PRACTICE – DIFFERENTIAL EQUATION

POWER SERIES SOLUTION

~~Dennis~~ Special case of Taylor series → Mclaurin Series.

A function can be expanded in this way.

$$\Rightarrow f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

where $x = 0$. & $f(x) = y$.

$\Rightarrow y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$

$\Rightarrow y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots$

$\Rightarrow y'' = 2c_2 + 6c_3 x + 12 c_4 x^2 + \dots$

Now to find the solution of a differential equation, we will substitute above values in the differential equation and solve accordingly.

Dennis zill book / Ex: 6.1.

35. $y' - 5y = 0.$

$\Rightarrow y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \dots$

$\Rightarrow y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + 6c_6 x^5 + \dots$

Substitute y & y' in the D.E.

$\Rightarrow c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \dots$
$- 5[c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + 6c_6 x^5 + \dots] = 0$

$\Rightarrow c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 - 5c_1 - 10c_2 x - 15c_3 x^2 - 20c_4 x^3 - 25c_5 x^4 - 30c_6 x^5 = 0$

$\Rightarrow$ ~~c_0 $-4c_1 x - 9c_2 x^2 - 14c_3 x^3 - 19c_4 x$~~

$\Rightarrow c_0 - 5c_1 + (c_1 - 10c_2)x + (c_2 - 15c_3)x^2 + (c_3 - 20c_4)x^3 + (c_4 - 25c_5)x^4 + (c_5 - 30c_6)x^5 = 0.$

Compare each term with R.H.S = 0.

$\Rightarrow \boxed{c_0 = 0}$ $\Rightarrow -5c_1 = 0 \Rightarrow \boxed{c_1 = 0}$ $c_2 = \frac{1}{10}c_1$

$\Rightarrow c_1 - 10c_2 = 0 \Rightarrow c_1 = 10c_2 \Rightarrow \boxed{c_2 = 0}$

$\Rightarrow c_2 - 15c_3 = 0 \Rightarrow c_2 = 15c_3 \Rightarrow \boxed{c_3 = 0}$

$\Rightarrow c_3 - 20c_4 = 0 \Rightarrow c_3 = 20c_4 \Rightarrow \boxed{c_4 = 0}$

$\Rightarrow c_4 - 25c_5 = 0 \Rightarrow c_4 = 25c_5 \Rightarrow \boxed{c_5 = 0}$

$\Rightarrow c_5 - 30c_6 = 0 \Rightarrow c_5 = 30c_6 \Rightarrow \boxed{c_6 = 0}$

Date ______ 20___

Substitute all c values in eq. of y. to get the solution

$\Rightarrow y = 0 + 10c_2x + 15c_3x^2 + 20c_4x^3 + 25c_5x^4 + 30c_6x^5 + \dots$

$\Rightarrow y = \cancel{10c_2}\ 5\left[2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + \dots\right]$

$\Rightarrow y = 0$

Dennis Zill book, Ex: 6.1

35. $y' - 5y = 0$

$\Rightarrow y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \dots$

$\Rightarrow y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + \dots$

Substitute y and y' in the D.E.

$\Rightarrow c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5$
$-5(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6) = 0.$

$\Rightarrow c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5$
$-5c_0 - 5c_1x - 5c_2x^2 - 5c_3x^3 - 5c_4x^4 - 5c_5x^5 -$
$5c_6x^6 = 0$

$\Rightarrow -5c_0 + c_1 + (2c_2 - 5c_1)x + (3c_3 - 5c_2)x^2 + (4c_4 - 5c_3)x^3$
$+ (5c_5 - 5c_4)x^4 + (6c_6 - 5c_5)x^5 = 0.$

Compare each term with R.H.S = 0.

$\Rightarrow -5c_0 + c_1 = 0 \Rightarrow \cancel{c_0 = 0}\ \boxed{c_1 = 5c_0}$

$\Rightarrow \cancel{\boxed{c_1 = 0}}$.

$\Rightarrow 2c_2 - 5c_1 = 0 \Rightarrow 2c_2 = 5(5c_0) \Rightarrow 2c_2 = 25c_0 \Rightarrow \boxed{c_2 = \frac{25c_0}{2}}$

$\Rightarrow 3c_3 - 5c_2 = 0 \Rightarrow 3c_3 = 5(c_0) = 3c_3 = 5\left(\frac{25c_0}{2}\right) \Rightarrow \boxed{c_3 = \frac{125}{6}c_0}$

$\Rightarrow 4c_4 - 5c_3 = 0 = 4c_4 - 5c_3 = 0 \Rightarrow c_4 = \frac{5}{4}\left(\frac{125}{6}c_0\right) \Rightarrow \boxed{c_4 = \frac{625}{24}c_0}$

$\Rightarrow 5c_5 - 5c_4 = 0 \Rightarrow \boxed{c_5 = 0}$

$\Rightarrow 6c_6 - 5c_5 = 0 \Rightarrow \boxed{c_6 = 0}$

Now, substitute all c values in y solution

$\cancel{\boxed{y = 0}}$.

$y = c_0 + 5c_0x + \frac{25}{2}c_0x^2 + \frac{125}{6}c_0x^3 + \frac{625}{24}c_0x^4 + \dots$

$y = c_0\left[1 + 5x + \frac{25}{2}x^2 + \frac{125}{6}x^3 + \frac{625}{24}x^4 + \dots\right]$

$c_0 + c_1 = 0$ $\quad \frac{dy}{y} = \frac{1}{14} dx$ $\quad \ln y = \frac{1}{4}x + c$

$\boxed{c_0 = -c_1}$

Date ____ 20

36. $4y' + y = 0$

$\Rightarrow y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \dots$

$\Rightarrow y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + 6c_6 x^5 + \dots$

Substitue y & y' in the D.E

$\Rightarrow 4(c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + 6c_6 x^5) +$
$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 = 0$

$\Rightarrow 4c_1 + 8c_2 x + 12c_3 x^2 + 16c_4 x^3 + 20c_5 x^4 + 24c_6 x^5 + \cancel{c_0} +$
$\cancel{c_0} + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 = 0$

$\Rightarrow c_0 + 4c_1 + (8c_2 + c_1)x + (12c_3 + c_2)x^2 + (16c_4 + c_3)x^3 +$
$(20c_5 + c_4)x^4 + (24c_6 + c_5)x^5 = 0$

Compare each term

$\Rightarrow \boxed{c_0 + 4c_1} = 0 \Rightarrow \boxed{c_1 = -\frac{1}{4}c_0}$

$\Rightarrow \cancel{4c_1 = 0} \Rightarrow \cancel{c_1 = 0}$

$\Rightarrow 8c_2 + c_1 = 0 \Rightarrow 8c_2 = \frac{1}{4}c_0 \Rightarrow \boxed{c_2 = \frac{1}{32}c_0}$

$\Rightarrow 12c_3 + c_2 = 0 \Rightarrow \boxed{c_3 = -\frac{1}{384}c_0}$

$\Rightarrow 16c_4 + c_3 = 0 \Rightarrow \boxed{c_4 = \frac{1}{6144}c_0}$

$20c_5 + c_4 = 0 \Rightarrow \boxed{c_5 = 0}$

$24c_6 + c_5 = 0 \Rightarrow \boxed{c_6 = 0}$

$c_0 + 4c_1 = 0$

$\boxed{c_0 = -4c_1} \Rightarrow c_1 = -\frac{1}{4}c_0$

$8c_2 + c_1 \Rightarrow 8c_2 = +\frac{1}{4}c_0$

$\boxed{c_2 = \frac{1}{32}c_0}$

$12c_3 = -c_2 \Rightarrow 12c_3 = -\frac{1}{32}c_0$

$\boxed{c_3 = -\frac{1}{384}c_0}$

$16c_4 = -c_3$

$c_4 = \frac{1}{384}c_0$

$\boxed{c_4 = \frac{1}{6144}c_0}$

Now substitute all c values in y.

$\boxed{y = c_0 - \frac{1}{4}c_0 x + \frac{1}{32}c_0 x^2 - \frac{1}{384}c_0 x^3 + \dots}$

37. $y' = xy$.

$\Rightarrow y' - xy = 0$.

$\Rightarrow y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5$

$\Rightarrow y'' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4$.

Substitute y & y' in the DE.

$\Rightarrow c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 - x(c_0 + c_1 x + c_2 x^2 +$
$c_3 x^3 + c_4 x^4 + c_5 x^5) = 0$.

$\Rightarrow c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 - c_0 x - c_1 x^2 - c_2 x^3 -$
$c_3 x^4 - c_4 x^5 - c_5 x^6 = 0$

Compare each term with 0.

$c_1 + (2c_2 - c_0)x + (3c_3 - c_1)x^2 + (4c_4 - c_2)x^3 +$
$\boxed{c_1 = 0}$ $(5c_5 - c_3)x^4 = 0$

Compare each term with $= 0$

$\boxed{c_1 = 0}$

$2c_2 - c_0 = 0 \Rightarrow \boxed{c_2 = \frac{1}{2}c_0}$

$3c_3 - c_1 = 0 \Rightarrow \boxed{c_3 = 0}$

$4c_4 - c_2 = 0 \Rightarrow c_4 = \frac{1}{4}c_2 \Rightarrow c_4 = \frac{1}{4}\left(\frac{1}{2}c_0\right) = c_4 = \frac{1}{8}c_0$

$5c_5 - c_3 = 0 \Rightarrow \boxed{c_5 = 0}$.

Substitute all c values in y.

$y = c_0 + (0)x + \frac{1}{2}c_0 x + (0)x^3 + \frac{1}{8}c_0 x^4 + O(x^5)$

$y = c_0 + \frac{1}{2}c_0 x + \frac{1}{8}c_0 x^4 + \cdots$

$y = c_0\left[1 + \frac{1}{2}x + \frac{1}{8}x^4 + \cdots\right]$

Ans

38. $(1+x)y' + y = 0$.

$\cancel{(1+x)}$ $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$

$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \cdots$

Substitute y & y' in the DE.

$\Rightarrow (1+x)\, c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 = 0$

$\Rightarrow c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + c_1 x + 2c_2 x^2 + 3c_3 x^3 + 4c_4 x^4 + 5c_5 x^5 + c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 = 0$

$\Rightarrow c_0 + c_1 + (2c_2 + 2c_1)x + (3c_3 + 3c_2)x^2 + (4c_4 + 4c_3)x^3 + (5c_5 + 5c_4)x^4 + \cdots = 0$

Compare each term with 0.

$c_0 + c_1 = 0 \Rightarrow \boxed{c_1 = -c_0}$

$2c_2 + 2c_1 = 0 \Rightarrow \cancel{2}c_2 = -\cancel{2}c_1 \Rightarrow \boxed{c_2 = +c_0}$

$\boxed{y = 0}$.

$3c_3 + 3c_2 = 0 \Rightarrow c_3 = -c_2 \Rightarrow \boxed{c_3 = -c_0}$

$y = c_0 - c_0 x + c_0 x^2 - c_0 x^3 + \cdots$

$\boxed{y = c_0\left[1 - x + x^2 - x^3 + \cdots\right]}$

Ans

$y = \cancel{\ }\, c_0 - \frac{1}{4}c_0 x + \frac{1}{32}c_0 x^2 - \frac{1}{384}c_0 x^3 + \frac{1}{6144}c_0 x^4 + \cdots$

$y = c_0\left[\cancel{\frac{1}{4}}\, 1 - \frac{1}{4}x + \frac{1}{32}x^2 - \frac{1}{384}x^3 + \frac{1}{6144}x^4 + \cdots\right]$

Date

Assignment Question:

$y'' - xy' = 0$

$\Rightarrow y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6$

$\Rightarrow y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5$

$\Rightarrow y'' = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4$

Substitute y' & y'' in D.E.

$\Rightarrow 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 - x(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5) = 0$

$\Rightarrow 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 - c_1x - 2c_2x^2 - 3c_3x^3 - 4c_4x^4 - 5c_5x^5 - 6c_6x^5 = 0$

$\Rightarrow 2c_2 + (6c_3 - c_1)x + (12c_4 - 2c_2)x^2 + (20c_5 - 3c_3)x^3 + (30c_6 - 4c_4)x^5 = 0$

Compare each term with 0.

$\Rightarrow 2c_2 = 0 \Rightarrow \boxed{c_2 = 0}$

$\Rightarrow 6c_3 - c_1 = 0 \Rightarrow \boxed{c_3 = \frac{1}{6}c_1}$

$\Rightarrow 12c_4 - 2c_2 = 0 \Rightarrow \boxed{c_4 = 0}$

~~$\Rightarrow 30c_6 - 4c_4 = 0 \Rightarrow$~~ $20c_5 - 3c_3 = 0 \Rightarrow 20c_5 = 3\left(\frac{1}{6}c_1\right)$

$\boxed{c_5 = \frac{1}{40}c_1}$

$\Rightarrow 30c_6 - 4c_4 = 0 \Rightarrow \boxed{c_6 = 0}$

Substitute all c values in ~~D.E~~ eq for y.

$y = c_0 + c_1x + (0)x^2 + \frac{1}{6}c_1x^3 + (0)x^4 + \frac{1}{40}c_1x^5 + (0)x^6 + \ldots$

$y = c_0 + c_1x + \frac{1}{6}c_1x^3 + \frac{1}{40}c_1x^5 + \ldots$

$y = c_0 + c_1\left[x + \frac{1}{6}x^3 + \frac{1}{40}x^5 + \ldots\right]$ Ans

Kent Nagle book Section 8.3

Ex: 21 Find a power series solution about $x = 0$

$y' + 2xy = 0$

$\Rightarrow y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6$

$\Rightarrow y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + \ldots$

Substitute y & y' in eq D.E

Date ______ 20__

$\Rightarrow c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + 2xc_0 +$
$2c_1x^2 + 2c_2x^3 + 2c_3x^4 + 2c_4x^5 + 2c_5x^6 + 2c_6x^7 = 0.$

$\Rightarrow c_1 + (2c_2 + 2c_0)x + (3c_3 + 2c_1)x^2 + (4c_4 + 2c_2)x^3$
$+ (5c_5 + 2c_3)x^4 + (6c_6 + 2c_4)x^5 = 0.$

Comparing the eq with $= 0$.

$\boxed{c_1 = 0}$

$2c_2 + 2c_0 = 0 \Rightarrow 2c_2 = -2c_0 \Rightarrow \boxed{c_2 = -c_0}$

$3c_3 + 2c_1 = 0 \Rightarrow \boxed{c_3 = -\frac{2}{3} c_1 = 0}$

$4c_4 + 2c_2 = 0 \Rightarrow 4c_4 = -2c_2 \Rightarrow \boxed{c_4 = -\frac{1}{2}(-c_0) = \frac{1}{2}c_0}$

$5c_5 + 2c_3 = 0 \Rightarrow 5c_5 = -2c_3 \Rightarrow c_5 = -\frac{2}{5}\left(-\frac{2}{3}c_1\right) \Rightarrow \boxed{c_5 = 0}$

$6c_6 + 2c_4 = 0 \Rightarrow 6c_6 = -2c_4 \Rightarrow c_6 = -\frac{1}{3}\left(\frac{1}{2}c_0\right) \Rightarrow \boxed{c_6 = -\frac{1}{6}c_0}$

Substituting all c values in eq for y.

$y = c_0 + (0)x + (-c_0)x^2 + (0)x^3 + \left(\frac{1}{2}c_0\right)x^4 + (0)x^5 + \left(-\frac{1}{6}c_0\right)x^6 + \ldots$

$y = c_0 - c_0x^2 + \frac{1}{2}c_0x^4 - \frac{1}{6}c_0x^6 + \ldots$

$y = c_0\left[1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \ldots\right]$

Ans

Ex:3: Find a general solution to $2xy'' + xy' + y = 0$, in the form of power series about the ordinary point $x=0$.

$\Rightarrow 2xy'' + xy' + y = 0.$

$\Rightarrow y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \ldots$

$\Rightarrow y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + \ldots$

$\Rightarrow y'' = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \ldots$

Substitute y, y' & y'' in the D.E.

$\Rightarrow 2x(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4) +$
$x(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5) +$
$c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + = 0.$

$\Rightarrow 4c_2x + 12c_3x^2 + 24c_4x^3 + 40c_5x^4 + 60c_6x^5 + c_1x + 2c_2x^2$
$+ 3c_3x^3 + 4c_4x^4 + 5c_5x^5 + 6c_6x^6 + c_0 + c_1x + c_2x^2 + c_3x^3 +$
$c_4x^4 + c_5x^5 + c_6x^6 = 0$

$\Rightarrow c_0 + (4c_2 + 2c_1)x + (12c_3 + 3c_2)x^2 + (24c_4 + 4c_3)x^3 + (40c_5 + 5c_4)x^4 + (60c_6 + 6c_5)x^5 = 0.$

Comparing each term.

$\Rightarrow \boxed{c_0 = 0}$

$\Rightarrow 4c_2 + 2c_1 = 0 \Rightarrow 4c_2 = -2c_1 = \boxed{c_2 = -\frac{1}{2}c_1}$

$\Rightarrow 12c_3 + 3c_2 = 0 \Rightarrow 12c_3 = -3c_2 \Rightarrow c_3 = -\frac{1}{4}(-\frac{1}{2}c_1)$

$\boxed{c_3 = \frac{1}{8}c_1}$

$\Rightarrow 24c_4 + 4c_3 = 0 \Rightarrow 24c_4 = -4c_3 \Rightarrow c_4 = -\frac{1}{6}(\frac{1}{8}c_1)$

$\boxed{c_4 = -\frac{1}{48}c_1}$

$\Rightarrow 40c_5 + 5c_4 = 0 \Rightarrow 40c_5 = -5c_4 = c_5 = -\frac{1}{8}(\frac{-1}{48}c_1)$

$\boxed{c_5 = \frac{1}{384}c_1}$

$\Rightarrow$ Substituting c values in y equation

$y = c_1x + (-\frac{1}{2}c_1)x^2 + (\frac{1}{8}c_2)x^3 + (-\frac{1}{48}c_1)x^4 + (\frac{1}{384}c_1)x^5 + \dots$

$y = c_1x - \frac{1}{2}c_1x^2 + \frac{1}{8}c_2x^3 - \frac{1}{48}c_1x^4 + \frac{1}{384}x^5 + \dots$

$y = c_1\left[x - \frac{1}{2}x^2 + \frac{1}{8}x^3 - \frac{1}{48}x^4 + \frac{1}{384}x^5 + \dots\right]$

Ans

Ex:4 : Find first few terms in a power series expansion about x_0, for a general solution to

$(1+x^2)y'' - y' + y = 0.$

$\Rightarrow y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \dots$

$\Rightarrow y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \dots$

$\Rightarrow y'' = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \dots$

Substitute y, y' & y'' in the D.E.

$\Rightarrow y'' + x^2y'' - y' + y = 0$

$\Rightarrow 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 2c_2x^2 + 6c_3x^3 + 12c_4x^4 + 20c_5x^5 - c_1 - 2c_2x - 3c_3x^2 - 4c_4x^3 - 5c_5x^4 + c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 = 0$

$\Rightarrow c_0 - c_1 + 2c_2 + (6c_3 - 2c_2 + c_1)x + (12c_4 + 2c_2 - 3c_3 + c_2)x^2 + (20c_5 + 6c_3 - 4c_4 + c_3)x^3 + \dots = 0$

Compute each term with 0.

Date ______ 20__

$\Rightarrow c_0 - c_1 + 2c_2 = 0 \Rightarrow c_2 = (c_0 + \cancel{2}c_1)/2$

$\Rightarrow 6c_3 - 2c_2 + c_1 = 0 \Rightarrow c_3 = \frac{2c_2 - c_1}{6}$

$c_3 = \frac{c_0 + \cancel{c_1} - \cancel{c_1}}{6} \Rightarrow \boxed{c_3 = \frac{c_0}{6}}$

$\Rightarrow 12c_4 + 3c_2 - 3c_3 + \cancel{c_2} = 0$

$\cancel{12c_4 + 3c_2 - 3\left(\frac{-1}{6}c_0\right) = 0}$

$\Rightarrow 12c_4 + 3\left(\frac{-c_0 + c_1}{2}\right) - 3\left(\frac{-c_0}{6}\right) = 0$

$\Rightarrow 12c_4 - \frac{3}{2}c_0 + \frac{3}{2}c_1 + \frac{1}{2}c_0 = 0$

$\Rightarrow 12c_4 \; c_0 + \frac{3}{2}c_1 = 0$

$\Rightarrow c_4 = \frac{+c_0 - \frac{3}{2}c_1}{12} \Rightarrow \boxed{\frac{+2c_0 - 3c_1}{24} = c_4}$

Substitute c terms in y equation

$$\boxed{y = c_0 + c_1x + \left(\frac{-c_0 + c_1}{2}\right)x^2 - \frac{c_0}{6}x^2 + \left(\frac{2c_0 - 3c_1}{24}\right)x^3 + \dots}$$

BERNOULLI'S EQUATION

(Non-linear Differential Equations

Equations reducible to linear form.

Form ⇒ $\frac{dy}{dx} + P(x)y = Q(x)y^n$; $y \neq 0, 1$

① $y' + y = y^{-3}$.

$P(x) = 1$, $Q(x) = 1$, $n = -3$.

non-linear.

divide the equation by y^{-3}.

$\frac{1}{y^{-3}} y' + \frac{y}{y^{-3}} = \cancel{\frac{y^{-3}}{y^{-3}}}$

$\frac{1}{y^{-3}} y' + \frac{1}{y^{-4}} = 1$

$\frac{u'}{4} + u = 1$

Let $\boxed{u = \frac{1}{y^{-4}}} = y^4$

$\frac{du}{dx} = 4y^3 y'$

$u' = \frac{4y'}{y^{-3}}$

$\boxed{\frac{u'}{4} = \frac{1}{y^{-3}} y'}$

$u' + 4u = 4$. —②

Equation is linear now.

Find Integrating factor.

⇒ $\mu(x) = e^{\int P(x)\,dx}$

$\mu(x) = e^{\int 4\,dx} \Rightarrow \boxed{e^{4x}}$

multiply $\mu(x)$ with eq ②

⇒ $e^{4x}u' + e^{4x}4u = 4e^{4x}$.

L.H.S is total differential, i.e. it can be enclosed in $d/dx\,(u.v)$.

⇒ $\frac{d}{dx}(e^{4x}.u) = 4e^{4x}$.

⇒ $d(e^{4x}.u) = 4e^{4x}\,dx$.

Integrating both sides.

⇒ $\int d(e^{4x}.u) = 4\int e^{4x}\,dx$

⇒ $e^{4x}u = \cancel{4}\,e^{4x}/\cancel{4} + c$

⇒ $e^{4x}u = e^{4x} + c$

÷ by e^{4x}

$$\frac{\cancel{e^{4x}}\,u}{\cancel{e^{4x}}} = \frac{\cancel{e^{4x}}}{\cancel{e^{4x}}} + \frac{c}{e^{4x}}$$

$$u = 1 + ce^{-4x}.$$

$$\because u = y^4$$

$$\boxed{y^4 = 1 + ce^{-4x}.}$$

Ans.

(7) $y' + \left(\frac{2}{x}\right) y = x^2 y^3.$

Non-linear D.E.

divide the equation by y^3.

$$\frac{1}{y^3} y' + \left(\frac{2}{x}\right) \frac{1}{y^{\cancel{3}}} \cancel{y} = \frac{x^2 \cancel{y^3}}{\cancel{y^3}}$$

$$\frac{1}{y^3} y' + \frac{2}{x} \cdot \frac{1}{y^2} = x^2.$$

Let $\boxed{u = \frac{1}{y^2}} = y^{-2}$

$$\frac{du}{dx} = -2y^{-3}y'$$

$$u' = -2\frac{y'}{y^3}$$

$$\boxed{-\frac{u'}{2} = \frac{1}{y^3} y'}$$

$$\frac{-u'}{2} + \frac{2}{x} \cdot u = x^2$$

$$+u' - \frac{4}{x} u = -2x^2. \quad (7)$$

$P(x) = -4/x$

Equation is now a linear DE.

Finding integrating factor

$$\mu(x) = e^{\int -4/x \, dx} \Rightarrow e^{-4\int \frac{1}{x} dx} \Rightarrow e^{-4 \ln x} \Rightarrow e^{\ln x^{-4}} \Rightarrow \boxed{x^{-4}}$$

multiply $\mu(x)$ ~~with~~ by eq (7)

$$+u' x^{-4} - \frac{4}{\cancel{x}} x^{-4} u = -2x^2 \cdot x^{-4}$$

$$+u' x^{-4} - 4x^{-5} u = -2x^{-2}.$$

RHS is total differential.

$$\frac{d}{dx}(x^{-4} \cdot u) = -2x^{-2}.$$

$$d(x^{-4} \cdot u) = -2x^{-2} \, dx.$$

integrate both sides

$$\int d(x^{-4} \cdot u) = -2\int x^{-2} \, dx.$$

$$x^{-4} u = +2x^{-1} + c.$$

$\div$ by x^{-4}

Date

$$\frac{x^{-4} u}{x^{-4}} = \frac{2x^{-1}}{x^{-4}} + \frac{c}{x^{-4}}$$

$$u = 2x^{3} + cx^{4}.$$

$$\because u = \frac{1}{y^2}$$

$$\boxed{\frac{1}{y^2} = 2x^3 + cx^4}$$ A.

(3) $y' - 5y = -\frac{5}{2} xy^3.$

Non-linear Equation - Bernoulli's.

divide the equation by y^3.

$$\frac{1}{y^3} y' - \frac{5\cancel{y}}{y^{\cancel{3}2}} = \frac{-5}{2} \frac{x\cancel{y^3}}{\cancel{y^3}}$$

$$\frac{1}{y^3} y' - \frac{5}{y^2} = -\frac{5}{2} x.$$

Let $\boxed{u = \frac{1}{y^2}} = y^{-2}$

$$\frac{du}{dx} = -2y^{-3} y'$$

$$u' = -2 \cdot \frac{1}{y^3} y'$$

$$\boxed{-\frac{u'}{2} = \frac{1}{y^3} y'}$$

$$-\frac{u'}{2} - 5u = -\frac{5}{2} x.$$

$$u' + 10u = 5x. \longrightarrow (2)$$

Equation is linear now.

$P(x) = 10$

Finding Integrating factor.

$$\mu(x) = e^{\int 10 \, dx} \Rightarrow e^{10x}$$

multiply the $\mu(x)$ by eq (2)

$$u' \cdot e^{10x} + 10u e^{10x} = 5x e^{10x}.$$

$$\frac{d}{dx}(e^{10x} u)$$

$$e^{10x} u + u 10e^{10x}$$

LHS of the eq is total differential.

$$\frac{d}{dx}(u' \cdot e^{10x}) = 5x e^{10x}.$$

$$d(u \cdot e^{10x}) = 5x e^{10x} \, dx.$$

Integrating both sides.

$$\int d(u \cdot e^{10x}) = 5\int x e^{10x} \, dx.$$

$$u \cdot e^{10x} = 5 \quad \because \text{Solve using by parts.}$$

$$\int f g' = fg - \int f' g$$

$$u \cdot e^{10x} = 5\left[\frac{x e^{10x}}{10} - \frac{1}{10}\int e^{10x}\,dx\right].$$ $f = x,\ g' = e^{10x}$

$f' = 1,\ g = \frac{e^{10x}}{10}$

$$u \cdot e^{10x} = 5\left[\frac{x e^{10x}}{10} - \frac{1}{10}\frac{e^{10x}}{10}\right].$$

$$u \cdot e^{10x} = \frac{x e^{10x}}{2} - \frac{1}{20}e^{10x} + c.$$

$$u = \frac{x}{2} - \frac{1}{20} + ce^{-10x}.$$

$\because u = 1/y^2$

$$\boxed{\frac{1}{y^2} = \frac{x}{2} - \frac{1}{20} + ce^{-10x}.}$$

(2) $y' + 3x^2y = 4x^2y^2$

Bernoulli's Equation.

divide the D.E by y^2.

$$\frac{1}{y^2}y' + \frac{3x^2\cancel{y}}{y^{\cancel{2}}} = \frac{4x^2\cancel{y^2}}{\cancel{y^2}}$$

$$\frac{1}{y^2}y' + 3x^2 \cdot \frac{1}{y} = 4x^2$$

Let $\boxed{u = \frac{1}{y}} = y^{-1}$

$\frac{du}{dx} = -y^{-2}y'$

$u' = -\frac{1}{y^2}y'$

$\boxed{-u' = \frac{1}{y^2}y'}$

$$-u' + 3x^2u = 4x^2$$

$$u' - 3x^2u = -4x^2 \quad \text{(2)}$$

Eq is in linear form

$P(x) = -3x^2$

Finding Integrating factor

$$\mu(x) = e^{\int -3x^2\,dx} \Rightarrow e^{-\cancel{3}\frac{x^3}{\cancel{3}}} = \boxed{e^{-x^3}}$$

multiply $\mu(x)$ by eq (2).

$$u'e^{-x^3} - 3x^2ue^{-x^3} = -4x^2e^{-x^3}.$$

LHS is total differential.

$$\frac{d}{dx}\left(e^{-x^3}.u\right) = -4x^2e^{-x^3}.$$

$$d\left(e^{-x^3}.u\right) = -4x^2e^{-x^3}\,dx.$$

Integrating both sides.

$$\int d(e^{-x^3} \cdot u) = -4\int x^2 e^{-x^3}\,dx$$

$$u \cdot e^{-x^3}$$

Let $v = -x^3$

$$\frac{dv}{dx} = -3x^2$$

$$-\frac{dv}{3x^2} = dx.$$

$$u \cdot e^{-x^3} = -4 \cdot \int \cancel{x^2}\, e^{v} \cdot \left(-\frac{dv}{3\cancel{x^2}}\right)$$

$$u \cdot e^{-x^3} = \frac{4}{3} e^{v} + c.$$

$$u \cdot e^{-x^3} = \frac{4}{3} e^{-x^3} + c.$$

$\because u = 1/y$. $\div$ both sides by e^{-x^3}.

$$\boxed{\frac{1}{y} = \frac{4}{3} + c e^{x^3}}$$

⑤ $(x^2 y^4 - 1)\,dx + x^3 y^3\,dy = 0.$

$$x^3 y^3 \cdot dy = -(x^2y^4 - 1)\,dx.$$

$$x^3 y^3 \frac{dy}{dx} = -x^2 y^4 + 1.$$

$\div$ by $x^3 y^3$.

$$\frac{dy}{dx} = \frac{-\cancel{x^2} y^{\cancel{4}}}{x^{\cancel{3}} \cancel{y^3}} + \frac{1}{x^3 y^3}.$$

$$\frac{dy}{dx} = -\frac{y}{x} + x^{-3} y^{-3}.$$

$$\frac{dy}{dx} + \frac{1}{x} y = x^{-3} y^{-3}$$ Bernoulli's eq.

divide the equation by y^{-3}.

$$\frac{1}{y^{-3}} y' + \frac{1}{x}\frac{y}{y^{-3}} = x^{-3}$$

$$\frac{1}{y^{-3}} y' + \frac{1}{x}\frac{1}{y^{-4}} = x^{-3}$$

Let $\boxed{u = \frac{1}{y^{-4}}} = y^4$

$$\frac{du}{dx} = 4y^3 y'$$

$$\boxed{\frac{u'}{4} = \frac{y'}{y^{-3}}}$$

$$\frac{u'}{4} + \frac{1}{x} u = x^{-3}$$

$$u' + \frac{4}{x} u = 4x^{-3} \quad -(2)$$

linear equation.

$$P(x) = 4 \cdot \frac{1}{x}$$

Finding integrating factor.

$$\mu(x) = e^{4\int 1/x \, dx} \Rightarrow e^{4\ln x} \Rightarrow e^{\ln x^4} \Rightarrow \boxed{x^4}$$

multiply the integrating factor by eq.-(2).

$$x^4 \cdot u' + \frac{4}{\cancel{x}} x^{\cancel{4}\,3} u = 4x^{-3} \cdot x^4$$

$$4x^3 u \qquad\qquad 4x^1$$

The left hand side of eq is total differential

$$\frac{d}{dx}(x^4 u) = 4x$$

$$d(x^4 u) = dx\,(4x)$$

Integrating both sides.

$$\int d(x^4 u) = 4\int x \, dx$$

$$x^4 u = \cancel{4}^2 \frac{x^{+2}}{\cancel{+2}} + c$$

$$x^4 u = +2x^{+2} + c$$

$$u = +2\frac{x^{+2}}{x^4} + cx^{-4}$$

$$u = +2x^{-2} + cx^{-4}$$

$$\because u = y^4$$

$$\boxed{y^4 = +2x^{-2} + cx^{-4}}$$

Equations Reducible to First order.

① $y' y'' = x$.   Case I (y is not present)

Let $v = y'$ and $v' = y''$

$v \cdot v' = x$.

$v \cdot \frac{dv}{dx} = x$.

$v \cdot dv = x \; dx$

Integrating both sides

$\int v \cdot dv = \int x \; dx$

$\frac{v^2}{2} = \frac{x^2}{2} + c$

$v^2 = 2\left(\frac{x^2}{2} + c\right)$

$v^2 = x^2 + c$.

$v = \pm\sqrt{x^2 + c}$.

$\because v = y' = \frac{dy}{dx}$.

$\frac{dy}{dx} = \pm\sqrt{x^2+c}$.

$dy = \pm\sqrt{x^2+c} \; dx$

Integrating both sides.

$\int dy = \pm\int \sqrt{x^2+c} \; dx$

$$\boxed{y = \pm\int \sqrt{x^2+c} \; dx.}$$

② $y y' = y''$

Let $v = y'$ and $v' = y''$

$\therefore \frac{d^2y}{dx^2} = \frac{dv}{dx}$

Let $v = y'$

$v \cdot \frac{dv}{dy} = y''$ or $= \frac{dv}{dy} \cdot \frac{dy}{dx}$ $\because \frac{dy}{dx} = v$

$$\boxed{\frac{d^2y}{dx^2} = v \cdot \frac{dv}{dy}}$$

Date ____ 20__

$y\cancel{x} = \cancel{v}\dfrac{dv}{dy}$

$y\,dy = dv.$

Integrating both sides.

$\int y\,dy = \int dv.$

$\dfrac{y^2}{2} = v + c.$

$v = \dfrac{y^2}{2} + c. = \dfrac{y^2+c}{2}$

$\because v = dy/dx.$

$\dfrac{dy}{dx} = \dfrac{y^2+c}{2}.$

$2\cancel{\dfrac{1}{2}}\dfrac{dy}{y^2+c} = \dfrac{\cancel{1}}{2}\,dx.$

$= \int \dfrac{1}{x^2+a^2}\,dx.$

$= \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + c.$

Integrating both sides

$2\cancel{\dfrac{1}{7}}\int \dfrac{1}{y^2+c}\,dy = \int dx.$

$\cancel{\dfrac{1}{2}}2.\ \dfrac{1}{c}\tan^{-1}\left(\dfrac{y}{c}\right) = x + c.$

$\dfrac{1}{2c}\tan^{-1}\left(\dfrac{y}{c}\right) = x + c.$

$\dfrac{y}{c} = \tan(2c\cancel{x}+c)$

$\boxed{y = c\tan(2cx+c)}$ A.

3) $yy'' + (y')^{2} = yy'$

$yy'' + (y')^2 = yy'$

Let $v = y'$, $v' = y''.$

or $y'' = \dfrac{dv}{dy}v.$

$y.v.\dfrac{dv}{dy} + v^2 = y.v.$

$\cancel{v}\left(y\dfrac{dv}{dy} + v\right) = y\cancel{v}$

Date

$$y\frac{dv}{dy} + v = y$$

$\div$ the eq by y.

$$\frac{dv}{dy} + \frac{1}{y}v = 1 \quad \text{linear Eq.}$$

$$\mu(y) = e^{\int \frac{1}{y}dy} \Rightarrow e^{\ln y} \Rightarrow \boxed{y}$$

$$y\frac{dv}{dy} + v = y$$

$$\frac{d}{dy}(yv) = y.$$

$$d(yv) = y\,dy$$

$$\int d(yv) = \int y\,dy$$

$$yv = \frac{y^2}{2} + c.$$

$$v = \frac{y}{2} + cy^{-1} = \frac{y+cy^{-1}}{2}$$

$$\because v = \frac{dy}{dx}.$$

$$\frac{dy}{dx} = \frac{y + cy^{-1}}{2}$$

$$\int \frac{2}{y+cy^{-1}} = \int dx$$

$$2\int \frac{1}{y+\frac{c}{y}}dy = x + c.$$

$$2\int \frac{y}{y^2+c}dy = x + c.$$

Let $u = y^2 + c$

$$\frac{du}{dy} = 2y.$$

$$\frac{du}{2y} = dy.$$

$$\cancel{2}\int \frac{\cancel{y}}{u} \frac{du}{2\cancel{y}} = x + c.$$

$$\int \frac{1}{u}du = x + c.$$

$$\ln(u) = x + c$$

$$\ln(y^2 + c) = x + c.$$

$$e \quad e$$

$$\boxed{y^2 + c = e^x \cdot c}$$

Date ____ 20 __

(ii) $yy'' = (y')^2$

let $v = y'$ and $v' = y''$

$\Rightarrow y'' = v\frac{dv}{dy}$

$y \cdot v \cdot \frac{dv}{dy} = (v)^2$

$\int \frac{dv}{v} = \int \frac{dy}{y}$

$e^{\ln v} = e^{\ln y + c}$

$v = e^{\ln y} e^{c}$

$v = y\, c$

$\because v = dy/dx$

$\frac{dy}{dx} = y\, c$

$\int \frac{dy}{y} = c \int dx$

$e^{\ln y} = e^{xc}$

$$\boxed{y = e^{xc}}$$

Self-Study Problems

(a) $xy'' = y'$

Let $v = y'$ and $v' = y''$

$x v' = v$

$x \frac{dv}{dx} = v$

$\int \frac{1}{v} dv = \int \frac{1}{x} dx$

$e^{\ln v} = e^{\ln x + c}$

$v = x + c$

$\frac{dy}{dx} = x + c$

$dy = x + c\ dx$

$\int dy = \int x + c\ dx$

$$\boxed{y = \frac{x^2}{2} + c}$$

Maths & code

Date 20

(b) $yy'' + (y')^2 = 0$

Let $v = y'$, $v' = y''$

$y'' = v \cdot \frac{dv}{dy}$.

$yv \cdot \frac{dv}{dy} + (v)^2 = 0$.

$v\left(y \frac{dv}{dy} + v\right) = 0$.

$y \frac{dv}{dy} + v = 0$.

$\div$ by y.

$\frac{dv}{dy} + \frac{v}{y} = 0$.

I.F $\Rightarrow e^{\int \frac{1}{y} dy} = e^{\ln y} = \boxed{y}$.

$y \frac{dv}{dy} + v = 0$.

$\frac{d}{dy}(yv) = 0$.

$\int d(yv) = 0$.

$yv = \cancel{0}\ c$

$\therefore v = dy/dx$

$y \cdot \frac{dy}{dx} = c$.

$$\boxed{\frac{y^2}{2} = xc.}$$

(c) $xy'' + y' = 4x$.

Let $v = y'$ & $v' = y''$

$xv' + v = 4x$.

$\frac{\cancel{x} \frac{dv}{dx} + \frac{v}{x} = \frac{4x}{x}}{\cancel{x}}$

$\frac{dv}{dx} + \frac{v}{x} = 4$

IF $e^{\int \frac{1}{x} dx} = e^{\ln x} = \boxed{x}$

$x \cdot \frac{dv}{dx} + \frac{v}{\cancel{x}} \cancel{x} = 4x$

$x \frac{dv}{dx} + v = 4x$

$\frac{d}{dx}(xv) = 4x$

$\int d(xv) = \int 4x\, dx$

$xv = \frac{4x^2}{2} + c$.

$v = 2x + cx^{-1}$

$\frac{dy}{dx} = 2x + cx^{-1}$

$dy = \int 2x + cx^{-1}\, dx$.

$y\ \cancel{dy} = x^2 + c\ln(x)$

$\boxed{y = \cancel{3x}}$

(d) $(x^2-1)y'' + 2y' = 0$

$x^2y'' - y'' + 2y' = 0$

Let $v = y'$ & $v' = y''$

$x^2v' - v' + 2v = 0$

$v'(x^2-1) = -2v$.

$\frac{dv}{dx}(x^2-1) = -2v$

$\int -\frac{1}{2v} dv = \int \frac{1}{x^2-1} dx$

$-\frac{1}{2}\ln v = \cancel{\frac{1}{2(1)}\tan^{-1}\left(\frac{x}{1}\right)} + c$

$\frac{1}{2(1)} \ln\left|\frac{x-1}{x+1}\right| + c$

$$\boxed{-\frac{1}{2}\ln v = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c.}$$

$e^{\ln v^{-1/2}} = e^{\ln\left(\frac{x-1}{x+1}\right)^{1/2}} + c$

$v^{-1/2}$

$$\frac{-2y^2+1}{y}$$ $\frac{du}{dy} = -4y$. $-\frac{du}{4y} = dy$.

Date ____ 20__

$x^2y'' + 3xy' = 2$.

let $v = y'$ & $v' = y''$

$x^2v' + 3xv = 2$.

$x(xv' + 3v) = 2$.

$$\frac{\cancel{x}v'}{\cancel{x}} + \frac{3v}{x} = \frac{2}{x \cdot x}$$

$$v' + \frac{3}{x}v = \frac{2}{x^2}.$$

$$e^{\int \frac{3}{x} dx} \Rightarrow e^{3\ln x} \Rightarrow e^{\ln x^3} \Rightarrow \boxed{x^3}$$

$$x^3v' + \frac{3}{\cancel{x}}x^{\cancel{3}}v = \frac{2}{\cancel{x^2}}x^3$$

$$x^3v' + 3x^2v = 2x.$$

$$\frac{d}{dx}(x^3 \cdot v) = 2x$$

$$\int d(x^3 \cdot v) = \int 2x\,dx$$

$$x^3 \cdot v = \frac{\cancel{2}x^2}{\cancel{2}} + c$$

$$\frac{\cancel{x^3} \cdot v}{\cancel{x^3}} = \frac{x^2}{x^3} + \frac{c}{x^3}.$$

$$v = x^{-1} + cx^{-3}$$

$$\frac{dy}{dx} = x^{-1} + cx^{-3}.$$

$$\int dy = \int \frac{1}{x}dx + c\int \frac{1}{x^3}dx.$$

$$\boxed{y = \ln x \cdot \cancel{+} - c\frac{x^{-2}}{2}}$$ A

$$\frac{dv}{dy} = 2y \cdot v - \frac{1}{y}v.$$

$$\frac{dv}{dy} - \left(2y - \frac{1}{y}\right)v = 0.$$

IF $\Rightarrow P(x) = 2y - \frac{1}{y}$.

$$e^{\int -2y + \frac{1}{y}dy}$$

$$e^{-y^2 + \ln y}.$$

$$\frac{e^{\cancel{y^2}}}{\cancel{e^y}} \quad e^{-y^2} \cdot e^{\ln y} \quad \boxed{ye^{-y^2}}$$

$$ye^{-y^2}\frac{dv}{dy} - \left(2y - \frac{1}{y}\right)ye^{-y^2}v = 0$$

$$ye^{-y^2}\frac{dv}{dy} - (2y^2 - 1)e^{-y^2}v = 0$$

$$\frac{d}{dy}(ye^{-y^2} \cdot v) = 0.$$

$$\int d(ye^{-y^2} \cdot v) = 0$$

$$ye^{-y^2} \cdot v = 0$$

$$ye^{-y^2} \cdot \frac{dy}{dx} = 0.$$

$$\int ye^{-y^2}dy = 0$$

Let $u = -y^2$.

$$\frac{du}{dy} = -2y \Rightarrow \boxed{\frac{du}{-2y} = dy}$$

$$\int \cancel{y} \cdot e^u \cdot \frac{du}{-2\cancel{y}} = 0.$$

$$\Rightarrow x^{-3+1} - \frac{1}{2}e^u = c.$$

$$\boxed{-\frac{1}{2}e^{-y^2} = c}$$ A

$$y'' = \left(2y - \frac{1}{y}\right)(y')^2.$$

let $v = y'$ & $y'' = v \cdot \frac{dv}{dy}$

$$\cancel{v}\frac{dv}{dy} = \left(2y - \frac{1}{y}\right)v^{\cancel{2}}$$

Second Order linear Non-homogenous DEs

Quiz question.

$y'' + 3y' + 2y = 2^x.$

Finding homogenous Solution

$y'' + 3y' + 2y = 0.$

Auxillary equation: $r^2 + 3r + 2 = 0.$

$$\frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2}$$

$$r_1 = \frac{-3+1}{2} = -1 \quad , \quad r_2 = \frac{-3-2}{2} = -2.$$

$$y_h = c_1 e^{-x} + c_2 e^{-2x}.$$

For particular solution

$y'' + 3y' + 2y = 2^x$ $\qquad \because 2^x \Rightarrow e^{\ln(2^x)}$

$y'' + 3y + 2y = e^{\ln(2)x}.$ $\qquad e^{x\ln(2)}$

$D^2y + 3Dy + 2y = e^{\ln(2)x}$ $\qquad$ or $e^{\ln(2)x}$

$y(D^2 + 3D + 2) = e^{\ln(2)x},$

$$y_p = \frac{1}{D^2+3D+2} e^{\ln(2)x},$$

$D \to a \Rightarrow D \to \ln(2).$

$$y_p = \frac{1}{(\ln(2))^2 + 3(\ln(2)) + 2} e^{\ln(2)x}.$$

$$y_p = \frac{1}{4.56} e 2^x$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{4.56} 2^x.}$$

Reduction of Order.

It is for linear Homogeneous Differential Equations

Let one of the solution is given

$\Rightarrow y = c_1 y_1(x) + c_2 y_2(x)$.

Formula: $y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$.

General form: $y'' + P(x)y' + Q(x)y = 0$

Q: If $y_1 = x^2$. Find a solution of $x^2y'' - 3xy' + 4y = 0$. Find the general solution.

$\Rightarrow x^2y'' - 3xy' + 4y = 0$

$\div$ the equation by x^2.

$\Rightarrow y'' - \frac{3\cancel{x}y'}{x^{\cancel{2}}} + \frac{4y}{x^2} = 0$

$\Rightarrow y'' - \frac{3}{x}y + \frac{4}{x^2}y = 0$.

$P(x) = -\frac{3}{x}$.

$y_2 = x^2 \int \frac{e^{-\int -3/x\,dx}}{(x^2)^2}\,dx$.

$y_2 = x^2 \int \frac{e^{3\int 1/x\,dx}}{x^4}\,dx$

$y = x^2 \int \frac{e^{3\ln x}}{x^4}\,dx$

$y = x^2 \int \frac{\cancel{e^{\ln x^3}}}{x^{\cancel{4}}}\,dx$

$y_2 = x^2 \int \frac{1}{x}\,dx$

$\boxed{y_2 = x^2 \ln x}$

$\boxed{y = c_1 x^2 + c_2 x^2 \ln x}$ Ans.

Assignment Question:

$y_1 = x$

$x^2 y'' - x(x+2) y' + (x+2) y = 0$

$\div$ the equation by x^2

$y'' - \frac{\cancel{x}(x+2)}{x^{\cancel{2}}} y' + \frac{x+2}{x^2} y = 0$

$y'' - \frac{x+2}{x} y' + \frac{x+2}{x^2} y = 0$

$P(x) = -\frac{x+2}{x}$

$y_2 = x \int \frac{e^{\cancel{-}\int \left(\cancel{-}\frac{x+2}{x}\right) dx}}{x^2} dx$

$y_2 = x \int \frac{e^{\int 1 + \frac{2}{x} dx}}{x^2}$

$y_2 = x \int \frac{e^{x + 2\ln x}}{x^2} dx$

$y_2 = x \int \frac{e^x \cdot \cancel{e}^{\cancel{\ln} x^2}}{x^2} dx$

$y_2 = x \int \frac{e^x \cdot \cancel{x^2}}{\cancel{x^2}} dx$

$y_2 = x e^x + c$

$\boxed{y = c_1 x + c_2 x e^x}$

$y'' + 8y' + 15y = 0 \qquad y_1 = e^{-3x}$

$P(x) = 8$

$y_2 = e^{-3x} \int \frac{e^{-\int 8 dx}}{(e^{-3x})^2} dx \Rightarrow e^{-3x} \int \frac{e^{-8x}}{e^{-6x}} dx$

$y_2 = e^{-3x} \int e^{-8x+6x} dx$

$y_2 = e^{-3x} \int e^{-2x} dx \Rightarrow e^{-3x} \left(-\frac{e^{-2x}}{2} \right)$

$y\cancel{2} = \cancel{e}^{-3x} \cancel{\frac{1}{14}} e^{\cancel{14}x}$

$\boxed{-\frac{1}{2} e^{-5x}}$

c_2

$y'' + 3y' + 2y = 0$; $y_1 = e^{-2x}$.

$p(x) = 3$.

$$y_2 = e^{-2x} \int \frac{e^{-\int 3\,dx}}{(e^{-2x})^2}\, dx.$$

$$y_2 = e^{-2x} \int \frac{e^{-3x}}{e^{-4x}}\, dx.$$

$$y_2 = e^{-2x} \int e^{+x}\, dx.$$

$$y_2 = e^{-2x} \cdot e^{+x}.$$

$$\boxed{y_2 = e^{-x}}$$

A

Date ______ 20

Non-homogeneous.

(7) $y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$

Finding homogeneous solution

A.E $\Rightarrow r^2 - 3r + 2 = 0$

$\dfrac{+3 \pm \sqrt{9-8}}{2} \Rightarrow \dfrac{3 \pm 1}{2} \Rightarrow r_1 = 2, \; r_2 = 1$

$y_h = c_1 e^{x} + c_2 e^{2x}$.

Finding particular solution

$D^2y - 3Dy + 2y = 3e^{-x} - 10\cos 3x$

$(D^2 - 3D + 2)y = 3e^{-x} - 10\cos 3x$.

$y_p = \dfrac{1}{D^2 - 3D + 2}\left(3e^{-x} - 10\cos 3x\right)$

$y_p = \dfrac{1}{D^2-3D+2} 3e^{-x} - \dfrac{1}{D^2-3D+2} 10\cos 3x$

$D \to -1 \qquad D^2 \to -9$.

$y_p = \dfrac{1}{(-1)^2 - 3(-1) + 2} 3e^{-x} - \dfrac{1}{-9 - 3D + 2} 10\cos 3x$

$1+3+2$

$y_p = \dfrac{1}{\cancel{6}\,2} \cancel{3}e^{-x} - \dfrac{1}{-3D - 7} 10\cos 3x$

$y_p = \dfrac{1}{2} e^{-x} - \left(-\dfrac{1}{3}\right) \dfrac{1}{D + \frac{7}{3}} 10\cos 3x$.

$y_p = \dfrac{1}{2} e^{-x} + \dfrac{1}{3} \dfrac{1}{D + \frac{7}{3}} \times \dfrac{D - \frac{7}{3}}{D - \frac{7}{3}} \cdot 10\cos 3x$

$y_p = \dfrac{1}{2} e^{-x} + \dfrac{1}{3} \dfrac{D - 7/3}{D^2 - \frac{49}{9}} \cdot 10\cos 3x$

$y_p = \dfrac{1}{2} e^{-x} + \dfrac{1}{3} \dfrac{D - 7/3}{-9 - \frac{49}{9}} 10\cos 3x$

$y_p = \dfrac{1}{2} e^{-x} + \dfrac{1}{\cancel{3}1} \dfrac{D - 7/3}{-130/\cancel{9}3} 10\cos 3x$

$y_p = \dfrac{1}{2} e^{-x} - \dfrac{3}{130} \left(D - \dfrac{7}{3}\right)\left(10\cos 3x\right)$

$y_p = \dfrac{1}{2} e^{-x} - \dfrac{3}{130}\left[-30\sin 3x - \dfrac{70}{3}\cos 3x\right]$

Date ______ 20 ___

$$y_p = \frac{1}{2}e^{-x} + \frac{9}{13}\sin 3x + \frac{7}{13}\cos 3x.$$

$$y = c_1e^{x} + c_2e^{2x} + \frac{1}{2}e^{-x} + \frac{9}{13}\sin 3x + \frac{7}{13}\cos 3x.$$

A

(8) $y'' + 6y' + 13y = e^{-3x}\cos 2x.$

$D_y^2 + 6D_y + 13_y = e^{-3x}\cos 2x$

$(D^2 + 6D + 13)y = e^{-3x}\cos 2x.$

$$y_p = \frac{1}{D^2 + 6D + 13} e^{-3x}\cos(2x)$$

$D \to D + a = D - 3$

$$y_p = e^{-3x}\frac{1}{(D-3)^2 + 6(D-3) + 13}\cancel{e^{-3x}}\cos(2x)$$

$$y_p = e^{-3x}\frac{1}{D^2 - \cancel{6D} + 9 + \cancel{6D} - 18 - 13}\cancel{e^{-3x}}\cos(2x)$$

$$y_p = e^{-3x}\frac{1}{D^2 - 22}\cos(2x)$$

$D^2 \to -4$

$$y_p = e^{-3x}\frac{1}{-4-22}\cos(2x).$$

$$y_p = e^{-3x}\frac{1}{-26}\cos(2x)$$

(9) $y'' + y = \tan x.$

$r^2 + 1 \quad 0.$

$r^2 = -1$

$\cancel{r} = \pm\sqrt{-1}$

$r = -i$

$y_h = c_1\cos(x) + c_2\sin(x).$

$y_1 = \cos(x)$ $\quad w(x) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$

$y_2 = \sin(x)$

$= \cos^2 x + \sin^2 x = 1$

$$\therefore \; y_p = -y_1 \int \frac{f(x) y_2}{w(x)} dx + y_2 \int \frac{f(x)\, y_1}{w(x)} dx$$

$$y_p = -\cos(x) \int \frac{\tan(x)\sin(x)\,dx}{1} + \sin(x) \int \frac{\tan(x)\cos(x)}{1} dx$$

$$y_p = -\cos(x) \int \frac{\sin(x)}{\cos(x)} \sin(x)\, dx + \sin(x) \int \frac{\sin(x)}{\cancel{\cos(x)}} \cdot \cancel{\cos(x)}\, dx$$

$$y_p = -\cos(x) \int \frac{\sin^2(x)\, dx}{\cos(x)} + \sin(x) \int \sin(x)\, dx.$$

$$y_p = -\cos(x) \int \frac{\sin^2(x)}{\cos(x)} dx + \sin(x) \left(-\cos(x)\right) + c.$$

Let $u = \sin(x)$

$$y_p = -\cos(x) \int \sin^2(x) \sec(x)\, dx - \sin(x)\cos(x).$$

$$\int (1 - \cos^2(x)) \sec(x)\, dx$$

$$\int \sec(x) - \cos(x)\, dx.$$

$$\int \sec(x) - \int \cos(x)\, dx$$

$$y_p = -\cos x \left[\ln|\sec(x) + \tan(x)| \right] - \sin(x)\cos(x).$$

Date ____ 20__

Reducible to first order

(b) $y y'' = 3(y')^2$

let $v = y'$ & $y'' = v'$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} \Rightarrow \frac{dv}{dy} \cdot \frac{dy}{dx}$$

$$y'' = v \cdot \frac{dv}{dy}. \qquad \Rightarrow \frac{dv}{dy} \cdot v$$

$$y \cdot \cancel{v} \frac{dv}{dy} = 3v^{\cancel{2}}$$

$$y \cdot \frac{dv}{dy} = 3v.$$

$$\int \frac{1}{v} dv = \int \frac{3}{y} dy.$$

$$\ln(v) = 3\ln(y) + c.$$

$$e^{\ln v} = e^{\ln y^3 + c}$$

$$v = e^{\ln y^3} e^c$$

$$v = y^3 c.$$

$$\frac{dy}{dx} = y^3 c.$$

$$\int \frac{1}{y^3} dy = \int c\, dx$$

$$\int y^{-3} dy = xc.$$

$$\frac{y^{-2}}{-2} = xc.$$

$$-\frac{1}{2} \cdot y^{-2} = xc.$$

l) $(\tan x)\, y'' = y'$

Let $v = y'$ ~~or~~ $v' = y''$

$(\tan x)\, v' = v$.

$\tan x \cdot \frac{dv}{dx} = v$.

Let $u = \sin x$

$\frac{du}{dx} = \cos x$

$\frac{du}{\cos x} = dx$

$\int \frac{1}{v}\, dv = \int \frac{1}{\tan x}\, dx$

$\ln v = \int \frac{\cos x}{\sin x}\, dx$

$\ln v = \int \frac{\cancel{\cos x}}{u} \cdot \frac{du}{\cancel{\cos x}}$

$e^{\ln v} = e^{\ln u + c} \Rightarrow e^{\ln(\sin x) + c}$.

$v = \sin(x) \cdot c$.

$\frac{dy}{dx} = \sin(x)\, c$. $\Rightarrow$

$\int dy = \int \sin(x)\, c\, dx$

$\boxed{y = -\cos(x) \cdot c}$

(b) $y y'' + (y')^2 = 0$

Let $v = y'$ & $y'' = v \frac{dv}{dy}$

$y \cdot v \cdot \frac{dv}{dy} + (v)^2 = 0$

$v\left(y \frac{dv}{dy} + v\right) = 0$

$y \frac{dv}{dy} + v = 0$

$v = -y \frac{dv}{dy}$

$-\frac{1}{y}\, dy = \frac{1}{v}\, dv$

$-\ln y + c = \ln v$

or $e^{\ln v} = e^{\ln y^{-1} + c}$

$v = y^{-1} c$.

$\frac{dy}{dx} = y^{-1} c$

$\int y\, dy = \int c\, dx$

$\boxed{\frac{y^2}{2} = xc}$

Date ______ 20 ___

$y'' = 2yy'$

let $y' = v$ & $y'' = v\dfrac{dv}{dy}$

$\cancel{v}\cdot\dfrac{dv}{dy} = 2y\cancel{v}$

$\int dv = \int 2y\,dy$

$v = \dfrac{2y^2}{2} + c$

$v = y^2 + c$

$\dfrac{dy}{dx} = y^2 + c$

$\int \dfrac{dy}{y^2+c^2} = \int dx \quad \because \int \dfrac{1}{x^2+a^2}dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$

$\dfrac{1}{c}\tan^{-1}\left(\dfrac{y}{c}\right) = x + c$

$\tan^{-1}\left(\dfrac{y}{c}\right) = xc + c$

$\dfrac{y}{c} = \tan(xc+c)$

$y = c\tan(x+c)$. A.

(1) $y^3 y'' = 1$

let $y' = v$ & $y'' = v\dfrac{dv}{dy}$.

$y^3 \cdot v \cdot \dfrac{dv}{dy} = 1$

$\int v\,dv = \int \dfrac{1}{y^3}dy$

$\dfrac{v^2}{2} = \int y^{-3}\,dy$

$\dfrac{v^2}{2} = \dfrac{1}{-2}\dfrac{1}{y^2} + c.$

$v^2 = 2\left(-\dfrac{1}{2}\dfrac{1}{y^2} + c\right)$

$v^2 = -\dfrac{1}{y^2} + c$

d) $(x^2-1)y'' + 2y' = 0$

Let $y' = v$ & $y'' = v'$

$(x^2-1)v' + 2v = 0.$

$(x^2-1)\frac{dv}{dx} + 2v = 0.$

$2v = -(x^2-1)\frac{dv}{dx}$

$\int \frac{2}{1-x^2}dx = \int \frac{1}{v}dv$

$2\int \frac{1}{1-x^2}dx = \ln v + c.$

$\because \int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

$\cancel{2}\cdot\frac{1}{\cancel{2}}\ln\left|\frac{1+x}{1-x}\right| = \ln v + c$

$\ln\left(\frac{1+x}{1-x}\right) = \ln v + c.$

$e^{\ln v} = e^{\ln\left(\frac{1+x}{1-x}\right)} + c$

$v = \frac{1+x}{1-x} + c.$

$\frac{dy}{dx} = \frac{1+x}{1-x} + c.$

(h) $y'' = 2y(y')^3$

Let $y' = v$ & $y'' = v\frac{dv}{dy}.$

$\cancel{v}\frac{dv}{dy} = 2y(v)^{\cancel{3}}$

$\frac{dv}{dy} = 2yv^2$

$\int \frac{1}{v^2}dv = \int 2y\,dy$

$\int v^{-2+1}dv = \frac{2y^2}{2} + c$

$\frac{v^{-1}}{-1} = y^2 + c$

$-\frac{1}{v} = y^2 + c$

$-\frac{1}{y^2+c} = v.$

$\frac{dy}{dx} = -\frac{1}{y^2+c}$

$\int y^2 + c\,dy = -1\int dx$

$\frac{y^3}{3} + cy = -x$

Variation of parameters

Date ______ 20 ___

$y'' + 4y = \tan(2x)$

for y_h

$r^2 + 4 = 0$

$r^2 + 2^2 = 0$

$\sqrt{r^2} = \sqrt{-2^2}$

$r = \pm 2i$

$y_h = c_1 \cos(2x) + c_2 \sin(2x)$

$$W(x) = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix}$$

$= 2\cos^2(2x) + 2\sin^2(2x) = 1$

$= 2$.

$$y_p = -\cos(2x)\int \frac{\tan(2x)\cdot\sin(2x)}{2}\,dx + \sin(2x)\int \frac{\tan(2x)\cos(2x)}{2}\,dx$$

$$y_p = -\cos(2x)\frac{1}{2}\int \frac{\sin(2x)}{\cos(2x)}\sin(2x)\,dx + \sin(2x)\frac{1}{2}\int \frac{\sin(2x)}{\cancel{\cos(2x)}}\cancel{\cos(2x)}\,dx$$

$$y_p = -\cos(2x)\frac{1}{2}\int \frac{\sin^2(2x)}{\cos(2x)}\,dx + \sin(2x)\cdot\frac{1}{2}\int \sin(2x)\,dx$$

$$y_p = -\cos(2x)\frac{1}{2}\int \sec(2x)\cdot\sin^2(2x)\,dx + \sin(2x)\frac{1}{2}(-\cos(2x)).$$

$$y_p = -\cos(2x)\frac{1}{2}\int (1-\cos^2 2x)\sec(2x)\,dx + -\frac{1}{2}\sin(2x)\cos(2x)$$

$\because \cos^2(2x)\cdot\frac{1}{\cancel{\cos(2x)}}$

$$y_p = -\cos(2x)\frac{1}{2}\int \sec 2x - \cos 2x\,dx - \frac{1}{2}\sin(2x)\cos(2x).$$

$$y_p = -\cos(2x)\cdot\frac{1}{2}\cdot\frac{\ln|\sec(2x)+\tan(2x)|}{2} - \frac{\sin(2x)}{2} - \frac{1}{2}\frac{\sin(2x)}{\cos(2x)}$$

$$y_p = -\cos(2x)\cdot\frac{1}{4}\ln|\sec(2x)+\tan(2x)| + \frac{\cos(2x)\cancel{\sin(2x)}}{\cancel{2}} - \frac{\cancel{\sin(2x)}}{\cancel{\cos(2x)}\,/2}$$

$$\boxed{y_p = -\frac{1}{4}\cos(2x)\cdot\ln|\sec(2x)+\tan(2x)|}$$

Ans

take $\int \sec 2x\,dx$

$= \frac{\ln|\tan(2x)+\sec(2x)|}{2}$

$u = \cos(2x)$ $\frac{\sin(2x)}{\cos(2x)}$

$\frac{du}{dx} = -2\sin(2x)$ $\frac{\sin^2(2x)}{u} \cdot \frac{du}{-2\sin(2x)}$

Date 20

$\int f g' = fg - \int f' g$

$f = \frac{1}{u}$ $g' = \sin(2x)$

$f' = \ln u$ $g =$

$\int \tan(2x)\sin(2x)\,dx$

$\int \frac{\sin(2x)}{\cos(2x)} \sin(2x)\,dx$

$\int \frac{\sin^2(2x)}{\cos(2x)}\,dx$

$\int \sec(2x)\cdot\sin^2(2x)\,dx$

$\int (1-\cos^2(2x))\sec(2x)\,dx$

$\int \sec(2x) - \cos(2x)\,dx$

$\int \sec(2x)\,dx - \int \cos(2x)$

$\int \sec(2x)\left(\frac{\tan(2x)+\sec(2x)}{\tan(2x)+\sec(2x)}\right) - \frac{\sin(2x)}{2}$

$\int \frac{\sec(2x)\tan(2x) + \sec^2(2x)}{\tan(2x)+\sec(2x)} - "$

let $u = \tan(2x) + \sec(2x)$

$\frac{du}{dx} = \frac{2\sec^2(2x)}{\cancel{2}} + \frac{2\sec(2x)\tan(2x)}{\cancel{2}}$

$\frac{du}{dx} = \frac{2(\sec^2(2x) + \sec(2x)\tan(2x))}{\cancel{2}}$

$\frac{\cancel{2}\,du}{2(\sec^2(2x)+\sec(2x)\tan(2x))} = dx$

$\int \frac{\sec(2x)\cdot\tan(2x)+\sec^2(2x)}{u} \cdot \frac{\cancel{2}\,du}{2(\sec^2(2x)+\sec(2x)\tan(2x))}$

$\frac{1}{2}\ln u$

$\boxed{\frac{1}{2}\ln(\sec x + \tan x)}$ A

Reduction of Order. (13)

Date ______ 20 ___

$6y'' + y' - y = 0 \quad ; \quad y_1 = e^{2x/3}.$

$\div$ by 6.

$y'' + \frac{1}{6}y' - \frac{1}{6}y = 0$

$P(x) = \frac{1}{6} \quad , \quad y_1 = e^{\frac{2x}{3}}.$

$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx.$

$y_2 = e^{\frac{2x}{3}} \int \frac{e^{-\int \frac{1}{6}\,dx}}{(e^{\frac{2x}{3}})^2}\,dx$

$y_2 = e^{\frac{2x}{3}} \int \frac{e^{-\frac{1}{6}x}}{e^{\frac{4x}{6}}}\,dx.$

$y_2 = e^{\frac{2x}{3}} \int \frac{e^{-\frac{1}{6}x}}{e^{4x/6}}\,dx$

$y_2 = e^{\frac{2x}{3}} \int \frac{e^{-1/6 x}}{e^{4x/6}}\,dx.$

$y_2 = e^{\frac{x}{3}} \int e^{-\frac{x}{6} - \frac{2x}{6}}\,dx$

$y_2 = e^{\frac{x}{3}} \int e^{-\frac{x}{2}}\,dx.$

$y_2 = e^{\frac{x}{3}} \cdot -2e^{-x/2}$

$y_2 = -\frac{2}{1}e^{-\frac{x}{6}}$

$y_2 = e^{\frac{x}{3}} \int \frac{e^{-\int \frac{1}{6}\,dx}}{(e^{\frac{x}{3}})^2}\,dx.$

$y_2 = e^{\frac{x}{3}} \int \frac{e^{-\frac{1}{6}x}}{e^{\frac{2x}{3}}}\,dx$

$y_2 = e^{\frac{x}{3}} \int e^{-5/6 x}\,dx.$

$y_2 = e^{\frac{x}{3}} \cdot -\frac{6}{5}e^{-\frac{5}{6}x} + c$

$\boxed{y_2 = -\frac{6}{5}e^{-\frac{1}{2}x}} + c$

$y = c_1 e^{\frac{x}{3}} - \frac{2}{1}c_2 e^{-x/6}$

$\boxed{y = c_1 e^{\frac{x}{3}} - \frac{6}{5}c_2 e^{-\frac{x}{2}}}$

) $(1-x^2)y'' + 2xy' = 0 \quad ; \quad y_1 = 1$

$\div$ by $1-x^2$

$y'' + \frac{2x}{1-x^2}y' = 0.$

$P(x) = \frac{2x}{1-x^2}$

$\int \frac{2x}{1-x^2}\,dx. \quad$ let $u = 1-x^2$

$\frac{du}{dx} = -2x$

$\int \frac{-2x}{u}\frac{du}{-2x}$

$\ln(1-x^2)$

$y_2 = 1\int \frac{e^{\ln(1-x^2)}}{1^2}\,dx$

$y_2 = \int 1-x^2\,dx.$

$y_2 = x - \frac{x^3}{3}$

$\boxed{y = c_1 + c_2\left(x - \frac{x^3}{3}\right)}$

(15) $(1-2x-x^2)y'' + 2(1+x)y' - 2y = 0$; $y_1 = x+1$

$\div$ by $1-2x-x^2$.

$$y'' + \frac{2(1+x)}{1-2x-x^2}y' - \frac{2y}{1-2x-x^2}y = 0.$$

$$P(x) = \frac{2+2x}{1-2x-x^2}$$

$$-\int \frac{2+2x}{1-2x-x^2}\,dx \qquad \text{Let } u = 1-2x-x^2$$

$$\frac{du}{dx} = -2-2x.$$

$$\cancel{+}\int \frac{2+2x}{u}\,\frac{du}{\cancel{-}(2+2x)} \qquad \frac{du}{dx} = -(2+2x)$$

$$\ln(u) = \ln(1-2x-x^2).$$

$$y_2 = x+1\int \frac{e^{\ln(1-2x-x^2)}}{(x+1)^2}\,dx.$$

$$y_2 = x+1\int \frac{1-2x-x^2}{x^2+2x+1}\,dx.$$

$$y_2 = x+1\int \frac{-(x^2+2x-1)}{x^2+2x+1}\,dx$$

$$= x+1\int \frac{-x^2-2x+1}{x^2+2x+1}\,dx$$

(7) $9y'' - 12y' + 4y = 0$. $y_1 = e^{2x/3}$

$$y'' - \frac{\cancel{12}^{4}}{\cancel{9}_{3}}y' + \frac{4}{9} = 0$$

$$P(x) = -\frac{4}{3}$$

$$-\frac{4}{3} - \frac{4}{6}$$

$$y_2 = e^{2x/3}\int \frac{e^{-\int -\frac{4}{3}dx}}{(e^{\frac{2x}{3}})^2}\,dx$$

$$y_2 = e^{\frac{2x}{3}}\int \frac{e^{+\frac{4}{3}x}}{e^{\frac{4}{6}x}}\,dx \Rightarrow y_2 = e^{\frac{2x}{3}}\int e^{\frac{2}{3}x}\,dx$$

$$y_2 = e^{\frac{2x}{3}}\cdot\frac{3}{2}e^{\frac{2}{3}x}$$

$$y = \frac{3}{2}e^{\frac{4}{3}x}$$

$$y_2 = e^{\frac{2x}{3}}\int e^{+\frac{2x}{3}}\,dx$$

$$y = e^{2x/3}\,\frac{3e^{+\frac{2}{3}x}}{2\cancel{-}\cancel{2}} + c$$

$$y = +\frac{3}{2}e^{-4x/3}$$

Date ____ 20 __

$y'' - 9y = 0 \; ; \; y_1 = e^{3x}$

$P(x) = 0$

$y_2 = e^{3x} \int \frac{e^{\int 0\, dx}}{(e^{3x})^2} dx$

$y_2 = e^{3x} \int \frac{e^{c}}{e^{6x}} dx$

$y_2 = e^{3x} e^{c} \int e^{-6x} dx.$

$y_2 = ce^{3x} - \frac{1}{6} e^{-6x}$

$\boxed{y_2 = \frac{-1}{6} e^{-3x}}$

$y'' + 16y = 0 \; ; \; y_1 = \cos 4x$

$P(x) = 0$

$y_2 = \cos(4x) \int \frac{e^{-\int 0\, dx}}{(\cos 4x)^2} dx$

$y_2 = \cos(4x) \int \frac{e^{-0+c}}{(\cos 4x)^2} dx.$

$y_2 = \cos(4x) e^{c} \int \frac{\cancel{e^{c}}\, 1}{\cos^2 4x} dx$

$y_2 = \cos(4x) e^{c} \int \sec^2 4x \, dx$

$y_2 = \cos(4x) e^{c} \cdot \tan(4x)$

$y_2 = \cancel{\cos(4x)}\, e^{c} \cdot \frac{\sin(4x)}{4}$

$y_2 = \frac{c}{4} \sin(4x)$ ~~4 cos(4x)~~

$4x^2 y'' + y = 0 \quad , \quad y = x^{1/2} \ln x.$

$4y'' + \frac{1}{4x^2} y = 0$

$P(x) = \cancel{\frac{1}{4x^2}} \qquad -\int \frac{1}{4x^2} dx \Rightarrow -\int 4x^{-2}$

$y_p = \cancel{y_1}\; x^{1/2} \ln(x) \int \frac{e^{-\int 0\, dx}}{(x^{1/2} \ln(x))^2} dx$

$y_p = x^{1/2} \ln(x) \int \frac{e^{c}}{x (\ln(x))^2} dx$ let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$y_p = x^{1/2} \ln(x)\, e^{c} \int \frac{1}{x (\ln x)^2} dx$ $\frac{x\,du}{\cancel{dx}} \cdot dx$

$\int \frac{1}{\cancel{x} u^2} \cdot \cancel{x}\, du$

$\int u^{-2+1} du$

$y_p = \quad -\frac{1}{u}.$

$y_p = x^{1/2} \cancel{\ln(x)}\, e^{c} \left\{ -\frac{1}{\cancel{\ln x}} \right.$

$y_p = -x^{1/2} \cdot e^{c}$

Cauchy - Euler's.

① $x^2y'' + 5xy' + 4y = x^4 \cdot \ln x$.

Let $x = e^z \rightarrow z = \ln x$.

and $x^2y'' = D^2y - D^4$ and $xy' = Dy$.

$D^2y - Dy + 5Dy + 4y = e^{4z} \cdot z$.

$D^2y + 4Dy + 4y = e^{4z} \cdot z$.

Finding y_h

A.E: $r^2 + 4r + 4 = 0$.

$\frac{-4 \pm \sqrt{16-16}}{2} \Rightarrow \frac{-4 \pm 0}{2} \Rightarrow \frac{-4}{2} = r_1 = r_2 = -2$

real & repeated.

$y_h = c_1 e^{r_1 z} + c_2 \cancel{z} e^{r_2 z}$

$y_h = c_1 e^{-2\ln x} + c_2 \ln x\, e^{-2\ln x}$

$y_h = c_1 e^{\ln x^{-2}} + c_2 \ln x\, e^{\ln x^{-2}}$

$y_h = c_1 x^{-2} + c_2 \ln x\, x^{-2}$ ✓

Finding y_p.

$D^2y + 4Dy + 4y = e^{4z} \cdot z$.

$(D^2 + 4D + 4)y = e^{4z}\, z$.

$y_p = \frac{1}{D^2+4D+4} e^{4 \cdot z} \cdot z$

$D \rightarrow D + a \Rightarrow D + 4$.

$y_p = \frac{1}{(D+4)^2 + 4(D+4) + 4} e^{4z} \cdot z$.

$y_p = e^{4z} \frac{1}{D^2 + 8D + 16 + 4D + 16 + 4} \cancel{e^{4z}} \cdot z$.

$y_p = e^{4z} \frac{1}{D^2 + 12D + 36} \cancel{e^{4z}} \cdot z$.

$y_p = \frac{e^{4z}}{36} \frac{1}{\left(1 + \frac{D^2 + 12D}{36}\right)} z$.

$y_p = \frac{e^{4z}}{36} \left(1 + \frac{D^2 + 12D}{36}\right)^{-1} z$

$y_p = \frac{e^{4z}}{36} \left[1 - \frac{D^2 + 12D}{36}\right] z$.

$= \frac{e^{4z}}{36} \left[z - \frac{1}{3}\right]$

$= \frac{z e^{4z}}{36} - \frac{e^{4z}}{108}$

$\because z = \ln x$

$= \frac{\ln x\, e^{4\ln x}}{36} - \frac{e^{4\ln x}}{108}$

$= \frac{\ln x\, \cancel{e}^{\ln x^4}}{36} - \frac{\cancel{e}^{\ln x^4}}{108}$

$= \frac{\ln x \cdot x^4}{36} - \frac{x^4}{108}$

$$\boxed{y_p = \frac{x^4}{36}\left(\ln x - \frac{1}{3}\right)}$$

$y = c_1 e^{-x^2} + c_2 \ln x\, x^{-2} + \frac{x^4}{36}\left(\ln x - \frac{1}{3}\right)$

✓

Date ______ 20 ___

$x^2y'' + 3xy' - 8y = \ln^2 x - \ln x$

let $x = e^z \rightarrow z = \ln x$.

and $x^2y'' = D^2y - Dy$ & $xy' = Dy$.

$D^2_y - Dy + 3Dy - 8 = z^2 - z$.

$D^2_y + 2Dy - 8 = z^2 - z$.

Find y_h

$r^2 + 2r - 8 = 0$.

$\dfrac{-2 \pm \sqrt{4 + 32}}{2} \Rightarrow \dfrac{-2 \pm \sqrt{+36}}{2} \Rightarrow \dfrac{-2 \pm \sqrt{36}}{2}$

$\Rightarrow \dfrac{\cancel{2}(-1 \pm 3)}{\cancel{2}} \Rightarrow -1 \pm 3 \Rightarrow \boxed{r_1 = \overset{-1+3}{2}}, \boxed{r_2 = \overset{-1-3}{-4}}$

$-1 + \frac{1}{2}$

$-\frac{1}{2}$

$y = c_1 e^{r_1 z} + c_2 e^{r_2 z}$

$y = c_1 e^{2\ln x} + c_2 e^{-4\ln x}$

$y = c_1 x^2 + c_2 x^{-4}$

Find y_p

$(D^2 + 2D - 8)y = z^2 - z$.

$y_p = \dfrac{1}{D^2 + 2D - 8} z^2 - z$.

$y_p = \dfrac{1}{-8\left(1 - \frac{D^2 + 2D}{8}\right)} z^2 - z$

$y_p = \dfrac{-1}{8}\left(1 - \dfrac{D^2 + 2D}{8}\right)^{-1} z^2 - z$.

$y_p = \dfrac{-1}{8}\left[1 + \dfrac{D^2 + 2D}{8} + \left(\dfrac{D^2 + 2D}{8}\right)^2 + \ldots\right] z^2 - z$.

$z^2 - z$

$2z - 1$

$2z$

$\cancel{y_p = -\frac{1}{8}}\ \frac{1}{8} D^2 + 2D\ (z^2 - z) \Rightarrow \frac{1}{\cancel{8}\,2}(\cancel{2} + 4z - \cancel{2}) = \boxed{\frac{1}{2}z}$

$\left(\dfrac{D^2 + 2D}{8}\right)^2 \Rightarrow \dfrac{D^2 + 2D}{8} \cdot \underbrace{\dfrac{D^2 + 2D}{8} \cdot z^2 - z}\ \Rightarrow \frac{1}{2}z$

$\frac{1}{2}z$

$\frac{1}{2}$

0

$\dfrac{1}{8}D^2 + 2D \cdot \dfrac{1}{2}z \Rightarrow \dfrac{1}{8}(0 + 1) = \dfrac{1}{8}$

$y_p = -\dfrac{1}{8}\left(z^2 - z + \dfrac{1}{2}z + \dfrac{1}{8}\right)$

$y_p = -\dfrac{1}{8}\left(z^2 - \dfrac{1}{2}z + \dfrac{1}{8}\right)$

$y_p = -\dfrac{1}{8}\left(z^2 - \dfrac{1}{2}z + \dfrac{1}{8}\right)$