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1. DIFFERENTIATIAL EQUATIONS OF FIRST ORDER & FIRST DEGREE

An equation which contains independent variable, and dependent variable and its derivatives is called differential equation.

Ex:- $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0.$

Order of D.E:- In the differential equation highest derivative is called order of D.E.

Degree of D.E:- In the differential equation highest derivative power is called degree of D.E.

Ex:- $\left(\frac{d^3y}{dx^3}\right)^5 + 4\left(\frac{dy}{dx}\right)^7 + y = 0.$

Order = 3 Degree = 5.

Solving of First Order DE.

1. Variable Seperable.
2. Homogeneous
3. Non-Homogeneous
4. Linear
5. Non-Linear
6. Exact
7. Non Exact.

1. Solve $(x^2+1)\frac{dy}{dx} + (y^2+1) = 0$.

Sol. Given

$$(x^2+1)\frac{dy}{dx} + (y^2+1) = 0$$

$$(x^2+1)\frac{dy}{dx} = -(y^2+1)$$

$$\frac{1}{1+y^2}\,dy = -\frac{1}{1+x^2}\,dx.$$

Variables are seperable.

Integrating on both sides.

$$\int \frac{1}{1+y^2}\,dy = -\int \frac{1}{1+x^2}\,dx.$$

$$\tan^{-1}y = -\tan^{-1}x + c.$$

$$\tan^{-1}x + \tan^{-1}y = c.$$

2. Solve $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

Sol. Given

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy} \quad \text{--- (1)}$$

Put $y = vx$.

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

$$① \Rightarrow \quad v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x(vx)}$$

$$v + x\frac{dv}{dx} = \frac{x^2(1+v^2)}{x^2 2v}$$

$$v + x\frac{dv}{dx} = \frac{1+v^2}{2v}.$$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$x\frac{dv}{dx} = \frac{1+v^2-2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1-v^2}{2v}.$$

$$\frac{1}{x}dx = \frac{2v}{1-v^2}dv.$$

Variables are seperable.
Integrating on both sides.

$$-\int \frac{-2v}{1-v^2}\,dv = \int \frac{1}{x}\,dx.$$

$$\int \frac{f'(x)}{f(x)} = \log|f(x)|$$

$$-\log|1-v^2| = \log x + \log c.$$

$$-\log|1-v^2| = \log cx. \qquad \log a + \log b = \log ab.$$

$$\log cx + \log\left|1-\frac{y^2}{x^2}\right| = 0$$

$$\log\left[cx\left(1-\frac{y^2}{x^2}\right)\right] = 0.$$

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3. Solve $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Sol. Given

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$$

Put

$$x = X+h \qquad y = Y+k$$

$$\frac{dy}{dx} = \frac{dY}{dX}$$

$$\frac{dY}{dX} = \frac{X+h+2Y+2k-3}{2X+2h+Y+k-3}$$

$$\frac{dY}{dX} = \frac{(X+2Y)+(h+2k-3)}{(2X+Y)+(2h+k-3)}$$

Put

$$h+2k-3=0 \Rightarrow h+2k=3 \quad \times 2$$

$$2h+k-3=0 \Rightarrow (-)2h+k=3$$

$$0+3k=3$$

$$\boxed{k=1} \text{ sub in eq.}$$

$$2h+1=3$$

$$2h=2$$

$$\boxed{h=1}$$

$$\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$$

Put $Y = VX$

$$\frac{dY}{dX} = V + X\frac{dV}{dX}.$$

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$$v + x\frac{dv}{dx} = \frac{x + 2vx}{2x + vx}$$

$$v + x\frac{dv}{dx} = \frac{x(1+2v)}{x(2+v)}$$

$$v + x\frac{dv}{dx} = \frac{1+2v}{2+v}$$

$$x\frac{dv}{dx} = \frac{1+2v}{2+v} - v.$$

$$x\frac{dv}{dx} = \frac{1-v^2}{2+v}.$$

$$\frac{2+v}{1-v^2}\,dv = \frac{1}{x}\,dx.$$

Variables are seperable.

Integrating on Both Sides.

$$\int \frac{2+v}{1-v^2}\,dv = \int \frac{1}{x}\,dx.$$

$$\int \frac{2}{1-v^2}\,dv + \int \frac{v}{1-v^2}\,dv = \int \frac{1}{x}\,dx. \qquad \boxed{\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|}$$

$$2\int \frac{1}{1-v^2}\,dv - \frac{1}{2}\int \frac{-2v}{1-v^2}\,dv = \int \frac{1}{x}\,dx.$$

$$\cancel{2}\,\frac{1}{\cancel{2}}\log\left|\frac{1+v}{1-v}\right| - \frac{1}{2}\log|1-v^2| = \log x + \log c$$

$$\log|1+v| - \log|1-v| - \frac{1}{2}\log|(1-v)(1+v)| = \log xc.$$

$$\log|1+v| - \log|1-v| - \frac{1}{2}\log|1-v| - \frac{1}{2}\log|1+v| = \log xc$$

$$\frac{1}{2}\log|1+v| - \frac{3}{2}\log|1-v| = \log xc$$

$$\frac{1}{2}\log|1+v|^{\cancel{1/2}} - \frac{3}{2}\log|1-v|^{\cancel{3/2}} = \log xc.$$

$$\log|1+v| - 3\log|1-v| = 2\log xc.$$

$$\log|1+v| - \log|1-v|^3 = \log(xc)^2$$

$$\log\left|\frac{1+v}{(1-v)^3}\right| = \log x^2c^2$$

$$\left|\frac{1+v}{(1-v)^3}\right| = x^2c^2$$

$$\frac{1+y/x}{\left(1-y/x\right)^3} = x^2c^2$$

$$\frac{\frac{x+y}{\cancel{x}}}{\frac{(x-y)^3}{x^{\cancel{3}}}} = x^2c^2$$

$$\frac{x+y}{(x-y)^3} = c^2$$

$h=1$ $k=$

$x = X+h \Rightarrow X+1$

$y = Y+k \Rightarrow Y+1$

$X = x-1$

$Y = y-1$

$$\frac{x-1+y-1}{(x-1-y+1)^3} = c^2$$

$$\frac{x+y-2}{(x-y)^3} = c^2$$

$$x+y-2 = c^2(x-y)^3.$$

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Linear differential equation

Type - I.

A Linear differential equation in the form

$$\frac{dy}{dx} + Py = Q \quad -①$$ [Linear in y]

$$I.F = e^{\int P dx}$$

General Solution of ①.

$$y[I.F] = \int Q . IF \, dx + c.$$

Type - II.

A linear differential equation in the form.

$$\frac{dx}{dy} + Px = Q \quad -①$$

$$I.F = e^{\int P dy}$$

General Solution of ①

$$x[I.F] = \int Q . I.F \, dy + c.$$

① Solve $x\frac{dy}{dx}+y=\log x.$

Given

$$x\frac{dy}{dx}+y=\log x.$$

$$\frac{dy}{dx}+\frac{1}{x}y=\frac{\log x}{x}. \quad -①$$

This is in the form $\frac{dy}{dx}+Py=Q$.

$$P=\frac{1}{x} \qquad Q=\frac{\log x}{x}.$$

$$I.F = e^{\int P dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x. \qquad \int\frac{1}{x}dx=\log x$$

$$I.F = x.$$

General solution of ① is

$$y(I.F)=\int Q(I.F)dx+c \qquad \int \log x\,dx = x\log x - x.$$

$$y.x=\int\frac{\log x}{\cancel{x}}\cancel{x}\,dx+c \qquad e^{\log a}=a.$$

$$xy=\int \log x\,dx+c$$

$$xy=x\log x-x+c.$$

$$\boxed{xy=x\log x-x+c}$$

2. Solve $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$

Given

$$(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$$

$$\frac{dy}{dx} + \frac{-1}{x+1}y = e^{3x}(x+1). \quad \text{--- (1)}$$

This is in the form $\frac{dy}{dx} + Py = Q$.

$$P = \frac{-1}{1+x} \qquad Q = e^{3x}(x+1).$$

$$I.F = e^{\int P dx} = e^{\int \frac{-1}{1+x} dx} = e^{-\log(1+x)} = e^{\log(1+x)^{-1}}$$

$$I.F = (1+x)^{-1}$$

General solution of (1) is. $\qquad \int \frac{1}{x} dx = \log x + c$

$$y(I.F) = \int Q(I.F)dx + c$$

$$y\cdot(1+x)^{-1} = \int e^{3x}(x+1)(1+x)^{-1}dx + c$$

$$y(1+x)^{-1} = \boxed{\int e^{3x} dx + c}$$

$$y(1+x)^{-1} = \frac{e^{3x}}{3} + c.$$

$$\boxed{\frac{y}{x+1} = \frac{e^{3x}}{3} + c.}$$

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3. Solve $\frac{dy}{dx} + 2xy = e^{-x^2}$

Given

$$\frac{dy}{dx} + 2xy = e^{-x^2} \quad \text{--- (1)}$$

This is in the form $\frac{dy}{dx} + Py = Q$.

$P = 2x \qquad Q = e^{-x^2}$

$$I \cdot F = e^{\int P dx} = e^{\int 2x\, dx} = e^{\frac{\cancel{2}x^2}{\cancel{2}}} = e^{x^2}$$

$$I \cdot F = e^{x^2}$$

General solution of (1) is

$$y(I \cdot F) = \int Q\,(I \cdot F)\,dx + c$$

$$ye^{x^2} = \int e^{-x^2} e^{x^2}\, dx + c.$$

$$ye^{x^2} = \int dx + c$$

$$\boxed{ye^{x^2} = x + c.}$$

4. Solve $\frac{dy}{dx} + y = e^{e^x}$

Given

$$\frac{dy}{dx} + y = e^{e^x} \quad \text{—} \; ①$$

This is in the form $\frac{dy}{dx} + Py = Q$.

$P = 1 \qquad Q = e^{e^x}$

$I.F = e^{\int P dx} = e^{\int 1 dx} = e^x$

$I.F = e^x$.

General solution of ① is

$y(I.F) = \int Q\,(I.F)\,dx + c.$

$y \cdot e^x = \int e^{e^x} e^x \, dx + c.$

$*ye^x = \int e^{e^x + x} \, dx + c.$ Put $e^x = t$

$e^x dx = dt$

$ye^x = \int e^t dt + c.$

$ye^x = e^t + c$

$$\boxed{ye^x = e^{e^x} + c}$$

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5) Solve $(1+y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$.

Given

$$(1+y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$$

$$(x - e^{\tan^{-1}y})\frac{dy}{dx} = -(1+y^2)$$

$$x - e^{\tan^{-1}y} = -(1+y^2)\frac{dx}{dy}$$

$$(1+y^2)\frac{dx}{dy} + x = e^{\tan^{-1}y}$$

$$\frac{dx}{dy} + x\frac{1}{1+y^2} = \frac{e^{\tan^{-1}y}}{1+y^2} \quad \text{—— (1)}$$

This is in the form $\frac{dx}{dy} + Px = Q$

$$P = \frac{1}{1+y^2} \cdot \quad Q = \frac{e^{\tan^{-1}y}}{1+y^2}$$

$I.F = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1}y}$

General solution of (1) is

$$x(I.F) = \int Q \cdot (I.F)dy + c$$

$$x e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2} \times e^{\tan^{-1}y} dy + c.$$

Put $\tan^{-1}y = t$.

$$\frac{1}{1+y^2}dy = dt.$$

$$x e^{\tan^{-1}y} = \int e^t \cdot e^t \, dt + c.$$

$x e^{\tan^{-1}y} = \int e^{2t} dt + c$

$x e^{\tan^{-1}y} = \frac{e^{2t}}{2} + c$

$x e^{\tan^{-1}y} = \frac{(e^{\tan^{-1}y})^2}{2} + c$

$$\boxed{x e^{\tan^{-1}y} = \frac{e^{2\tan^{-1}y}}{2} + c}$$

$\int \frac{1}{1+x^2} dx = \tan^{-1}x + c$

$\int e^t dt = e^t + c$

6) Solve $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{\sin 2x}{\log x}$

Given

$$\frac{dy}{dx} + \frac{1}{x\log x} y = \frac{\sin 2x}{\log x} \quad \text{—(1)}$$

This is in the form $\frac{dy}{dx} + Py = Q$.

$P = \frac{1}{x\log x}$ $\quad Q = \frac{\sin 2x}{\log x}$

$I.F = e^{\int P dx} = e^{\int \frac{1}{x\log x} dx} = e^{\int \frac{1}{t} dt} = e^{\log t} = e^{\log\log x}$

$\log x = t$

$\frac{1}{x} dx = dt$

$I.F = \log x$.

General solution of (1) is

$\int \frac{f'(x)}{f(x)} = \log|f(x)| + c$

$y(I.F) = \int Q(I.F) dx + c$

$y \cdot \log x = \int \frac{\sin 2x}{\log x} \log x \, dx + c$.

$y \log x = \int \sin 2x \, dx + c$.

$\int \sin x \, dx = -\cos x + c$

$$\boxed{y \log x = -\frac{\cos 2x}{2} + c}$$

7 Solve $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$

Given

$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$$

$$\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} = \frac{dy}{dx} \quad \text{—— ①}$$

$$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

This is in the form $\frac{dy}{dx} + Py = Q$.

$$P = \frac{1}{\sqrt{x}} \qquad Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

$$I.F = e^{\int P\,dx} = e^{\int \frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}} \qquad \int \frac{1}{\sqrt{x}} = 2\sqrt{x} + c$$

General solution of ① is.

$$y\cdot(I.F) = \int Q \cdot I.F\,dx + c$$

$$y\cdot e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} e^{2\sqrt{x}}\,dx + c$$

$$ye^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}}\,dx + c.$$

$$\boxed{ye^{2\sqrt{x}} = 2\sqrt{x} + c.}$$

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8 Solve $(1+y^2)dx = (\tan^{-1}y - x)dy$

Given.

$$(1+y^2)dx = (\tan^{-1}y - x)dy \quad \text{—} \; (*)$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{1}{1+y^2}\cdot x = \frac{\tan^{-1}y}{1+y^2} \quad \text{—} \; (1)$$

This is in the form $\frac{dx}{dy} + Px = Q.$

$$P = \frac{1}{1+y^2} \qquad Q = \frac{\tan^{-1}y}{1+y^2}$$

$$\text{I.F} = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

$$\text{I.F} = e^{\tan^{-1}y}$$

General solution of (1) is

$$x(\text{I.F}) = \int Q(\text{I.F})dy + c$$

$$xe^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2}\, e^{\tan^{-1}y}\, dy + c.$$

Put $\tan^{-1}y = t$

$\frac{1}{1+y^2}dy = dt$

$$xe^{\tan^{-1}y} = \int t\cdot e^t + c.$$

$$xe^{\tan^{-1}y} = te^t - e^t + c.$$

$\int uv = u\int v - \int u'\int v$

$$xe^{\tan^{-1}y} = (\tan^{-1}y - 1)e^{\tan^{-1}y} + c.$$

$$\boxed{xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c}$$

Non-Linear Differential Equation

An equation of the form $\frac{dy}{dx} + Py = Qy^n$ is called Bernoulli's equation

$$\frac{dy}{dx} + Py = Qy^n \quad \text{--- (1)}$$

$$\frac{1}{y^n}\frac{dy}{dx} + P\frac{y}{y^n} = Q$$

$$\boxed{y^{-n}\frac{dy}{dx} + Py^{1-n} = Q} \quad \text{--- (2)}$$

Put $y^{1-n} = t$

$$(1-n)\,y^{-n}\frac{dy}{dx} = \frac{dt}{dx}$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dt}{dx}$$

$$\frac{dt}{dx} + Pt = Q \quad \text{--- (3)}$$

$$I.F. = e^{\int P\,dx}$$

General solution of (3) is

$$t\cdot[I.F] = \int Q\,[I.F]\,dx.$$

1 Solve $x\frac{dy}{dx} + y = x^3 y^6$

Given

$$x\frac{dy}{dx} + y = x^3 y^6$$

$$\frac{dy}{dx} + \frac{y}{x} = x^2 y^6 \quad \text{—— ①}$$

This is in the form $\frac{dy}{dx} + Py = Qy^n$ $\quad P = \frac{1}{x} \quad Q = x^2 \quad n = 6.$

$$y^{-6}\frac{dy}{dx} + \frac{1}{x} y^{-5} = x^2 \quad \text{—— ②}$$

$$y^{-5} = t$$

$$-5y^{-6}\frac{dy}{dx} = \frac{dt}{dx}$$

$$y^{-6}\frac{dy}{dx} = \frac{-1}{5}\frac{dt}{dx}.$$

$$-\frac{1}{5}\frac{dt}{dx} + \frac{1}{x}t = x^2$$

$$\frac{dt}{dx} - \frac{5}{x}t = -5x^2 \quad \text{—— ③}$$

This is in the form $\frac{dt}{dx} + Pt = Q$

$$P = \frac{-5}{x} \quad Q = -5x^2$$

$$I.F = e^{\int P dx} = e^{\int -\frac{5}{x}dx} = e^{-5\int \frac{1}{x}dx} = e^{-5\log x} = e^{\log x^{-5}}$$

$$\boxed{I.F = x^{-5}.}$$

General solution of ③ is

$$t \cdot I.F = \int Q \cdot (I.F)\, dx$$

$$t \cdot x^{-5} = \int -5x^2 x^{-5}\, dx.$$

$$t x^{-5} = -\int 5x^{-3}\, dx$$

$t x^{-5} = -5\int x^{-3}\,dx + c$

$t x^{-5} = -5\frac{x^{-2}}{-2} + c$

$t x^{-5} = \frac{5}{2}x^{-2} + c$

$$\boxed{y^{-5}x^{-5} = \frac{5}{2}x^{-2} + c}$$

$$\frac{1}{x^5y^5} = \frac{5}{2x^2} + c$$

2 Solve $xy(1+xy^2)\frac{dy}{dx} = 1$

Given

$xy(1+xy^2)\frac{dy}{dx} = 1$

$(xy + x^2y^3)\frac{dy}{dx} = 1$

$\frac{dx}{dy} = xy + x^2y^3$

$\frac{dx}{dy} - yx = x^2y^3$ — ①

This is in the form $\frac{dx}{dy} + Px = Qx^n$

$P = -y$ $Q = y^3$ .. $n = 2$

$\frac{1}{x^2}\frac{dx}{dy} - y\frac{x}{x^2} = y^3$

$\frac{1}{x^2}\frac{dx}{dy} - y\frac{1}{x} = y^3$

$$x^{-2}\frac{dy}{dx} - yx^{-1} = y^3 \quad \text{— ②}$$

$$x^{-1} = t$$

$$-x^{-2}\frac{dx}{dy} = \frac{dt}{dy}$$

$$x^{-2}\frac{dx}{dy} = -\frac{dt}{dy}$$

$$-\frac{dt}{dy} - yt = y^3$$

$$\frac{dt}{dy} + yt = -y^3 \quad \text{— ③}$$

This is in the form $\frac{dt}{dy} + Pt = Q$

$$P = y \qquad Q = -y^3$$

$$I\cdot F = e^{\int P\,dy} = e^{\int y\,dy} = e^{y^2/2}$$

$$I\cdot F = e^{y^2/2}$$

General solution of ③ is

$$t\cdot[I\cdot F] = \int Q\cdot[I\cdot F]\,dy + c$$

$\int uv = u\int v - \int u'\int v + \int u''\iint v$

$$te^{y^2/2} = \int -y^3\cdot e^{y^2/2}\,dy + c$$

$$te^{y^2/2} = -\int e^{y^2/2}\,y^2\cdot y\,dy + c$$

$y^2/2 = v$

$\frac{2y\,dy}{2} = dv$

$y\,dy = dv$

$$te^{y^2/2} = -\int e^v\,2v + c$$

$$te^{y^2/2} = -2ve^v + \int 2e^v + c \Rightarrow te^{y^2/2} = -2ve^v + 2e^v + c$$

$$te^{y^2/2} = 2[-ve^v + e^v] + c$$

$$\frac{1}{x}e^{y^2/2} = 2\left[-\frac{y^2}{2}e^{y^2/2} + e^{y^2/2}\right] + c$$

3 Solve $\frac{dz}{dx} + \left(\frac{z}{x}\right)\log z = \frac{z}{x}(\log z)^2$

Given

$$\frac{dz}{dx} + \left(\frac{z}{x}\right)\log z = \frac{z}{x}(\log z)^2 \quad \text{--- (1)}$$

sub $\log z = t$

$$\frac{d}{dx}\log z = \frac{d}{dx}t$$

$$\frac{1}{z}\frac{dz}{dx} = \frac{dt}{dx}$$

$$\frac{dz}{dx} = z\frac{dt}{dx}$$

$$(1) \Rightarrow \quad z\frac{dt}{dx} + \frac{z}{x}t = \frac{z}{x}t^2.$$

$$\boxed{\frac{dt}{dx} + \frac{1}{x}t = \frac{1}{x}t^2} \quad \text{--- (2)}$$

This is in the form $\frac{dt}{dx} + Pt = Qt^n$

$$P = \frac{1}{x} \qquad Q = \frac{1}{x} \qquad n = 2.$$

$$\frac{1}{t^2}\frac{dt}{dx} + \frac{1}{x}\frac{t}{t^2} = \frac{1}{x}$$

$$\boxed{t^{-2}\frac{dt}{dx} + \frac{1}{x}t^{-1} = \frac{1}{x}} \quad \text{--- (3)}$$

Put $t^{-1} = m.$

$$\frac{d t^{-1}}{dx} = \frac{dm}{dx}$$

$$(-)t^{-2}\frac{dt}{dx} = \frac{dm}{dx}$$

$$t^{-2}\frac{dt}{dx} = -\frac{dm}{dx}$$

(3) $\Rightarrow$ $-\frac{dm}{dx} + \frac{1}{x} m = \frac{1}{x}$

$$\boxed{\frac{dm}{dx} - \frac{1}{x} m = -\frac{1}{x}} \quad \text{--- (4)}$$

This is in the form $\frac{dm}{dx} + Pm = Q$.

$P = -\frac{1}{x} \qquad Q = -\frac{1}{x}$.

$$I.F = e^{\int P\,dx} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1}$$

$$\boxed{I.F = \frac{1}{x}}$$

General Solution is

$$m(I.F) = \int Q \cdot (I.F) + c$$

$$m \frac{1}{x} = \int -\frac{1}{x} \frac{1}{x} + c$$

$$m \frac{1}{x} = -\int \frac{1}{x^2} + c$$

$$t^{-1} \frac{1}{x} = -\left(\frac{x^{-2+1}}{-2+1}\right) + c$$

$$t^{-1} \frac{1}{x} = -\left(\frac{x^{-1}}{-1}\right) + c$$

$$t^{-1} \frac{1}{x} = \frac{1}{x} + c$$

$$\frac{1}{xt} = \frac{1}{x} + c$$

$$\boxed{\frac{1}{x \log z} = \frac{1}{x} + c.}$$

4 $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

Given

$$\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$$

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + x\frac{\sin 2y}{\cos^2 y} = x^3$$

$$\sec^2 y\frac{dy}{dx} + x\frac{2\sin y\cos y}{\cos^2 y} = x^3$$

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3 \quad \text{--- } (1)$$

$$\tan y = t$$

$$\sec^2 y\frac{dy}{dx} = \frac{dt}{dx}.$$

$$\frac{dt}{dx} + 2xt = x^3 \quad \text{--- } (2)$$

This is in the form $\frac{dt}{dx} + Pt = Q.$

$$P = 2x \quad Q = x^3$$

$$I.F = e^{\int P dx} = e^{\int 2x\,dx} = e^{\cancel{2}x^2/\cancel{2}} = e^{x^2}$$

$$I.F = e^{x^2}$$

General solution of (2)

$$t(I.F) = \int Q(I.F) + c.$$

$$te^{x^2} = \int x^3 \cdot e^{x^2} dx + c$$

$$te^{x^2} = \int x^2 e^{x^2} x\,dx + c.$$

$\cancel{x}^2 = m$

$\frac{d}{dm} x^2 = \frac{d}{dm} m$

$2x\,dx = dm$

$x\,dx = \frac{dm}{2}$

$te^{x^2} = \int m\, e^m \frac{dm}{2} + c$

$te^{x^2} = \frac{1}{2}\int m\, e^m\, dm + c$

ILATE.

$te^{x^2} = \frac{1}{2}\left[me^m - \int 1.\, e^m\, dm\right] + c$

$te^{x^2} = \frac{1}{2}\left[me^m - e^m\right] + c$

$\tan y\, e^{x^2} = \frac{1}{2}\left[x^2 e^{x^2} - e^{x^2}\right] + c$

$\tan y\, e^{x^2} = \frac{x^2}{2} e^{x^2} - \frac{e^{x^2}}{2} + c$

5. Solve $y'+y=y^2$; $y(0)=-1$

Given

$y'+y=y^2$, $y(0)=-1$

$\frac{dy}{dx}+y=y^2$ —— ①

This is in the form

$\frac{dy}{dx}+Py=Qy^n$.

divide eq ① both sides with y^2

$y^{-2}\frac{dy}{dx}+y^{-1}=1$ —— ②

$y^{-1}=t$

$-\frac{dy}{dx}y^{-2}=\frac{dt}{dx} \Rightarrow \frac{dy}{dx}y^{-2}=-\frac{dt}{dx}$

From eq ② $-\frac{dt}{dx}+t=1$

$$\boxed{\frac{dt}{dx}-t=-1} \text{ —— ③}$$

This is in the form $\frac{dt}{dx}+Pt=Q$

$P=-1$ $Q=-1$.

$I.F=e^{\int P dx}=e^{\int -1\, dx}=e^{-x}$

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$I.F=e^{-x}$

General Solution is

$$t(I.F) = \int Q(I.F)\,dx + c.$$

$$t(e^{-x}) = \int -1\,(e^{-x})\,dx + c$$

$$te^{-x} = \frac{\cancel{-}e^{-x}}{\cancel{-1}} + c.$$

$$te^{-x} = e^{-x} + c$$

$$y^{-1}e^{-x} = e^{-x} + c.$$

$$\frac{e^{-x}}{y} = e^{-x} + c. \quad \text{—— (4)}$$

Put $x = 0$ & $y = -1$, we get

$$\frac{1}{-1} = \cancel{1+c}.$$

$$\frac{e^{-0}}{-1} = e^{-0} + c$$

$$\frac{1}{-1} = 1 + c$$

$$-1 = 1 + c$$

$$c = -2.$$

(4) $\Rightarrow$ $$\frac{e^{-x}}{y} = e^{-x} - 2$$

6. Solve $2yy' + y^2 \sin x = \sin x$; $y(0) = \sqrt{2}$

Sol. Given

$$2yy' + y^2 \sin x = \sin x$$

$$2y\frac{dy}{dx} + y^2 \sin x = \sin x$$

$$\frac{dy}{dx} + y\frac{\sin x}{2} = \frac{\sin x}{2y}$$

$$\frac{dy}{dx} + \frac{\sin x}{2}y = \frac{\sin x}{2}y^{-1} \quad \text{--- } (1)$$

It is in the form $\frac{dy}{dx} + Py = Qy^n$

$P = \frac{\sin x}{2}$ $\quad Q = \frac{\sin x}{2}$ $\quad n = -1$

Divide (1) with y^{-1} both sides.

$$y\frac{dy}{dx} + \frac{\sin x}{2}y^2 = \frac{\sin x}{2} \quad \text{--- } (2)$$

Put $y^2 = t$

$$2y\frac{dy}{dx} = \frac{dt}{dx}$$

$$y\frac{dy}{dx} = \frac{1}{2}\frac{dt}{dx}$$

$$\frac{1}{\cancel{2}}\frac{dt}{dx} + \frac{\sin x}{\cancel{2}}t = \frac{\sin x}{\cancel{2}} \quad \text{--- } (3)$$

It is in the form

$$\frac{dt}{dx} + \sin x\, t = \sin x \quad \text{--- } (3)$$

It is in the form $\frac{dt}{dx} + Pt = Q$

$P = \sin x \quad Q = \sin x.$

$I.F = e^{\int P dx} = e^{\int \sin x \, dx} = e^{-\cos x}$

General solution of ③ is

$t \cdot (I.F) = \int Q \cdot I.F \, dx + c.$

$t(e^{-\cos x}) = \int \sin x \, e^{\cos x} dx + c.$

Put $\cos x = m$

$-\sin x dx = dm$

$t e^{-\cos x} = \int -e^{-m} dm + c.$

$t e^{-\cos x} = + e^{m} + c.$

$y^2 e^{-\cos x} = + e^{-\cos x} + c. \quad$ —④

Put $x = 0, \ y = \sqrt{2}$ in ④

$(\sqrt{2})^2 e^{-\cos 0} = e^{-\cos 0} + c$

$2 e^{-1} = e^{-1} + c$

$\frac{2}{e} = \frac{1}{e} + c$

$c = \frac{2}{e} - \frac{1}{e} \Rightarrow c = \frac{1}{e}.$

④ $\Rightarrow y^2 e^{-\cos x} = e^{-\cos x} + \frac{1}{e}.$

8) $(x^3y^2+xy)\,dx = dy$

Given $(x^3y^2+xy)dx = dy$

$$x^3y^2+xy = \frac{dy}{dx}.$$

$$\frac{dy}{dx} - xy = x^3y^2 \quad \text{——} (1)$$

This is in the form $\frac{dy}{dx}+Py = Qy^n$

$$P=-x \quad Q=x^3$$

Divide eq with y^2 on bothsides

$$\frac{dy}{dx}\frac{1}{y^2} - x\frac{y}{y^2} = x^3$$

$$\frac{dy}{dx}y^{-2} - xy^{-1} = x^3 \quad \text{——} (2)$$

Let $y^{-1} = t$

$$\frac{d}{dx}y^{-1} = \frac{d}{dx}t$$

$$-y^{-2}\frac{dy}{dx} = \frac{dt}{dx}$$

$$y^{-2}\frac{dy}{dx} = -\frac{dt}{dx}.$$

$$-\frac{dt}{dx} - xt = x^3$$

$$\frac{dt}{dx} + xt = -x^3 \quad \text{——} (3)$$

This is in the form $\frac{dt}{dx}+Pt = Q.$

$$I.F = e^{\int P dx} = e^{\int x dx} = e^{\frac{x^2}{2}}$$

$$I.F = e^{x^2/2}$$

General solution of ③ is

$$t(I.F) = \int Q(I.F) dx + c.$$

$$t e^{x^2/2} = \int -x^3 e^{x^2/2} dx + c.$$

Put $\frac{x^2}{2} = m$

$$\frac{2x}{2} dx = dm$$

$$x dx = dm.$$

$$t e^{x^2/2} = \int -x^2 e^{x^2/2} x dx + c.$$

$$t e^{x^2/2} = -\int 2m e^m dm + c$$

ILATE

$$t e^{x^2/2} = -2 \int \underset{u}{m} \underset{v}{e^m} dm + c.$$

$$t e^{x^2/2} = -2 [m e^m - \int 1 . e^m dm] + c$$

$$t e^{x^2/2} = -2 [m e^m - \int e^m dm] + c$$

$$t e^{x^2/2} = -2 [m e^m - e^m] + c$$

$$t e^{x^2/2} = -2 m e^m + 2 e^m + c$$

$$y^{-1} e^{x^2/2} = -2 \frac{x^2}{2} e^{x^2/2} + 2 e^{x^2/2} + c$$

$$\frac{e^{x^2/2}}{y} = -2 x^2 e^{x^2/2} + 2 e^{x^2/2} + c.$$

Exact Differential Equations

The equation $Mdx + Ndy = 0$ is to be an Exact Differential Equation if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

General solution of $Mdx + Ndy = 0$ is

$\int M\,dx + \int N\,dy = \text{constant}$

[y constant] [terms not containing x]

① Solve $(hx+by+f)dy + (ax+hy+g)dx = 0$

Sol. Given

$$(hx+by+f)dy + (ax+hy+g)dx = 0$$

$$\boxed{(ax+hy+g)dx + (hx+by+f)dy = 0} \quad \text{—①}$$

This is in the form $Mdx + Ndy = 0$.

$M = ax+hy+g$	$N = hx+by+f$
$\frac{\partial M}{\partial x} = \frac{\partial}{\partial x}(ax+hy+g)$	$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(hx+by+f)$
$= 0 + h(1) + 0$	$= h(1) + 0 + 0$
$= h$	$= h$

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial x}.$$

① is Exact Differential Equation

$\therefore$ General Solution of ① is

$$\underset{[y\ \text{constant}]}{\int M\,dx} + \underset{[\text{terms not containing } x]}{\int N\,dy} = \text{constant}$$

$$\int (ax+hy+g)\,dx + \int (by+f)\,dy = c$$

$$a\frac{x^2}{2} + hyx + gx + \frac{by^2}{2} + fy = c$$

$$ax^2 + 2hxy + 2gx + by^2 + 2fy = 2c$$

② Solve $\left[y\left(1+\frac{1}{x}\right) + \cos y\right]dx + \left[x + \log x - x\sin y\right]dy = 0$

Given

$$\left[y\left(1+\frac{1}{x}\right) + \cos y\right]dx + \left[x + \log x - x\sin y\right]dy = 0 \quad \text{—①}$$

This is in the form $M\,dx + N\,dy = 0$.

$M = y\left(1+\frac{1}{x}\right) + \cos y$	$N = x + \log x - x\sin y$
$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[y\left(1+\frac{1}{x}\right) + \cos y\right]$	$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[x + \log x - x\sin y\right]$
$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y$	$\frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

① is Exact differential Equation.

General solution of ① is

$\int M\,dx + \int N\,dy = \text{constant}$

[y constant] [terms not contain x]

$\int y(1+\frac{1}{x}) + \cos y\, dx + \cancel{\int x + \log x - x\sin y\, dy}\ \int 0\, dy = c$

y const.

$y\int 1+\frac{1}{x}dx + \cos y\int 1\, dx = c$

$y[x+\log x] + x\cos y = c.$

3 Solve $(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$

Given

$(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$ — ①

This is in the form $M\,dx + N\,dy = 0.$

$M = y\cos x + \sin y + y$	$N = \sin x + x\cos y + x$
$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y\cos x + \sin y + y)$	$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(\sin x + x\cos y + x)$
$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$	$\frac{\partial N}{\partial x} = \cos x + \cos y + 1$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

① is a Exact D.E.

General Solution of ① is

$$\int M\,dx + \int N\,dy = \text{constant}$$

[y const] [terms not contain x]

$$\int y\cos x + \sin y + y\,dx + \int 0\,dy = c$$

$$y\int \cos x\,dx + \sin y\int 1\,dx + y\int 1\cdot dx = c$$

$$y\sin x + x\sin y + xy = c.$$

4. Solve $(y^2 - 2xy)\,dx - (x^2 - 2xy)\,dy = 0$

Given

$$(y^2 - 2xy)\,dx - (x^2 - 2xy)\,dy = 0 \quad \text{—①}$$

This is in the form $M\,dx + N\,dy = 0$

$M = y^2 - 2xy$	$N = -(x^2 - 2xy)$
$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y^2 - 2xy)$ $= 2y - 2x$	$\frac{\partial N}{\partial y} = \frac{\partial}{\partial y}(2xy - x^2)$ $= 2y - 2x$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$$

① is a exact DE.

General Solution of ① is

$$\int M\,dx + \int N\,dy = \text{constant}$$

[y const] [terms not contain x]

$$\int y^2 - 2xy\,dx + \int 0\,dy = c.$$

$y^2\int dx - 2y\int x\,dx = c$

$y^2x - 2y\frac{x^2}{2} = c$

$xy^2 - x^2y = c$

$xy(y-x) = c.$

5. Solve $(5x^4+3x^2y^2-2xy^3)dx+(2x^3y-3x^2y^2-5y^4)dy = 0$

Given

$(5x^4+3x^2y^2-2xy^3)dx + (2x^3y-3x^2y^2-5y^4)dy = 0$ — (1)

This is in the form $Mdx + Ndy = 0$

$M = 5x^4+3x^2y^2-2xy^3$	$N = 2x^3y-3x^2y^2-5y^4$
$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(5x^4+3x^2y^2-2xy^3)$ $= 3x^2(2y) - 2x(3y^2)$ $= 6x^2y - 6xy^2$	$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2x^3y-3x^2y^2-5y^4)$ $= 2y(3x^2) - 3y^2(2x)$ $= 6x^2y - 6xy^2$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(1) is a exact D.E

General Solution of (1) is

$\int M\,dx + \int N\,dy = \text{constant}$

[y constant] [terms not contain x]

$$\int 5x^4 + 3x^2y^2 - 2xy^3\, dx + \int 5y^4\, dy = c$$

$$\int 5x^4 dx + \int 3x^2y^2 dx - \int 2xy^3 dx + \int 5y^4 dy = c$$

$$5\int x^4 dx + 3y^2 \int x^2 dx - 2y^3 \int x\, dx + 5\int y^4 dy = c$$

$$\cancel{5}\left(\frac{x^5}{\cancel{5}}\right) + \cancel{3}y^2\left(\frac{x^3}{\cancel{3}}\right) - \cancel{2}y^3\left(\frac{x^2}{\cancel{2}}\right) + \cancel{5}\left(\frac{y^5}{\cancel{5}}\right) = c$$

$$x^5 + y^2x^3 - y^3x^2 + y^5 = c$$

$$x^5 + y^5 + x^2y^2(x-y) = c$$

X6. Solve $(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$

Given

$$(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0 \quad \text{— (1)}$$

This is in the form $Mdx + Ndy = 0$

$M = y^2 e^{xy^2} + 4x^3$	$N = 2xye^{xy^2} - 3y^2$

7 Solve $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$

Given
$$(x^2y-2xy^2)dx+(x^3-3x^2y)dy=0 \quad \text{—} ①$$

This is in the form $Mdx+Ndy=0$

$M=x^2y-2xy^2$	$N=-x^3+3x^2y$
$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(x^2y-2xy^2)$	$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(3x^2y-x^3)$
$=x^2(1)-2x(2y)$	$=6xy-3x^2$
$=x^2-4xy$	$=6xy-3x^2$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial y}$$

① is not an exact DE.

6. Solve $(y^2e^{xy^2}+4x^3)dx+(2xye^{xy^2}-3y^2)dy=0$

Given
$$(y^2e^{xy^2}+4x^3)dx+(2xye^{xy^2}-3y^2)dy=0 \quad \text{—} ①$$

This is in the form $Mdx+Ndy=0$

$M=y^2e^{xy^2}+4x^3$	$N=2xye^{xy^2}-3y^2$
$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(y^2e^{xy^2}+4x^3)$	$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(2xye^{xy^2}-3y^2)$
$=\frac{\partial}{\partial y}(y^2e^{xy^2})+\frac{\partial}{\partial y}(4x^3)$	$=\frac{\partial}{\partial x}(2xye^{xy^2})-\frac{\partial}{\partial y}(3y^2)$

$\frac{\partial M}{\partial y}$

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$$\frac{\partial M}{\partial y} = y^2\frac{\partial}{\partial y}e^{xy^2} + e^{xy^2}\frac{\partial}{\partial y}y^2 + 0$$

$$= y^2 e^{xy^2}\frac{\partial}{\partial y}(xy^2) + e^{xy^2}(2y)$$

$$= y^2 e^{xy^2} x(2y) + e^{xy^2}(2y)$$

$$= 2y\, e^{xy^2}(xy^2+1)$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}($$

$$\frac{\partial N}{\partial x} = 2y\left[x\frac{\partial}{\partial x}e^{xy^2} + e^{xy^2}\frac{\partial}{\partial x}x\right] + 0$$

$$= 2y\left[x e^{xy^2}\frac{\partial}{\partial x}xy^2 + e^{xy^2}(1)\right]$$

$$= 2y\left[x e^{xy^2}y^2 + e^{xy^2}\right]$$

$$= 2y e^{xy^2}(xy^2+1)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

(1) is an exact D.E

General solution of (1) is

$$\underset{y\ \text{const}}{\int M\,dx} + \underset{\substack{\text{terms not}\\ \text{contain } x}}{\int N\,dy} = \text{constant}$$

$$\int y^2 e^{xy^2}dx + \int 4x^3 dx + \int -3y^2 dy = c$$

$$\cancel{y^2}\frac{e^{xy^2}}{\cancel{y^2}} + \cancel{4}\frac{x^4}{\cancel{4}} - \cancel{3}\frac{y^3}{\cancel{3}} = c$$

$$e^{xy^2} + x^4 - y^3 = c.$$

$Mdx + Ndy = 0.$

Exact

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow$ **General Solution.**

$\int M dx + \int N dy = \text{constant}$

(y const) [terms not contain x]

Non Exact

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\Rightarrow$ Calculate Integrating Factor.

	I.F.
M, N are Homogeneous	$\frac{1}{Mx+Ny}$.
$y f(xy) dx + x g(xy) dy = 0$.	$\frac{1}{Mx-Ny}$
$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$	$e^{\int f(x) dx}$
$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$	$e^{\int g(y) dy}$

Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Sol. Given

$$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0 \quad \text{—} ①$$

This is in the form $Mdx + Ndy$

$M = x^2y - 2xy^2$ $\qquad$ $N = -x^3 + 3x^2y$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^2y - 2xy^2)$ $\qquad$ $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(3x^2y - x^3)$

$= x^2(1) - 2x(2y)$ $\qquad$ $= 3y(2x) - 3x^2$

$= x^2 - 4xy$ $\qquad$ $= 6xy - 3x^2$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

① is not an exact DE.

M,N are ~~① is a~~ homogeneous

$$I.F = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)x + (-x^3 + 3x^2y)y}$$

$$= \frac{1}{\cancel{x^3y} - 2x^2y^2 - \cancel{x^3y} + 3x^2y^2} = \frac{1}{x^2y^2}$$

$$I.F = \frac{1}{x^2y^2}.$$

Multiply I.F on both sides of ①.

$$\frac{1}{x^2y^2}(x^2y-2xy^2)dx - \frac{1}{x^2y^2}(x^3-3x^2y)\,dy = 0$$

$$\frac{x^2y}{x^2y^2}dx - \frac{2xy^2}{x^2y^2}dx - \frac{x^3}{x^2y^2}dy + \frac{3x^2y}{x^2y^2}dy = 0$$

$$\frac{1}{y}dx - \frac{2}{x}dx - \frac{x}{y^2}dy + \frac{3}{y}dy = 0$$

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx - \left(\frac{x}{y^2} + \frac{3}{y}\right)dy = 0 \quad \text{— (2)}$$

This is the form $M_1dx + N_1dxy = 0$

(2) is an exact D.E

General solution of (2) is

$$\int M_1 dx + \int N_1 dy = \text{constant}$$

[y constant] [terms not contain x]

$$\int\left(\frac{1}{y} - \frac{2}{x}\right)dx - \int \frac{3}{y}\,dy = \cancel{con}\ c$$

$$\int \frac{1}{y}dx - 2\int \frac{1}{x}dx + 3\int \frac{1}{y}dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

9. Solve $(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0$.

Given

$$(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0 \quad \text{--- (1)}$$

This is in the form. $M\,dx + N\,dy = 0$.

M	N
$M = (xy\sin xy + \cos xy)y$	$N = (xy\sin xy - \cos xy)x$
$M = xy^2\sin xy + y\cos xy$	$N = x^2y\sin xy - x\cos xy$
$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[xy^2\sin xy + y\cos xy]$	$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[x^2y\sin xy - x\cos xy]$
$= x\frac{\partial}{\partial y}y^2\sin xy + \frac{\partial}{\partial y}y\cos xy$	$= y\frac{\partial}{\partial x}x^2\sin xy - \frac{\partial}{\partial x}x\cos xy$
$= x[y^2\cos xy(x) + \sin xy(2y)] + [y(-\sin xy)(x) + \cos xy(1)]$	$= y[x^2\cos xy(y) + \sin xy(2x)] + [x(-\sin xy)y + \cos xy(1)]$
$= x^2y^2\cos xy + 2xy\sin xy - xy\sin xy + \cos xy$	$= x^2y^2\cos xy + 2xy\sin xy + xy\sin xy - \cos xy$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

(1) is not exact D.E.

The (1) is in the form $yf(xy)\,dx + xg(xy)\,dy = 0$.

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{(xy\sin xy + \cos xy)yx - xy(xy\sin xy - \cos xy)}$$

$$\frac{1}{\cancel{x^2y^2\sin xy} + xy\cos xy - \cancel{x^2y^2\sin xy} + xy\cos xy}$$

$$IF = \frac{1}{2xy\cos xy}$$

Multiply ① with I.F on both sides.

$$\frac{1}{2xy\cos xy}\, y(xy\sin xy + \cos xy)dx + \frac{1}{2xy\cos xy}\, x(xy\sin xy - \cos xy)dy = 0$$

$$\frac{xy\sin xy\,dx}{2x\cos xy} + \frac{\cos xy\,dx}{2x\cos xy} + \frac{xy\sin xy\,dy}{2y\cos xy} - \frac{\cos xy}{2y\cos xy}dy = 0$$

$$\frac{y}{2}\tan xy\,dx + \frac{1}{2x}dx + \frac{x}{2}\tan xy\,dy - \frac{1}{2y}dy = 0$$

$$\left(\frac{y}{2}\tan xy + \frac{1}{2x}\right)dx + \left(\frac{x}{2}\tan xy - \frac{1}{2y}\right)dy = 0 \quad \text{— ②}$$

This is in the form $M_1 dx + N_1 dy = 0$

② is an exact D.E.

General Solution of ② is

$$\int M_1 dx + \int N_1 dy = \text{constant}$$

(y constant) [terms not contain x]

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$$\int \frac{y}{2}\tan xy\,dx + \int \frac{1}{2x}dx - \int \frac{1}{2y}dy = c$$

$$\frac{y}{2}\int \tan xy\,dx + \frac{1}{2}\int \frac{1}{x}dx - \frac{1}{2}\int \frac{1}{y}dy = c$$

$$\int \tan ax = \frac{1}{a}\log(\sec ax - \tan ax)$$

$$\cancel{\frac{y}{2}}\left[\cancel{\frac{1}{y}}\log(\sec xy - \tan xy)\right] + \cancel{\frac{1}{2}}\log x - \cancel{\frac{1}{2}}\log y = \cancel{\frac{1}{2}}\log c$$

$$\log(\sec xy - \tan xy) + \log x - \log y = \log c.$$

$$\log(\sec xy - \tan xy) + \log\left(\frac{x}{y}\right) = \log c$$

$$\log\left(\frac{(\sec xy - \tan xy)x}{y}\right) = \log c$$

$$\frac{(\sec xy - \tan xy)x}{y} = c.$$

$$x(\sec xy - \tan xy) = cy$$

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3. Solve $\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx+\frac{1}{4}(x+xy^2)dy=0$

Given

$$\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx+\frac{1}{4}(x+xy^2)dy=0 \quad \text{--- (1)}$$

This is in the form $Mdx+Ndy=0$.

M	N
$M = y+\frac{y^3}{3}+\frac{x^2}{2}$	$N = \frac{x}{4}+\frac{xy^2}{4}$
$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)$	$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{4}+\frac{xy^2}{4}\right)$
$= 1+\frac{\cancel{3}y^2}{\cancel{3}}+0$	$= \frac{1}{4}+\frac{y^2}{4}$
$= 1+y^2$	$= \frac{1}{4}(1+y^2)$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

(1) is not an exact D.E.

To find I.F

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}$$

$$1+y^2-\frac{1}{4}(1+y^2) = 1+y^2\left(1-\frac{1}{4}\right) = 1+y^2\left(\frac{3}{4}\right) = \frac{3}{4}(1+y^2)$$

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} = \frac{3}{4}(1+y^2)$$

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$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{\frac{1}{4}x(1+y^2)}(1+y^2)\frac{3}{4}$$

$$=\frac{3}{x}=f(x)$$

$$I.F=e^{\int f(x)dx}=e^{\int\frac{3}{x}dx}=e^{3\log x}$$

$$=e^{\log x^3}=x^3.$$

$$I.F=x^3$$

Multiply ① Both sides with I.F.

$$x^3\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx+\frac{1}{4}x^3(x+xy^2)dy=0$$

$$\left[x^3y+\frac{(xy)^3}{3}+\frac{x^5}{2}\right]dx+\left[\frac{1}{4}x^4+(x^2y)^2\right]dy=0. \quad — ②$$

This is in the form $M_1dx+N_1dy=0$.

② is an exact D.E.

General solution of ② is

$$\int M_1dx+\int N_1dy=\text{constant}$$

[y constant] [terms not contain x.]

$$\int x^3y\,dx+\int\frac{x^3y^3}{3}dx+\int\frac{x^5}{2}dx+0=c$$

$$y\int x^3dx+\frac{y^3}{3}\int x^3dx+\frac{1}{2}\int x^5dx=c.$$

$$y\frac{x^4}{4}+\frac{y^3x^4}{12}+\frac{1}{12}x^6=c.$$

4. Solve $y^2dx + (x^2 - xy - y^2)\,dy = 0$

Given

$$y^2dx + (x^2 - xy - y^2)\,dy = 0 \quad \text{—} ①$$

This is in the form $Mdx + Ndy = 0$

M	N
$M = y^2$	$N = x^2 - xy - y^2$
$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y^2)$	$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2 - xy - y^2)$
$\frac{\partial M}{\partial y} = 2y$	$\frac{\partial N}{\partial x} = 2x - y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

① is not an exact D.E

M, N are homogeneous.

$$I.F = \frac{1}{Mx+Ny} = \frac{1}{x(y^2) + y(x^2 - xy - y^2)} = \frac{1}{\cancel{xy^2} + x^2y - \cancel{xy^2} - y^3}$$

$$I.F = \frac{1}{x^2y - y^3}$$

Multiply I.F on bothsides of ①

$$① \Rightarrow y^2dx + (x^2 - xy - y^2)dy = 0$$

$$\frac{1}{x^2y - y^3}\left(y^2dx + (x^2 - xy - y^2)dy\right) = \frac{1}{x^2y - y^3}\,(0)$$

$$\frac{y^2}{(x^2-y^2)y}\,dx + \left[\frac{x^2-y^2-xy}{(x^2-y^2)y}\right]dy = 0$$

$$\frac{y}{x^2-y^2}\,dx + \frac{\cancel{x^2-y^2}}{\cancel{(x^2-y^2)}y}\,dy - \frac{xy}{(x^2-y^2)y}\,dy = 0$$

$$\frac{y}{x^2-y^2}\,dx + \frac{1}{y}\,dy - \frac{x}{x^2-y^2}\,dy = 0$$

$$\frac{y}{x^2-y^2}\,dx + \left(\frac{1}{y} - \frac{x}{x^2-y^2}\right)dy = 0 \quad \text{— (2)}$$

This is in the form $M_1 dx + N_1 dy = 0$.

General solution of (2) is

$$\int M_1\,dx + \int N_1\,dy = \text{constant}$$

y constant   terms not contain x

$$\int \frac{y}{x^2-y^2}\,dx + \int \frac{1}{y}\,dy = c$$

$$y\int \frac{1}{x^2-y^2}\,dx + \int \frac{1}{y}\,dy = c$$

$$y\left[\frac{-1}{2y}\log\left|\frac{x+y}{x-y}\right|\right] + \log y = \log c.$$

$$\log\left|\frac{x+y}{x-y}\right|^{-1/2} + \log y = \log c$$

$$\log\left(\sqrt{\frac{x-y}{x+y}}\cdot y\right) = \log c$$

$$y\sqrt{\frac{x-y}{x+y}} = c.$$

5. Solve $y(1+xy)dx + x(1-xy)dy = 0$

Given

$$y(1+xy)dx + x(1-xy)\,dy = 0 \quad \text{—— ①}$$

This is in the form $Mdx + Ndy = 0$

M	N
$M = y(1+xy)$	$N = x(1-xy)$
$M = y + xy^2$	$N = x - x^2y$
$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y+xy^2)$	$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x - x^2y)$
$\frac{\partial M}{\partial y} = 1+2xy$	$\frac{\partial N}{\partial x} = 1 - 2xy$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

① is not an exact D.E

① is in the form $yf(xy)dx + xg(xy)dy = 0$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{(y+xy^2)x - (x-x^2y)y} = \frac{1}{\cancel{xy} + x^2y^2 - \cancel{xy} + x^2y^2}$$

$$I.F = \frac{1}{2x^2y^2}$$

Multiply ① with I.F on both sides

$$① \Rightarrow (y+xy^2)dx + (x - x^2y)\,dy = 0.$$

$$\frac{1}{2x^2y^2}\left[y(1+xy)dx + x(1-xy)dy\right] = \frac{1}{2x^2y^2}(0)$$

$$\left(\frac{1+xy}{2x^2y}\right)dx + \left(\frac{1-xy}{2xy^2}\right)dy = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0 \quad \text{--- (2)}$$

This is in the form $M_1 dx + N_1 dy = 0$

(2) is an exact D.E

General solution of (2) is

$$\int M_1\, dx + \int N_1\, dy = \text{constant}$$

y constant   terms not contain x

$$\int \frac{1}{2x^2y}\,dx + \int \frac{1}{2x}\,dx - \int \frac{1}{2y}\,dy = c$$

$$\frac{1}{2y}\int \frac{1}{x^2}\,dx + \frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{y}\,dy = c$$

$$\frac{1}{2y}\left(\frac{-1}{x}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = \frac{1}{2}\log c.$$

$$\frac{-1}{xy} + \log x - \log y = \log c.$$

$$\log\frac{x}{y} - \frac{1}{xy} = \log c.$$

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6. Solve $(xy^3+y)dx + 2(x^2y^2+x+y^4)dy = 0$

Given

$$(xy^3+y)dx + 2(x^2y^2+x+y^4)dy = 0 \quad \text{--- (1)}$$

This is in the form $Mdx + Ndy = 0$

M	N
$M = xy^3+y$	$N = 2(x^2y^2+x+y^4)$
$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(xy^3+y)$	$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(2(x^2y^2+x+y^4)\right)$
$\frac{\partial M}{\partial y} = x(3y^2)+1$	$\frac{\partial N}{\partial x} = 2(y^2(2x)+1+0)$
$\frac{\partial M}{\partial y} = 3xy^2+1$	$\frac{\partial N}{\partial x} = 4xy^2+2$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

(1) is not an exact D.E.

<u>To Find I.F</u>

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3xy^2+1-(4xy^2+2) = 3xy^2+1-4xy^2-2$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -xy^2-1.$$

$$\frac{-1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-1}{xy^3+y}(-xy^2-1) = \frac{-1}{\cancel{(xy^2+1)}y}\left[-\cancel{(xy^2+1)}\right]$$

$$= \frac{1}{y} = g(y).$$

$$I.F = e^{\int g(y)dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y.$$

$I.F = y.$

Multiply ① with IF on both sides

$① \Rightarrow (xy^3+y)dx + 2(x^2y^2+x+y^4)dy = 0$

$y[(xy^3+y)dx + 2(x^2y^2+x+y^4)dy] = 0(y)$

$y(xy^3+y)dx + 2y(x^2y^2+x+y^4)dy = 0$

$(xy^4+y^2)dx + 2(x^2y^3+xy+y^5)dy = 0$ — ②

This is in the form $M_1 dx + N_1 dy = 0$

② is an exact D.E

General solution of ② is

$\int\limits_{y\text{ constant}} M_1 dx + \int\limits_{\text{terms not contain } x} N_1 dy = \text{constant}$

$\int(xy^4+y^2)dx + 2\int \cancel{x}y^5 dy = \cancel{c}$

$\int xy^4 dx + \int y^2 dx + 2\int y^5 dy = c$

$y^4\int x dx + y^2\int 1 \cdot dx + 2\int y^5 dy = c$

$y^4 \frac{x^2}{2} + y^2 x + 2\frac{y^6}{\cancel{6}_3} = c.$

$\frac{x^2y^4}{2} + \cancel{2}xy^2 + \frac{y^6}{3} = c.$

$3x^2y^4 + 6xy^2 + 2y^6 = 6c.$

7 Solve $(2xy)dy - (x^2+y^2+1)dx = 0$

Given

$$2xy\,dy - (x^2+y^2+1)dx = 0$$

$$(x^2+y^2+1)dx - 2xy\,dy = 0 \quad \text{---} \quad ①$$

This is in the form $M\,dx + N\,dy = 0$

M	N
$M = x^2+y^2+1$	$N = -2xy$
$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2+1)$	$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-2xy)$
$\frac{\partial M}{\partial y} = 2y$	$\frac{\partial N}{\partial x} = -2y.$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

① is not an exact D.E.

To Find I.F

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y-(-2y) = 2y+2y = 4y.$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4y$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-1}{\cancel{2}xy}(\overset{2}{\cancel{4}}y) = -\frac{2}{x} = f(x)$$

$$I.F = e^{\int f(x)dx} = e^{\int \frac{2}{x}dx} = e^{-2\int \frac{1}{x}dx}$$

$$I.F = e^{-2\log x} = e^{\log x^{-2}}$$

$$I.F = x^{-2}$$

Multiply ① with I.F on both sides.

$$\frac{2xy}{x^2}\,dy - \frac{x^2+y^2+1}{x^2}\,dx = 0$$

$$2\frac{y}{x}\,dy - \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx = 0.$$

$$\left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx - 2\frac{y}{x}\,dy = 0 \quad \text{—— ②}$$

This is in the form $M_1 dx + N_1 dy = 0$.

② is an exact D.E.

General solution of ② is.

$$\underset{y\ \text{constant}}{\int M_1\,dx} + \underset{\substack{\text{terms not}\\ \text{contain } x}}{\int N_1\,dy} = \text{constant}$$

$$\int\left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx - \int 0\,dy = c$$

$$\int 1\,dx + y^2\int\frac{1}{x^2}\,dx + \int\frac{1}{x^2}\,dx = c$$

$$x + y^2\left(\frac{-1}{x}\right) + \left(\frac{-1}{x}\right) = c$$

$$x - \frac{y^2}{x} - \frac{1}{x} = c.$$