# Extra Insight into the iOS App

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**Abstract**

This document consists of examples. It also includes derivations for the monthly compound interest rate and ten-year minimum payment.

# 1 Examples

## 1.1 Increment size

If $p_{\max} = \$5{,}000$, $p_{\min} = \$3{,}000$ and $N = 20$,

$$
\begin{aligned}
\Delta N &= \frac{5{,}000 - 3{,}000}{20} \\
&= \frac{2{,}000}{20} \\
&= \$100/\texttt{increment}
\end{aligned}
$$

If $p_{\max} = \$6{,}500$, $p_{\min} = \$4{,}500$ and $N = 40$,

$$
\begin{aligned}
\Delta N &= \frac{6{,}500 - 4{,}500}{40} \\
&= \frac{2{,}000}{40} \\
&= \$50/\texttt{increment}
\end{aligned}
$$

If $p_{\max} = \$10{,}000$, $p_{\min} = \$0$ and $N = 50$,

$$
\begin{aligned}
\Delta N &= \frac{10{,}000 - 0}{50} \\
&= \frac{10{,}000}{50} \\
&= \$200/\texttt{increment}
\end{aligned}
$$

## 1.2 Annual interest rate

If APR $= 7\%$,

$$r = 7 \div 100$$
$$= 0.07/\texttt{year}$$

If APR $= 6.8\%$,

$$r = 6.8 \div 100$$
$$= 0.068/\texttt{year}$$

If APR $= 1.75\%$,

$$r = 1.75 \div 100$$
$$= 0.0175/\texttt{year}$$

## 1.3 Monthly interest rate

If APR $= 7\%$,

$$i = \frac{0.07}{12}$$
$$= 0.00583.../\texttt{month}$$

If APR $= 6.8\%$,

$$i = \frac{0.068}{12}$$
$$= 0.00566.../\texttt{month}$$

If APR $= 3.45\%$,

$$i = \frac{0.0345}{12}$$
$$= 0.002875/\texttt{month}$$

If APR $= 7\%$ compounded,

$$
\begin{aligned}
i &\approx \left(1 + \frac{0.07}{365.25}\right)^{\frac{365.25}{12}} - 1 \\
&= (1.000191...)^{\frac{365.25}{12}} - 1 \\
&= 1.00584... - 1 \\
&= 0.00584.../\texttt{month}
\end{aligned}
$$

If APR $= 6.8\%$ compounded,

$$
\begin{aligned}
i &\approx \left(1 + \frac{0.068}{365.25}\right)^{\frac{365.25}{12}} - 1 \\
&= (1.000186...)^{\frac{365.25}{12}} - 1 \\
&= 1.00568... - 1 \\
&= 0.00568.../\texttt{month}
\end{aligned}
$$

If APR $= 3.45\%$ compounded,

$$
\begin{aligned}
i &\approx \left(1 + \frac{0.0345}{365.25}\right)^{\frac{365.25}{12}} - 1 \\
&= (1.0000944...)^{\frac{365.25}{12}} - 1 \\
&= 1.00287... - 1 \\
&= 0.002879.../\texttt{month}
\end{aligned}
$$

## 1.4 Absolute minimum monthly payment

If $p = \$2,000$, APR $= 6.8\%$ and $\alpha = 0.25$,

$$
\begin{aligned}
a_{\min_n} &= \left\lfloor 0.25\left(2,000 \cdot 0.00566...\right) \times 100 \right\rfloor \div 100 + 0.01 \\
&= \left\lfloor 0.25\left(11.333...\right) \times 100 \right\rfloor \div 100 + 0.01 \\
&= \left\lfloor 283.3... \right\rfloor \div 100 + 0.01 \\
&= 283 \div 100 + 0.01 \\
&= 2.83 + 0.01 \\
&= \$2.84
\end{aligned}
$$

If $p = \$2{,}700$, APR $= 2.61\%$ and $\alpha = 0.2$,

$$
\begin{aligned}
a_{\min_n} &= \left\lfloor 0.2\,(2{,}700 \cdot 0.00217\underline{5}) \times 100 \right\rfloor \div 100 + 0.01 \\
&= \left\lfloor 0.2\,(5.872...) \times 100 \right\rfloor \div 100 + 0.01 \\
&= \left\lfloor 117.4... \right\rfloor \div 100 + 0.01 \\
&= 117 \div 100 + 0.01 \\
&= 1.17 + 0.01 \\
&= \$1.18
\end{aligned}
$$

If $p = \$2{,}700$, APR $= 2.61\%$ compounded and $\alpha = 0.2$,

$$
\begin{aligned}
a_{\min_n} &= \left\lfloor 0.2\,(2{,}700 \cdot 0.00217\underline{7}...) \times 100 \right\rfloor \div 100 + 0.01 \\
&= \left\lfloor 0.2\,(5.878...) \times 100 \right\rfloor \div 100 + 0.01 \\
&= \left\lfloor 117.5... \right\rfloor \div 100 + 0.01 \\
&= 118 \div 100 + 0.01 \\
&= 1.18 + 0.01 \\
&= \$1.19
\end{aligned}
$$

If $p = \$35{,}221$, APR $= 7\%$ and $\alpha = 0$,

$$
\begin{aligned}
a_{\min_n} &= \left\lfloor 0\,(35{,}221 \cdot 0.00583...) \times 100 \right\rfloor \div 100 + 0.01 \\
&= \left\lfloor 0\,(15.75) \times 100 \right\rfloor \div 100 + 0.01 \\
&= \left\lfloor 0 \right\rfloor \div 100 + 0.01 \\
&= 0 \div 100 + 0.01 \\
&= 0 + 0.01 \\
&= \$0.01
\end{aligned}
$$

## 1.5 Ten-year minimum monthly payment

If $p = \$35{,}221$, APR $= 7\%$ and $\alpha = 0$,

$$i = 0.00583... > 0$$
$$\alpha = 0 \ (given)$$
$$\text{First case, proceeding...}$$

$$a_{\min_{120}} = \left\lceil \frac{35{,}221}{120} \times 100 \right\rceil \div 100$$
$$= \lceil 29{,}350.8... \rceil \div 100$$
$$= 29{,}351 \div 100$$
$$= \$293.51$$

If $p = \$26{,}970$, APR $= 3.45\%$ compounded and $\alpha = 0$,

$$i \approx 0.002879... > 0$$
$$\alpha = 0 \ (given)$$
$$\text{First case, proceeding...}$$

$$a_{\min_{120}} = \left\lceil \frac{26{,}970}{120} \times 100 \right\rceil \div 100$$
$$= \lceil 22{,}475 \rceil \div 100$$
$$= 22{,}475 \div 100$$
$$= \$224.75$$

If $p = \$24{,}190$, APR $= 0\%$ and $\alpha = 0.5$,

$$i = 0$$
$$\alpha = 0.5 \ (given)$$
$$\text{Second case, proceeding...}$$

$$a_{\min_{120}} = \left\lceil \frac{24{,}190}{120} \times 100 \right\rceil \div 100$$
$$= \lceil 20{,}158.3... \rceil \div 100$$
$$= 20{,}159 \div 100$$
$$= \$201.59$$

If $p = \$13{,}500$, APR $= 6.8\%$ and $\alpha = 0.5$,

$$i = 0.00566... > 0$$
$$\alpha = 0.5 \text{ (given)}$$
$$\text{Third case, proceeding...}$$

$$
\begin{aligned}
a_{\min_{120}} &= \left\lceil \frac{0.5\,(13{,}500 \cdot 0.00566...)\,(1 + 0.5 \cdot 0.00566...)^{120}}{(1 + 0.5 \cdot 0.00566...)^{120} - 1} \times 100 \right\rceil \div 100 \\
&= \left\lceil \frac{0.5\,(76.50)\,(1.00283...)^{120}}{(1.00283...)^{120} - 1} \times 100 \right\rceil \div 100 \\
&= \left\lceil \frac{0.5\,(76.50)\,(1.404...)}{(1.404...) - 1} \times 100 \right\rceil \div 100 \\
&= \left\lceil \frac{53.713...}{(1.404...) - 1} \times 100 \right\rceil \div 100 \\
&= \left\lceil 13{,}286.4... \right\rceil \div 100 \\
&= 13{,}287 \div 100 \\
&= \$132.87
\end{aligned}
$$

If $p = \$20{,}000$, APR $= 5.56\%$ compounded and $\alpha = 0.7$,

$$i \approx 0.00464... > 0$$
$$\alpha = 0.7 \text{ (given)}$$
$$\text{Third case, proceeding...}$$

$$
\begin{aligned}
a_{\min_{120}} &= \left\lceil \frac{0.7\,(20{,}000 \cdot 0.00464...)\,(1 + 0.7 \cdot 0.00464...)^{120}}{(1 + 0.7 \cdot 0.00464...)^{120} - 1} \times 100 \right\rceil \div 100 \\
&= \left\lceil \frac{0.7\,(92.874...)\,(1.00325...)^{120}}{(1.00325...)^{120} - 1} \times 100 \right\rceil \div 100 \\
&= \left\lceil \frac{0.7\,(92.874...)\,(1.476...)}{(1.476...) - 1} \times 100 \right\rceil \div 100 \\
&= \left\lceil \frac{95.968...}{(1.476...) - 1} \times 100 \right\rceil \div 100 \\
&= \left\lceil 20{,}154.8... \right\rceil \div 100 \\
&= 20{,}155 \div 100 \\
&= \$201.55
\end{aligned}
$$

If $p = \$20,000$, APR $= 5.56\%$ and $\alpha = 0.7$,

$$i = 0.00463... > 0$$
$$\alpha = 0.7 \text{ (given)}$$

Third case, proceeding...

$$
\begin{aligned}
a_{\min_{120}} &= \left\lceil \frac{0.7\,(20{,}000 \cdot 0.00463...)\,(1 + 0.7 \cdot 0.00463...)^{120}}{(1 + 0.7 \cdot 0.00463...)^{120} - 1} \times 100 \right\rceil \div 100 \\[2mm]
&= \left\lceil \frac{0.7\,(92.666...)\,(1.00324...)^{120}}{(1.00324...)^{120} - 1} \times 100 \right\rceil \div 100 \\[2mm]
&= \left\lceil \frac{0.7\,(92.666...)\,(1.474...)}{(1.474...) - 1} \times 100 \right\rceil \div 100 \\[2mm]
&= \left\lceil \frac{95.669...}{(1.474...) - 1} \times 100 \right\rceil \div 100 \\[2mm]
&= \lceil 20{,}146.5... \rceil \div 100 \\[2mm]
&= 20{,}147 \div 100 \\[2mm]
&= \$201.47
\end{aligned}
$$

If $p = \$5,600$, APR $= 4\%$ and $\alpha = 0.25$,

$$i = 0.003333... > 0$$
$$\alpha = 0.25 \text{ (given)}$$

Third case, proceeding...

$$
\begin{aligned}
a_{\min_{120}} &= \left\lceil \frac{0.25\,(5{,}600 \cdot 0.00333...)\,(1 + 0.25 \cdot 0.00333...)^{120}}{(1 + 0.25 \cdot 0.00333...)^{120} - 1} \times 100 \right\rceil \div 100 \\[2mm]
&= \left\lceil \frac{0.25\,(18.666...)\,(1.000833...)^{120}}{(1.000833...)^{120} - 1} \times 100 \right\rceil \div 100 \\[2mm]
&= \left\lceil \frac{0.25\,(18.666...)\,(1.105...)}{(1.105...) - 1} \times 100 \right\rceil \div 100 \\[2mm]
&= \left\lceil \frac{5.157...}{(1.105...) - 1} \times 100 \right\rceil \div 100 \\[2mm]
&= \lceil 4{,}905.8... \rceil \div 100 \\[2mm]
&= 4{,}906 \div 100 \\[2mm]
&= \$49.06
\end{aligned}
$$

## 1.6   Algorithm

**Example 1.6a.** *If $p = \$2{,}000$, APR $= 7\%$, $\alpha = 0.33$ and $a = \$550$,*

$$B_0 = 2{,}000\,;$$

$$O_0 = 0\,;$$

$$m = 1$$

$< \text{First iteration} >$

Check: $\quad B_0 - \left\{ a - \left\lfloor \alpha \left( B_0 \cdot i \right) \times 100 \right\rceil \div 100 \right\} \overset{?}{>} 0$

$\qquad 2{,}000 - \left\{ 550 - \left\lfloor 0.33 \left( 2{,}000 \cdot 0.00583... \right) \times 100 \right\rceil \div 100 \right\} \overset{?}{>} 0$

$\qquad 2{,}000 - \left\{ 550 - 3.85 \right\} \overset{?}{>} 0$

$\qquad 2{,}000 - 546.15 \overset{?}{>} 0$

$\qquad 1{,}453.85 > 0$

$\qquad$ Proceeding...

$$B_1 = B_0 - \left\{ a - \left\lfloor \alpha \left( B_0 \cdot i \right) \times 100 \right\rceil \div 100 \right\}$$

$$= \$1{,}453.85\,;$$

$$O_1 = O_0 + \left\lfloor \left( B_0 \cdot i \right) \times 100 \right\rceil \div 100 - \left\lfloor \alpha \left( B_0 \cdot i \right) \times 100 \right\rceil \div 100$$

$$= 0 + \left\lfloor \left( 2{,}000 \cdot 0.00583... \right) \times 100 \right\rceil \div 100 - \left\lfloor 0.33 \left( 2{,}000 \cdot 0.00583... \right) \times 100 \right\rceil \div 100$$

$$= 0 + 11.67 - 3.85$$

$$= \$7.82\,;$$

$$m = 1 + 1 = 2 \text{ months}$$

$< \text{Second iteration} >$

Check: $\quad B_1 - \left\{ a - \left\lfloor \alpha \left( B_1 \cdot i \right) \times 100 \right\rceil \div 100 \right\} \overset{?}{>} 0$

$\qquad 1{,}453.85 - \left\{ 550 - \left\lfloor 0.33 \left( 1{,}453.85 \cdot 0.00583... \right) \times 100 \right\rceil \div 100 \right\} \overset{?}{>} 0$

$\qquad 1{,}453.85 - \left\{ 550 - 2.80 \right\} \overset{?}{>} 0$

$\qquad 1{,}453.85 - 547.20 \overset{?}{>} 0$

$\qquad 906.65 > 0$

$\qquad$ Proceeding...

$$B_2 = B_1 - \left\{ a - \left\lfloor \alpha \left( B_1 \cdot i \right) \times 100 \right\rceil \div 100 \right\}$$

$$= \$906.65\,;$$

$$O_2 = O_1 + \lfloor (B_1 \cdot i) \times 100 \rfloor \div 100 - \lfloor \alpha (B_1 \cdot i) \times 100 \rfloor \div 100$$

$$= 7.82 + \lfloor (1{,}453.85 \cdot 0.00583...) \times 100 \rfloor \div 100 - \lfloor 0.33 (1{,}453.85 \cdot 0.00583...) \times 100 \rfloor \div 100$$

$$= 7.82 + 8.48 - 2.80$$

$$= \$13.50 \; ;$$

$$m = 2 + 1 = 3 \, \text{months}$$

$$< \text{Third iteration} >$$

$$\text{Check}: \quad B_2 - \left\{ a - \lfloor \alpha (B_2 \cdot i) \times 100 \rfloor \div 100 \right\} \overset{?}{>} 0$$

$$906.65 - \left\{ 550 - \lfloor 0.33 (906.65 \cdot 0.00583...) \times 100 \rfloor \div 100 \right\} \overset{?}{>} 0$$

$$906.65 - \left\{ 550 - 1.75 \right\} \overset{?}{>} 0$$

$$906.65 - 548.25 \overset{?}{>} 0$$

$$358.40 > 0$$

Proceeding...

$$B_3 = B_2 - \left\{ a - \lfloor \alpha (B_2 \cdot i) \times 100 \rfloor \div 100 \right\}$$

$$= \$358.40 \; ;$$

$$O_3 = O_2 + \lfloor (B_2 \cdot i) \times 100 \rfloor \div 100 - \lfloor \alpha (B_2 \cdot i) \times 100 \rfloor \div 100$$

$$= 13.50 + \lfloor (906.65 \cdot 0.00583...) \times 100 \rfloor \div 100 - \lfloor 0.33 (906.65 \cdot 0.00583...) \times 100 \rfloor \div 100$$

$$= 13.50 + 5.29 - 1.75$$

$$= \$17.04 \; ;$$

$$m = 3 + 1 = 4 \, \text{months}$$

$$< \text{Fourth iteration} >$$

$$\text{Check}: \quad B_3 - \left\{ a - \lfloor \alpha (B_3 \cdot i) \times 100 \rfloor \div 100 \right\} \overset{?}{>} 0$$

$$358.40 - \left\{ 550 - \lfloor 0.33 (358.40 \cdot 0.00583...) \times 100 \rfloor \div 100 \right\} \overset{?}{>} 0$$

$$358.40 - \left\{ 550 - 0.69 \right\} \overset{?}{>} 0$$

$$358.40 - 549.31 \overset{?}{>} 0$$

$$-190.91 \not> 0$$

Halt!

$$n = 4 \, \text{months}$$

$$a_f = B_3 + \lfloor (B_3 \cdot i) \times 100 \rfloor \div 100 + O_3$$
$$= 358.40 + \lfloor (358.40 \cdot 0.00583...) \times 100 \rfloor \div 100 + 17.04$$
$$= 358.40 + 2.09 + 17.04$$
$$= \$377.53 \; ;$$

$$B_4 = B_3 - \left\{ a_f - \lfloor (B_3 \cdot i) \times 100 \rfloor \div 100 - O_3 \right\}$$
$$= 358.40 - \left\{ 377.53 - \lfloor (358.40 \cdot 0.00583...) \times 100 \rfloor \div 100 - 17.04 \right\}$$
$$= 358.40 - \left\{ 377.53 - 2.09 - 17.04 \right\}$$
$$= 358.40 - 358.40$$
$$= \$0$$

Monthly Balance :

| | | | | | | |
|---|---|---|---|---|---|---|
| 2000.00 | + | $2000.00 \cdot 0.00583...$ | − | 550.00 | = | 1453.85 |
| 1453.85 | + | $1453.85 \cdot 0.00583...$ | − | 550.00 | = | 906.65 |
| 906.65 | + | $906.65 \cdot 0.00583...$ | − | 550.00 | = | 358.40 |
| 358.40 | + | $358.40 \cdot 0.00583...$ | − | 377.53 | = | 0.00 |

Breakdown of Pay :

| | | | | | |
|---|---|---|---|---|---|
| 546.15 Prin. | + | 3.85 Int. | = | 550 |
| 547.20 Prin. | + | 2.80 Int. | = | 550 |
| 548.25 Prin. | + | 1.75 Int. | = | 550 |
| 358.40 Prin. | + | 19.13 Int. | = | 377.53 |

**Example 1.6b.** *If $p = \$5{,}000$, $APR = 3.45\%$, $\alpha = 0.25$ and $a = \$200$,*

$B_0 = 5{,}000$ ;

$O_0 = 0$ ;

$m = 1$

$< \text{First iteration} >$

Check: $B_0 - \left\{ a - \lfloor \alpha \left( B_0 \cdot i \right) \times 100 \rfloor \div 100 \right\} \stackrel{?}{>} 0$

$5{,}000 - \left\{ 200 - \lfloor 0.25 \left( 5{,}000 \cdot 0.002875 \right) \times 100 \rfloor \div 100 \right\} \stackrel{?}{>} 0$

$5{,}000 - \left\{ 200 - 3.59 \right\} \stackrel{?}{>} 0$

$5{,}000 - 196.41 \stackrel{?}{>} 0$

$4{,}803.59 > 0$

Proceeding...

$B_1 = B_0 - \left\{ a - \lfloor \alpha \left( B_0 \cdot i \right) \times 100 \rfloor \div 100 \right\}$

$\quad = \$4{,}803.59$ ;

$O_1 = O_0 + \lfloor \left( B_0 \cdot i \right) \times 100 \rfloor \div 100 - \lfloor \alpha \left( B_0 \cdot i \right) \times 100 \rfloor \div 100$

$\quad = 0 + \lfloor \left( 5{,}000 \cdot 0.002875 \right) \times 100 \rfloor \div 100 - \lfloor 0.25 \left( 5{,}000 \cdot 0.002875 \right) \times 100 \rfloor \div 100$

$\quad = 0 + 14.38 - 3.59$

$\quad = \$10.79$ ;

$m = 1 + 1 = 2 \text{ months}$

$< \text{Second iteration} >$

Check: $B_1 - \left\{ a - \lfloor \alpha \left( B_1 \cdot i \right) \times 100 \rfloor \div 100 \right\} \stackrel{?}{>} 0$

$4{,}803.59 - \left\{ 200 - \lfloor 0.25 \left( 4{,}803.59 \cdot 0.002875 \right) \times 100 \rfloor \div 100 \right\} \stackrel{?}{>} 0$

$4{,}803.59 - \left\{ 200 - 3.45 \right\} \stackrel{?}{>} 0$

$4{,}803.59 - 196.55 \stackrel{?}{>} 0$

$4{,}607.04 > 0$

Proceeding...

$B_2 = B_1 - \left\{ a - \lfloor \alpha \left( B_1 \cdot i \right) \times 100 \rfloor \div 100 \right\}$

$\quad = \$4{,}607.04$ ;

$$O_2 = O_1 + \left\lfloor (B_1 \cdot i) \times 100 \right\rfloor \div 100 - \left\lfloor \alpha \left( B_1 \cdot i \right) \times 100 \right\rfloor \div 100$$

$$= 10.79 + \left\lfloor (4{,}803.59 \cdot 0.002875) \times 100 \right\rfloor \div 100 - \left\lfloor 0.25 \left( 4{,}803.59 \cdot 0.002875 \right) \times 100 \right\rfloor \div 100$$

$$= 10.79 + 13.81 - 3.45$$

$$= \$21.15 \ ;$$

$$m = 2 + 1 = 3 \text{ months}$$

$$< \texttt{Third iteration} >$$

$$\texttt{Check:} \quad B_2 - \left\{ a - \left\lfloor \alpha \left( B_2 \cdot i \right) \times 100 \right\rfloor \div 100 \right\} \overset{?}{>} 0$$

$$4{,}607.04 - \left\{ 200 - \left\lfloor 0.25 \left( 4{,}607.04 \cdot 0.002875 \right) \times 100 \right\rfloor \div 100 \right\} \overset{?}{>} 0$$

$$4{,}607.04 - \left\{ 200 - 3.31 \right\} \overset{?}{>} 0$$

$$4{,}607.04 - 196.69 \overset{?}{>} 0$$

$$4{,}410.35 > 0$$

$$\texttt{Proceeding...}$$

$$B_3 = B_2 - \left\{ a - \left\lfloor \alpha \left( B_2 \cdot i \right) \times 100 \right\rfloor \div 100 \right\}$$

$$= \$4{,}410.35 \ ;$$

$$O_3 = O_2 + \left\lfloor (B_2 \cdot i) \times 100 \right\rfloor \div 100 - \left\lfloor \alpha \left( B_2 \cdot i \right) \times 100 \right\rfloor \div 100$$

$$= 21.15 + \left\lfloor (4{,}607.04 \cdot 0.002875) \times 100 \right\rfloor \div 100 - \left\lfloor 0.25 \left( 4{,}607.04 \cdot 0.002875 \right) \times 100 \right\rfloor \div 100$$

$$= 21.15 + 13.25 - 3.31$$

$$= \$31.09 \ ;$$

$$m = 3 + 1 = 4 \text{ months}$$

$$\vdots$$

$$< \texttt{Twenty-sixth iteration} >$$

$$\texttt{Check:} \quad B_{25} - \left\{ a - \left\lfloor \alpha \left( B_{25} \cdot i \right) \times 100 \right\rfloor \div 100 \right\} \overset{?}{>} 0$$

$$47.26 - \left\{ 200 - \left\lfloor 0.25 \left( 47.26 \cdot 0.002875 \right) \times 100 \right\rfloor \div 100 \right\} \overset{?}{>} 0$$

$$47.26 - \left\{ 200 - 0.03 \right\} \overset{?}{>} 0$$

$$47.26 - 199.97 \overset{?}{>} 0$$

$$-152.71 \not> 0$$

$$\texttt{Halt!}$$

$$n = 26 \text{ months}$$

$$a_{\mathrm{f}} = B_{25} + \left\lfloor (B_{25} \cdot i) \times 100 \right\rceil \div 100 + O_{25}$$

$$= 47.26 + \left\lfloor (47.26 \cdot 0.002875) \times 100 \right\rceil \div 100 + 141.77$$

$$= 47.26 + 0.14 + 141.77$$

$$= \$189.17 \; ;$$

$$B_{26} = B_{25} - \left\{ a_{\mathrm{f}} - \left\lfloor (B_{25} \cdot i) \times 100 \right\rceil \div 100 - O_{25} \right\}$$

$$= 47.26 - \left\{ 189.17 - \left\lfloor (47.26 \cdot 0.002875) \times 100 \right\rceil \div 100 - 141.77 \right\}$$

$$= 47.26 - \left\{ 189.17 - 0.14 - 141.77 \right\}$$

$$= 47.26 - 47.26$$

$$= \$0$$

Monthly Balance :

| | | | | | | |
|---|---|---|---|---|---|---|
| 5000.00 | + | 5000.00 · 0.00287... | − | 200.00 | = | 4803.59 |
| 4803.59 | + | 4803.59 · 0.00287... | − | 200.00 | = | 4607.04 |
| 4607.04 | + | 4607.04 · 0.00287... | − | 200.00 | = | 4410.35 |
| 4410.35 | + | 4410.35 · 0.00287... | − | 200.00 | = | 4213.52 |
| ⋮ | | ⋮ | | ⋮ | | ⋮ |
| 47.26 | + | 47.26 · 0.00287... | − | 189.17 | = | 0.00 |

Breakdown of Pay :

| | | | | |
|---|---|---|---|---|
| 196.41 Prin. | + | 3.59 Int. | = | 200 |
| 196.55 Prin. | + | 3.45 Int. | = | 200 |
| 196.69 Prin. | + | 3.31 Int. | = | 200 |
| 196.83 Prin. | + | 3.17 Int. | = | 200 |
| ⋮ | | ⋮ | | ⋮ |
| 47.26 Prin. | + | 141.91 Int. | = | 189.17 |

**Example 1.6c.** *If $p = \$13{,}500$, APR $= 6.8\%$, $\alpha = 0.5$ and $a = a_{\min_{120}} = \$132.87$,*

$B_0 = 13{,}500$ ;

$O_0 = 0$ ;

$\quad m = 1$

$<$ First iteration $>$

Check: $\quad B_0 - \left\{ a - \lfloor \alpha\,(B_0 \cdot i) \times 100 \rfloor \div 100 \right\} \overset{?}{>} 0$

$13{,}500 - \left\{ 132.87 - \lfloor 0.5\,(13{,}500 \cdot 0.00566...) \times 100 \rfloor \div 100 \right\} \overset{?}{>} 0$

$13{,}500 - \left\{ 132.87 - 38.25 \right\} \overset{?}{>} 0$

$13{,}500 - 94.62 \overset{?}{>} 0$

$13{,}405.38 > 0$

Proceeding...

$B_1 = B_0 - \left\{ a - \lfloor \alpha\,(B_0 \cdot i) \times 100 \rfloor \div 100 \right\}$

$\quad = \$13{,}405.38$ ;

$O_1 = O_0 + \lfloor (B_0 \cdot i) \times 100 \rfloor \div 100 - \lfloor \alpha\,(B_0 \cdot i) \times 100 \rfloor \div 100$

$\quad = 0 + \lfloor (13{,}500 \cdot 0.00566...) \times 100 \rfloor \div 100 - \lfloor 0.5\,(13{,}500 \cdot 0.00566...) \times 100 \rfloor \div 100$

$\quad = 0 + 76.50 - 38.25$

$\quad = \$38.25$ ;

$\quad m = 1 + 1 = 2$ months

$<$ Second iteration $>$

Check: $\quad B_1 - \left\{ a - \lfloor \alpha\,(B_1 \cdot i) \times 100 \rfloor \div 100 \right\} \overset{?}{>} 0$

$13{,}405.38 - \left\{ 132.87 - \lfloor 0.5\,(13{,}405.38 \cdot 0.00566...) \times 100 \rfloor \div 100 \right\} \overset{?}{>} 0$

$13{,}405.38 - \left\{ 132.87 - 37.98 \right\} \overset{?}{>} 0$

$13{,}405.38 - 94.89 \overset{?}{>} 0$

$13{,}310.49 > 0$

Proceeding...

$B_2 = B_1 - \left\{ a - \lfloor \alpha\,(B_1 \cdot i) \times 100 \rfloor \div 100 \right\}$

$\quad = \$13{,}310.49$ ;

$O_2 = O_1 + \lfloor (B_1 \cdot i) \times 100 \rfloor \div 100 - \lfloor \alpha\,(B_1 \cdot i) \times 100 \rfloor \div 100$

$\quad = 38.25 + \lfloor (13{,}405.38 \cdot 0.00566...) \times 100 \rfloor \div 100 - \lfloor 0.5\,(13{,}405.38 \cdot 0.00566...) \times 100 \rfloor \div 100$

$\quad = 38.25 + 75.96 - 37.98$

$\quad = \$76.23$ ;

$\quad m = 2 + 1 = 3$ months

< Third iteration >

Check: $B_2 - \left\{ a - \left\lfloor \alpha \left( B_2 \cdot i \right) \times 100 \right\rceil \div 100 \right\} \overset{?}{>} 0$

$13{,}310.49 - \left\{ 132.87 - \left\lfloor 0.5 \left( 13{,}310.49 \cdot 0.00566... \right) \times 100 \right\rceil \div 100 \right\} \overset{?}{>} 0$

$13{,}310.49 - \left\{ 132.87 - 37.71 \right\} \overset{?}{>} 0$

$13{,}310.49 - 95.16 \overset{?}{>} 0$

$13{,}215.33 > 0$

Proceeding...

$B_3 = B_2 - \left\{ a - \left\lfloor \alpha \left( B_2 \cdot i \right) \times 100 \right\rceil \div 100 \right\}$

$\quad = \$13{,}215.33 \; ;$

$O_3 = O_2 + \left\lfloor \left( B_2 \cdot i \right) \times 100 \right\rceil \div 100 - \left\lfloor \alpha \left( B_2 \cdot i \right) \times 100 \right\rceil \div 100$

$\quad = 76.23 + \left\lfloor \left( 13{,}310.49 \cdot 0.00566... \right) \times 100 \right\rceil \div 100 - \left\lfloor 0.5 \left( 13{,}310.49 \cdot 0.00566... \right) \times 100 \right\rceil \div 100$

$\quad = 76.23 + 75.43 - 37.71$

$\quad = \$113.95 \; ;$

$m = 3 + 1 = 4$ months

$\vdots$

< 120$^{\text{th}}$ iteration >

Check: $B_{119} - \left\{ a - \left\lfloor \alpha \left( B_{119} \cdot i \right) \times 100 \right\rceil \div 100 \right\} \overset{?}{>} 0$

$131.70 - \left\{ 132.87 - \left\lfloor 0.5 \left( 131.70 \cdot 0.00566... \right) \times 100 \right\rceil \div 100 \right\} \overset{?}{>} 0$

$131.70 - \left\{ 132.87 - 0.37 \right\} \overset{?}{>} 0$

$131.70 - 132.50 \overset{?}{>} 0$

$-0.80 \not> 0$

Halt!

$n = 120$ months

$a_{\text{f}} = B_{119} + \left\lfloor \left( B_{119} \cdot i \right) \times 100 \right\rceil \div 100 + O_{119}$

$\quad = 131.70 + \left\lfloor \left( 131.70 \cdot 0.00566... \right) \times 100 \right\rceil \div 100 + 2{,}443.24$

$\quad = 131.70 + 0.75 + 2{,}443.24$

$\quad = \$2{,}575.69 \; ;$

$B_{120} = B_{119} - \left\{ a_{\text{f}} - \left\lfloor \left( B_{119} \cdot i \right) \times 100 \right\rceil \div 100 - O_{119} \right\}$

$\quad = 131.70 - \left\{ 2{,}575.69 - \left\lfloor \left( 131.70 \cdot 0.00566... \right) \times 100 \right\rceil \div 100 - 2{,}443.24 \right\}$

$\quad = 131.70 - \left\{ 2{,}575.69 - 0.75 - 2{,}443.24 \right\}$

$\quad = 131.70 - 131.70$

$\quad = \$0$

Monthly Balance :

$$13500.00 \quad + \quad 13500.00 \cdot 0.00566\ldots \quad - \quad 132.87 \quad = \quad 13405.38$$

$$13405.38 \quad + \quad 13405.38 \cdot 0.00566\ldots \quad - \quad 132.87 \quad = \quad 13310.49$$

$$13310.49 \quad + \quad 13310.49 \cdot 0.00566\ldots \quad - \quad 132.87 \quad = \quad 13215.33$$

$$13215.33 \quad + \quad 13215.33 \cdot 0.00566\ldots \quad - \quad 132.87 \quad = \quad 13119.90$$

$$\vdots \qquad \qquad \vdots \qquad \qquad \vdots \qquad \qquad \vdots$$

$$131.70 \quad + \quad 131.70 \cdot 0.00566\ldots \quad - \quad 2575.69 \quad = \quad 0.00$$

Breakdown of Pay :

$$94.62 \text{ Prin.} \quad + \quad 38.25 \text{ Int.} \quad = \quad 132.87$$

$$94.89 \text{ Prin.} \quad + \quad 37.98 \text{ Int.} \quad = \quad 132.87$$

$$95.16 \text{ Prin.} \quad + \quad 37.71 \text{ Int.} \quad = \quad 132.87$$

$$95.43 \text{ Prin.} \quad + \quad 37.44 \text{ Int.} \quad = \quad 132.87$$

$$\vdots \qquad \qquad \vdots \qquad \qquad \vdots$$

$$131.70 \text{ Prin.} \quad + \quad 2443.99 \text{ Int.} \quad = \quad 2575.69$$

**Example 1.6d.** *If $p = \$4{,}000$, $APR = 6.8\%$, $\alpha = 0.5$ and $a = \$5{,}000$,*

$$B_0 = 4{,}000 \; ;$$

$$O_0 = 0 \; ;$$

$$m = 1$$

Check: $\quad B_0 - \left\{ a - \left\lfloor \alpha \left( B_0 \cdot i \right) \times 100 \right\rfloor \div 100 \right\} \overset{?}{>} 0$

$\qquad 4{,}000 - \left\{ 5{,}000 - \left\lfloor 0.5 \left( 4{,}000 \cdot 0.00566... \right) \times 100 \right\rfloor \div 100 \right\} \overset{?}{>} 0$

$\qquad 4{,}000 - \left\{ 5{,}000 - 11.33 \right\} \overset{?}{>} 0$

$\qquad 4{,}000 - 4{,}988.67 \overset{?}{>} 0$

$\qquad -988.67 \not> 0$

Halt!

$$n = 1 \, \text{month}$$

$$a_{\text{f}} = B_0 + \left\lfloor \left( B_0 \cdot i \right) \times 100 \right\rfloor \div 100 + O_0$$

$$= 4{,}000 + \left\lfloor \left( 4{,}000 \cdot 0.00566... \right) \times 100 \right\rfloor \div 100 + 0$$

$$= 4{,}000 + 22.67 + 0$$

$$= \$4{,}022.67 \; ;$$

Check: $\quad a - a_{\text{f}}$

$$= 5{,}000 - 4{,}022.67$$

$$= 977.33 > 0$$

$$R = a - a_{\text{f}} = 977.33 \; ;$$

"Refunded $977.33"

Monthly Balance:

$\quad 4000.00 \quad + \quad 4000.00 \cdot 0.00566... \quad - \quad 4022.67 \quad = \quad 0.00$

Breakdown of Pay:

$\quad 4000.00 \, \text{Prin.} \quad + \quad 22.67 \, \text{Int.} \quad = \quad 4022.67$

**Example 1.6e.** *If $p = \$4,000$, $APR = 6.8\%$, $\alpha = 0$ and $a = \$4,000$,*

$$B_0 = 4,000 \; ;$$

$$O_0 = 0 \; ;$$

$$m = 1$$

Check : $B_0 - \left\{ a - \lfloor \alpha (B_0 \cdot i) \times 100 \rfloor \div 100 \right\} \overset{?}{>} 0$

$4,000 - \left\{ 4,000 - \lfloor 0 (4,000 \cdot 0.00566...) \times 100 \rfloor \div 100 \right\} \overset{?}{>} 0$

$4,000 - \left\{ 4,000 - 0 \right\} \overset{?}{>} 0$

$4,000 - 4,000 \overset{?}{>} 0$

$0 \ngtr 0$

Halt!

$$n = 1 \text{ month}$$

$$a_\text{f} = B_0 + \lfloor (B_0 \cdot i) \times 100 \rfloor \div 100 + O_0$$

$$= 4,000 + \lfloor (4,000 \cdot 0.00566...) \times 100 \rfloor \div 100 + 0$$

$$= 4,000 + 22.67 + 0$$

$$= \$4,022.67 \; ;$$

Check : $a - a_\text{f}$

$$= 4,000 - 4,022.67$$

$$= -22.67 < 0$$

$$E = |a - a_\text{f}| = 22.67 \; ;$$

"Pay Extra $22.67"

Monthly Balance :

4000.00 + 4000.00 · 0.00566... − 4022.67 = 0.00

Breakdown of Pay :

4000.00 Prin. + 22.67 Int. = 4022.67

## 1.7 Length of repayment

For $n = 4$,

$$l_y = \left\lfloor \frac{4}{12} \right\rfloor$$
$$= \left\lfloor 0.333... \right\rfloor$$
$$= 0 \text{ years}$$

$$l_m = 4 - 12 \cdot 0$$
$$= 4 - 0$$
$$= 4 \text{ months}$$

For $n = 19$,

$$l_y = \left\lfloor \frac{19}{12} \right\rfloor$$
$$= \left\lfloor 1.583... \right\rfloor$$
$$= 1 \text{ year}$$

$$l_m = 19 - 12 \cdot 1$$
$$= 19 - 12$$
$$= 7 \text{ months}$$

For $n = 27$,

$$l_y = \left\lfloor \frac{27}{12} \right\rfloor$$
$$= \left\lfloor 2.25 \right\rfloor$$
$$= 2 \text{ years}$$

$$l_m = 27 - 12 \cdot 2$$
$$= 27 - 24$$
$$= 3 \text{ months}$$

## 1.8    Total payments, savings and change in savings

**Example 1.8a.** *Assume $p = \$13{,}500$, $APR = 6.8\%$ and $\alpha = 0.5$.*

*Total payments,*

Let $a = 300$

$$\therefore n = 49 \text{ months}$$
$$B_{48} = \$61.91$$
$$O_{48} = \$961.95$$

$$T(a) = (48)a + B_{48} + \left\lfloor (B_{48} \cdot i) \times 100 \right\rfloor \div 100 + O_{48}$$
$$T(300) = (48)300 + 61.91 + \left\lfloor (61.91 \cdot 0.00566...) \times 100 \right\rfloor \div 100 + 961.95$$
$$= 14{,}400 + 61.91 + 0.35 + 961.95$$
$$= \$15{,}424.21$$

Let $a = a_{\min_{120},\, \alpha=1} = 155.36$

$$\therefore n = 120 \text{ months}$$
$$B_{119} = \$154.16$$
$$O_{119} = \$0$$

$$T_{\max} = (119)a_{\min,\, \alpha=1} + B_{119} + \left\lfloor (B_{119} \cdot i) \times 100 \right\rfloor \div 100 + O_{119}$$
$$= (119)155.36 + 154.16 + \left\lfloor (154.16 \cdot 0.00566...) \times 100 \right\rfloor \div 100 + 0$$
$$= 18{,}487.84 + 154.16 + 0.87 + 0$$
$$= \$18{,}642.87$$

*Savings,*

$$s = T_{\max} - T(300)$$
$$= 18{,}642.87 - 15{,}424.21$$
$$= \$3{,}218.66$$

*Total payments,*

Let $a = 725$

$$\therefore n = 20 \text{ months}$$
$$B_{19} = \$113.61$$
$$O_{19} = \$388.62$$

$$T(a) = (19)a + B_{19} + \lfloor (B_{19} \cdot i) \times 100 \rfloor \div 100 + O_{19}$$
$$T(725) = (19)725 + 113.61 + \lfloor (113.61 \cdot 0.00566...) \times 100 \rfloor \div 100 + 388.62$$
$$= 13{,}775 + 113.61 + 0.64 + 388.62$$
$$= \$14{,}277.87$$

$$T_{\max} = \$18{,}642.87 \text{ (page 20)}$$

*Savings,*

$$s = T_{\max} - T(725)$$
$$= 18{,}642.87 - 14{,}277.87$$
$$= \$4{,}365$$

*Change in savings,*

Let $s_1 = \$3{,}218.66$ *(page 20)*
$$s_2 = \$4{,}365$$

$$\Delta s = |\, 4{,}365 - 3{,}218.66 \,|$$
$$= \$1{,}146.34$$

**Example 1.8b.** *Assume $p = \$5{,}000$, $APR = 3.45\%$ and $\alpha = 0.25$.*

*Total payments,*

Let $a = 700$

$$\therefore n = 8 \text{ months}$$
$$B_7 = \$114.63$$
$$O_7 = \$43.90$$

$$T(a) = (7)a + B_7 + \lfloor (B_7 \cdot i) \times 100 \rfloor \div 100 + O_7$$
$$T(700) = (7)700 + 114.63 + \lfloor (114.63 \cdot 0.002875) \times 100 \rfloor \div 100 + 43.90$$
$$= 4{,}900 + 114.63 + 0.33 + 43.90$$
$$= \$5{,}058.86$$

Let $a = a_{\min_n, \alpha=1} = 14.39$

$$\therefore n = 2{,}387 \text{ months}$$
$$B_{2,386} = \$0.37$$
$$O_{2,386} = \$0$$

$$T_{\max} = (2{,}386)a_{\min, \alpha=1} + B_{2,386} + \lfloor (B_{2,386} \cdot i) \times 100 \rfloor \div 100 + O_{2,386}$$
$$= (2{,}386)14.39 + 0.37 + \lfloor (0.37 \cdot 0.002875) \times 100 \rfloor \div 100 + 0$$
$$= 34{,}334.54 + 0.37 + 0 + 0$$
$$= \$34{,}334.91$$

*Savings,*

$$s = T_{\max} - T(700)$$
$$= 34{,}334.91 - 5{,}058.86$$
$$= \$29{,}276.05$$

*Change in savings,*

Let $s_1 = \$4{,}365$ *(page 21)*
$$s_2 = \$29{,}276.05$$

$$\Delta s = \mid 29{,}276.05 - 4{,}365 \mid$$
$$= \$24{,}911.05$$

**Example 1.8c.** *Assume* $p = \$2{,}000$, $APR = 7\%$ *and* $\alpha = 0.33$.

*Total payments,*

Let $a = 550$

$$\therefore n = 4 \text{ months}$$
$$B_3 = \$358.40$$
$$O_3 = \$17.04$$

$$T(a) = (3)a + B_3 + \lfloor (B_3 \cdot i) \times 100 \rfloor \div 100 + O_3$$
$$T(550) = (3)550 + 358.40 + \lfloor (358.40 \cdot 0.00583...) \times 100 \rfloor \div 100 + 17.04$$
$$= 1{,}650 + 358.40 + 2.09 + 17.04$$
$$= \$2{,}027.53$$

Let $a = a_{\min_{120}, \alpha=1} = 23.23$

$$\therefore n = 120 \text{ months}$$
$$B_{119} = \$21.65$$
$$O_{119} = \$0$$

$$T_{\max} = (119)a_{\min, \alpha=1} + B_{119} + \lfloor (B_{119} \cdot i) \times 100 \rfloor \div 100 + O_{119}$$
$$= (119)23.23 + 21.65 + \lfloor (21.65 \cdot 0.00583...) \times 100 \rfloor \div 100 + 0$$
$$= 2{,}764.37 + 21.65 + 0.13 + 0$$
$$= \$2{,}786.15$$

*Savings,*

$$s = T_{\max} - T(550)$$
$$= 2{,}786.15 - 2{,}027.53$$
$$= \$758.62$$

*Change in savings,*

Let $s_1 = \$29{,}276.05$ *(page 22)*
$$s_2 = \$758.62$$

$$\Delta s = |\, 758.62 - 29{,}276.05 \,|$$
$$= \$28{,}517.43$$

*Total payments,*

Let $a = 1{,}000$

$\therefore n = 3 \text{ months}$
$B_2 = \$5.78$
$O_2 = \$11.75$

$$T(a) = (2)a + B_2 + \left\lfloor (B_2 \cdot i) \times 100 \right\rfloor \div 100 + O_2$$
$$T(1{,}000) = (2)1{,}000 + 5.78 + \left\lfloor (5.78 \cdot 0.00583...) \times 100 \right\rfloor \div 100 + 11.75$$
$$= 2{,}000 + 5.78 + 0.03 + 11.75$$
$$= \$2{,}017.56$$

$$T_{\max} = \$2{,}786.15 \text{ (page 23)}$$

*Savings,*

$$s = T_{\max} - T(1{,}000)$$
$$= 2{,}786.15 - 2{,}017.56$$
$$= \$768.59$$

*Change in savings,*

Let $s_1 = \$758.62$ *(page 23)*
$s_2 = \$768.59$

$$\Delta s = \left| 768.59 - 758.62 \right|$$
$$= \$9.97$$

*Total payments,*

Let $a = 200$

$$\therefore n = 11 \text{ months}$$
$$B_{10} = \$21.42$$
$$O_{10} = \$43.50$$

$$T(a) = (10)a + B_{10} + \lfloor (B_{10} \cdot i) \times 100 \rceil \div 100 + O_{10}$$
$$T(200) = (10)200 + 21.42 + \lfloor (21.42 \cdot 0.00583...) \times 100 \rceil \div 100 + 43.50$$
$$= 2{,}000 + 21.42 + 0.12 + 43.50$$
$$= \$2{,}065.04$$

$$T_{\max} = \$2{,}786.15 \text{ (page 23)}$$

*Savings,*

$$s = T_{\max} - T(200)$$
$$= 2{,}786.15 - 2{,}065.04$$
$$= \$721.11$$

*Change in savings,*

$$\text{Let } s_1 = \$768.59 \text{ (page 24)}$$
$$s_2 = \$721.11$$

$$\Delta s = |\, 721.11 - 768.59 \,|$$
$$= \$47.48$$

# 2 Derivation of Monthly Compound Interest Rate

**Definition 2.1.** *If interest is compounded, it is compounded daily. Let daily payment balance be a function of each day's previous balance and the annual interest rate:*

$$B_d = B_{d-1} + B_{d-1}\frac{r}{365.25},$$

*for number of days, $d = 1, 2, 3, \ldots, 30, \frac{365.25}{12}$, where $\frac{365.25}{12}$ is the average number of days per month. Treat the last day is a partial day.*

**Theorem 2.2.** *Monthly compound interest rate will be a function of the annual rate:*

$$i = \left(1 + \frac{r}{365.25}\right)^{\frac{365.25}{12}} - 1$$

*Proof.* Using definition 2.1, find average monthly balance, $B_{\frac{365.25}{12}}$, and simplify.

$$\begin{aligned}
B_1 &= B_0 + B_0\frac{r}{365.25} \\
&= p + p\frac{r}{365.25} \\
&= p\left(1 + \frac{r}{365.25}\right)
\end{aligned}$$

$$\begin{aligned}
B_2 &= B_1 + B_1\frac{r}{365.25} \\
&= p\left(1 + \frac{r}{365.25}\right) + p\left(1 + \frac{r}{365.25}\right)\frac{r}{365.25} \\
&= p\left(1 + \frac{r}{365.25}\right)\left(1 + \frac{r}{365.25}\right) \\
&= p\left(1 + \frac{r}{365.25}\right)^2
\end{aligned}$$

$$\begin{aligned}
B_3 &= B_2 + B_2\frac{r}{365.25} \\
&= \cdots = p\left(1 + \frac{r}{365.25}\right)^2\left(1 + \frac{r}{365.25}\right) \\
&= p\left(1 + \frac{r}{365.25}\right)^3
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
B_{\frac{365.25}{12}} &= B_{\frac{365.25}{12}-1} + B_{\frac{365.25}{12}-1}\frac{r}{365.25} \\
&= \cdots = p\left(1 + \frac{r}{365.25}\right)^{\frac{365.25}{12}} \quad\quad\quad\quad\quad (2.2a)
\end{aligned}$$

Write equation 2.2a as a function of $i$.

$$B_{\frac{365.25}{12}} = p(1+i) \tag{2.2b}$$

Combine equations 2.2a and 2.2b to solve for $i$.

$$p(1+i) = p\left(1 + \frac{r}{365.25}\right)^{\frac{365.25}{12}}$$

$$i = \left(1 + \frac{r}{365.25}\right)^{\frac{365.25}{12}} - 1 \qquad \square$$

Numerically, do not round any terms in the equation or solution itself, otherwise solutions to equations that depend on the interest rate may lose precision. The solution will be a decimal, not a percentage; this is because $i$ is only meant to be an interim calculation.

*Remark* 2.3. The researcher is only aware of one instance in which interest is compounded: when loans are capitalized as students enter repayment. Nevertheless, he has mentioned interest being compounded, purely for the purpose of mathematical exploration.

# 3   Derivation of Ten-Year Minimum Payment

**Definition 3.1.** *Student loan payments are made in monthly installments. Let the monthly principal balance, be a function of each month's previous balance, the interest rate, minimum payment and proportion of interest that is paid, such that:*

$$B_m = B_{m-1} - \left[ a_{\min} - \alpha \left( B_{m-1} \cdot i \right) \right], \tag{3.1}$$

*for $m = 1, 2, 3, \ldots, 120$ and $0 \leq \alpha \leq 1$, where 120 is the number of months in ten years. We want final balance, $B_{120} = 0$.*

**Theorem 3.2.** *Minimum monthly payment within ten years will depend on $i$ and $\alpha$:*

$$a_{\min_{120}} = \begin{cases} \left\lceil \dfrac{p}{120} \times 100 \right\rceil \div 100 & \text{if } i > 0 \text{ and } \alpha = 0 \\[2ex] \left\lceil \dfrac{p}{120} \times 100 \right\rceil \div 100 & \text{if } i = 0 \\[2ex] \left\lceil \dfrac{\alpha \left( p \cdot i \right) \left( 1 + \alpha \cdot i \right)^{120}}{\left( 1 + \alpha \cdot i \right)^{120} - 1} \times 100 \right\rceil \div 100, \text{for } \alpha \cdot i \neq 0 & \text{if } i > 0 \text{ and } 0 < \alpha \leq 1 \end{cases}$$

*Proof.* Simplify equation 3.1, if possible, using each case of $i$ and $\alpha$.

$$B_m = \begin{cases} B_{m-1} - a_{\min} & \text{if } i > 0 \text{ and } \alpha = 0 & (3.2\text{a}) \\[1ex] B_{m-1} - a_{\min} & \text{if } i = 0 & (3.2\text{b}) \\[1ex] B_{m-1} - \left[ a_{\min} - \alpha \left( B_{m-1} \cdot i \right) \right] & \text{if } i > 0 \text{ and } 0 < \alpha \leq 1 & (3.2\text{c}) \end{cases}$$

Using cases 3.2a and 3.2b, find $B_{120}$ and simplify.

$$B_1 = B_0 - a_{\min}$$
$$= p - a_{\min}$$

$$B_2 = B_1 - a_{\min}$$
$$= (p - a_{\min}) - a_{\min}$$
$$= p - 2a_{\min}$$

$$B_3 = B_2 - a_{\min}$$
$$= \cdots = p - 3a_{\min}$$

$$\vdots$$

$$B_{120} = B_{119} - a_{\min}$$
$$= p - 120a_{\min}$$

Set $B_{120} = 0$ to solve for $a_{\min}$.

$$0 = p - 120 a_{\min}$$
$$p = 120 a_{\min}$$
$$a_{\min} = \frac{p}{120}$$

Numerically, terms in the solution may span more than two decimal places, but cents span only two. Also, we need to ensure that one repays enough money each month. So, round terms in the solution *up* to the nearest two decimal places.

$$a_{\min} = \left\lceil \frac{p}{120} \times 100 \right\rceil \div 100$$

Let $a_{\min} = a_{\min_{120}}$ to differentiate it from the absolute minimum payment.

$$a_{\min_{120}} = \left\lceil \frac{p}{120} \times 100 \right\rceil \div 100 \qquad \qquad \square$$

Using case 3.2c, find $B_{120}$ and simplify.

$$
\begin{aligned}
B_1 &= B_0 - \left[ a_{\min} - \alpha \left( B_0 \cdot i \right) \right] \\
&= B_0 + \alpha \left( B_0 \cdot i \right) - a_{\min} \\
&= p + \alpha \left( p \cdot i \right) - a_{\min} \\
&= p \left( 1 + \alpha \cdot i \right) - a_{\min}
\end{aligned}
$$

$$
\begin{aligned}
B_2 &= B_1 - \left[ a_{\min} - \alpha \left( B_1 \cdot i \right) \right] \\
&= B_1 + \alpha \left( B_1 \cdot i \right) - a_{\min} \\
&= \left[ p \left( 1 + \alpha \cdot i \right) - a_{\min} \right] + \alpha \left( \left[ p \left( 1 + \alpha \cdot i \right) - a_{\min} \right] \cdot i \right) - a_{\min} \\
&= \left[ p \left( 1 + \alpha \cdot i \right) - a_{\min} \right] \left[ 1 + \alpha \cdot i \right] - a_{\min} \\
&= p \left( 1 + \alpha \cdot i \right)^2 - \left( 1 + \alpha \cdot i \right) a_{\min} - a_{\min}
\end{aligned}
$$

$$
\begin{aligned}
B_3 &= B_2 - \left[ a_{\min} - \alpha \left( B_2 \cdot i \right) \right] \\
&= \cdots = \left[ p \left( 1 + \alpha \cdot i \right)^2 - \left( 1 + \alpha \cdot i \right) a_{\min} - a_{\min} \right] \\
&\qquad + \alpha \left( \left[ p \left( 1 + \alpha \cdot i \right)^2 - \left( 1 + \alpha \cdot i \right) a_{\min} - a_{\min} \right] \cdot i \right) - a_{\min} \\
&= \left[ p \left( 1 + \alpha \cdot i \right)^2 - \left( 1 + \alpha \cdot i \right) a_{\min} - a_{\min} \right] \left[ 1 + \alpha \cdot i \right] - a_{\min} \\
&= p \left( 1 + \alpha \cdot i \right)^3 - \left( 1 + \alpha \cdot i \right)^2 a_{\min} - \left( 1 + \alpha \cdot i \right) a_{\min} - a_{\min}
\end{aligned}
$$

$$\vdots$$

$$B_{120} = B_{119} - \left[a_{\min} - \alpha\left(B_{119} \cdot i\right)\right]$$
$$= \cdots = \left[p\left(1+\alpha\cdot i\right)^{119} - \left(1+\alpha\cdot i\right)^{118} a_{\min} - \left(1+\alpha\cdot i\right)^{117} a_{\min}\right.$$
$$\left. - \cdots - a_{\min}\right]\left[1+\alpha\cdot i\right] - a_{\min}$$
$$= p\left(1+\alpha\cdot i\right)^{120} - \left(1+\alpha\cdot i\right)^{119} a_{\min} - \left(1+\alpha\cdot i\right)^{118} a_{\min}$$
$$- \cdots - \left(1+\alpha\cdot i\right) a_{\min} - a_{\min}$$
$$= p\left(1+\alpha\cdot i\right)^{120} - a_{\min}\sum_{m=1}^{120}\left(1+\alpha\cdot i\right)^{m-1}$$

Set $B_{120} = 0$ to solve for $a_{\min}$.

$$0 = p\left(1+\alpha\cdot i\right)^{120} - a_{\min}\sum_{m=1}^{120}\left(1+\alpha\cdot i\right)^{m-1}$$

$$p\left(1+\alpha\cdot i\right)^{120} = a_{\min}\sum_{m=1}^{120}\left(1+\alpha\cdot i\right)^{m-1}$$

$$p\left(1+\alpha\cdot i\right)^{120}\times\left(1+\alpha\cdot i\right) = a_{\min}\sum_{m=1}^{120}\left(1+\alpha\cdot i\right)^{m}$$

$$p\left(1+\alpha\cdot i\right)^{120} - p\left(1+\alpha\cdot i\right)^{120}\times\left(1+\alpha\cdot i\right) = a_{\min}$$
$$- \left(1+\alpha\cdot i\right)^{120} a_{\min}$$

$$p\left(1+\alpha\cdot i\right)^{120}\left[1 - \left(1+\alpha\cdot i\right)\right] = a_{\min}\left[1 - \left(1+\alpha\cdot i\right)^{120}\right]$$

$$p\left(1+\alpha\cdot i\right)^{120}\left[\alpha\cdot i\right] = a_{\min}\left[\left(1+\alpha\cdot i\right)^{120} - 1\right]$$

$$a_{\min} = \frac{\alpha\left(p\cdot i\right)\left(1+\alpha\cdot i\right)^{120}}{\left(1+\alpha\cdot i\right)^{120} - 1}, \text{ for } \alpha\cdot i \neq 0$$

Numerically, do not round any terms in the equation; round only those in the solution. However, again terms in the solution may span more than two decimal places, and we need to ensure that one repays enough money each month. So, round terms in the solution *up* to the nearest two decimal places.

$$a_{\min} = \left\lceil\frac{\alpha\left(p\cdot i\right)\left(1+\alpha\cdot i\right)^{120}}{\left(1+\alpha\cdot i\right)^{120} - 1}\times 100\right\rceil \div 100, \text{ for } \alpha\cdot i \neq 0$$

Let $a_{\min} = a_{\min_{120}}$ to differentiate it from the absolute minimum payment.

$$a_{\min_{120}} = \left\lceil\frac{\alpha\left(p\cdot i\right)\left(1+\alpha\cdot i\right)^{120}}{\left(1+\alpha\cdot i\right)^{120} - 1}\times 100\right\rceil \div 100, \text{ for } \alpha\cdot i \neq 0 \qquad \square$$

*Remark* 3.3. The reason for using $\alpha\,(B_{m-1}\cdot i)$ in equation 3.1, instead of the corresponding expression $\lfloor \alpha\,(B_{m-1}\cdot i) \times 100 \rceil \div 100$ of section 2 of "Deeper Insight into the iOS App", is because, if $i > 0$ and $0 < \alpha \leq 1$ and one uses the latter expression to find a simplified equation for $B_{120}$, factoring $p + \lfloor \alpha\,(p\cdot i) \times 100 \rceil \div 100 - a$ from the `nint` function in $B_2$ will not be possible. Subsequent equations (i.e., $B_3$, $B_4$, $B_5$, …) will continue to expand with no way to simplify them even.

*Remark* 3.4. The caveat to using the former expression, though, is that each numerical solution of $a_{\min_{120}}$ for case 3.2c could still deviate by one cent or more from each solution had they been computed by using the latter expression. For example, for default parameters (i.e., $p_{\min} = \$2{,}000$, $p_{\max} = \$10{,}000$, $N = 40$, APR $= 4.53\%$ [not compounded] and $\alpha = 1$) no numerical solutions deviate. If APR $= 5\%$, all other parameters being the default, one solution deviates by a cent; the correct solution for $p = \$4{,}600$ should be $a_{\min_{120}} = \$48.79$, not $\$48.80$. If one broadens or alters parameters, more solutions may deviate. Roughly 2.14% of all solutions, for case 3.2c, do in fact deviate, by at most one cent, 56% one cent high and 44% one cent low.[1] Open "Ten-Year_Minimum_Errors.ipynb" in Jupyter Notebook or Jupyter Lab, customize the cell, and run it to see for one self. For one's convenience, the iOS app automatically checks and corrects for such errors in precision. (None of the latter four numerical solutions in section 1.5 deviated.)

*Remark* 3.5. Simplifying the fractional part

$$\frac{(1+\alpha\cdot i)^{120}}{(1+\alpha\cdot i)^{120}-1}, \text{ for } \alpha\cdot i \neq 0$$

is not possible. $\alpha$ is at most equal to 1, and APR is probably at most 10% (i.e., $i$ is at most 0.00836... [compounded] ).

$$\frac{(1+1\cdot 0.00836...)^{120}}{(1+1\cdot 0.00836...)^{120}-1} = \frac{(1.00836...)^{120}}{(1.00836...)^{120}-1} = \frac{2.717...}{2.717...-1} = 1.582...$$

In fact, all other quotients for values of $\alpha$ and $i$ will exceed 1.582... Only if all quotients were equal to 1, could one cancel the numerator and denominator from the fractional part.

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[1]Based on over one million computations of $a_{\min_{120}}$ given $p_{\min} = \$0$, $p_{\max} = \$100{,}000$ and $N = 100{,}000$, provided $\alpha = 1$ for APR $= 4.45\%$, APR $= 5\%$ and APR $= 7.4\%$ (all not compounded) and APR $= 2.98\%$, APR $= 4.45\%$ and APR $= 5\%$ (all compounded), provided $\alpha = 0.25$, $\alpha = 0.71$ and $\alpha = 0.2$ for APR $= 4.45\%$ (not compounded) and provided $\alpha = 0.6$ for APR $= 4.45\%$ (compounded).