Sir Mudabbar
FF-132.

DIFFERENTIAL EQUATIONS

26-10-23

Geometerical, Methamatical and physical interpretations of an equation are <u>important</u>.

Cause Learning Outcome:

To model a physical phenomenan → Mathematical model.

$\Sigma F = 0$ ⇒ Sum of all forces acting on a body

$F_{net} = -W$ $\quad\because a_y = \frac{dv}{dt}$

$ma_y = -mg$

$a_y = -g$. $\quad$ let say $v = \frac{dh}{dt}$.

$$\boxed{\frac{dv}{dt} = -g.}$$

$$\frac{d}{dt}\left(\frac{dh}{dt}\right) = \frac{d^2h}{dt^2}$$

$$\boxed{\frac{d^2h}{dt^2} = -g}$$ Differential Equation-

1- Deterministic Model. (closed form solution in have no error).
2- Probabilistic / Stochastic Model (the solution has error)

$y = ax + b + E$ (Probabilistic Model) (Error).
Linear regression

Books $\quad$ Go to Libgen and write author's name

① Dennis Zill
② Henry Edward Penny $\quad$ Rai Singhania
③ Kent Nagle $\quad$ for problem solving
④ William Boyce.

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30-10-23

Origin of Differential Equations

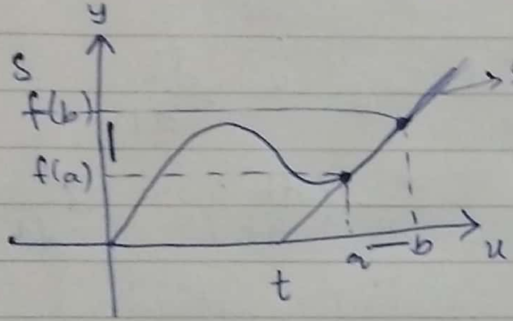
Differential → infinitesimaly small (as small as possible) or think

$v = \boxed{\frac{ds}{dt}}$ (rate of change of displacement)

differential coefficient or derivative

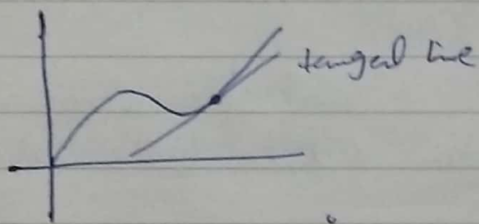
difference Δx — finite change instantaneous

differential dx — infinitesimaly small change



$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$ (average rate of change)

$$\lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \frac{ds}{dt}$$



instantaneous change $\frac{ds}{dt}$

Implicit Differentiation

$y^2 = 2x$

$2y \frac{dy}{dx} = 2$

$$\boxed{\frac{dy}{dx} = \frac{1}{y}}$$ differential equation.

$y = 2x$

$x^2 + c$ (Integral) not necessarily unique

x^2 (Antiderivative) unique

Order of Differential Equations

Number of times an equation is differentiated.

$\frac{d}{dx}(y^2 = 2x)$ $\qquad x^2$

$\frac{d}{dx}$ derivative operator.

$2x \times \frac{d}{dx}(x)$

$2y\frac{dy}{dx} = 2x\frac{d}{dx}(x)$

$2x.$

$2.\quad \cancel{2}y\frac{dy}{dx} = \cancel{2}x.$

$$\boxed{\frac{dy}{dx} = x/y}$$

$(\sin x)^2 \Rightarrow 2(\sin x)^{2-1}.\frac{d}{dx}(\sin x).\frac{d}{dx}(x) \Rightarrow \boxed{2\sin x.\cos x.1}$

$(\sin x^3)^2 \Rightarrow 2(\sin x^3)^{2-1}.\frac{d}{dx}(\sin x^3).\frac{d}{dx}(x^3).\frac{d}{dx}(x).$

$$\boxed{2\sin x^3 . \cos x^3 (3x^2)(1)}$$

$\frac{d}{dx}(\sin y^3)^2. \Rightarrow 2(\sin y^3).\frac{dy}{dx}(\sin y^3).\frac{dy}{dx}(y^3).\frac{dy}{dx}(y)$

$\Rightarrow 2\sin y^3 . \cos y^3 . (3y^2). \ 1$

$\Rightarrow 2\sin y^3 . \cos y^3 . 3y^2 \left(\frac{dy}{dx}\right)$

Order Highest derivative that occurs in the equation.

$\left(\frac{dy}{dx}\right)^3$ First order

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ Second order.

Degree of DE Power of the highest derivative, when equation is radical free. (~~non~~ negative integer or non-integer powers)

$\left(\frac{d^2y}{dx^2}\right)^3$ degree

Ex:

$$\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^5 = 0$$

order = 3
degree = 2.

$$\left\{\left[\frac{d^3y}{dx^3}\right]^{1/\cancel{3}}\right\}^{\cancel{3}} = (y)^3$$ take cube on b/s.

$$\frac{d^3y}{dx^3} = 0.$$ order = 3.
degree = 1

FORMATION OF ODEs 02-11-2023

ordinary differential equations

Q) $y = e^{mx}$ ① (where m is arbitrary constant)

m can be any $\mathbb{R}$.

$$\Rightarrow \frac{dy}{dx} = e^{mx}\frac{d}{dx}(mx)$$

$$" = e^{mx}(m)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = me^{mx}} \quad \text{—— ②}$$

÷ inq ① by ②.

$$\Rightarrow \frac{y}{y'} = \frac{e^{\cancel{mx}}}{me^{\cancel{mx}}}$$

$$\Rightarrow \frac{y}{y'} = \frac{1}{m} \Rightarrow \boxed{m = \frac{y'}{y}.}$$

~~put~~ substitute m in ②.

$$\Rightarrow \cancel{y'} = \frac{\cancel{y'}}{y}e^{(y'/y)x}.$$

$$1 = \frac{e^{(y'/y)x}}{y} \Rightarrow y = e^{(y'/y)(x)} \quad \text{—— ③}$$

take ln on b/s on ③

$$\ln y = \ln e^{(y'/y)x}.$$

$$\ln y = \left(\frac{y'}{y}\right)x. \quad \because \ln e = 1$$

$$\boxed{y' = \frac{y}{x}\ln y}$$

Ans

Q) $x^2 + y^2 = a^2$ where "a" is arbitrary constant)

$2x + 2y\left(\frac{dy}{dx}\right) = 0 \Rightarrow \frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y}$

$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{x}{y}}$ ODE

Q) $x^2+y^2+2gx = 0$ (1) (where 'g' is arbitrary const)

$2x + 2y\left(\frac{dy}{dx}\right) + 2g = 0.$

General Eq of a circle

$x^2+y^2+2gx+2fy+c=0$

center $(-g, -f)$.

$r = \sqrt{g^2+f^2-c}$

$\frac{dy}{dx} = \frac{-2x-2g}{2y}$

$\boxed{\frac{dy}{dx} = -\frac{x}{y} - \frac{g}{y}}$ (2)

$\rightarrow y' = \frac{-x-g}{y}$

$yy' = -x-g.$

$\boxed{g = -x - yy'}$ (3).

Substitute g in eq (1).

$x^2+y^2+2(-x-yy')x=0$

$x^2+y^2-2x^2-2xyy'=0$

$x^2+y^2-2x^2 = 2xyy'$

$\boxed{y' = \frac{x^2+y^2-2x^2}{2xy}}$

eq (1) $\Rightarrow y^2 = -x^2-2gx.$

$\div$ ing eq (1) by (2)

$\cancel{\frac{y^2}{y'} = \frac{-x^2-2gx}{\frac{-x-g}{y}}}$

$\cancel{\frac{y^2}{y'} = \frac{-x^2y-2gx}{-x-g}}$

Substitute g in eq (2)

$\cancel{y' = -\frac{x}{y} + \frac{x+yy'}{y}}$

$\cancel{y' = \frac{-x+x+yy'}{y} = \frac{yy'}{y}}$

$\boxed{\frac{dy}{dx} = \frac{-x^2+y^2}{2xy}}$

ODE

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Q) $y = a\sin x + b\cos x + x\sin x$. (a, b are arbitrary constant).

$$\frac{dy}{dx} = a\cos x - b\sin x + x\cos x + \sin x.$$

$$\frac{d^2y}{dx^2} = -a\sin x - b\cos x + \cos x - x\sin x + \cos x.$$

$$\frac{d^2y}{dx^2} = -(a\sin x + b\cos x + x\sin x) + 2\cos x$$

$$\boxed{\frac{d^2y}{dx^2} = -y + 2\cos x}$$

ODE.

Q) $y = e^x(A\cos x + B\sin x)$ — ① A & B are arbitrary constants.

$$\frac{dy}{dx} = e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x)$$

$A\cos x + B\sin x$.

$-A\sin x + B\cos x$.

$$\frac{dy}{dx} = e^x(-A\sin x + B\cos x + A\cos x + B\sin x) \text{ — ①}$$

$$\frac{d^2y}{dx^2} = e^x(-A\cos x - B\sin x - A\sin x + B\cos x) + e^x(-A\sin x + B\cos x + A\cos x + B\sin x)$$

$$\frac{d^2y}{dx^2} = e^x(-\cancel{A\cos x} - \cancel{B\sin x} - A\sin x + B\cos x - A\sin x + B\cos x + \cancel{A\cos x} + \cancel{B\sin x})$$

$$d^2y/dx^2 = e^x(-2A\sin x + 2B\cos x).$$

$$y'' = -2e^x(A\sin x - B\cos x).$$

$$\boxed{y'' = -2Ae^x\sin x + 2Be^x\cos x} \text{ — ③}$$

Answer $\Rightarrow$ $\boxed{y'' - 2y' + 2y = 0.}$

$\Rightarrow -2Ae^x\sin x + 2Be^x\cos x - 2(e^x\{-A\sin x + B\cos x + A\cos x + B\sin x\}) + 2(e^x\{A\cos x + B\sin x\}) = 0.$

$\Rightarrow \cancel{-2Ae^x\sin x} + \cancel{2Be^x\cos x} + \cancel{2e^xA\sin x} - \cancel{2Be^x\cos x} - \cancel{2Ae^x\cos x} - \cancel{2Be^x\sin x} + \cancel{2Ae^x\cos x} + \cancel{2Be^x\sin x} = 0$

$0 = 0$

6-11-23

Q) $x^2 + y^2 + 2gx + 2fy + c = 0$ —— ①

form diff. eqn. of the given eqn. where g, f and c are arbitiary constants

Solve

diff ① $2x + 2y\left(\frac{dy}{dx}\right) + 2g + 2fy' = 0$ —— ②.

2 is common

$x + yy' + g + fy' = 0$.

$y \cdot y'$
$y \cdot y'' + y' \cdot y'$
$yy'' + y'^2$

$y \cdot y''$
$y \cdot y''' + y'' \cdot y'$

diff ② $1 + yy'' + (y')^2 + fy'' = 0$ — ③.

diff ③ $0 + yy''' + y''y' + 2y' \cdot y'' + fy''' = 0$ — ④

~~÷ eqn ③ by ④~~.

③ $\Rightarrow$ $1 + yy'' + (y')^2 = -fy''$ $\longrightarrow$ (iii)

④ $\Rightarrow$ $yy''' + \underbrace{y''y' + 2y'y''}_{3y'y''} = -fy'''$ $\longrightarrow$ (iv)

÷ eqn (iii) by (iv)

$$\frac{1 + yy'' + (y')^2}{yy''' + 3y'y''} = \frac{\cancel{f}y''}{\cancel{f}y'''}$$

$\Rightarrow$ $y'''\left(1 + yy'' + (y')^2\right) = y''\left(yy''' + 3y'y''\right)$.

$\Rightarrow$ $y''' + \cancel{yy''y'''} + (y')^2 y''' = \cancel{yy''y'''} + 3y'(y'')^2$.

$\Rightarrow$ $y''' + (y')^2 y''' = 3y'(y'')^2$.

$\Rightarrow$ $\boxed{y''' + (y')^2 y''' - 3y'(y'')^2 = 0}$

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ORDINARY
FIRST ORDER DIFFERENTIAL EQUATIONS (ODEs)

Types: $\frac{dy}{dx} + Q(x)\,y = P(x)$.

$\frac{dy}{dx} = f(x)\,g(y)$.

Separable ODES: can be separated by x and y.

$\rightarrow \frac{dy}{dx} = f(x)\,g(y)$.

$\frac{dy}{dx} = \frac{x}{y} \Rightarrow \underbrace{x}_{f(x)} \cdot \underbrace{\frac{1}{y}}_{g(y)}$. | $\frac{dy}{dx} = e^{x+y} \Rightarrow \underbrace{e^{x}}_{f(x)} \cdot \underbrace{e^{y}}_{g(x)}$

$\frac{dy}{dx} = x+y$

(not separable).

Ex: $\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x}$. apply integral.

$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$.

$\Rightarrow \ln y = \ln x + c$. $\rightarrow$ or $\ln y - \ln x = c$.

$\Rightarrow \ln y = \ln x + \ln c$. $\qquad \ln\left(\frac{y}{x}\right) = c$

$\Rightarrow e^{\ln y} = e^{\ln(xc)}$ $\qquad \cancel{e}^{\ln\left(\frac{y}{x}\right)} = \cancel{e}^{c}$

$\Rightarrow \cancel{e}^{\ln(y)} = \cancel{e}^{\ln(xc)}$.

$\Rightarrow \boxed{y = xc}$ $\qquad \boxed{\frac{y}{x} = c}$

Q: $\frac{dy}{dx} = xy^2 + x$ $\qquad \because \int \frac{dx}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

$\frac{dy}{dx} = x(y^2+1)$.

$\frac{dy}{y^2+1} = x\cdot dx \Rightarrow \int \frac{1}{y^2+1}\,dy = \int x\,dx$.

$\frac{1}{1}\tan^{-1}\frac{y}{1} = \frac{x^2}{2} + c$

$$\Rightarrow \boxed{\tan^{-1} y = \frac{x^2}{2} + c}$$

$$\Rightarrow \boxed{y = \tan\left[\frac{x^2}{2} + c\right]}.$$

$\frac{1}{x} = \ln x$

$\ln e^{4}$

Q) : $\frac{dy}{dx} = \frac{x e^{x^2+y^2}}{y}$.

$\Rightarrow \frac{dy}{dx} = x e^{x^2} \cdot \frac{e^{y^2}}{y}$.

$dy \cdot \frac{y}{e^{y^2}} = dx\, x e^{x^2}$

let $u = y^2$

$\int \frac{y}{e^{y^2}} \cdot dy = \int x e^{x^2} \cdot dx$.

$\frac{du}{dy} = 2y$

$\int \frac{\cancel{y}}{e^{u}} \cdot \frac{du}{2\cancel{y}} = \int \cancel{x} e^{t} \frac{dt}{2\cancel{x}}$

$\frac{du}{2y} = dy$

$\frac{1}{2} \int \frac{1}{e^{u}} du = \frac{1}{2} \int e^{t} dt$ -

let $t = x^2$

$\frac{dt}{dx} = 2x$

$\frac{1}{2} \int e^{-u} du. = \frac{1}{2} e^{t} + c$

$= \frac{1}{2} e^{x^2} + c.$

$\frac{dt}{2x} = dx.$

$\frac{1}{2}(-e^{-u})$

$$\boxed{-\frac{1}{2} e^{-y^2} = \frac{1}{2} e^{x^2} + c}$$

Closed form solution.

Reduceable to Separable.

Q. $\frac{dy}{dx} = \cos(x+y)$

Let $\boxed{u = x + y}$.

$\frac{du}{dx} = 1 + \frac{dy}{dx}$.

$$\boxed{\frac{dy}{dx} = \frac{du}{dx} - 1}$$

$\Rightarrow \frac{du}{dx} - 1 = \cos(u)$.

$\Rightarrow \frac{du}{dx} = 1 + \cos(u)$

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$\Rightarrow \frac{du}{dx} = 1 + \cos(u).$

$\frac{c}{s} = \cot$

$\frac{\cos u}{\sin u} \cdot \frac{1}{\sin u}$

$\Rightarrow \frac{du}{1+\cos(u)} = dx.$

$\Rightarrow \frac{(1-\cos u)\, du}{(1+\cos u)(1-\cos u)} = dx.$

$\because \operatorname{cosec}^2 u = -\cot u$

$\Rightarrow \left(\frac{1-\cos u}{\sin^2 u}\right) du = dx$

$\because \cot u \operatorname{cosec} u$

$= -\operatorname{cosec} u.$

$\Rightarrow \left(\frac{1}{\sin^2 u} - \frac{\cos u}{\sin^2 u}\right) du = dx.$

$\Rightarrow \int [\operatorname{cosec}^2 u - \operatorname{cosec} u \cot u]\, du = \int dx$

$\Rightarrow -\cot u - (-\operatorname{cosec} u) = x + C.$

$\Rightarrow -\cot u + \operatorname{cosec} u = x + c.$

$\Rightarrow \quad -\cot u - (-\operatorname{cosec} u) = x + c.$

$\Rightarrow \quad -\cot u + \operatorname{cosec} u = x + c.$

13-11-23

Homogenous First Order ODEs:

Homogenous functions.

$f(kx, ky) = k^n f(x,y).$ n: degree of homogenity.

or $f(x, y) = x^n f(y/x).$

Example: $f(x,y) = x^2 + y^2.$ $\because f(kx, ky) = k^n f(x,y).$

$f(kx, ky) = (kx)^2 + (ky)^2.$

$" \quad = k^2 x^2 + k^2 y^2$

$f(kx, ky) = k^2(x^2 + y^2)$ → Function is homogenous

→ degree of homogenity = 2.

Q) $f(x,y) = e^{x+y}$

$f(kx,ky) = e^{kx+ky}$

$f(kx,ky) = e^{k(x+y)} \neq k^n f(x,y) \Rightarrow$ not homogeneous.

Q) $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

Let $y = vx \Rightarrow v = y/x$

$\frac{dy}{dx} = v(1) + x\frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{x^2+y^2}{xy}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2+(v\cdot x)^2}{x\cdot x v} \Rightarrow \frac{\cancel{x^2}(1+v^2)}{\cancel{x^2}v}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^2}{v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{v} - v \Rightarrow x\frac{dv}{dx} = \frac{1+\cancel{v^2}-\cancel{v^2}}{v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1}{v}$

$\Rightarrow \int v\, dv = \int \frac{1}{x}\, dx$

$\Rightarrow \frac{v^2}{2} = \ln x + \ln c$

$\Rightarrow \frac{v^2}{2} = \ln(cx)$ taking antilog.

$\Rightarrow e^{v^2/2} = e^{\ln(cx)}$

$\Rightarrow e^{v^2/2} = cx$

$\Rightarrow \boxed{e^{\frac{1}{2}\left(\frac{y}{x}\right)^2} = cx}$ Implicit.

- one way to inspect polynomial ODE is to check powers of numerator & denominators, if it is same then it is homog..

Explicit form
when one variable is represented as a dependent variable

Implicit form
· mixed variable

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Q) $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right) \Rightarrow f(kx, ky) = \frac{Ky}{Kx} + \tan\left(\frac{Ky}{Kx}\right)$

$f(kx, ky) = k^0\left[\frac{y}{x} + \tan\left(\frac{y}{x}\right)\right]$

Let $y = vx$. $\Rightarrow v = \frac{y}{x}$.

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{v\cancel{x}}{\cancel{x}} + \tan\left(\frac{vx}{x}\right)$

$\Rightarrow v + x\frac{dv}{dx} = v + \tan(v)$

$\Rightarrow x\frac{dv}{dx} = \cancel{v} + \tan(v) - \cancel{v}$

$\Rightarrow x\frac{dv}{dx} = \tan(v)$.

$\Rightarrow \int \frac{1}{\tan(v)}\,dv = \int \frac{1}{x}\,dx$.

$\Rightarrow \ln \sin(v) = \ln x + \ln c$.

$\Rightarrow \ln \sin(v) = \ln(cx)$ anti log.

$\Rightarrow e^{\ln \sin(v)} = e^{\ln(cx)}$.

$\Rightarrow \sin(v) = cx$.

$\Rightarrow \boxed{\sin\left(\frac{y}{x}\right) = cx.}$

Equations Reducible to Homogenous. forms.

Q) $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$ (not homogenous).

Let $x = X + h$ and $y = Y + k$.

$\frac{dx}{dX} = 1$, $\frac{dy}{dY} = 1 \Rightarrow \frac{\frac{dx}{dX}}{\frac{dy}{dY}} = 1$

$\Rightarrow \frac{X + h + 2(Y + k) - 3}{2(X + h) + Y + k - 3}$

$\frac{dx}{dX} \times \frac{dY}{dy} = 1$

$\rightarrow \frac{dY}{dX} = \frac{dy}{dx}$ ①

$$\Rightarrow \frac{X+h+2Y+2k-3}{2X+2h+Y+k-3} \quad \text{—} \quad (A)$$

$\Rightarrow \quad h+2k-3=0 \quad \Rightarrow \quad h+2k=3$

$\Rightarrow \quad 2h+k-3=0. \quad \Rightarrow \quad 2(2h+k=3)$

$\Rightarrow h+2k=3$

$\Rightarrow 4h+2k=6$

$-3h=-3$

$\boxed{h=1}$

$1+2k=3$

$2k=2$

$\boxed{k=1}$

put h and k in eq (A).

$$\Rightarrow \frac{X+1+2Y+2(1)-3}{2X+2(1)+Y+1-3}$$

$$\Rightarrow \frac{X+2Y}{2X+Y} \quad \text{—} \quad (2).$$

now, we have, by eq (1) and (2).

$$\Rightarrow \frac{dY}{dX} = \frac{X+2Y}{2X+Y}$$

Let $Y = VX. \Rightarrow V = Y/X.$

$$\frac{dY}{dV} = V + X\frac{dV}{dX}$$

$$\Rightarrow V + X\frac{dV}{dX} = \frac{X+2VX}{2X+VX}.$$

$$\Rightarrow V + X\frac{dV}{dX} = \frac{\cancel{X}(1+2V)}{\cancel{X}(2+V)}$$

$$\Rightarrow V + X\frac{dV}{dX} = \frac{1+2V}{2+V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1+2V}{2+V} - V.$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1+\cancel{2V}-\cancel{2V}-V^2}{2+V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1-V^2}{2+V.}$$

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$$\Rightarrow \int \frac{2+V}{1-V^2}\, dV = \int \frac{1}{X}\, dX$$

$$= \ln X + \ln C.$$

Exact Equations

$f(x,y)\,dx + g(x,y)\,dy = 0$ or $M\,dx + N\,dy = 0$.

$$\frac{dy}{dx} = \frac{x}{y}$$

$$x\,dx - y\,dy = 0$$

if $f_x = g_x \Rightarrow$ exact equations

test for exactness.

Q) $e^{3y}\,dx + 3xe^{3y}\,dy = 0.$

Let $M = e^{3y}$, $N = 3xe^{3y}$.

$$\frac{\partial M}{\partial y} = \frac{\partial(e^{3y})}{\partial y}, \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(3xe^{3y}\right).$$

$$M_y = \frac{\partial M}{\partial y} = 3e^{3y} \qquad N_x = \frac{\partial N}{\partial y} = 3e^{3y}.$$

as $M_y = N_x \Rightarrow$ hence equation is Exact.

$$\Rightarrow \underbrace{\int e^{3y}\,dx}_{y \text{ term constant}} + \underbrace{\cancel{\int 3xe^{3y}\,dy}}_{x \text{ term vanish}} = 0.$$

we are vanishing one of the term because end result will be same

$$\Rightarrow e^{3y}\int dx = 0$$

$$\Rightarrow \boxed{xe^{3y} = c.}$$

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Q: $[2xy\,dx + dy]\,e^{x^2} = 0.$

$\Rightarrow 2xy\,e^{x^2}\,dx + e^{x^2}\,dy = 0.$

Let $M = 2xye^{x^2}$, $N = e^{x^2}$

$M_y = 2xe^{x^2}$, $N_x = 2xe^{x^2}$

$M_y = N_x \Rightarrow$ Exact

$\Rightarrow \cancel{\int 2xye^{x^2}\,dx} + \int e^{x^2}\,dy = 0.$

y term — x term

$\Rightarrow$ vanish — constant

~~let $u = x^2$, $\frac{du}{dx} = 2x$.~~

$\Rightarrow$ $\cancel{e^{x^2}} \cancel{+ c = 0}$

$e^{x^2}\int dy = 0.$

$\Rightarrow$ $\boxed{y\,e^{x^2} = c}$

$\Rightarrow$

$$e^{x^2}\int dy = 0.$$

$\Rightarrow$ $\boxed{y e^{x^2} = C}$

$\frac{du}{du} = \cancel{dx}$

16-11-23

LINEAR DIFF EQNS. OF ORDER ONE:

$\frac{dy}{dx} + x^2 y = e^x$ Linear.

$\left(\frac{dy}{dx}\right)^2 + xy = 1.$ Non-Linear

$y\frac{dy}{dx} + x^2 = 1$ Non-Linear

$\frac{dy}{dx} + x \sin y = 1$ Non-linear

$\downarrow$

$\sin(xy) \rightarrow$ non-linear

Standard form of Linear diff eqn.

$$\frac{dy}{dx} + P(x)y = Q(\cancel{x,y})$$

① $y' + xy = 1.$

$P(x) = x, \quad Q(x) = 1$

② $(\cos x)\, y' + xy = \cos x.$

$\div$ by $\cos x.$

$$\left(\frac{\cancel{\cos x}}{\cancel{\cos x}}\right) y' + \frac{x y}{\cos x} = \frac{\cancel{\cos x}}{\cancel{\cos x}}$$

$$y' + \frac{x}{\cos x}\, y = 1$$

$$P(x) = \frac{x}{\cos x}, \quad Q(x) = 1$$

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$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$I.F = e^{\int P(x)\,dx}$$

Integrating Factor.

$$I.F = f(x)$$

$$f(x)\left[\frac{dy}{dx} + P(x)\,y = Q(x)\right]$$

Exact Diff Eqn. or Total Diff.

$$f(x)\,y' + f(x)\,P(x)\,y = f(x)\,Q(x)$$

$\Rightarrow$ now left part is total diff

$$\because \frac{d}{dx}\left[f(x)\,y\right]$$

$$\Rightarrow \frac{d}{dx}(xy).$$

$$\Rightarrow y + x\frac{dy}{dx}$$

$$\Rightarrow \frac{d}{dx}\left[\underset{u\ \cdot\ v}{f(x)\,y}\right] = f(x)\,y' + f(x)\,P(x)\,y.$$

$$\cancel{f(x)\,y'} + y\,f'(x) = \cancel{f(x)\,y'} + f(x)\,P(x)\,y.$$

$$\cancel{y}\,f'(x) = f(x)\,P(x)\,\cancel{y}$$

$$f'(x) = f(x)P(x).$$

$$\frac{d\,f(x)}{dx} = f(x)\,P(x).$$

$$\int \frac{d\,f(x)}{f(x)} = \int P(x)\,dx.$$

$$e^{\ln f(x)} = e^{\int P(x)\,dx}$$

antilog.

$$\boxed{f(x) = e^{\int P(x)\,dx}}$$

This is a derivation for I.F.

General form

$$y' + P(x)\,y = Q(x)$$

$$I.F = e^{\int P(x)\,dx}.$$

Ex: $\cos x \dfrac{dy}{dx} + (\sin x)\,y = \cos x.$

$\div$ by $\cos x$ both sides.

$$y' + \left(\frac{\sin x}{\cos x}\right) y = 1.$$

$$y' + \tan x\ y = 1.$$

multiply whole eqn. by I.F.

$$(\sec x)\,y' + (\sec x \tan x)\,y = \sec x.$$

$$e^{\int \tan x\,dx}$$

$$e^{\ln \sec x}$$

$$\sec x.$$

hence, $\frac{d}{dx}\left[(\sec x)y\right] = \sec x.$

$$\int d(\sec x \cdot y) = \int \sec x \, dx.$$

$$\boxed{y \sec x = \ln|\sec x + \tan x| + c.}$$

LINEAR DIFF EQN. 20-11-23

Ex: $\frac{dx}{dt} = \frac{1}{x+t}$ Non-linear in x.

$\frac{dt}{dx} = x + t$ Linear in t.

$P(x) = +1$, $Q(x) = x$.

$\frac{dt}{dx} - t = x.$; $P(x) = -1.$

$e^{\int -dx} = e^{-x}.$

$e^{-x}\left(\frac{dt}{dx}\right) - e^{-x}t = e^{t}x.$

$\int \frac{d}{dx}\left[(e^{-x})t\right] = e^{t}x. \Rightarrow d\left[(e^{-x})t\right] = e^{t}\int x\,dx.$

$$\boxed{te^{-x} = \frac{x^2}{2}e^{t} + c.}$$

E

EQUATIONS REDUCIBLE TO EXACT:

→ $M(x,y) + N(x,y)\,dy = 0.$

if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ → Exact

if $\frac{\partial x}{\partial y} \neq \frac{\partial N}{\partial x}$ → Non-exact.

$\overset{M}{2xy}\,dx + \overset{N}{x^2}\,dy = 0.$

$M_y = 2x$, $N_x = 2x.$

Exact.

$2y\,dx + x\,dy = 0$

$M_y = 2$, $N_x = 1$

Non-Exact.

PAPERWORK

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Finding Integrating Factor:

$\because$ $M\,dx + N\,dy = 0$ and $M_y \neq N_x$.

Let $\mu \rightarrow$ an I.F., multiplied by eqn will made eqn exact (claim).

$$\mu(x)\left[M\,dx + N\,dy\right] = 0$$

$$[\mu(x)M]\,dx + [\mu(x)N]\,dy = 0.$$

$$\frac{\partial}{\partial y}[\mu(x)M] = \frac{\partial}{\partial x}[\mu(x)N] \Rightarrow \text{This should be satisfied.}$$

$$\mu(x)M_y = \mu(x)N_x + \cancel{N_x} + N(\mu(x))' \qquad \because \frac{\partial M}{\partial y} = M_y$$

$$\mu(x)M_y - \mu(x)N_x = \cancel{N_x} + N(\mu(x))'$$

$$\mu(x)[M_y - N_x] = \cancel{N_x} + N(\mu(x))'$$

$$\mu(x)\left[\frac{M_y - N_x}{N}\right] = (\mu(x))'$$

$$\frac{d\mu(x)}{dx} = \left[\frac{M_y - N_x}{N}\right]\mu(x).$$

$$\int \frac{d\mu(x)}{\mu(x)} = \int \left[\frac{M_y - N_x}{N}\right] dx.$$

$$e^{\ln \mu(x)} = e^{\int\left(\frac{M_y - N_x}{N}\right) dx}.$$

$$\boxed{\mu(x) = e^{\int\left(\frac{M_y - N_x}{N}\right) dx}}$$

iff $\Rightarrow$ $\frac{M_y - N_x}{N}$ will give a function of x. Then it'll be an integrating factor.

Q1) $2y\,dx + x\,dy = 0.$

$M = 2y \qquad N = x.$

$M_y = 2 \qquad N_x = 1 \quad ; M_y \neq N_x.$

$M_y - N_x = 2 - 1 = 1$

$\frac{M_y - N_x}{N} = \frac{1}{x}. \Rightarrow I.F = e^{\int \frac{1}{x} dx}$

$I.F. = \cancel{e}^{\cancel{\ln x}}.$

$\boxed{I.F = x.}$

if $M(y) \Rightarrow e^{\int \frac{N_x - M_y}{M} dy}.$

$\frac{N_x - M_y}{M} = \frac{1-2}{2y} = \frac{-1}{2y}. \Rightarrow I.F = e^{-\frac{1}{2}\int \frac{1}{y} dy}$

$I.F. = e^{-\frac{1}{2}\ln y}.$

$I.F = \cancel{e}^{\cancel{\ln y}^{-1/2}}$

$\boxed{I.F = y^{-1/2}.}$

Now multiple I.F with main eqn. to check it'll made it exact or not.

Multiple Integrating Factors can be found.

$[2y\,dx + x\,dy]\,y^{-1/2} = 0.$

$2y \cdot y^{-1/2} dx + xy^{-1/2}\,dy = 0.$

$2y^{1/2} dx + xy^{-1/2}\,dy = 0.$

$M_y = \frac{1}{\cancel{x}}\cancel{x} y^{-1/2}, \quad N_x = y^{-1/2}.$

$M_y = y^{-1/2}, \quad N_x = y^{-1/2}$

$M_y = N_x$ ✓.

Then solve, transformed eqn with the method to solve exact eqn.

$\Rightarrow 2y^{1/2}dx + xy^{-1/2}\,dy = 0.$

$2y^{1/2}\int dx + \cancel{x\int y^{-1/2}\,dy} = 0.$

y ten constant $\qquad$ x tem vanish.

$2y^{1/2}x. = 0.$

$2xy^{1/2} = C.$

$\boxed{xy^{1/2} = c}$

$\int 2y^{1/2}\,dx + \int xy^{-1/2}\,dy = c$

$2y^{1/2}\int dx + x\int y^{-1/2}\,dy.$

$2xy^{1/2} + 2x\,y^{1/2} = c$

$4xy^{1/2} = c$

$xy^{1/2} = c$

PAPERWORK

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Q. ~~$e^x \sin y$~~ $x^2y\,dx - (x^3-y^3)\,dy = 0$.

$M = x^2y$ $\qquad N = -x^3 + y^3$.

$M_y = x^2$ $\qquad N_x = -3x^2$

Not-exact, we'll reduce it to exact by calculating an integrating factor.

$\therefore \mu(x) = e^{\int \left(\frac{M_y - N_x}{N}\right) dx}$

$\frac{M_y - N_x}{N} = \frac{x^2-(-3x^2)}{-x^3+y^3} \Rightarrow \int \frac{x^2+3x^2}{-x^3+y^3}\,dx$ $\qquad$ let $u = -x^3+y^3$, $\frac{du}{dx} = -3x^2+3y^2$

$\Rightarrow \int \frac{4x^2}{-x^3+y^3}\,dx$. $\qquad$ Let $u = -x^3+y^3$.

$\frac{du}{dx} = -3x^2 \Rightarrow -\frac{du}{3x^2} = dx$.

$\int \frac{4x^2}{u} \cdot \left(\frac{-1}{3x^2}\right) du$

$-\frac{4}{3}\int \frac{1}{u}\,du \Rightarrow \frac{-4}{3} \ln u + c$.

$\Rightarrow -\frac{4}{3} \ln(-x^3+y^3) + c$.

$\mu(x) = e^{-\frac{4}{3}\ln(-x^3+y^3)}$

$= e^{\ln(-x^3+y^3)^{-\frac{4}{3}}}$

$\mu(x) = (-x^3+y^3)^{-4/3} \Rightarrow$ I.F

Checking: $\Rightarrow x^2y\,dx - (x^3-y^3)\,dy = 0$

$(-x^3+y^3)^{-4/3}x^2y\,dx - (x^3-y^3)(-x^3+y^3)^{-4/3}\,dy = 0$.

Not exact

$\therefore \mu(y) = e^{\int \left(\frac{N_x - M_y}{M}\right) dy}$

$\frac{N_x - M_y}{M} = \frac{-3x^2-x^2}{x^2y} = \frac{-4x^2}{x^2y} = \frac{-4}{y}$.

$-4\int \frac{1}{y}\,dy \Rightarrow -4\ln y \Rightarrow \ln y^{-4}$.

$e^{\ln y^{-4}} \Rightarrow \mu(y) = y^{-4}$

$\Rightarrow (y^{-4})(x^2y)\,dx + (-x^3+y^3)(y^{-4})\,dy = 0$

(1) — $(x^2y^{-3})\,dx + (-x^3y^{-4} + y^{-1})\,dy = 0$

$M = x^2y^{-3}$ $\qquad N = -x^3y^{-4} + y^{-1}$

$M_y = -3x^2y^{-4}$ $\qquad N_x = -3x^2y^{-4}$

$M_y = N_x \Rightarrow$ Exact

Eq (1)

$\int (x^2y^{-3})\,dx + \int (-x^3y^{-4} + y^{-1})\,dy = 0$

$\frac{x^3y^{-3}}{3} + \left(\frac{+x^3y^{-3}}{3} + \ln y\right) = C$

$\frac{x^3y^{-3}}{3} + \frac{x^3y^{-3}}{3} + \ln y$

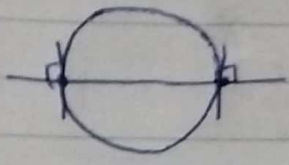
$2x^3y^{-3} + 3\ln y + 3c = 0$

x^3y^{-3}

ORTHOGONAL TRAJECTORIES:

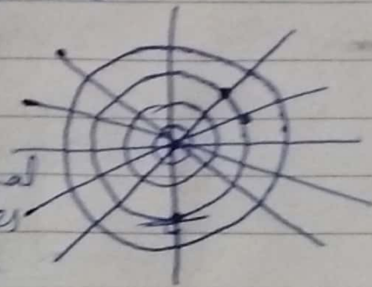
27-11-23

cut perpendiculary, the curves each other.



Self-orthogonal

Family of straight lines passing through the origin are orthogonal to the family of circles centered at origin.



ellipses orthogonal to parabolas.

slope of two $\perp$ lines $= m_1 \times m_2 = -1$.

Ex: $x^2 + y^2 = r^2$. (Family of circles center at origin).

$\Rightarrow$ we have to check which family is orthogonal to family of circles.

differentiate it.

$\frac{d}{dx}(x^2) + \left(\frac{d}{dy}\right)y^2 = \left(\frac{d}{dx}\right)r^2$.

$\cancel{2}x + \cancel{2}yy' = 0. \Rightarrow \boxed{y' = -\frac{x}{y}.} = m_1$

$m_1 = \frac{dy}{dx}$, $m_2 = ?$

$\therefore m_1 \times m_2 = -1$

$\frac{dy}{dx} \times m_2 = -1. \Rightarrow$

$m_1 \times m_2 = -1$

$-\frac{x}{y} \times m_2 = -1$

$\boxed{m_2 = -\frac{dy}{dx}} \Rightarrow \boxed{m_2 = \frac{y}{x}}$

$+\frac{dx}{dy} = +\frac{x}{y} \Rightarrow \frac{dx}{dy} = \frac{x}{y}$.

$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y}$

$\Rightarrow e^{\ln(x \cdot c)} = e^{\ln y}$

$\Rightarrow \boxed{xC = y} \rightarrow$ eq of line passing through origin.

PAPERWORK

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Ex: $y = ax^2$ — (1) Eqn of parabola.

$\frac{dy}{dx} = 2ax$.

$a = \frac{y'}{2x}$

$y' = 2ax$ — (2)

$y = \frac{y'}{2\cancel{x}} x^{\cancel{2}} \Rightarrow y = \frac{xy'}{2}$

$y' = \frac{2y}{x} = m_1$

$m_1 = \frac{dy}{dx}$, $m_2 = ?$

$\because m_1 \times m_2 = -1$

$\frac{2y}{x} \times m_2 = -1$.

$\frac{dy}{dx} \times m_2 = -1$.

$\boxed{m_2 = \frac{-x}{2y}}$

$m_2 \frac{dy}{dx}$

$\boxed{m_2 = -\frac{dx}{dy}}$

$-\frac{dx}{dy} = \frac{2y}{x}$

$\int -x \cdot dx = 2\int y \, dy$

$-\frac{x^2}{2} = y^2 + c$.

$y^2 + \frac{x^2}{2} = c$.

$\frac{y^2}{c} + \frac{x^2}{2c} = 1$.

$\boxed{\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1}$ std. eq. of Ellips.

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Self Orthogonal Trajectories.

Ex: $y^2 = 4a(x+a)$.

$\Rightarrow y^2 = 4ax + 4a^2$ — ①

$\Rightarrow 2yy' = 4a$ — ②.

$\Rightarrow a = \frac{yy'}{2}$

$\Rightarrow y^2 = 4\left(\frac{yy'}{2}\right)\left(x + \frac{yy'}{2}\right)$.

$\Rightarrow y^2 = 2yy'\left(x + \frac{yy'}{2}\right)$.

$\Rightarrow \boxed{y^2 = 2xyy' + y^2(y')^2}$

$y^2 = 2xyy' + y^2(y')^2$

$\because \frac{dy}{dx} \xrightarrow{\text{Orthogonal trajectory}} -\frac{dx}{dy}$.

$\cancel{y^2} = \cancel{y}\{2xy' + y(y')^2\}$.

$y = 2xy' + y(y')^2$ — ①. put $y' = -\frac{dx}{dx}$,

$y = 2x\left(-\frac{dx}{dy}\right) + y\left(-\frac{dx}{dy}\right)^2$.

$y = -2x\left(\frac{dx}{dy}\right) + y\left(\frac{dx}{dy}\right)^2$.

$y + 2x\left(\frac{dx}{dy}\right) = y\left(\frac{dx}{dy}\right)^2$.

$\frac{dx}{dy} = x'$

$y + 2xx' = y(x')^2$.

$\frac{y}{(x')^2} + \frac{2x\cancel{x'}}{(x')^{\cancel{2}}} = y$.

$y\cdot\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) = y$

or $\boxed{y(y')^2 + 2xy = y.}$ — ②

Since, both eq ① and ② differential eqns are same as we've taken slopes for both, so they're self-orthogonal.

PAPERWORK

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Q1) $r = a(1-\cos\theta)$ — ① polar coordinates.

$x = r\cos\theta$

$y = r\sin\theta$.

$\frac{dr}{d\theta} = a(0-(-\sin\theta))$.

$\frac{dr}{d\theta} = a\sin\theta$ — ②.

eq① $\Rightarrow r = a(1-\cos\theta)$.

$a = r/(1-\cos\theta)$ → put in eq ②.

$$\boxed{\frac{dr}{d\theta} = \frac{r}{1-\cos\theta}(\sin\theta)}$$ m_1.

$\Rightarrow -r^2\frac{d\theta}{dr} = m_2 = \frac{dr}{d\theta}$

$\Rightarrow -r^{\cancel{2}}\frac{d\theta}{dr} = \frac{\cancel{r}}{1-\cos\theta}(\sin\theta)$

$\Rightarrow -r\frac{d\theta}{dr} = \frac{\sin\theta}{1-\cos\theta}$.

$\frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)} = \frac{\sin^{\cancel{2}}\theta}{\cancel{\sin\theta}(1+\cos\theta)}$

$\frac{\sin\theta}{1+\cos\theta}$

$\Rightarrow \int\frac{1-\cos\theta}{\sin\theta}.d\theta = \int.\frac{-dr}{r}$

~~Let $u = 1-\cos\theta$.~~

~~$\frac{du}{d\theta} = \sin\theta$.~~

~~Let~~ Rationalize.

$\Rightarrow \int\frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)}d\theta$ "

$\Rightarrow \int.\frac{\sin^{\cancel{2}}\theta}{\cancel{\sin\theta}(1+\cos\theta)}d\theta$. "

$\Rightarrow \int-\frac{\sin\theta}{1+\cos\theta}d\theta = \int\frac{dr}{r}$

Let $u = 1+\cos\theta$

$\frac{du}{d\theta} = -\sin\theta$

$\frac{du}{-\sin\theta} = d\theta$.

$\Rightarrow \int-\frac{\cancel{\sin\theta}}{u}.\left(\frac{-du}{\cancel{\sin\theta}}\right) = \int\frac{1}{r}dr$.

$\Rightarrow \int 1/u\ .du. = \int\frac{1}{r}dr$.

$\Rightarrow \ln u = \ln r + c$.

$\Rightarrow \ln(1+\cos\theta) = \ln r + \ln c$

$\Rightarrow e^{\ln(1+\cos\theta)} = e^{\ln(rc)}$.

$\Rightarrow 1+\cos\theta = rc$.

$\Rightarrow r = a(1+\cos\theta)$

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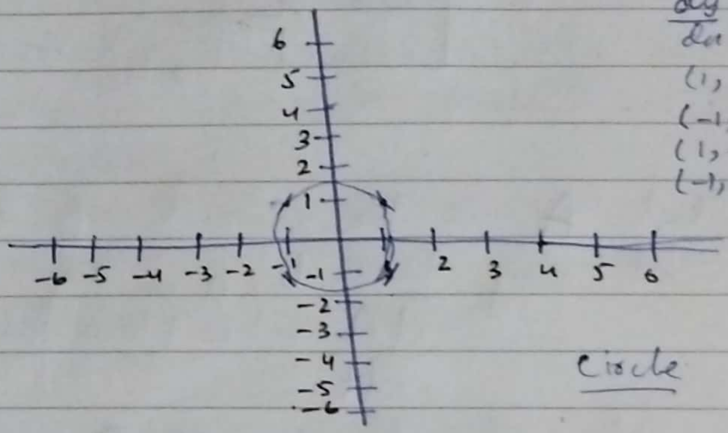
Slope. Fields

$$\frac{dy}{dx} = \frac{-x}{y}$$

Solution of the given DE is $\Rightarrow x^2 + y^2 = c^2$.



$\frac{dy}{dx} = \frac{-x}{y}$.

$(1,1) = -1$
$(-1,1) = 1$
$(1,-1) = 1$
$(-1,-1) = -1$

Circle

if $x = y$, then slope is $45°$

$\tan\theta = \frac{y}{x} = \frac{1}{1} = 1$

$\theta = \tan^{-1}(1)$

$\boxed{\theta = 45°}$

APPLICATIONS OF FIRST ORDER ODEs 30-11-23

Growth & Decay Models.

Malthusian Law:

"Rate of change of population at any time is directly proportional to the current population"

$$\frac{dP}{dt} \propto P(t).$$

$$\boxed{\frac{dP}{dt} = r P}$$, where r is "growth rate"

$\frac{dN}{dt} \propto N(t) \Rightarrow \boxed{\frac{dN}{dt} = -\lambda N}$ $N \Rightarrow$ no. of atoms.

radioactive decay.

PAPERWORK

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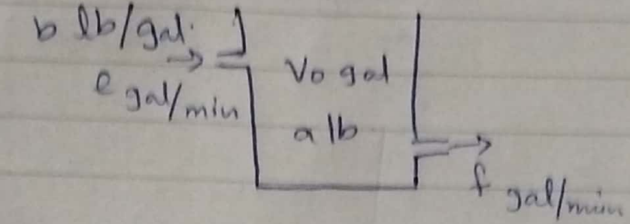
Dilution Model. (Mixing Problem).



Brine Solution : V
Amount of Salt : a.
Inflow rate : e.
Outflow rate : f.
Amount of brine inflowing : b.

Let 'A' is the amount of salt in the container at any time 't'

$$\frac{dA}{dt} = \begin{pmatrix}\text{Amount}\\ \text{inflow}\\ \text{rate}\end{pmatrix} - \begin{pmatrix}\text{Amount}\\ \text{outflow}\\ \text{rate}\end{pmatrix} \quad -(1).$$

rate of change of A

→ Amount inflow rate:

$$\left(b \frac{lb}{\cancel{gal}}\right)\left(e \frac{\cancel{gal}}{min}\right) = \boxed{be \frac{lb}{min}}$$

$$\boxed{\begin{aligned} Q &= \frac{V}{t} \\ V &= Qt \\ V &= et \end{aligned}}$$

→ Amount out flow rate:

$$\left[\frac{A}{V_0 + et - ft}\right] \times f \qquad \frac{\cancel{gal}}{min} \times \frac{lb}{\cancel{gal}}$$ → amount of salt flowing out.

$$\frac{Af}{V_0 + et - ft} \quad \frac{lb}{min}$$

Hence eq (1) becomes.

$$\boxed{\frac{dA}{dt} = be - \frac{Af}{V_0 + t(e-f)}}$$ linear or separable.

Autonomous D.E: equation contains only dependent variable.

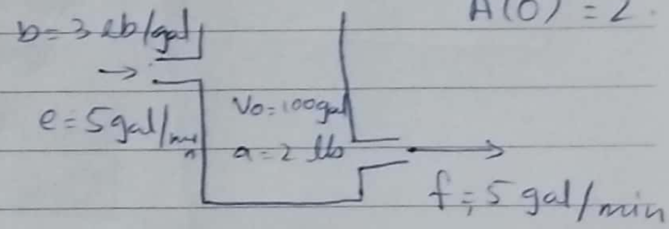
Q) A container contains 100 gal of brine solution with 2 lb of salt dissolved in it at t = 0. If another brine solution is flowing in to the container at the rate of 5 gal/min ~~entering~~ containing 3 lb/gal of salt, and brine is flowing out at the same rate. Find the amount of salt present in the container at any time 't'.

solve

$A(0) = 2$

$V_0 = 100$ gal

$a = 2$ lb



$$\frac{dA}{dt} = be - \frac{Af}{V_0 + t(e-f)}$$

$$\frac{dA}{dt} = (3)(5) - \frac{A(5)}{100 + 0(5-3)}$$

$$\frac{dA}{dt} = 15 - \frac{5A}{100} \Rightarrow \text{autonomous DE}$$

$$\frac{dA}{dt} = 15 - 0.05A$$

$$-(-15 + 0.05A)$$

$$\int \left(\frac{0.05}{-15 + 0.05A}\right) dA = -0.005\int dt$$

Let $u = 0.05A - 15 \Rightarrow \frac{du}{dA} = 0.05 \Rightarrow dA = \frac{du}{0.05}$

$$\int \frac{\cancel{0.05}}{u} \frac{du}{\cancel{0.05}} = -0.05\int dt$$

$$\ln u = -0.05t + c$$

$$\cancel{\ln}(0.05A - 15) = -0.05t + c$$

$$0.05A - 15 = e^{(-0.05t + c)}$$

$$0.05A - 15 = e^{-0.05t} \cdot e^{c} \to c$$

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$$0.05A - 15 = e^{-0.05t} c$$

$$0.05A = 15 + e^{-0.05t} c.$$

$$A = \frac{15}{0.05} + \frac{c}{0.05} e^{-0.05t}$$

$$\boxed{A = 300 + c e^{-0.05t}}$$

$\because A(0) = 2.$

$$2 = 300 - c e^{-0.05(0)}$$

$$2 = 300 - c e^{0}$$

$$c = 300 - 2.$$

$$\boxed{c = 298}$$

$$\boxed{A = 300 - 298 e^{-0.05t}}$$ Amount of salt at any time 't'

INTEGRATION FORMULAS

$\int 1\,dx = x + c$

$\int a\,dx = ax + c$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$\int \sin x\,dx = -\cos x + c$

$\int \cos x\,dx = \sin x + c$

$\int \sec^2 x\,dx = \tan x + c$

$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

$\int \sec x \tan x\,dx = \sec x + c$

$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

$\int \frac{1}{x}\,dx = \ln|x| + c$

$\int e^x\,dx = e^x + c$

$\int a^x\,dx = \frac{a^x}{\ln a} + c \; ; a > 0, a \neq 1$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$

$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$

$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + c$

$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a} \log\left|\frac{(x-a)}{(x+a)}\right| + c$

$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a} \log\left|\frac{(a+x)}{(a-x)}\right| + c$

$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log|x + \sqrt{x^2-a^2}| + c$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln|x + \sqrt{x^2+a^2}| + c$

DIFFERENTIATION FORMULAS

$(d/dx)(x^n) = nx^{n-1}$

$\frac{d}{dx}(a^x) = a^x \ln a$

$(d/dx)(u \cdot v) = uv' + vu'$

$\frac{d}{dx}(e^x) = e^x$

$\left(\frac{d}{dx}\right)\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

$\left(\frac{d}{dx}\right)(\sin x) = \cos x$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\left(\frac{d}{dx}\right)(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

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$$\boxed{C = 298}$$

$$\boxed{A = 300 - 298\, e^{-0.05t}}$$

Amount of salt at any time 't'

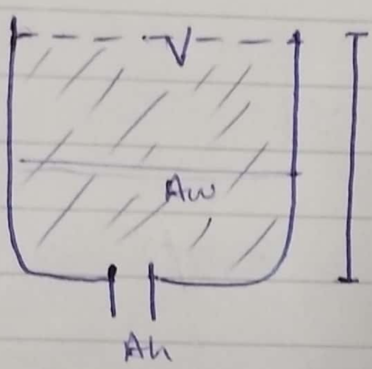
TORICELLI'S LAW

04-12-23

related to flow problems.

$$\left|\frac{dy}{dx}\right| - \cos x = x$$

Nonlinear.



A_w = Area of cross-section

A_h = Area of hole.

$\because Q = \frac{V}{t}$ ← volume

$Q = A\frac{l}{t} = v$

$\boxed{Q = Av}$ ← velocity

Q = flowrate

K.E = P.V (Energy is conserved)

$\frac{1}{2}mv^2 = mgh$

$v^2 = 2gh$

$\boxed{v = \sqrt{2gh}}$ ← flux → flow out

flux: flow of something w.r.t to area.

$Q = Av$

$\frac{dV}{dt} = -A_h v$

-ve sign as volume is decreasing.

$$\boxed{\frac{dV}{dt} = -A_h\sqrt{2gh}}$$

$\boxed{v = \sqrt{2gh}}$, $\boxed{Q = Av}$, $\boxed{\frac{dV}{dt} = -A_h v}$

By using chainrule

$\Rightarrow \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$] as volume is also related to height

$\boxed{\frac{dV}{dt} = -A_h \sqrt{2gh}}$

Assuming $V/h \Rightarrow A$ it is constant.

$\frac{dV}{dh} = A_w$.

$\Rightarrow \frac{dV}{dh} \cdot \frac{dh}{dt} = -A_h \sqrt{2gh}$

$\Rightarrow A_w \cdot \frac{dh}{dt} = -A_h \sqrt{2gh}$.

$\Rightarrow \boxed{\frac{dh}{dt} = -\frac{A_h}{A_w} \sqrt{2gh}}$

Analytic Function.

$z = f(x,y) \rightarrow$ continuous

① if $z_x, z_y \rightarrow$ continuous necessary

② Cauchy Reimann $\hookrightarrow$ sufficient

necessary & sufficient

① → Linear / Non-linear.

② → Solution of DE.

③ → Mixing problem.

Formation of ODE.

Existence & Uniqueness.

A slowtion for a differential equation exist or not.

If exist, is it a unique solution or not.

Lipchitz found a way to identify this.

it says if a certain criteria passes, then it is exist & unique, but if criteria fails, then it doesn't know → sufficient condition.

① sufficient con

② necessary con

③ sufficient & con condition

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Sufficient Condition.

Exactness & Uniqueness.

$$\frac{dy}{dx} = 2\sqrt{y} \qquad \underset{x}{y(0)} = \underset{y}{0} \Rightarrow \text{Initial condition}$$

$$f(x,y) = 2\sqrt{y}.$$

$$\Rightarrow \frac{dy}{dx} = 2\sqrt{y}.$$

$$\Rightarrow \int \frac{1}{\sqrt{y}}\, dy = 2\int x\, dx.$$

$$\Rightarrow \cancel{2}\sqrt{y} = \cancel{2}x + c$$

$$\Rightarrow \sqrt{y} = x + c.$$

$$\Rightarrow y = (x+c)^2. \qquad \Rightarrow \boxed{y = x^2}$$

$$\Rightarrow 0 = (0+c)^2 \qquad \because y(0) = 0.$$

$$\Rightarrow \boxed{c = 0}$$

$y = x^2$ will satisfy the D.E. $\Rightarrow \frac{dy}{dx} = 2\sqrt{y}$.

$$\frac{dy}{dx} = 2x.$$

$$2x = 2\sqrt{x^2}$$

$$\boxed{2x = 2x} \qquad \text{Satisfied.}$$

and $\boxed{y = 0}$, ~~x = 0~~. $dy/dx = 0$.

$$\Rightarrow 0 \qquad \because \frac{dy}{dx} = 2\sqrt{y}.$$

$$\boxed{0 = 0} \qquad \text{Not unique.}$$

$\frac{dy}{dx} = f(x,y) \rightarrow$ continuous

$\frac{\partial f}{\partial y} \longrightarrow$ continuous

} in a domain, then the solution will be unique in that domain.

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Q) $\dfrac{dy}{dx} = x^{1/2}\, y^{1/2}.$ → function is continuous with initial condition.

$y(0) = 0.$

$\dfrac{\partial f}{\partial y} = \dfrac{x^{1/2}}{2\sqrt{y}}.$

if we put $y=0$, then it'll be undefined, not continuous.

$\dfrac{dy}{dx} = \sqrt{x}\,\sqrt{y}.$

$\displaystyle\int \frac{dy}{\sqrt{y}} = \int \sqrt{x}\, dx.$

if $y(1)=0$ $\dfrac{\partial f}{\partial y}$ is still dis-continuous.

$2\sqrt{y} = \dfrac{2}{3} x^{3/2} + C.$

put $y=0$, $x=1$.

$2\sqrt{0} = \dfrac{2}{3}(1)^{3/2} + C$

$0 = \dfrac{2}{3} + C.$

$\boxed{C = -\dfrac{2}{3}}$

$\boxed{2\sqrt{y} = \dfrac{2}{3} x^{3/2} - \dfrac{2}{3}.}$

$\frac{1}{2} + \frac{3}{2} = \frac{4}{2} = 2$

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FINALS

18-12-23

BERNOULLI'S EQUATION

Equations reducable to linear-form.

Standard Form: $\frac{dy}{dx} + P(x)y = Q(x)y^n$; $n \neq 0, n \neq 1$

To linearise the eqn.
Divide whole eqn by $1/y^n$.

$$\frac{1}{y^n} y' + \frac{P(x)}{y^{n-1}} = Q(x). \quad \text{— (A)}$$

Let $\boxed{u = \frac{1}{y^{n-1}}}$

$$\frac{du}{dx} = (-n+1)\frac{1}{y^n} \cdot y'$$

$$u' = \frac{du}{dx} = (1-n)\, y' \cdot \frac{1}{y^n}$$

$$\boxed{\frac{u'}{1-n} = \frac{1}{y^n} y'}$$

Substitute both into (A)

$$\left(\frac{1}{1-n}\right)u' + P(x)\,u = Q(x)$$

$$\boxed{u' + (1-n)P(x)\,u = (1-n)\,Q(x)}$$

Q) $y' + y = y^{-3}$.

$y' + P(x)y = Q(x)\,y^n$ $\quad P(x) = 1,\ Q(x) = 1.$

$n = -3.$

$$\frac{1}{y^{-3}} y' + \frac{1}{y^{-3-1}} = 1$$

y^4

$4y^3$

Let $\boxed{u = \frac{1}{y^{-4}}}$ $\quad \frac{du}{dx} = (\ \ )\, y' \cdot \frac{1}{y^{-4}}$

$$\frac{du}{dx} = 4y^3 y' \Rightarrow u' = 4y^3 y'$$

$$\boxed{\frac{u'}{4} = y^3 y'}$$

$$\frac{u'}{4} + u = 1$$

$$u' + 4u = 4 \quad \text{--- (B)}$$

Linear Equation: $P(x) = 4$, $Q(x) = 4$.

Finding Integrating factor

$$\mu(x) = e^{\int P(x)\,dx}$$

$$\mu(x) = e^{\int 4\,dx} = \boxed{e^{4x}}$$

Mutiple it with eq (B)

$$e^{4x} u' + e^{4x} \cdot 4u = e^{4x} 4.$$

LHS is now total differential.

$$\frac{d}{dx}\left(e^{4x} u\right) = e^{4x} \cdot 4$$

$$d\left(e^{4x} u\right) = 4e^{4x}\,dx.$$

Applying integration.

$$\int d\left(e^{4x} u\right) = \int 4 e^{4x}\,dx. \qquad 4\int e^{4x} \Rightarrow 4\frac{e^{4x}}{4} + c = e^{4x} + c$$

$$e^{4x} u = e^{4x} + c.$$

$$\frac{\cancel{e^{4x}}\, u}{\cancel{e^{4x}}} = \frac{\cancel{e^{4x}}}{\cancel{e^{4x}}} + \frac{c}{e^{4x}}.$$

$$\boxed{u = 1 + c e^{-4x}}$$

Back-substituting $u = 1/y^{-4} = y^4$.

$$\boxed{y^4 = 1 + c e^{-4x}}$$

$$z = f(x,y) \Rightarrow dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$M(x,y)\,dx + N(x,y)\,dy.$$

$$\Rightarrow \frac{\partial f}{\partial x} = M(x,y).$$

$$\int \partial f = \int M(x,y)\,dx$$

$$f = \int M(x,y)\,dx + g(y).$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right] + g'(y).$$

$$\frac{\partial f}{\partial y} = N(x,y).$$

$$\boxed{N(x,y) = \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right] + g'(y)}$$

$$\boxed{M(x,y) = \frac{\partial}{\partial x}\left[\int N(x,y)\,dy\right] + g'(x)}$$

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Q) $(x^2y)\,dx - (x^3 - y^3)\,dy = 0.$

$M_y = x^2 \qquad N_x = -3x^2. \qquad \mu(y) = y^{-4}$

$[x^2y^{-4} \cdot y]\,dx - [x^3y^{-4} - y^{-1}]\,dy = 0.$

$x^3y^{-3}\,dx - \underbrace{[x^3y^{-4} - y^{-1}]}_{1}\,dy = 0.$

$\frac{\partial f}{\partial x} = M = x^2y^{-3}.$

$\int \partial f = \int x^2y^{-3}\,dx + g(y).$

$f = \frac{x^3y^{-3}}{3} + g(y).$

$\frac{\partial f}{\partial y} = \frac{-\cancel{3}\,x^3y^{-3}}{\cancel{3}} + g'(y) \Rightarrow \underbrace{-x^3y^{-4} + g'(y)}_{2}$

equating 1 and 2.

$\cancel{-x^3y^{-4}} + g'(y) = \cancel{-x^3y^{-4}} + y^{-1}$

$g'(y) = y^{-1}$

$\frac{dg}{dy} = 1/y$

$\int dg = \int \frac{1}{y}\,dy$

$\boxed{g = \ln(y)}$

SECOND ORDER 28-12-23

EQUATIONS REDUCIBLE TO FIRST ORDER.

$$f(x, y, y', y'') = 0$$

Case 1: if y is not present in D.E.

$$F(x, y', y'') = 0. \Rightarrow F(x, v, v') = 0.$$

$v = \frac{dy}{dx}$, $v' = \frac{d^2y}{dx^2}$ $\Rightarrow$ Substitutions.

Q) $y'y'' = x\cancel{(y')^2}$

Case - 1.

Let $v = y'$ and $v' = y''$

$$v \cdot \frac{dv}{dx} = x.$$

$$\int v \cdot dv = \int x \, dx$$

$$\frac{v^2}{2} = \frac{x^2}{2} + C.$$

$$v^2 = x^2 + C \Rightarrow \boxed{v = \pm\sqrt{x^2 + C}.}$$

$\Rightarrow$ $v = y'$

$$\frac{dy}{dx} = \pm\sqrt{x^2 + C}.$$

$$\int dy = \pm \int \sqrt{x^2 + C} \, dx$$

$$y = \pm \int \sqrt{x^2 + C} \, dx.$$

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Case 2 $F(y, y', y'') = 0$ (x is absent)

Subtitute: $v = y'$

$v' = \frac{d^2y}{dx^2}$

$y'' = v' = \frac{dv}{dx} \Rightarrow \boxed{\frac{dv}{dy}\,\frac{dy}{dx}} = y' \cdot \frac{dv}{dy}$

$\downarrow$

$v \cdot \frac{dv}{dy}$

Ex: $\frac{dv}{dt} = \frac{d^2x}{dt^2}$, $\frac{dv}{dx} \cdot \frac{dx}{dt} \Rightarrow v \cdot \frac{dv}{dx}$.

Q. $y y' = y''$

Let $v = y'$ and $\frac{dv}{dy} \cdot v = y''$.

$\int y \cdot \cancel{v} = \int \cancel{v} \frac{dv}{dy}$.

$\int y \, dy = \int dv$.

$\frac{y^2}{2} = v + c$.

$\boxed{v = \frac{y^2}{2} + c}$

$\because v = dy/dx$.

$\frac{dy}{dx} = \frac{y^2}{2} + c$.

$\frac{1}{2} \int \frac{dy}{y^2 + a^2} = dx$.

$\boxed{\frac{1}{2a} \tan^{-1}\left(\frac{y}{a}\right) = x + c}$

$\tan^{-1}\left(\frac{y}{a}\right) = 2ax + c$.

$\frac{y}{a} = \tan(2ax + c)$.

$\boxed{y = a \tan(2ax + c)}$

$\frac{dy}{dx} = \frac{y^2 + c}{2}$

$\int \frac{2}{y^2 + c} dy = \int dx$

$2 \; \frac{1}{c} \tan^{-1}\left(\frac{y}{c}\right) = x + c$

$\tan^{-1}\left(\frac{y}{c}\right) = x + c$

$\frac{y}{c} = \tan(cx + c)$

$\boxed{y = c \tan(cx + c)}$.

$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

Second Order ODEs

Classification => Homogenous, Non-homogenous, linear, non-linear,
Equations with variable coefficients & constant coefficients

① Linear Homogenous SODE with constant coefficient.

General form: $ay'' + by' + cy = 0$ — ①

Homogenous means, right side of eqn. is $= 0$.

=> First Order linear Homogenous.

G.F => $y' + P(x)y = 0$.

Solution => $\boxed{y = Ae^{rx}}$ — ②.

$y' = rAe^{rx}$

$y'' = r^2Ae^{rx}$

Substitute y' & y'' in eqn ①

$r^2Ae^{rx} + brAe^{rx} + cAe^{rx} = 0$.

$Ae^{rx}(r^2 + br + c) = 0$.

$r^2 + br + c = 0$.

↳ Auxillary / Characteristic equation

For any D.E, it is linear, if there are two solutions for that D.E, then the linear combination of both solutions will satisfy that D.E

↳ $y = c_1e^{-x}$

$y = c_2e^{x}$

$y = c_1e^{-x} + c_2e^{x}$.

Ex: $y'' + 3y' + 2y = 0$ — Ⓐ.

$r^2 + 3r + 2y = 0$. ⟶ $\frac{-3 \pm \sqrt{9-8}}{2}$

$\frac{-3 \pm 1}{2} \Rightarrow \frac{-3+1}{2}, \frac{-3-1}{2}$

$\frac{-2}{2}, \frac{-4}{2} \Rightarrow \boxed{-1, -2}$

$r_1 = -2$, $r_2 = -1$

$y_1 = c_1e^{-2x}$, $y_2 = c_2e^{-x}$.

Principle of Superposition.

$\boxed{y = c_1e^{-2x} + c_2e^{-x}}$

$y' = \frac{dy}{dx} = -2c_1e^{-2x} - c_2e^{-x}$.

$y'' = 4c_1e^{-2x} + c_2e^{-x}$.

Put y' & y'' in eq Ⓐ

$\Rightarrow 4c_1e^{-2x} + c_2e^{-x} + 3(-2c_1e^{-2x} - c_2e^{-x}) + 2(c_1e^{-2x} + c_2e^{-x}) = 0$

$\Rightarrow \cancel{4c_1e^{-2x}} + \cancel{c_2e^{-x}} - \cancel{6c_1e^{-2x}} - \cancel{3c_2e^{-x}} + \cancel{2c_1e^{-2x}} + \cancel{2c_2e^{-x}} = 0$

$\Rightarrow 0 = 0$ ✓

SECOND ORDER D.E's

1-1-24.

General Form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

Case I. Real & Distinct $y = c_1e^{r_1x} + c_2e^{r_2x}$

divide by a.

$$\frac{d^2y}{dx^2} + \frac{b}{a}\frac{dy}{dx} + \frac{c}{a}y = 0.$$

$\because$ Sum of roots. $r_1 + r_2 = -\frac{b}{a}$

Product of roots. $r_1r_2 = \frac{c}{a}$.

$$\frac{d^2y}{dx^2} - r_1\frac{dy}{dx} - r_2\frac{dy}{dx} + r_1r_2y = 0.$$

$\frac{d}{dx}$ is a linear operator, or so it preserves addition & scalar multiplication

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) - r_1\frac{d}{dx}(y) - r_2\frac{d}{dx}(y) + r_1r_2y = 0.$$

$$\frac{d}{dx}\left(\frac{dy}{dx} - r_1y\right) - r_2\left(\frac{dy}{dx} - r_1y\right) = 0$$

Let $\boxed{u = \frac{dy}{dx} - r_1y.}$ — ①

$$\frac{du}{dx} - r_2u = 0.$$

Solution to FOLHDE

$\therefore$ $\boxed{u = Ae^{r_2x}.}$

Put in ①.

$$Ae^{r_2x} = \frac{dy}{dx} - r_1y. \text{ — ②}$$

Linear First Order D.E

Finding Integrating factor

$$I.F = e^{-r_1x}.$$

multiply I.F by eq ②

$$Ae^{r_2x}e^{-r_1x} = \frac{d}{dx}\left(ye^{-r_1x}\right).$$

$$Ae^{(r_2-r_1)x} = \frac{d}{dx}\left(ye^{-r_1x}\right)$$

Case I: roots are equal.

$$Ae^0 = \frac{d}{dx}\left(ye^{-r_1x}\right)$$

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$$A = \frac{d}{dx}\left(y e^{-r_1 x}\right).$$

$$A\int dx = \int d\left(y e^{-r_1 x}\right)$$

$$y e^{-r_1 x} = Ax + B.$$

$$y = \frac{Ax}{e^{-r_1 x}} + \frac{B}{e^{-r_1 x}}.$$

$$\boxed{y = Axe^{r_1 x} + Be^{r_1 x}}.$$ Real & repeated roots.

Case II: roots are not equal.

$$A e^{(r_2 - r_1)x} = \frac{d}{dx}\left(y e^{-r_1 x}\right).$$

$$\int A e^{(r_2 - r_1)x}\, dx = \int d\left(y e^{-r_1 x}\right)$$

$$\left(\frac{A}{r_2 - r_1}\right) e^{(r_2 - r_1)x} + B = y e^{-r_1 x}.$$

$\because \frac{A}{r_2 - r_1} \Rightarrow$ constant

$\because \frac{A}{r_2 - r_1}$ is constant

$$A e^{r_2 x} e^{-r_1 x} + B = y e^{-r_1 x}.$$

$$\frac{A e^{r_2 x} \cancel{e^{-r_1 x}}}{\cancel{e^{-r_1 x}}} + \frac{B}{e^{-r_1 x}} = y$$

$$\boxed{y = Ae^{r_2 x} + Be^{r_1 x}}$$ Real & distinct roots

Q) $y'' + 2y' + y = 0.$

Auxillary equation.

$$r^2 + 2r + 1 = 0.$$

$$(r+1)^2 = 0.$$

$$(r+1)(r+1) = 0$$

$$\boxed{r_1 = -1,\ r_2 = -1}$$

roots are equal, Case I. $\Rightarrow y = Axe^{r_1 x} + Be^{r_1 x}.$

$$\boxed{y = Ae^{-x} + Bxe^{-x}}$$

Case III: Imaginary Roots

$$y = [c_1 \cos \beta x + c_2 \sin \beta x] e^{\alpha x}$$

Q) $y'' + 4y' + 5y = 0$.

Auxillary Equation

$$r^2 + 4r + 5 = 0.$$

$$r = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$r = \frac{-4 \pm \sqrt{-4}}{2} \Rightarrow \frac{-4 \pm 2i}{2} \Rightarrow \frac{\cancel{2}(-2 \pm i)}{\cancel{2}}$$

$$r = -2+i, \; -2-i$$

Complex roots.

Euler's Identity $= \boxed{e^{i\pi} + 1 = 0}$

$\Rightarrow$ Oiler

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{in\theta} = \cos n\theta + i\sin\theta$$

$$\theta = \pi$$

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$e^{i\pi} = -1$$

$$e^{i\pi} + 1 = 0$$

roots are unequal.

$$y = c_1 e^{(-2-i)x} + c_2 e^{(-2+i)x}.$$

$$y = c_1 e^{-2x} e^{-ix} + c_2 e^{-2x} e^{ix}.$$

$$y = e^{-2x}(c_1 e^{-ix} + c_2 e^{ix}).$$

$$y = e^{-2x}\left[c_1\{\cos(-x) + i\sin(-x)\} + c_2\{\cos(x) + i\sin(x)\}\right].$$

$\because \cos(-x) = \cos(x)$

$\sin(-x) = -\sin x$

$$y = e^{-2x}\left[c_1\cos(x) - c_1 i\sin(x) + c_2\cos(x) + c_2 i\sin(x)\right].$$

~~$y = e^{-2x}[2c_1\cos(x)$~~

$$\boxed{y = e^{-2x}\left[A\cos(x) + B\sin(x)\right]}$$

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Linear Non-Homogenous Second-Order D.E:

General Form : $ay'' + by' + cy = f(x)$.

Q) $y'' + 3y' + 2y = e^x$

$\Rightarrow \frac{d^2}{dx^2}(y) + 3\frac{d}{dx}(y) + 2y = e^x$.

Let $\frac{d^2}{dx^2} = D^2$ and $\frac{d}{dx} = D$.

$\Rightarrow D^2 y + 3Dy + 2y = e^x$.

$\Rightarrow (D^2 + 3D + 2)y = e^x$.

$\Rightarrow y = \left[\frac{1}{D^2+3D+2}\right] e^x$

$\frac{1}{D^2+3D+2} \Rightarrow$ operator

operator is applied on e^x

Case I (i)

$f(x) = e^{ax}$ then $D \xrightarrow{\text{replace}} a$. If denominator is not calculating to 0.

$\Rightarrow y = \left[\frac{1}{1^2+3(1)+2}\right] e^x = \boxed{\frac{e^x}{6} = y}$ → Particular Solution.

Superposition Principle = $y = y_h + y_p$.

y_h → y-homogenous or complentary Solution

y_p → y. Particular Solution.

If eq is homogenous. Solution

$y'' + 3y' + 2y = 0$

$y = C_1 e^{-x} + C_2 e^{-2x}$.

$$\boxed{y = C_1 e^{-x} + C_2 e^{-2x} + \frac{e^x}{6}}$$

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Case I (ii)

$f(x) = e^{ax}$; $D \to a$ & $f(D \to a) = \frac{f(x)}{0}$.

$y'' - 2y' + y = e^{x}$.

$\Rightarrow y_p = \frac{1}{D^2 - 2D + 1}(e^x)$

$\Rightarrow y_p = \frac{1}{1^2 - 2(1) + 1 = 0}(e^x) = ?$

$\Rightarrow y_h = c_1 e^x + c_2 x e^x$.

differentiate denominator w.r.t D and multiply by x.

$y_p = (x)\frac{1}{2D-2}(e^x)$ (check if replacing D by a, again evaluate the denominator to 0 or not)

$\to 2(1) - 2 = 0$ again differentiate.

$\Rightarrow y_p = (x^2)\frac{1}{2}(e^x)$

$\Rightarrow \boxed{y = c_1 e^x + c_2 x e^x + (x^2)\frac{1}{2}(e^x)}$

Derivation:

$\because \frac{d}{dx}\left(\frac{dy}{dx} - r_1 y\right) - r_2\left(\frac{dy}{dx} - r_1 y\right) = e^x$.

Let $u = \frac{dy}{dx} - r_1 y$.

$\frac{du}{dx} - r_2 u = e^x$ linear F.O.D.E.

$I.F = e^{-r_2 x}$.

$\frac{d}{dx}\left(u \cdot e^{-r_2 x}\right) = e^x e^{-r_2 x}$.

$\int d\left(u e^{-r_2 x}\right) = \int e^{x(1-r_2)} dx$.

$u e^{-r_2 x} = \frac{1}{1-r_2} e^{x(1-r_2)} + c$

$$\boxed{u = \frac{1}{1-r_2}\frac{e^{x(1-r_2)}}{e^{-r_2 x}} + \frac{c}{e^{-r_2 x}}}$$

4-1-23.

Case 2 when $f(x)$ is $\sin(ax)$ or $\cos(ax)$

then $D^2 \to -(a^2)$.

Q) $y'' + 4y = \sin(3x)$

$a = 3$.

$-(a)^2 = -9$

$D^2 y + 4y = \sin(3x)$

$(D^2 + 4) y = \sin(3x)$.

$y_p = \frac{1}{D^2+4} \sin(3x)$

$y_p = \frac{1}{-9+4} \sin(3x)$

$y_p = -\frac{1}{5} \sin(3x)$

Particular Solution.

$y'' + 4y = 0$.

$r^2 + 4 = 0$.

$r^2 = -4$.

$\boxed{r = \pm 2i}$

$\Rightarrow y_h = A\cos(2x) + B\sin(2x)$.

Homogeneous Solution.

$$\boxed{y = A\cos(2x) + B\sin(2x) - \frac{1}{5}\sin(3x)}$$

General solution.

Find y_h here.

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Q) $y'' + 5y' + 4y = \cos\left(\frac{3}{2}x\right)$

$D^2y + 5Dy + 4y = \cos\left(\frac{3}{2}x\right)$.

$a = \frac{3}{2}$

$(D^2 + 5D + 4)y = \cos\left(\frac{3}{2}x\right)$

$-(a^2) = -\left(\frac{3}{2}\right)^2 = -\frac{9}{4}$

$y = \frac{1}{D^2+5D+4} \cos\left(\frac{3}{2}x\right)$

$y = \frac{1}{-\left(\frac{3}{2}\right)^2 + 5D + 4} \cos\left(\frac{3}{2}x\right)$.

$-\frac{9}{4} + 4^{\times 4}$

$\frac{-9+16}{4} = \boxed{\frac{7}{4}}$

$y = \frac{1}{5D + \frac{7}{4}} \cos\left(\frac{3}{2}x\right)$

$y = \frac{1}{5} \frac{1}{\left(D + \frac{7}{20}\right)} \cos\left(\frac{3}{2}x\right)$

taking conjugate.

$y = \frac{1}{5} \frac{(D - 7/20)}{(D + 7/20)(D - 7/20)} \cos\left(\frac{3}{2}x\right)$

$y = \frac{1}{5} \frac{(D - 7/20)}{D^2 - \left(\frac{7}{20}\right)^2} \cos\left(\frac{3}{2}x\right)$

$y = \frac{1}{5} \frac{(D - 7/20)}{-\left(\frac{3}{2}\right)^2 - \left(\frac{7}{20}\right)^2} \cos\left(\frac{3}{2}x\right)$

$-\frac{9^{\times 100}}{4} - \frac{49}{400}$

$\frac{-900-49}{400}$

$-\frac{949}{400}$

$y = \frac{1}{5 \times \left(-\frac{949}{400_{80}}\right)} (D - 7/20) \cos\left(\frac{3}{2}x\right)$.

$$\boxed{y_p = -\frac{80}{949}\left[-\frac{3}{2}\sin\left(\frac{3}{2}x\right) - \frac{7}{20}\cos\left(\frac{3}{2}x\right)\right]}$$

find y_h

$\Rightarrow y'' + 5y' + 4y = 0.$

$r^2 + 5r + 4 = 0.$

$\frac{-5 \pm \sqrt{25-16}}{2} \Rightarrow \frac{-5 \pm \sqrt{9}}{2} = \frac{-5 \pm 3}{2}$

$\frac{-5+3}{2}, \frac{-5-3}{2} \Rightarrow \frac{-\cancel{2}}{\cancel{2}}, \frac{-8}{\cancel{2}} \Rightarrow \boxed{\underset{r_1}{-1}, \underset{r_2}{-4}}$

case : real & distinct.

$y = Ae^{r_2 x} + Be^{r_1 x}.$

$$\boxed{y_h = Ae^{-4x} + Be^{-x}}$$

$$\boxed{y = Ae^{-4x} + Be^{-x} - \frac{80}{949}\left[\frac{-3}{2}\sin\left(\frac{3}{2}x\right) - \frac{7}{20}\cos\left(\frac{3}{2}x\right)\right]}$$

Q) $y'' + 25y = \sin(5x).$

$D^2y + 25y = \sin(5x).$

$(D^2+25)y = \sin(5x)$

$y = \frac{1}{D^2+25}\sin(5x).$

$y = x\frac{1}{2D}\sin(5x).$

$y = \frac{x}{2}\frac{1}{D}\sin(5x)$

$y = \frac{x}{2}\int \sin(5x)\,dx.$

$$\boxed{y_p = -\frac{x}{10}\cos(5x)}$$

$a = 5$

$-(a^2) = -25$

$-25+25 = 0$

Not allowed ~~so~~ differentiate denominator & multiply with x.

$D \to$ derivative.

$\frac{1}{D} \to$ integral.

Another approach

Mul & Div by D.

$y = \frac{x}{2}\frac{D}{D^2} \times \sin(5x)$

$y = \frac{x}{2}\frac{D}{-25} \sin(5x)$

$y = \frac{x}{2(-\cancel{25})}\cancel{5}\cos(5x)$

$\boxed{y = \frac{-x}{10}\cos(5x)}$

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Find y_h.

$y'' + 25y = 0$.

$r^2 + 25 = 0$.

$r^2 = -25$.

$\boxed{r = \pm 5i}$

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8 - 1 - 24

Case 3: if $f(x)$ is a polynomial.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 x^0.$$

polynomial is an algebraic function, where n is a +ve integer

Q) $y'' + 3y' + 2y = x^3 - 2x^2 + 1$

For homogenous Solution.

A. E $\Rightarrow r^2 + 3r + 2 = 0.$

$$y_h = c_1 e^{-x} + c_2 e^{-2x}.$$

For particular Solution.

$\Rightarrow y'' + 3y' + 2y = x^3 - 2x^2 + 1.$

$\Rightarrow D^2 y + 3Dy + 2y = x^3 - 2x^2 + 1$

$\Rightarrow (D^2 + 3D + 2)\, y = x^3 - 2x^2 + 1$

$$\Rightarrow y_p = \frac{1}{D^2 + 3D + 2} (x^3 - 2x^2 + 1).$$

$$\Rightarrow y_p = \frac{1}{2} \frac{1}{\left(1 + \frac{D^2 + 3D}{2}\right)} (x^3 - 2x^2 + 1)$$

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$$\Rightarrow y_p = \frac{1}{2}\left(1+\frac{D^2+3D}{2}\right)^{-1}\left[x^3-2x^2+1\right]$$

$\because (1+x)^{-1} = 1-x+x^2-x^3+x^4 \dots$ (Binomial expansion).

$$\Rightarrow y_p = \frac{1}{2}\left[1-\frac{D^2+3D}{2}+\left(\frac{D^2+3D}{2}\right)^2-\left(\frac{D^2+3D}{2}\right)^3+\dots\right]\left[x^3-2x^2+1\right]$$

$$\Rightarrow y_p = \frac{1}{2}\left[x^3-2x^2+1-\frac{1}{2}(9x^2-6x-4)+\frac{27}{2}x-\frac{81}{4}\right]$$

$$\Rightarrow y_p = \frac{1}{2}\left[x^3-\frac{13}{2}x^2+\frac{33}{2}x-\frac{69}{4}\right]$$

$$\Rightarrow \boxed{y_p = \frac{x^3}{2}-\frac{13}{4}x^2+\frac{33}{4}x-\frac{69}{8}}$$

for $\frac{1}{2}(D^2+3D)(x^3-2x^2+1)$

$\Rightarrow \frac{1}{2}\left[6x-4+9x^2-12x\right]$

$\Rightarrow \frac{1}{2}(9x^2-6x-4)$.

for $\left(\frac{D^2+3D}{2}\right)\left(\frac{D^2+3D}{2}\right)(x^3-2x^2+1)$

$\Rightarrow \frac{1}{2}(D^2+3D)\left[\frac{1}{2}(9x^2-6x-4)\right]$

$\Rightarrow \frac{1}{4}(\cancel{18}+54x-\cancel{18})$

$\Rightarrow \frac{27}{2}x$

for $\left(\frac{D^2+3D}{2}\right)^3[x^3-2x^2+1]$

$\frac{1}{2}(D^2+3D)\left(\frac{27}{2}x\right)$

$\frac{1}{2}\left(\frac{81}{2}\right) \Rightarrow \boxed{\frac{81}{4}}$

Verification.

$$y'_p = \frac{3}{2}x^2-\frac{13}{2}x+\frac{33}{4}.$$

$$y''_p = \frac{6x}{2}-\frac{26}{4}.$$

$\frac{9}{2}-\frac{13}{2}$

$\frac{9-13}{2} = -\frac{4}{2}$

$$\left(\frac{6x}{2}-\frac{26}{4}\right)+\left(\frac{9}{2}x^2-\frac{78}{4}x+\frac{99}{4}\right)$$

$$+\left(x^3-\frac{13}{2}x^2+\frac{33}{2}x-\frac{69}{4}\right)$$

$$= x^3-2x^2+1$$

Satisfied

$0 = \frac{6}{2}-\frac{78}{4}+\frac{33}{2}$

$\frac{12-78+66}{4}$

78
12
-66
99
-33

$-\frac{26}{4}+\frac{99}{4}-\frac{69}{4} = 1$

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Case 4: when $f(x)$ is mixture of ~~algebraic~~ polynomial & exponential function.

$$f(x) = \left(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 x^0\right) e^{ax}.$$

Shift $D \to (D + a)$

Q) $y'' + 3y' + 2y = x e^{2x}.$

$$D^2 y + 3Dy + 2y = xe^{2x}.$$

$$(D^2 + 3D + 2)y = xe^{2x}.$$

$$y_p = \frac{1}{D^2+3D+2}\left(xe^{2x}\right) \Rightarrow y_p = e^{2x} \cdot \frac{1}{\{(D+2)^2+3(D+2)+2\}} \cdot x$$

$$\Rightarrow y_p = e^{2x} \frac{1}{D^2+4D+4+3D+6+2} x.$$

$$\Rightarrow y_p = e^{2x} \frac{1}{D^2+7D+12} x$$

$$\Rightarrow y_p = \frac{1}{12} \frac{1}{\left(1+\frac{D^2+7D}{12}\right)} e^{2x} x.$$

$$\Rightarrow y_p = \frac{e^{2x}}{12}\left(1+\frac{D^2+7D}{12}\right)^{-1}(x)$$

$$\because (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$\Rightarrow y_p = \frac{e^{2x}}{12}\left[1 - \left(\frac{D^2+7D}{12}\right) + \left(\frac{D^2+7D}{12}\right)^2 - \cdots\right](x).$$

$$\Rightarrow y_p = \frac{e^{2x}}{12}\left[x - \frac{7}{12}\right]$$

$$\Rightarrow y_p = \frac{xe^{2x}}{12} - \frac{7e^{2x}}{144}.$$

$$\frac{xe^{2x}}{3} + \frac{5}{36}e^{2x} + \frac{3xe^{2x}}{6} - \frac{3}{72}e^{2x} + \frac{2xe^{2x}}{12} - \frac{7(2)}{144}e^{2x}$$

$$\boxed{xe^{2x}}$$

Verifie

Verification.

$$y'_p = \frac{2xe^{2x} + e^{2x}}{12} - \frac{\cancel{14}\,7\, e^{2x}}{\cancel{144}\,72}$$

$$y'_p = \frac{xe^{2x}}{6} + \frac{e^{2x}}{12} - \frac{7}{72}e^{2x}. \Rightarrow \boxed{\frac{xe^{2x}}{6} - \frac{1}{72}e^{2x}}$$

$$y''_p = \frac{2xe^{2x}}{6} + \frac{e^{2x}}{6} + \frac{2e^{2x}}{12} - \frac{14}{72}e^{2x}.$$

$$y''_p = \frac{xe^{2x}}{3} + \frac{e^{2x}}{6} + \frac{e^{2x}}{6} - \frac{7}{36}e^{2x}.$$

$$\frac{xe^{2x}}{3} + \frac{e^{2x}}{3} - \frac{7}{36}e^{2x}. \Rightarrow \boxed{\frac{xe^{2x}}{3} + \frac{5}{36}e^{2x}}$$

CamScanner

Date: 8-1-24.

Q) $y'' - 2y + y = xe^x \sin x$.   Let $y'' = D^2$ any $y' = D$.

$D^2y - 2Dy + y = xe^x \sin x$.

$(D^2 - 2D + 1)y = xe^x \sin x$.

$y_p = \frac{1}{D^2 - 2D + 1} (xe^x \sin x)$.

$y_p = \frac{1}{(D-1)^2} (xe^x \sin x)$ shif $D \to (D+a)$

$a = 1$

$y_p = (e^x) \frac{1}{(D \cancel{+1} - \cancel{1})^2} (x \sin x)$

$y_p = (e^x) \frac{1}{D^2} (x \sin x)$ $\because \frac{1}{D} = \int \Rightarrow \frac{1}{D^2} = \iint$

$y_p = (e^x) \iint (x \sin x)\, dx$ $\because \int fg' = fg - \int f'g$

using by parts

$f = x \qquad g' = \sin x$

$f' = 1 \qquad g = -\cos x$

$y_p = e^x \int \left(-x\cos x + \int \cos x (1)\, dx\right) dx$

$y_p = e^x \int (-x\cos x + \sin x)\, dx$

$y_p = e^x \left[-\int (x\cos x)\, dx + \int (\sin x)\, dx\right]$.

$\cancel{y_p = e^x}$ $f = x \qquad g' = \cos x$

$f' = 1 \qquad g = \sin x$

$y_p = e^x \left[-\left(x\sin x - \int (1)(\sin x)\, dx\right) + (-\cos x)\right]$

$y_p = e^x \left[-x\sin x + (-\cos x) - \cos x\right]$.

$y_p = e^x \left[-x\sin x - \cos x - \cos x\right]$.

$y_p = e^x \left[-x\sin x - 2\cos x\right]$.

$\boxed{y_p = e^x(-x\sin x) + e^x(2\cos x)}$.

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Date:

For $y_h \Rightarrow y'' - 2y + y = 0$

$r^2 - 2r + 1 = 0$

$r^2 - r - r + 1 = 0$

$r(r-1) - 1(r-1) = 0$

$(r-1)(r-1) = 0$

$\boxed{r = 1}$

$(r-1)^2 = 0$

$\boxed{r = 1}$

$$\frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)}$$

$$\frac{4 \pm \sqrt{0}}{2} \rightarrow \frac{4}{2} = 2.$$

case I : roots are equal

$$\boxed{y_h = Axe^x + Be^x}$$

$y = y_h + y_p$

$$\boxed{y = Axe^x + Be^x + e^x(-x\sin x) + e^x(2\cos x)}$$

11-1-24.

Variation of Parameters.

$$y_p = -y_1 \int \frac{f(x)\, y_2}{W(x)}\, dx + y_2 \int \frac{f(x)\, y_1}{W(x)}\, dx.$$

$\because y'' + ay' + by = f(x).$

$y_h = C_1 y_1(x) + C_2 y_2(x)$

Wronskian:

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

for rational functions
inverse trigonometric
logarithmic
↳ wronskian works.

Q) $y'' + 9y = \operatorname{cosec}(3x)$

Find y_h.

$r^2 + 3^2 = 0.$

$\boxed{r = \pm 3i}$

$$y_h = A\underbrace{\cos(3x)}_{y_1} + B\underbrace{\sin(3x)}_{y_2}.$$

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Date:

$y_1 = \cos(3x)$ $\qquad$ $y_2 = \sin(3x)$.

$W(x) \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix}$ $\Rightarrow 3\cos^2(3x) - (-3\sin^2(3x))$

$\Rightarrow 3\cos^2(3x) + 3\sin^2 3x$.

$\Rightarrow 3(\underbrace{\cos^2(3x) + \sin^2(3x)}_{1}) = \boxed{3}$

$\boxed{W(x) = 3}$

Find $y_P = -y_1 \int \frac{f(x)\, y_2}{W(x)} dx + y_2 \int \frac{f(x)\, y_1}{W(x)} dx.$

$y_p = -\cos(3x) \int \frac{\operatorname{cosec}(3x)\sin(3x)}{3} dx + \sin(3x) \int \frac{\operatorname{cosec}(3x)\cos(3x)}{3} dx$

$y_p = -\cos(3x) \int \frac{1}{\cancel{\sin(3x)}} \cdot \frac{\cancel{\sin(3x)}}{3} dx + \sin(3x) \int \frac{1}{3} \frac{1}{\sin(3x)} \frac{3\cos(3x)}{3} dx$

$\times$ & $\div$ by 3

$y_p = -\cos(3x)\left(\frac{x}{3}\right) + \frac{\sin(3x)}{3} \int \frac{1}{\sin(3x)} \frac{3\cos(3x)}{3} dx$

$y_p = -\cos(3x)\left(\frac{x}{3}\right) + \frac{\sin(3x)}{3} \int \frac{1}{3u} \cancel{\cos(3x)} \cdot \frac{du}{\cancel{3\cos(3x)}}$

$y_p = -\cos(3x)\left(\frac{x}{3}\right) + \frac{\sin(3x)}{9} \ln|u| + c.$

Let $u = \sin(3x)$

$\frac{du}{dx} = 3\cos(3x)$

$dx = \frac{du}{3\cos(3x)}$

$$\boxed{y_p = -\cos(3x)\left(\frac{x}{3}\right) + \frac{\sin(3x)}{9} \ln|\sin(3x)| + c.}$$

Q) $y'' + 3y' + 2y = \frac{e^x}{1+e^x}.$

$r^2 + 3r + 2 = 0$.

$\frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2}$

$\frac{-3+1}{2}, \frac{-3-1}{2}$

$\boxed{-1, -2}$

$r_1 = -1$, $r_2 = -2$.

$y = A\underbrace{e^{-2x}}_{y_1} + B\underbrace{e^{-x}}_{y_2}.$

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Date:

$y_1 = e^{-x}$, $y_2 = e^{-2x}$

$$w(x) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} \Rightarrow e^{-x}(-2e^{-2x}) - (e^{-2x})(-e^{-x})$$

$$\Rightarrow -2e^{-3x} + e^{-3x}$$

$$\Rightarrow \boxed{-e^{-3x}}$$

$$y_p = -e^{-x} \int \frac{e^x}{1+e^x} \cdot \frac{e^{-2x}}{(-e^{-3x})} dx - e^{-2x} \int \frac{e^x}{1+e^x} \cdot \frac{e^{-x}}{(-e^{-3x})} dx.$$

$$y_p = e^{-x} \int \frac{e^{x-2x-(-3x)}}{1+e^x} dx - e^{-2x} \int \frac{e^{x-x-(-3x)}}{1+e^x} dx.$$

$$y_p = e^{-x} \int \frac{e^{2x}}{1+e^x} dx - e^{-2x} \int \frac{e^{3x}}{1+e^x} dx.$$

Let $u = 1+e^x \Rightarrow e^x = u-1$. Let $u = 1+e^x = e^x = u-1$

$\frac{du}{dx} = e^x$ $e^{2x} = (u-1)^2$

$du = dx\, e^x$.

$$y_p = e^{-x} \int \frac{e^x\, e^x}{1+e^x} dx. - e^{-2x} \int \frac{e^{2x}\, e^x}{1+e^x} dx.$$

$$y_p = e^{-x} \int \frac{u-1}{u} du - e^{-2x} \int \frac{(u-1)^2}{u} du.$$

$$y_p = e^{-x} \int 1 - \frac{1}{u} du. - e^{-2x} \int \left(\frac{u^2}{u} - \frac{2u}{u} + \frac{1}{u}\right) du$$

$$y_p = e^{-x} (u - \ln|u|) - e^{-2x} \int (u - 2 + 1/u) du.$$

$$y_p = e^{-x} [1+e^x - \ln(1+e^x)] - e^{-2x} \left[\frac{u^2}{2} - 2u + \ln|u|\right].$$

$$\boxed{y_p = e^{-x}[1+e^x - \ln(1+e^x)] - e^{-2x}\left[\frac{e^{2x}}{2} - 2(1+e^x) + \ln|1+e^x|\right]}$$

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Date:

Q) $y'' - 2y' + y = \frac{e^x}{x}$

$r^2 - 2r + r = 0$

$r_1 = 1, r_2 = 1$

$y_h = c_1 e^{\cancel{2}x} + c_2 x e^{r_1 x}$

$y_h = c_1 e^x + c_2 x e^x$

$\uparrow y_1 \qquad \uparrow y_2$

$y_1 = e^x, \; y_2 = xe^x$

$w(x) = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}$

$e^x \cdot xe^x + e^{2x} - xe^{2x}$

$\cancel{xe^{2x}} + e^{2x} - \cancel{xe^{2x}}$

$w(x) = \boxed{e^{2x}}$

$y_p = -e^x \int \frac{\cancel{x}\cancel{e^x}}{\cancel{e^{2x}}} \cdot \frac{\cancel{e^x}}{\cancel{x}} \, dx + xe^x \int \frac{\cancel{e^x}}{\cancel{e^{2x}}} \cdot \frac{\cancel{e^x}}{x} \, dx$

$y_p = -e^x \int 1 \, dx + xe^x \int \frac{1}{x} \, dx$

$y_p = -e^x \cdot x + xe^x \cdot \ln|x|$

$\boxed{y_p = -xe^x + xe^x \ln|x|}$

Second Order DEs — Auxillary Eq: $r^2 + br + c = 0$

Linear Homogenous:

① Solutions are real and distinct $\Rightarrow y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

② Solutions are real and repeated $\Rightarrow y = c_1 x e^{r_1 x} + c_2 e^{r_2 x}$

③ Solutions are complex $\Rightarrow e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$

Linear Non-homogenous:

① RHS of D.E. has exponential function.
replace $D \to a$, if it undefines the eq, then differentiate denumenator wrt to D & multiply by x.

② RHS of D.E. has $\sin ax$ or $\cos ax$.
replace $D \to -(a)^2$, if required, take conjugate of D terms..

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(3) RHS of DE has polynomial function.
In denumenator, take the constant term common, then inverse it, then expand it using binomial expansion & then apply operators on polynomial function.
Equation will be expanded, times polynomial degree.

(4) RHS of DE has combination of polynomial, exponential & trigonometric functions.
replace $D \to D+a$ by place exponential term at first
If it remains, solve it according to the term at the right side of D operator terms, if it is sin / cos, then $D \to -(a^2)$, if polynomial, the expansion.

(5) Variety of Parameters.
RHS of eq has inverse trigonometric, logarithmic, rational functions.

Use wronskian method, with formula.

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad y_p = -y_1 \int \frac{f(x)\, y_2\, dx}{w(x)} + y_2 \int \frac{f(x)\, y_1\, dx}{w(x)}$$

$$y_h = c_1 \underbrace{e^{r_1 x}}_{y_1} + c_2 \underbrace{e^{r_2 x}}_{y_2}$$

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Date: 15-1-24

CAUCHY-EULER O.D.E.

Linear-homogenous Second order D.E. with variable coefficients.

General form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_0 y = f(x).$$

equidimensional: order of derivative in a term is consistent with variable's power.

Let $x = e^z \Rightarrow \underline{z = \log x}$. Substitution we use to solve.

or let $y = x^m$.

Q) $x^2 y'' + xy - 4y = 0$

Let $y = x^m$

$y' = m x^{m-1}$

$y'' = m(m-1) x^{m-1}$

$x^2 m(m-1) x^{m-2} + x m x^{m-1} - 4x^m = 0.$

$m(m-1) x^{m-\cancel{2}+\cancel{2}} + m x^{m-\cancel{1}+\cancel{1}} - 4x^m = 0$

$m(m-1) x^m + m x^m - 4x^m = 0.$

$(m(m-1) + m\cancel{x^m} - 4\cancel{x^m}) x^m = 0.$

$m(m-1) + m - 4 = 0.$

$m^2 - \cancel{m} + \cancel{m} - 4 = 0$

$m^2 = 4.$

$\boxed{m = \pm 2}$

$y = x^m \Rightarrow y_1 = x^2, \quad y_2 = x^{-2}.$

$\boxed{y = c_1 x^2 + c_2 x^{-2}}$

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Date:

Q) $x^2y'' - 3xy' + 4y = 0.$

Let $y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}.$

$x^2 m(m-1)x^{m-2} - 3x\, m x^{m-1} + 4x^m = 0.$

$m(m-1)x^{m-2+2} - 3mx^{m-1+1} + 4x^m = 0.$

$m(m-1)x^m - 3mx^m + 4x^m = 0.$

$(m(m-1) - 3m + 4)x^m = 0.$

$m^2 - m - 3m + 4 = 0.$

$m^2 - 4m + 4 = 0.$

$(m-2)^2 = 0.$

$m_1 = 2, \; m_2 = 2.$

$y = x^m$

$y_1 = x^2, \quad y_2 = x^2.$

~~y = c~~

Roots are real & repeated.

$y = c_1 x^m + c_2 (\ln x) x^m.$

$$\boxed{y = c_1 x^2 + c_2 (\ln x) x^2}$$

Let $x = e^z \rightarrow z = \ln x$

$\therefore y = c_1 e^{rz} + c_2 z e^{rz}$

$y = c_1 e^{r\ln x} + c_2 \ln x\, e^{r\ln x}$

$y = c_1 e^{\ln x^r} \cancel{+ c_2 \ln x\, e^{\ln x^r}}$

$$\boxed{y = c_1 x^r + c_2 (\ln x) x^r}$$

Q) $x^2y'' + y = 0$

$x^2 m(m-1)x^{m-2} + x^m = 0.$

$m(m-1)x^{m-2+2} + x^m = 0$

$m(m-1)x^m + x^m = 0$

$(m(m-1) + 1)x^m = 0.$

$m^2 - m + 1 = 0.$

$m = \dfrac{1 \pm \sqrt{1-4}}{1} \Rightarrow \boxed{m = \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2} i}$

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Date:

complex roots.

$\because y = e^{\frac{1}{2}z}\left[A\cos\left(\frac{\sqrt{3}}{2}z\right) + B\sin\left(\frac{\sqrt{3}}{2}z\right)\right].$ $\quad \because x^z = e^z$ $\downarrow$ $z = \ln x.$

$y = e^{\frac{1}{2}\ln x}\left[A\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + B\sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right].$

$$\boxed{y = x^{1/2}\left[A\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + B\sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right]}$$

$\Rightarrow y = x^{\alpha}\left[A\cos(\beta\ln x) + B\sin(\beta\ln x)\right]$ → case when roots are complex.

Q) ~~x^2~~ Linear Non-homogenous. with variable coefficients.

Q) $x^2 y'' + 2xy' = \ln x.$

$x = e^z \rightarrow z = \ln x.$

$z = \ln x$ differentiate w.r.t to x.

$\boxed{\frac{dz}{dx} = \frac{1}{x}}$

$\therefore \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}.$

$\rightarrow \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz}.$

or $\boxed{x\frac{dy}{dx} = \frac{dy}{dz}}$

$xy' = \frac{dy}{dz} = D$

$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{dy}{dz}\cdot\frac{dz}{dx}\right]$

$= \frac{d}{dx}\left[\frac{1}{x}\cdot\frac{dy}{dz}\right]$

$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$

for $\frac{1}{x}\frac{d}{dx}\frac{dy}{dz}.$

$\frac{1}{x}\frac{d}{dz}\left(\frac{dz}{dx}\cdot\frac{dy}{dx}\right).$

$\frac{1}{x}\left(\frac{d}{dz}\frac{dy}{dx}\right)\frac{dz}{dx}.$

$\frac{1}{x}\frac{d^2y}{dz^2}\cdot\frac{1}{x} = \boxed{\frac{1}{x^2}\frac{d^2y}{dz^2}}$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

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$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right].$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}.$$

$$\boxed{x^2y'' = D^2 - D = D(D-1)}.$$

Q) $x^2y'' + 2xy' = \ln x.$

Substitute: $x^2y'' = D_y(D_y-1)$, $xy' \Rightarrow Dy$

$D_y(D_y-1) + 2D_y = \cancel{\ln x}\ z$ $\quad\because x = e^z$

$D_y^2 - D_y + 2D_y = z.$ $\quad z = \ln x.$

$(D^2+D)y = z.$ $\quad y_h =$

$$y_p = \frac{1}{D^2+D}(z). \Rightarrow \frac{1}{D(D-1)}(z)$$

$y_p = \frac{1}{D}(\cancel{D}1+D)^{-1}(z).$ $\quad (1+x)^n = 1 - x + x^2 - x^3 + \dots$

$y_p = \frac{1}{D}\left[1 - (\cancel{D}1+D)^1 + \dots\right](z).$

$(\cancel{D}1+D)(z) = 1$

$y_h \Rightarrow x^2y'' + 2xy' = 0.$

$D^2 + D$

$r^2 + r = 0$

$r(r+1) = 0$

$\boxed{r=0}, \boxed{r=-1}.$

$y_p = \frac{1}{D}[z - 1]$

$\boxed{y_h = c_1e^{01} + c_2e^{-z}}$

$\boxed{y_h = c_1 + c_2e^{-z}}$

$y_p = \int (z-1)\, dz.$

$$\boxed{y_p = \frac{z^2}{2} - z}$$

$y_h = c_1 + c_2 \cancel{e}^{+\ln x^{-1}}$

$\boxed{y_h = c_1 + c_2x^{-1}}$

$\because z = \ln x,$

$$y_p = \frac{(\ln x)^2}{2} - \ln x$$

$\boxed{y_1 = 1}$

$\boxed{y_2 = x^{-}}$

$y = y_h + y_p.$

$$\boxed{y = c_1 + c_2x^{-1} + \frac{(\ln x)^2}{2} - \ln x}$$

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Linear Homogenous Eq. With Constant Coefficients. Date: 25-01-24

Reduction of Order

$y = c_1 y_1(x) + c_2 y_2(x)$.

General form:

$y'' + P(x) y' + Q(x) y = 0$. — Linear homogenous SODE.

$y_1 \neq \alpha y_2$.

$\frac{y_1}{y_2} \neq \alpha$.

then

$\frac{y_1(x)}{y_2(x)} = u(x)$

$y_1(x) = y_2(x)\, u(x)$

y_1 & y_2 are linearly independent

$\Rightarrow y'' + P(x) y' + Q(x) y = 0$ — ①

Let one solution is known.

$\Rightarrow y_2 = y_1 u$

$\Rightarrow y_2' = y_1 u' + y_1' u$

$\Rightarrow y_2'' = y_1 u'' + y_1' u' + y_1' u' + y_1'' u$.

$\Rightarrow y_2'' = y_1 u'' + 2y_1' u' + y_1'' u$.

Substitute in ①.

$\Rightarrow y_1 u'' + 2y_1' u' + y_1'' u + P(x)[y_1 u' + y_1' u] + Q(x)[y_1 u] = 0$

$\Rightarrow y_1 u'' + 2y_1' u' + \boxed{y_1'' u} + P(x) y_1 u' + \boxed{P(x) y_1' u} + \boxed{Q(x) y_1 u} = 0$.

$\Rightarrow u\underbrace{[y_1'' + P(x) y_1' + Q(x) y_1]}_{0} + y_1 u'' + 2y_1' u' + P(x) y_1 u' = 0$

$= 0$.

$\Rightarrow$ Since, it is also the solution of ①, it will satisfy eq $= 0$

$\Rightarrow y_1 u'' + 2y_1' u' + P(x) y_1 u' = 0$.

divide by y_1

$\Rightarrow \frac{\cancel{y_1} u''}{\cancel{y_1}} + \frac{2y_1' u'}{y_1} + \frac{P(x) \cancel{y_1} u'}{\cancel{y_1}} = 0$

$\Rightarrow u'' + 2u' \frac{y_1'}{y_1} + P(x) u' = 0$.

$\Rightarrow u'' + u' \left[2 \frac{y_1'}{y_1} + P(x) \right] = 0$

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$\Rightarrow u'' + u'\left[2\frac{y_1'}{y_1} + P(x)\right] = 0.$

Let $u' = v$, $u'' = v'$

$\Rightarrow v' + v\left[2\frac{y_1'}{y_1} + P(x)\right] = 0.$

$\Rightarrow \frac{dv}{dx} = -v\left[2\frac{y_1'}{y_1} + P(x)\right] = 0.$

$\Rightarrow \int \frac{dv}{v} = -\int\left[2\frac{y_1'}{y_1} + P(x)\right] dx.$

$\Rightarrow \ln v = -2\int \frac{y_1'(x)}{y_1(x)}\,dx \cancel{*} - \int P(x)\,dx.$

$\Rightarrow \cancel{e}^{\ln v} = e^{-2\ln y_1} e^{-\int P(x)\,dx}.$

$\Rightarrow v =$ $\cancel{e^{[-2\ln y_1 - \int P(x)\,dx]}}$

$\Rightarrow \cancel{e}^{\ln v} = e$

$\Rightarrow v = \cancel{e}^{\ln y_1^{-2}} \cdot e^{-\int P(x)\,dx}$

$\Rightarrow \boxed{v = y_1^{-2} \cdot e^{-\int P(x)\,dx}}.$

$\cancel{\Rightarrow}$ $\because v = u' = du/dx.$

$\Rightarrow \frac{du}{dx} = \frac{1}{y_1^2} e^{-\int P(x)\,dx}.$

$\Rightarrow \int du = \int \frac{e^{\cancel{\int P(x)\,dx}}}{y_1^2}\,dx.$

$\Rightarrow u = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx.$

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$$\Rightarrow \quad u = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx.$$

$$\because \quad y_2 = y_1 \cdot u$$

$$u = \frac{y_2}{y_1}$$

$$\Rightarrow \quad \frac{y_2}{y_1} = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx.$$

$$\Rightarrow \quad \boxed{y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx.}$$

Q) $y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$, Find the general solution of the equation.

$$\because \quad y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx.$$

divide main eq with x^2.

$$\Rightarrow \quad \frac{\cancel{x^2}y''}{\cancel{x^2}} - \frac{3\cancel{x}y'}{x\cancel{x}} + \frac{4y}{x^2} = 0.$$

$$\Rightarrow \quad y'' - \frac{3}{x}y' + \frac{4y}{x^2} = 0.$$

$$P(x) = -\frac{3}{x} \quad \text{and} \quad y_1 = x^2.$$

$$\Rightarrow \quad y_2 = x^2 \int \frac{e^{-\int(-3/x)\,dx}}{(x^2)^2}\,dx.$$

$$\Rightarrow \quad y_2 = x^2 \int \frac{e^{3\ln x}}{x^4}\,dx$$

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$$\Rightarrow \quad y_2 = x^2 \int \frac{\cancel{x^3}}{x^{\cancel{4}}} \, dx.$$

$$\Rightarrow \quad y_2 = x^2 \int \frac{1}{x} \, dx.$$

$$\Rightarrow \quad \boxed{y_2 = x^2 \ln x}.$$

$$\because \quad y = C_1 y_1 + C_2 y_2$$

$$\boxed{y = C_1 x^2 + C_2 x^2 \ln x}.$$

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$\Rightarrow y_2 = x^2 \int \frac{\cancel{x}}{x^{\cancel{x}}} dx.$

$\Rightarrow y_2 = x^2 \int \frac{1}{x} dx.$

$\Rightarrow \boxed{y_2 = x^2 \ln x}.$

$\because y = C_1 y_1 + C_2 y_2$

$\boxed{y = C_1 x^2 + C_2 x^2 \ln x}.$

29-01-23

POWER SERIES SOLUTION

Special case of Taylor Series → Mclaurin Series.

a function can be expanded in this way.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$

Q) $y' - xy = 0$

derivative of y.

Substitute y and y'

$$[c_1 + 2c_2 x + 3c_3 x^2 + \cdots] - x[c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots] = 0$$

$$c_1 + x(2c_2 - c_0) + x^2(3c_3 - c_1) + x^3(4c_4 - c_2) + \cdots = 0$$

comparing each term with R.H.S (0).

$\boxed{c_1 = 0}$

$2c_2 - c_0 = 0 \Rightarrow \boxed{c_2 = \frac{1}{2}c_0}$

$3c_3 - c_1 = 0$

$c_3 = \frac{1}{3}\underset{0}{c_1} \Rightarrow c_3 = \frac{1}{3}\left(\frac{1}{2}c_0\right) \Rightarrow \boxed{c_3 = \frac{1}{6}c_0}$

$\boxed{c_3 = 0}$

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$4c_4 - c_2 = 0$

$c_4 = \frac{1}{4} c_2$

$c_4 = \frac{1}{4}\left(\frac{1}{2} c_0\right)$

$\boxed{c_4 = \frac{1}{8} c_0}$

$y = \cancel{c_0} + \frac{1}{2} c_0 x + \frac{1}{6} \cancel{c_0 x^2} + \frac{1}{24} c_0 \cancel{x^3} + \dots$

$$\boxed{y = c_0 + \frac{1}{2} c_0 x^2 + \frac{1}{8} c_0 x^4 + \dots}$$

$$\boxed{y = c_0\left[1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \dots\right]}.$$

$\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$e^{x^2/2} = 1 + \frac{x^2}{2} + \frac{(x^2/2)^2}{2!} + \dots$

$e^{x^2/2} = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \dots \rightarrow e^{x^2/2} \Rightarrow \boxed{c_0 e^{x^2/2} = y}$

Q) $y'' + xy' = 0$. Take atleast 6 terms of y.

$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6$

$y' = 0 + c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \dots$

$y'' = 0 + 0 + 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \dots$

$[2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3] + x[c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 \dots]$

$2c_2 + x(6c_3 + c_1) + x^2(12c_4 + 2c_2) + x^3(3c_3 + 20c_5) =$

Comparing.

$2c_2 = 0 \Rightarrow \boxed{c_2 = 0}$

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$y'' + xy' = 0$

$6c_3 + c_1 = 0$

$\boxed{c_3 = -\frac{1}{6}c_1}$

$12c_4 + + 2c_2 = 0$

$2(6c_4 + c_2) = 0$

$6c_4 + c_2 = 0$

$6c_4 + 0 = 0$

$\boxed{c_4 = 0}$

$3c_3 + 20c_5 = 0$

$c_5 = -\frac{3}{20}c_3$

$c_5 = -\frac{\cancel{3}}{20}\left(-\frac{1}{\cancel{6}_2}c_1\right)$

$\boxed{c_5 = +\frac{1}{40}c_1}$

$y = c_0 + c_1x - \frac{1}{6}c_1x^3 + \frac{1}{40}c_1x^5 + \cdots$

$\boxed{y = c_0 + c_1\left[x - \frac{1}{6}x^3 + \frac{1}{40}x^5 + \cdots\right]}$

Ans

Fin

Q 1) (i) Bernoulli's Equation.
(ii) Non-linear second order ODE's Reducible to first order

Q2) Second order ODES (All cases (i) operator D, (ii) variation of parameters)

Q3) Cauchy Euler ODEs.

Q4) Reduction of Order.

Q5) Power Series Solution.

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Date: 1 - 2 - 24

PARTIAL DIFFERENTIAL EQUATIONS

An equation containing functions of more than one variable and their partial derivatives.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \; ; \text{ where } u = f(x, t)$$

$T = f(\underbrace{x, y, z}_{\text{space}}, \underbrace{t}_{\text{time}})$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} \quad \text{(Linear Partial D.E.)}.$$

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

$z = f(x, y)$

$P = f(\underbrace{M}_{\text{material}}, \underbrace{c}_{\text{cost}})$ Production

General form of **Linear First-order Partial D.E**

$$a(x,y)\frac{\partial z}{\partial x} + b(x,y)\frac{\partial z}{\partial y} = (z)\cdot c(x,y) + d(x,y)$$

dependent variable z in it's partial derivatives should be linear (have degree = 1).

Semi-linear P D E (First order).

General form:

$$a(x,y)\frac{\partial z}{\partial x} + b(x,y)\frac{\partial z}{\partial y} = c(x,y,z) + d(x,y)$$

Every linear is ~~not~~ Semi-linear but not conversely

means for $c(x,y,z) \Rightarrow z$ can be in any form.

$z e^{xy}$, $\ln(xyz)$, $\sin(z)xy$

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First Order Quasi-Linear PDEs.

General form.

$$a(x,y,z)\frac{\partial z}{\partial x} + b(x,y,z)\frac{\partial z}{\partial y} = c(x,y,z) + d(x,y)$$

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