Date ______ 20__

QUESTION No. 1

Let V be the subset of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $u = (u_1, u_2)$ and $v = (v_1, v_2)$:

$$u + v = (u_1 + v_1, u_2 + v_2), \quad ku = (0, ku_2).$$

a) Compute $u+v$ and ku for $u = (-1, 2)$, $v = (3, 4)$ and $k = 3$.

Solution

$\because u + v = (u_1 + v_1, u_2 + v_2)$.

$u + v = (-1+3, 2+4)$.

$\boxed{u + v = (2, 6)}$

$\because ku = (0, ku_2)$.

$ku = (0, (3)(2))$.

$\boxed{ku = (0, 6)}$

b) In words, explain why V is closed under addition and scalar multiplication.

V is closed under addition because $V \subseteq \mathbb{R}$ and the operation of addition $(u_1 + v_1, u_2 + v_2) = u + v$ results in another ordered pair of real numbers, which is still in V.

Similarly, V is closed under scalar multiplication operation on V, $ku = (0, ku_2)$ also results in another pair of real numbers, which is still in V, with $k \in \mathbb{R}$.

Date ________ 20__

(c) Since addition on V is the standard addition operation on $\mathbb{R}^2$, certain vector space axioms for $\mathbb{R}^2$. Which axioms are they?

Solution

$u, v, w \in \mathbb{R}$

Axiom 1: Closure under addition.

$\Rightarrow$ $u = (u_1, u_2)$ and $v = (v_1, v_2)$.

$\Rightarrow$ $u \in V$ and $v \in V$ then $u+v \in V$.

$\Rightarrow$ $u+v = (u_1+v_1, u_2+v_2)$.

Since, $u, v \in \mathbb{R}$ then $(u+v) \in \mathbb{R}$.

Axiom 2: $u+v = v+u$.

$\Rightarrow$ $u = (u_1, u_2)$ and $v = (v_1, v_2)$

according to axiom

$\Rightarrow$ $u+v = v+u$.

$\Rightarrow$ $(u_1, u_2) + (v_1, v_2) = (v_1, v_2) + (u_1, u_2)$.

$\Rightarrow$ $(u_1+v_1, u_2+v_2) = (v_1+u_1, v_2+u_2)$

Axiom 3: $u+(v+w) = (u+v)+w$.

$\Rightarrow$ $u = (u_1, u_2)$, $v = (v_1, v_2)$ and $w = (w_1, w_2)$.

according to axiom.

$\Rightarrow$ $u+(v+w) = (u+v)+w$.

$\Rightarrow$ $(u_1, u_2) + ((v_1, v_2) + (w_1, w_2)) = ((u_1, u_2) + (v_1, v_2)) + (w_1,$

$\Rightarrow$ $(u_1, u_2) + (v_1+w_1, v_2+w_2) = (u_1+v_1, u_2+v_2) + (w_1$

$\Rightarrow$ $(u_1+v_1+w_1, u_2+v_2+w_2) = (u_1+v_1+w_1, u_2+v_2+$

Since addition of real numbers is associative theref this axiom holds.

Date ________ 20 ___

Axiom 4: zero vector $\vec{O} = (0,0)$.

$\Rightarrow$ $O + u = u$.

$\Rightarrow$ $(0,0) + (u_1, u_2) = (u_1, u_2)$.

$\Rightarrow$ $(0+u_1 ; 0+u_2) = (u_1, u_2)$.

$\Rightarrow$ $(u_1, u_2) = (u_1, u_2)$.

Therefore, axiom holds. and O is additive identity.

Axiom 5:

For each u in V, there exists $-u$, such that

$\Rightarrow$ $u + (-u) = 0$.

$\Rightarrow$ $(u_1, u_2) + (-u_1, -u_2) = (0,0)$.

$\Rightarrow$ $(u_1 - u_1, u_2 - u_2) = (0,0)$

$\Rightarrow$ $(0, 0) = (0,0)$.

Therefore, axiom holds and $-u$ is additive for u.

All addition axioms, Axiom 1-5 holds for V.

d) Show that axiom 7,8,9 hold.

Axiom 7: $k(u+v) = ku + kv$; $k \in \mathbb{R}$.

$\Rightarrow$ $k((u_1, u_2) + (v_1, v_2)) = k(u_1, u_2) + k(v_1, v_2)$.

$\Rightarrow$ $k(u_1+v_1, u_2+v_2) = (ku_1, ku_2) + (kv_1, kv_2)$.

$\Rightarrow$ $(ku_1 + kv_1, ku_2 + kv_2) = (ku_1 + kv_1, ku_2 + kv_2)$

Therefore, axiom holds.

Axiom 8: $(k+m)u = ku + mu \quad ; k, m \in \mathbb{R}$

$\Rightarrow \quad (k+m)(u_1, u_2) = k(u_1, u_2) + m(u_1, u_2).$

$\Rightarrow \quad (k+m)(u_1, u_2) = (k+m)(u_1, u_2).$

Therefore, axiom holds.

Axiom 9: $k(mu) = (km)u.$

$k(m.(u_1, u_2)) = km(u_1, u_2).$

$k(mu_1, mu_2) = (kmu_1, kmu_2)$

$(kmu_1, kmu_2) = (kmu_1, kmu_2)$

Therefore, axiom holds.

(e) Show that Axiom 10 fails and hence V is not a vector space under under the given operation.

Axiom 10: $1u = u$

According to the definition given for scalar multiplication, for the operation $ku = (0, ku_2)$, there is no element/vector in V that satisfies the condition for all u in V.

$\Rightarrow \quad 1.u = u.$

$\because k = 1$

$(0, 1.u_2) = (u_1, u_2).$

$(0, u_2) \neq (u_1, u_2).$

There is no element 1 in $\mathbb{R}$ Such that $1u =$ for all u in V.

Therefore, V does not satisfy Axiom 10 and V is not a vector space, under the given operations of addition and scalar multiplication

Date ____ 20__

QUESTION No. 3 – 7

Determine whether each set equipped with the given operation is a vector space. For those that are not vector spaces, identify the vector space axioms that fails.

QUESTION No. 3

The set of all real numbers with the standard operations of addition and multiplication.

The set of all $\mathbb{R}$ with standard operations of addition and multiplication.

1. Axiom 1: Closure under addition.
Let $a, b \in \mathbb{R}$ then $a + b \in \mathbb{R}$.
Pf we have $a = 1$, $b = 2 \in \mathbb{R}$.
then $a + b \Rightarrow 1 + 2 = 3 \in \mathbb{R}$.

Axiom 1 holds

2. Axiom 2: Commutativity of addition.
Let $a, b \in \mathbb{R}$, $a + b = b + a$.
Let $a = 4$, $b = 2$
$4 + 2 = 2 + 4$
$6 = 6$.

Axiom holds

Teacher's Signature ____ RAZA Paper Products

3. Axiom 3 : Associativity of addition.

$$(a+b)+c = a+(b+c) \quad , a,b,c \in$$

Let $a = 2$, $b = 1$, $c = 4$.

$$(2+1)+4 = 2+(1+4).$$
$$3+4 = 2+5$$
$$7 = 7$$

Axiom holds

4. Axiom 4: Additive Identity $\Rightarrow$ 0 :

$$a + 0 = a.$$

Let $a = 3$

$$3+0 = 3$$
$$3 = 3$$

Hence, 0 is additive identity & Axiom holds for $\mathbb{R}$.

5. Axiom 5 : Additive Inverse

$$a + (-a) = 0.$$

Let $a = 2$

$$2 + (-2) = 0$$
$$2 - 2 = 0$$
$$0 = 0$$

Hence, $-a$ is additive inverse for a, Axiom holds for $\mathbb{R}$.

6. Axiom 6 : Closure under scalar multiplication.

$$ka \;; \quad a \in \mathbb{R} \text{ \& } k \in \mathbb{R}.$$

Let $a = 2$ and $k = 3$.

$$(3)(2) = 6 \in \mathbb{R}.$$

Axiom holds.

Axiom 7: $k(a+b) = ka + kb$.

Let $a, b \in \mathbb{R}$ and $k \in \mathbb{R}$.

$a = 2, b = 4, k = 1$

$$1(2+4) = (1)(2) + (1)(4).$$
$$1(6) = 2 + 4$$
$$6 = 6$$

Axiom holds.

Axiom 8: $(k+m)a = ka + ma$; $a \in \mathbb{R}$ & $k, m \in \mathbb{R}$.

Let $a = 2$, $k = 3$ & $m = 5$.

$$(3+5)2 = (3)(2) + (5)(2).$$
$$(8)(2) = 6 + 10$$
$$16 = 16$$

Axiom holds

9. Axiom 9: $k(ma) = (km)a$; $k, m \in \mathbb{R}$ & $a \in \mathbb{R}$.

Let $k = 2$, $m = 4$ and $a = 3$.

$$2((4)(3)) = ((2)(4))3.$$
$$2(12) = (8)3$$
$$24 = 24$$

Axiom holds

10. Axiom 10: $1a = a$; $a \in \mathbb{R}$.

Let $a = 2$

$$1(2) = 2$$
$$2 = 2.$$

Hence, 1 is Multiplicative Identity & Axiom holds for $\mathbb{R}$.

All the axioms hold for $\mathbb{R}$ with standard operat of addition and multiplication which $\mathbb{R}$ a vector space.

Question No. 5

The set of all pairs of real numbers of the form (x, y), where $x \geq 0$, with the standard operations on $\mathbb{R}^2$.

We take $u = (x, y)$, $v = (x, y)$, $w = (x, y)$.

1. Axiom 1: Closure under addition
For $x, y \in \mathbb{R}$. then $x + y \in \mathbb{R}$

2. Axiom 2: $u + v = v + u$
For any $x, y \in \mathbb{R}$ with $x \geq 0$, it holds.

3. Axiom 3: $u + (v + w) = (u + v) + w$
For any $x, y \in \mathbb{R}$; $x \geq 0$, associativity of addition holds.

4. Axiom 4: $0 + u = u$
$u = (x, y)$; $x, y \in \mathbb{R}$. , additive identity is 0 and it holds.

5) Axiom 5: $u + (-u) = 0$.
$u = (x, y)$ then $-u = (-x, -y)$ by $x \geq 0$
Since x can't be ~~X~~ negative, then axiom for additive inverse does not hold.

Since, axiom 5 fails to hold, therefore set of all pairs of (x, y) where $x \geq 0$ with the standard operations of $\mathbb{R}^2$ is not a vector space.

QUESTION No. 7

The set of all triples of real numbers with the standard vector addition but with scalar multiplication defined by

$$k(x, y, z) = (k^2 x, k^2 y, k^2 z)$$

$u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$, $w = (x_3, y_3, z_3)$.

Axiom 1: Closure ~~under~~ addition. $u, v \in \mathbb{R}$, $u+v \in \mathbb{R}$.

$u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$

$u+v = (x_1+x_2, y_1+y_2, z_1+z_2)$.

Axiom holds.

Axiom 2: $u+v = v+u$

$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_2, y_2, z_2) + (x_1, y_1, z_1)$

$(x_1+x_2, y_1+y_2+z_1+z_2) = (x_2+x_1, y_2+y_1, z_2+z_1)$

Axiom holds.

Axiom 3: $u+(v+w) = (u+v)+w$.

$(x_1, y_1, z_1) + ((x_2, y_2, z_2) + (x_3, y_3, z_3)) = ((x_1, y_1, z_1) + (x_2, y_2, z_2)) + (x_3, y_3, z_3)$

$(x_1, y_1, z_1) + (x_2+x_3, y_2+y_3, z_2+z_3) = (x_1+x_2, y_1+y_2, z_1+z_2) + (x_3, y_3, z_3)$

$(x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3) = (x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3)$

Axiom holds

Date ______ 20___

4. Axiom 4: zero vector, $0 + u = u$. Additive Identity

$\Rightarrow \quad 0 + u = u. \qquad 0 = (0,0,0).$

$\Rightarrow \quad (0,0,0) + (x_1, y_1, z_1) = (x_1, y_1, z_1).$

$\Rightarrow \quad (0+x_1, 0+y_1, 0+z_1) = (x_1, y_1, z_1)$

$\Rightarrow \quad (x_1, y_1, z_1) = (x_1, y_1, z_1).$

Axiom holds.

5. Axiom 5: $u + (-u) = 0$ Additive Inverse

$\Rightarrow \quad (x_1, y_1, z_1) + (-x_1, y_1, z_1) = (0,0,0).$

$\Rightarrow \quad (\cancel{x_1} - \cancel{x_1}, \cancel{y_1} - \cancel{y_1}, \cancel{z_1} - \cancel{z_1}) = (0,0,0),$

$\Rightarrow \quad (0,0,0) = (0,0,0)$

Axiom holds.

closure under.

6. Axiom 6: ku Scalar Multiplication.

$\Rightarrow \quad k \in R.$

$\Rightarrow \quad ku = k(x_1, y_1, z_1) = (k^2x_1, k^2y_1, k^2z_1)$

Axiom holds

7. Axiom 7: $k(u+v) = ku + kv. \quad ; k \in \mathbb{R}$

$\Rightarrow \quad k\left((x_1, y_1, z_1) + (x_2, y_2, z_2)\right) = (k^2x_1, k^2y_1, k^2z_1) + (k^2x_2, k^2y^2, k^2z_2)$

$\Rightarrow \quad k^2(x_1+x_2, y_1+y_2, z_1+z_2) = (k^2(x_1+x_2), k^2(y_1+y_2), k^2(z_1+z_2)).$

$\Rightarrow \quad (k^2(x_1+x_2), k^2(y_1+y_2), k^2(z_1+z_2)) = (k^2(x_1+x_2), k^2(y_1+y_2), k^2(z_1+z_2)).$

Axiom holds

Date ______ 20__

Axiom 8: $(k+m)u = ku + mu$. $k, m \in \mathbb{R}$.

$(k+m)(x_1, y_1, z_1) = (k^2x_1, k^2y_1, k^2z_1) + (k^2x_2, k^2y_2, k^2z_2$

let $k+m = i$ and $i \cdot u = i^2 u$

$(k+m)^2 (x_1, y_1, z_1) = (k^2x_1, k^2y_1, k^2z_1) + (k^2x_2, k^2y_2, k^2z_2$

$\because (k+m)^2 \neq k^2 + m^2$.

Axiom fails!

Since, axiom 8 fails to hold, therefore set of all triples with standad vector and scalar multiplication operations. of $\mathbb{R}$ is not a vector space.