QUESTION 01

Identify the equations that are linear and, non-linear.

(i) $\frac{dy}{dx} = xy$

Linear Equation

(ii) $y \frac{dy}{dx} + xy = 1$

Non-linear Equation.

(iii) $\left|\frac{dy}{dx}\right| - \cos x = x.$

Non-linear Equation.

QUESTION 02

Solve the following differential equations.

(i) $x^2 y\, dx + \left(\frac{x^3}{3}\right) dy = 0$

Solve

$\Rightarrow$ $x^2 y\, dx + \left(\frac{x^3}{3}\right) dy = 0.$

Given equation is exact differential equation.

$\because M(x,y)\, dx + N(x,y)\, dy = 0.$

Let $M = x^2 y$ and $N = \frac{x^3}{3}.$

$\Rightarrow$ $M_y = x^2$ $N_x = \frac{\cancel{3} x^2}{\cancel{3}} = x^2$

Date 20

$\Rightarrow \quad M_y = N_x$

Hence, the given equation is Exact.

$\Rightarrow \quad x^2 y \, dx + \frac{x^3}{3} \, dy = 0$

Applying Integration.

$\Rightarrow \quad \int x^2 y \, dx + \int \frac{x^3}{3} \, dy = 0$

$\Rightarrow \quad y \int x^2 \, dx + \cancel{\frac{x^3}{3} \int dy} = 0$

y term constant   x term vanish

$\Rightarrow \quad \boxed{\frac{x^3 y}{3} = c}$ Answer

(ii) $\frac{dy}{dx} + \frac{y}{x} = e^x$

Solve

$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = e^x$ —— (1)

The given equation is linear differential equation

$\because \quad \frac{dy}{dx} + P(x) \cdot y = Q(x)$

$\Rightarrow \quad P(x) = \frac{1}{x}, \quad Q(x) = e^x$

$\because$ The integrating factor, I.F $= e^{\int P(x) \cdot dx}$

Finding integrating factor.

$\Rightarrow \quad e^{\int P(x) \cdot dx} = e^{\int (\frac{1}{x}) dx}$

$\Rightarrow \quad \int \frac{1}{x} \, dx = \ln(x) \Rightarrow \cancel{e}^{\cancel{\ln}(x)} = x$

$\Rightarrow \quad$ I.F $= x$

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Date ______ 20

Now, multiplying the I.F with eq ①.

$\Rightarrow$ $\cancel{e^x}\, x \dfrac{dy}{dx} + \cancel{x} \cdot \dfrac{y}{\cancel{x}} = x e^x$

$\Rightarrow$ $x \dfrac{dy}{dx} + y = x e^x.$

$\because \dfrac{d}{dx}[(x)y] = x \dfrac{dy}{dx} + y.$

$\Rightarrow$ $\dfrac{d}{dx}[(x)y] = x e^x.$

$\Rightarrow$ $d[(x)y] = x e^x\, dx.$

$\Rightarrow$ Applying integration

$\Rightarrow$ $\int d(x)y = \int x e^x\, dx.$

$\Rightarrow$ $xy = \int x e^x\, dx. \quad$ — ②

Solve $\int x e^x\, dx$, using integration by parts.

Let $f = x \qquad g' = e^x.$

$f' = 1 \qquad g = e^x.$

$\therefore \int f g' = fg - \int f' g.$

$\Rightarrow$ $\int x e^x\, dx = x e^x - \int e^x\, dx.$

$\Rightarrow$ $\int x e^x\, dx = x e^x - e^x + C.$

Substitute it back to our equation ②

$\Rightarrow$ $\boxed{xy = x e^x - e^x + c}$

Answer.

(iii) $\dfrac{dy}{dx} = \dfrac{xy}{x^2 + y^2}.$

Solve

$\dfrac{dy}{dx} = \dfrac{xy}{x^2 + y^2}.$

Given equation is homogenous differential equation

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since, the degree of numerator and denominator is same, 2.

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{x^2+y^2} \quad \text{--- (1)}$$

Let $y = vx$. $\quad \because u.v = uv' + vu'$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substitute it in equation (1).

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x(vx)}{x^2+(vx)^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2 v}{x^2+x^2v^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{\cancel{x^2} v}{\cancel{x^2}(1+v^2)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^2} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{\cancel{v} - \cancel{v} - v^3}{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^3}{1+v^2}$$

Now the equation becomes separable.

$$\Rightarrow -\frac{1+v^2}{v^3}\,dv = +\frac{1}{x}\,dx$$

Applying integration

$$\Rightarrow -\left[\int \frac{1+v^2}{v^3}\,dv\right] = \int \frac{1}{x}\,dx$$

$$\Rightarrow -\left[\int \frac{1}{v^3}\,dv + \int \frac{\cancel{v^2}}{v^{\cancel{3}}}\,dv\right] = \int \frac{1}{x}\,dx$$

Date ______ 20 ____

$\Rightarrow -\left[\int \frac{1}{v^3}\,dv + \int \frac{1}{v}\,dv\right] = \int \frac{1}{x}\,dx.$

$\Rightarrow -\left[\int v^{-3}\,dv + \int \frac{1}{v}\,dv\right] = \int \frac{1}{x}\,dx.$

$\Rightarrow -\left[\frac{v^{-2}}{-2} + \ln(v)\right] = \ln(x) + \ln(c)$

$\Rightarrow -\left[-\frac{1}{2v^2} + \ln(v)\right] = \ln(x) + \ln(c)$

$\Rightarrow \frac{1}{2v^2} - \ln(v) = \ln(xc)$

$\Rightarrow \frac{1}{2v^2} = \ln(xc) + \ln(v)$

$\Rightarrow \frac{1}{2v^2} = \ln(vxc)$

$\because y = vx \Rightarrow v = y/x.$

$\Rightarrow \frac{1}{2\left(\frac{y}{x}\right)^2} = \ln\left(\frac{y}{\cancel{x}}.\cancel{x}.c\right)$

$\Rightarrow \frac{x^2}{2y^2} = \ln(yc).$

$\Rightarrow \frac{x^2}{2y^2} - \ln(y) + \ln(c) \Rightarrow c$ since it's constant

$\Rightarrow \boxed{\frac{x^2}{2y^2} - \ln y = c}$ Answer

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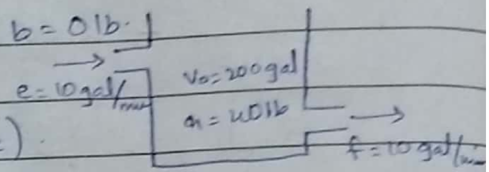
QUESTION 03

A tank initially holds 200 gal of a brine solution containing 40lb of salt. At $t=0$, fresh water is poured into the tank at the rate of 10 gal/min, while the well-stirred mixture leaves at the same rate. Find the amount of salt in the tank at any time t.



Solve

Given: $V_0 = 200$ gal. (volume of brine)

$a = 40$ lb (salt in container).

$e = 10$ gal/min (inflow of water).

$f = 10$ gal/min (outflow of mixture).

$S(t) \Rightarrow$ Amount of salt at time 't'

$S(0) = 40$ lb.

$b = 0$ lb (amount of salt in fresh water).

Let 'S' ist the amount of salt in the container at time 't'

$$\Rightarrow \frac{dS}{dt} = \begin{pmatrix}\text{Amount of} \\ \text{inflow rate}\end{pmatrix} - \begin{pmatrix}\text{Amount of} \\ \text{outflow rate}\end{pmatrix} \quad \text{—①}$$

$\frac{dS}{dt} \downarrow$ rate of change of salt S.

Amount of inflow rate.

$$\Rightarrow \left(b \frac{lb}{gal}\right)\left(e \frac{gal}{min}\right) = \boxed{be \ \frac{lb}{min}} \quad \text{—②}$$

Amount of outflow rate.

$\therefore Q = \frac{V}{t}$

$V = Qt$

$V = et$.

$$\Rightarrow \left[\frac{S}{V_0 + et - ft}\right] \times f \ \frac{gal}{min} \times \frac{lb}{gal}$$

$$\Rightarrow \boxed{\frac{Sf}{V_0 + t(e-f)} \ \frac{lb}{min}} \quad \text{—③}$$

Substitute both equation 2 and 3 in 1.

$$\Rightarrow \boxed{\frac{dS}{dt} = be - \frac{S f}{V_0 + t(e-f)}}$$

Now, substitute all values given.

$$\Rightarrow \frac{dS}{dt} = 10)(10) - \frac{S(10)}{200 + 0(10-10)}$$

$$\Rightarrow \frac{dS}{dt} = - \frac{S\,10}{200}$$

$$\Rightarrow \frac{dS}{dt} = - \frac{S}{20}$$

The equation is separable now.

$$\Rightarrow \frac{1}{S}\, dS = \frac{-1}{20}\, dt$$

Applying integration.

$$\Rightarrow \int \frac{1}{S}\, dS = \frac{-1}{20} \int dt$$

$$\Rightarrow \ln(S) = -\frac{1}{20} t + c$$

$$\Rightarrow e^{\cancel{\ln}(S)} = e^{\left(-\frac{1}{20} t + c\right)}$$

$$\Rightarrow S = e^{\left(-\frac{1}{20} t + c\right)}$$

$$\Rightarrow S = e^{-\frac{1}{20} t} \cdot e^{c}$$

$\because e^c = c$ since it is constant

$$\Rightarrow \boxed{S = e^{-\frac{1}{20} t}\, c}$$

or

$$\Rightarrow \boxed{S(t) = e^{-\frac{1}{20} t}\, c}$$

$\because S(0) = 40$; $t = 0$.

$$\Rightarrow 40 = e^{-\frac{1}{20}(0)}\, c$$

$$\Rightarrow 40 = e^0 \cdot c$$

$\because e^0 = 1$

$$\Rightarrow \boxed{40 = c}$$

$$\Rightarrow \boxed{S(t) = 40 e^{-\frac{1}{20} t}}$$

$\Rightarrow$ Amount of salt in the tank at any time 't'.

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