Chain Rule:

Ex1: $z = x^2 y$, $x = t^2$, $y = t^3$, Find dz/dt

solution

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dz}{dt} = 2xy \cdot 2t + x^2 \cdot 3t^2$$

$$\frac{dz}{dt} = 2(t^2)(t^3) \cdot 2t + (t^2)^2 \cdot 3t^2$$

$$\frac{dz}{dt} = 4t^6 + 3t^6$$

$$\frac{dz}{dt} = 7t^6$$

alternatively

$$z = x^2 y$$

$$z = (t^2)^2 \cdot t^3$$

$$z = t^7$$

$$\frac{dz}{dt} = 7t^6$$

Ex: 2 $w = xy$; $x = \cos t$ & $y = \sin t$. what's derivative value at $t = \pi/2$?

Sol

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dw}{dt} = -y \cdot \sin t + x \cdot \cos t$$

$$\frac{dw}{dt} = -(\sin t)\cdot(\sin t) + (\cos t)\cdot(\cos t)$$

$$\frac{dw}{dt} = -\sin^2 t + \cos^2 t$$

put $t = \pi/2$

$$\left.\frac{dw}{dt}\right|_{\pi/2} = -\left(\sin\left(\frac{\pi}{2}\right)\right)^2 + \left(\cos\left(\frac{\pi}{2}\right)\right)^2$$

$$-(1)^2 + (0)^2$$

$$\left.\frac{dw}{dt}\right|_{\pi/2} = -1$$

Ans

Functions of 3 Vars.

Ex! Suppose that $w = \sqrt{x^2 + y^2 + z^2}$, $x = \cos\theta$, $y = \sin\theta$, $z = \tan\theta$. Use the chain rule to find $dw/d\theta$, when $\theta = \pi/4 = 45°$

Sol

$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x} \cdot \frac{dx}{d\theta} + \frac{\partial w}{\partial y} \cdot \frac{dy}{d\theta} + \frac{\partial w}{\partial z} \cdot \frac{dz}{d\theta}$$

$$\frac{dw}{d\theta} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} \cdot (-\sin\theta) + \frac{2y}{2\sqrt{x^2+y^2+z^2}} (\cos\theta) + \frac{2z}{2\sqrt{x^2+y^2+z^2}} (\sec^2\theta)$$

$$\frac{dw}{d\theta} = -\frac{2x}{2\sqrt{x^2+y^2+z^2}} (\underbrace{\sin \pi/4}_{\sin 45}) + \frac{2y}{2\sqrt{x^2+y^2+z^2}} (\underbrace{\cos \pi/4}_{\cos 45}) + \frac{2z}{2\sqrt{x^2+y^2+z^2}} (\underbrace{\sec^2 \pi/4}_{\sec^2 45})$$

$\cancel{dw/d\theta =}$ $\cos 45 = \frac{\sqrt{2}}{2}$, $\sin 45 = \frac{\sqrt{2}}{2}$, $\tan 45 = 1$

$$\frac{dw}{d\theta} = -\frac{\cancel{2}\left(\frac{\sqrt{2}}{\cancel{2}}\right)}{2\sqrt{x^2+y^2+z^2}} \cdot \left(\frac{\sqrt{2}}{2}\right) + \frac{\cancel{2}\left(\frac{\sqrt{2}}{\cancel{2}}\right)}{2\sqrt{x^2+y^2+z^2}} \cdot \left(\frac{\sqrt{2}}{2}\right) + \frac{2(1)}{2\sqrt{x^2+y^2+z^2}} \cdot (2)$$

$$\frac{dw}{d\theta} = -\frac{\sqrt{2}}{2\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2+(1)^2}} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2+(1)^2}} \left(\frac{\sqrt{2}}{2}\right) + \frac{2 \cdot 2}{2\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2+(1)^2}}$$

$$dw/d\theta = -\,{}^{1/\cancel{2}}\!\big/(\sqrt{2})\cancel{4}2 + {}^{\cancel{1}}\!\big/(\sqrt{2})42 + {}^{2}\cancel{4}\big/\cancel{2}\sqrt{2}$$

$$\frac{dw}{d\theta} = -\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{2}{\sqrt{2}}$$

$$\frac{dw}{d\theta} = \sqrt{2}$$

Ans

PAPERWORK

CamScanner

Ex. 2 Find dw/dt, if $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$. what is the derivatives value at $t = 0$?

Sol.
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$\frac{dw}{dt} = y\cdot(-\sin t) + x\cdot(\cos t) + 1\cdot 1$$

$$\frac{dw}{dt} = -\sin t\cdot\sin t + \cos t\cdot\cos t + 1$$

$$\frac{dw}{dt} = -\sin^2 t + \cos^2 t + 1$$

$$\left.\frac{dw}{dt}\right|_{t=0} = -(\sin(0))^2 + (\cos(0))^2 + 1$$

$$" = -0 + 1 + 1$$

$$\left.\frac{dw}{dt}\right|_{t=0} = 2$$

Ans

Ex: 1 Given that $z = e^{xy}$, $x = 2u+v$, $y = \frac{u}{v}$ find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ using the chain rule.

Sol
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = (y e^{xy})\cdot(2) + (x e^{xy})\cdot(1/v)$$

$$\frac{\partial z}{\partial u} = \left(\frac{u}{v} e^{(2u+v)\left(\frac{u}{v}\right)}\right)(2) + \left((2u+v) e^{(2u+v)\left(\frac{u}{v}\right)}\right)\frac{1}{v}$$

$$\frac{\partial z}{\partial u} = e^{\frac{2u^2+uv}{v}}\left(\frac{2u}{v} + \frac{2u+v}{v}\right)$$

$$\frac{\partial z}{\partial u} = e^{2u^2+uv/v}\left(\frac{4u+v}{v}\right)$$

Ans.

$y = \frac{u}{v} \Rightarrow uv^{-1}$

$\frac{\partial y}{\partial v} = -uv^{-2} = \frac{-u}{v^2}$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = (y e^{xy})\cdot(1) + (x e^{xy})\cdot\left(\frac{-u}{v^2}\right)$$

$$\frac{\partial z}{\partial v} = \left(\frac{u}{v} e^{(2u+v)\left(\frac{u}{v}\right)}\right) + \left((2u+v) e^{(2u+v)\left(\frac{u}{v}\right)}\right)\cdot\frac{-u}{v^2}$$

$$\frac{\partial z}{\partial v} = e^{\frac{2u^2+uv}{v}}\left(\frac{u}{v}^{\times v} - \frac{2u^2-uv}{v^2}\right)$$

$$\frac{\partial z}{\partial v} = e^{\frac{2u^2+uv}{v}}\left(\frac{\cancel{uv} - 2u^2 - \cancel{uv}}{v^2}\right) \Rightarrow \boxed{e^{\frac{2u^2+uv}{v}}\left(\frac{-2u^2}{v^2}\right)}$$

Ans

Ex-3: Express $\partial w/\partial r$ & $\partial w/\partial s$ in terms of r & s if $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$.

Sol: $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$

$\frac{\partial w}{\partial r} = (1) \cdot \left(\frac{1}{s}\right) + (2) \cdot (2r) + (2z) \cdot (2)$

$\frac{\partial w}{\partial r} = \frac{1}{s} + 4r + 4(2r)$

$\frac{\partial w}{\partial r} = \frac{1}{s} + 4r + 8r$

$\frac{\partial w}{\partial r} = \frac{1 + 4sr + 8sr}{s}$

$\frac{\partial w}{\partial r} = \frac{1 + 12sr}{s}$ or $\frac{1}{s} + 12r$ Ans.

$x = \frac{r}{s} = r s^{-1}$

$\frac{\partial x}{\partial s} = -r s^{-2}$

$\partial x = -\frac{r}{s^2}$

$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$

$\frac{\partial w}{\partial s} = (1) \cdot \left(\frac{-r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z) \cdot (0)$

$\frac{\partial w}{\partial s} = \frac{-r}{s^2} + \frac{2}{s}$

$\frac{\partial w}{\partial s} = \frac{-r + 2s}{s^2}$ or $\frac{-r}{s^2} + \frac{2}{s}$ Ans.

$w(x, y, z)$

$\partial w / \partial x$

Ex 3. Suppose that $w = xy + yz$, $y = \sin x$, $z = e^x$. Find $\frac{dw}{dx}$

Sol: $\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dx}$

$\frac{dw}{dx} = y + (x + z) \cdot (\cos x) + (y) \cdot (e^x)$

$\frac{dw}{dx} = \sin x + (x + e^x)(\cos x) + (\sin x)(e^x)$

$\frac{dw}{dx} = \sin x + x\cos x + e^x \cos x + e^x \sin x$. Ans.

Ex: Suppose that $w = x^2 + y^2 - z^2$ & $x = \rho \sin\phi \cos\theta$,
$y = \rho \sin\phi \sin\theta$, $z = \rho\cos\phi$.
Find $\frac{\partial w}{\partial \rho}$; $\frac{\partial w}{\partial \theta}$.

Sol

$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \rho}$$

$$\frac{\partial w}{\partial \rho} = (2x)(\sin\phi\cos\theta) + (2y)(\sin\phi\sin\theta) + (-2z)(\cos\phi)$$

$$\frac{\partial w}{\partial \rho} = 2\rho\sin^2\phi\cos^2\theta + 2\rho\sin^2\phi\sin^2\theta - 2\rho\cos^2\phi.$$

$$\frac{\partial w}{\partial \rho} = 2\rho\sin^2\phi(\cos^2\theta + \sin^2\theta) - 2\rho\cos^2\phi$$

$$\partial w/\partial\rho = 2\rho\sin^2\phi(1) - 2\rho\cos^2\phi$$

$$\partial w/\partial\rho = 2\rho(\sin^2\phi - \cos^2\phi)$$

$$\partial w/\partial\rho = 2\rho(\cos 2\phi)$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \theta}$$

$$\frac{\partial w}{\partial \theta} = (2x)(-\rho\sin\phi\sin\theta) + (2y)(\rho\sin\phi\cos\theta) + (-2z)(0).$$

$$\frac{dw}{\partial \theta} = -2x\rho\sin\phi\sin\theta + 2y\rho\sin\phi\cos\theta + 0.$$

$$\frac{dw}{d\theta} = -2\,\cancel{\rho^2\sin^2\phi\cos\theta\sin\theta} + 2\,\cancel{\rho\sin^2\phi\cos\theta\sin\theta}$$

$$\frac{dw}{d\theta} = 0.$$