Stoke's theorem :-

(Transformation between Line Integral and Surface Integral)

Statement: Let s be a open surface bounded by a closed, non intersecting curve C. If $\bar{F}$ is any differentiable vector point function then

$\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\, \bar{F}\cdot \bar{n}\, ds$ where C is traversed in the positive direction and $\bar{n}$ is unit outward drawn normal at any point of the surface.

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① Verify stoke's theorem for the function $\bar{F} = x^2\bar{i} + xy\bar{j}$ integrated round the square in the plane $z=0$ whose sides are along the lines $x=0,\ y=0,\ x=a,\ y=a$.

Sol: Given $\bar{F} = x^2\bar{i} + xy\bar{j}$

S: Square in the plane $z=0$ with sides $x=0$, $y=0,\ x=a,\ y=a$.

To verify stoke's theorem, we have to prove $\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\, \bar{F}\cdot\bar{n}\, ds$

i.e., $I_C = I_S$

To find I_S i.e., $\int_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds$

$$\operatorname{Curl}\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xy & 0 \end{vmatrix}$$

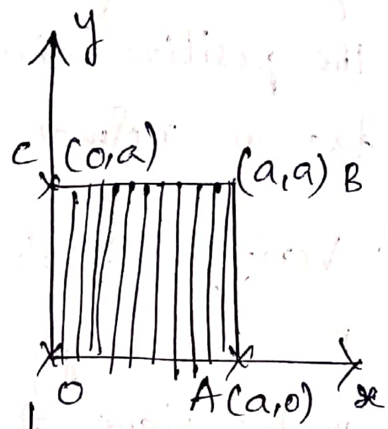
$$= \bar{k}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= y\bar{k}$$

Here $\bar{n} = \bar{k}$

$\therefore \operatorname{Curl}\bar{F}\cdot\bar{n} = y.$

Let R is the projection on xy plane.



then $\int_S \operatorname{curl}\bar{F}\cdot\bar{n}\,ds$

$$= \iint \operatorname{curl}\bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \int_{x=0}^{a}\int_{y=0}^{a} y\,dy\,dx$$

y : $y=0$ to $y=a$

x limits : $x=0$ to $x=a$

$$= \int_{x=0}^{a} \left.\frac{y^2}{2}\right]_0^a dx = \frac{a^2}{2}\int_0^a dx$$

$$= \frac{a^2}{2}\,x\Big]_0^a = \boxed{\frac{a^3}{2} = I_S} \quad \text{—— (A)}$$

To find I_C i.e $\oint \bar{F}\cdot d\bar{r}$

Here $I_C = I_{OA} + I_{AB} + I_{BC} + I_{CO}$

$\bar{F} = x^2\,\bar{i} + xy\,\bar{j}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\bar{F}\cdot d\bar{r} = x^2\,dx + xy\,dy.$

$\therefore\; I_C = \oint_C \bar{F}\cdot d\bar{r} = \oint_C x^2\,dx + xy\,dy.$

i) Along OA:- Equation of OA: $y=0$

$dy = 0$

$$I_{OA} = \int_0^A x^2\,dx + xy\,dy = \int_0^A x^2\,dx$$

$$= \left.\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}.$$

ii) Along AB:- Equation of AB: $x=a$

$\Rightarrow dx = 0.$

$$I_{AB} = \int_A^B x^2\,dx + xy\,dy = \int_A^B ay\,dy$$

$$= \left.\frac{ay^2}{2}\right]_0^a = \frac{a^3}{2}$$

iii) Along BC: Equation of BC: $y=a$

$\Rightarrow dy = 0$

$$I_{BC} = \int_B^C x^2\,dx + xy\,dy = \int_B^C x^2\,dx$$

$$= \left.\frac{x^3}{3}\right]_a^0 = -\frac{a^3}{3}$$

iv) Along CO: Equation of CO: $x=0$

$\Rightarrow dx = 0$

$$I_{CO} = \int_C^O x^2\,dx + xy\,dy = 0.$$

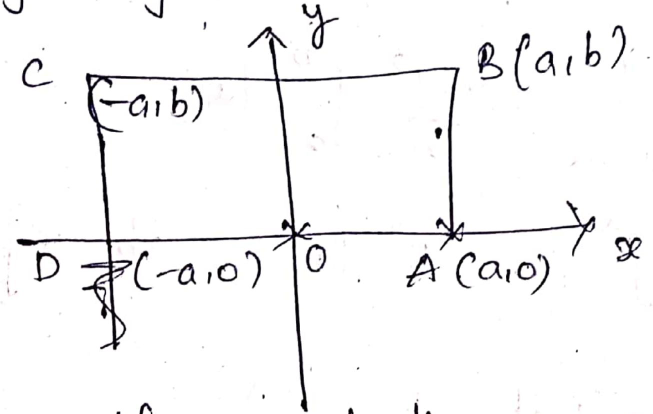
$$\therefore\; I_C = \frac{\cancel{a^3}}{\cancel{3}} + \frac{a^3}{2} - \frac{\cancel{a^3}}{\cancel{2}} + 0 = \frac{a^3}{2} \quad \text{——— (B)}$$

from (A) & (B), Stoke's theorem is verified.

(2) Verify Stoke's theorem, for $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ taken round the rectangle bounded by the lines $x = \pm a, y=0, y=b$.

<u>Sol:</u> Given $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$

S: Rectangle bounded by the lines $x = \pm a$, $y=0$, $y=b$.



To verify Stoke's theorem, we have to prove $\int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = \oint_C \bar{F}\cdot d\bar{r}$

$$I_S = I_C$$

<u>To find I_S</u>

$$\text{Curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x^2+y^2) & -2xy & 0 \end{vmatrix}$$

$$= 0\bar{i} - 0\bar{j} + \bar{k}\left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2+y^2)\right]$$

$$= \bar{k}\left[-2y-2y\right] = -4y\bar{k}$$

Here $\bar{n} = \bar{k}$ [$\because$ Rectangle is present in xy-plane]

$$\text{Curl}\,\bar{F}\cdot\bar{n} = -4y.$$

Let R is the projection of S on xy-plane then $I_S = \int_S \operatorname{curl} \bar{F} \cdot \bar{n}\, ds$

$$= \iint_R \operatorname{curl} \bar{F} \cdot \bar{n}\, dx\,dy$$

$$= \int_{x=-a}^{a} \int_{y=0}^{b} -4y\, dy\, dx$$

$$= \int_{x=-a}^{a} -4 \frac{y^2}{2} \Big]_0^b dx$$

$$= \int_{x=-a}^{a} -2b^2\, dx = -2b^2 x \Big]_{-a}^{a}$$

$$I_S = -4ab^2 \quad \text{—— (A)}$$

To find I_C

Here $I_C = I_{AB} + I_{BC} + I_{CD} + I_{DA}$.

~~i) Along AB: Equation of AB: $y=0$~~

$$I_C = \oint_C \bar{F} \cdot d\bar{r}$$

Given $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$

$$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

$$\bar{F} \cdot d\bar{r} = (x^2+y^2)\, dx - 2xy\, dy.$$

$$I_C = \oint_C \bar{F} \cdot d\bar{r} = \oint_C (x^2+y^2)\, dx - 2xy\, dy$$

i) Along AB: Equation of AB: $x=a$

$$\Rightarrow dx = 0$$

$$I_{AB} = \int_A^B (x^2+y^2)\,dx - 2xy\,dy$$

$$= \int_A^B -2ay\,dy = -\frac{\cancel{2}ay^2}{\cancel{2}}\Big]_0^b = -ab^2$$

ii) **Along BC**: Equation of BC: $y = b$

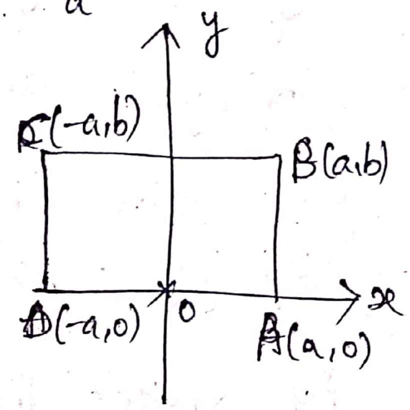
$dy = 0$

$$I_{BC} = \int_B^C (x^2+y^2)\,dx - 2xy\,dy$$

$$= \int_B^C (x^2+b^2)\,dx = \frac{x^3}{3} + b^2x\Big]_a^{-a}$$

$$= -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$$

$$= -\frac{2a^3}{3} - 2ab^2$$



iii) Along CD:- Eq'n of CD

$x = -a$

$\Rightarrow dx = 0$

$$I_{CD} = \int_C^D (x^2+y^2)\,dx - 2xy\,dy$$

$$= \int_b^0 +2ay\,dy = \frac{\cancel{2}ay^2}{\cancel{2}}\Big]_b^0 = -ab^2$$

iv) Along DA: Eq'n of DA: $y = 0$

$dy = 0$

$$I_{DA} = \int_D^A (x^2+y^2)\,dx - 2xy\,dy = \int_D^A x^2\,dx$$

$$= \frac{x^3}{3}\Big]_{-a}^{a} = \frac{a^3}{3} + \frac{a^3}{3} = \frac{2a^3}{3}$$

$$\therefore I_C = -ab^2 - \cancel{\frac{2a^3}{3}} - 2ab^2 - ab^2 + \cancel{\frac{2a^3}{3}} = -4ab^2 \quad (B)$$

from (A) & (B), Stoke's theorem is verified.

(3) Verify Stoke's theorem for $\bar{F} = (x^2 - y^2)\bar{i} + 2xy\bar{j}$ over the box bounded by the planes $x = 0, x = a, y = 0, y = b$.

(4) Verify Stoke's theorem for $\bar{F} = y^2\bar{i} - 2xy\bar{j}$ taken round the rectangle bounded by $x = \pm b, y = 0, y = a$.

(5) Evaluate by Stoke's theorem $\int_C e^x dx + 2y\,dy - dz$ where C is the curve $x^2 + y^2 = 9$ and $z = 2$

Sol: Given $I_C = \int_C e^x dx + 2y\,dy - dz$

C: curve $x^2 + y^2 = 9$ and $z = 2$

By Stoke's theorem, $\int_S \text{Curl}\bar{F} \cdot \bar{n}\, ds = \oint_C \bar{F} \cdot d\bar{r}$

$$I_S = I_C.$$

Comparing given I_C with $\oint_C \bar{F} \cdot d\bar{r}$

Here $\bar{F} \cdot d\bar{r} = e^x dx + 2y\,dy - dz$

$\Rightarrow \quad = (e^x\bar{i} + 2y\bar{j} - \bar{k}) \cdot (dx\bar{i} + dy\bar{j} + dz\bar{k})$

$\Rightarrow \bar{F} = e^x\bar{i} + 2y\bar{j} - \bar{k}$

$$\text{Curl}\,\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x & 2y & -1 \end{vmatrix}$$

$= \bar{i}[0] - \bar{j}[0] + \bar{k}[0]$

$= \bar{0}$

$\because I_C = I_S$

$= \int_S \text{curl}\,\bar{F} \cdot \bar{n}\, ds$

$= \int_S \bar{0} \cdot \bar{n}\, ds = \int_S 0\, ds = 0.$

Gauss divergence theorem:-

(Transformation between Surface Integral and Volume Integral)

Statement:- Let S be a closed surface enclosing a volume V. If $\bar{F}$ is continuously differentiable vector point function, then

$$\int_V \text{div}\,\bar{F}\, dv = \int_S \bar{F} \cdot \bar{n}\, ds,$$

where $\bar{n}$ is the outward drawn normal vector at any point of S.

① Verify Gauss divergence theorem for $\bar{F} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$ over cube formed by the planes $x=0$, $x=a$, $y=0$, $y=b$, $z=0$, $z=c$.

Sol: Given $\bar{F} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$

and V: Cube formed by the planes $x=0$, $x=a$, $y=0$, $y=b$, $z=0$, $z=c$

To verify Gauss divergence theorem, we have to prove $\int_V \text{div}\,\bar{F}\, dv = \int_S \bar{F} \cdot \bar{n}\, ds$

i.e., $I_V = I_S$.

To find I_v i.e., $\int_V \operatorname{div}\bar{F}\, dv$

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(x^2\bar{i} + y^2\bar{j} + z^2\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

$$I_v = \int_V (2x+2y+2z)\, dv$$

$$= \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} 2x+2y+2z \; dz\, dy\, dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} 2xz + 2yz + z^2\Big]_0^c \, dy\, dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} 2xc + 2yc + c^2 \, dy\, dx$$

$$= \int_{x=0}^{a} 2xcy + cy^2 + c^2y\Big]_0^b \, dx$$

$$= \int_{x=0}^{a} \left(2xbc + b^2c + bc^2\right) dx$$

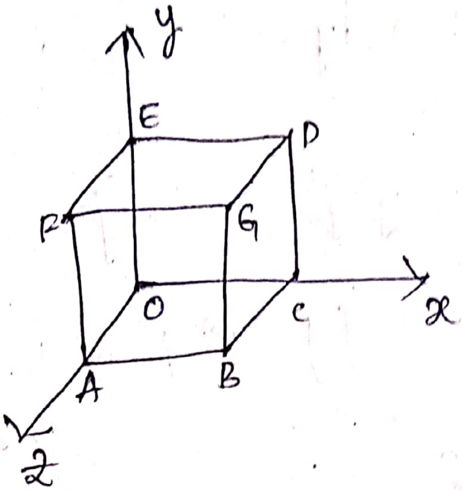
$$= x^2bc + b^2c\cdot x + bc^2x\Big]_0^a$$

$$= a^2bc + ab^2c + abc^2$$

$$I_v = abc\,[a+b+c] \quad \text{——— (A)}$$

To find I_s

Here $I_s = I_{s_1} + I_{s_2} + I_{s_3} + I_{s_4} + I_{s_5} + I_{s_6}$



i) On S_1: $x=0$

$\Rightarrow dx=0$

$\bar{n}=-\bar{i}$

$\bar{F}\cdot\bar{n} = x^2\bar{i}+y^2\bar{j}+z^2\bar{k}\cdot -\bar{i} = -x^2=0$

$\Rightarrow I_{S_1} = \int_{S_1} \bar{F}\cdot\bar{n}\, ds = \boxed{0 = I_{S_1}}$

ii) On S_2: $x=a \Rightarrow dx=0$

here $\bar{n}=\bar{i}$

$\bar{F}\cdot\bar{n} = x^2\bar{i}+y^2\bar{j}+z^2\bar{k}\cdot\bar{i} = x^2=a^2$

$I_{S_2} = \int_{S_2} \bar{F}\cdot\bar{n}\, ds = \int_{y=0}^{b}\int_{z=0}^{c} a^2 \frac{\cancel{dy}\,dz\,dy}{|\bar{n}\cdot\bar{i}|}$

$= \int_{y=0}^{b} a^2 z\Big]_0^c dy = \int_{y=0}^{b} a^2 c\, dy$

$= a^2 c\, y\Big]_0^b = \boxed{a^2bc = I_{S_2}}$

iii) On S_3: $y=0 \Rightarrow dy=0$

here $\bar{n}=-\bar{j}$

$\bar{F}\cdot\bar{n} = (x^2\bar{i}+y^2\bar{j}+n^2\bar{j})\cdot -\bar{j} = -y^2=0$

$\Rightarrow \boxed{I_{S_3}=0}$

iv) On S_4 ! $y = b \Rightarrow dy = 0$

here $\bar{n} = \bar{j}$

$$\bar{F}\cdot\bar{n} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k} \cdot \bar{j} = y^2 = b^2$$

$$I_{S_4} = \int_{S_4} \bar{F}\cdot\bar{n}\, ds = \iint \bar{F}\cdot\bar{n}\, \frac{dx\,dz}{|\bar{n}\cdot\bar{j}|}$$

$$= \int_{x=0}^{a} \int_{z=0}^{c} b^2 \, dz\, dx$$

$$= \int_{x=0}^{a} b^2 z \Big]_0^c dx = \int_{x=0}^{a} b^2 c \, dx$$

$$= x b^2 c \Big]_0^a = \boxed{ab^2c = I_{S_4}}$$

v) On S_5 ! $z = 0 \Rightarrow dz = 0$

here $\bar{n} = -\bar{k}$

$$\bar{F}\cdot\bar{n} = (x^2\bar{i} + y^2\bar{j} + z^2\bar{k})\cdot -\bar{k} = -z^2 = 0$$

$$\boxed{I_{S_5} = 0}$$

vi) On S_6 ! $z = c \Rightarrow dz = 0$

here $\bar{n} = \bar{k}$

$$\bar{F}\cdot\bar{n} = (x^2\bar{i} + y^2\bar{j} + z^2\bar{k})\cdot\bar{k} = z^2 = c^2$$

$$I_{S_6} = \int_S \bar{F}\cdot\bar{n}\, ds = \iint \bar{F}\cdot\bar{n}\, \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} c^2\, dy\, dx = \int_{x=0}^{a} c^2 y \Big]_0^b dx$$

$$= \int_{x=0}^{a} c^2 b\, dx = bc^2 x\Big]_0^a = abc^2$$

$\therefore I_S = 0 + a^2bc + ab^2c + 0 + 0 + abc^2$

$= abc(a+b+c)$. — (B)

Hence, Gauss ~~divergence~~ theorem is verified.

(2) Verify Gauss divergence theorem for $\bar{F} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}$ taken over the surface of the cube bounded by the planes $x = y = z = a$ and coordinate planes.

Sol: Given $\bar{F} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}$

S: Surface of the cube bounded by the planes $x = a, y = a, z = a$ and coordinate planes $(x = 0, y = 0, z = 0)$.

To verify Gauss divergence theorem, we have to prove $\int_V \operatorname{div}\bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds$.

i.e., $I_V = I_S$

To find I_V

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}$$

$$= \frac{\partial}{\partial x}(x^3 - yz) + \frac{\partial}{\partial y}(-2x^2y) + \frac{\partial}{\partial z}(z)$$

$$= 3x^2 - 2x^2 + 1 = x^2 + 1$$

$$I_V = \int_V \operatorname{div}\bar{F}\, dv = \iiint_V (x^2+1)\, dx\, dy\, dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} (x^2+1)\, dx\, dy\, dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a} \left(\frac{x^3}{3} + x\right)_0^a dy\, dz.$$

$$I_V = \int_{z=0}^{a} \int_{y=0}^{a} \left[\frac{a^3}{3}+a\right] dy\, dz$$

$$= \left(\frac{a^3}{3}+a\right) \int_{z=0}^{a} y\Big]_0^a dz$$

$$= \left(\frac{a^3}{3}+a\right) \int_{z=0}^{a} a\, dz = \left(\frac{a^3}{3}+a\right) az\Big]_0^a$$

$$= \left(\frac{a^3}{3}+a\right) a^2 = \frac{a^5}{3}+a^3.$$

$$\boxed{\therefore\ I_V = \frac{a^5}{3}+a^3} \text{ ——— } (A)$$

<u>To find I_S</u> i.e., $\int_S \bar{F}\cdot\bar{n}\, ds$.

Here $I_S = I_{S_1} + I_{S_2} + I_{S_3} + I_{S_4} + I_{S_5} + I_{S_6}$

i) On S_1, $x=0 \Rightarrow dx=0$

here $\bar{n} = -\bar{i}$

$$\bar{F}\cdot\bar{n} = \left[(x^3-yz)\bar{i} - 2x^2y\bar{j} + 2\bar{k}\right]\cdot -\bar{i}$$

$$= -x^3+yz = yz.$$

$$I_{S_1} = \int_{S_1} \bar{F}\cdot\bar{n}\, ds = \iint \bar{F}\cdot\bar{n} \frac{dy\, dz}{|\bar{n}\cdot\bar{i}|}$$

$$= \int_{z=0}^{a}\int_{y=0}^{a} yz\, dy\, dz$$

$$= \int_{z=0}^{a} \frac{y^2}{2} z\Big]_0^a dz$$

$$= \int_{z=0}^{a} \frac{a^2}{2} z\, dz = \frac{a^2}{2}\cdot\frac{z^2}{2}\Big]_0^a = \frac{a^4}{4}$$

2) On S_2 : $x = a \Rightarrow dx = 0$

here $\bar{n} = \bar{i}$

$$F \cdot \bar{n} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k} \cdot \bar{i}$$

$$= (x^3 - yz) = a^3 - yz$$

$$I_{S_2} = \int F \cdot \bar{n}\, ds = \iint F \cdot \bar{n} \frac{dy\,dz}{|\bar{n} \cdot \bar{i}|}$$

$$= \int_{z=0}^{a} \int_{y=0}^{a} (a^3 - yz)\, dy\, dz$$

$$= \int_{z=0}^{a} \left(a^3 y - \frac{y^2}{2} z\right)_0^a dz$$

$$= \int_{z=0}^{a} \left(a^4 - \frac{a^2}{2} z\right) dz$$

$$= \left[a^4 z - \frac{a^2}{2} \cdot \frac{z^2}{2}\right]_0^a = a^5 - \frac{a^4}{4}$$

(3) On S_3 : $y = 0 \Rightarrow dy = 0$

here $\bar{n} = -\bar{j}$

$$F \cdot \bar{n} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k} \cdot -\bar{j}$$

$$= 2x^2y = 0$$

$$\therefore I_{S_3} = 0$$

(4) On S_4 : $y = a \Rightarrow dy = 0$

here $\bar{n} = \bar{j}$

$$F \cdot \bar{n} = \left[(x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}\right] \cdot \bar{j}$$

$$= -2x^2y = -2x^2a$$

$$I_{S_4} = \int F \cdot \bar{n}\, ds = \iint F \cdot \bar{n} \frac{dx\,dz}{|\bar{n} \cdot \bar{j}|}$$

$$I_{S_4} = \int_{z=0}^{a}\int_{x=0}^{a} -2x^2 a \, dx \, dz$$

$$= \int_{z=0}^{a} -\frac{2x^3}{3}\cdot a \Big]_0^a dz$$

$$= \int_{z=0}^{a} -\frac{2a^4}{3} dz = -\frac{2a^4}{3} z \Big]_0^a$$

$$\boxed{I_{S_4} = \frac{-2a^5}{3}}$$

(5) On S_5 : $z=0 \Rightarrow dz=0$

Here $\bar{n} = -\bar{k}$

$$F\cdot\bar{n} = \left((x^3 - yz)\bar{i} - 2yx^2\bar{j} + z\bar{k}\right)\cdot -\bar{k}$$

$$= -z = 0$$

$$\boxed{\therefore I_{S_5} = 0}$$

(6) On S_6 ! $z = a \Rightarrow dz = 0$

here $\bar{n} = \bar{k}$

$$F\cdot\bar{n} = \left[(x^3 - yz)\bar{i} - 2x^2 y\bar{j} + z\bar{k}\right]\cdot\bar{k}$$

$$= z = a$$

$$I_{S_6} = \int F\cdot\bar{n}\, ds = \iint F\cdot\bar{n}\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \int_{y=0}^{a}\int_{x=0}^{a} a\, dx\, dy = \int_{y=0}^{a} ax\Big]_0^a dy$$

$$= \int_{y=0}^{a} a^2 dy = a^2 y\Big]_0^a = a^3 = I_{S_6}$$

$$\therefore I_S = \cancel{\frac{a^4}{4}} + a^5 - \cancel{\frac{a^4}{4}} - \frac{2a^5}{3} + a^3$$

$$= \frac{a^5}{3} + a^3 \quad \text{(B)}$$ Here (A) & (B) are equal.

$\therefore$ Gauss divergence theorem verified.

(3) Show that $\int_S (ax\bar{i} + by\bar{j} + cz\bar{k}) \cdot \bar{n}\, ds = \frac{4\pi}{3}(a+b+c)$

where S is the surface of the sphere

$x^2 + y^2 + z^2 = 1$

Sol: Given $I_S = \int_S (ax\bar{i} + by\bar{j} + cz\bar{k}) \cdot \bar{n}\, ds$

Here S is surface of the sphere $x^2+y^2+z^2=1$.

Comparing given integral with $\int_S \bar{F} \cdot \bar{n}\, ds$

Here $\bar{F} = ax\bar{i} + by\bar{j} + cz\bar{k}$.

By Gauss divergence theorem,

$$\int_S \bar{F} \cdot \bar{n}\, ds = \int_V \operatorname{div} \bar{F}\, dv$$

i.e., $I_S = I_V$

$$\operatorname{div} \bar{F} = \nabla \cdot \bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot (ax\bar{i} + by\bar{j} + cz\bar{k})$$

$$= \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)$$

$$= a+b+c.$$

$$I_S = I_V = \int_V \operatorname{div} \bar{F}\, dv$$

$$= \int_V (a+b+c)\, dv$$

$$= (a+b+c)V,$$

V is the volume of the sphere $x^2+y^2+z^2=1$

$$= (a+b+c)\frac{4}{3}\pi r^3$$

here $r=1$

$$= \frac{4}{3}\pi(a+b+c).$$

(4) Verify divergence theorem for $\bar{F} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$ over the surfaces of the solid cut off by the plane $x+y+z=a$ in the first octant.

Sol: Given $\bar{F} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$

S: Surface of the solid cutt off by plane $x+y+z=a$ in the first octant

To verify Gauss divergence theorem, we have to prove $\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv$

i.e., $I_S = I_V$

To find I_V

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot(x^2\bar{i} + y^2\bar{j} + z^2\bar{k})$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

$$I_V = \int_V \operatorname{div}\bar{F}\,dv$$

$$= \iiint \operatorname{div}\bar{F}\,dx\,dy\,dz$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}(2x+2y+2z)\,dz\,dy\,dx$$

z limits

$z=0$ to $z=a-x-y$

y limits (Put $z=0$)

$y=0$ to $y=a-x$

x-limits (Put $y, z=0$)

$x=0$ to $x=a$.

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\left[2xz + 2yz + z^2\right]_0^{a-x-y} dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} 2x(a-x-y) + 2y(a-x-y) + (a-x-y)^2 \, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} (a-x-y)\left[2x+2y+a-x-y\right] dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} (a-(x+y))(a+(x+y)) \, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} \left[a^2-(x+y)^2\right] dy\, dx$$

$$= \int_{x=0}^{a} \left[a^2y - \frac{(x+y)^3}{3}\right]_0^{a-x} dx$$

$$= \int_{x=0}^{a} a^2(a-x) - \frac{[x+(a-x)]^3}{3} - 0 + \frac{x^3}{3} \, dx$$

$$= \int_{x=0}^{a} a^3 - a^2x - \frac{a^3}{3} + \frac{x^3}{3} \, dx$$

$$= \left[a^3x - \frac{a^2x^2}{2} - \frac{a^3}{3}x + \frac{x^4}{12}\right]_0^a$$

$$= a^4 - \frac{a^4}{2} - \frac{a^4}{3} + \frac{a^4}{12} = \frac{12a^4 - 6a^4 - 4a^4 + a^4}{12}$$

$$= \frac{3a^4}{12} = \boxed{\frac{a^4}{4} = I_V} \text{ --- (A)}$$

To find I_S: i.e., $\int_S \bar{F} \cdot \bar{n} \, ds$

To find $\bar{n}$ $\quad \bar{n} = \frac{\nabla \phi}{|\nabla \phi|}$ [unit normal vector to the surface]

Here $\phi : x+y+z-a=0$.

$$\frac{\partial \phi}{\partial x} = 1 \quad \Big| \quad \frac{\partial \phi}{\partial y} = 1 \quad \Big| \quad \frac{\partial \phi}{\partial z} = 1$$

$$\therefore \nabla \phi = \bar{i} + \bar{j} + \bar{k}$$

$$|\nabla \phi| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\therefore \bar{n} = \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{3}}.$$

$$F \cdot \bar{n} = \left(x^2 \bar{i} + y^2 \bar{j} + z^2 \bar{k}\right) \cdot \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{3}}$$

$$= \frac{x^2}{\sqrt{3}} + \frac{y^2}{\sqrt{3}} + \frac{z^2}{\sqrt{3}}$$

Let R is the projection of surface on xy plane. then put $z=0$ in the surface eq'n to find the limits.

y limits: $y=0$ to $y = a - x$.

x limits $x=0$ to $x=a$ [Put $y=0$ $z=0$]

$$\therefore I_s = \int_S F \cdot \bar{n}\, ds = \iint F \cdot \bar{n} \frac{dx\,dy}{|\bar{n} \cdot k|}$$

$$= \iint_R \frac{x^2 + y^2 + z^2}{\cancel{\sqrt{3}}} \cdot \frac{dx\,dy}{\cancel{1/\sqrt{3}}}$$

$$= \iint_R (x^2 + y^2 + z^2)\, dx\, dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} x^2 + y^2 + (a-x-y)^2\, dy\, dx$$

$$= \int_{x=0}^{a} \left[x^2 y + \frac{y^3}{3} + \frac{(a-x-y)^3}{-3} \right]_0^{a-x} dx$$

$$= \int_{x=0}^{a} x^2(a-x) + \frac{(a-x)^3}{3} + 0 + \frac{(a-x)^3}{+3} dx$$

$$= \int_{x=0}^{a} \left[ax^2 - x^3 + \frac{2}{3}(a-x)^3 \right] dx$$

$$= \frac{ax^3}{3} - \frac{x^4}{4} + \frac{2}{3}\frac{(a-x)^4}{-4} \Big]_0^a$$

$$= \frac{a^4}{3} - \frac{a^4}{4} + 0 + \frac{1}{6}a^4$$

$$= \frac{4a^4 - 3a^4 + 2a^4}{12} = \frac{3a^4}{12} = \frac{a^4}{4} \quad \text{— (B)}$$

Here from (A) & (B) $I_S = I_V$

$\therefore$ Gauss divergence theorem is verified.