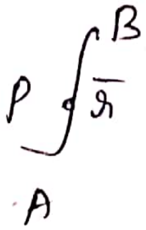
Unit - V

Vector Integration

Let $\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ be any continuous vector function defined on a curve C.



Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ be the position vector of any point p on the curve c.

Then $\int_C \bar{F}\cdot d\bar{r} = \int (f_1\bar{i} + f_2\bar{j} + f_3\bar{k}) \cdot dx\bar{i} + dy\bar{j} + dz\bar{k}$

$= \int f_1 dx + f_2 dy + f_3 dz$

is called the Line integral.

Note :- If $\bar{F}$ represents the force acting on a particle moving along an arc AB then the work done during the small displacement δr is $\bar{F}\cdot d\bar{r}$

Hence the total work done by $\bar{F}$ during the displacement from A to B is $\int_A^B \bar{F}\cdot d\bar{r}$

Hence mathematically work done is identical to Line integral.

① If $\bar{F} = 3xy\bar{i} - y^2\bar{j}$ then evaluate $\int_C \bar{F}\cdot d\bar{r}$ where c is the curve in xy plane $y = 2x^2$ from (0,0) to (1,2)

Sol: Given $\bar{F} = 3xy\bar{i} - y^2\bar{j}$

C : Curve in the xy plane $y = 2x^2$ from O(0,0) to A(1,2).

To find $\int_C \bar{F}\cdot d\bar{r}$

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ (here $z=0$ since curve lies in xy plane)

$\Rightarrow \bar{r} = x\bar{i} + y\bar{j}$

$\Rightarrow d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

Now $\bar{F}\cdot d\bar{r} = (3yx\,\bar{i} - y^2\bar{j})\cdot(dx\,\bar{i} + dy\,\bar{j})$

$= 3xy\,dx - y^2 dy$

[given $y = 2x^2 \Rightarrow \frac{dy}{dx} = 4x \Rightarrow dy = 4x\,dx$]

$= 3x(2x^2)\,dx - (2x^2)^2 4x\,dx$

$= 6x^3 dx - 16x^5 dx$

$= (6x^3 - 16x^5)\,dx.$

Now, $\int_C \bar{F}\cdot d\bar{r} = \int_O^A (6x^3 - 16x^5)\,dx$

$= \int_0^1 (6x^3 - 16x^5)\,dx$

$= \left[\frac{6.x^4}{4} - \frac{16x^6}{6}\right]_0^1$

$= \frac{6}{4} - \frac{16}{6} = \frac{-7}{6}$

(2) If $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$ then evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve C in xy plane $y = x^2$ from the point (1,1) to (2,8)

Sol: Given $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$

C : $y = x^2$ from the point A(1,1) to B(2,8)

To find $\int_C \bar{F}\cdot d\bar{r}$

Let $\bar{r} = x\bar{i} + y\bar{j}$ [here $z = 0$]

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

Now, $\bar{F} \cdot d\bar{r} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j} \cdot dx\bar{i} + dy\bar{j}$

$= (5xy - 6x^2)\,dx + (2y - 4x)\,dy$

$\left[\because \text{Given } y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dy = 2x\,dx\right]$

$= (5x \cdot x^2 - 6x^2)\,dx + (2x^2 - 4x)\,2x\,dx$

$= (5x^3 - 6x^2 + 4x^3 - 8x^2)\,dx$

$= (9x^3 - 14x^2)\,dx$

$\int_C \bar{F} \cdot d\bar{r} = \int_A^B (9x^3 - 14x^2)\,dx$

$= \left[\frac{9x^4}{4} - \frac{14x^3}{3}\right]_1^2$

$= \frac{9}{4}(2)^4 - \frac{14}{3}(2)^3 - \frac{9}{4} + \frac{14}{3}$

$= \frac{72}{4} - \frac{112}{3} - \frac{9}{4} + \frac{14}{3} = \frac{-13}{12}$

(3) If $\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$ and c is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$, evaluate $\int_C \bar{F} \cdot d\bar{r}$

Sol: Given $\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$

C: $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$

To find $\int_C \bar{F} \cdot d\bar{r}$

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\bar{F} \cdot d\bar{r} = (xy\bar{i} - z\bar{j} + x^2\bar{k}) \cdot (dx\bar{i} + dy\bar{j} + dz\bar{k})$

$= xy\,dx - z\,dy + x^2\,dz.$

$\left[\text{given } x = t^2;\ y = 2t,\ z = t^3\right.$
$\left.\Rightarrow dx = 2t\,dt;\ dy = 2dt,\ dz = 3t^2dt\right]$

$\therefore \bar{F}\cdot d\bar{r} = 2t^3 \cdot 2t\,dt - t^3 \cdot 2dt + t^4 \cdot 3t^2dt$

$= [4t^4 - 2t^3 + 3t^6]dt$

Now $\int_C \bar{F}\cdot d\bar{r} = \int_{t=0}^{1} (4t^4 - 2t^3 + 3t^6)\,dt$

$= \left.\frac{4t^5}{5} - \frac{2t^4}{4} + \frac{3t^7}{7}\right]_0^1$

$= \frac{4}{5} - \frac{2}{4} + \frac{3}{7} = \frac{51}{70}$

(4) If $\bar{F} = 3xy\bar{i} - 5z\bar{j} + 10xz\bar{k}$, then evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve $x = t^2+1,\ y = 2t^2,\ z = t^3$ from $t=1$ to $t=2$.

Sol: Given $\bar{F} = 3xy\bar{i} - 5z\bar{j} + 10xz\bar{k}$

$C: x = t^2+1,\ y = 2t^2,\ z = t^3$ from $t=1$ to $t=2$

To find $\int_C \bar{F}\cdot d\bar{r}$

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\Rightarrow d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\bar{F}\cdot d\bar{r} = [3xy\bar{i} - 5z\bar{j} + 10xz\bar{k}]\cdot(dx\bar{i} + dy\bar{j} + dz\bar{k})$

$= 3xy\,dx - 5z\,dy + 10xz\,dz$

$\left[\because \text{given } x = t^2+1,\ y = 2t^2,\ z = t^3\right.$
$\left.\Rightarrow dx = 2t\,dt,\ dy = 4t\,dt,\ dz = 3t^2dt\right]$

$= 3(2t^4 + 2t^2)\cdot 2t\,dt - 5t^3\cdot 4t\,dt + 10(t^5 + t^3)3t^2dt$

$= (12t^5 + 12t^3 - 20t^4 + 30t^7 + 30t^5)\,dt$

$= (30t^7 + 42t^5 + 12t^3 - 20t^4)\,dt$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_1^2 (30t^7 + 42t^5 - 20t^4 + 12t^3)\, dt$$

$$= \left[\frac{30t^8}{8} + \frac{42t^6}{6} - \frac{20t^5}{5} + \frac{12t^4}{4}\right]_1^2$$

$$= \frac{30}{8}(2^8) + \frac{42}{6}2^6 - \frac{20(2)^5}{5} + \frac{12}{4}(2)^2 - \frac{30}{8} - \frac{42}{6} + \frac{20}{5} - \frac{12}{4}$$

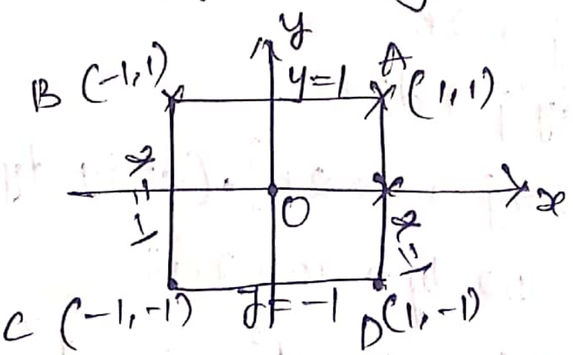
$$= \frac{1377}{4}$$

Excercises If $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$, then evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve c in xyplane $y = x^3$ from $(1,1)$ to $(2,8)$.

(5) Evaluate the line integral $\int_C (x^2 + xy)\,dx + (x^2 + y^2)\,dy$ where c is a square formed by the lines $x = \pm 1$ and $y = \pm 1$.

Sol: Let $I = \int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$

C: Square formed by the lines $x = \pm 1$ & $y = \pm 1$



$$I_C = I_{AB} + I_{BC} + I_{CD} + I_{DA}.$$

To find I_{AB}

Equation of AB: $y = 1$

$\Rightarrow dy = 0$

$$\therefore I_{AB} = \int_A^B (x^2+xy)\,dx + (x^2+y^2)\,dy$$

$$= \int_A^B (x^2+x)\,dx + 0$$

$$= \left.\frac{x^3}{3}+\frac{x^2}{2}\right]_1^{-1} = \frac{-1}{3}+\frac{1}{2}-\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{-2}{3}$$

To find I_{BC}

Equation of BC: $x=-1 \Rightarrow dx=0$.

$$I_{BC} = \int_B^C (x^2+xy)\,dx + (x^2+y^2)\,dy$$

$$= 0 + \int_B^C (1+y^2)\,dy = \left. y+\frac{y^3}{3}\right]_1^{-1}$$

$$= -1-\frac{1}{3} - \left(1+\frac{1}{3}\right) = -\frac{8}{3}$$

To find I_{CD}

Equation of CD: $y=-1 \Rightarrow dy=0$

$$I_{CD} = \int_C^D (x^2+xy)\,dx + (x^2+y^2)\,dy$$

$$= \int_C^D (x^2-x)\,dx + 0 = \left.\frac{x^3}{3}-\frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{3}-\frac{1}{2}-\left[\frac{-1}{3}-\frac{1}{2}\right] = +\frac{2}{3}.$$

To find I_{DA}: Equation of DA is $x=1 \Rightarrow dx=0$

$$I_{DA} = \int_D^A (x^2+xy)\,dx + (x^2+y^2)\,dy$$

$$= \int_D^A (1+y^2)\,dy = \left. y+\frac{y^3}{3}\right]_{-1}^{1}$$

$$= 1+\frac{1}{3}-\left(-1-\frac{1}{3}\right) = \frac{8}{3}.$$

$$\therefore I_C = \frac{-2}{3}-\frac{8}{3}+\frac{2}{3}+\frac{8}{3} = 0$$

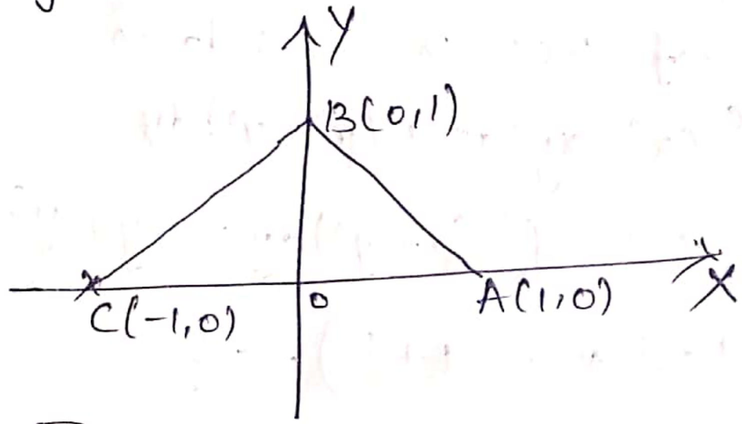
Scanned with CamScanner

(b) Compute the line integral $\int y^2 dx - x^2 dy$ round the triangle whose vertices are $(1, 0)$, $(0,1)$ and $(-1,0)$ in xy plane.

Sol: Let $I_C = \int_C y^2 dx - x^2 dy$

C: triangle with vertices $A(1,0)$, $B(0,1)$ & $C(-1,0)$

~~Here~~



Here $I_C = I_{AB} + I_{BC} + I_{CA}$.

To find I_{AB}:

Slope of AB $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{1}{-1} = -1$

Eq'n of AB: $(y-0) = -1(x-1)$

$y = -x+1 \Rightarrow dy = -dx$.

$$\therefore I_{AB} = \int_A^B y^2 dx - x^2 dy$$

$$= \int_A^B (-x+1)^2 dx + x^2 dx$$

$$= \int_A^B (+x^2 - 2x + 1 + x^2) dx$$

$$= \left.\frac{x^3}{3} - \frac{2x^2}{2} + x + \frac{x^3}{3}\right]_1^0$$

$$= -\frac{1}{3} + \frac{2}{2} \mp 1 \mp \frac{1}{3} = -2/3$$

To find I_{BC}

Slope of BC $= \frac{1-0}{0+1} = 1$

Eq'n of BC $\Rightarrow y = 1(x+1)$

$\Rightarrow y = x+1 \Rightarrow dy = dx$

$$I_{BC} = \int_B^C y^2 dx - x^2 dy$$

$$= \int_B^C (x+1)^2 dx - x^2 dx$$

$$= \left.\frac{(x+1)^3}{3} - \frac{x^3}{3}\right]_0^{-1} = 0 + \frac{1}{3} - \frac{1}{3} + 0$$

$$= 0$$

To find I_{CA} Equation of CA is $y=0$

$\Rightarrow dy = 0$

$$\therefore I_{CA} = \int_C^A y^2 dx - x^2 dy$$

$$= \int 0 = \underline{\underline{0}}$$

$$\therefore I_C = -\frac{2}{3} + 0 + 0 = \underline{\underline{-\frac{2}{3}}}$$

(8) Exercise: If $F = (x^2+y^2)\bar{i} - 2xy\bar{j}$ then evaluate $\oint_C \bar{F}\cdot d\bar{r}$, where C is the rectangle bounded by $y=0$, $y=b$, $x=0$, $x=a$.

Ans: $-2ab^2$

(7) Find the work done in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$

Sol: Given $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$

C: Straight line from $O(0,0,0)$ to $A(2,1,3)$

To find Work done

$$\text{Work done} = \int_C \bar{F}\cdot d\bar{r}$$

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

Eq'n of Straight line from $O(0,0,0)$ to $A(2,1,3)$ is given by $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

$$\Rightarrow \frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t_{(say)}$$

$$\Rightarrow x = 2t, \quad y = t, \quad z = 3t.$$

$$\Rightarrow dx = 2dt, \quad dy = dt, \quad dz = 3dt.$$

Now, $\bar{F}\cdot d\bar{r} =$

$$= (3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k})\cdot(dx\bar{i}+dy\bar{j}+dz\bar{k})$$

$$= 3x^2dx + (2xz-y)dy + zdz$$

$$= 12t^2\cdot 2dt + (12t^2-t)dt + 3t\cdot 3dt$$

$$= (24t^2+12t^2-t+9t)dt$$

$$= (36t^2+8t)\,dt$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 (36t^2+8t)\,dt \quad \left[\because y=t;\ \text{limits of } y \text{ and } t \text{ are same}\right]$$

$$= \left[\frac{36t^3}{3} + \frac{8t^2}{2}\right]_0^1$$

$$= \frac{36}{3} + \frac{8}{2} = \underline{\underline{16}}$$

Exercise:- Find the work done in moving a particle in the force field $\bar{F} = 3x^2\,\bar{i} + \bar{j} + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$. $\left[\text{Ans: } \frac{27}{2}\right]$

(2) Find the workdone by the force $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$ when it moves a particle along the arc of the curve $\bar{r} = \cos t\,\bar{i} + \sin t\,\bar{j} - t\,\bar{k}$ from $t=0$ to $t=2\pi$.

$\left[\text{Ans:} -(-\pi)\right]$

(8) Find the work done by the Force $\bar{F} = (2y+3)\bar{i} + xz\bar{j} + (yz-x)\bar{k}$ when it moves a particle from the point $(0,0,0)$ to $(2,1,1)$ along the curve $x = 2t^2,\ y = t,\ z = t^3$.

Sol: Given $\bar{F} = (2y+3)\bar{i} + xz\bar{j} + (yz-x)\bar{k}$

C: $x = 2t^2,\ y = t,\ z = t^3$ from $\overset{O}{(0,0,0)}$ to $\overset{A}{(2,1,1)}$

To find Work done :- Work done $= \int_C \bar{F}\cdot d\bar{r}$

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\Rightarrow d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$.

Now $\bar{F}\cdot d\bar{r} = (2y+3)dx + xz\,dy + (yz-x)dz$.

[$\because$ given $x = 2t^2,\ y = t,\ z = t^3$

$\Rightarrow dx = 4t\,dt,\ dy = dt,\ dz = 3t^2 dt$]

$\Rightarrow \bar{F}\cdot d\bar{r} = (2t+3)4t\,dt + 2t^5 dt + [t^4 - 2t^2)3t^2 dt$

$= [8t^2 + 12t + 2t^5 + 3t^6 - 6t^4]dt$

$= (3t^6 + 2t^5 - 6t^4 + 8t^2 + 12t)dt$.

$\therefore$ Work done $= \int_O^A (3t^6 + 2t^5 - 6t^4 + 8t^2 + 12t)dt$

$= \left[\frac{3t^7}{7} + \frac{2t^6}{6} - \frac{6t^5}{5} + \frac{8t^3}{3} + \frac{12t^2}{2}\right]_0^1$ [$\because y = t$ The limits of y and t are same]

$= \frac{3}{7} + \frac{2}{6} - \frac{6}{5} + \frac{8}{3} + \frac{12}{2} = \frac{288}{35}$

Conservative force field :- A force field $\bar{F}$ is said to be conservative, if the workdone is independent of the path and vice-versa.

Note: If $\bar{F}$ is Conservative then curl $\bar{F} = \bar{0}$

(2) then $\exists$ a scalar potential function ϕ such that $\bar{F} = \text{grad}\,\phi$

(9) Prove that the force field given by
$\bar{F} = 2xyz^3\,\bar{i} + x^2z^3\,\bar{j} + 3x^2yz^2\,\bar{k}$ is conservative
Find the work done by moving a particle from
$(1,-1,2)$ to $(3,2,-1)$ in this force field.

Sol: Given $\bar{F} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k}$

Let $A(1,-1,2)$ and $B(3,2,-1)$

To find $\bar{F}$ conservative

$$\text{Curl}\,\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$= \bar{i}\left[3x^2z^2 - 3x^2z^2\right] - \bar{j}\left[6xyz^2 - 6xyz^2\right] + \bar{k}\left[2xz^3 - 2xz^3\right]$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k} = \bar{0}$$

$\therefore \bar{F}$ is Conservative.

To find work done :- Work done $= \int_C \bar{F}\cdot d\bar{r}$

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\bar{F}\cdot d\bar{r} = 2xyz^3\,dx + x^2z^3\,dy + 3x^2yz^2\,dz$

$$\int_C \bar{F}\cdot d\bar{r} = \int_A^B 2xyz^3dx + x^2z^3dy + 3x^2yz^2dz.$$

$$= \int_A^B yz^3(2x\,dx) + x^2z^3dy + x^2y(3z^2dz)$$

$$= \int_A^B 1\, d(x^2yz^3)$$

$$= \left. x^2yz^3 \right]_{(1,-1,2)}^{(3,2,-1)}$$

$$= 3^2\cdot 2\cdot(-1)^3 - 1^2\cdot(-1)\,2^3$$

$$= -18 + 8 = \underline{-10}$$

Scanned with CamScanner

(10) If $\bar{F} = (4xy - 3x^2z^2)\bar{i} + 2x^2\bar{j} - 2x^3z\bar{k}$, prove that $\int_C \bar{F} \cdot d\bar{r}$. i.e. work done is independent of the curve joining two points.

Sol: Given $\bar{F} = (4xy - 3x^2z^2)\bar{i} + 2x^2\bar{j} - 2x^3z\bar{k}$

To find work done independent of path

$$\text{Curl } \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \bar{i}[0 - 0] - \bar{j}[-6x^2z + 6x^2z] + \bar{k}[4x - 4x]$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k} = \bar{0}.$$

$\therefore$ $\bar{F}$ is conservative.

Green's theorem in a plane:-

(Transformation between Line integral and Double integral)

If R is a closed region in xy plane bounded by a simple closed curve c and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\boxed{\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

where c is traversed in the positive (anticlockwise) direction.