CSCI 567, Fall'18
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Homework #5 Written Assignments

Due: 11:59 pm, November 18, 2018

Instructions

Submission: Assignment submission will be via courses.uscden.net. By the submission date, there will be a folder set up in which you can submit your files. Please be sure to follow all instructions outlined here.

You can submit multiple times, but only *the last submission* counts. As a results, if you finish some problems, you might want to submit them first, and update later when you finish the rest. You are encouraged to do so. This way, if you forget to finish the homework on time or something happens (remember Murphy's Law), you still get credit for whatever you have turned in.

Problem sets must be typewritten or neatly handwritten when submitted. In both cases, your submission must be a single PDF. It is strongly recommended that you typeset with LaTeX. There are many free integrated LaTeX editors that are convenient to use (e.g Overleaf, ShareLaTeX). Choose the one(s) you like the most. This tutorial Getting to Grips with LaTeX is a good start if you do not know how to use LaTeX yet.

Please also follow the rules below:

- The file should be named as `firstname_lastname_USCID.pdf` e.g., `Don_Quijote_de_la_Mancha_8675309045.pdf`).
- Do not have any spaces in your file name when uploading it.
- Please include your name and USC ID in the header of the report as well.

Collaboration: You may discuss with your classmates. However, you need to write your own solutions and submit separately. Also in your report, you need to list with whom you have discussed for each problem. Please consult the syllabus for what is and is not acceptable collaboration. Review the rules on academic conduct in the syllabus: a single instance of plagiarism can adversely affect you significantly more than you could stand to gain.

Notes on notation:

- Unless stated otherwise, scalars are denoted by small letter in normal font, vectors are denoted by small letters in bold font and matrices are denoted by capital letters in bold font.
- $\|.\|$ means L2-norm unless specified otherwise i.e. $\|.\| = \|.\|_2$

Problem 1 Hidden Markov Models

Recall a hidden Markov model is parameterized by

- initial state distribution $P(Z_1 = s) = \pi_s$
- transition distribution $P(Z_{t+1} = s' \mid Z_t = s) = a_{s,s'}$
- emission distribution $P(X_t = o \mid Z_t = s) = b_{s,o}$

1.1 Suppose we observe a sequence of outcomes $x_1, \ldots, x_T$ and would like to predict the next state Z_{T+1}, that is, we want to figure out for each possible state s,

$$P(Z_{T+1} = s \mid X_{1:T} = x_{1:T}).$$

Write down how one can compute this probability using the the forward message:

$$\alpha_s(T) = P(Z_T = s, X_{1:T} = x_{1:T}).$$

1.2 More generally, suppose based on the same observation $x_1, \ldots, x_T$ we would like to predict the state at time $T + k$ for $k \geq 1$, that is, we want to figure out for each possible state s,

$$P(Z_{T+k} = s \mid X_{1:T} = x_{1:T}).$$

Write down how one can compute this probability by establishing a recursive form. In other words, express the above probability in terms of $P(Z_{T+k-1} = s' \mid X_{1:T} = x_{1:T})$ and the model parameters.

Problem 2 Principal Component Analysis

In the class we showed that PCA is finding the directions with most variance. In this problem, you will show that PCA is in fact also minimizing reconstruction error in some sense.

Specifically, suppose we have a dataset $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^D$ with zero mean, and we would like to compress it into a one-dimensional dataset $c_1, \ldots, c_N \in \mathbb{R}$. To reconstruct the dataset (approximately), we also keep a direction vector $\mathbf{v} \in \mathbb{R}^D$ with unit norm (i.e. $\|\mathbf{v}\|_2 = 1$) so that the reconstructed dataset is $c_1\mathbf{v}, \ldots, c_N\mathbf{v} \in \mathbb{R}^D$.

The way we find $c_1, \ldots, c_N$ and $\mathbf{v}$ is to minimize the reconstruction error in terms of L2 distance, that is, we solve

$$\operatorname*{arg\,min}_{c_1,\ldots,c_N,\mathbf{v}:\|\mathbf{v}\|_2=1} \sum_{n=1}^{N} \|\mathbf{x}_n - c_n\mathbf{v}\|_2^2.$$

Prove that the solution is exactly the following

1) $\mathbf{v}$ is the first principal component of the dataset;

2) $c_n = \mathbf{x}_n^T\mathbf{v}$ for each $n = 1, \ldots, N$.

Problem 3 Convergence of value iteration

Recall that the value function V of a Markov Decision Process is defined as

$$V(s) = \max_{a\in\mathcal{A}} \left(r_s(a) + \gamma \sum_{s'\in\mathcal{S}} P_a(s,s')V(s') \right)$$

where r is the reward function, P is the transition probability, $\mathcal{A}$ is the space of all possible actions and $\gamma \in (0,1)$ is some discount factor. Value iteration (approximately) finds this value function by starting with some initial guess $V_0(s)$ for each s and then repeatedly doing the following Bellman update

$$V_k(s) = \max_{a \in \mathcal{A}} \left(r_s(a) + \gamma \sum_{s' \in \mathcal{S}} P_a(s,s') V_{k-1}(s') \right).$$

3.1 Prove that V_k is getting closer and closer to V in the following sense:

$$\max_s |V_k(s) - V(s)| \leq \gamma \max_s |V_{k-1}(s) - V(s)|.$$

You can use the fact that for any two functions $f, g : \mathcal{A} \to \mathbb{R}$, we have

$$|\max_a f(a) - \max_a g(a)| \leq \max_a |f(a) - g(a)|,$$

although you are also encouraged to prove this fact.

3.2 Now suppose we initialize $V_0(s)$ to be 0 for all s, and assume $|r_s(a)| \leq 1$ for all s and a. Prove

$$\max_s |V_k(s) - V(s)| \leq \frac{\gamma^k}{1-\gamma}$$