{8}^{4} y^2 / \cancel{2} \right|_2^4$

$\left[\{80(4) - 4(4)^2\} - \{80(2) - 4(2)^2\} \right]$

$320 - 64 - 160 + 16$

$= 112$ Ans.

Fubinis Theorem. $\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$

Ex. 2: Evaluate $\iint_R y \sin(xy)\, dA$, where $R = [1,2] \times [0,\pi]$.

$\iint_R y \cdot \sin(xy)\, dA = \int_0^\pi \int_1^2 y \cdot \sin(xy)\, dx\, dy$

$= \int_0^\pi \left[\cancel{y}\left(-\cos(xy) \cdot \frac{1}{\cancel{y}}\right) \right]_1^2 dy$

$= \int_0^\pi \left| -\cos(xy) \right|_1^2$

$= \int_0^\pi \left[-\cos(2y) + \cos(y) \right] dy$

$= \left| -\frac{1}{2}\sin(2y) + \sin(y) \right|_0^\pi$

$= -\frac{1}{2}\sin(2\pi) + \sin(0)$

$-\frac{1}{2}\sin 360 + 0$

$= \boxed{0}$

② Reversing the order of integration.

$$\iint_R y\sin(xy)\,dA = \int_1^2 \int_0^\pi x\sin(xy)\,dy\,dx.$$

$\sin(xy)'$

Integration by parts $\Rightarrow \int u\,dv = uv - \int v\,du$

let $u = y$ | $dv =$

$du = dy$ | $v = \dfrac{-\cos(xy)}{x}$

$$= \int_1^2 \left[-\frac{y\cos(xy)}{x} - \int -\frac{\cos(xy)}{x}\cdot dy \right]_0^\pi dx$$

$$= \int_1^2 \left[-\frac{\pi\cos(\pi x)}{x} + \left\{ \frac{1}{x} \frac{\sin(xy)}{x}\Big|_0^\pi \right\} \right] dx$$

$$= \int_1^2 \left[-\frac{\pi\cos(\pi x)}{x} + \frac{1}{x^2}\sin(\pi x) \right] dx$$

$$\Rightarrow \int_1^2 \left(\frac{-\pi\cos(\pi x)}{x} + \frac{\sin(\pi x)}{x^2} \right) dx.$$

$$= \int_1^2 \frac{(-\pi\cos(\pi x)}{x} dx + \int_1^2 \left(\frac{\sin(\pi x)}{x^2} \right) dx.$$

let $u = -1/x \Rightarrow du = +1/x^2$

and $v = \dfrac{\cancel{\pi}\sin(\pi x)}{\cancel{\pi}}$ $dv = \pi\cos(\pi x)$

$$= -\frac{\sin(\pi x)}{x}\Big|_1^2 - \cancel{\int_1^2 \frac{\sin(\pi x)}{x^2} dx} + \cancel{\int_1^2 \frac{\sin(\pi x)}{x^2} dx}$$

$$= -\frac{\sin 2\pi}{2} + \frac{\sin \pi}{1}$$

$$= \boxed{0}$$

Q Use a double integral to find V of the solid that is bounded above by the plane $z = 4 - x - y$ & below by rectangle $R = [0,1] \times [0,2]$.

$$\iint_R (4 - x - y)\,dA = \int_0^2 \int_0^1 (4 - x - y)\,dx\,dy.$$

$$= \int_0^2 \left. 4x - \frac{x^2}{2} - xy \right|_0^1 dy.$$

$$= \int_0^2 \left[4(1) - \frac{1^2}{2} - 1(y) \right] - [0]\,dy$$

$$= \int_0^2 \left(4 - \frac{1}{2} - y \right) dy$$

$$= \int_0^2 \left(\frac{7}{2} - y \right) dy$$

$$= \left| \frac{7}{2} - \frac{y^2}{2} \right|_0^2$$

$$= \left[\left\{ \frac{7}{2}(2) - \frac{(2)^2}{2} \right\} - \{0\} \right]$$

$$= 7 - \frac{4^2}{2}$$

$$= 7 - 2$$

$$= \boxed{5}$$

$$\boxed{V = 5}$$

Ex 2 Find V of region bounded above by elliptical paraboloid $z = 10 + x^2 + 3y^2$ & below rectangle $R: 0 \le x \le 1, 0 \le y \le 2$.

Sol

$$\iint_R (10 + x^2 + 3y^2)\, dA = \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\, dy\, dx$$

$$= \int_0^1 \left| 10y + x^2 y + \cancel{3} \frac{y^3}{\cancel{3}} \right|_0^2 dx$$

$$= \int_0^1 \left[\{ 10(2) + x^2(2) + 2^3 \} - \{0\} \right] dx .$$

$$= \int_0^1 (20 + 2x^2 + 8)\, dx$$

$$= \left| 20x + \frac{2}{3}x^3 + 8x \right|_0^1 .$$

$$= \left[\{ 20(1) + \frac{2}{3}(1)^3 + 8(1) \} - \{0\} \right].$$

$$= 20 + \frac{2}{3} + 8 .$$

$$V = \frac{86}{3} \text{ Ans.}$$

Ex 3 Evaluate the double integral $\iint_R y^2 x\, dA$. over the $R = \{(x, y) : -3 \le x \le 2, 0 \le y \le 1\}$.

Sol

$$\iint_R y^2 x\, dA = \int_0^1 \int_{-3}^2 y^2 x\, dx\, dy .$$

$$= \int_0^1 \left| \frac{y^2 x^2}{2} \right|_{-3}^2 dy .$$

$$= \int_0^1 \left[\left\{ \frac{y^2 2^2}{2} \right\} - \left\{ \frac{y^2(-3)^2}{2} \right\} \right] dy .$$

$$= \int_0^1 2y^2 - \frac{9y^2}{2}\, dy$$

$$= \int_0^1 \frac{-5y^2}{2}\, dy .$$

$$= \left| \frac{-5y^3}{6} \right|_0^1$$

$$= \frac{-5(1)^3}{6} - 0$$

$$V = -5/6 \text{ Ans.}$$

Q: Evaluate the integral

(ii) $\int_{-1}^1 \int_{-x^2}^{x} (x^2 - y)\, dy\, dx$. Type-I

Sol

$$\Rightarrow \int_{-1}^1 \left| x^2 y - \frac{y^2}{2} \right|_{-x^2}^{x} dx .$$

$$\Rightarrow \int_{-1}^1 \left[\left\{ x^2 \cdot x - \frac{(x)^2}{2} \right\} - \left\{ x^2(-x^2) - (-x^2)^2/2 \right\} \right] dx$$

$$\Rightarrow \int_{-1}^1 \left[x^3 - \frac{x^2}{2} + x^4 + \frac{x^4}{2} \right] dx$$

$$\Rightarrow \int_{-1}^{1} \left(x^3 - x^2/2 + 3x^4/2\right) dx$$

$$\Rightarrow \left| x^4/4 - x^3/6 + 3x^5/10 \right|_{-1}^{1}$$

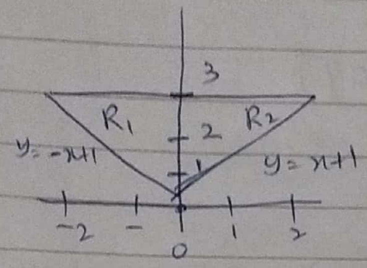
$$\Rightarrow \left[\left\{ 1^4/4 - 1^3/6 + 3\,1^5/10 \right\} - \left\{ (-1)^4/4 - (-1)^3/6 + 3(-1)^5/10 \right\}\right]$$

$$\Rightarrow \left[\left\{ 1/4 - 1/6 + 3/10 \right\} - \left\{ 1/4 + 1/6 - 3/10 \right\}\right].$$

$$\Rightarrow 1/4 - 1/6 + 3/10 - 1/4 - 1/6 + 3/10$$

$$\Rightarrow \boxed{4/15}$$

Q Evaluate the iterated integral $\iint_R (2x-y^2)\, dA$ over the triangular region enclosed b/w two lines $y=-x+1$, $y=x+1$, $y=3$.



upper boundary is $y=3$ whereas the lower boundary is divided, $y=-x+1$ to the left in $y=x+1$ to the right of x-axis

Type – I

$$\iint_R (2x-y^2)\, dA = \iint_{R_1} (2x-y^2)\, dy\, dx + \iint_{R_2} (2x-y^2)\, dy\, dx$$

$$\Rightarrow \int_{-2}^{0}\int_{-x+1}^{3} (2x-y^2)\, dy\, dx + \int_{0}^{2}\int_{x+1}^{3} (2x-y^2)\, dy\, dx$$

$$\Rightarrow \int_{-2}^{0} \left| 2xy - \frac{y^3}{3} \right|_{-x+1}^{3} dx + \int_{0}^{2} \left| 2xy - \frac{y^3}{3} \right|_{x+1}^{3} dx.$$

$$\Rightarrow \int_{-2}^{0} \left[\left\{2x(3) - \frac{(3)^3}{3}\right\} - \left\{2x(-x+1) - \frac{(-x+1)^3}{3}\right\}\right] dx + \int_0^2 \left[\left\{2x(3) - \frac{(3)^3}{3}\right\} - \left\{2x(x+1) - \frac{(x+1)^3}{3}\right\}\right] dx$$

$$\Rightarrow \int_{-2}^{0} \left[\left(6x - \cancel{\tfrac{27}{3}}^{\,9}\right) - \left\{-2x^2 + 2x - \frac{(-x^3+1+3x^2-3x)}{3}\right\}\right] dx + \int_0^2 \left[\left(6x - \cancel{\tfrac{27}{3}}^{\,9}\right) - \left\{2x^2 + 2x - \frac{(x^3+1+3x^2+3x)}{3}\right\}\right] dx$$

$$= \int_{-2}^{0} \left[6x - 9 - \frac{(-6x^2 + 6x + x^3 - 1 - 3x^2 + 3x)}{3}\right] dx + \int_0^2 \left[6x - 9 - \frac{(6x^2 + 6x - x^3 - 1 - 3x^2 + 3x)}{3}\right] dx.$$

$$= \int_{-2}^{0} \frac{(18x - 27 + 6x^2 - 6x - x^3 + 1 + 3x^2 - 3x)}{3}\, dx + \int_0^2 \frac{(18x - 27 - 6x^2 - 6x + x^3 + 1 + 3x^2 - 3x)}{3}\, dx.$$

$$= \frac{1}{3}\int_{-2}^{0} (9x - 26 + 9x^2 - x^3)\, dx + \frac{1}{3}\int_0^2 (15x - 26 - 3x^2 + x^3)\, dx$$

$$= \frac{1}{3}\left| \frac{9x^2}{2} - 26x + \frac{9x^3}{3} - \frac{x^4}{4} \right|_{-2}^{0} + \frac{1}{3}\left| \frac{15x^2}{2} - 26x - \frac{3x^3}{3} + \frac{x^4}{4} \right|_0^2$$

$$= \frac{1}{3}\left[\left\{\frac{9(0)^2}{2} - 26(0) + 3(0)^3 - \frac{(0)^4}{4}\right\} - \left\{\frac{9(-2)^2}{2} - 26(-2) + 3(-2)^3 - \frac{(-2)^4}{4}\right\}\right] +$$

$$\frac{1}{3}\left[\left\{\frac{15(2)^2}{2} - 26(2) - (2)^3 + \frac{(2)^4}{4}\right\} - \{0\}\right].$$

$$= \frac{1}{3}\left[0 - \{18 + 52 - 24 - 4\}\right] + \frac{1}{3}\left[\{30 - 52 - 8 + 4\}\right]$$

$$= \frac{1}{3}(-42) + \frac{1}{3}(-26) \Rightarrow -14 - \frac{26}{3} \Rightarrow \boxed{\frac{-68}{3}}$$

Q Find the volume of the solid bounded by cylinder $x^2+y^2=4$. in the plane $y+z=4$ in $z=0$

S: $\because V = \iint_R f(x,y)\, dA.$

$y+z=4$

Type-I Region or $z = 4-y$

$V = \iint_R f(x,y)\, dy\, dx.$

$x^2+y^2=4 \Rightarrow y^2 = 4-x^2 \Rightarrow y = \pm\sqrt{4-x^2} \Rightarrow -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

let $y=0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2. \Rightarrow -2 \le x \le 2$

$$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\, dy\, dx.$$

$$\Rightarrow \int_{-2}^{2} \left| 4y - \frac{y^2}{2} \right|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx.$$

$$\Rightarrow \int_{-2}^{2} \left[\left\{ 4\sqrt{4-x^2} - \frac{(\sqrt{4-x^2})^2}{2} \right\} - \left\{ -4\sqrt{4-x^2} - \frac{(-\sqrt{4-x^2})^2}{2} \right\} \right] dx.$$

$$\Rightarrow \int_{-2}^{2} \frac{8\sqrt{4-x^2} - 4 + x^2}{2} - \left(\frac{-8\sqrt{4-x^2} - 4 + x^2}{2} \right) dx$$

$$\Rightarrow \int_{-2}^{2} \left(8\sqrt{4-x^2} \cancel{-4} + \cancel{x^2} + 8\sqrt{4-x^2} + \cancel{4} - \cancel{x^2} / 2 \right) dx.$$

$$\Rightarrow \int_{-2}^{2} \left({}^{8}\cancel{16}\sqrt{4-x^2} / \cancel{2}_1 \right) dx.$$

$$\Rightarrow \int_{-2}^{2} (8\sqrt{4-x^2})\, dx.$$

$$\Rightarrow 8\int_{-2}^{2} \sqrt{4-x^2}\, dx. \quad \Rightarrow 8\int_{-2}^{2} \sqrt{4-x^2}\, dx$$

converting it to polar coordinates

$\because \sqrt{a^2-x^2} \Rightarrow$ let $x = a\sin\theta$

$\sqrt{2^2-x^2} \Rightarrow \quad x = 2\sin\theta$

$dx/d\theta = 2\cos\theta$

$\boxed{dx = 2\cos\theta\, d\theta}$

$\because x=-2,\ x=2$

$\boxed{x = 2\sin\theta}$

$\sin\theta = \frac{x}{2}$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$\theta = \sin^{-1}\left(\frac{-2}{2}\right) = \frac{-\pi}{2}$

$\theta = \sin^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{2}$

$$\cancel{\Rightarrow 8 \left| \left(-\frac{2x}{2\sqrt{4-x^2}} \right) \right|_{-2}^{2}}$$

$$\cancel{\Rightarrow 8 \left[\left\{ \frac{-2(2)}{2\sqrt{4-2^2}} \right\} - \left(\frac{-2(-2)}{2\sqrt{4-(-2)^2}} \right) \right].}$$

$$\cancel{\Rightarrow 8 \left[\frac{-4^2}{2\sqrt{4-4}} \right.}$$

$$\Rightarrow 8\int_{-\pi/2}^{\pi/2} \sqrt{4-(2\sin\theta)^2}\ 2\cos\theta\, .\, d\theta$$

$$\Rightarrow 8\int_{-\pi/2}^{\pi/2} \sqrt{4-4\sin^2\theta}\ 2\cos\theta\, d\theta.$$

$$\Rightarrow 8\int_{-\pi/2}^{\pi/2} \sqrt{4(1-\sin^2\theta}\ .\ 2\cos\theta\, d\theta.$$

PAPERWORK

$$\Rightarrow 8\int_{-\pi/2}^{\pi/2} 2\sqrt{1-\sin^2\theta}\cdot 2\cos\theta\, d\theta \quad \because 1-\sin^2\theta = \cos^2\theta$$

$$\Rightarrow 16\int_{-\pi/2}^{\pi/2} \sqrt{\cos^2\theta}\cdot 2\cos\theta\, d\theta$$

$$\Rightarrow 32\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

since integration of $\cos^2\theta$ could be computed then

$\because \cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = \cos^2\theta - (1-\cos^2\theta)$

$\cos 2\theta = \cos^2\theta - 1 + \cos^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$$\boxed{\cos^2\theta = \frac{\cos 2\theta + 1}{2}}$$

$$\Rightarrow 32\int_{-\pi/2}^{\pi/2} \frac{\cos 2\theta + 1}{2}\, d\theta$$

$$\Rightarrow \frac{\cancel{32}^{16}}{\cancel{2}}\int_{-\pi/2}^{\pi/2} \cos 2\theta + 1\, d\theta$$

$$\Rightarrow 16\left|\frac{\sin 2\theta}{2} + \theta\right|_{-\pi/2}^{\pi/2}$$

$$\Rightarrow 16\left[\left\{\frac{\sin \cancel{2}(\pi/\cancel{2})}{2} + \pi/2\right\} - \left\{\frac{\sin \cancel{2}(-\pi/\cancel{2})}{2} - \pi/2\right\}\right]$$

$$\Rightarrow 16\left[\frac{\sin 180}{2}^{0} + \frac{\pi}{2} - \left\{\frac{\sin(-180)}{2}^{0} - \pi/2\right\}\right]$$

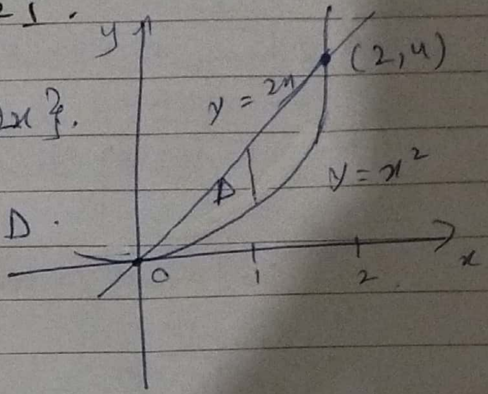
$$\Rightarrow 16\left[\left(0+\frac{\pi}{2}\right) - 0 + \frac{\pi}{2}\right]$$

$$\Rightarrow 16\left[\frac{\pi}{2} + \frac{\pi}{2}\right]$$

$$\Rightarrow 16\left[\frac{\cancel{2}\pi}{\cancel{2}}\right]$$

$$\Rightarrow \boxed{16\pi} \quad A.$$

Ex2 Find the volume of solid that lies under the paraboloid $z = x^2+y^2$ & above region D in xy-plane bounded by line $y=2x$ & parabola $y = x^2$. Type I.



Sol

$$D = \{(x,y) \mid 0 \le x \le 2,\ x^2 \le y \le 2x\}.$$

V under $z = x^2+y^2$ & above D.

$$V = \iint_D (x^2+y^2)\, dA = \int_0^2\int_{x^2}^{2x} x^2+y^2 \; dy\, dx.$$

$$\Rightarrow \int_0^2 \left| x^2y + \frac{y^3}{3} \right|_{x^2}^{2x} dx$$

$$\Rightarrow \int_0^2 \left[\left\{ x^2(2x) + \frac{(2x)^3}{3} \right\} - \left\{ x^2(x^2) + \frac{(x^2)^3}{3} \right\} \right] dx$$

$$\Rightarrow \int_0^2 \left[\left(2x^3 + \frac{8x^3}{3}\right) - \left(x^4 + \frac{x^6}{3} \right) \right] dx$$

$$\Rightarrow \int_0^2 \left(\frac{6x^3 + 8x^3}{3} - x^4 - \frac{x^6}{3} \right) dx$$

$$\Rightarrow \int_0^2 \left(\frac{14x^3}{3} - x^4 - \frac{x^6}{3} \right) dx$$

$$\Rightarrow \left| \frac{\cancel{14}^{\,7}\, x^4}{\cancel{12}_{\,6}} - \frac{x^5}{5} - \frac{x^7}{21} \right|_0^2$$

$$\Rightarrow \left[\left\{ \frac{7}{6}(2)^4 - \frac{(2)^5}{5} - \frac{(2)^7}{21} \right\} - \{0\} \right]$$

$$\Rightarrow \boxed{\frac{216}{35}} \text{ Ans.}$$

D can also be written as type II region

since $y = 2x$, $y = x^2$

$x = \frac{1}{2}y$, $x = \sqrt{y}$

$$D = \{(x,y) \mid \sqrt{y} \le x \le \tfrac{1}{2}x,\ 0 \le y \le 4\}$$

$$V = \int_0^4 \int_{\sqrt{y}}^{1/2\, x} x^2+y^2 \; dx\, dy = \boxed{\frac{216}{35}}$$