Algebraic G^1 connection retaining fixed twist vectors

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Given three cubic B-spline curves (C, C_1 and C_2) with the *same* knot vector, find a sextic-by-linear B-spline surface S that (i) interpolates C at $v = 0$, (ii) interpolates $C_1 - C$ (*first* derivatives) at the endpoints, (iii) interpolates $C_1' - C'$ (*twist vectors*) at the endpoints, and (iv) is defined as a combination of the direction blend $D = \frac{1}{2}(C_1 - C_2)$ and C'. (Let us assume that the parameter interval is $[0, 1]$ for simplicity.)

In other words, we define the cross derivatives as

$$T(u) = D(u) \cdot \alpha(u) + C'(u) \cdot \beta(u), \tag{1}$$

where α and β are suitable scalar functions. Then the surface will be given as

$$S(u, v) = P(u) + T(u) \cdot v, \tag{2}$$

which can be represented as a sextic-by-linear B-spline when α is at most cubic and β is at most quartic.

With the above formula, (i) is obviously satisfied. From (ii), we have

$$D(0) \cdot \alpha(0) + C'(0) \cdot \beta(0) = C_1(0) - C(0), \tag{3}$$

so $[\alpha(0), \beta(0)]^T$ is the coordinate vector of $C_1(0) - C(0)$ in the $(D(0), C'(0))$ planar coordinate system. (Same for the $u = 1$ parameter.)

From (iii), we also have the same for the u-derivative:

$$D'(0) \cdot \alpha(0) + D(0) \cdot \alpha'(0) + C''(0) \cdot \beta(0) + C'(0) \cdot \beta'(0) = C_1'(0) - C'(0), \tag{4}$$

which leads to

$$D(0) \cdot \alpha'(0) + C'(0) \cdot \beta'(0) = C_1'(0) - C'(0) - D'(0) \cdot \alpha(0) - C''(0) \cdot \beta(0), \tag{5}$$

so $[\alpha'(0), \beta'(0)]^T$ is the coordinate vector of $C_1'(0) - C'(0) - D'(0) \cdot \alpha(0) - C''(0) \cdot \beta(0)$ in the $(D(0), C'(0))$ planar coordinate system. (Note that this means that this vector should *be* in that plane!)

After these observations, we can define the α, β scalar functions as a planar cubic Bézier curve with the control points

$$\begin{aligned} P_0 &= (\alpha(0), \beta(0)), & P_1 &= P_0 + \frac{1}{3}(\alpha'(0), \beta'(0)), \\ P_2 &= P_3 - \frac{1}{3}(\alpha'(1), \beta'(1)), & P_3 &= (\alpha(1), \beta(1)). \end{aligned} \tag{6}$$

Finally, the control points of the surface are computed by the B-spline multiplication method of Che et al. (2011).

Coordinate computation

Given three co-planar vectors (u, v and w), we can express u as $\alpha v + \beta w$ by solving a LSQ linear equation (when the vectors are co-planar the solution will be exact). So instead of solving the (seemingly) overdetermined

$$\begin{bmatrix} v_x & w_x \\ v_y & w_y \\ v_z & w_z \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}, \tag{7}$$

we solve

$$\begin{bmatrix} v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} \begin{bmatrix} v_x & w_x \\ v_y & w_y \\ v_z & w_z \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}, \tag{8}$$

which can be simplified to

$$\begin{bmatrix} \|v\|^2 & \langle v, w \rangle \\ \langle v, w \rangle & \|w\|^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \langle v, u \rangle \\ \langle w, u \rangle \end{bmatrix}. \tag{9}$$

The inverse of the matrix on the left side is

$$\frac{1}{\|v\|^2 \cdot \|w\|^2 - \langle v, w \rangle^2} \begin{bmatrix} \|w\|^2 & -\langle v, w \rangle \\ -\langle v, w \rangle & \|v\|^2 \end{bmatrix}, \tag{10}$$

from which we get

$$\begin{aligned} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} &= \frac{1}{\|v\|^2 \cdot \|w\|^2 - \langle v, w \rangle^2} \begin{bmatrix} \|w\|^2 & -\langle v, w \rangle \\ -\langle v, w \rangle & \|v\|^2 \end{bmatrix} \begin{bmatrix} \langle v, u \rangle \\ \langle w, u \rangle \end{bmatrix} \\ &= \frac{1}{\|v\|^2 \cdot \|w\|^2 - \langle v, w \rangle^2} \begin{bmatrix} \|w\|^2 \langle v, u \rangle - \langle v, w \rangle \langle w, u \rangle \\ \|v\|^2 \langle w, u \rangle - \langle v, w \rangle \langle v, u \rangle \end{bmatrix}. \end{aligned} \tag{11}$$

Bibliography

Che et al. (2011): *The product of two B-spline functions.* Advanced Materials Research, Vol. 186, pp. 445–448.