ASSIGNMENT No 3.

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TOPIC :- DIRECTIONAL DERIVATIVES AND GRADIENT

QUESTION # 1

Find Duf at P.

$f(x,y) = (1+xy)^{3/2}$; $P(3,1)$; $u = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$.

Solve

$\because D_u f(x_0, y_0) = \frac{d}{ds}\left[f(x_0 + su_1, y_0 + su_2) \right]_{s=0}$

$u = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j \Rightarrow u_1 = \frac{1}{2}, u_2 = \frac{1}{\sqrt{2}}$

$D_u f(3,1) = \frac{d}{ds}\left[f\left(3 + s\frac{1}{\sqrt{2}}, 1 + s\frac{1}{\sqrt{2}}\right)\right]_{s=0}$ —— ①

$f\left(3 + s/\sqrt{2}, 1 + s/\sqrt{2}\right) = ?$

$f(x,y) = (1+xy)^{3/2}$.

$f\left(3 + s/\sqrt{2}, 1 + s/\sqrt{2}\right) = \left\{1 + (3 + s/\sqrt{2})(1 + s/\sqrt{2})\right\}^{3/2}$.

$f\left(3 + s/\sqrt{2}, 1 + s/\sqrt{2}\right) = \left(1 + 3 + 3s/\sqrt{2} + s/\sqrt{2} + s^2/2\right)^{3/2}$

$f\left(3 + s/\sqrt{2}, 1 + s/\sqrt{3}\right) = \left(4 + 4s/\sqrt{2} + s^2/2\right)^{3/2}$

Putting above equation into equation ①

$D_u f(3,1) = \frac{d}{ds}\left[\left(4 + \frac{4s}{\sqrt{2}} + \frac{s^2}{2}\right)^{3/2} \right]_{s=0}$.

$$D_u f(3,1) = \left| \frac{3}{2}\left(4 + \frac{4s}{\sqrt{2}} + \frac{s^2}{2}\right)^{1/2} \cdot \left(\frac{4}{\sqrt{2}} + \frac{\cancel{2}s}{\cancel{2}}\right) \right|_{s=0}$$

$$D_u f(3,1) = \frac{3}{2}\left(4 + \frac{4(0)}{\sqrt{2}} + \frac{(0)^2}{2}\right)^{1/2}\left(\frac{4}{\sqrt{2}} + 0\right)$$

$$D_u f(3,1) = \frac{3}{2}(4)^{1/2}\left(\frac{4}{\sqrt{2}}\right)$$

$$D_u f(3,1) = \frac{3}{\cancel{2}}(\cancel{2})\left(\frac{4}{\sqrt{2}}\right)$$

$$D_u f(3,1) = \frac{12}{\sqrt{2}} \qquad \because \frac{12}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancel{12}^{\,6}\sqrt{2}}{\cancel{2}} = 6\sqrt{2}$$

$$D_u f(3,1) = 6\sqrt{2}.$$

Ans

QUESTION # 11

Find the directional derivative of f at P in the direction of a.

$f(x,y) = y^2 \ln x$; $P(1,4)$; $a = -3i + 3j$.

Solve

$\because D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u$.

To find $\nabla f(x_0, y_0) = f_x(x,y)i + f_y(x,y)j$.

$$\nabla f(x_0, y_0) = \frac{y^2}{x} i + 2y \cdot \ln x\, j.$$

$$\nabla f(1,4) = \frac{(4)^2}{1} i + 2(4) \cdot \ln 1\, j.$$

$$\nabla f(1,4) = 16i + 8 \cdot (0)\, j$$

$$\nabla f(1,4) = 16i.$$

To find $u = \dfrac{\vec{a}}{|a|}$

$$u = \frac{-3i+3j}{\sqrt{(-3)^2+(3)^2}} = \frac{-3i+3j}{\sqrt{9+9}} = \frac{-3i+3j}{\sqrt{18}} = \frac{-3i+3j}{3\sqrt{2}}$$

$$u = -\frac{\cancel{3}}{\cancel{3}\sqrt{2}}i + \frac{\cancel{3}}{\cancel{3}\sqrt{2}}j$$

$$u = -\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$$

$$D_uf(1,4) = (16i)\cdot\left(-\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j\right) \qquad \because i\cdot i = j\cdot j = k\cdot k = 1$$

$$D_uf(1,4) = \frac{-16}{\sqrt{2}} \qquad -\frac{16}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\overset{8}{\cancel{16}}\sqrt{2}}{\cancel{2}} = -8\sqrt{2}$$

$$D_uf(1,4) = -8\sqrt{2}.$$

Ans

QUESTION # 19

Find the directional derivative of f at P in the direction of a vector making the counterclockwise angle θ with the positive x-axis.

$f(x,y) = \sqrt{xy}$; $P(1,4)$; $\theta = \pi/3$

Solve

$$\because D_uf(x_0,y_0) = f_x(x_0,y_0)\cos\theta + f_y(x_0,y_0)\sin\theta.$$

$$f_x(x_0,y_0) = \frac{1}{2\sqrt{xy}}\cdot y = \frac{y}{2\sqrt{xy}}$$

$$f_x(1,4) = \frac{\overset{2}{\cancel{4}}}{\cancel{2}\sqrt{1\times 4}} = \frac{2}{\sqrt{4}} = \frac{\cancel{2}}{\cancel{2}} = 1.$$

$$f_y(x_0,y_0) = \frac{1}{2\sqrt{xy}}\cdot x = \frac{x}{2\sqrt{xy}}.$$

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$$f_y(1,4) = \frac{1}{2\sqrt{(1\times4)}} = \frac{1}{2\sqrt{4}} = \frac{1}{2\times2} = \frac{1}{4}$$

$$D_uf(1,4) = 1\cdot\cos\left(\frac{\pi}{3}\right) + \frac{1}{4}\sin\left(\frac{\pi}{3}\right)$$

$$D_uf(1,4) = \cos(60^\circ) + \frac{1}{4}\sin(60^\circ)$$

$$D_uf(1,4) = \frac{1}{2} + \frac{1}{4}\times\frac{\sqrt{3}}{2}$$

$$D_uf(1,4) = \frac{1}{2} + \frac{\sqrt{3}}{8}$$

Ans

QUESTION # 34

Find ∇z or ∇w.

$z = 7\sin(6x/y)$

$6xy^{-1}$
$-6xy^{-2}$
$-6x/y^2$

Solve

$$\nabla z = z_x i + z_y j$$

$$\nabla z = 7\cos(6x/y)\cdot(6/y)\, i + 7\cos(6x/y)(-6x/y^2) j$$

$$\nabla z = \frac{42}{y}\cos(6x/y)\, i - \frac{42x}{y^2}\cos(6x/y) j$$

Ans

$w = xe^{8y}\sin(6z)$

Solve

$$\nabla w = w_x i + w_y j + w_z k$$

$$\nabla w = e^{8y}\sin(6z) i + 8xe^{8y}\sin(6z) j + 6xe^{8y}\cos(6z)$$

Ans

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QUESTION # 45

Find the gradient of f at the indicated point.

$f(x, y, z) = y \ln(x+y+z)$; $(-3, 4, 0)$

Solve

$\because \nabla f(x,y,z) = f_x(x,y,z)i + f_y(x,y,z)j + f_z(x,y,z)k.$

$f_x(x,y,z) = y \cdot \frac{1}{x+y+z} \cdot (1+0+0) = \frac{y}{x+y+z}$

$f_x(-3,4,0) = \frac{4}{-3+4+0} = 4.$ —★

$f_y(x,y,z) = \ln(x+y+z)\, f_y(y) + (y)\, f_y(\ln(x+y+z))$

$f_y(x,y,z) = \ln(x+y+z)(1) + (y) \cdot \frac{1}{x+y+z} (0+1+0).$

$f_y(x,y,z) = \ln(x+y+z) + \frac{y}{x+y+z}$

$f_y(-3,4,0) = \ln(-3+4+0) + \frac{4}{-3+4+0}.$

$f_y(-3,4,0) = \ln(1) + 4.$

$f_y(-3,4,0) = 0 + 4$

$f_y(-3,4,0) = 4.$ —★

$f_z(x,y,z) = y \cdot \frac{1}{x+y+z} (0+0+1).$

$f_z(x,y,z) = \frac{y}{x+y+z}$

$f_z(-3,4,0) = \frac{4}{-3+4+0}$

$f_z(-3,4,0) = 4.$ —★

$$\nabla f(-3,4,0) = 24i + 4j + 4k$$ Ans

QUESTION # 57

Find a unit vector in the direction in which f increases most rapidly at P, and find the rate of change of f at P in that direction.

$f(x,y,z) = x^3z^2 + y^3z + z - 1$; $P(1,1,-1)$

Solve

The function f increases most rapidly in the direction of ∇f at $(1,1,-1)$. The gradiant there is

$\nabla f(x,y,z) = f_x(x,y,z)i + f_y(x,y,z)j + f_z(x,y,z)k$

$f_x(x,y,z) = 3x^2z^2$

$f_x(1,1,-1) = 3(1)^2(-1)^2 = 3.$

$f_y(x,y,z) = 3y^2z$

$f_y(1,1,-1) = 3(1)^2(-1) = -3.$

$f_z(x,y,z) = 2x^3z + y^3 + 1$

$f_z(1,1,-1) = 2(1)^3(-1) + (1^3) + 1 = -2+1+1 = 0.$

$$\nabla f(1,1,-1) = 3i - 3j$$

finding unit vector in the direction where f increases

$$u = \frac{\nabla f}{|\nabla f|} = \frac{3i-3j}{\sqrt{(3)^2+(-3)^2}} = \frac{3i-3j}{3\sqrt{2}} = \frac{3}{3\sqrt{2}}i - \frac{3}{3\sqrt{2}}j$$

$$u = \frac{1}{2}i - \frac{1}{2}j.$$

The rate of change of f at P in this direction

$$|\nabla f| = \sqrt{(3)^2+(-3)^2} = 3\sqrt{2}.$$

QUESTION #61

Find a unit vector in the direction in which f decreases most rapidly at P, and find the rate of change of f at P in that direction.

$f(x,y) = 20 - x^2 - y^2$; $P(-1,-3)$.

Solve

$\nabla f(x,y) = f_x(x,y)i + f_y(x,y)j$

$f_x(x,y) = -2x$

$f_x(-1,-3) = -2(-1) = 2.$

$f_y(x,y) = -2y$

$f_y(-1,-3) = -2(-3) = 6.$

$\nabla f(-1,-3) = 2i + 6j$

Since, f decreases in this direction

$-\nabla f(-1,-3) = -(2i + 6j).$

finding unit vector in the direction where f decrease

$u = \dfrac{-\nabla f}{|\nabla f|} = -\left(\dfrac{2i + 6j}{\sqrt{(2)^2 + (6)^2}}\right) = -\left(\dfrac{2}{2\sqrt{10}} i + \dfrac{6}{2\sqrt{10}} j\right)$

$u = -\dfrac{1}{\sqrt{10}} i - \dfrac{3}{\sqrt{10}} j.$

The rate of change of f at P in this direction

$-|\nabla f| = -\sqrt{(2)^2 + (6)^2} = -2\sqrt{10}.$

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TOPIC :- LAGRANGE MULTIPLIERS:

QUESTION:

The plane $x+y+z=1$ cuts the cylinder $x^2+y^2=1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.

Solve

We have to find extreme values of

$$f(x,y,z) = x^2+y^2+z^2.$$

Above function is the distance from (x,y,z) to the origin subject to the constraints.

Lagrange Multipliers Three variable with two constraints

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \quad g_1(x,y,z)=0$$
$$g_2(x,y,z)=0.$$

$$g_1(x,y,z) = x^2+y^2-1=0 \quad \text{— ①}$$
$$g_2(x,y,z) = x+y+z-1=0. \quad \text{— ②}$$
$$\therefore \nabla f = f_x(x,y,z)i + f_y(x,y,z)j + f_z(x,y,z)k.$$
$$\nabla f = 2xi + 2yj + 2zk. \quad \text{— ③}$$

Now, we'll put eq 1, 2 and 3 in Lagrange's equation

$$2xi+2yj+2zk = \lambda(g_{1x}i+g_{1y}j+g_{1z}k) + \mu(g_{2x}i+g_{2y}j+g_{2z}k)$$
$$2xi+2yj+2zk = \lambda(2xi+2yj) + \mu(i+j+k)$$
$$2xi+2yj+2zk = (2\lambda x+\mu)i + (2\lambda y+\mu)j + \mu k.$$

Equating above equation:

$$2x = 2\lambda x+\mu, \quad 2y = 2\lambda y+\mu, \quad 2z = \mu.$$

The scalar equations in above equations yield

$$2x = 2\lambda x + 2z \Rightarrow \cancel{2}x = \cancel{2}(\lambda x+z)$$
$$x = \lambda x + z \Rightarrow z = x-\lambda x \Rightarrow \boxed{z=(1-\lambda)x} \text{ — ④}$$

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$2y = 2\lambda y + 2z \Rightarrow \cancel{2}y = \cancel{2}(\lambda y + z)$

$y = \lambda y + z \Rightarrow z = y - \lambda y \Rightarrow \boxed{z = y(1-\lambda)}$ — (5)

Equations 4 and 5 are satisfied simultaneously if either $\lambda = 1$ or $z = 0$ or $\lambda \neq 1$ and $x = y = \dfrac{z}{(1-\lambda)}$

So, if $z=0$, then solving equation 1 and 2 simultaneously to find corresponding points on ellipse gives points $(1,0,0)$ and $(0,1,0)$.

If $x = y$, equation 1 and 2 give

$x^2 + y^2 - 1 = 0.$

$x^2 + x^2 - 1 = 0$

$2x^2 - 1 = 0$

$2x^2 = 1$

$x^2 = 1/2$

Square $\sqrt{\ }$ on both sides.

$x = \pm 1/\sqrt{2}$

or $x = \pm \sqrt{2}/2$

$x + y + z - 1 = 0$

$x + x + z - 1 = 0$

$2x + z - 1 = 0$

$z = 1 - 2x.$

$\because x = \pm\sqrt{2}/2$

$z = 1 - \cancel{2}(\pm\sqrt{2}/\cancel{2})$

$z = 1 - \sqrt{2}$ or $1 + \sqrt{2}$

$z = 1 \pm \sqrt{2}$

The corresponding points on the ellipse are $P_1\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1-\sqrt{2}\right)$ and $P_2\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1+\sqrt{2}\right)$

Although P_1 or P_2 give local maxima of f on ellipse, P_2 is farther from origin than P_1

The points on ellipse closest to origin are $(1,0,0)$ and $(0,1,0)$. The points on the ellipse farthest from the origin is P_2.