Computing derivatives based on surface curvatures

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At a corner the following quantities are known:

- First derivative (S_u)
- First cross-derivative (S_v)
- Normal vector ($n = (S_u \times S_v)/\|S_u \times S_v\|$)
- Principal directions ($e_{\max}$, $e_{\min}$)
- Principal curvatures ($\kappa_{\max}$, $\kappa_{\min}$)

Based on the above information, we need to compute S_{uu}, S_{vv} and S_{uv}.

Let ϑ_u (ϑ_v) be the oriented angles between S_u (S_v) and $e_{\max}$. Then, according to Euler's theorem

$$\begin{aligned}\kappa_u &= \kappa_{\max}\cos^2\vartheta_u + \kappa_{\min}\sin^2\vartheta_u,\\ \kappa_v &= \kappa_{\max}\cos^2\vartheta_v + \kappa_{\min}\sin^2\vartheta_v,\end{aligned} \tag{1}$$

where κ_u and κ_v are the normal curvatures in the u and v directions. We also know that

$$L = \kappa_u E, \qquad\qquad N = \kappa_v G, \tag{2}$$

or, equivalently,

$$S_{uu}n = \kappa_u\|S_u\|^2, \qquad\qquad S_{vv}n = \kappa_v\|S_v\|^2, \tag{3}$$

which constrains the "height" of the second derivatives. Similarly, from

$$M^2 = LN - K(EG - F^2), \tag{4}$$

or, equivalently,

$$S_{uv}n = \pm\sqrt{(S_{uu}n)(S_{vv}n) - \kappa_{\max}\kappa_{\min}\|S_u \times S_v\|^2}, \tag{5}$$

we can determine the height of the twist vector; its position in that plane can be assigned using the parallelogram rule or some other heuristics.

Computing oriented angles

First express both vectors in their common plane, so $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$. Then the angle can be computed by

$$\vartheta = \operatorname{atan2}(x_1 y_2 - y_1 x_2, x_1 x_2 + y_1 y_2).$$

But note that we do not really need oriented angles, as all (co)sines are squared.