VECTOR FIELD:

$F(x,y) = f(x,y)i + g(x,y)j$ → 2 space

$F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$ → 3 space

Inverse square field.

$F(r) = \frac{c}{\|r\|^3}\cdot r$, r is radius vector an c is a constant

$\therefore F(x,y) = \frac{c}{(x+y)^{3/2}}\cdot(xi+yj)$. $r_2 = xi + yj$

$r_3 = xi + yj + zk$.

$\therefore F(x,y,z) = \frac{c}{(x+y+z)^{3/2}}\cdot(xi+yj+zk)$

Inverse square fields are conservative in any region that does not contain the origin

E.g. $\phi(x,y) = -\frac{c}{(x^2+y^2)^{1/2}}$. → is a potential function

$\nabla\phi(x,y) = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j$.

$-\frac{c}{(x^2+y^2)^{1/2}} = -c\cdot\frac{1}{(x^2+y^2)^{1/2}}$

$= -c\cdot(x^2+y^2)^{-1/2}$

$= -c\cdot -\frac{1}{2}(x^2+y^2)^{-3/2}\cdot 2x$

$= \frac{cx}{(x^2+y^2)^{3/2}}$

$= \frac{cx}{(x^2+y^2)^{3/2}}i + \frac{cy}{(x^2+y^2)^{3/2}}j$

$= \frac{c}{(x^2+y^2)^{3/2}}(xi+yj)$

$= F(x,y)$.

Ex. Show that divergence of inverse square field

$F(x,y,z) = \frac{c}{(x^2+y^2+z^2)^{3/2}}(xi+yj+zk)$ is zero.

Ans let $r = (x^2+y^2+z^2)^{1/2}$, $\Rightarrow \left((x^2+y^2+z^2)^{1/2}\right)^3 \Rightarrow \boxed{r^3}$

$F(x,y,z) = \frac{c}{r^3}(xi+yj+zk)$

$F(x,y,z) = \frac{cxi+cyj+czk}{r^3} = \frac{cxi}{r^3} + \frac{cyj}{r^3} + \frac{czk}{r^3}$.

$\operatorname{div} F = \nabla\cdot F = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot\left(\frac{cx}{r^3}i + \frac{cy}{r^3}j + \frac{cz}{r^3}k\right)$

$\Rightarrow \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot\left[c\left(\frac{x}{r^3}i + \frac{y}{r^3} + \frac{z}{r^3}k\right)\right]$.

$\Rightarrow C\left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right]$ — (1)

$\because r = (x^2+y^2+z^2)^{1/2}$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x$$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\underbrace{(x^2+y^2+z^2)^{1/2}}_{=r}}\cdot \cancel{2}x$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

$$\frac{\partial r}{\partial y} = \frac{1}{\cancel{2}\underbrace{(x^2+y^2+z^2)^{1/2}}_{=r}}\cdot \cancel{2}y$$

$$\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$$

$$\frac{\partial r}{\partial z} = \frac{1}{\cancel{2}\underbrace{(x^2+y^2+z^2)^{1/2}}_{=r}}\cdot \cancel{2}z$$

$$\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$$

$$(1) \Rightarrow C\left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right]$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) \quad \because \frac{u}{v} = v\cdot du - u\cdot dv / v^2$$

$$= \frac{r^3\cdot(1)\cancel{\partial x/\partial x} - x\cdot 3r^2\,(\partial r/\partial x)}{(r^3)^2} = \frac{r^3 - 3r^{\cancel{2}}x\,(x/\cancel{r})}{r^6}$$

$$= \frac{r^3 - 3rx^2}{r^6} \Rightarrow \frac{\cancel{r^3}}{r^6} - \frac{3\cancel{r}x^2}{r^6} \Rightarrow \frac{1}{r^3} - \frac{3x^2}{r^5}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{1}{r^3} - \frac{3x^2}{r^5}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) = \frac{1}{r^3} - \frac{3y^2}{r^5}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{1}{r^3} - \frac{3z^2}{r^5}$$

$$\text{div } F = C\left[\frac{1}{r^3} - \frac{3x^2}{r^5} + \frac{1}{r^3} - \frac{3y^2}{r^5} + \frac{1}{r^3} - \frac{3z^2}{r^5}\right].$$

$$\text{div} F = C\left[\frac{3}{r^3} - \left(\frac{3x^2+3y^2+3z^2}{r^5}\right)\right].$$

$$\text{div} F = C\left[\frac{3}{r^3} - \left\{\frac{3(x^2+y^2+z^2)}{r^5}\right\}\right] \qquad \because (x^2+y^2+z^2)^{1/2} = r$$

so $x^2+y^2+z^2 = r^2$.

$$\text{div} F = C\left[\frac{3}{r^3} - \frac{3\cancel{r^2}}{r^5}\right] \Rightarrow C\left[\frac{3}{\cancel{r^3}} - \frac{3}{\cancel{r^3}}\right] \Rightarrow C(0)$$

$$\boxed{\text{div } F = 0}$$ proved.

Q. Find the divergence & curl of following vector field.

$F(x,y,z) = xz^3 i + 2y^4x^2 j + 5z^2 y k$.

Sol. $\text{div} F = \nabla \cdot F = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot (xz^3 i + 2y^4x^2 j + 5z^2 y k)$

$\text{div} F = \frac{\partial}{\partial x}(xz^3) + \frac{\partial}{\partial y}(2y^4z^2) + \frac{\partial}{\partial z}(5z^2y)$.

$\text{div} F = z^3 + 8y^3z^2 + 10zy$ — scalar ✓

$\text{Curl} = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$ $\quad F_1 = xz^3 i$, $F_2 = 2y^4x^2 j$, $F_3 = 5z^2yk$

$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz^3 & 2y^4x^2 & 5z^2y \end{vmatrix}$

$\nabla \times F = i\{\partial/\partial y(5z^2y) - \partial/\partial z(2y^4z^2)\} - j\{\partial/\partial x(5z^2y) - \partial/\partial z(xz^3)\} + k\{\partial/\partial x(2y^4x^2) - \partial/\partial y(xz^3)\}$.

$\nabla \times F = i(5z^2 - 0) - j(0 - 3xz^2) + k(4xy^4 - \cancel{3xz^2})$ 0

$\nabla \times F = 5z^2 i + 3xz^3 j + 4xy^4 k$. — vector ✓.

Q. Find the divergence & curl of the vector field.

$F(x,y,z) = x^2y i + 2y^3z j + 3z k$.

Sol. $\text{div} F = \nabla \cdot F = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot (x^2y i + 2y^3z j + 3z k)$.

$\text{div} F = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(2y^3z) + \frac{\partial}{\partial z}(3z)$.

$\text{div} F = 2xy + 6y^2z + 3$. — scalar ✓.

$\text{curl} = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & 2y^3z & 3z \end{vmatrix}$

$\nabla \times F = i\{\partial/\partial y(3z) - \partial/\partial z(2y^3z)\} - j\{\partial/\partial x(3z) - \partial/\partial z(x^2y)\} + k\{\partial/\partial x(2y^3z) - \partial/\partial y(x^2y)\}$

$\nabla \times F = i(0 - 2y^3) - j(0 - 0) + k(0 - x^2)$.

$\nabla \times F = -2y^3 i - x^2 k$. — vector ✓.

Vector field again

Q Confirm that "ϕ" is a potential function or not.

(i) $\phi(x,y) = 2y^2 + 3x^2 y - xy^3$

$F(x,y) = 2xi + 6yj + 8zk.$

Sol

If $F = \nabla\phi$, it is a potential function else not.

$\therefore \nabla\phi = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j\right) \phi(x,y).$

$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j\right)(2y^2 + 3x^2 y - xy^3).$

$= \frac{\partial}{\partial x} i (2y^2 + 3x^2 y - xy^3) + \frac{\partial}{\partial y} j (2y^2 + 3x^2 y - xy^3).$

$\nabla\phi = (0 + 6xy - \cancel{3x}y^3) i + (4y + 3x^2 - 3xy^2) j.$

since $F \neq \nabla\phi$, ϕ is not a potential function

(ii) $\phi(x,y,z) = x\sin z + y\sin x + z\sin y.$

Sol $F(x,y,z) = (\sin z + y\cos x) i + (\sin x + z\cos y) j + (\sin y + x\cos z) k.$

$\therefore \nabla\phi = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \phi(x,y,z).$

$= \frac{\partial}{\partial x}(x\sin z + y\sin x + z\sin y) i + \frac{\partial}{\partial y}(x\sin z + y\sin x + z\sin y) j$

$+ \frac{\partial}{\partial z}(x\sin z + y\sin x + z\sin y) k.$

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$= (\sin z + y\cos x + 0)i + (0 + \sin x + z\cos y)j + (x\cos z + 0 + \sin y)k$

$\nabla\phi = (\sin z + y\cos x)i + (\sin x + z\cos y)j + (\sin y + x\cos z)k$

Since $F = \nabla\phi$, it is a potential function ✓

Conservative & Non-Conservative fields.

$F(x,y) = f(x,y)i + g(x,y)j$.

if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ (conservative)

else non-conservative

it is valid for 2-space.

$F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$.

if $\nabla \times F = 0$ (conservative)

else non-conservative.

for 3-space.

Q. Whether the field is conservative or not.

(i) $F(x,y) = 2xy^3 i + (1+3x^2y^2)j$

$f(x,y) = 2xy^3$ $\qquad$ $g(x,y) = 1+3x^2y^2$

$\frac{\partial f}{\partial y} = 6xy^2$ $\qquad$ $\frac{\partial g}{\partial x} = 6xy^2$.

$\partial f/\partial y = \partial g/\partial x$, it is a conservative vector field.