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Limits

Limits along curve in parametric form.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f(x(t), y(t)).$$

$$\lim_{(x,y,z)\to(0,0,0)} f(x,y,z) = \lim_{t\to 0} f(x(t), y(t), z(t)).$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

Example 01:-

$$f(x,y) = -\frac{xy}{x^2+y^2}$$

a) Find the limit along x-axis.

$$(x,y) = (t,0).$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f(t,0) = \lim_{t\to 0} \frac{-(t)(0)}{(t)^2+(0)^2}$$

$$\Rightarrow \lim_{t\to 0} 0 = 0$$

b) along y-axis

$$(x,y) = (0,t).$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f(0,t) = \lim_{t\to 0} \frac{-(0)(t)}{(0)^2+(t)^2}$$

$$\Rightarrow \lim_{t\to 0} 0 = 0.$$

c) the line $y=x$.

$$(x,y) = (t,t).$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0} \frac{-(t)(t)}{(t)^2+(t)^2}$$

$$\Rightarrow \lim_{t\to 0} f = \frac{-\cancel{t^2}}{2\cancel{t^2}} = -\frac{1}{2}$$

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(d) the line $y = -x$. $(x,y) = (t,-t)$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f(t,-t) = \lim_{t\to 0} \frac{-(t)(-t)}{(t)^2+(-t)^2}$$

$$\Rightarrow \lim_{t\to 0} f(t,-t) = \frac{\cancel{t^2}}{2\cancel{t^2}} = \frac{1}{2}$$

(e) the parabola $y = x^2$

$(x,y) = (t,t^2)$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f(t,t^2) = \lim_{t\to 0} \frac{(t)(t^2)}{(t)^2+(t^2)^2}$$

$$\Rightarrow \lim_{t\to 0} f(t,t^2) = \frac{t^3}{\cancel{t^2}(1+t^2)}$$

$$\Rightarrow \lim_{t\to 0} f(t,t^2) = \frac{t}{1+t^2}$$ applying limit

$$\Rightarrow \lim_{t\to 0} f(t,t^2) = \frac{0}{1+0^2}$$

$$\Rightarrow \lim_{t\to 0} f(t,t^2) = 0.$$

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Ex. 2

If $f(x,y) = \frac{xy}{x^2+y^2}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution

along x-axis, $y=0$.

$$f(x,0) = \frac{x(0)}{x^2+0^2}$$

$$f(x,0) = 0.$$

Therefore, $f(x,0) \to 0$ as $(x,y) \to (0,0)$ along x-axis

$$\lim_{(x,y)\to(0,0)} f(x,0) = 0$$

along y-axis, $x = 0$.

$$f(0,y) = \frac{(0)y}{0^2+y^2}$$

$$f(0,y) = 0.$$

So, $f(0,y) \to 0$ as $(x,y) \to (0,0)$ along y-axis

$$\lim_{(x,y)\to(0,0)} f(0,y) = 0.$$

Although, from these paths, limits are identical. Lets now approach $(0,0)$ along another line $y = x$ for $x \neq 0$.

$$f(x,x) = \frac{(x)(x)}{x^2+x^2}$$

$$f(x,x) = \frac{x^2}{x^2+x^2} = \frac{\cancel{x^2}}{2\cancel{x^2}}$$

$$f(x,x) = \frac{1}{2}.$$

Therefore, $f(x,x) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along $y = x$.

$$\lim_{(x,y)\to(0,0)} f(x,x) = \frac{1}{2}.$$

Since, limits are not identical as compared to other two paths, hence limit doesn't exist.

Ex: 7

Find $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.

Solution

Let $x = r\cos\theta \quad y = r\sin\theta$.

$r^2 = x^2 + y^2$.

$\lim_{(r,\theta)\to(0,\theta)} r^2 \ln r^2 \qquad \because \ln a^x \Rightarrow x\ln a$.

$\lim_{(r,\theta)\to(0,\theta)} \dfrac{2\ln r}{1/r^2}$

using L hopital's rule.

$\lim_{r\to 0} \dfrac{\cancel{2}/r}{-\cancel{2}/r^{\cancel{2}}}$

$\lim_{r\to 0} -r$

applying limit.

$\Rightarrow -0$

$\Rightarrow \boxed{0}$

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$\epsilon - \delta$ Definition of limit.

To each $\epsilon > 0$, there exist a +ve value for δ, such that

$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$

① $\lim_{x \to 2} f(x) = 2x - 3 = 1$.

Given $|x-2| < \delta$.

$\because$ $|f(x) - L| < \epsilon$.

$|2x - 3 - 1| < \epsilon$

$|2x - 4| < \epsilon$.

$|2(x-2)| < \epsilon$

$2|x-2| < \epsilon$

from condition mul by 2 b/s.

$2|x-2| < 2\delta$.

$|f(x) - L| < 2\delta$

$|f(x) - L| < \cancel{2}\left(\frac{\epsilon}{\cancel{2}}\right)$

$|f(x) - L| < \epsilon$

$\therefore$ $2\delta = \epsilon$

$\delta = \frac{\epsilon}{2}$

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② For two vars.

if $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$

$\Rightarrow |f(x) - L| < \epsilon$.

$\lim_{(x,y)\to(1,2)} 7x - 2y = 3$

Given $\sqrt{(x-a)^2+(y-b)^2} < \delta$.

$\because |f(x) - L| < \epsilon$.

$|7x - 2y - 3| < \epsilon$.

$|7(x-1+1) - 2y - 3| < \epsilon$.

$|7(x-1) + 7 - 2y - 3| < \epsilon$

$|7(x-1) - 2y + 4| < \epsilon$

$|7(x-1) - 2(y-2)| < \epsilon$

$\because |z_1 - z_2| \leq |z_1| + |z_2|$

$\leq |7(x-1)| + |2(y-2)|$.

$\leq 7|(x-1)| + 2|y-2|$ ✓ ①

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for $|x-1|$

$$|x-1| = \sqrt{(x-1)^2} \leq \sqrt{(x-1)^2+(y-2)^2}.$$

or $|x-1| < \sqrt{(x-1)^2+(y-2)^2} < \delta.$

so $|x-1| < \delta.$

for $|y-2|$

$$|y-2| = \sqrt{(y-2)^2} \leq \sqrt{(x-1)^2+(y-2)^2}$$

or $|y-2| < \sqrt{(x-1)^2+(y-2)^2} < \delta.$

so $|y-2| < \delta.$

If we make eq (A) in this way.

$7\delta + 2\delta$

$9\delta.$

$|f(x,y)-L| < 9\delta$ $\qquad 9\delta = \epsilon$

$|f(x,y)-L| < \cancel{9}\left(\frac{\epsilon}{\cancel{9}}\right)$ $\qquad \delta = \frac{\epsilon}{9}$

$|f(x,y)-L| < \epsilon$

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Find $\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2+y^2}$, if it exists.

Given condition.

$\Rightarrow \sqrt{(x-0)^2+(y-0)^2} < \delta$

$\Rightarrow \sqrt{x^2+y^2} < \delta$.

$\because |f(x) - 2| < \epsilon$

$\left|\frac{4xy}{x^2+y^2} - 0\right| < \epsilon$.

$\left|\frac{4xy}{x^2+y^2}\right| < \epsilon$.

or $\frac{4|x|y}{x^2+y^2} < \epsilon$.

$\because \frac{y^2}{x^2+y^2} \leq 1$

$\frac{4|x|y^2}{x^2+y^2} \leq 4|x| = 4\sqrt{x^2} \leq 4\sqrt{x^2+y^2}$

so $4|x| \leq 4\sqrt{x^2+y^2} < 4\delta$.

$4|x| \leq 4\delta$.

$|f(x) - L| \le 4\delta \qquad 4\delta = \epsilon$

$|f(x) - L| < 4\left(\frac{\epsilon}{4}\right) \qquad \delta = \frac{\epsilon}{4}$

$|f(x) - L| < \epsilon$

Find $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

Given $\sqrt{x^2+y^2} < \delta$.

$\therefore |f(x) - L| < \epsilon$.

$\left| \frac{3x^2y}{x^2+y^2} - 0 \right| < \epsilon$.

$\left| \frac{3x^2y}{x^2+y^2} \right| < \epsilon$

or $\frac{3x^2|y|}{x^2+y^2} < \epsilon$.

$\because \frac{x^2}{x^2+y^2} \le 1$

$\frac{3x^2|y|}{x^2+y^2} \le 3|y| = 3\sqrt{y^2} \le 3\sqrt{x^2+y^2}$

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$\sqrt{x^2+y^2} < 3\delta$.

so $3|x| \leq \sqrt{x^2+y^2}$

$3|x| < 3\delta$.

$|f(x)-L| < 3\delta$. $3\delta = \epsilon$

$\delta = \frac{\epsilon}{3}$

$|f(x)-L| < \cancel{3}\left(\frac{\epsilon}{\cancel{3}}\right)$

$|f(x)-L| < \epsilon$

Limits at different path

Ex: 2.

If $f(x,y) = \frac{xy}{x^2+y^2}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

path 1: along x- axis

$y = 0$.

$f(x,0) = \frac{x(0)}{x^2+0} = 0$.

Ex: 3

If $f(x,y) = \frac{xy^2}{x^2+y^4}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

path 1: along a non-vertical line through origin $\to y = mx$.

$f(x,mx) = \frac{x(mx)^2}{x^2+(mx)^4}$

$f(x,mx) = \frac{m^2x^3}{x^2+m^4x^4}$

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$$f(x, mx) = \frac{m^2 \cancel{x^3}}{\cancel{x^2}(1 + m^4 x^2)}$$

$$f(x, mx) = \frac{m^2 x}{1 + m^4 x^2}.$$

$$\lim_{(x, mx) \to (0,0)} f(x, mx) = \frac{m^2 (0)}{1 + m^4 (0)} = 0.$$

Therefore, $f(x, mx) \to 0$ as $(x, y) \to (0, 0)$ along nonvertical line through origin.

path 2 : $x = y^2$

$$f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4}$$

$$f(y^2, y) = \frac{y^4}{y^4 + y^4} = \frac{\cancel{y^4}}{2\cancel{y^4}}$$

$$f(y^2, y) = \frac{1}{2}.$$

$$\lim_{(y^2, y) \to (0,0)} f(y^2, y) = 1/2.$$

Therefore, $f(y^2, y) \to 1/2$ as $(x, y) \to (0, 0)$ along $y = x^2$.

Since, for both paths limits are not identical at $\lim_{(x,y) \to (0,0)}$, hence limit doesn't exist.

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Ex: 5.
Shows that
$$f(x,y) \begin{cases} \frac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

For $y = mx$, $x \neq 0$.

$f(x, mx) = 2$

$$f(x,y)\Big|_{y=mx} = \frac{2xy}{x^2+y^2}\Big|_{y=mx}.$$

$$\Rightarrow \frac{2x(mx)}{x^2+m^2x^2} \Rightarrow \frac{2m\cancel{x^2}}{\cancel{x^2}(1+m^2)}$$

$$\Rightarrow \frac{2m}{1+m^2}.$$

from this, we conclude that limit changes with each value of m, limit doesn't exist ∴ function isn't continuous.