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NUMERICAL COMPUTING

Numerical Analysis:

A branch of mathematics that deals with development of efficient methods to solve mathematical problem numerically.

$f(x) = \sin x \quad , \quad x = 30°$

Solve by Taylor Series.

Surprise quizzes

$$f(x) = f'(x) + \frac{f''(x)}{2!} + \frac{f'''(x)}{3!} + \cdots$$

0.49997 → approx. answer.

1- Creation of problem solving approach
2- Analysis method (study error and efficiency)
3- Implementation of method in a computer code.

Why study Numerical Analysis/Computation:

1- Familiarize yourself with the process of implementation of a method to solve a problem.
2- Identify which methods can be applied to problem.
3- Understand the limitations of different methods.
4- ~~Problem~~ Mistake analysis, Error analysis

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Measuring Errors.

Why?

Accuracy of numerical results

Stoping criteria of iterative algorithms.

Numerical method is an algorithm which is programmed

Analytical method is to solve the problem in getting exact solution

True Error:

True Error = True ~~Error~~ Value (Analytical) Solution − Approximate Value (Numerical Solution).

- True value is also called the exact value

Example:

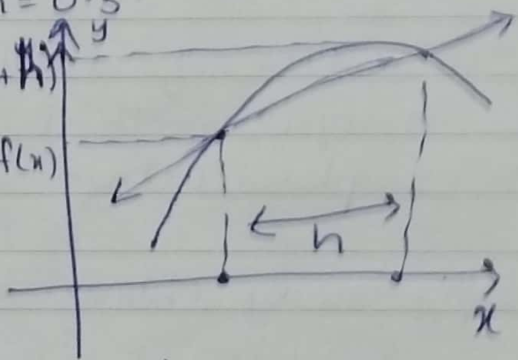
derivative of a function $f'(x)$ can be approximated by the equation

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

If $f(x) = 7e^{0.5x}$ and $h = 0.3$.

$$f'(x) \approx \frac{f(x+h) - 7e^{0.5x}}{0.3}$$

$$f'(x) \approx \frac{7e^{0.5(x+h)} - 7e^{0.5x}}{0.3}$$



(a) $$f'(2) \approx \frac{7e^{0.5(2+0.3)} - 7e^{0.5(2)}}{0.3}$$

$$f'(2) \approx \frac{7e^{1.15} - 7e}{0.3}$$

$$f'(2) \approx \frac{22.107 - 19.027}{0.3}$$

$$\boxed{f'(2) \approx 10.266}$$

$$f'(x) = 7e^{0.5x}\frac{d}{dx}(0.5x)$$

$$= 7e^{0.5x}(0.5)$$

$$= 3.5e^{0.5x}$$

$$f'(2) = 3.5e^{0.5\times 2}$$

$$\boxed{f'(2) = 9.513.}$$

True Error = 9.513 − 10.263.

$\boxed{\text{True Error} = -0.75}$

$\boxed{\text{RTE} = -0.078}$

Relative True Error (ϵ_t)

$$\epsilon_t = \frac{\text{True Error}}{\text{True value}}$$

$\boxed{\text{RTE in percentage} = -7.8\ \%}$

$$f'(2) \approx \frac{7e^{0.5(2+0.1)} - 7e^{0.5(2)}}{0.1}$$

$$f'(2) \approx \frac{7e^{1.05} - 7e^{1}}{0.1}$$

$$f'(2) \approx \frac{20.003 - 19.027}{0.1}$$

$$\boxed{f'(2) \approx 9.76}$$

True Error = 9.513 − 9.76

$\boxed{\text{True Error} = -0.247}$

Relative True Error (ϵ_T) = $\dfrac{\text{True Error}}{\text{True Value}}$

$$R.T.E = \frac{-0.247}{9.513}$$

$\boxed{R.T.E = -0.025}$

Approximate Error (E_a)

E_a = Present Approximation − Previous Approximation.

$E_a = 9.76 - 10.266$

$\boxed{E_a = -0.506}$

When we don't have true values, then we can find approximate error to quantify the error.

Sources of Error

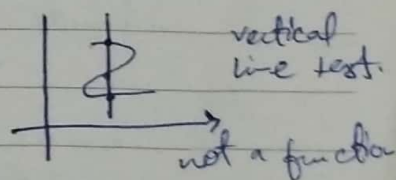
1. Numerical Error.
   - → Round off Error
   - → Truncation Error.

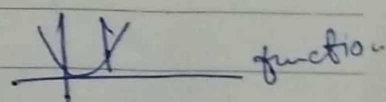
Bisection Method.

02-11-23

Bisection means cut into half.

Check that there exist atleast one root where $y=0$, in a function by taking two points from the domain.





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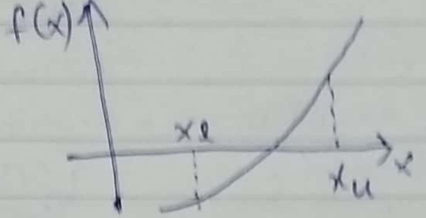
Algorithm of Bisection Method.

Step 1:

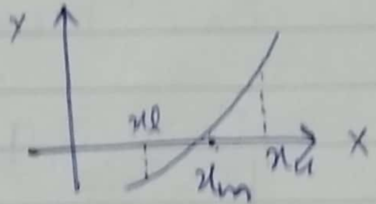
choose x_l and x_u as two guesses for the root such that $f(x_l) < 0$, or in other words, $f(x)$ changes sign between x_l and x_u. This was demonstrated in a figure below.



Step 2:

Estimate the root, x_m of the equation $f(x)=0$. as the point b/w x_l & x_u as

$$x_m = \frac{x_l + x_u}{2}$$ (mid point).



Step 3: Now check the following.

a) If $f(x_l)\,f(x_m) < 0$ => root lies b/w x_l & x_m then do, $x_l = x_l$; $x_u = x_m$

b) if $f(x_l)\,f(x_m) > 0$ => root lies b/w x_m & x_u. then do, $x_l = x_m$; $x_u = x_u$

c) if $f(x_l)\,f(x_m) = 0$ => root is x_m, Stop the algorithm, if this is true.

Step 4:

Find the new estimate of the root.

$$x_m = \frac{x_l + x_u}{2}$$

Find the absolute relative approximate error.

$$|\epsilon_a| = \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| \times 100. \qquad |\epsilon_a| \le |\epsilon_s|$$

where, x_m^{old} = previous estimate of root.

x_m^{new} = current estimate of root.

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Step 5:

Compare ARAE : $|\epsilon_a|$ with pre-specified tolerance. $\{\epsilon_s\}$

is $|\epsilon_a| > \epsilon_s$? ⟶ Yes ⟶ Next step for new guesses.

⟶ No ⟶ Stop algorithm.

<u>Example</u>

$f(x) = x^4 + 2x^3 - x - 1$. ; $x_l = 0$ & $x_u = 1$.

$f(x_l) = 0^4 + 2(0)^3 - 0 - 1$

$f(x_l) = -1$

$f(x_u) = (1)^4 + 2(1)^2 - 1 - 1$

$f(x_u) = 1 + 2 - 1 - 1$

$f(x_u) = +1$

$f(x_l)\,f(x_u) < 0$

$(-1)(1) < 0$

$-1 < 0$.

root lies b/w x_l & x_u

now find mid point:

$x_m = 0 + 1 / 2 = 1/2 = 0.5$

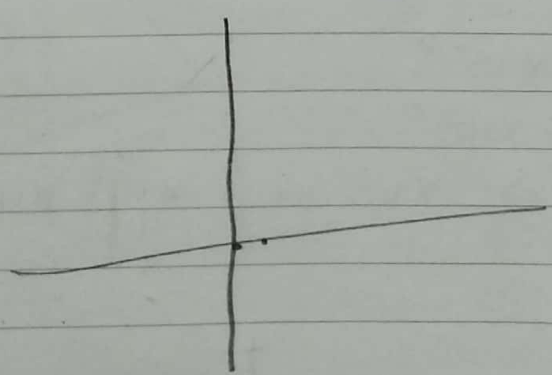
now $x_l = 0$, $x_u = 0.5$

$f(x_l) = -1$ $f(x_u) = -1.1875$.

7-11-2023

<u>Ex: 1</u>

equation for depth: $x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$



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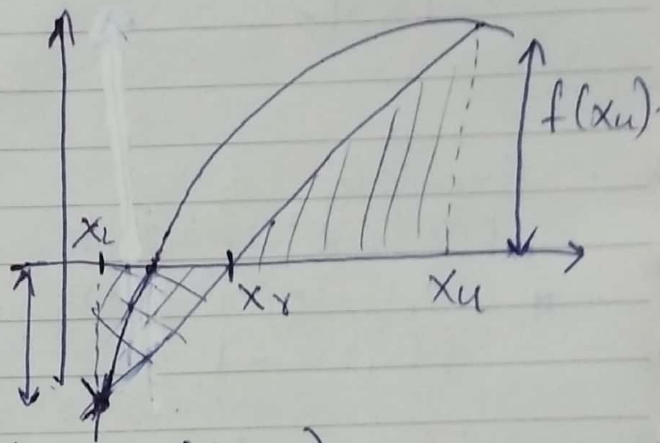
FALSE-POSITION METHOD FOR SOLVING NON-LINEAR EQUATIONS:

Intermediate Value Theorem..

$f(x_u)\, f(x_L) < 0$ . . .



$$\frac{f(x_L)}{x_r - x_L} = \frac{f(x_u)}{x_r - x_u}$$

$\Rightarrow (x_r - x_u)\, f(x_L) = (x_r - x_L)\, f(x_u)$

$\Rightarrow x_r f(x_L) - x_u f(x_L) = x_r f(x_u) - x_L f(x_u)$.

$\Rightarrow x_r f(x_L) - x_r f(x_u) = x_u f(x_L) - x_L f(x_u)$.

$\Rightarrow x_r \{ f(x_L) - f(x_u) \} = x_u f(x_L) - x_L f(x_u)$.

$$\Rightarrow x_r = \frac{x_u f(x_L) - x_L f(x_u)}{f(x_L) - f(x_u)}.$$

Ex: 1: $x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$.

$x_L = 0 \qquad x_u = 0.1$

$f(x_L) = 3.993 \times 10^{-4} \qquad f(x_u) = -2.507 \times 10^{-4}$

$\underline{1^{st}}$

$$x_r = \frac{(0.1)(3.993 \times 10^{-4}) - 0\,(-2.507 \times 10^{-4})}{3.993 \times 10^{-4} - (-2.507 \times 10^{-4})}$$

$x_r = 0.0614$.

$f(x_r) = 8.732144 \times 10^{-6}$.

$f(x_L)\, f(x_r) = 3.486 \times 10^{-9}$.

$f(x_L)\, f(x_r) > 0 \Rightarrow x_L = x_r,\ x_u = x_u$.

$x_L = 0.0614$ $\qquad$ $x_u = 0.1$

2nd

$f(x_L) = 8.732144 \times 10^{-9}$ $\qquad$ $f(x_u) = -2.507 \times 10^{-4}$

$$x_r = \frac{(0.1)(8.732144\times10^{-9}) - (0.0614)(-2.507\times10^{-4})}{(8.732144\times10^{-9}) - (-2.507\times10^{-4})}$$

$x_r = 0.0593$

$f(x_r) = 2.7607007 \times 10^{-5}$

$f(x_L)\, f(x_r) > 0 \Rightarrow x_L = x_r, \; x_u = x_u$

3rd

$x_L = 0.0593$ $\qquad$ $x_u = 0.1$

$f(x_L) = 2.7607007 \times 10^{-5}$ $\qquad$ $f(x_u) = -2.507 \times 10^{-4}$

$$x_r = \frac{(0.1)(2.7607007\times10^{-5}) - (0.0593)(-2.507\times10^{-4})}{2.7607007\times10^{-5} - (-2.507\times10^{-4})}$$

$x_r = 0.0633$

$f(x_r) = \cancel{3.745053\times10^{-3}}$ -8.200713×10^{-6}

$f(x_L)\, f(x_r) < 0$

$x_L = x_L, \quad x_u = x_r$

$x_L = 0.0593, \quad x_u = 0.0633$

NEWTON RAPHSON METHOD

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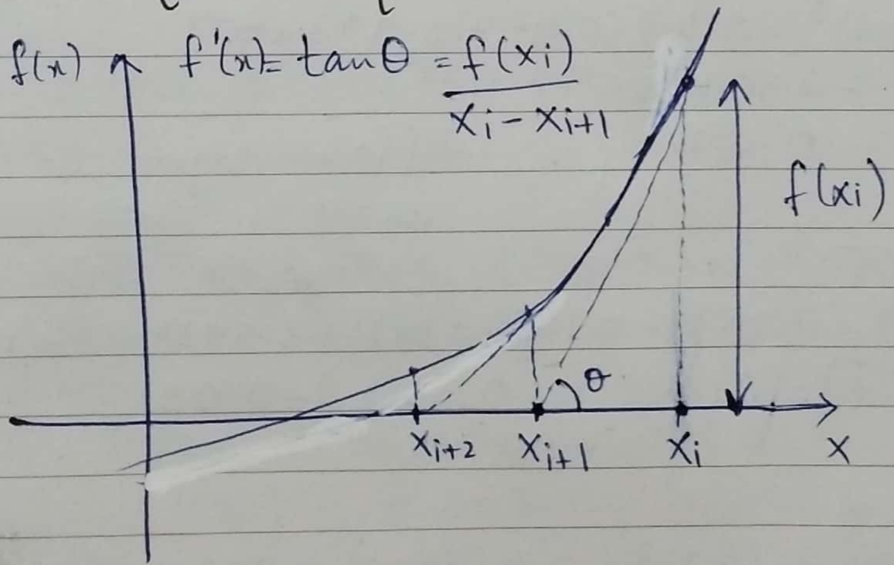

$$f'(x) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_i - x_{i+1} = \frac{f(x_i)}{f'(x)}$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x)}}$$

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$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

STEPS:

1. ~~Evaluate $f'(x)$ analytically.~~
2. Calculate

1. Assume an initial value x_i such that the value of $f(x_i)$ and $f''(x_i)$ have the same sign.
2. Evaluate $f'(x)$ analytically.
3. Evaluate $f(x_i)$ and $f'(x_i)$.
4. Use the equation $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ to estimate the new root.
5. Find the absolute relative approximate error $|\epsilon_a|$ as $|\epsilon_a| = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100$.

Example 1:

$f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$.

$f'(x) = 3x^2 - 0.33x$

$f''(x) = 6x - 0.33$.

① Let $x_i = 0.1$.

$f(x_i) = f(0.1) = (0.1)^3 - 0.165(0.1)^2 + 3.993 \times 10^{-4}$.

$f(0.1) = -2.507 \times 10^{-4}$.

$f''(x_i) = f''(0.1) = 0.27$. differen signs

no.

Let $x_i = 0.05$.

$f(x_i) = f(0.05) = (0.05)^3 - 0.165(0.05)^2 + 3.993 \times 10^{-4} = 1.118 \times 10^{-4}$

$f''(x_i) = f(0.05) = 6(0.05) - 0.33 = -0.03$.

Let $x_i = 0.2$.

$f(x_i) = f(0.2) = (0.2)^3 - 0.165(0.2)^2 + 3.993 \times 10^{-4} = 1.79 \times 10^{-3}$
93

$f''(x_i) = f''(0.2) = 6(0.2) - 0.33 = 0.87$

$f(x_i)$ and $f''(x_i)$ are positive.

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② $f'(x) = f'(0.2) = 3(0.2)^2 - 0.33(0.2)$
$f'(0.2) = 0.054.$

③ $f(x_i) = 1.7993 \times 10^{-3}.$
$f'(x_i) = 0.054.$

④ $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = 0.2 - \frac{1.7993 \times 10^{-3}}{0.054.}$
$x_{i+1} = 0.1666796296.$

⑤ $|\epsilon a| = \left|\frac{0.166 - 0.2}{0.2}\right| \times 100$
$|\epsilon a| = \cancel{0.1666\ 0185\ 2 \times 10}$
$|\epsilon a| = \boxed{19.99\ \%}$

2ND Iter. $x_i = 0.166.$
② $f'(x_i) = 0.028342019.$
③ $f(x_i) = 4.459 \times 10^{-4}.$
④ $x_{i+1} = 0.166 - \frac{4.459 \times 10^{-4}}{0.028342019} = 0.1509468055$
⑤ $|\epsilon a| = \left|\frac{0.150 - 0.166}{0.150}\right| \times 100 \boxed{10.42\%}$

3RD Iter. $x_i = 0.1509.$
② $f'(x_i) = 0.0185423 6846.$
③ $f(x_i) = 7.909 \times 10^{-5}.$
④ $x_{i+1} = 0.1509 - \frac{7.909 \times 10^{-5}}{0.0185423 6846}$
$x_{i+1} = 0.1466814989$
⑤ $|\epsilon a| = \left|\frac{0.146 - 0.1509}{0.146}\right| \times 100 =$
$\boxed{2.90\%}$

1

① Let $x = -0.1.$
$f(-0.1) = (-0.1)^3 - 0.165(-0.1) + 3.993 \times 10^{-3} = -0.00225$
-2.2507×10^{-3}

$f''(-0.1) = 6(-0.1) - 0.33 = -0.93.$
So the point -0.1 fulfills the condition.

② Now evaluate $f'(0.1).$
$f'(0.1) = 3(-0.1)^2 - 0.33(-0.1) = 0.063.$

③ $f(x_i) = -2.2507 \times 10^{-3}.$
$f'(x_i) = 0.063.$

④ $x_{i+1} = -0.1 + \frac{2.2507 \times 10^{-3}}{0.063} = -0.0642.$

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(5) Now evaluate ϵ_a

$$|\epsilon_a| = \left| \frac{-0.0642-(+0.1)}{-0.0642} \right| \times 100.$$

$$|\epsilon_a| = 55.76\%.$$

2ND Iter

Now $x_i = -0.0642$.

(2) $f'(x) = f'(-0.0642) = 3(-0.0642)^2 - 0.33(-0.0642)$

$f'(-0.0642) = 0.0335$

(3) $f(x_i) = f(-0.0642) = (-0.0642)^3 - 0.165(-0.0642) + 3.993\times10^{-4}$

$f(-0.0642) = -5.453\times10^{-4}$

(4) $$x_{i+1} = -0.0642 - \frac{(-5.453\times10^{-4})}{0.0335}$$

$x_{i+1} = -0.0479$.

(5) $$|\epsilon_a| = \left| \frac{-0.0479-(-0.0642)}{-0.0479} \right| \times 100.$$

$|\epsilon_a| = 34.02\%$.

3rd Iter

Now $x_i = -0.0479$.

(2) $f'(x) = f'(-0.0479) = 3(-0.0479)^2 - 0.33(-0.0479)$

$f'(-0.0479) = \cancel{8.917\times10^{-5}}\ 0.0226.$

(3) $f(x_i) = f(-0.0479) = (-0.0479)^3 - 0.165(-0.0479) + 3.993\times10^{-4}$.

$f(-0.0479) = -8.917\times10^{-5}$

(4) $$x_{i+1} = -0.0479 - \frac{(-8.917\times10^{-5})}{0.0226} = -0.04395.$$

(5) $$|\epsilon_a| = \left| \frac{-0.04395-(-0.0479)}{-0.04395} \right| \times 100 = 8.98\%.$$

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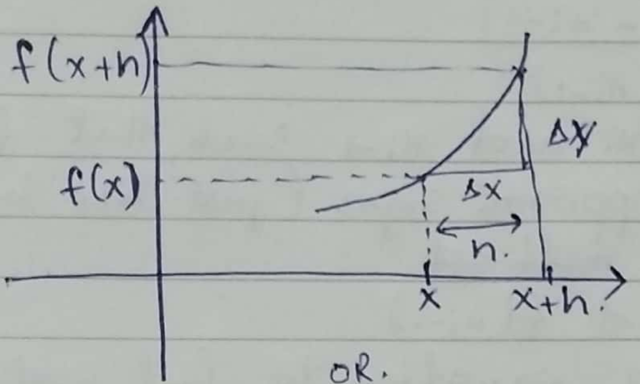
SECANT METHOD.

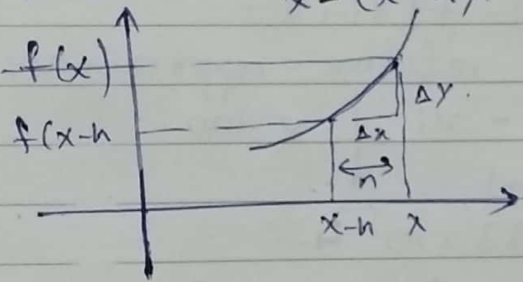
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$



$$\frac{dy}{dx} \neq \frac{\Delta y}{\Delta x}.$$

OR.

$$f'(x) = \lim_{h \to 0} \frac{f(x) - f(x-h)}{x - (x-h)}.$$



Newton's Method.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad (1)$$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}. \quad (2)$$

Substituting. (2) in (1).

$$x_{i+1} = x_i - \frac{f(x_i) - (x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}.$$

Newton's Method: $x_{i+1} = x_i - \frac{f(x_i)}{f(x_i)}$ — (1).

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}. \quad \text{—— (2).}$$

Substituting eq (2) in (1)

$$x_{i+1} = x_i - \frac{f(x_i)}{\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}}$$

$$x_{i+1} = x_i - \frac{x_i f(x_i) - x_{i-1} f(x_i)}{f(x_i) - f(x_{i-1})} \Rightarrow \boxed{x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}}$$

$$x_{i+1} = \frac{\cancel{x_i f(x_i)} - x_i f(x_{i-1}) + x_i f(x_i) - x_{i-1} f(x_i)}{f(x_i) - f(x_{i-1})}.$$

$$\cancel{(x_{i+1}) \{ f(x_i) - f x_i} = \cancel{\frac{-x_i (f x_{i-1}) - x_i \; f(x_{i-1})}{f(x_i - f(x_{i-1})}}$$

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$$\cancel{x_{i+1}} = \frac{\cancel{f(x_{i-1})}(-x_i - \cancel{x_{i-1}})}{\cancel{f(x_i) - f(x_{i-1})}}$$

$$x_{i+1} = x_i - \frac{f(x_i)\,(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}.$$

Step 1: Choose two points x_i and x_{i-1} such that $f(x_i)$ and $f(x_{i-1})$ have opposite signs (just like bisection and false position method.

Step 2: Calculate $f(x_i)$ and $f(x_{i-1})$.

Step 3: Use the following equation to find next point

$$x_{i+1} = x_i - \frac{f(x_i)\,(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}.$$

Step 4: Find the absolute relative approximate error.

$$|\epsilon_a| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100.$$

1st Iter Ex: $f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$.

① Let $x_i = 0.1$ Let $x_{i-1} = 0$.

$f(0.1) = -2.507\times10^{-4}$ $f(0) = 3.993\times10^{-4}$

② $f(0.1) = -2.507\times10^{-4}$ $f(0) = 3.993\times10^{-4}$.

③ $$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} = 0.1 - \frac{(-2.507\times10^{-4})(0.1-0)}{(-2.507\times10^{-4}) - (3.993\times10^{-4})}$$

$$\boxed{x_{i+1} = 0.06143076923}$$

④ $$\boxed{|\epsilon_a| = 62.78\%}$$

2ND Iter: $x_i = x_{i+1} = 0.06143076923$ $x_{i-1} = x_i = 0.1$

① $f(x_i) = \cancel{8.602\times10^{-4}}$ 8.456×10^{-6} $f(x_{i-1}) = -2.507\times10^{-4}$.

② →

③ $$x_{i+1} = 0.0614.. - \frac{(8.456\times10^{-6})(0.0614...-0.1)}{(8.456\times10^{-6}) - (-2.507\times10^{-4})}$$ $= \cancel{0.04744075199}$ 0.06268924449

(iv) $|\epsilon_a| = \cancel{36.98\%}\ 2.00\%$

3rd Iter. $x_i = x_{i+1} = 0.0628924449$ $\qquad x_{i-1} = x_i = 0.061443076923$

(1) $f(x_i) = -4.582 \times 10^{-6}$ $\qquad f(x_{i-1}) = 8.456 \times 10^{-6}$

(2) $\Rightarrow$

(3) $$x_{i+1} = 0.062\ldots - \frac{(-4.582 \times 10^{-6})(0.062\ldots - 0.0614\ldots)}{(-4.582 \times 10^{-6}) - (8.456 \times 10^{-6})}$$

$x_{i+1} = 0.06237876198$

(iv) $|\epsilon_a| = 0.82\%$

System of Linear Equations

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$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3.$$

Gauss - Jacobi
Gauss - Siedel

Diagonally Dominant:

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

* If this condition does not met, then we can change order of the equations.

Step ①.

$$x = \frac{b_1 - a_{12}y - a_{13}z}{a_{11}}$$

$$y = \frac{b_2 - a_{21}x - a_{23}z}{a_{22}}$$

$$z = \frac{b_3 - a_{31}x - a_{32}y}{a_{33}}$$

Step ②

Suppose initial values

$x_0 =$ $\quad y_0 =$ $\quad z_0 =$

Jacobi	Siedel
$x_1(y_0, z_0)$	$x_1(y_0, z_0)$
$y_1(x_0, z_0)$	$y_1(x_1, z_0)$
$z_1(x_0, y_0)$	$z_1(x_1, y_1)$

Step ③

Relative Approximate Error.

$$|\epsilon_{ax}| = \left|\frac{x_{NEW} - x_{OLD}}{x_{NEW}}\right| \times 100$$

$$|\epsilon_{ay}| = \left|\frac{y_{NEW} - y_{OLD}}{x_{NEW}}\right| \times 100$$

$$|\epsilon_{az}| = \left|\frac{{}_{NEW} - {}_{OLD}}{x_{OLD}}\right| \times 100$$

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Example: $2x - y + 3z = 9$ ①
$x - 3y - 2z = 0$ ②
$3x + 2y - z = -1$ ③

Flip eqn ① and 3
$3x + 2y - z = -1$
$x - 3y - 2z = 0$
$2x - y + 3z = 9$

Step ①.
$$x = \frac{-1 - 2y + z}{3}$$
$$y = \frac{-x + 2z}{3} = \frac{x - 2z}{3}$$
$$z = \frac{9 - 2x + y}{3}$$

Step 2. $x_0 = 0$, $y_0 = 0$, $z_0 = 0$.

Guass - Jacobi.

$x_1(\overset{y_0}{0}, \overset{z_0}{0})$.

1st
$$x_1 = \frac{-1 - 2(0) + (0)}{3} = \frac{-1}{3}$$
$$y_1 = \frac{0 - 2(0)}{3} = 0$$
$$z_1 = \frac{9 - 2(0) + 0}{3} = \frac{9}{3} = 3$$
$$|\epsilon_{ax}| = \left| \frac{-\frac{1}{3} - 0}{-\frac{1}{3}} \right| \times 100 = 100\%$$
$$|\epsilon_{ay}| = \left| \frac{0 - 0}{0} \right| \times 100 = 0\%$$
$$|\epsilon_{az}| = \left| \frac{3 - 0}{3} \right| \times 100 = 100\%$$

2ND. $x_2(\overset{y_1}{0}, \overset{z_1}{3})$.
$$x_2 = \frac{-1 - 2(0) + 3}{3} = \frac{2}{3}$$
$$y_2 = \frac{-\frac{1}{3} - 2(3)}{3} = \frac{-19}{9}$$
$$z_2 = \frac{9 - 2(-\frac{1}{3}) + 0}{3} = \frac{29}{9}$$

Gauss - Siedel.

1st.
$$x_1 = \frac{-1 - 2(0) + (0)}{3} = \frac{-1}{3}$$
$$y_1 = \frac{-\frac{1}{3} - 2(0)}{3} = \frac{-1}{9}$$
$$z_1 = \frac{9 - 2(-\frac{1}{3}) - \frac{1}{9}}{3} = \frac{86}{27}$$
$$|\epsilon_{ax_1}| = \left| \frac{-1/3 - 0}{-1/3} \right| \times 100 = 100\%$$
$$|\epsilon_{ay_1}| = \left| \frac{-\frac{1}{9} - 0}{-1/9} \right| \times 100 = 100\%$$
$$|\epsilon_{az_1}| = \left| \frac{86/27 - 0}{86/27} \right| \times 100 = 100\%$$

2ND.
$$x_2 = \frac{-1 - 2(\frac{-1}{9}) - (\frac{86}{27})}{3} = \frac{65}{81}$$
$$y_2 = \frac{\frac{65}{81} - 2(\frac{86}{27})}{3} = \frac{-451}{243}$$
$$z_3 = \frac{9 - 2(\frac{65}{81}) + \frac{451}{243}}{3}$$

$$|\epsilon_{ax_2}| = \left| \frac{\frac{2}{3} + \frac{1}{3}}{2/3} \right| \times 100 = 150\%$$

$$|\epsilon_{ay_2}| = \left| \frac{-19/9 - 0}{-19/9} \right| \times 100 = 100\%$$

$$|\epsilon_{az_2}| = \left| \frac{29/9 - 3}{29/9} \right| \times 100 = 6.89\%$$

Example:

$83x - 11y - 4z = 95$ ①

$7x + 52y + 13z = 104$. ②

$3x + 8y + 29z = 71$. ③

Assignment 2.

Explain

$x = \frac{95 + 11y + 4z}{83}$

$y = \frac{104 - 7x - 13z}{52}$

$z = \frac{71 - 3x - 8y}{29}$.

Compare the two methods & ~~decide~~ which method is better

$x_0 = 0 \quad y_0 = 0 \quad z_0 = 0$.

Guass - Jacobi.

1st $x_1(y_0, z_0)$

$$x_1 = \frac{95 + 11(0) + 4(0)}{83} = \frac{95}{83}.$$

$y_1(x_0, z_0)$.

$$y_1 = \frac{104 - 7(0) - 13(0)}{52} = 2$$

$z_1(y_0, z_0)$

$$z_1 = \frac{71 - 3(0) - 8(0)}{29} = \frac{71}{29}.$$

$$|\epsilon_{ax_1}| = \left| \frac{95/83 - 0}{95/83} \right| \times 100 = 100\%.$$

$$|\epsilon_{ay_1}| = \left| \frac{2 - 0}{2} \right| \times 100 = 100\%.$$

$$|\epsilon_{az_1}| = \left| \frac{71/29 - 0}{71/29} \right| \times 100 = 100\%.$$

Guass - Siedel.

1st $x_1(y_0, z_0)$

$$x_1 = \frac{95 + 11(0) + 4(0)}{83} = \frac{95}{83}.$$

$y_1(x_1, z_0)$

$$y_1 = \frac{104 - 7\left(\frac{95}{83}\right) - 13(0)}{52}$$

$y_1 = 1.845$.

$z_1(y_1, x_1)$

$$z_1 = \frac{71 - 3\left(\frac{95}{83}\right) - 8(1.845)}{29} = 1.820$$

$$|\epsilon_{ax_1}| = \left| \frac{95/83 - 0}{95/83} \right| \times 100 = 100\%.$$

$$|\epsilon_{ay_1}| = \left| \frac{1.845 - 0}{1.845} \right| \times 100 = 100\%.$$

$$|\epsilon_{az_1}| = \left| \frac{1.820 - 0}{1.820} \right| \times 100 = 100\%$$

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30-Nov-23

Guass - Elimination:

Q) $x + y + z = 4$ —— (1)

$x + 4y + 3z = 8$ —— (2)

$x + 6y + 2z = 6.$ —— (3)

Eq(2) - Eq(1) , Eq(3) - Eq(1).

$\Rightarrow x + y + z = 4.$

$3y + 2z = 4.$

$5y + z = 2$

Eq(2)/3. , $5 \times$ Eq(2) - Eq(3)

$\Rightarrow x + y + z = 4.$

$y + \frac{2}{3}z = \frac{4}{3}$

$0 + \frac{7}{3}z = \frac{14}{3}.$

$\frac{3}{7}$(Eq(3)). , $(\frac{-2}{3})$ Eq(3) + Eq(2).

$\Rightarrow x + y + z = 4.$ $\Rightarrow x + 0 + 2 = 4 \Rightarrow x = 4 - 2 = \boxed{x = 2}$

$y + \frac{2}{3}z = \frac{4}{3}$ $\Rightarrow y + \frac{2}{3}(2) = \frac{4}{3} \Rightarrow y + \frac{4}{3} = \frac{4}{3}$ $\boxed{y = 0}$

$\boxed{z = 2}$ $\boxed{z = 2}$

$\frac{10}{3} - 1$

$\frac{20}{3} - 2$

$x = 2 , y = 0 , z = 2.$

POWER METHOD.

$[A](x) = \lambda(x).$

where λ is the eigenvalue and X is the given eigen-vector

Step 1: Choose the initial vector such that the largest element is unity (1).

Step 2: This normalized vector $v^{(0)}$ is pre-multiplied by the given matrix. [A].

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Step 3: The resultant vector $v^{(1)}$ is again normalized.

Step 4: The iteration is continued and the new normalized vector is multiplied by matrix [A] until the required accuracy is obtained.

Find the eigenvalue of the vector and associated eigen matrix.

$$A = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix}$$

$$v^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow A\, v^{(0)}$$

1st Iteration

$$\begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 14 \end{bmatrix} \rightarrow U^{(1)}$$

(Divide by largest element.)

Normalize the vector.

$$U^{(1)} = \underbrace{14}_{q_1} \underbrace{\begin{bmatrix} 7/14 \\ 12/14 \\ 1 \end{bmatrix}}_{v^{(1)}} = q_1\, v^{(1)}$$

$q_1 \rightarrow$ Eigen value.
$V \rightarrow$ Eigen vector.

2nd Iteration.

$$\begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{bmatrix} 7/14 \\ 12/14 \\ 1 \end{bmatrix} = \begin{bmatrix} 39/7 \\ 67/7 \\ 171/14 \end{bmatrix} \text{ or } \begin{bmatrix} 5.571 \\ 9.571 \\ 12.214 \end{bmatrix} \rightarrow U^{(2)}.$$

Normalize the vector.

$$U^{(2)} = \underbrace{\frac{171}{14}}_{q_2} \underbrace{\begin{bmatrix} 26/57 \\ 134/171 \\ 1 \end{bmatrix}}_{v^{(2)}}$$

Find error.

$$|\epsilon_a| = \left| \frac{q_2 - q_1}{q_2} \right| \times 100$$

$$|\epsilon_a| = \left| \frac{\frac{171}{14} - 14}{171/14} \right| \times 100$$

$$\boxed{|\epsilon_a| = 14.16\%}$$

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3rd Iteration $\begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{bmatrix} 24/57 \\ 134/171 \\ 1 \end{bmatrix} = \begin{bmatrix} 100/19 \\ 523/57 \\ 2041/171 \end{bmatrix}$ 5.26 9.17 11.93

$\downarrow v^{(3)}$

Normalize the vector $v^{(3)}$

$= \frac{2041}{171} \begin{bmatrix} (100/19)(171/2041) \\ (523/57)(171/2041) \\ 1 \end{bmatrix} = \begin{bmatrix} 900/2041 \\ 1569/2041 \\ 1 \end{bmatrix}$

α_3 $\qquad v^{(3)}$.

Finding Error : $|\epsilon_a| = \left| \dfrac{\frac{2041}{171} - \frac{171}{14}}{\frac{2041}{171}} \right| \times 100$

$\boxed{|\epsilon_a| = 2.33\,\%}$

4th Iteration. $\begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{bmatrix} 900/2041 \\ 1569/2041 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{10584}{2041} \\ \frac{1424}{157} \\ 11.860 \end{bmatrix}$

$\downarrow v^{(4)}$.

Normalize the vector $v^{(4)}$

$= \frac{11.860}{1} \begin{bmatrix} \frac{10589}{2041}/11.860 \\ \frac{1424}{157}/11.860 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4374 \\ 0.7647 \\ 1 \end{bmatrix} v^{(4)}$

α^{4}.

Finding Error : $|\epsilon_a| = \left| \dfrac{11.860 - \frac{2041}{171}}{11.860} \right|$

$\therefore \boxed{|\epsilon_a| = 0.638\,\%}$

5-12-23.

Interpolation:

Much data is given & we have to find an point within them. The effect of all other points have on them.

(x_0, y_0) ✓✓ (x, y) ✓ (x_1, y_1) ✓✓.   ✓ Given, $x \to$ we suppose.

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} \quad \to \text{Linear Interpolation.}$$

Interpolate : point lie in between range of

Extrapolate : outside of the given point

INTERPOLATION METHODS

① NEWTON FORWARD DIFFERENCE INTERPOLATION:

Step 1

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		
		Δy_2			
x_3	y_3				

diff b/w x is h } i.e. h is a constant

$$P = \frac{x - x_0}{h}.$$

Q

	0	1	2	3	4	5	6
x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
y	0.003	0.067	0.148	0.248	0.370	0.518	0.697

$h = 0.2$

Solve

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	0.003				
		0.064			
0.3	0.067		0.017		
		0.081		0.002	
0.5	0.148		0.019		0.001
		0.1		0.003	
0.7	0.248		0.022		0.001
		0.122		0.004	
0.9	0.370		0.026		0.001
		0.148		0.005	
1.1	0.518		0.031		
		0.179			
1.3	0.697				

$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0.001		
	0	
0.001		0
	0	
0.001		

$$Y = Y_0 + p\Delta Y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

Find the value of y at x = 0.15.

In forward difference we use x_0 & y_0 values.

$$\because p = \frac{x - x_0}{h} \Rightarrow \frac{0.15 - 0.1}{0.2} = 0.25$$

0.00065625

$$Y = 0.003 + 0.25(0.064) + \frac{0.25(0.25-1)}{2!}(0.017) - \frac{0.25(0.25-1)(0.25-2)\,0.002}{3!} + \frac{0.25(0.25-1)(0.25-2)(0.25-3)\,0.001}{4!} + 0 + 0$$

−0.0000375976 5

$$Y = 0.003 + 0.016 + (-0.00159375) + 0.000109375 + ($$

$$\boxed{Y = 0.017}$$

Ans

7-12-23

NEWTON BACKWARD DIFFERENCE

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y + \ldots$$

$$p = \frac{x - x_n}{h}$$

Find the value of y at x = 1.2.

$$P = \frac{1.2 - 1.3}{0.2} = \frac{-0.1}{0.2} = \boxed{-0.5}$$

$$y = 0.697 + (-0.5)(0.179) + \frac{(-0.5)(-0.5+1)}{2!}(0.031) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(0.005) + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}(0.001)$$

$$y = 0.697 - 0.0895 - 0.003875 - 0.0003125 - 0.0000390625$$

$$\boxed{y = 0.6032734375}$$

Forward difference is used for starting values (near the start)

Backward difference is used for ending values (near the end).

erere
alues.

9765

7-12-23

x	1	2	3	4	5	6	7	8.
y	1	8	27	64	125	216	343	512

$h = 1$

Solve

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$	$\nabla^7 y$
1	1	7	12	6	0	0	0	0
2	8	19	18	6	0	0	0	
3	27	37	24	6	0	0		
4	64	61	30	6	0			
5	125	91	36	6				
6	216	127	42					
7	343	169						
8	512							

Estimate the value

$f(1.5)$ $\qquad p = \frac{x - x_0}{h} = \frac{1.5 - 1}{1} = \boxed{0.5}$

$\cancel{y = y_0 + (0.5)}$

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

$$y = 1 + (0.5)(7) + \frac{(0.5)(0.5-1)}{2!}(12) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(6)$$

$$y = 1 + 3.5 - 1.5 + 0.375$$

$$\boxed{y = 3.375}$$

$f(6.5)$ $\qquad p = \frac{x - x_n}{h} = \frac{6.5 - 8}{1} = -1.5$

$$y = y_n + p\nabla y_n + \frac{(p)(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$y = 512 + (-1.5)(169) + \frac{(-1.5)(-1.5+1)}{2!}(42) + \frac{(-1.5)(-1.5+1)(-1.5+2)}{3!}(6).$$

$$y = 512 - 253.5 + 15.75 + 0.375$$

$$\boxed{y = 274.625}$$

FINALS

19-12-23

Divided Difference Method:

→ used for data that is not equally spaced.

Δx is not constant

★ differences are slopes

Divided Difference Table

i	x_i	$y_i / f(x_i)$	1st difference	2nd difference	3rd difference
0	x_0	$f(x_0)$			
			$f(x_1, x_0)$		
1	x_1	$f(x_1)$		$f(x_2, x_1, x_0)$	
			$f(x_2, x_1)$		$f(x_3, x_2, x_1, x_0)$
2	x_2	$f(x_2)$		$f(x_3, x_2, x_1)$	
			$f(x_3, x_2)$		
3	x_3	$f(x_3)$			

→ Using the table we can write a 3rd order polynomial.

f

$$f_3(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

where $b_0 = y_0$.

$$b_1 = f(x_1, x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f(x_2, x_1, x_0) = \frac{f(x_2, x_1) - f(x_1, x_0)}{x_2 - x_0}$$

$$b_3 = f(x_3, x_2, x_1, x_0) = \frac{f(x_3, x_2, x_1) - f(x_2, x_1, x_0)}{x_3 - x_0}$$

Find ln 2

Q)

x_i	$y_i = \ln x$	1st diff	2nd diff	3rd diff
1	0			
		0.462		
4	1.386		-0.059	
		0.223		0.0078
5	1.609		-0.02	
		0.183		
6	1.792			

$$f_3(2) = \cancel{b_0 + b} \; 0 + 0.462(1) + (-0.059)(1)(-2) + (0.0078)(1)(-2)(-4)$$

$$\boxed{f_3(2) = 0.6424}$$

PAPERWORK

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Lagrange's Interpolation

x_i	y_i
x_0	y_0
x_1	y_1
x_2	y_2
$\vdots$	$\vdots$

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}\, y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} + y_1 + \frac{(x-x_0)(x-x_1)(x_1-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)}\, y_2 + \cdots$$

Q.

	x_i	y_i
0	1	-3
1	3	0
2	4	30
3	6	132

find $f(5)$.

$$y = f(5) = \frac{(5-3)(5-4)(5-6)}{(1-3)(1-4)(1-6)} \times (-3) + \frac{(5-1)(5-4)(5-6)}{(3-1)(3-4)(3-6)}\,(0) + \frac{(5-1)(5-3)(5-6)}{(4-1)(4-3)(4-6)}\,(30) + \frac{(5-1)(5-3)(5-4)}{(6-1)(6-3)(6-4)}\,(132)$$

$$y = f(5) = \left(-\tfrac{1}{5}\right) + 0 + 40 + 35.2$$

$$\boxed{y = f(5) = 75}$$

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26-12-23

GUASS'S INTERPOLATION: (FORWARD)

→ Used for data with equally spaced x-values and to find values near the center of the data.

i	x_i	y_i	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-2	x_{-2}	y_{-2}	Δy_{-2}	$\Delta^2 y_{-2}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-2}$
-1	x_{-1}	y_{-1}	Δy_{-1}	$\Delta^2 y_{-1}$	$\Delta^3 y_{-1}$	
0	x_0	y_0	Δy_0	$\Delta^2 y_0$		
1	x_1	y_1	Δy_1			
2	x_2	y_2				

Formula:

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{p(p-1)(p+1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots$$

Question:

i	x_i	y_i	Δy	$\Delta^2 y$	$\Delta^3 y$	~~$\Delta^4 y$~~
-1	25	0.25	0.05	-0.02	0.03	
0	30	0.3	0.03	0.01		
1	35	0.33	0.04			
2	40	0.37				

Find $f(32)$

$$p = \frac{x - x_0}{h}, \quad h = 5 \longrightarrow p = \frac{32-30}{5} = \frac{2}{5} = 0.4$$

$$y(32) = 0.3 + (0.4)(0.03) + \frac{(0.4)(0.4-1)}{2!}(-0.02) + \frac{(0.4)(0.4-1)(0.4+1)(0.03)}{3!}$$

$$y(32) = 0.3 + 0.012 + 2.4\times10^{-3} + (-1.68\times10^{-3})$$

$$\boxed{y(32) = 0.31272}$$

PAPERWORK

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Question

i	x_i	y_i	Δy_0	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-2	1	+1	-2	4	-8	16
-1	2	-1	2	-4	8	
0	3	1	-2	4		
1	4	-1	2			
2	5	1				

$f(3.5)$

$h = 1$, $P = \dfrac{x - x_0}{h} = \dfrac{3.5 - 3}{1} = \dfrac{1}{2} = 0.5$

$$y(3.5) = 1 + (0.5)(-2) + \frac{(0.5)(0.5-1)}{2!}(-4) + \frac{(0.5)(0.5-1)(0.5+1)}{3!}(8)$$
$$+ \frac{(0.5)(0.5-1)(0.5+1)(0.5-2)(16)}{4!}$$

$y(3.5) = 1 + (-1) + 0.5 + (-0.5) + (0.375)$

$$\boxed{y(3.5) = \cancel{+0.125}\ 0.375}$$

Question

~~i~~ ~~x_i~~ ~~y_i~~

GUASS'S BACKWARD FORMULA

$$y(x) = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2} +$$
$$\frac{p(p+1)(p-1)(p+2)}{4!}\Delta^4 y_{-2} + \cdots \Delta^5 y_{-3}$$

Question

~~i~~ ~~x_i~~ ~~y_i~~

Question

	x_i	y_i	Δy_0	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-3	1940	17	3	4	-6	7	-9.
-2	1950	20	7	-2	1	-2	
-1	1960	27	5	-1	-1		
0	1970	32	4	-2			
1	1980	36	2.				
2	1990	38.					

$f(1976)$.

$h = 10$, $p = \dfrac{x - x_0}{h} = \dfrac{1976 - 1970}{10} = \dfrac{6}{10} = 0.6$.

$$y(1976) = 32 + (0.6)(5) + \frac{(0.6)(0.6+1)}{2!}(-1) + \frac{(0.6)(0.6+1)(0.6-1)}{3!}(1)$$

$$+ \frac{(0.6)(0.6+1)(0.6-1)(0.6+2)}{4!}(-2) + \frac{(0.6)(0.6+1)(0.6-1)(0.6+2)(0.6-2)}{5!} \times (-9$$

$$y(1976) = 32 + 3 + (-0.48) + (-0.064) + (0.0832) + (-0.104832).$$

$$\boxed{y(1976) = 34.43.}$$

DERIVATIVE USING STERLING'S CENTRAL DIFFERENCE.

28-12-23

Formula:

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots +$$

where $p = \dfrac{x - x_0}{h}$

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}.$$

$$\Downarrow$$

$$\frac{dp}{dx} = \frac{1}{h}.$$

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Formula:

$$\frac{dy}{dx} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + \frac{6P}{12}\left(\Delta^3 y_{-1} + \Delta^3 y_{-2}\right) + \frac{6P^2}{12}\Delta^4 y_{-2} + \cdots\right].$$

Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x-2h$	y_{-2}	Δy_{-2}	$\Delta^2 y_{-2}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-2}$
$x-h$	y_{-1}	Δy_{-1}	$\Delta^2 y_{-1}$	$\Delta^3 y_{-1}$	
x_0	y_0	Δy_0	$\Delta^2 y_0$		
$x+h$	y_1	Δy_1			
$x+2h$	y_2				

Example: Find the solution using the sterling's formula.

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
	20	49225	-909.	-171	-59.	-21.
	25	48316	-1080	-230	-80	
x_0	30	47236	-1310	-310.		
	35	45926	-1620.			
	40	44306				

Find $f'(28)$:

$$P = \frac{x - x_0}{h} = \frac{28-30}{5} = \frac{-2}{5} = -0.4.$$

$h = 5$

$$\frac{dy}{dx} = \frac{1}{5}\left[-230 + \frac{6(-0.4)}{12}\left(-80 + (-59)\right)\right.$$

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$$\frac{dy}{dx} = \frac{1}{5}\left[\frac{-1310+(-1080)}{2} - \frac{1}{6}\left(\frac{-80+(-59)}{2}\right)\right]$$

$$\frac{dy}{dx} = \frac{1}{5}\left[1195 - \frac{1}{6}\left(\frac{-139}{2}\right)\right].$$

$$\frac{dy}{dx} = \frac{1}{5}\left(-1195 + \frac{139}{12}\right)$$

$$\frac{dy}{dx} = \frac{1}{5}\left(-\frac{14201}{12}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{-14201}{60} \approx -236.68}$$

2-1-2024

DERIVATIVE USING BESSEL'S CENTRAL DIFFERENCE

Formula for values in between the given points.

$$\frac{dy}{dx} = \frac{1}{h}\left\{ \Delta y_0 + \frac{2p-1}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{3p^2-2p+1/2}{3!}\left(\Delta^3 y_{-1}\right) + \frac{4p^3-6p-2p+2}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \dots \right\}$$

If we have to find the value at a given point that we have in the given values then we set $x = x_0$, so, $p = \frac{x - x_0}{h} = 0$ ↳ not necessary

So, the formula becomes

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left\{ \Delta y_0 - \frac{1}{2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{1}{2}\Delta^3 y_{-1} - \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \dots \right\}$$

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Difference Table

	X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
-3	3.47	0.193	0.002	0.001	-0.001	0	0.003	-0.010
x_{-2}	3.48	0.195	0.003	0	-0.001	0.003	-0.007	
x_{-1}	3.49	0.198	0.003	-0.001	0.002	-0.004		
$x_0 \rightarrow$	3.50	0.201	0.002	0.001	-0.002			
1	3.51	0.203	0.003	-0.001				
2	3.52	0.206	0.002					
3	3.53	0.208						

Find $f'(3.5)$

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left\{\Delta y_0 - \frac{1}{2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{1}{12}\left(\Delta^3 y_{-1}\right) + \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots\right\}$$

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{0.01}\left\{0.002 - \frac{1}{2}\left(\frac{-0.001 + 0.001}{2}\right) + \frac{1}{12}(0.002) - \frac{1}{12}\left(\frac{+0.003 + (-0.004)}{2}\right) + \cdots\right\}$$

$$\left(\frac{dy}{dx}\right)_{x_0} = 100\left\{0.002 - \frac{1}{2}(0) + \frac{1}{12}(0.002) - \frac{1}{12}\left(-5\times10^{-4}\right) + \cdots\right\}$$

$$\left(\frac{dy}{dx}\right)_{x_0} = 100\left(\cancel{3.125\times10^{-3}}\ 2.208\times10^{-3}\right)$$

$$\left(\frac{dy}{dx}\right)_{x_0} = \cancel{0.2125}\ \boxed{0.22}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta y_0 - \frac{1}{2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) - \frac{1}{2}\Delta^3 y_{-1} - \right.$$
$$\left. - \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots \right].$$

$$\frac{d^2y}{dx^2} = \frac{1}{0.01^2}\left[0.002 - \frac{1}{2}\left(\frac{-0.001 + 0.001}{2}\right) + \frac{1}{12}(0.002) * \right.$$
$$\left. - \frac{1}{12}\left(\frac{0.003 + (-0.004)}{2}\right) + \cdots \right].$$

$$\frac{d^2y}{dx^2} = \boxed{-9.58}$$

Quiz Question.

x	y
20	2854
24	3162
28	3544
32	3992

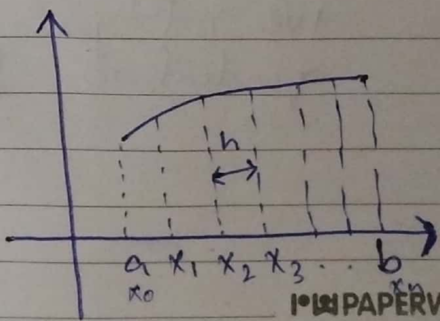
$f'(25) = 91.065$

NEWTON - COTES QUADRATURE FORMULA 4-1-24

* Integrating newton forward difference to find the quadrature.

Let $I = \int_a^b y\, dx$ where $y = f(x)$

$\underset{x_0}{a} = x_0, x_1, x_2, \cdots, x_n = \underset{x_n}{b}$



$a = x_0$, $b = x_0 + nh$

$$p = \frac{x - x_0}{h}$$

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$$p = \frac{x - x_0}{h}, \quad \text{so} \quad x = x_0 + ph$$
$$dx = h\,dp.$$

$$I = \int_{x_0}^{x_0+nh} f(x_0 + ph)\,h\,dp \qquad \text{when } x = x_0 \Rightarrow p = 0$$
$$x = x_0 + nh \Rightarrow p = n.$$

$$I = h \int_0^n y_p \, dp.$$

Now, Applying Newton Forward Interpolation, so that y_p is replaced by the polynomial.

$$I = h \int_0^n \Big\{ y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots \Big\}\, dp.$$

$$= h\Big[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\Big[\frac{n^3}{3} - \frac{n^2}{2}\Big]\Delta^2 y_0 + \frac{1}{6}\Big[\frac{n^4}{4} - n^3 + n^2\Big]\Delta^3 y_0$$
$$+ \frac{1}{24}\Big[\frac{n^5}{5} + \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2\Big\}\Delta^4 y_0 + \cdots \Big]$$

This is called the Newton-Cotes Quadrature Formula used for equidistant values of x. (h is constant) By using this formula and assigning suitable +ve integer values of n, we derive other quadrature formulas.

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9-1-24

TRAPIZOIDAL RULE :(A Linear approximation)

When we substitute $n=1$ and neglect second higher order differences.

For the first subinterval

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$y_0 + \frac{y_1}{2} - \frac{y_0}{2}$

$\frac{y_1}{2} - \frac{y_0}{2}$

$\frac{2y_0 - y_0}{2} = \frac{y_0}{2}$

$$'' \quad = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$\int_{x_0}^{x_0+h} f(x)\cdot dx = \frac{h}{2}\left[y_0 + y_1\right]$$

Similarly

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}\left[y_1 + y_2\right]$$

$\vdots$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[y_{n-1} + y_n\right]$$

Adding the result of all intervals

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left\{(y_0+y_n) + 2(y_1+y_2+\cdots y_{n-1})\right\}}$$

SIMPSON'S 1/3 RULE: (A Quadratic Formula)

When $n=2$ in Newton-Cotes Formula. After neglecting 3rd order & higher terms.

$$\int_{x_0}^{x_0+2h} f(x)\,dx = h\left\{2y_0 + 2\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - 2\right)\Delta^2 y_0\right\}$$

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$$\int_{x_0}^{x_0+2h} f(x)\,dx = h\left\{2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right\}$$

$$\int_{x_0}^{x_0+2h} f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Similarly

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

$$\vdots$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

After adding all intervals

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2})\right]}$$

Simpson's 1/3 rule is applied when n is an even number.

Simpson's 3/8 Rule: (A Cubic Approximation)

Substitute $n=3$ in Newton - Cotes Formula & neglect 4^{th} & higher order differences.

$$\boxed{\int_{x_0+(n-3)h}^{x_0+3h} f(x)\,dx = \frac{3h}{8}\left\{(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3})\right\}}$$

For Simpson's $\frac{3}{8}$ rule n is a multiple of 3.

Integrate x^3 by Trapizoidal, 1/3 or 3/8 rule: by choosing appropriate value of n in each case and compare the value by definite integrals. The limits of integration are between $a = 0$ and $b = 4$.

$\int_a^b f(x)\,dx$.

n is no. of intervals.

Using Trapizoidal rule

$n = 3$, $h = \frac{b-a}{n} = \frac{4-0}{3} = \frac{4}{3}$

i	0	1	2	3.
x	0	4/3	8/3	4
y	0	64/27	512/27	64.

$$\because \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left\{(y_0+y_n) + 2(y_1+y_2+\cdots y_{n-1})\right\}$$

$$\Rightarrow \int_0^4 x^3\,dx = \frac{4/3}{2}\left\{(0+64)+2\left(\frac{64}{27}+\frac{512}{27}\right)\right\}.$$

$$\Rightarrow \int_0^4 x^3\,dx = 2/3\left(\frac{320}{3}\right) = \frac{640}{9} \approx \boxed{71.11}$$

Using Simpson's 1/3 rule.

$n = 4$, $h = \frac{b-a}{n} = \frac{4-0}{4} = 1$

i →	0	1	2	3	4
x	0	1	2	3	4
y	0	1	8	27	64.

$$\because \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots+y_{n-1})+2(y_2+y_4+y_6+\cdots y_{n-2})\right]$$

$$\int_0^4 x^3\,dx = \frac{1}{3}\left[(0+64)+4(1+27)+2(8)\right] \Rightarrow \frac{1}{3}(192) = \boxed{64}$$

Using Simpson's 3/8 rule

$n = 6$, $h = \frac{b-a}{n} = \frac{4-0}{6} = \frac{2}{3}$

i →	0	1	2	3	4	5	6
x →	0	2/3	4/3	2	8/3	10/3	4
y →	0	8/27	64/27	8	512/27	1000/27	64.

Using Simpson's 3/8 rule.

$$\because \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left\{(y_0+y_n)+3(y_1+y_2+y_4+\cdots+y_{n-1})+2(y_3+y_6+y_9+\cdots+y_{n-3})\right\}$$

$$\int_0^4 x^3\,dx = \frac{3(2/3)}{8}\left\{(0+64)+3\left(\frac{8}{27}+\frac{64}{27}+\frac{512}{27}+\frac{1000}{27}\right)+2(8)\right\}$$

$$= \frac{1}{4}(256) = \boxed{64}$$

11-1-24

Q) $\int_0^1 e^x \, dx$. Evaluate the integral by Simpson's using Simpson 1/3 or 3/8 rule.
Divide the interval into 6 subintervals.

i	0	1	2	3	4	5	6
x	0	1/6	1/3	1/2	2/3	5/6	1
y	1	1.181	1.395	1.648	1.947	2.300	2.718.

$n = 6$.

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6} =$$

Using Simpson's 1/3 rule:

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots+y_{n-1})+2(y_2+y_4+y_6+\cdots y_{n-2})\right]$$

$$\int_0^1 e^x\,dx = \frac{1/6}{3}\left[(1+2.718)+4(1.181+1.648+2.300)+2(1.395+1.947)\right]$$

$$\int_0^1 e^x\,dx = \frac{1}{18}\left[3.718+4(5.129)+2(3.342)\right].$$

$$\int_0^1 e^x\,dx = \frac{1}{18}\left[3.718+20.516+6.684\right].$$

$$\int_0^1 e^x\,dx = \frac{1}{18}(30.918)$$

$$\boxed{\int_0^1 e^x\,dx = 1.717}$$

Using Simpson's 3/8 Rule.

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left\{(y_0+y_n) + 3(y_1+y_2+y_4+y_5+y_7+\cdots y_{n-1}) + 2(y_3+y_6+y_9+\cdots y_{n-3})\right\}.$$

$$\int_0^1 e^x\,dx = \frac{3(\frac{1}{6})}{8}\left\{(1+2.718) + 3(1.181+1.395+1.947+2.300) + 2(1.648)\right\}.$$

$$\int_0^1 e^x\,dx = \frac{1}{16}\left\{3.718 + 3(6.823) + 3.296\right\}.$$

$$\int_0^1 e^x\,dx = \frac{1}{16}\left\{3.718 + 20.469 + 3.296\right\},$$

$$\int_0^1 e^x\,dx = \frac{1}{16}(27.483)$$

$$\boxed{\int_0^1 e^x\,dx = 1.717}$$

Q: Given new data.

	$t(s) \to t(h)$	v (km/h).
0	0.0	0.0
1	2.0	5.5
2	3.0	7.5
3	4.0	10
4	5.0	5

$h = 1.0$, $n = 4$.

$D = \int v\,dt$.

Integrate using trapizoidal rule & Simpon's 1/3 rule to find the distance covered by the vehicle.

n is not multiple of 3. we can apply trapizoidal & Simpson's 1/3 rule.

Using trapizoidal rule:

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left\{(y_0+y_n) + 2(y_1+y_2+y_3+\cdots+y_{n-1})\right\}.$$

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$$\int_{0.0}^{5.0} t(h)\,dh = \frac{1.0}{2}\left\{(0.0+5)+2(5.5+7.5+10)\right\}.$$

$$\int_{0.0}^{5.0} t(h)\,dh = \frac{51}{2} = \underline{25.5}$$

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Using Simpson's 1/3 rule.

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots+y_{n-1})+2(y_2+y_4+\cdots y_{n-2})\right]$$

$$\int_{0.0}^{5.0} t(h)\,dh = \frac{1.0}{3}\left[(0.0+5)+4(5.5+10)+2(7.5)\right].$$

$$\int_{0.0}^{5.0} t(h)\,dh = \frac{1}{3}(82) \approx \underline{27.33}$$

TAYLOR SERIES METHOD:

16-1-24.

Let $f(x)$ be a solution of an equation.

$$\frac{dy}{dx} = f(x,y)$$

with $y(x_0) = y_0$.

Expanding by Taylor series.

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \cdots$$

This can be written as the sum of 'y_0' and putting.
$x - x_0 = h$.
So, $x_1 = x_0 + h$.

$$\Rightarrow f(x) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

Similarly for y_{n+1}.

$$\Rightarrow y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots$$

Note: Taylor series method is applicable when various derivatives of $f(x)$ exist. Value of $(x - x_0)$ in the equation $y = f(x)$. near x_0 should be very small so the series can have a minimum error.

Example: $\frac{dy}{dx} = x + y$, Solve numerically $y(1) = 0$

upto $x = 1.2$ with $h = 0.1$

Solution:

From the given data we have $y(1) = 0$.

$x_0 = 1$, $y_0 = 0$, $h = 0.1$ So, $x_1 = x_0 + h = 1.1$

$x_2 = x_1 + h = 1.2$

$$\because f(x) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

We have y_0 but we have to find derivatives.

$$\Rightarrow \frac{dy}{dx} = y_0' = x + y \Rightarrow y_0' = 1 + 0 = \boxed{1 = y_0'}$$

$$\Rightarrow \frac{dy}{dx} = x + y$$

$$\Rightarrow \frac{d^2y}{dx^2} = 1 + \frac{dy}{dx} \Rightarrow 1 + 1 = \boxed{2 = y_0''}$$

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$$\Rightarrow \frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \frac{d^3y}{dx^3} = \frac{d^2y}{dx^2} = y_0'' = \boxed{2 = y_0''}$$

$$y_1(1.1) = 0 + \frac{0.1}{1!} + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \ldots$$

$$y_1(1.1) = 0.11034.$$

$$\because y' = x + y.$$

$$y_1' = 1.1 + 0.11034 = \boxed{1.21034.}$$

$$y_1'' = 1 + y_1' = 1 + 1.21034 = \boxed{2.21034}$$

$$y_1''' = y_1'' = \boxed{2.21034.}$$

$$y_2(1.2) = \cancel{0} + 0.11034 + \frac{0.1}{1!}(1.21034) + \frac{(0.1)^2}{2!}(2.21034) + \frac{(0.1)^3}{3!}(2.21034)$$

$$\boxed{y_2(1.2) = 0.242794409}.$$

Answer

EULER'S METHOD:

23-01-24.

Numerical method for solving a differential equation (1st order and 1st degree). Consider a first order, first degree equation $\frac{dy}{dx} = f(x, y)$, with a given initial ~~value~~ condition x_0, y_0. Now, to find the value of y_n at x_n divide the interval into n-subintervals having a width of $h = \frac{b-a}{n}$.

$$\int_{x_{n-1}}^{x_n} dy = \int_{x_{n-1}}^{x_n} f(x,y)\, dx.$$

$$\int_{y_{n-1}}^{y_n} dy = \int_{x_{n-1}}^{x_n} f(x,y)\, dx.$$

$$y_n - y_{n-1} = \int_{x_{n-1}}^{x_n} f(x,y)\, dx.$$

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1}) \int_{x_{n-1}}^{x_n} dx.$$

$$\boxed{y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})}.$$

This is Euler's formula.

Example: Solve $\frac{dy}{dx} = 1 - y$, with the initial condition $x = 0$, $y = 0$, using Euler's ~~alg~~ formula at $x = 0.1, 0.2, 0.3$.

<u>Solve</u>

$n = 3$

$h = \frac{b-a}{n} = \frac{0.3 - 0}{3} = \underline{0.1}$

$x_0 = 0$

$x_1 = 0.1$

$x_2 = 0.2$

$x_n = 0.3.$

$\Rightarrow x_n = x_{n-1} + h$

$\Rightarrow f(x,y) = \frac{dy}{dx} = 1 - y.$

$\Rightarrow f(x_0, y_0) = f(0,0) = 1 - 0 = \boxed{1}.$

$\because y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$

$\Rightarrow y_1 = y_0 + h f(x_0, y_0).$

$y_1 = 0 + 0.1(1).$

$\boxed{y_1 = 0.1}$

$\Rightarrow f(x_1, y_1) = dy/dx \Rightarrow f(0.1, 0.1) = 1 - 0.1 = \boxed{0.9}$

$\Rightarrow y_2 = y_1 + h f(x_1, y_1)$

$y_2 = 0.1 + (0.1)(0.9)$

$\boxed{y_2 = 0.19}$

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$\Rightarrow f(x_2, y_2) = dy/dx = 1 - y$

$f(0.2, 0.19) = 1 - 0.19 = \boxed{0.81}$

$\Rightarrow y_3 = y_2 + h f(x_2, y_2)$.

$y_3 = 0.19 + (0.1)(0.81)$

$\boxed{y_3 = 0.271}$

i	x	y
0	0	0
1	0.1	0.1
2	0.2	0.19
3	0.3	0.271.

Modified Euler's Method:

From Euler's Method

$$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$$

Here $y(x_n) = y_n$ denotes the initial value by using the above equation. The other approximate value $y_n^{(0)}$ can be calculated.

$$y_n^{(0)} = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f(x_n, y_n)\right]$$

then

Replacing $f(x_n, y_n)$ by $f(x_n, y_n)^{(0)}$ then

$$y_n^{(1)} = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + \underbrace{f(x_n, y_n)^{(0)}}\right]$$

update this value in each iteration

$$y_n^{(k)} = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f(x_n, y_n)^{(k-1)}\right]$$

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Example: Solve $\frac{dy}{dx} = x^2+y$, $y=1$ when $x=0$ by Euler's Modified Method, find y at $x = 0.02$ when $h=0.01$.

$x_0 = 0$.

$x_1 = x_0+h = 0+0.01 = 0.01$

$x_2 = x_1+h = 0.01+0.01 = 0.02$.

$$f(x,y) = \frac{dy}{dx} = x^2+y.$$

$\Rightarrow f(x_0,y_0) = f(0,1) = (0)^2+1 = 1$

$\Rightarrow y_1 = y_0 + h f(x_0,y_0)$

$y_1 = 1+(0.01)(1)$

$\boxed{y_1 = 1.01}$

$\Rightarrow f(0.02, 1.01) = (0.02)^2 + 1.01 = 1.0104$

$\Rightarrow y_2 = y_1 + h f(x_1,y_1)$

$y_2 = 1.01+(0.01)(1.0104)$

$\boxed{y_2 = 1.020104}$

i	x	y.
0	0	1
1	0.01	1.01
2	0.02	1.020104

$y_1^{(0)} = y_0 + h f(x_0,y_0)$

$y_1^{(0)} = 1+0.01(0^2+1)$

$\boxed{y_1^{(0)} = 1.01}$.

Now apply Euler's Modified Formula.

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0)+f(x_1,y_1)\right].$$

$$y_1^{(1)} = 1+\frac{0.01}{2}\left[(0^2+1)+(0.01^2+0.01)\right].$$

$$y_1^{(1)} = 1.01005$$

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$$y_1^{(2)} = 1 + \frac{0.01}{2}\left[(0^2+1) + (0.01)^2 + 1.01005\right]$$

$$\boxed{y_1^{(2)} = 1.01005075}$$

Stabiliz y_1 values => number are same at 2 decimal places.

$$y_2^{(0)} = y_1 + hf(x_1, y_1)$$

$$y_2^{(0)} = 1.01005075 + 0.01\, f(0.01,\ 1.01005075)$$

$$y_2^{(0)} = 1.01005075 + 0.01\,(0.01^2 + 1.01005075)$$

$$y_2^{(0)} = \cancel{1.011009075}\ 1.020152250$$

Now apply Euler's Modified Formula.

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2)\right]$$

$$y_2^{(1)} = 1.01005075 + \frac{0.01}{2}\left[\{(0.01)^2 + 1.01005075\} + \{(0.02)^2 + 1.020152250\}\right]$$

$$\boxed{y_2^{(1)} = 1.020204265}$$

i	x	y
0	0	1
1	0.01	1.01005075
2	0.02	1.020204265

30-1-24

Runge-Kutta Method

Fourth order R-K Method is most commonly used method for solving first order differential equations.
This method is developed to avoid the computation of higher order derivatives in Taylor Series.

$\frac{dy}{dx} = f(x,y)$ is the first order differential equation where h denotes the step length.
x_0, y_0 are the initial values (initial conditions), then the first increment Δy in y is computed by.

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

$$k_1 = hf(x_0, y_0).$$

$$y_1 = y_0 + \Delta y_0.$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right).$$

$$k_4 = hf(x_0 + h, y_0 + k_3).$$

Q. Using the R-K method find an approximate value of y at $x = 0.2$, if $dy/dx = x + y^2$ given that $y(0) = 1$ & $h = 0.1$.

Solve

$$f(x,y) = \frac{dy}{dx} = x + y^2.$$

$$k_1 = hf(x_0, y_0)$$

$$k_1 = (0.1)\{0 + (1)^2\}$$

$$k_1 = 0.1$$

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$$k_2 = (0.1)\left\{\left(0+\frac{0.1}{2}\right)+\left(1+\frac{0.1}{2}\right)^2\right\}.$$

$$\underline{k_2 = 0.11525}$$

$$k_3 = (0.1)\left\{\left(0+\frac{0.1}{2}\right)+\left(1+\frac{0.11525}{2}\right)^2\right\}.$$

$$\underline{k_3 = 0.11685}$$

$k_1 = 0.1$
$k_2 = 0.11525$
$k_3 = 0.11685$
$k_4 = 0.13473$

$$k_4 = (0.1)\left\{(0+0.1)+(1+0.11685)^2\right\}.$$

$$\underline{k_4 = 0.13473.}$$

$$\Delta y_0 = \frac{1}{6}\left(0.1+2(0.11525)+2(0.11685)+0.13473\right)$$

$$\Delta y_0 = \frac{1}{6}(0.69893).$$

$$\underline{\Delta y_0 = 0.11648}$$

$$y_1 = y_0 + \Delta y_0.$$

$$y_1 = 1 + 0.11648.$$

$$\boxed{y_1 = 1.11648}$$

$$x_1 = 0.1 \quad , \quad y_1 = 1.11648$$

$$k_1 = hf(x_1, y_1).$$

$$k_1 = (0.1)\left\{0.1 + (1.11648)^2\right\}.$$

$$\underline{k_1 = 0.13465}$$

$$k_2 = (0.1)\left\{\left(0.1+\frac{\cancel{0.1}\ \cancel{0.13465}\ 0.1}{2}\right)+\left(1.11648+\frac{\cancel{0.1}\ 0.13465}{2}\right)^2\right\}.$$

$$\underline{k_2 = 0.15583.}$$

$$k_3 = 0.1\left\{\left(0.1+\frac{0.1}{2}\right)+\left(1.11648+\frac{0.15513}{2}\right)\right\}$$

$$\underline{k_3 = 0.15757.}$$

$k_4 = 0.1 \{ (0.1 + 0.1) + (1.11648 + 0.15757)^2 \}$.

$k_4 = 0.18232$

$\Delta y_2 = \frac{1}{6} \{ 0.13465 + 2(0.15513) + 2(0.15757) + 0.18232$

$\Delta y_2 = \frac{1}{6} (0.94237)$.

$\Delta y_2 = 0.15706$.

$y_2 = y_1 + \Delta y_1$

$y_2 = 1.11648 + 0.15706$

$y_2 = 1.27354$.

$x_2 = 0.2$, $y_2 = 1.27354$.

i	x	y
0	0	1
1	0.1	1.11648
2	0.2	1.27354.

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