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Class: BSCS 5B

Course: Differential Equations

Course Instructor:

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Assignment no. 1

Date 20

Question 01

Identify the equations that are linear and, non-linear.

(i) $\frac{dy}{dx} = xy$

Linear Equation

(ii) $y\frac{dy}{dx} + xy = 1$

Non-linear Equation.

(iii) $\left|\frac{dy}{dx}\right| - \cos x = x.$

Non-linear Equation.

Question 02

Solve the following differential equations.

(i) $x^2y\,dx + \left(\frac{x^3}{3}\right)dy = 0$

Solve

$\Rightarrow$ $x^2y\,dx + \left(\frac{x^3}{3}\right)dy = 0.$

Given equation is exact differential equation.

$\because M(x,y)\,dx + N(x,y)\,dy = 0.$

Let $M = x^2y$ and $N = \frac{x^3}{3}.$

$\Rightarrow$ $M_y = x^2$ $\qquad N_x = \frac{3x^2}{3} = x^2$

$\Rightarrow$ $My = Nx$.

Hence, the given equation is Exact.

$\Rightarrow$ $x^2 y\, dx + \frac{x^3}{3}\, dy = 0$.

Applying Integration.

$\Rightarrow$ $\int x^2 y\, dx + \int \frac{x^3}{3}\, dy = 0$.

$\Rightarrow$ $y \int x^2\, dx + \frac{x^3}{3} \int dy = 0$.

y term constant, x term vanish

$\Rightarrow$ $\boxed{\frac{x^3 y}{3} = c}$ Answer

(ii) $\frac{dy}{dx} + \frac{y}{x} = e^x$

Solve

$\Rightarrow$ $\frac{dy}{dx} + \frac{y}{x} = e^x$. — (1)

The given equation is linear differential equation

$\because$ $\frac{dy}{dx} + P(x).y = Q(x)$.

$\Rightarrow$ $P(x) = \frac{1}{x}$, $Q(x) = e^x$.

$\because$ The integrating factor, $I.F = e^{\int P(x).dx}$.

Finding integrating factor.

$\Rightarrow$ $e^{\int P(x).dx} = e^{\int (\frac{1}{x}) dx}$

$\Rightarrow$ $\int \frac{1}{x}\, dx = \ln(x)$ $\Rightarrow$ $e^{\ln(x)} = x$.

$\Rightarrow$ $I.F = x$

Now, multiplying the I.F with eq (1).

$$\Rightarrow e^x x \frac{dy}{dx} + x \cdot \frac{y}{x} = x e^x$$

$$\Rightarrow x \frac{dy}{dx} + y = x e^x.$$

$$\because \frac{d[(x)y]}{dx} = x \frac{dy}{dx} + y.$$

$$\Rightarrow \frac{d[(x)y]}{dx} = x e^x.$$

$$\Rightarrow d[(x)y] = x e^x dx.$$

Applying integration

$$\Rightarrow \int d(x)y = \int x e^x dx.$$

$$\Rightarrow xy = \int x e^x dx. \quad \text{—(2)}$$

Solve $\int x e^x dx$, using integration by parts.

Let $f = x$ $\quad g' = e^x$.

$f' = 1$ $\quad g = e^x$.

$$\because \int f g' = fg - \int f' g.$$

$$\Rightarrow \int x e^x dx = x e^x - \int e^x dx.$$

$$\Rightarrow \int x e^x dx = x e^x - e^x + C.$$

Substitute it back to our equation (2)

$$\Rightarrow \boxed{xy = x e^x - e^x + c}$$

Answer.

(iii) $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$.

Solve

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}.$$

Given equation is homogeneous differential equation

since, the degree of numerator and denomenator is same, 2.

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{x^2+y^2} \quad -①$$

Let $y = vx$ $\quad \because u.v = uv' + vu'$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substitute it in equation ①.

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x(vx)}{x^2+(vx)^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2v}{x^2+x^2v^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2v}{x^2(1+v^2)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^2} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v - v - v^3}{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^3}{1+v^2}$$

Now the equation becomes separable.

$$\Rightarrow -\frac{1+v^2}{v^3}\,dv = \pm\frac{1}{x}\,dx.$$

Applying integration

$$\Rightarrow -\left[\int \frac{1+v^2}{v^3}\,dv\right] = \int \frac{1}{x}\,dx.$$

$$\Rightarrow -\left[\int \frac{1}{v^3}\,dv + \int \frac{v^2}{v^3}\,dv\right] = \int \frac{1}{x}\,dx$$

$$\Rightarrow -\left[\int \frac{1}{v^3}\,dv + \int \frac{1}{v}\,dv\right] = \int \frac{1}{x}\,dx$$

$$\Rightarrow -\left[\int v^{-3}\,dv + \int \frac{1}{v}\,dv\right] = \int \frac{1}{x}\,dx$$

$$\Rightarrow -\left[\frac{v^{-2}}{-2} + \ln(v)\right] = \ln(x) + \ln(c)$$

$$\Rightarrow -\left[\frac{-1}{2v^2} + \ln(v)\right] = \ln(x) + \ln(c)$$

$$\Rightarrow \frac{1}{2v^2} - \ln(v) = \ln(xc)$$

$$\Rightarrow \frac{1}{2v^2} = \ln(xc) + \ln(v)$$

$$\Rightarrow \frac{1}{2v^2} = \ln(vxc)$$

$$\because y = vx \Rightarrow v = y/x.$$

$$\Rightarrow \frac{1}{2\left(\frac{y}{x}\right)^2} = \ln\left(\frac{y}{x} \cdot x \cdot c\right)$$

$$\Rightarrow \frac{x^2}{2y^2} = \ln(yc).$$

$$\Rightarrow \frac{x^2}{2y^2} = \ln(y) + \ln(c) \Rightarrow c \quad \text{since it's constant}$$

$$\Rightarrow \boxed{\frac{x^2}{2y^2} - \ln y = c.}$$

Answer

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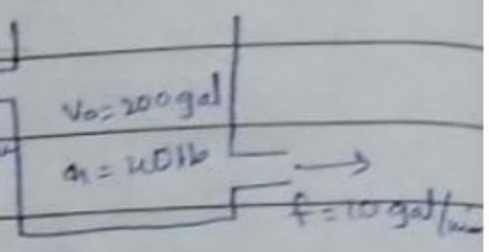
Question 03

A tank initially holds 200 gal of a brine solution containing 40lb of salt. At t=0, fresh water is poured into the tank at the rate of 10 gal/min, while the well-stirred mixture leaves at the same rate. Find the amount of salt in the tank at any time t.



Solve

Given: $V_0 = 200$ gal. (volume of brine)

$a = 40$ lb (salt in container).

$e = 10$ gal/min (inflow of water).

$f = 10$ gal/min (outflow of mixture).

$S(t)$ => Amount of salt at time 't'

$S(0) = 40$ lb.

$b = 0$lb (amount of salt in fresh water).

Let 'S' ist the amount of salt in the container at time 't'

$$\Rightarrow \frac{dS}{dt} = (\text{Amount of inflow rate}) - (\text{Amount of outflow rate}) \quad -①$$

$\frac{dS}{dt}$ ↓ rate of change of salt S.

Amount of inflow rate.

$$\Rightarrow \left(b \frac{lb}{gal}\right)\left(e \frac{gal}{min}\right) = \boxed{be \frac{lb}{min}} \quad -②$$

Amount of outflow rate.

$$\Rightarrow \left[\frac{S}{V_0 + et - ft}\right] \times f \frac{gal}{min} \times \frac{lb}{gal}$$

$$\therefore Q = \frac{V}{t}$$

$$V = Qt$$

$$V = et.$$

$$\Rightarrow \boxed{\frac{Sf}{V_0 + t(e-f)} \frac{lb}{min}} \quad -③$$

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Substitute both equation 2 and 3 in 1.

$$\Rightarrow \boxed{\frac{dS}{dt} = be - \frac{Sf}{V_0 + t(e-f)}}$$

Now, substitute all values given.

$$\Rightarrow \frac{dS}{dt} = (10)(10) - \frac{S(10)}{200 + 0(10-10)}$$

$$\Rightarrow \frac{dS}{dt} = -S\frac{10}{200}.$$

$$\Rightarrow \frac{dS}{dt} = -\frac{S}{20}.$$ The equation is separable now.

$$\Rightarrow \frac{1}{S}\,dS = \frac{-1}{20}\,dt.$$

Applying integration.

$$\Rightarrow \int \frac{1}{S}\,dS = \frac{-1}{20}\int dt.$$

$$\Rightarrow \ln(S) = -\frac{1}{20}t + c.$$

$$\Rightarrow e^{\ln(S)} = e^{(-\frac{1}{20}t + c)}.$$

$$\Rightarrow S = e^{-\frac{1}{20}t + c}$$

$$\Rightarrow S = e^{-\frac{1}{20}t} \cdot e^{c}. \quad \because e^{c} = c \text{ since it is constant.}$$

$$\Rightarrow \boxed{S = e^{-\frac{1}{20}t}\, c.}$$

or

$$\Rightarrow \boxed{S(t) = e^{-\frac{1}{20}t}\, c.}$$

$\because S(0) = 40 \; ; t = 0.$

$$\Rightarrow 40 = e^{-\frac{1}{20}(0)}\, c.$$

$$\Rightarrow 40 = e^{0} \cdot c \quad \because e^{0} = 1$$

$$\Rightarrow \boxed{40 = c.}$$

$$\Rightarrow \boxed{S(t) = 40e^{-\frac{1}{20}t}}$$ $\Rightarrow$ Amount of salt in the tank at any time 't'.

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