Volume Integrals :- If V is the volume bounded by a surface, then the integration evaluated over the volume is called volume integration.

i) If ϕ is any scalar point function

then $\int_V \phi \, dv = \iiint \phi \, dx\,dy\,dz$

ii) If $\bar{F} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$ is any vector point function, then

$$\int_V \bar{F} dV = \bar{i} \iiint f_1 \, dx\,dy\,dz + \bar{j} \iiint f_2 \, dx\,dy\,dz + \bar{k} \iiint f_3 \, dx\,dy\,dz.$$

Ex:- ① If $\phi = 45x^2y$ evaluate $\iiint_V \phi \, dv$ where V is the closed region bounded by the planes $4x + 2y + z = 8$, $y=0$, $z=0$, $x=0$.

Sol: Given $\phi = 45x^2y$.

let $I_V = \iiint_V \phi \, dv$

where V = closed region bounded by the planes $4x + 2y + z = 8$, $y=0$, $z=0$, $x=0$.

limits :- $z = 0$ to $\boxed{z = 8 - 2y - 4x.}$

y limits :- $y = 0$ to $4x + 2y + z = 8$ (Put $z=0$)

$4x + 2y = 8$

$2y = 8 - 4x$

$\boxed{y = 4 - 2x}$

x limits : $x = 0$ to $4x + 2y + z = 8$ (Put $z=0$, $y=0$)

$\Rightarrow 4x = 8$

$\boxed{x = 2}$

$$\therefore I_V = \int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{8-2y-4x} 45x^2y \, dz \, dy \, dx$$

[Keep x and y constant]

$$= \int_{x=0}^{2} \int_{y=0}^{4-2x} 45x^2y \; z \Big]_0^{8-2y-4x} dy \, dx.$$

$$= \int_{x=0}^{2} \int_{y=0}^{4-2x} 45x^2y \, (8-2y-4x) \, dy \, dx.$$

$$= \int_{x=0}^{2} \int_{y=0}^{4-2x} (360x^2y - 90x^2y^2 - 180x^3y) \, dy \, dx.$$

[Keep x constant]

$$= \int_{x=0}^{2} \left[\overset{180}{\cancel{360}}x^2 \frac{y^2}{\cancel{2}} - \overset{30}{\cancel{90}}x^2 \frac{y^3}{\cancel{3}} - \overset{90}{\cancel{180}}x^3 \frac{y^2}{\cancel{2}} \right]_0^{4-2x} dx.$$

$$= \int_{x=0}^{2} 180x^2(4-2x)^2 - 30x^2(4-2x)^3 - 90x^3(4-2x)^2 \, dx$$

$$= \int_{x=0}^{2} 180x^2[16+4x^2-16x] - 30x^2[64-8x^3-96x+48x^2] - 90x^3[16+4x^2-16x] \, dx$$

$$= \int_{x=0}^{2} 2880x^2 + 720x^4 - \cancel{2880x^3} - 1920x^2 + 240x^5 + \cancel{2880x^3} - \cancel{1440x^4} - 1440x^3 - 360x^5 + \cancel{1440x^4} \, dx$$

$$= \left[\frac{2880x^3}{3} + \frac{720x^5}{5} - \frac{1920x^3}{3} - \frac{120x^6}{6} - \frac{1440x^4}{4} \right]_0^2$$

$$= \frac{2880 \times 8}{3} + \frac{720}{5} \times 32 - \frac{1920}{3} \times 2^3 - \frac{120 \times 2^6}{6} - \frac{1440 \times 2^4}{4}$$

$$= \underline{\underline{128}}$$

(2) If $\overline{F} = 2xz\overline{i} - x\overline{j} + y^2\overline{k}$ evaluate $\int_V \overline{F}\, dv$ where V is the region bounded by the surfaces $x=0, x=2, y=0, y=6, z=x^2, z=4$

Sol: Given $\overline{F} = 2xz\overline{i} - x\overline{j} + y^2\overline{k}$

V: region bounded by the surfaces $x=0, x=2,$ $y=0, y=6, z=x^2, z=4$

$$\int_V \overline{F}\, dv = \iiint_V \overline{F}\, dx\, dy\, dz$$

$$= \iiint_V 2xz\overline{i} - x\overline{j} + y^2\overline{k}\, dx\, dy\, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{6} \int_{z=x^2}^{z=4} \left(2xz\overline{i} - x\overline{j} + y^2\overline{k}\right) dz\, dy\, dx.$$

[Integrate with respect to z, keep x and y constant]

$$= \int_{x=0}^{2} \int_{y=0}^{6} \left[\frac{\cancel{2}xz^2}{\cancel{2}}\overline{i} - xz\overline{j} + y^2 z\overline{k}\right]_{x^2}^{4} dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{6} (16x - x^5)\overline{i} + (x^3 - 4x)\overline{j} + (4y^2 - x^2y^2)\overline{k}\, dy\, dx$$

[Now integrate with respect to y by keeping x constant]

$$= \int_{x=0}^{2} \left[(16x - x^5)y\overline{i} + (x^3 - 4x)y\overline{j} + \left(\frac{4y^3}{3} - \frac{x^2y^3}{3}\right)\overline{k}\right]_0^6 dx$$

$$= \int_{x=0}^{2} (96x - 6x^5)\overline{i} + (6x^3 - 24x)\overline{j} + (288 - 72x^2)\overline{k}\, dx$$

$$I_v = \int_{x=0}^{2} (96x - 6x^5)\bar{i} + (6x^3 - 24x)\bar{j} + (288 - 72x^2)\bar{k}\, dx.$$

$$= \left[\left(\frac{96x^2}{2} - \frac{6x^6}{6}\right)\bar{i} + \left(\frac{6x^4}{4} - \frac{24x^2}{2}\right)\bar{j} + \left(288.x - \frac{72x^3}{3}\right)\bar{k}\right]_0^2$$

$$= \left(96\times 2 - \frac{\cancel{6}\times 64}{\cancel{6}}\right)\bar{i} + \left(\frac{6\times 16}{4} - \frac{24\times 4}{2}\right)\bar{j} + \left(288\times 2 - \frac{72\times 8}{3}\right)\bar{k}$$

$$I_v = (128\bar{i} - 24\bar{j} + 384\bar{k})$$

Surface Integral :-

If $\bar{F}$ is a continuous vector point function defined over a closed surface S, then the integration of the vector $\bar{F}$ over the surface is called a surface integral and is denoted by $\int_S \bar{F}.\bar{n}\, ds$, where $\bar{n}$ is a unit outward drawn normal vector of the surface

Evaluation of Surface integral.

Surface integral can be evaluated, using double integration over R, where R is projection of given surface on xy (or) yz (or) zx planes.

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(I) $\int_S \bar{F}\cdot\bar{n}\, ds = \iint_R \bar{F}\cdot\bar{n} \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

where R is the projection of given surfaces on xy-plane.

(II) $\int_S \bar{F}\cdot\bar{n}\, ds = \iint_R \bar{F}\cdot\bar{n} \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

where R is the projection of given surfaces on yz-plane

(III) $\int_S \bar{F}\cdot\bar{n}\, ds = \iint_R \bar{F}\cdot\bar{n} \frac{dx\,dz}{|\bar{n}\cdot\bar{j}|}$

Where R is the projection of given surface S on xz-plane.

Ex: Evaluate $\int_S \bar{F}\cdot\bar{n}\, ds$, where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and S is the part of the surface of the plane $2x + 3y + 6z = 12$ located in the first octant.

Sol: Let $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$

$$I_S = \int_S \bar{F}\cdot\bar{n}\, ds$$

$\phi = 2x + 3y + 6z - 12 = 0$. located in the first octant.

To find $\bar{n}$ $\quad \bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\frac{\partial\phi}{\partial x} = 2 \quad \Big| \quad \frac{\partial\phi}{\partial y} = 3 \quad \Big| \quad \frac{\partial\phi}{\partial z} = 6$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = 2\bar{i} + 3\bar{j} + 6\bar{k}.$$

$$\bar{n} = \frac{2\bar{i} + 3\bar{j} + 6\bar{k}}{\sqrt{4+9+36}} = \frac{2}{7}\bar{i} + \frac{3}{7}\bar{j} + \frac{6}{7}\bar{k}$$

$F \cdot \bar{n} = (18z\,\bar{i} - 12\bar{j} + 3y\,\bar{k}) \cdot \left(\frac{2}{7}\bar{i} + \frac{3}{7}\bar{j} + \frac{6}{7}\bar{k}\right)$

$= \frac{36}{7}z - \frac{36}{7} + \frac{18y}{7}$

Let R is the projection of g·S on yz plane.

Put $x = 0$. then We have $3y + 6z = 12$

y limits : $y = 0$ to $3y = 12 - 6z$

$y = 4 - 2z$

z limits : $z = 0$ to $6z = 12$ $\left[\text{Put } y = 0\right]$

$z = 2$.

$\therefore I_s = \iint_R F \cdot \bar{n} \frac{dy\,dz}{|\bar{n} \cdot \bar{i}|}$

$= \int_{z=0}^{2} \int_{y=0}^{4-2z} \frac{36}{7}z - \frac{36}{7} + \frac{18}{7}y \cdot \frac{dy\,dz}{\frac{2}{7}}$

$= \int_{z=0}^{2} \int_{y=0}^{4-2z} (18z - 18 + 9y)\,dy\,dz$

(Keep z constant)

$= \int_{z=0}^{2} \left[18zy - 18y + \frac{9y^2}{2}\right]_0^{4-2z} dz$

$= \int_{z=0}^{2} 18z(4-2z) - 18(4-2z) + \frac{9(4-2z)^2}{2}\,dz$

$= \left[\frac{72z^2}{2} - \frac{36z^3}{3} - 72z + \frac{36z^2}{2} + \frac{9}{2}\frac{(4-2z)^3}{3\times -2}\right]_0^2$

$= \frac{72\times 2^2}{2} - \frac{36\times 2^3}{3} - 72\times 2 + \frac{36}{2}\times 2^2 - 0 + \frac{9}{12}\,64$

$= 24$

② Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ if $\bar{F} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$, and S is the surface of the cylinder $x^2+y^2=9$ Contained in the first octantant between the planes $z=0$ and $z=2$

Sol: Given $\bar{F} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$

S: surface of the cylinder $x^2+y^2=9$ contained in the first octant between the planes $z=0$ & $z=2$

To find Let $I_s = \iint_S \bar{F}\cdot\bar{n}\,ds$

$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

Let ϕ: $x^2+y^2-9=0$, $z=0$, $z=2$

$\frac{\partial\phi}{\partial x} = 2x$; $\frac{\partial\phi}{\partial y} = 2y$; $\frac{\partial\phi}{\partial z} = 0$

$\nabla\phi = 2x\bar{i} + 2y\bar{j}$

$|\nabla\phi| = \sqrt{4x^2+4y^2} = \sqrt{4\times 9} = 6$

∴ Outward unit normal vector $= \frac{2x\bar{i}+2y\bar{j}}{6}$

$= \frac{x}{3}\bar{i} + \frac{y}{3}\bar{j}$

Now $\bar{F}\cdot\bar{n} = (yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k})\cdot(\frac{x}{3}\bar{i} + \frac{y}{3}\bar{j})$

$= \frac{xyz}{3} + \frac{2y^3}{3}$

Let R is projection of S on xz plane then put $y=0$ in $\phi \Rightarrow x^2-9=0$

$x^2=9$

$x=3$

∴ x limits $x=0$ to $x=3$

& z limits $z=0$ to $z=2$

$$\therefore I_S = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dz}{|\bar{n}\cdot\bar{j}|}$$

$$\bar{n}\cdot\bar{j} = \left(\frac{x}{3}\hat{i}+\frac{y}{3}\hat{j}\right)\cdot\hat{j} = \frac{y}{3}$$

$$|\bar{n}\cdot\bar{j}| = \sqrt{\frac{y^2}{9}} = \frac{y}{3}$$

$$= \int_{x=0}^{3}\int_{z=0}^{2}\left(\frac{xyz}{3}+\frac{2y^3}{3}\right)\frac{dx\,dz}{y/3}$$

$$= \int_{x=0}^{3}\int_{z=0}^{2}\left(xz+2y^2\right)dz\,dx$$

$$= \int_{x=0}^{3}\int_{z=0}^{2}\left[xz+2(9-x^2)\right]dz\,dx$$

$$= \int_{x=0}^{3}\left[\frac{xz^2}{2}+18z-2x^2z\right]_0^2 dx$$

$$= \int_{x=0}^{3}\left(2x+36-4x^2\right)dx$$

$$= \left[x^2+36x-\frac{4x^3}{3}\right]_0^3 = 9+36(3)-4(9)$$

$$\boxed{I_S = 81}$$

Ex:-(3) Evaluate $\int \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = z\hat{i}+x\hat{j}-3y^2z\hat{k}$ and S is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$

[Ans :- 90]

Ex:-(4) Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 12x^2y\hat{i}-3yz\hat{j}+2z\hat{k}$ and S is the portion of the plane $x+y+z=1$ included in the first octant.

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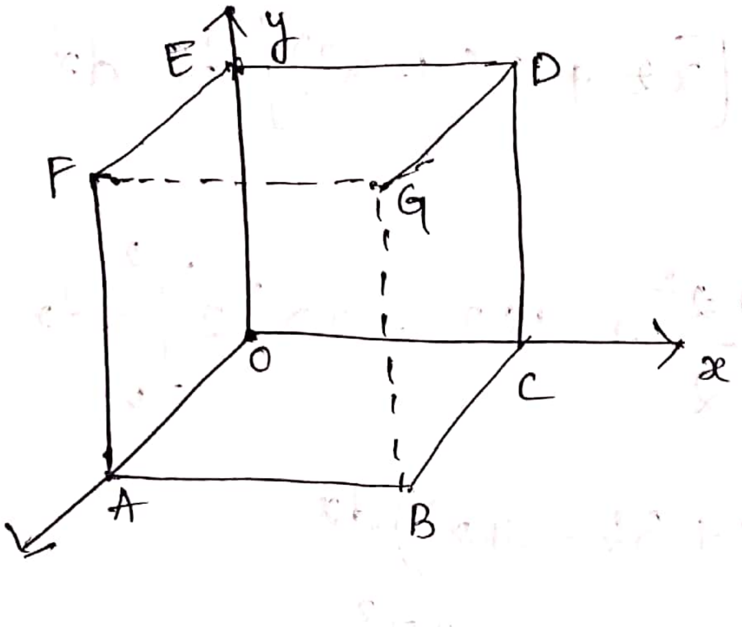
(5) If $\overline{F} = 4xz\,\overline{i} - y^2\overline{j} + yz\overline{k}$, evaluate $\int_S \overline{F}\cdot\overline{n}\,ds$ where S is the surface of the cube bounded by $x=0$, $x=a$, $y=0$, $y=a$, $z=0$, $z=a$.

Sol: Given $\overline{F} = 4xz\overline{i} - y^2\overline{j} + yz\overline{k}$

Let $I_S = \int_S \overline{F}\cdot\overline{n}\,ds$

S: Surface of the cube bounded by $x=0$, $x=a$, $y=0$, $y=a$, $z=0$, $z=a$.



To find I_S: Here $I_S = I_{S_1} + I_{S_2} + I_{S_3} + I_{S_4} + I_{S_5} + I_{S_6}$

i) On S_1, $x=0$ (i.e., AOEF).

$\Rightarrow dx = 0$.

Here $\overline{n} = -\overline{i}$

$\overline{F}\cdot\overline{n} = -4xz$

$$I_{S_1} = \int_{S_1} \overline{F}\cdot\overline{n}\,ds = \int_{y=0}^{a}\int_{z=0}^{a} -4xz\,dy\,dz = 0$$

a a

$$\boxed{\therefore I_{S_1} = 0}.$$

ii) On S_2, $x = a$ [i.e., BCDG]

$dx = 0$, here $\bar{n} = \bar{i}$ & $\bar{F}\cdot\bar{n} = 4xz$.

$$I_{S_2} = \int_{S_2} \bar{F}\cdot\bar{n}\, ds = \iint_{S_2} \bar{F}\cdot\bar{n} \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$$

$$= \int_{y=0}^{a}\int_{z=0}^{a} 4xz\, dy\, dz = \int_{y=0}^{a}\int_{z=0}^{a} 4az\, dz\, dy$$

$$= \int_{y=0}^{a} 4a\frac{z^2}{2}\Big]_0^a dy = \int_{y=0}^{a} 2a^3\, dy$$

$$= 2a^3 .\, y\Big]_0^a = \boxed{2a^4. = I_{S_2}}$$

iii) On S_3 : $y = 0$ [i.e, OABC]

$dy = 0$

here $\bar{n} = -\bar{j}$

$\bar{F}\cdot\bar{n} = y^2$ & $|\bar{n}\cdot\bar{j}| = 1$

$\Rightarrow \bar{F}\cdot\bar{n} = 0$

$$\Rightarrow I_{S_3} = \int_{S_3} \bar{F}\cdot\bar{n}\, ds = 0 \Rightarrow \boxed{I_{S_3} = 0}$$

iv) On S_4 : $y = a$ (EFGD)

$dy = 0$

here $\bar{n} = \bar{j}$

$\bar{F}\cdot\bar{n} = -y^2$, $|\bar{n}\cdot\bar{j}| = 1$

$= -a^2$

$$I_{S_4} = \int_{S_4} F\cdot\bar{n}\,ds = \int_{x=0}^{a}\int_{z=0}^{a} -a^2\,dz\,dx$$

$$= -a^2\int_{x=0}^{a} z\Big]_0^a dx = -a^3\int_{x=0}^{a} dx$$

$$\boxed{I_{S_4} = -a^4}$$

v) On S_5 : $z=0$ (DEOC)

$dz=0$

$\bar{n} = -\bar{k}$

$F\cdot\bar{n} = -yz$ & $|\bar{n}\cdot\bar{k}| = 1$

$F\cdot\bar{n} = 0$

$$\Rightarrow I_{S_5} = \int_{S_5} \bar{F}\cdot\bar{n}\,ds = \boxed{0 = I_{S_5}}$$

vi) On S_6 : $z=a$ (ABGF)

$dz=0$

here $\bar{n} = \bar{k}$

$F\cdot\bar{n} = yz$ & $|\bar{n}\cdot\bar{k}| = 1$

$= ay$

$$\therefore I_{S_6} = \int_{S_6} \bar{F}\cdot\bar{n}\,ds = \iint \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \int_{x=0}^{a}\int_{y=0}^{a} ay\,dy\,dx = \int_{x=0}^{a} \frac{ay^2}{2}\Big]_0^a dx$$

$$= \int_{x=0}^{a} \frac{a^3}{2}\,dx = \frac{a^3}{2}x\Big]_0^a$$

$$= \frac{a^4}{2}$$

$$\therefore I_s = 0 + 2a^4 + 0 - a^4 + \frac{a^4}{2} = \frac{3a^4}{2}$$