TRIPLE INTEGRALS

Ex: 1:- Evaluate the triple integral $\iiint_G 12xy^2z^3\, dV$ over the rectangular box G defined by inequalities $-1 \le x \le 2, 0 \le y \le 3, 0 \le z \le 2$.

Sol: $\iiint_G 12xy^2z^3\, dV = \int_{-1}^{2}\int_{0}^{3}\int_{0}^{2} 12xy^2z^3\, dz\, dy\, dx$

$\Rightarrow \int_{-1}^{2}\int_{0}^{3} \frac{\cancel{12}^{3}xy^2z^4}{\cancel{4}} \Big|_0^2 \, dy\, dx$

$\Rightarrow \int_{-1}^{2}\int_{0}^{3} \left[\{3xy^2(2)^4 - 3xy^2(0)^4\}\right] dy\, dx$

$\Rightarrow \int_{-1}^{2}\int_{0}^{3} 48xy^2\, dy\, dx$

$\Rightarrow \int_{-1}^{2} \cancel{48}^{16}xy^3/\cancel{3}_1 \Big|_0^3 dx$

$\Rightarrow \int_{-1}^{2} \left[16x(3)^3 - 16x(0)^3\right] dx$

$\Rightarrow \int_{-1}^{2} 432x\, dx$

$\Rightarrow \cancel{432}^{216}x^2/\cancel{2}_1 \Big|_{-1}^{2}$

$\Rightarrow \left[216(2)^2 - 216(-1)^2\right]$

$\Rightarrow \boxed{648}$ Ans.

Ex: 2:- Evaluate the triple integral $\iiint_B xyz^2\, dV$, where B is the rectangular box $B = \{(x,y,z) \mid 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$.

Sol: $\iiint_B xyz^2\, dV = \int_0^3\int_{-1}^{2}\int_0^1 xyz^2\, dx\, dy\, dz$

$\Rightarrow \int_0^3\int_{-1}^{2} \frac{x^2yz^2}{2}\Big|_0^1 dy\, dz$

$\Rightarrow \int_0^3\int_{-1}^{2} \left[\frac{1^2yz^2}{2} - 0\right] dy\, dz$

$\Rightarrow \int_0^3\int_{-1}^{2} \frac{yz^2}{2}\, dy\, dz$

$\Rightarrow \int_0^3 \frac{y^2z^2}{2\cdot 2}\Big|_{-1}^{2}$

$\Rightarrow \int_0^3 \left[\left\{\frac{2^2z^2}{4} - \frac{(-1)^2z^2}{4}\right\}\right] dz$

$\Rightarrow \int_0^3 \left[z^2 - \frac{z^2}{4}\right] dz$

$\Rightarrow \int_0^3 \frac{3z^2}{4}\, dz$

$\Rightarrow \cancel{3}\frac{z^3}{4\times\cancel{3}}\Big|_0^3$

$\Rightarrow \left[\frac{3^3}{4} - \frac{0^3}{4}\right]$

$\Rightarrow \boxed{\frac{27}{4}}$

$5-x=1$
$5-1=x$
$x=4$

Ex. Use a triple integral to find the volume of the solid within the cylinder $x^2+y^2=9$ & b/w the plane $z=1$ & $x+z=5$.

Sol. cylinder $x^2+y^2=9$ plane $z=1$ & $x+z=5$

$\Rightarrow y^2 = 9-x^2$ $z=1$ & $z=5-x$.

$y = \pm\sqrt{9-x^2}$

$-3 \le x \le 3, \quad -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}, \quad 1 \le z \le 5-x.$

$$V = \iiint dV = \iint\left[\int_1^{5-x} dz\right] dA.$$

$$\Rightarrow \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_1^{5-x} dz\,dy\,dx.$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\Big|_1^{5-x} dy\,dx.$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$\Rightarrow \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (5-x-1)\,dy\,dx$$

$$\Rightarrow \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-x)\,dy\,dx$$

$$\Rightarrow \int_{-3}^{3} 4y - xy\Big|_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx.$$

$$\Rightarrow \int_{-3}^{3}\left[\left\{4\sqrt{9-x^2} - x\sqrt{9-x^2}\right\} - \left\{-4\sqrt{9-x^2} + x\sqrt{9-x^2}\right\}\right] dx$$

$$\Rightarrow \int_{-3}^{3}\left[4\sqrt{9-x^2} - x\sqrt{9-x^2} + 4\sqrt{9-x^2} - x\sqrt{9-x^2}\right] dx$$

$$\Rightarrow \int_{-3}^{3}\sqrt{9-x^2}\,(4-x+4-x)\,dx.$$

$$\Rightarrow \int_{-3}^{3}(8-2x)\left(\sqrt{9-x^2}\right) dx$$

$$\Rightarrow \int_{-3}^{3}\left(8\sqrt{9-x^2} - 2x\sqrt{9-x^2}\right) dx.$$

$\because a^2-x^2 = x = a\sin\theta.$

$\sqrt{3^2-x^2} \Rightarrow x = 3\sin\theta. \Rightarrow \frac{dx}{d\theta} = 3\cos\theta \Rightarrow dx = 3\cos\theta\,d\theta.$

for limits. $x = 3\sin\theta \Rightarrow \sin\theta = x/3 \Rightarrow \theta = \sin^{-1}(x/3)$

$\theta = \sin^{-1}(-3/3) = -\pi/2$

$\theta = \sin^{-1}(3/3) = \pi/2.$

let $9-x^2=u \Rightarrow \frac{du}{dx} = -2x$

$\Rightarrow \frac{du}{-2x} = dx$

$$\Rightarrow 8\int_{-\pi/2}^{\pi/2}\sqrt{9-(3\sin\theta)^2}\cdot 3\cos\theta\,d\theta - \int_{-3}^{3} 2x\sqrt{u}\,\frac{du}{-2x}$$

$$\Rightarrow 8\int_{-\pi/2}^{\pi/2}\sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\,d\theta + \int_{-3}^{3} u^{1/2}\,du$$

$$\Rightarrow 8\int_{-\pi/2}^{\pi/2}\sqrt{9(1-\sin^2\theta)}\cdot 3\cos\theta\,d\theta + \frac{2u^{3/2}}{3}\Big|_{-3}^{3}$$

$$\Rightarrow 8\int_{-\pi/2}^{\pi/2} 3\sqrt{\cos^2\theta}\cdot 3\cos\theta\,d\theta + \left[\frac{2}{3}(9-x^2)^{3/2}\right]_{-3}^{3}$$

$$\Rightarrow 24\times 3\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$

hence $\cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$

$\Rightarrow 72\int_{-\pi/2}^{\pi/2} \dfrac{\cos 2\theta + 1}{2}\, d\theta$

$\Rightarrow \dfrac{72}{2}\int_{-\pi/2}^{\pi/2} \cos 2\theta + 1 \, d\theta$

$\Rightarrow 36 \left\{ -\dfrac{\sin 2\theta}{2} + \theta \Big|_{-\pi/2}^{\pi/2} \right\}$

$\Rightarrow 36 \left[\left\{ -\dfrac{\sin 2(\pi/2)}{2} + \dfrac{\pi}{2} \right\} - \left\{ -\dfrac{\sin 2(-\pi/2)}{2} - \dfrac{\pi}{2} \right\} \right]$

$\Rightarrow 36 \left\{ -0 + \dfrac{\pi}{2} \right\} - \left\{ -0 - \dfrac{\pi}{2} \right\}$

$\Rightarrow 36 \left\{ \dfrac{\pi}{2} + \dfrac{\pi}{2} \right\}$

$\Rightarrow 36 \left\{ \dfrac{2\pi}{2} \right\}$

$\Rightarrow \boxed{36\pi}$

$+ \dfrac{2}{3}\left[\left\{ (9 - 3^2)^{3/2} \right\} - \left\{ (9 - (-3)^2)^{3/2} \right\} \right]$

$+ \dfrac{2}{3}\left[0 - 0 \right].$

$+ \quad 0$