LINE INTEGRALS:

if C is a curve which is smoothly parametized.

In 2-space, $\gamma(t) = x(t)i + y(t)j \quad (a \le t \le b)$.

then $\displaystyle\int_c f(x,y)\,ds = \int_a^b f(x(t), y(t))\,\|\gamma'(t)\|\,dt.$

for 3-space, $\gamma(t) = x(t)i + y(t)j + z(t)k.$

then $\displaystyle\int_c f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\,\|\gamma'(t)\|\,dt.$

Ex: 1 Using the given parametization, evaluate the integral $\int_c (1+xy^2)\,ds$.

(a) $C: \gamma(t) = ti + 2tj \quad (0 \le t \le 1)$

&

$\gamma'(t) = i + 2j$

$\|\gamma'(t)\| = \sqrt{1^2+2^2} = \sqrt{1+4} = \sqrt{5}.$

$\displaystyle\int_c (1+xy^2)\,ds = \int_0^1 [1+(2t)^2 t]\sqrt{5}\,dt.$

$\displaystyle = \int_0^1 (1+4t^3)\sqrt{5}\,dt.$

$\displaystyle = \sqrt{5}\int_0^1 1+4t^3\,dt \qquad t+t^4$

$= \sqrt{5}\left| t + \cancel{4}\frac{t^4}{\cancel{4}} \right|_0^1$

$= \sqrt{5}\left[\{(1) + (1)^4\} - \{(0) + (0)^3\}\right].$

$= \sqrt{5}\,(1 + 1 - 0 - 0)$

$= \sqrt{5}\,(2)$

$\displaystyle\int_c (1+xy)^2\,ds = 2\sqrt{5}$

(b) $C: \gamma(t) = (1-t)i + (2-2t)j \quad (0 \le t \le 1).$

&

$\gamma'(t) = -1i + (-2)j$

$\gamma'(t) = -i - 2j.$

$\|\gamma'(t)\| = \sqrt{(-1)^2+(-2)^2} = \sqrt{1+4} = \sqrt{5}.$

$\displaystyle\int_c (1+xy^2)\,ds = \int_0^1 [1+(1-t)(2-2t)^2]\sqrt{5}\,dt$

$= \cancel{\sqrt{5}\int_0^1 [1+(1-t)(4-8t+4t^2)\,dt}$ ×

$\displaystyle = \sqrt{5}\int_0^1 [1+(1-t)2^2(1-t)^2\,dt$

$\displaystyle = \sqrt{5}\int_0^1 1+4(1-t)^3\,dt$

$= \sqrt{5}\left| t + \cancel{4}\frac{(1-t)^4}{\cancel{4}(-1)} \right|_0^1$

$= \sqrt{5}\left[\{(1) - (1-1)^4\} - \{(0) - (1-0)^4\}\right].$

$= \sqrt{5}\,(1 - (0)^4 - 0 + 1)$

$= \sqrt{5}\,(1 - 0 + 1)$

$= \sqrt{5}\,(2)$

$\displaystyle\int_c (1+xy)^2\,ds = 2\sqrt{5}$

PAPERWORK

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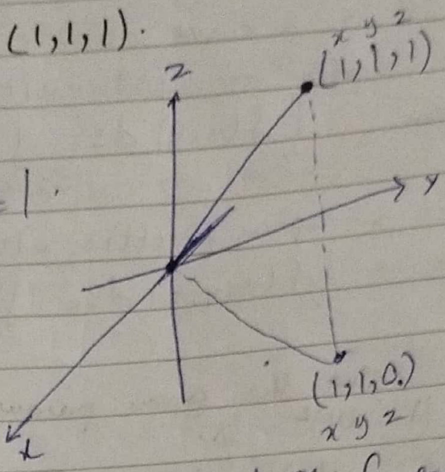
Ex 2 Integrate $f(x,y,z) = x - 3y^2 + z$ over the line segment C joining the origin to the point $(1,1,1)$.

Sol



Choosing simplest parametization.

$\gamma(t) = ti + tj + tk, \quad 0 \le t \le 1.$

$\gamma'(t) = i + j + k.$

The components have continuous first derivatives.

$|\gamma'(t)| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

and $|\gamma'(t)| = \sqrt{3}$ is never 0,

so the parametization is smooth. ∴ The integral of f over C is

$$\int_C f(x,y,z)\, ds = \int_0^1 f(t,t,t)\,\|\gamma'(t)\|\, dt.$$

$f(t,t,t) = t - 3t^2 + t.$

$$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\, dt$$

$$= \sqrt{t}\left|\frac{t^2}{2} - \frac{\cancel{3}t^3}{\cancel{3}} + \frac{t^2}{\cancel{2}}\right|_0^1$$

$$= \sqrt{t}\left|\frac{\cancel{2}t^2}{\cancel{2}} - t^3\right|_0^1$$

$$= \sqrt{t}\left[\{1^2 - 1^3\} - \{0^2 - 0^3\}\right]$$

$$= \sqrt{t}\,(0)$$

$$\int_C f(x,y,z)\, ds = 0 \quad Ans.$$

A line integral of f w.r.t to s along C doesn't depend on an orientation of C.

$x = x(t), \quad y = y(t) \quad (a \le t \le b).$

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$ (2-space).

$$\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$ (3-space)

Ex: 1: Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper-half of the unit circle $x^2 + y^2 = 1$.

Sol parametric equations to represent C. a unit circle can be parametized by means of equations.

$x = \cos t, \quad y = \sin t.$

The upper-half of circle is decribed by parameter interval $0 \le t \le \pi$.

Therefore, $\int_c (2+x^2y)\, ds = \int_0^{\pi} f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$f(\underset{\cos t}{x(t)}, \underset{\sin t}{y(t)}) = \cancel{2+t^2t} = \cancel{2+t^3} \Rightarrow 2+\cos^2 t \sin t$

$x = \cos t \qquad y = \sin t$

$dx/dt = -\sin t \qquad dy/dt = \cos t$

$$\int_c (2+x^2y)\, ds = \int_0^{\pi} (2+\cos^2 t \sin t) \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$

$$= \int_0^{\pi} (2+\cos^2 t \sin t) \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^{\pi} 2 + \cos^2 t \sin t\, dt \qquad \because \sin^2 t + \cos^2 t = 1$$

$$= \int_0^{\pi} 2\, dt + \int_0^{\pi} \cos^2 t \sin t\, dt \qquad \text{let } \boxed{u = \cos t}$$

$\frac{du}{dt} = -\sin t$

$\boxed{\frac{du}{-\sin t} = dt}$

$$= 2t \Big|_0^{\pi} + \int_0^{\pi} u^2 \cdot \cancel{\sin t}\, \frac{du}{-\cancel{\sin t}}$$

$$= 2[\pi - 0] + \left(-u^3/3\right)\Big|_0^{\pi}$$

$$= 2\pi - \left(\cos^3 t/3\right)\Big|_0^{\pi}$$

$$= 2\pi - \left[\left\{\frac{\cos^3 \pi}{3} - \frac{\cos^3 0}{3}\right\}\right].$$

$$= 2\pi - \left(\frac{-1}{3} - 1/3\right)$$

$$= 2\pi - (-2/3)$$

$$\int_c (2+x^2y)\, ds = 2\pi + 2/3$$

Ex: 2 Evaluate the line integral $\int_c (xy+z^3)\, ds$ from $(1,0,0)$ to $(-1,0,\pi)$ along the helic C that is represented by the parametric equations.

$x = \cos t, \; y = \sin t, \; z = t \qquad (0 \le t \le \pi).$

$$\int_c (xy+z^3)\, ds = \int_a^b f(x(t), y(t), z(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$f(\cos t, \sin t, t) = \cos t \sin t + t^3.$

$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t, \quad \frac{dz}{dt} = 1$

$$\int_c (xy+z^3)\, dz = \int_0^{\pi} (\cos t \sin t + t^3) \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} \cdot dt$$

$$= \int_0^{\pi} (\cos t \sin t + t^3) \sqrt{\sin^2 t + \cos^2 t + 1}\, dt$$

$$= \int_0^{\pi} \cos t \sin t + t^3 \cdot \sqrt{1+1}\, dt$$

$$= \int_0^{\pi} (\cos t \sin t + t^3) \sqrt{2}\, dt$$

$= \sqrt{2} \int_0^{\pi} \cos t \sin t + t^3 \quad dt$

$= \sqrt{2} \left[\int_0^{\pi} \cos t \sin t \, dt + \int_0^{\pi} t^3 \, dt \right]$

let $\boxed{u = \sin t} \Rightarrow \frac{du}{dt} = \cos t \Rightarrow \boxed{\frac{du}{\cos t} = dt}$

$= \sqrt{2} \left[\int_0^{\pi} \cos t \cdot u \frac{du}{\cos t} + \int_0^{\pi} t^3 \, dt \right]$

$= \sqrt{2} \left[\left| \frac{u^2}{2} \right|_0^{\pi} + \left| \frac{t^4}{4} \right|_0^{\pi} \right]$

$= \sqrt{2} \left[\left| \frac{\sin^2 t}{2} \right|_0^{\pi} + \left\{ \frac{\pi^4}{4} - \frac{0^4}{4} \right\} \right]$

$= \sqrt{2} \left[\left\{ \frac{\sin^2 \pi}{2} - \frac{\sin^2 0}{2} \right\} + \frac{\pi^4}{4} \right]$

$= \sqrt{2} \left(0 - 0 + \frac{\pi^4}{4} \right)$

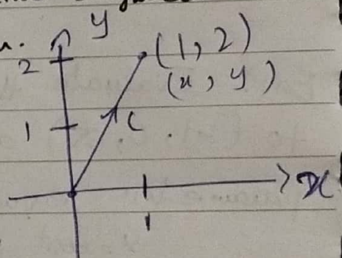
$\int_C (xy + z^3) \, ds = \frac{\sqrt{2} \, \pi^4}{4}$. Ans.

Line Integrals w.r.t. x, y And z.

$\int_C f(x,y,z) \, dz = \int_a^b f(x(t), y(t), z(t)) \, z'(t) \, dt$. (3-space

$x = x(t), \quad y = y(t), \quad z = z(t) \quad (a \le t \le b)$

Ex: Evaluate $\int_C 3xy \, dy$, where C is the line segment joining (0,0) to (1,2) with given orientation.



(a) Oriented from (0,0) to (1,2)

Sol: Using parametrization

$x = t, \quad y = 2t \quad (0 \le t \le 1)$

$\int_C 3xy \, dy = \int_0^1 3(t)(2t)(2) \, dt$

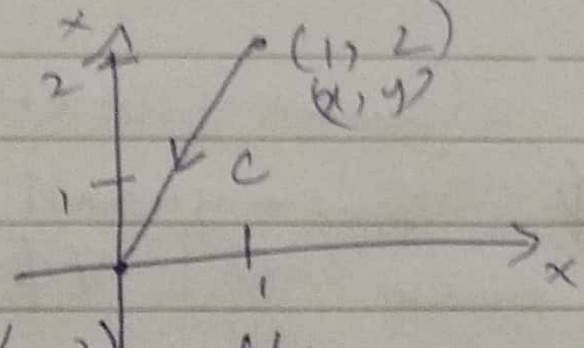
$= \int_0^1 12t^2 \, dt$

$= \frac{\cancel{12}^{4} t^3}{\cancel{3}} \Big|_0^1$

$= [4\{1^3 - 0^3\}]$

$\int_C 3xy \, dy = 4$ Ans.

(b) Oriented from (1,2) to (0,0).



Using paramettization

$x = 1 - t$, $y = 2 - 2t$ $(0 \le t \le 1)$.

$$\int_C 3xy \, dy = \int_0^1 3(1-t)(2-2t)(-2) \, dt$$

$$= \int_0^1 -6(1-t) \, 2(1-t) \, dt$$

$$= \int_0^1 -12(1-t)^2 \, dt.$$

$$= -12 \int_0^1 (1-t)^2 \, dt.$$

$$= -12 \left[-\frac{(1-t)^3}{3} \Big|_0^1 \right].$$

$$= 12 \left[\cancel{\frac{(1-1)^3}{3}} - \frac{(1-0)^3}{3} \right].$$

$$= \cancel{12}^{4} \left(-\frac{1}{\cancel{3}} \right)$$

$$\int_C 3xy \, dy = -4.$$

A.

Line Integral:

$\int_C f(x,y)\,dx + g(x,y)\,dy = \int_C f(x,y)\,dx + \int_C g(x,y)\,dx.$

Ex Evaluate $\int_C 2xy\,dx + (x^2+y^2)\,dy$ along the circula arc given by $x = \cos t,\ y = \sin t$ $(0 \le t \le \pi/2)$.

$\because \int f(x,y)\,dx = \int_a^b f(x(t), y(t))\, x'(t)\,dt$

$\int_C 2xy\,dx = \int_0^{\pi/2} 2\cos t \sin t \cdot (-\sin t)\,dt$

$= \int_0^{\pi/2} -2\cos t \sin^2 t\,dt.$

$= -2\int_0^{\pi/2} \cos t \sin^2 t\,dt$ Let $\boxed{u = \sin t}$ $\Rightarrow \frac{du}{dt} = \cos t$ $\boxed{\frac{du}{\cos t} = dt}$

$= -2\int_0^{\pi/2} \cancel{\cos t}\, u^2 \frac{du}{\cancel{\cos t}}$

$= -2\left[\left|\frac{u^3}{3}\right|_0^{\pi/2}\right]$

$= -2\left[\left|\frac{\sin^3 t}{3}\right|_0^{\pi/2}\right].$

$= -2\left[\left\{\frac{\sin^3 \pi/2}{3} - \frac{\sin^3 0}{3}\right\}\right].$

$= -2\left[\frac{(\sin 90)^3}{3} - 0\right].$

$= -2\left(\frac{(1)^3}{3}\right) \Rightarrow 2\cancel{\left(\frac{\frac{3\sqrt{3}}{8}}{3}\right)} \Rightarrow 2\cancel{\left(\frac{\sqrt{3}}{8}\right)}$

$\boxed{\int_C 2xy\,dx = -\frac{2}{3}}$

$\int_C x^2+y^2\,dy = \int_0^{\pi/2} (\cos^2 t + \sin^2 t)(\cos t)\,dt.$

$= \int_0^{\pi/2} \cos t\,dt$

$= \int_0^{\pi/2} \cos t\,dt$

$= \sin t \Big|_0^{\pi/2}$

$= \sin \pi/2 - \sin 0$

$= \sin 90$

$\boxed{\int_C x^2+y^2\,dy = 1}$

$\int_C 2xy\,dx + (x^2+y^2)\,dy = \int_C 2xy\,dx + \int_C (x^2+y^2)\,dy.$

$= -\frac{2}{3} + 1^{\times 3}$

$\int_C 2xy\,dx + (x^2+y^2)\,dy = \frac{1}{3}$

PAPERWORK

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$\sin^2 x + \cos^2 x = 1$
$\sin^2 x = 1 - \cos^2 x$

Q Evaluate $\int_c (3x^2+y^2)\,dx + 2xy\,dy$ along the circular arc C given by $x = \cos t$, $y = \sin t$). $(0 \le t \le \pi/2)$.

Sol

$$\int_c (3x^2+y^2)\,dx + 2xy\,dy = \int_C (3x^2+y^2)\,dx + \int_c 2xy\,dy$$

$$\int_c (3x^2+y^2)\,dx = \int_0^{\pi/2} 3\cos^2 t + \sin^2 t \cdot (-\sin t)\,dt$$

$$= \int_0^{\pi/2} -3\cos^2 t \sin t - \sin^3 t \,dt$$

$$= -3\int_0^{\pi/2} \cos^2 t \sin t\,dt - \int_0^{\pi/2} \sin^3 t\,dt$$

let $\boxed{u = \cos t}$

$\frac{du}{dt} = -\sin t$

$\boxed{\frac{du}{-\sin t} = dt}$

let $u = \cos t$

$\sin^2 t \cdot \sin t$

$$= -3\int_0^{\pi/2} u^2 \cancel{\sin t}\,\frac{du}{-\cancel{\sin t}} - \int_0^{\pi/2} (1-\cos^2 t)(\sin t)\,dt$$

$$= 3\left|\frac{u^3}{3}\right|_0^{\pi/2} - \int_0^{\pi/2} (1-u^2)\,\cancel{\sin t}\,\frac{du}{-\cancel{\sin t}}$$

$$= 3\left[\left|\frac{\cos^3 t}{3}\right|_0^{\pi/2}\right]. - \int_0^{\pi/2} u^2 - 1\,du$$

$$= 3\left[(\cos(\pi/2))^3/3 - (\cos(0))^3/3\right] - \left[\frac{u^3}{3} - u\right|_0^{\pi/2}\Big]$$

$$= 3\,(0 - 1/3) \qquad - \left[\left|\frac{\cos^3 t}{3} - \cos t\right|_0^{\pi/2}\right]$$

$$= \cancel{3}(-1/\cancel{3}) \qquad - \left[\left\{\frac{(\cos(\pi/2))^3}{3} - \cos(\pi/2)\right\} - \left\{\frac{(\cos(0))^3}{3} - \cos(0)\right\}\right]$$

$$= -1 \qquad - \left[(0-0) - \left(\frac{1}{3} - 1\right)\right].$$

$$= -1 \qquad - \left[0 - \left(-\frac{2}{3}\right)\right]$$

$$= -1 - \frac{2}{3}$$

$-1 - \frac{2}{3}$, $\frac{-3-2}{3} = -\frac{5}{3}$

$$\boxed{\int_c (3x^2+y^2)\,dx = -\frac{5}{3}}$$

$$\int_c 2xy\,dy = \int_0^{\pi/2} 2\cos t \sin t \cdot (\cos t)\,dt$$

$$= -2\int_0^{\pi/2} \cos^2 t \sin t\,dt$$

let $u = \cos t$

$du/dt = -\sin t$

$du/-\sin t = dt$

$$= -2\int_0^{\pi/2} u^2\,\cancel{\sin t}\,du/-\cancel{\sin t}$$

$$= +2\left|\frac{u^3}{3}\right|_0^{\pi/2}$$

$$= +2\left[\left|\frac{\sin^3 t}{3}\right|_0^{\pi/2}\right]$$

$$= +2\left[\frac{(\sin(\pi/2))^3}{3} - \frac{(\sin(0))^3}{3}\right]$$

$$= +2\left(\frac{1}{3} - 0\right)$$

$$\boxed{\int_c 2xy\,dy = +2/3}$$

$$\int_c (3x^2+y^2)\,dx + (2xy)\,dy = \int_c (3x^2+y^2)\,dy + \int_c 2xy\,dy$$

$$= \frac{-5}{3} + \frac{2}{3}$$

$$\int_c (3x^2+y^2)\,dx + (2xy)\,dy = -1$$

Ans.

OR $\Rightarrow \int_c (3x^2+y^2)\,dx + 2xy\,dy = \int_0^{\pi/2} \{(3\cos^2 t + \sin^2 t)(-\sin t)\} + \{2\cos t \sin t \cdot \cos t\}\,dt$

$$= \int_0^{\pi/2} \left(-3\cos^2 t \sin t - \sin^3 t + 2\cos^2 t \sin t\right) dt$$

$$= \int_0^{\pi/2} \left(-\cos^2 t \sin t - \sin^3 t\right) dt$$

$$= \int_0^{\pi/2} \{-\cos^2 t \sin t - (\sin^2 t)(\sin t)\}\,dt$$

$$= \int_0^{\pi/2} \sin t\,(-\cos^2 t - \sin^2 t)\,dt$$

$$= \int_0^{\pi/2} \{-\sin t\,(\cos^2 t + \sin^2 t)\}\,dt$$

$$= -\int_0^{\pi/2} \sin t\,(1)\,dt$$

$$= -\int_0^{\pi/2} \sin t\,dt$$

$$= -\left[-\cos t \Big|_0^{\pi/2}\right]$$

$$= +\cos \pi/2 - \cos 0$$

$$\int_c (3x^2+y^2)\,dx + 2xy\,dy = -1$$

Ans