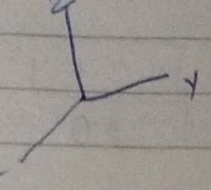
UNIT: 2

02/11/22



LIMITS

Parametric form : $(x,y) \to \{x(t), y(t)\}$.
variables represent in different formation.

Limits along curves

① $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{t\to 0} f(x(t), y(t))$.

$\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z) = \lim\limits_{t\to 0} f\big(x(t), y(t), z(t)\big)$

Ex: 01

$$f(x,y) = -\frac{xy}{x^2+y^2}$$

(a) along x-axis $(y=0)$.

$(x,y) = (t,0)$.

$$f(t,0) = \lim_{t\to 0} (t,0) - \frac{(t)(0)}{t^2+0^2}$$

$$\lim_{t\to 0} \left(-\frac{0}{t^2}\right)$$

$$\boxed{\lim_{t\to 0} (0) = 0}$$

(b) along y-axis. $(x=0)$.

$(x,y) = (0,t)$

$$f(0,t) = \lim_{t\to 0} (0,t) - \frac{(0)(t)}{0^2+t^2}.$$

$$\lim_{t\to 0} -\frac{0}{t^2}$$

$$\boxed{\lim_{t\to 0} 0 = \boxed{0}}$$

(c) the line $y = x$.

$(x,y) = (t,t)$.

$$f(t,t) = \lim_{t\to 0} (t,t) = \frac{(t)(t)}{t^2+t^2}$$

$$= \lim_{t\to 0} -\frac{\cancel{t^2}}{2\cancel{t^2}}$$

$$= \lim_{t\to 0} = \boxed{-\frac{1}{2}}$$

(d) line $y = -x$ $\qquad -y = x$

$(x, y) = (+t, -t)$.

$$f(t, -t) = \lim_{t \to 0} -\frac{(t)(-t)}{(t)^2 + (-t)^2}$$

$$= \lim_{t \to 0} \frac{t^2}{2t^2}$$

$$= \lim_{t \to 0} = \boxed{1/2}$$

(e) $y = x^2$

$(x, y) = (t, t^2)$.

$$f(t, t^2). \lim_{t \to 0} -\frac{(t)(t^2)}{(t)^2 + (t^2)^2}$$

$$= \lim_{t \to 0} -\frac{t^3}{t^2 + t^4} \Rightarrow -\frac{t^{\cancel{3}}}{\cancel{t^2}(1+t^2)} \Rightarrow -\frac{t}{1+t^2}$$

$$= \lim_{t \to 0} \boxed{-\frac{t}{1+t^2}}$$

$$= \frac{-(0)}{1+0^2}$$

$$f(t, t^2) = \boxed{0}$$

Ex: 2

$$\lim_{(x,y) \to (1,4)} [5x^3y^2 - 9] = \lim_{(x,y) \to (1,4)} [5x^3y^2] - \lim_{(x,y) \to (1,4)} 9.$$

$$= 5(1)^3(4)^2 - 9$$

$$= 5(1)(16) - 9$$

$$= 80 - 9$$

$$\lim_{(x,y) \to (1,4)} [5x^3y^2 - 9] = 71$$

Definition of limit b/w ϵ - δ (delta).
epsilon

- to each $\epsilon > 0$ there exist a +ve number δ such that when $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$

Definition of limit b/w ϵ epsilon − δ delta.

To each $\epsilon > 0$ there exist a positive number $\exists\ \delta$ such that when $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$

$$|x - a| < \delta$$

$$\pm (x-a) < \delta$$

$+(x-a) < \delta$

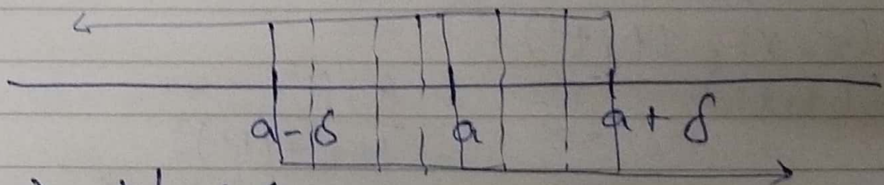
$x - a < \delta$

$x < a + \delta$

$-(x-a) < \delta$

$-x + a < \delta$

$x > a - \delta$



$$|f(x) - L| < \epsilon$$

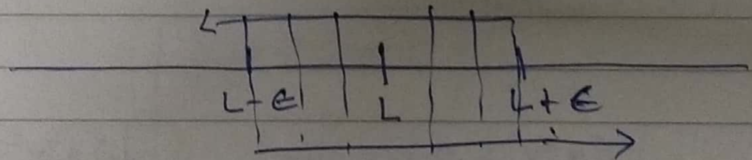
$$\pm (f(x) - L) < \epsilon$$

$+f(x) - L < \epsilon$

$\boxed{f(x) < L + \epsilon}$

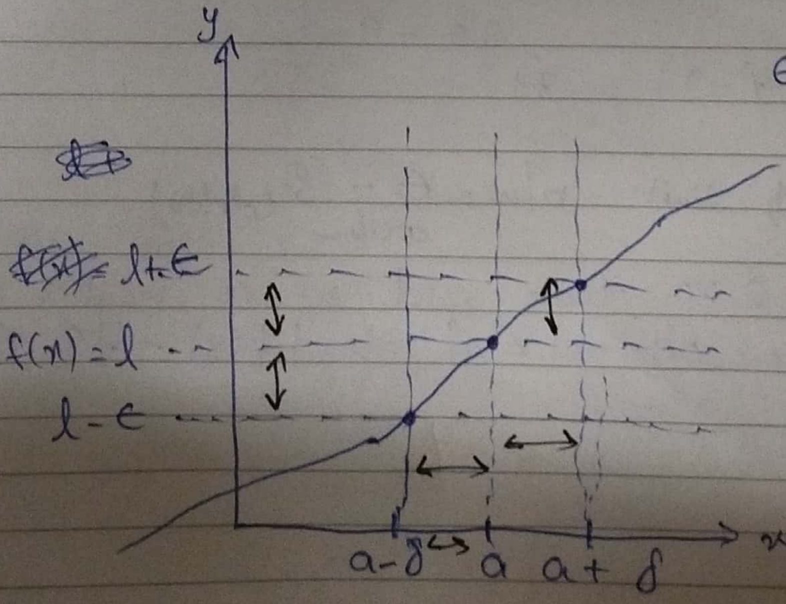
$-(f(x) - L) < \epsilon$

$\boxed{f(x) > L - \epsilon}$



$$\boxed{L - \epsilon < f(x) < L + \epsilon}$$

$$\boxed{a - \delta < x < a + \delta}$$



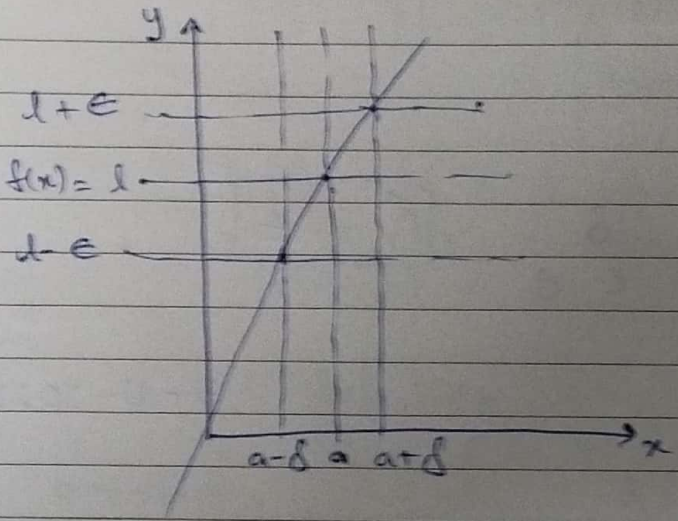
ϵ and δ are very smallest values (negligible).

change in values of ϵ & δ can be, due to different functions

if $x = a$
and $x = a - \delta$
so $y = a - \delta$
same for $x = a + \delta$
$y = a + \delta$

if a is L
$\delta = \epsilon$
$a - \delta$

if $y = 2x$ $\longrightarrow$ $x = a$

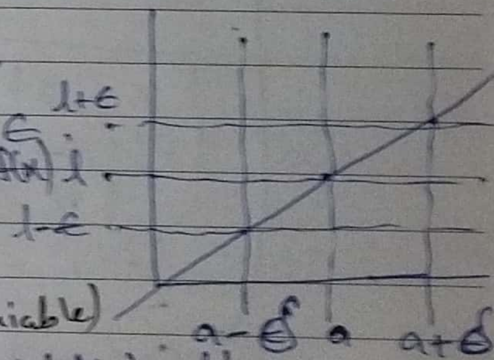


$y = 2a$
as $x = a - \delta$
$y = 2(a - \delta)$
$y = 2a - 2\delta$
and $x = a + \delta$
$y = 2(a + \delta)$
$y = 2a + 2\delta$
$\epsilon = 2\delta$
$2a + 2\delta = L + \epsilon$
$2a - 2\delta = L - \epsilon$

$y = \frac{x}{2}$
$x = a$
$y = \frac{a}{2}$
as $x = a - \delta$
$y = \frac{a - \delta}{2}$
$= L - \epsilon$

$x = a + \delta$
$y = \frac{a + \delta}{2}$
$= L + \epsilon$

$\epsilon = \frac{\delta}{2}$



$\epsilon - \delta$ definition of limit (on 1 variable)
$\lim_{x \to a} f(x) = l$, if $|x - a| < \delta \Rightarrow |f(x) - l| < \epsilon$.

Ex $\lim_{x \to 2} 2x - 3 = 1$

$|x - 2| < \delta$
$|f(x) - l| < \epsilon$
$|2x - 3 - 1|$
$|2x - 4|$ $\cancel{< \delta}$
$|2(x - 2)| \to ①$

$\to ①\ 2|x - 2| < 2\delta$
$\delta = \frac{\epsilon}{2}$
$|f(x) - l| < 2\delta$
$|f(x) - l| < \cancel{2}\left(\frac{\epsilon}{\cancel{2}}\right)$
$\boxed{|f(x) - l| < \epsilon}$
Proved!

PAPERWORK

CamScanner

$\lim_{x \to a} f(x) = l$, if $|x-a| < \delta \Rightarrow |f(x) - L| < \epsilon$

$$\lim_{x\to a}\left(\frac{x^2-a^2}{x-a}\right) = 2a.$$

$|x - \cancel{a}a| < \delta$

$|f(x) - l|$

$\left|\frac{x^2-a^2}{\cancel{x-a}} - 2a\right|$

$\left|\frac{(x+a)\cancel{(x-a)}}{\cancel{x-a}} - 2a\right|$

$|x + a - 2a|$

$|x-a| < \delta$ $\boxed{\delta = \epsilon}$

$|f(x) - l| < \epsilon$

3-11-2022

Q. $\lim_{x\to 2} x^2 = 4 \Rightarrow x = 2$

Given $|x-a| < \delta$

$|x^2 - \cancel{2}^2| < \delta$

$|f(x) - l|$

$|x^2-4|$

$|(x+2)(x-2)|$

$(x+2)\,|x-2| < (x+2)\delta$

$(x+2)(x-2)$

$\cancel{a^2 - 2ab + b^2}$

$\cancel{x^2 - 2x\,+}$

$|x-2| < \delta$

$|f(x) - l|$ x

$|x-2| < \delta$ $\delta = \epsilon$

$|f(x) - l| < \epsilon$

Given $|x-a| < \delta$

$|f(x) - l|$

$|x^2-4|$

$|(x)^2 - (2)^2|$

$|(x+a)(x-a)|$

$|x+a|\,|x-a|$

let $\delta = 1$

$\Rightarrow |x-2| < 1$

$\pm(x-2) < 1$

$x - 2 < 1$ $-x+2 < 1$

$x < 1+2$ $x < 2-1$

$x < 3$ $x > 1$

$1 < x < 3$

$\{2, 4, 6, 8, 10, 12, 14\}$

$\boxed{8/2} = 4 - 1 = \boxed{3}$

$x < 3$

$x + 2 < 3 + 2$

$\boxed{x + 2 < 5}$

$|f(x) - l|$

$|(x+2)(x-2)| < \delta$

5δ

$|f(x) - l| < 5\delta$

$\boxed{\delta = \frac{\epsilon}{5}}$

$|f(x) - l| < \delta \frac{(\epsilon)}{\delta}$

$|f(x) - l| < \epsilon$

$\epsilon - \delta$ definition (two-variable).

$\lim_{(x,y) \to (a,b)} f(x,y) = L$ if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$

$\Rightarrow |f(x,y) - L| < \epsilon$

Q $\lim_{(x,y) \to (1,2)} 7x - 2y = 3$

let $\sqrt{(x-1)^2 + (y-2)^2} < \delta$ — ① (According to given condition)

$|f(x,y) - L|$

① — $|7x - 2y - 3|$

$\Rightarrow |7(x-1+1) - 2(y-2+2) - 3|$

$\Rightarrow |7(x-1) + 7 - 2(y-2) - 4 - 3|$

$\Rightarrow |7(x-1) - 2(y-2)|$

same

complex numbers property

$\because |z_1 - z_2| \leq |z_1| + |z_2|$

alternative way

— ① $|7x - 2y - 3|$

$\Rightarrow |7(x-1+1) - 2y - 3|$

$\Rightarrow |7(x-1) + 7 - 2y - 3|$

$\Rightarrow |7(x-1) - 2y + 4|$

$\Rightarrow |\underbrace{7(x-1)}_{z_1} - \underbrace{2(y-2)}_{z_2}|$

$\leq |7(x-1)| + |2(y-2)|$

$\leq 7|(x-1)| + 2|y-2|$ — Ⓐ

let $|x-1| = \sqrt{(x-1)^2} <$

$\sqrt{(x-1)^2 + (y-2)^2}$

$|x-1| \leq \sqrt{(x-1)^2 + (y-2)^2} < \delta$

$\boxed{|x-1| < \delta}$

PAPERWORK

10/11/22

let $|y-2| = \sqrt{(y-2)^2} \leq \sqrt{(x-1)^2+(y-2)^2}$

$|y-2| \leq \sqrt{(x-1)^2+(y-2)^2}$

$\boxed{|y-2| < \delta}$

now, eq. (A) will be $7\delta + 2\delta$

9δ $\quad 9\delta = \epsilon$

$|f(x,y) - L| < 9\delta$ $\quad \delta = \frac{\epsilon}{9}$

$|f(x,y) - L| < \cancel{9}(\epsilon/\cancel{9})$

$\boxed{|f(x,y) - L| < \epsilon}$

Polynomial

$ax^n \; (\because n(+Z))$

Props. of limits in 2-Vars.

Ex: 2

$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ $\quad \because$ conjugate method.

$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \times \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$ $\quad \rightarrow (a-b)(a+b) = a^2 - b^2$

$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{(x^2 - xy)(\sqrt{x} + (\sqrt{y}))}{(\sqrt{x})^2 - (\sqrt{y})^2}$

$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{(x\cancel{(x-y)})(\sqrt{x} + \sqrt{y})}{\cancel{x-y}}$

$\Rightarrow \lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y})$

applying limit

$\Rightarrow 0(\sqrt{0} + \sqrt{0})$

$\Rightarrow 0.$

$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = 0$

Ans

16/11/22

Limits in Polar Coordinates

$z = x + iy$ (Cartesian formation).

in polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

(r, θ).

Ex: 7

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

$x, x(t)$

$y, y(t)$.

$x = r\cos\theta$, $y = r\sin\theta$.

$r^2 = x^2+y^2$

$$\lim_{r\to 0} r^2 \ln r^2$$

$(x,y) \to (x(t), y(t)) \to t \to 0$

$\because \ln a^n = n \ln a$.

$$\lim_{r\to 0} r^2\, 2\ln r.$$

$\because u \cdot v = v\,du + u\,dv$

$$\lim_{r\to 0} \frac{2\ln r}{1/r^2}$$ $\Rightarrow$ now, Del. hopital rule

differentiate numerator & denominator & separately.

$$\lim_{r\to 0} \frac{\cancel{2}/r}{-\cancel{2}/r^3}$$

$2\ln r$ (taking derivative)

$\boxed{2 \cdot 1/r}$

$$\lim_{r\to 0} \frac{1/r}{-1/r^3} \Rightarrow \frac{1}{\cancel{r}} \times (-r^{\cancel{3}})$$

$\frac{1}{r^2} = r^{-2}$ (taking derivative).

$= -2r^{-2-1}$

$= -2r^{-3}$

$$\lim_{r\to 0} -r^2$$ applying limit.

$-(0)^2$

$\boxed{0}$

$\frac{d}{dr}\frac{1}{r^2} = \frac{-2}{r^3}$

$a^2 + 2ab + b^2$

composition

$g(x) = x + 2$ $\quad f(x) = 2x^2$

$f \circ g(x) = f(g(x)) = 2(x+2)^2$

$= 2(x^2 + 4x + 4)$

$= 2x^2 + 8x + 8$