Relative Maxima or Minima. (Absolute Extrema)

Ex. 1: Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$. Then

~~Solution~~

$f_x(x,y) = 2x - 2$ $\quad$ $f_y(x,y) = 2y - 6$.

$0 = 2x - 2$ $\quad$ $0 = 2y - 6$

$x = 2/2$ $\quad$ $y = 6/2$

$x = 1$ $\quad$ $y = 3$.

P.Ds are equal to 0 at $x = 1$ & $y = 3$, so the only critical point is $(1,3)$.

By completing square.

$f(x,y) = x^2 + y^2 - 2x - 6y + 14$.

$= x^2 - 2x + y^2 - 6y + 14$.

$= \underline{x^2 - 2x + 1^2} - 1^2 + \underline{y^2 - 6y + (3)^2} - (3)^2 + 14$.

$= (x-1)^2 - 1 + (y-3)^2 - 9 + 14$.

$f(x,y) = (x-1)^2 + (y-3)^2 + 4$.

Since $(x-1)^2 \geq 0$ & $(y-3)^2 \geq 0$, we have $f(x,y) \geq 4$ for all values of x & y. Therefore $f(1,3) = 4$. is a local minimum & in fact absolute minimum of f

Ex:1: Find the local extreme values of $f(x,y) = x^2 + y^2 - 4y + 9$.

Solution

$f_x(x,y) = 2x$ $\quad$ $f_y(x,y) = 2y - 4$.

$0 = 2x$ $\quad$ $0 = 2y - 4$

$\boxed{x = 0}$ $\quad$ $y = 4/2$

$(x,y) = (0, 2)$ critical points. $\quad$ $\boxed{y = 2}$

The only possibility is the point $(0,2)$. where value of f is 5. e.g. $f(0,2) = 0^2 + 2^2 - 4(2) + 9 = \boxed{5}$

Since, $f(x,y) = x^2 - (y-2)^2 + 5$ is never less than 5, we see that the critical point $(0,2)$ gives a local minimum

Second derivative Test for local extreme values.

(i) Local maximum at (a,b) if $f_{xx} < 0$ & $f_{xx} f_{yy} - f_{xy}^2 > 0$

(ii) Local minimum at (a,b) if $f_{xx} > 0$ & $f_{xx} f_{yy} - f_{xy}^2 > 0$.

(iii) Saddle point at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$.

(iv) Inconclusive at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 = 0$.

$f_{xx} f_{yy} - f_{xy}^2 \rightarrow$ Discriminant

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Ex: 1. Find the local extreme values of the function.
$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4.$

Solution

$f_x(x,y) = y - 2x - 2$ $\quad\quad$ $f_y(x,y) = x - 2y - 2$

$y - 2x - 2 = 0$ $\quad\quad$ $x - 2y - 2 = 0$

$2(y - 2x - 2) = 0$

$\cancel{2y} - 4x - 4 = 0$

$\underline{-\cancel{2y} + x - 2 = 0}$

$-3x - 6 = 0$

$3x = -6$

$\boxed{x = -2}$

$y - 2x - 2 = 0$

$y - 2(-2) = 2$

$y = -4 + 2$

$\boxed{y = -2}$

Critical points $(x,y) = (-2,-2)$.

Therefore, $(-2,-2)$ is the only point where f have extreme values. To find, whether these points are max or min.

$f_{xx}(x,y) = -2$, $f_{yy}(x,y) = -2$, $f_{xy}(x,y) = 1$

$f_{xx} f_{yy} - f_{xy}^2 \Rightarrow (-2)(-2) - (1)^2 = 4 - 1 = \boxed{3}$

Since, we have $f_{xx} < 0$ an $f_{xx} f_{yy} - f_{xy}^2 > 0$

$(-2,-2)$ are local maximum for f.

The value f at this point

$f(-2,-2) = (-2)(-2) + (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4.$

$f(-2,-2) = 8.$

Ex. 2: Find the local maximum and local minimum and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.

Sol

$f(x,y) = x^4 + y^4 - 4xy + 1$

$f_x(x,y) = 4x^3 - 4y$ $\quad\quad$ $f_y(x,y) = 4y^3 - 4x$

$4x^3 - 4y = 0$ $\quad\quad$ $4y^3 - 4x = 0$

$4(x^3 - y) = 0$ $\quad\quad$ $4(y^3 - x) = 0$

$x^3 - y = 0$ — ① $\quad\quad$ $y^3 - x = 0$ — ②

$y = x^3$ — put in eq ② $\quad\quad$ $(x^3)^3 - x = 0$

$x^9 - x = 0$

$x^9 - x = 0$

$x(x^8 - 1) = 0$

$\boxed{x = 0}$, $x^8 - 1 = 0$ $\quad a^2 - b^2$

$(x^4)^2 - (1)^2 = 0$

$(x^4 + 1)(x^4 - 1) = 0$ — ①

— ① $x^4 + 1 = 0$, $x^4 - 1 = 0$.

$x^4 = -1$, $(x^2+1)(x^2-1) = 0$

$(x^2)^2 = -1$

$x^2 + 1 = 0$, $x^2 - 1 = 0$

$\boxed{x^2 = -1}$ ✗, $x^2 = 1$

$\boxed{x = \pm 1}$

$x=0$, $x=1$, $x-1$

eq ① $x^3 - y = 0$ ⇒

for $x=0$ ⇒ $y=0$

for $x=1$ ⇒ $y=1$

for $x=-1$ ⇒ $y=-1$

There are three critical points → $(0,0),(1,1),(-1,-1)$.

Finding which points are which values

$f_{xx} = 12x^2$ $\quad f_{yy} = 12y^2$ $\quad f_{xy} = -4$.

$\boxed{f_{xx} \cdot f_{yy} - f_{xy}^2} = D$

C.P	f_{xx} ~~L.Min~~	D ~~L.Max~~	Concl.
(0,0)	$0 = 0$	$-16 < 0$	S
(1,1)	$12 > 0$	$128 > 0$	Min
(-1,1)	$12 > 0$	$128 > 0$	Min

for $(0,0)$

$f_{xx}(0,0) = 12(0)^2 = 0$.

$f_{yy}(0,0) = 12(0)^2 = 0$.

$f_{xy}(0,0) = -4$.

$D = 0 \cdot 0 - (-4)^2 = -16$

for $(1,1)$

~~$144 - 144 - (-4)^2$~~ =

$12 \times 12 - (-4)^2 = 128$

for $(-1,-1)$

$12 \times 12 - (-4)^2 = 128$.

Ex: 3: Find the local extreme values of $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

Sol → $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

$f_x(x,y) = -6x + 6y$.

$-6x + 6y = 0$

$6(-x+y) = 0$

$-x + y = 0$

$x = y$.

$f_y(x,y) = 6y - 6y^2 + 6x$.

$6y - 6y^2 + 6x = 0$.

put $x = y$.

$6y - 6y^2 + 6y = 0$

$12y - 6y^2 = 0$.

$6(2y - y^2) = 0$

$2y - y^2 = 0$

$y(2-y) = 0$

$\boxed{y=0}$, $2 - y = 0$

$\boxed{y=2}$

as $x = y$

critical points are $(0,0),(2,2)$.

$f_{xx} = -6$ $\quad f_{yy} = 6 - 12y$ $\quad f_{xy} = 6$

for $(0,0)$

$f_{xx} = -6$

$f_{yy} = 6$

$D = -36 - 36 = -72$

for $(2,2)$

$f_{xx} = -6$

$f_{yy} = -18$

$D = 72$

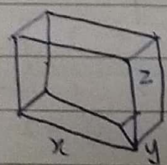
CP	f_{xx}	D	Conclusion
(0,0)	$-6 < 0$	$-72 < 0$	Saddle
(2,2)	$-6 < 0$	$72 < 0$	L.max.

Q. A rectangular box without lid is to be made from $12m^2$ of card board. Find the maximum volume of such a box.

&

Let length = x, Breadth = y and height = z of the box, so its volume will be $\boxed{V = xyz}$ — Max

V can be expressed as function of x & y by using the fact that area of four sides & bottom of box is $\boxed{2xz + 2yz + xy = 12}$. — ①



two sides each have area xz
two sides each have area yz
base has area xy.

Solving eq ① for z.

$$2(xz+yz) = 12 - xy$$

$$z(x+y) = \frac{12-xy}{2}$$

$$\boxed{z = \frac{12-xy}{2(x+y)}}$$ put this in main eq

$$V = xy\left(\frac{12-xy}{2(x+y)}\right)$$

$$V = \frac{12xy - x^2y^2}{2(x+y)}.$$

$$\frac{\partial V}{\partial x} = \frac{\{2(x+y)\cdot(12y-2xy^2)\} - \{(12xy-x^2y^2)(2)\}}{[2(x+y)]^2}$$

$$= \frac{\cancel{2}[\{(x+y)(12y-2xy^2)\} - \{(12xy-x^2y^2)\}]}{\cancel{4}^2(x+y)^2}$$

$$= 12\cancel{xy} - 2x^2y^2 + 12y^2 - 2xy^3 - 12\cancel{xy} + x^2y^2 / 2(x+y)^2$$

$$= -x^2y^2 + 12y^2 - 2xy^3 / 2(x+y)^2.$$

$$\frac{\partial V}{\partial x} = y^2(-x^2 + 12 - 2xy) / 2(x+y)^2 \quad \text{— ②}$$

$$\frac{\partial V}{\partial y} = \frac{\{2(x+y)\cdot(12x-2x^2y)\} - \{(12xy-x^2y^2)(2)\}}{[2(x+y)^2]}$$

$$= \cancel{2}[\{(x+y)(12x-2x^2y)\} - \{(12xy-x^2y^2)\}] / \cancel{4}^2(x+y)^2$$

$$= 12x^2 - 2x^3y + 12\cancel{xy} - 2x^2y^2 - 12\cancel{xy} + x^2y^2 / 2(x+y)^2$$

$$= 12x^2 - 2x^3y - x^2y^2 / 2(x+y)^2$$

$$\frac{\partial V}{\partial y} = x^2(12 - 2xy - y^2) / 2(x+y)^2 \quad \text{— ③}$$

$$\frac{\partial V}{\partial x} = \frac{y^2(12-2xy-x^2)}{2(x+y)^2}\ , \quad \frac{\partial V}{\partial y} = \frac{x^2(12-2xy-y^2)}{2(x+y)^2}$$

$$\frac{y^2(12-2xy-x^2)}{2(x+y)^2} = 0 \qquad \frac{x^2(12-2xy-y^2)}{2(x+y)^2} = 0$$

$$12-2xy-x^2=0 \quad \text{—(i)} \qquad 12-2xy-y^2=0. \quad \text{—(ii)}$$

$$\cancel{12}-\cancel{2xy}-x^2=0$$
$$\cancel{12}-\cancel{2xy}-y^2=0$$
$$-\quad + \quad +$$
$$-x^2+y^2=0$$
$$x^2=y^2$$
$$\boxed{x=y}$$

put $x=y$

$$12-2(y)y-y^2=0$$
$$12-2y^2-y^2=0$$
$$12-3y^2=0$$
$$4\cancel{12}=\cancel{3}y^2$$
$$y^2=4$$
$$\boxed{y=\pm 2}$$

Since V can't be -ve hence $y=2$.

critical points

$x=2,\ y=2$

$(x,y)=(2,2)$.

OR. If we put $x=y$ in eqv (I) or (II)

$$\frac{y^2(12-2y.y-y^2)}{2(y+y)^2}=0$$

$$\frac{y^2(12-2y^2-y^2)}{2(2y)^2}=0$$

$$\frac{\cancel{y^2}(12-3y^2)}{8\cancel{y^2}}=0$$

$$12-3y^2=0 \qquad \text{or} \qquad 12-3x^2=0.$$
$$4\cancel{12}=\cancel{3}y^2 \qquad 4\cancel{12}=\cancel{3}x^2$$
$$y^2=4 \qquad x^2=4$$
$$y=\pm 2 \qquad x=\pm 2$$
$$\boxed{y=2} \qquad (x,y)=(2,2)\ \boxed{x=2}$$

$\because$ Max Volume, $\frac{12xy-x^2y^2}{2(x+y)} = \frac{12(2)(2)-(2)^2(2)^2}{2(2+2)}$

$$= \frac{48-16}{8} = \frac{32}{8}\ 4$$

$$\boxed{\text{Max. Volume} = 8}$$

Shortest Distance From P. To The Plane

Q. Find the shortest distance from the point $(1,0,-2)$ to the plane $x+2y+z=4$.

Solution

The distance from any point (x,y,z) to the point $(1,0,-2)$ is

$$d = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

let $(x_0, y_0, z_0) = (1, 0, -2)$.

$$d = \sqrt{(x-1)^2 + (y-0)^2 + (z+2)^2}.$$

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$$

but if (x,y,z) lies on the plane $x+2y+z=4$ then $\boxed{z = 4-x-2y}$

So, we have

$$d = \sqrt{(x-1)^2 + y^2 + (4-x-2y+2)^2}$$

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2} \quad \text{— (i)}$$

we can minimize d by minimizing the simpler expression.

by taking square on both sides of eq (i).

$$\Rightarrow d^2 = (x-1)^2 + y^2 + (6-x-2y)^2$$

By taking its P.D with respect to x and y

$$f_x = 2(x-1) + 2(6-x-2y)$$

$$= 4x + 4y - 14 = 0 \quad \text{— (ii)}$$

$$f_y = 2y - 4(6-x-2y)$$

$$= 4x + 10y - 24 = 0 \quad \text{— (iii)}$$

solving (ii) & (iii) by subtracting.

$$\cancel{4x} + 4y - 14 = 0$$

$$\cancel{4x} + 10y - 24 = 0$$

$$\underline{- \quad - \quad +}$$

$$-6y + 10 = 0$$

$$6y = 10$$

$$y = 10/6$$

$$\boxed{y = 5/3}$$

now put y in eq (i).

$$4x + 4\left(\frac{5}{3}\right) - 14 = 0$$

$$4x + \frac{20}{3} = 14$$

$$4x = 14 - \frac{20}{3}$$

$$4x = 42 - 20/3$$

$$x = 22/3 \times 4 \times 2$$

$$\boxed{x = \frac{11}{6}}$$

critical points $\Rightarrow (x,y) = \left(\frac{11}{6}, \frac{5}{3}\right)$.

so, the critical points $(11/6, 5/3)$

$f_{xx} = 4 > 0$ ✓

$f_{yy} = 10$ at $(x,y) = (11/6, 5/3)$.

$f_{xy} = 4$. local minimum.

$D = (f_{xx})(f_{yy}) - (f_{xy})^2$

$D = 4 \times 10 - 4^2$

$D = 40 - 16$

$\boxed{D = 24 > 0}$ ✓

since points are minimum, so we can find shortest distance using eq. 1.

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$$

$$d = \sqrt{\left(\frac{11}{6}-1\right)^2 + \left(\frac{5}{3}\right)^2 + \left(6-\frac{11}{6}-2\left(\frac{5}{3}\right)\right)^2}$$

$$d = \frac{5\sqrt{6}}{6} = 2.041$$

we know that at C.P $(11/6, 5/3)$ function of distance will be min.

So, we can find the S.D by using this point

$$\boxed{d = \frac{5}{6}\sqrt{6}}$$

The shortest distance from $(1,0,-2)$ to the plane $x + 2y + z = 4$ is $\frac{5}{6}\sqrt{6}$.

CamScanner