Fourier's Methods in Optics & Image Processing

Fourier transform is a versatile mathematical technique that finds applications in numerous fields of science and technology today. First devised by Joseph Fourier to explain heat conduction in the year 1822 [1], it is a powerful tool that decomposes any periodic function into simple harmonics. Fourier transform shares an integral connection with far-field diffraction which is often overlooked when introduced for the first time to physics undergraduates and high-school students [2]. The diffraction pattern formed when light impinges on a micrometre-scale aperture, is mathematically the Fourier transform of the aperture. We show that a worthwhile approach to understanding this connection would be by simulating far-field diffraction by means of a Fourier transform program. This simple activity can allow students to utilise the known properties of Fourier transforms and simulate diffraction patterns of arbitrary apertures that are not easily available in laboratories. At the same time, diffraction experiments can be used to teach properties of Fourier transforms, thus allowing students to appreciate it as something more than an abstract mathematical tool. Fourier transform also finds uses in image processing; a set of tools that are essential in many branches of science including astrophysics, medicine and high-energy physics. This article explores some image processing techniques that utilize Fourier transforms.

When Fourier transform (FT) was first introduced in the context of heat transfer, there was some criticism concerning its validity [3], although it has now found applications in fields far from what Fourier initially developed it for. The dual nature of the Fourier integral happens to be one of its most appealing features. When Fourier transform is applied to a function, the function can be retrieved back by applying the inverse Fourier transform on it. This dual nature is of great importance in quantum mechanics where physical quantities like position and momentum are complementary which means, one cannot know with absolute precision the position and momentum of a particle simultaneously. This is the famous Heisenberg's uncertainty principle. The momentum space is essentially the Fourier transform of the position space [4]. This dual nature is also utilized in processing images where image enhancement functions are applied to the Fourier transform of the image and then reverted back by applying the inverse Fourier transform.

Diffraction: An optical analogue of Fourier transform

A point source of light gives out spherical wave-fronts which constitute a collection of coherent secondary point sources as per the Huygens- Fresnel principle [5]. These secondary wave-fronts, upon impinging an aperture or opening, spread out to form a bright and dark pattern in a phenomena known as diffraction. When the distance between the source and the opening (or aperture) is very large, these spherical wave-fronts are considered to be approximately planar. Additionally, if the screen on which the pattern is formed (known as the projection screen) is also at a large distance from the aperture, then the phenomena is termed Fraunhofer diffraction after the German physicist Joseph von Fraunhofer. It is then the electric field strength (intensity) at different locations on the projection screen due to the individual contributions of the secondary point sources at the aperture, that we are interested in obtaining.

When light waves are obstructed by an aperture, the diffraction pattern that we obtain on the projection screen is mathematically, a *Fourier transform*. This is not an obvious statement to most students that have studied optics and Fourier transforms separately, in entirely different contexts. Thus, this section attempts to justify this analogue using a simple mathematical formalism that is based on Hecht's book [6].

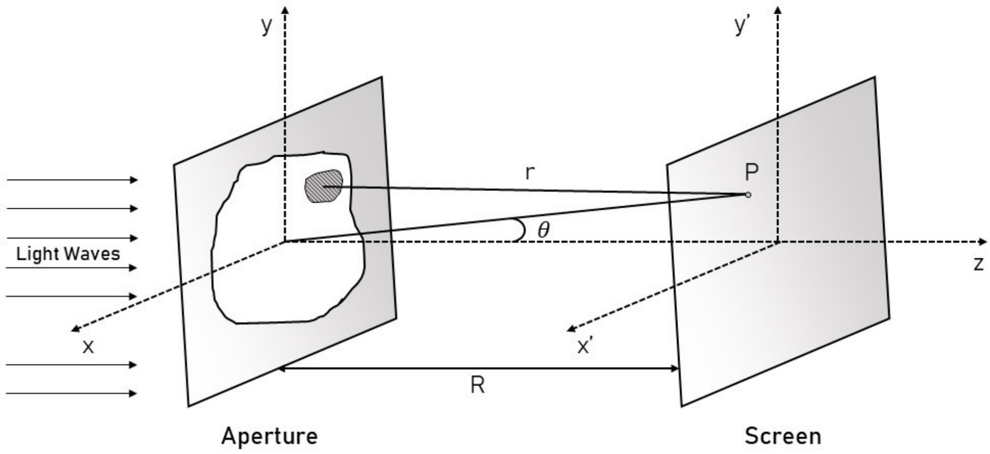


Fig. 1: Light waves striking an aperture of arbitrary shape, forming a diffraction pattern on the screen.

The field at point P [Fig. (1)] on the projection screen due to a small area dS on the aperture is given by,

$$dE = \left(\frac{\epsilon_A}{r}\right) e^{i(kr-\omega t)} dS \tag{1}$$

The field strength of the oscillators at the aperture drops off as $\frac{1}{r}$ due to its spherical nature where r is the distance of dS from the point P and ϵ_A is the field strength per unit area. Analysing the set-up in Cartesian co-ordinate system, r comes out as,

$$r = R\sqrt{1 + \frac{y^2 + x^2}{R^2} - 2\left(\frac{yy' + xx'}{R}\right)} \tag{2}$$

Since we are interested in far-field diffraction, R tends to infinity (for all practical purposes, this means that a converging lens is placed between the aperture and the projection screen) which gives an approximate expression for r as,

$$r = R\left[1 - \left(\frac{yy' + xx'}{R^2}\right)\right]^{1/2} \tag{3}$$

The contribution of the whole aperture to the field at any point on the screen is,

$$E(x', y') = \frac{\epsilon_A e^{i(kR-\omega t)}}{R} \iint_A e^{-ik\left(\frac{yy'+xx'}{R}\right)} dx\, dy \tag{4}$$

The expression for the field on the screen already resembles that of the Fourier's integral, however, some modifications have to be made for the reader to realize that it is *exactly* the integral proposed by Fourier.

The exponential pre-factor in $E(x', y')$ includes information about the phase of the electric field distribution at the screen, which is of little practical importance to us as we are interested in the amplitude or strength of the field on the projection screen. Thus, this phase information along with the $1/R$ drop-off of the field and the field strength per unit area ϵ_A can be encompassed into one expression as,

$$A(x, y) = A_o e^{i(kR-\omega t)} \tag{5}$$

This is the often termed as the aperture function which when transferred inside the integral in equation (4) takes the form,

$$E(x', y') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(x, y)\, e^{-ik\left(\frac{yy'+xx'}{R}\right)} dx\, dy \tag{6}$$

The Fourier tranform of any function $f(x, y)$ is usually written in standard textbooks [7] as,

$$\mathcal{F}\{f(x,y)\} = F(\mu,\nu) = \int_{-\infty}^{-\infty}\int_{-\infty}^{-\infty} f(x,y)\, e^{-i2\pi(\mu x+\nu y)}\, dx\, dy \tag{7}$$

Equation (6) and (7) have a similar mathematical form which tells the reader that the electric field contribution at any point on the projection screen placed very far from a small opening is the Fourier transform of the aperture function. In other words, the electric field distribution of the obstructed light waves on the projection screen is the frequency spectrum of the aperture in space. Rewriting this in the usual notation of Fourier transforms,

$$E(x',y') = \mathcal{F}\{A(x,y)\} \tag{8}$$

The Fourier transform, as we know, gives the amplitudes of the constituent sinusoids (w.r.t spatial frequency) that make up the input signal. The energy associated with a harmonic wave is proportional to the square of its amplitude. This means the square of the transform gives a measure of the distribution of energy at each and every component frequency. Therefore, the modulus of the Fourier transform is often called the power spectrum and is identical to the irradiance distribution of the diffraction pattern on the projection screen.

With Fourier transforms, one can obtain the diffraction pattern of various aperture shapes. Take for example, a square opening whose diffraction pattern is in Fig. 2. In order to appreciate the pattern formed due a square opening, consider a 1D 'top-hat' function as described in equation (9). The steps to obtain the FT of this function is relatively simple; it is a sinc function as shown in equation (10). The top-hat is analogous to a thin slit. The diffraction pattern of a thin-slit is well-known from diffraction experiments [ref. Fig. 3].

$$A(x) = \begin{cases} 1 & \text{if } -\frac{W}{2} < x < \frac{W}{2}, \\ 0 & \text{if } elsewhere, \end{cases} \tag{9}$$

$$\mathcal{F}\{A(x)\} = F(\mu) = W\,\frac{sin(\pi\,\mu\,W)}{\pi\,W\,\mu} \tag{10}$$

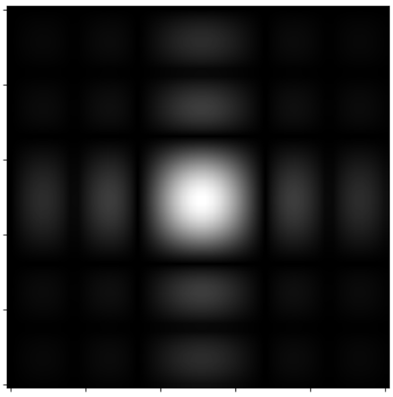


Fig. 2: Diffraction pattern of square aperture

Two 1D rectangular functions taken along two perpendicular axes when multiplied together give rise to a rectangular aperture. FT has an interesting property called *separability* [2] shown in eqn. (12). According to this property, the diffraction pattern of a rectangular aperture is two sinc functions multiplied together as we have already seen in Fig. 2.

$$A(x,y) = A(x)\,A(y) \tag{11}$$

$$\mathcal{F}\{A(x,y)\} = F(\mu,\nu) = W\,\frac{sin(\pi\,\mu\,W)}{\pi\,W\,\mu}\,W\,\frac{sin(\pi\,\nu\,W)}{\pi\,W\,\nu} \tag{12}$$

The FT of these aperture functions depends on their width; the wider the aperture, the smaller is the distance between consecutive maxima and minima [8]. This is known as the *scaling* property of Fourier transforms.

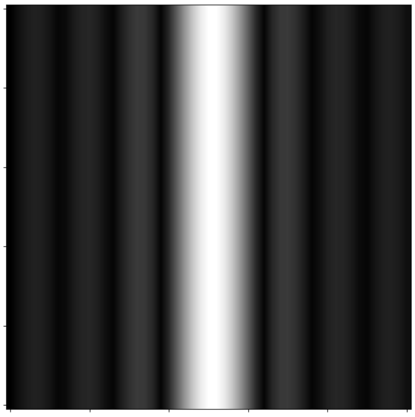
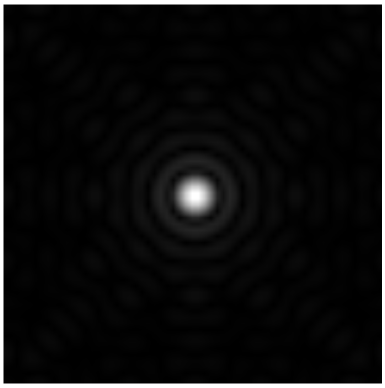


Fig. 3: Diffraction pattern of thin slit

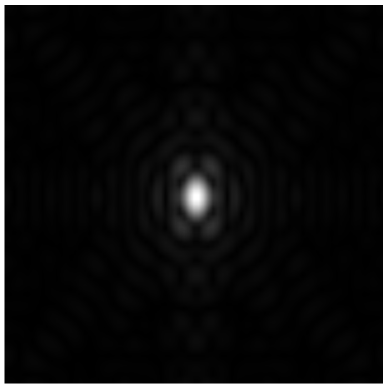
Linearity is yet another interesting property of Fourier transforms [9] which in the context of optics, is essentially the superposition principle of electromagnetic waves. The keyhole-shaped aperture in Fig. 4c, is an amalgamation of a circle and a rectangle. The linearity property of FT dictates that the diffraction pattern of this aperture as shown in Fig. 4d, is a superposition of the diffraction pattern of a circular aperture (Fig. 4b) and a rectangular aperture (Fig. 2). Thus, on one hand far-field diffraction provides a means to study properties of Fourier transforms and on the other, these properties can help us guess the diffraction pattern of arbitrary apertures. This is an interesting and useful connection that exists between Fourier transform and far-field diffraction.

(a)



(b)

(c)



(d)

Fig. 4: (a)Circular Aperture (b)Diffraction pattern of circular aperture (c)Keyhole shaped aperture (d)Diffraction pattern of keyhole aperture

All diffraction patterns in this article were generated using a simple discrete Fourier transform (DFT) algorithm. As the name suggests, it is a Fourier transform applied to a set of discrete points in a sample. The function has to be made discrete for the computer to be able to process it. Faster, and more efficient algorithms compared to the DFT have been invented since the 1960's called Fast Fourier Transfer (FFT) algorithms. These algorithms are much more appropriate for larger computations involving 2D images.

A simple activity such as this, that generates the diffraction pattern of arbitrary shapes can be a useful simulation exercise for students and at the same time, allow them to appreciate the connection between Fraunhofer diffraction and Fourier transform.

Image Processing using Fourier's methods

The transformation of an image into digital pixels, its enhancement and restoration has found applications in various fields of science today [10]. Every day, a large number of digital images are manipulated by means of mathematical methods and computer programming, in order to aid experiments in high-energy physics [11];restore and enhance images in astrophysics [12]; to detect and study diseases in the field of medicine ([13],[14]).

Image enhancement is the most common requirement in most industries today. Blurry images, low-contrast images, degraded images, they all need to be restored and enhanced in order to analyse them and make them more visually appealing to the human eye. Common photo editing tools in use today allow the user to apply various filtering techniques like smoothing and sharpening to enhance images. There are many ways of filtering so to speak but only two 'domains' in which it can be applied to an image. An image has two domains: the 'spatial' domain and the 'frequency' domain. The reader should by now anticipate Fourier's methods to come into the picture and recall the mathematical equation for the Fourier transformation of a function to be as follows,

$$\mathcal{F}\big\{f(x,y)\big\} = F(\mu,\nu) = \int_{-\infty}^{-\infty}\int_{-\infty}^{-\infty} f(x,y)\, e^{-i2\pi(\mu x+\nu y)}\, dx\, dy \tag{13}$$

On integrating the right hand side of equation (13), only the frequency variables μ and ν remain hence the original image is called the *spatial* domain and the Fourier transformation of the image is called the *frequency* domain.

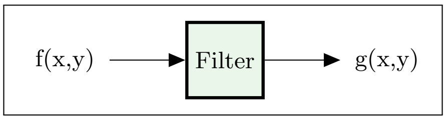


Fig. 5: Filtering in spatial domain

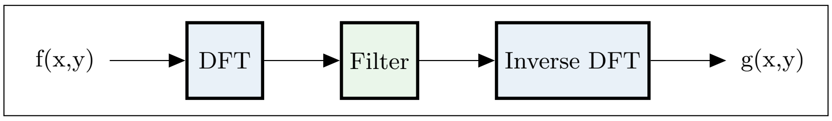


Fig. 6: Filtering in frequency domain

While spatial filtering techniques rely on manipulating the pixels of the original image matrix, frequency domain filtering techniques alter the pixels in the frequency domain of the image. Smoothing and sharpening of an image can be done easily by operating on the original image itself [ref. Fig. (5)]. The averaging filter is a basic smoothing filter that computes the average value of the neighbouring pixels of any given pixel in the image, achieving a blurring effect by removing noise and thus, beautifying the image. Contrastingly, a sharpening filter highlights the outlines of objects in an out-of-focus or blurry image. This filter works by differentiating intensity values in the input image and enhancing any sudden changes in the pixel values.

Smoothing and sharpening can also be performed on the Fourier transform of the image [ref. Fig. (6)]. Regions where the intensity values are rapidly changing are high frequency regions whereas, areas where the intensity values are not changing as rapidly are low frequency regions. Thus, a low-pass filter that passes lower frequencies and attenuates higher frequencies would serve the purpose of smoothing the image. On the other hand, a filter that attenuates lower frequencies would sharpen the image. A high pass filter essentially removes the DC term in the Fourier transform of an image. The DC term is $F(0,0)$ which is essentially the average of the original image $f(x,y)$ (notice the central bright spot in Fig.7b). So, when this zero frequency region is removed from the Fourier transform (Fig.7d), the edges are highlighted and the overall brightness of the image is diminished as seen in figure (7c).

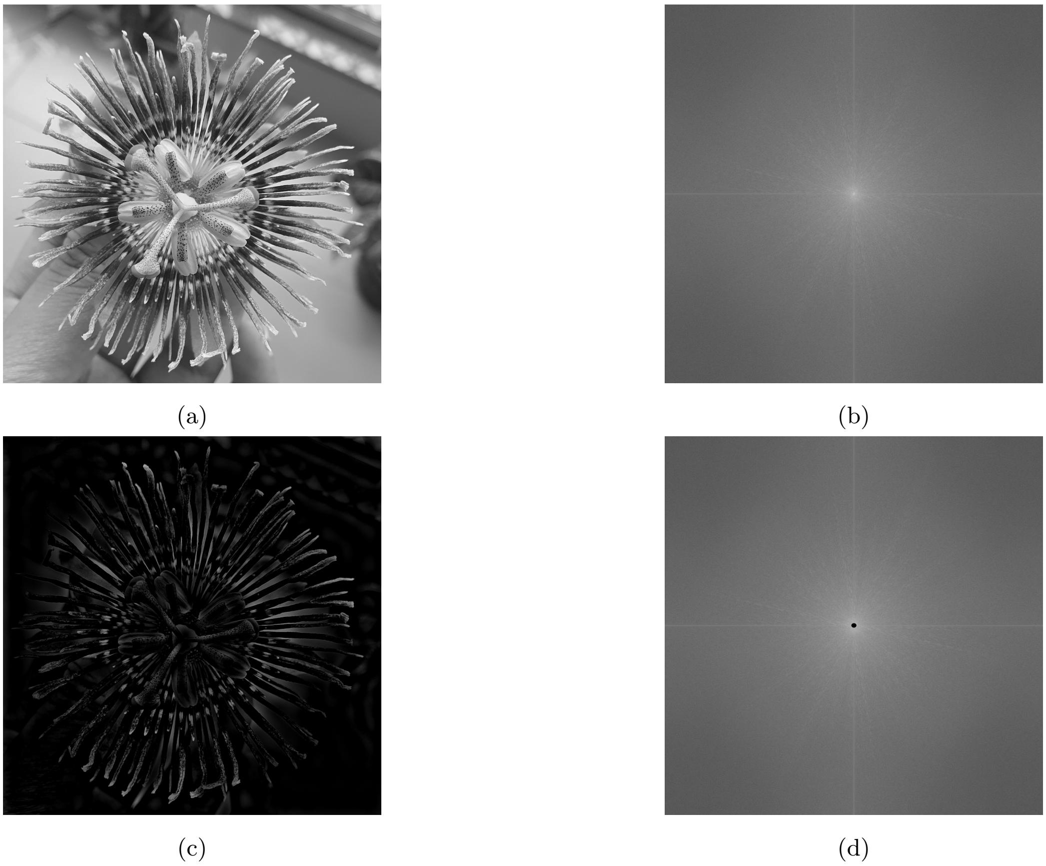


Fig. 7: (a) Grayscale image of Passiflora plant (b)Fourier transform of the image (c)Sharpened version of image (d) Ideal High Pass filter applied to Fourier transform of image

Filtering is just one of the ways Fourier transforms can be applied to 2D images. Images that have some kind of periodic noise (for e.g., sinusoidal), can also be easily removed from the FT of the image.

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