(10) If $\bar{F} = (4xy - 3x^2z^2)\bar{i} + 2x^2\bar{j} - 2x^3z\bar{k}$, prove that $\int_C \bar{F}\cdot d\bar{r}$. ie, work done is independent of the curve joining two points.

Sol: Given $\bar{F} = (4xy - 3x^2z^2)\bar{i} + 2x^2\bar{j} - 2x^3z\bar{k}$

To find work done independent of path

$$\text{curl } \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \bar{i}[0-0] - \bar{j}[-6x^2z + 6x^2z] + \bar{k}[4x - 4x]$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k} = \bar{0}.$$

$\therefore$ $\bar{F}$ is conservative.

Green's theorem in a plane:-

(Transformation between Line integral and Double integral)

If R is a closed region in xy plane bounded by a simple closed curve c and if M and N are continuous functions of x and y having continuous derivatives in R, then

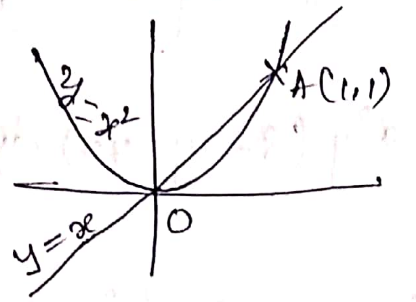
$$\boxed{\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

where c is traversed in the positive (anticlock-wise) direction.

Ex① Verify Green's theorem for $\int_C (xy+y^2)\,dx + x^2 dy$ where C is bounded by $y=x$ and $y=x^2$

Sol: Given $I_C = \int_C (xy+y^2)\,dx + x^2 dy$

C: bouded by $y=x$ and $y=x^2$



Put $y=x$ in $y=x^2$

$\Rightarrow x = x^2$

$\Rightarrow x - x^2 = 0$

$\Rightarrow x(1-x)=0$

$\Rightarrow x=0,\ x=1$

$\Rightarrow \because y=x$

$\Rightarrow y=0$ and $y=1$

Intersection points: $(0,0)$ and $(1,1)$

To verify Green's theorem, we have to prove

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

i.e., $I_C = I_R$.

To find I_C :- Here $I_C = I_{OA} + I_{AO}$

Along OA :- Equation of OA : $y = x^2$

$\Rightarrow dy = 2x\,dx.$

$$I_{OA} = \int_O^A (xy+y^2)\,dx + x^2 dy$$

$$= \int_O^A \left[x\cdot x^2 + (x^2)^2\right] dx + x^2 \cdot 2x\,dx$$

$$= \int_O^A (3x^3 + x^4)\,dx$$

$$= \left.\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Along AO :- Equation of AO is $y=x$

$\Rightarrow dy = dx$

$$I_{AO} = \int_A^0 (xy+y^2)\,dx + x^2 dy$$

$$= \int_A^0 (x^2+x^2+x^2)\,dx = \int_A^0 3x^2 dx$$

$$= \left.\frac{3x^3}{3}\right]_1^0 = -1$$

$$\therefore \boxed{I_C = \frac{19}{20} - 1 = \frac{-1}{20}} \quad \text{——— (A)}$$

To find I_R i.e., $\iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dx\,dy$

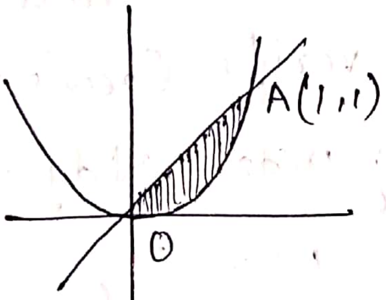
Comparing given integral with $\int M\,dx + N\,dy$.

$M = xy + y^2$ and $N = x^2$

$$\frac{\partial M}{\partial y} = x + 2y \qquad \frac{\partial N}{\partial x} = 2x.$$

$$\therefore I_R = \iint_R 2x - (x+2y)\,dx\,dy$$

$$= \iint_R (x-2y)\,dx\,dy$$



$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x-2y)\,dy\,.dx.$$

[Keep x constant]

R: y limits $y=x^2$ to $y=x$

x limits $x=0$ to $x=1$

$$= \int_{x=0}^{1} \left. x \cdot y - \frac{2y^2}{2} \right]_{x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left[\cancel{x^2} - \cancel{x^2} - x^3 + x^4\right] dx = \left. -\frac{x^4}{4} + \frac{x^5}{5} \right]_0^1$$

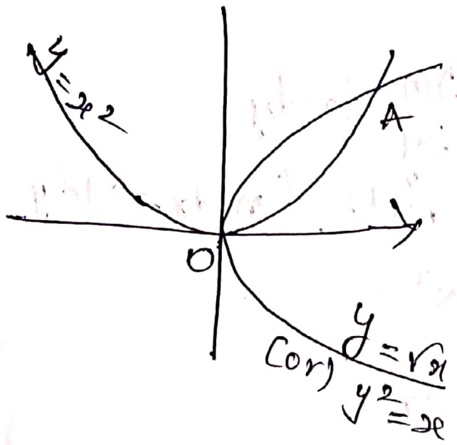
$$= -\frac{1}{4} + \frac{1}{5} = \boxed{\frac{-1}{20} = I_R} \quad \text{——— (B)}$$

from (A) and (B), Green's theorem is verified.

② Verify Green's theorem in plane for $\oint (3x^2-8y^2)dx + (4y-6xy)dy$ where C is the region bounded by $y=\sqrt{x}$ and $y=x^2$

Sol: Given $I = \oint_C (3x^2-8y^2)dx + (4y-6xy)dy$

C: region bounded by $y=\sqrt{x}$ and $y=x^2$



Put $y=\sqrt{x}$ in $y=x^2$

$\Rightarrow \sqrt{x}=x^2$

$\Rightarrow x=x^4$

$\Rightarrow x-x^4=0$

$\Rightarrow x(1-x^3)=0$

$\Rightarrow x=0 \quad x^3=1$

$\Rightarrow x=1$

$\because y=\sqrt{x}$

$\Rightarrow y=0 \text{ \& } y=1$

Points of intersection : $O(0,0), (1,1) A$

To verify Green's theorem, we have to prove that

$$\oint_C M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dx\,dy$$

i.e., $I_C = I_R$

To <u>find I_C</u> : Here $I_C = I_{OA} + I_{AO}$

<u>Along OA</u> (Parabola $y=x^2$)

Equation of OA : $y=x^2$

$\Rightarrow dy = 2x\,dx$.

$$I_{OA} = \int_O^A (3x^2-8y^2)dx + (4y-6xy)dy$$

$$= \int_0^A (3x^2-8x^4)dx + (4x^2-6x^3)2x\,dx$$

$$= \int_0^A (3x^2+8x^3-20x^4)\,dx$$

$$= \left. x^3+2x^2-4x^5 \right]_0^1 = 1+2-4 = -1$$

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Along AO :- Eq'n of AO is $y=\sqrt{x} \Rightarrow dy = \frac{1}{2\sqrt{x}}\,dx$

$$I_{AO} = \int_A^0 (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

$$= \int_A^0 (3x^2-8x)\,dx + (4\sqrt{x}-6x\sqrt{x})\frac{1}{2\sqrt{x}}\,dx$$

$$= \int_A^0 (3x^2-8x+2-3x)\,dx$$

$$= \left[\frac{3x^3}{3} - \frac{8x^2}{2} + 2x - \frac{3x^2}{2}\right]_1^0$$

$$= -\left[1-4+2-\frac{3}{2}\right] = \frac{5}{2}$$

$$\therefore I_C = I_{OA} + I_{AO} = -1+\frac{5}{2} = \frac{3}{2} \quad \text{——— (A)}$$

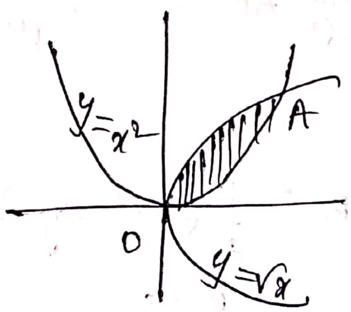
To find I_R i.e., $\iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dx\,dy$

Comparing given integral with $\int_C M\,dx + N\,dy$

$M = 3x^2-8y^2 \qquad N = 4y-6xy$

$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$

$$\therefore I_R = \iint_R -6y+16y\,dy\,dx$$



$$= \iint_R 10y\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} 10y\,dy\,dx$$ [Keep x constant]

y limits: $y=x^2$ to $y=\sqrt{x}$

x limits: $x=0$ to $x=1$

$$= \int_{x=0}^{1} \left[5y^2\right]_{x^2}^{\sqrt{x}}\,dx$$

$$= \int_{x=0}^{1} (5x-5x^4)\,dx = \left[\frac{5x^2}{2} - x^5\right]_0^1$$

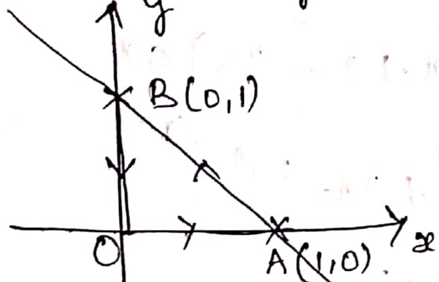
$$= 5/2 - 1 = 3/2 \quad \text{——— (B)}$$

from (A) & (B), Green's theorem is verified.

(3) Verify Green's theorem for $\int_c (3x^2-8y^2)dx+(4y-6xy)dy$ where c is the region bounded by $x=0, y=0, x+y=1$

Sol: Given $I_{c} = \int_c (3x^2-8y^2)dx+(4y-6xy)dy$

C: region bounded by $x=0, y=0, x+y=1$



To verify Green's theorem, we have to prove

$$\oint_c M\,dx+N\,dy = \iint_R \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\,dx\,dy$$

i.e., $I_c = I_R$.

To find I_c Here $I_c = I_{OA}+I_{AB}+I_{BO}$

Along OA :- Equation of OA (x-axis) is $y=0$
$\Rightarrow dy=0$

$$\therefore I_{OA} = \int_O^A (3x^2-8y^2)dx+(4y-6xy)\,dy$$

$$= \int_0^A 3x^2\,dx = \frac{3x^3}{3} = x^3\Big]_0^1 = 1$$

Along AB :- Equation of AB $x+y=1$
$\Rightarrow y=1-x$
$\Rightarrow dy=-dx.$

$$I_{AB} = \int_A^B (3x^2-8y^2)\,dx+(4y-6xy)\,dy$$

$$= \int_A^B \left[3x^2-8(1-x)^2\right]dx+\left[4(1-x)-6x(1-x)\right](-dx)$$

$$= \int_A^B \left[3x^2-8(1-x^2)^2+4x-4+6x-6x^2\right]dx.$$

$$= x^3-\frac{8(1-x)^3}{-3}+2x^2-4x+\frac{6x^2}{2}-\frac{6x^3}{3}\Big]_1^0$$

$$= \frac{8}{3}-1-2+4-3+2 = \frac{8}{3}$$

Along BO: Equation of BO (y-axis) is $x=0$
$dx=0$

$\therefore I_{BO} = \int_B^0 (3x^2-8y^2)dx + (4y-6xy)\,dy$

$= \int_B^0 4y\,dy = \frac{4y^2}{2} = 2y^2 \Big]_1^0 = -2.$

$\therefore I_c = 1 + \frac{8}{3} - 2 = \frac{5}{3}$ —— (A)

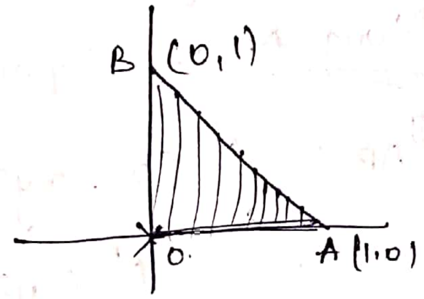
<u>To find I_R</u> i.e., $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Compare the given integral with $\int M\,dx + N\,dy$

Here $M = 3x^2 - 8y^2$ $\quad N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$ $\quad \frac{\partial N}{\partial x} = -6y.$

$I_R = \iint_R (-6y + 16y)\,dx\,dy$



$= \iint_R 10y\,dx\,dy$

$= \int_{x=0}^{1} \int_{y=0}^{1-x} 10y\,dy\,dx.$ [Keep x constant] $\quad$ y limits $y=0$ to $y=1-x$; x limits $x=0$ to $x=1$

$= \int_{x=0}^{1} 5y^2 \Big]_0^{1-x} dx$

$= \int_{x=0}^{1} 5(1-x)^2\,dx = \frac{5(1-x)^3}{-3}\Big]_0^1$

$= 0 - \frac{5}{-3} = \frac{5}{3}$ —— (B)

from (A) & (B), $I_c = I_R$

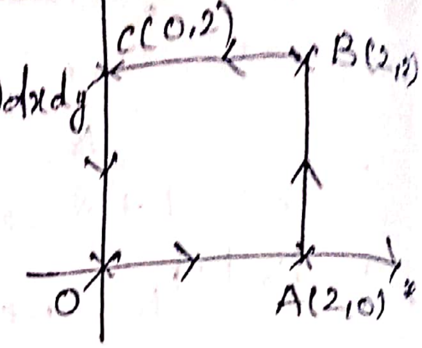
Hence Green's theorem is verified.

(4) Verify Green's theorem, in the plane for $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$ where C is a square with vertices (0,0), (2,0), (2,2), (0,2)

<u>Sol:</u> Given $I_c = \oint_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$

C: square with vertices (0,0), (2,0), (2,2) (0,2)

To verify Green's theorem, we have to prove $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$



i.e., $I_C = I_R$.

To find I_C:- ~~Here I_C~~

Here $I_C = I_{OA} + I_{AB} + I_{BC} + I_{CO}$

Along OA:- Equation of OA is $y = 0 \Rightarrow dy = 0$

$$I_{OA} = \int_O^A (x^2 - xy^3) dx + (y^2 - 2xy) dy$$

$$= \int_0^A x^2 dx = \left.\frac{x^3}{3}\right]_0^2 = 8/3$$

Along AB:- Equation of (AB) is $x = 2$ $\Rightarrow dx = 0$.

$$I_{AB} = \int_A^B (x^2 - xy^3) dx + (y^2 - 2xy) dy$$

$$= \int_A^B (y^2 - 4y) dy = \left.\frac{y^3}{3} - \frac{4y^2}{2}\right]_0^2$$

$$= \frac{8}{3} - \frac{8}{\cancel{2}} = \frac{-16}{3}$$

Along BC:- Equation of BC is $y = 2 \Rightarrow dy = 0$

$$\therefore I_{BC} = \int_B^C (x^2 - xy^3) dx + (y^2 - 2xy) dy$$

$$= \int_B^C (x^2 - 8x) dx = \left.\frac{x^3}{3} - \frac{8x^2}{2}\right]_2^0$$

$$= -\frac{8}{3} + \cancel{8}\,16 = \cancel{\frac{16}{3}}\ \frac{40}{3}$$

Along CO:- Equation of CO is $x = 0$ $\Rightarrow dx = 0$

$$I_{CO} = \int_C^O (x^2 - xy^3) dx + (y^2 - 2xy) dy$$

$$= \int_C^O y^2 dy = \left.\frac{y^3}{3}\right]_2^0 = -\frac{8}{3}$$

$$\therefore I_C = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = \frac{24}{3} = 8. \longrightarrow (A)$$

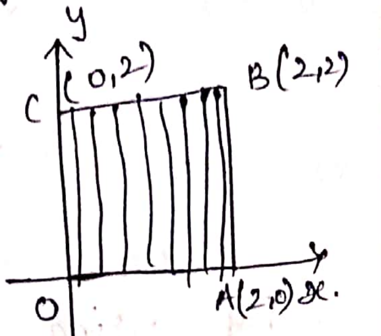
To find I_R i.e., $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Compare the given integral with $\int M dx + N dy$

Here $M = x^2 - xy^3$ $N = y^2 - 2xy$

$$\frac{\partial M}{\partial y} = -3xy^2 \qquad \frac{\partial N}{\partial x} = -2y$$

$$I_R = \iint_R (-2y + 3xy^2)\, dx\,dy$$



$$= \int_{x=0}^{2} \int_{y=0}^{2} (-2y + 3xy^2)\, dy\, dx. \quad \text{[Keep } x \text{ constant]}$$

y limits $y=0$ to $y=2$

x limits $x=0$ to $x=2$

$$= \int_{x=0}^{2} \left[-\frac{2y^2}{2} + \frac{3xy^3}{3} \right]_0^2 dx$$

$$= \int_{x=0}^{2} (-4 + 8x)\, dx$$

$$= \left[-4x + \frac{8x^2}{2} \right]_0^2 = -8 + 16 = \underline{\underline{8}} \quad \text{——— (B)}$$

from (A) & (B), $I_C = I_R$.

Hence Green's theorem is verified.

(5) Using Green's theorem evaluate

$\int_C (2xy - x^2)\, dx + (x^2 + y^2)\, dy$, where "C" is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

Sol: Given $I_C = \int_C (2xy - x^2)\, dx + (x^2 + y^2)\, dy$

C: closed curve of the region bounded by

$y = x^2$ and $y^2 = x$

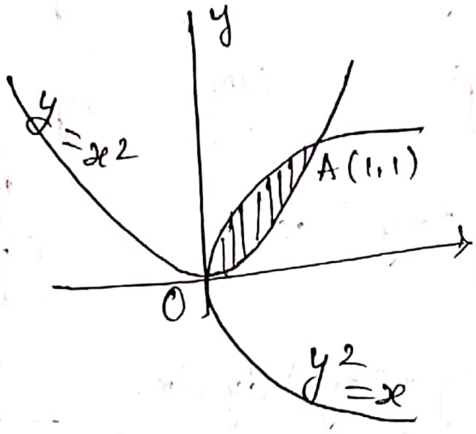
Put $y = x^2$ in $y^2 = x \Rightarrow (x^2)^2 = x \Rightarrow x^4 = x$

$x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0$ (or) $x^3 - 1 = 0$

$x=0$, y $x=1$

$\because$ $y=x^2 \Rightarrow y=0$ for $x=0$ and $y=1$ for $x=1$

Point of intersections $(0,0)$ & $(1,1)$



y limits:
$y=x^2$ to $y=\sqrt{x}$

x limits
~~x (0,0) to (1,1)~~
$x=0$ to $x=1$

By using Green's theorem. $I_c = I_R$

i.e., $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Comparing given interal with $\oint_C M\,dx + N\,dy$

Here $M = 2xy - x^2$ $\qquad N = x^2 + y^2$

$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x$

$\therefore I_c = I_R = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$= \iint_R (2x - 2x)\,dx\,dy$

$= \iint_R 0\,dx\,dy = 0.$

$\boxed{\therefore I_c = 0}$

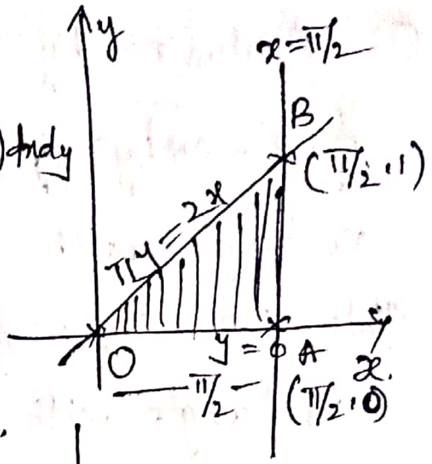
(6) Evaluate by Green's theorem $\oint_C (y - \sin x)dx + \cos x\,dy$ where C is the triangle enclosed by the lines $y=0$, $x=\pi/2$, ~~$\pi/y = 2$~~ $\pi y = 2x$.

Sol: Given $I_c = \oint_C (y - \sin x)dx + \cos x\,dy$

C: Triangle enclosed by the lines $y=0$, $x=\pi/2$, $\pi y = 2x$.

By using Green's theorem, $I_C = I_R$

i.e., $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$



Comparing given integral with

$\oint_C M dx + N dy$

Here . $M = y - \sin x$, $N = \cos x$.

$\frac{\partial M}{\partial y} = 1$ $\qquad \frac{\partial N}{\partial x} = -\sin x$.

y limits

$y = 0$ to $y = \frac{2x}{\pi}$

x limits.

$x = 0$ to $x = \pi/2$

$\therefore I_C = I_R = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} dx dy$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{2x/\pi} (-\sin x - 1) dx dy \quad [\text{Keep } x \text{ constant}]$$

$$= \int_{x=0}^{\pi/2} \left[(-\sin x - 1) \cdot y\right]_0^{2x/\pi} dx.$$

$$= -\frac{2}{\pi} \int_0^{\pi/2} x(\sin x + 1) dx$$

$$= -\frac{2}{\pi} \int_0^{\pi/2} (x\sin x + x) dx$$

$$= -\frac{2}{\pi} \left\{ \left[-x\cos x - \int 1(-\cos x) dx\right]_0^{\pi/2} + \left.\frac{x^2}{2}\right]_0^{\pi/2} \right\}$$

$$= -\frac{2}{\pi} \left\{ -x\cos x + \sin x + \left.\frac{x^2}{2}\right]_0^{\pi/2} \right\}$$

$$= -\frac{2}{\pi} \left\{ 0 + 1 + \frac{\pi^2}{8} \right\}$$

$$= -\frac{2}{\pi} - \frac{\pi}{4}.$$

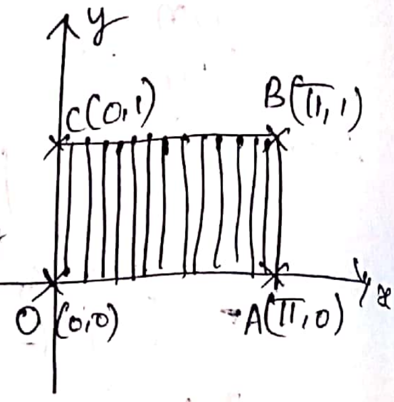
(7) Evaluate by Green's theorem,

$\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$, where c is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,1)$, $(0,1)$

Sol: Let $I_C = \oint (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$

C: retangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,1)$, $(0,1)$

By Using Green's theorem,



$I_C = I_R$

i.e., $\oint M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Comparing given integral with

$\oint_C M\,dx + N\,dy$

Here $M = (x^2 - \cosh y)$ $N = y + \sin x$.

Now, $\frac{\partial M}{\partial y} = 0 - \sinh y$ $\frac{\partial N}{\partial x} = \cos x$.

$\Rightarrow I_C = I_R = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$= \int_{x=0}^{\pi} \int_{y=0}^{1} (\cos x + \sinh y)\,dy\,dx$ [Keep x constant]

from graph
y limits: $y = 0$ to $y = 1$
x limits: $x = 0$ to $x = \pi$

$= \int_{x=0}^{\pi} \left. y\cos x + \cosh y \right]_0^1 dx$

$= \int_{x=0}^{\pi} (\cos x + \cosh 1 - \cosh 0)\,dx$

$= \left. \sin x + x\cosh 1 - x\cosh 0 \right]_0^{\pi}$

$= \pi\cosh 1 - \pi\cosh 0$

$= \pi\cosh 1 - \pi$