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Financial Engineering

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ASSIGNMENT - 3

Qu 1 Given sample space

$$\Omega = \{ (u,u), (u,d), (d,u), (d,d) \}$$

$$F_1 = \{ \Omega, \phi, \{(u,u)\}, \{(u,d), (d,u), (d,d)\} \}$$

Further,

$$F_2 = \{ \phi, \Omega, (u,u), (u,d), (d,u), (d,d), \{(u,d), (u,u)\}, \{(u,d), (d,d)\}, \{(d,u), \{(u,u), (d,u), (d,u)\}, \{(u,u), (u,d), (d,d)\}, \{(u,u), (d,d), (d,u)\}, \{(u,d), (d,u), (d,d)\} \}$$

is the largest σ field.

both F_1 and F_2 statisfy all 3 property of σ field.

Qu: 2 We know that,

X & Y are IId random variable each having uniform distribution on the interval $(\bar{I}, \bar{u})$

$E(X) = 0$ $Var(X) = 1$

$E(Y) = 0$ $Var(Y) = 1$

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$E(x^2) = 1$ $\quad E(xy) = E(x)\,E(y) = 0$

$E(y^2) = 1$

now, $\quad z(t) = \cos(xt + y)$

$E(z(t)) = E(\cos(xt + y))$

and

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \cdots$$

$$E(z(t)) = E\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \cdots\right]$$

$$= 1 - \frac{1}{2}E\{x^2t^2 + y^2 + 2xyt\} + \cdots$$

$$= 1 - \frac{1}{2}(t^2+1) + \frac{1}{4!}\left(E(xt+y)^4\right)$$

$$\Rightarrow \frac{1}{2} - \frac{t^2}{2} + \frac{1}{4!}\{E(xt+y)^4\}$$

now, the condition that a wide sense stationary process must hold are :-

1) $E(z(t))$ must be independent of t

2) $\mathrm{Cov}(z(t), z(s))$ depends only on the time difference $(t-s)$ for all $t > s$

3) $E(x^2(t)) < \infty$ (finite 2nd order moment)

But we can see clearly from (1) that the first condition isn't satisfied as

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$E(Z(t)) \propto t^2$ (depends on t^2)

Hence, this is not a wide sense stationary process.

Que 3 Let $\{W(t), t\geq 0\}$ be a Brownian motion, prove

Solution Brownine motion is defined as a stochastic process $\{X(t), t\geq 0\}$ if :-

1) $X(0)=0$

2) $\{X(t), t\geq 0\}$ has stationary & independent increments

3) for every $t\geq 0$, $X(t)$ is normally distributed with ~~mean~~ 0 & variance $t^2 t$.

now, we are given a brownine motion ~~with~~ $W(t)$ and we need to prove the first condition to brownine process.

now, $0 < s \leq t \Rightarrow \frac{1}{t} \leq \frac{1}{s}$

Thus

$$Cov(\hat{W}_s, \hat{W}_t) = Cov\left(s\, W_{1/s}, t\, W_{1/t}\right)$$

$$= st\, Cov(W_{1/s}, W_{1/t}) = s \cdot t \times \frac{1}{t} = s$$

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now, to show the continuity of $\hat{\omega}_t$, it is sufficient to show the continuity of $\hat{\omega}_t$ at $t=0$

But it given that $(t\, \omega(1/t))_{t=0} = 0$

also, $\lim_{t \to 0} \hat{\omega}_t = \lim_{t \to 0} \omega_{1/t} = \lim_{t \to \infty} \frac{\omega_t}{t} = 0$

Hence, from continuity we can state that the now process $\hat{\omega}_t$ is also a brownine motion (wiener process)

Qn σ feild definition :-

$\mathcal{F}$ set of subsets of Ω, F is a σ-feild if :-

1) $\Omega \subset F$

2) F is closed under complement if $A \in F$ & $A^c \in F$

3) F is closed under union, if $A_1, A_2, A_3 \ldots \in F$ then $\bigcup_i A_i \in F$

Given $\{a, b, c, d\} = \Omega$

condition $F_1 \subset F_2 \subset F_3 \subset F_4$

let $F_1 = \{\phi, \Omega\}$

$F_2 = \{\phi, \{a\}, \{b, c, d\}, \Omega\}$

let $F_3 = \{\phi, \Omega, \{a\}, \{b, c, d\}, \{b\},$

$\{a, c, d\}, \{a, b, d\}, \{a, b\}, \{a, c\},$

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$\{b,c\}, \{c,b\}\}$.

Here, $F_1 \subset F_2 \subset F_3 \subset F_4$ all are σ - feilds satisfy the required condition.

Ans: $Y_n = e^{\left(\sum_{i=n}^{n} X_i - \frac{n}{2}\right)}$

$W(n) = \sum_{1}^{n} X_i$

we get $Y_n = \exp(W_n - n/2)$

let $0 \leq s \leq n$, $X_1, X_2, \ldots X_n$ are r.v. i.i.d

$W(n) - W(s)$ is independent of F_3 where F_3 is filter where $s > 0$.

We have, $E\left(e^{W(n)}/F_s\right) = e^{W(n)} E\left(e^{W(n) - W(s)}\right)$

Given, $W(t) - W(s)$ has normal distribution with mean $= 0$ & Variance $= 1$ (let $n-s = 1$)

we get $E\left(e^{W(t) - W(s)}\right)$

$$E\left(e^{W(n)}/F_3\right) = e^{W(s)} \cdot e^{1/2}$$

$$E\left(e^{W(n) - n/2}/F_s\right) = e^{-n/2} E\left(e^{W(n)}/F_3\right)$$

$$= e^{W(s)} = s/2$$

Hence Y_n is a martingale.

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Qn Given that $Z(t)$ is normally distributed r.v. with

$E(Z) = 0$, $Var(Z) = 1$

$Var(Z) = E(Z^2) - (E(Z))^2$

$\Rightarrow E(Z^2) - 0$

or $E(Z^2) = 1$

now,

$X = \sqrt{t}\, Z$

$E(X) = E[\sqrt{t}\, Z] = \sqrt{t}\, E(Z) = 0$

now, $Var(X) = (\sqrt{t})^2 \Rightarrow Var(X) = t$

Hence, $X(t)$ is a Brownian motion was Brownian motion has variance of the $Var(B(t)) = \sigma^2 t$ at mean $E(B(t)) = \mu\sqrt{t}$

where $B(t)$ is any brownian motion.

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Qn: X and Y are IID and R.V. each having uniform distribution b/w the intervals α & β

& $z = x + y$

So $E\left(\frac{X}{2}\right) = E\left(\frac{Y}{2}\right)$ —— (1)

$E\left(\frac{X}{2}\right) = E\left(\frac{z-Y}{2}\right) = E(z/2) - E(Y/2)$

and

$E\left(\frac{X}{2}\right) + E\left(\frac{Y}{2}\right) = z$ $\left[E(z/2) = z\right]$

$2E\left(\frac{X}{2}\right) = z$

Or we can say that :-

$$\boxed{E\left(\frac{X}{2}\right) = \frac{z}{2}}$$

Hence proved

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Q4 X & Y are i.i.d r.v. each having uniform distribution b/w the intervals 0 & 1.

& $Z = X + Y$

So, $E\left(\frac{X}{Z}\right) = E\left(\frac{Y}{Z}\right)$ ——— (i)

$$E\left(\frac{X}{Z}\right) = E\left(\frac{Z-Y}{Z}\right) = E(Z/Z) - E(Y/Z)$$

$$E\left(\frac{X}{Z}\right) + E\left(\frac{Y}{Z}\right) = Z \qquad (E(Z/Z) = Z)$$

$$2E\left(\frac{X}{Z}\right) = Z$$

$$\boxed{E(X/Z) = \frac{Z}{2}}$$

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Q13 Given that stock Price, $S_0 = \$50$ Time, T (in years) $= 2$

Expected return $\mu = 0.18$ volatility $\sigma = 30\%$

We use probability distribution of the stock price in 2 years using log normal distribution

$$\ln S_T = \phi\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)\times T, \sigma^2 \times T\right)$$

$$= \phi\left(\ln 50 + \left(0.18 - \frac{0.09}{2}\right)\times 2, 0.3^2 \times 2\right)$$

$$\ln S_T = \phi(4.18, 0.18)$$

The mean of stock price, $E(S_T)$ is given by:-

$$E(S_T) = S_0 e^{\gamma T}$$

$$= 50 e^{0.18\times 2} = \boxed{\$71.67 = E(S_T)}$$

standard deviation of stock price, σ_{S_T} is given by

$$\sigma_{S_T} = S_0 e^{\gamma T}\sqrt{e^{\sigma^2 \times T} - 1} = 50 e^{0.18\times 2}\sqrt{e^{0.09\times 2} - 1}$$

$$\boxed{\sigma_{S_T} = \$31.83}$$

95% confidence interval for $\ln S_T$ are :-

By normal table for critical value at $\alpha/2 = \frac{0.05}{2} = 0.025$ is

1.96

$\Rightarrow$ $4.18 \pm 1.96 \times 0.42$

3.35, 5.01

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Corresponding 95% confience interval for S_T are

$e^{3.35}$ & $e^{5.01}$

$= 28.52$ & 150.44

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Q7 (ii) $\lambda = 1$, $\{N(t), t \geq 0\}$ be a poisson process

(i) $\{N(t) - t, t \geq 0\}$

$$E[N(t) - t / F_s] = E[N(t) - N(s) + N(s) - (t + s) - s / F_s]$$

$$= E(N(t) - N(s) / F_s) + E(N(s) / F_s) - E(t - s / F_s) - E(s / F_s)$$

$$= \cancel{\lambda(t-s)} + N(s) - \cancel{(t-s)} - s \qquad (\lambda = 1)$$

$$= N(s) - s$$

$\therefore$ $E(N(t) - t / F_s)$ is a function of s.

Hence, it is martingale

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Q8 Let $\{W(t), t \geq 0\}$ be a Wiener process. ~~Itself~~ ~~$\{W(t)\}$~~

Is $\exp\{\sigma W(t) - \frac{\sigma^2}{2} t\}$ a martingale where σ is +ve constant.

Let $0 < s < t$. Since, $W(t) - W(s)$ is independent of F_s

and $W(s)$ is F_s - measurable.

$$E(e^{\sigma W(t)} / F_s) = E\left[e^{\sigma(W(t) - W(s))} \cdot e^{\sigma W(s)} / F_s\right]$$

$$= e^{\sigma W(s)} E\left[e^{\sigma(W(t) - W(s))} / F_s\right]$$

$$= e^{\sigma W(s)} E\left[e^{\sigma(W(t) - W(s))}\right]$$

$$= e^{\sigma W(s)} \cdot e^{\sigma^2 \frac{(t-s)}{2}}$$

Now
~~Also~~

$$E\left[e^{\sigma W(t) - \frac{\sigma^2}{2} t} / F_s\right] = e^{-\frac{\sigma^2}{2} t} E\left[e^{\sigma W(t)} / F_s\right]$$

$$= e^{-\frac{\sigma^2}{2} t} e^{\frac{\sigma^2}{2}(t-s)} e^{\sigma W(s)}$$

$$= e^{\frac{\sigma^2}{2}(0-s)} \exp(2\sigma W(s) - \sigma^2 s)$$

$\forall \; 0 \leq s < t$
$t > s > 0$

Hence, $\exp\{\sigma W(t) - \frac{\sigma^2}{2} t\}$ is martingale.

$-t \; +t \; -s$

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Q9 SDE of $W^2(t)$

Using Ito - Doeblin formulas we get

$$d W^2(t) = dt + 2 W(t) \, d W(t)$$

Using equivalent Integral equation with condition $W(0) = u$

$$W^2(t) = t + 2 \int_0^t W(t) \, d W(t)$$

$$\int_0^t W(t) \, d W(t) = \frac{1}{2} W^2(t) - \frac{1}{2} t$$

$\therefore$ $W(t)$ is Ito process