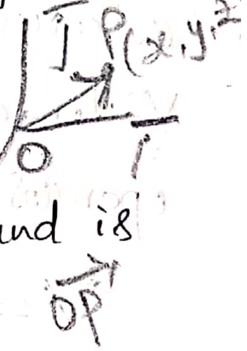
Vector Differentiation

Vector: A vector is a physical quantity which has both magnitude and direction.



Ex: $x\bar{i} + y\bar{j} + z\bar{k} = \bar{a}$ is a vector and its magnitude is denoted by $|\bar{a}|$ and is given by $|\bar{a}|$ (or) $a = \sqrt{x^2+y^2+z^2}$

$\bar{i} - 2\bar{j} + 3\bar{k} \rightarrow \sqrt{1+4+9} = \sqrt{14}$

Scalar point function:- If to each point $P(x,y,z)$ of a region E in space there corresponds a scalar denoted by $\phi(x,y,z)$, then $\phi(x,y,z)$ is called a scalar point function.

Eg: 1, Temperature at any instant
2, Density of a body.
3, Potential due to gravitational matter.

Vector point function:- If to each point $P(x,y,z)$ there corresponds a definite vector denoted by $\bar{F}(x,y,z)$, then it is called the vector point function.

Eg: 1, The velocity of a moving fluid at any instant.
2, The gravitational intensity of force.

$(1,0,1)$ $= 1+0+1 = 2$

Ex: ① $\phi(x,y,z) = x^2+y^2+z^2$ (Scalar point function)

② $\bar{F}(x,y,z) = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$ (Vector point function)

$(1,0,1)$ $= \bar{i} + \bar{k}$

Identify the nature of following functions

1, xyz^2 (S.P.F)
2, $xy^2\bar{i} - y^2\bar{j} + z\bar{k}$ (V.P.F)
3, $(x+2z)\bar{i} - \bar{k}$
4, $xy+yz+zx$

Note: Differentiation of a vector point function follows the same rules as those of ordinary Calculus.

Vector differential Operator:- The vector differential operator is denoted by ∇ and is defined as (Del)

$$\nabla = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)$$

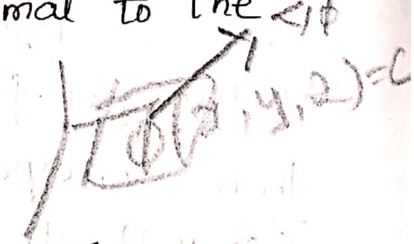
Grad
div
curl.

Del applied to Scalar point function:-

Gradient:- The gradient of a function ϕ is a vector function denoted by $\nabla\phi$ (or) grad ϕ and defined as

$$\text{grad}\,\phi \text{ (or) } \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

Geometrically, $\nabla\phi$ denoted normal to the surface $\phi(x,y,z) = c$.



Eg:- ①

Note: unit vector along $\bar{a}$ (vector) $= \frac{\bar{a}}{|\bar{a}|}$

Eg:- ① Find a unit normal vector to the surface $x^3+y^3+3xyz=3$ at the point $(1,2,-1)$

Sol: Given surface $x^3+y^3+3xyz=3$ and $P(1,2,-1)$

$$\Rightarrow x^3+y^3+3xyz-3=0$$

Let $\phi(x,y,z) = x^3+y^3+3xyz-3$

To find Normal vector

Normal Vector at $P = [\nabla\phi]_P$

By def $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(x^3+y^3+3xyz-3)$$

$$= \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(y^3) + 3\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial x}(3)$$

$$= 3x^2 + 0 + 3yz - 0$$

$$= 3x^2 + 3yz$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}(x^3) + \frac{\partial}{\partial y}(y^3) + 3\frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial y}(3)$$

$$= 0 + 3y^2 + 3xz - 0$$

$$= 3y^2 + 3xz$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial}{\partial z}(x^3) + \frac{\partial}{\partial z}(y^3) + 3\frac{\partial}{\partial z}(xyz) - \frac{\partial}{\partial z}(3)$$

$$= 0 + 0 + 3xy - 0$$

$$= 3xy.$$

$$\therefore \nabla\phi = \bar{i}(3x^2+3yz) + \bar{j}(3y^2+3xz) + \bar{k}(3xy)$$

$\therefore$ Normal vector at $p = (\nabla\phi)_{P(1,2,-1)}$

$$= \bar{i}(3.1^2+3.2.(-1)) + \bar{j}(3.2^2+3(1)(-1)) + \bar{k}(3.1.2)$$

$$= -3\bar{i} + 9\bar{j} + 6\bar{k}$$

unit normal vector $= \frac{\nabla\phi}{|\nabla\phi|} = \frac{-3\bar{i}+9\bar{j}+6\bar{k}}{\sqrt{(-3)^2+9^2+6^2}}$

$$= \frac{-3\bar{i}+9\bar{j}+6\bar{k}}{\sqrt{126}}$$

Eg:- (2) Find a unit vector normal to the surface $xy^3z^2=4$ at the point $(-1,-1,2)$

Sol: Given surface $xy^3z^2=4$

$xy^3z^2-4=0$ ~~and~~

Let $\phi(x,y,z) = xy^3z^2-4$ and $P(-1,-1,2)$

To find Normal

Normal vector at $P = (\nabla\phi)_P$.

By def $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

Now $\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(xy^3z^2-4)$

$= y^3z^2$

$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}(xy^3z^2-4)$

$= 3xy^2z^2$

$\frac{\partial\phi}{\partial z} = \frac{\partial}{\partial z}(xy^3z^2-4)$

$= 2xy^3z$

$\therefore \nabla\phi = \bar{i}(y^3z^2) + \bar{j}(3xy^2z^2) + \bar{k}(2xy^3z)$

Normal vector at $P = (\nabla\phi)_{P(-1,-1,2)}$

$= \bar{i}((-1)^3 \cdot 2^2) + \bar{j}(3 \cdot -1 \cdot -1^2 \cdot 2^2) + \bar{k}(2 \cdot -1 \cdot -1^3 \cdot 2)$

$= -4\bar{i} - 12\bar{j} + 4\bar{k}$

$$\text{unit normal} = \frac{\nabla\phi}{|\nabla\phi|_P}$$

$$= \frac{-4\bar{i} - 12\bar{j} + 4\bar{k}}{\sqrt{(-4)^2 + (-12)^2 + 4^2}}$$

$$= \frac{-4\bar{i} - 12\bar{j} + 4\bar{k}}{\sqrt{176}}$$

③ If $f(x,y,z) = 3x^2y - y^3z^2$, then find grad f at the point $(1,-2,-1)$

<u>Sol:</u> Given $f(x,y,z) = 3x^2y - y^3z^2$

and $P(1,-2,-1)$.

<u>To find grad f at P</u>

$$\text{grad} f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}.$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial x}(y^3z^2) = 6xy$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(3x^2y) - \frac{\partial}{\partial y}(y^3z^2) = 3x^2 - 3y^2z^2$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(3x^2y) - \frac{\partial}{\partial z}(y^3z^2) = -2y^3z$$

$$\therefore \nabla f = \bar{i}(6xy) + \bar{j}(3x^2 - 3y^2z^2) + \bar{k}(-2y^3z)$$

$$(\nabla f)_{P(1,-2,-1)} = \bar{i}(6.1.-2) + \bar{j}(3.1^2 - 3(-2)^2(-1)^2) + \bar{k}(-2(-2)^3(-1))$$

$$= -12\bar{i} - 9\bar{j} - 16\bar{k}$$

(4a) Find $\nabla\phi$, if $\phi = \log(x^2+y^2+z^2)$

Sol: Given $\phi = \log(x^2+y^2+z^2)$

$$\frac{\partial\phi}{\partial x} = \frac{1}{x^2+y^2+z^2}\frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$= \frac{2x+0+0}{x^2+y^2+z^2}$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{x^2+y^2+z^2}\frac{\partial}{\partial y}(x^2+y^2+z^2)$$

$$= \frac{2y}{x^2+y^2+z^2}$$

$$\frac{\partial\phi}{\partial z} = \frac{2z}{x^2+y^2+z^2}$$

By def $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$$= \frac{2x\,\bar{i}}{x^2+y^2+z^2} + \frac{2y\,\bar{j}}{x^2+y^2+z^2} + \frac{2z\,\bar{k}}{x^2+y^2+z^2}$$

$$= \frac{1}{x^2+y^2+z^2}\left(2x\bar{i}+2y\bar{j}+2z\bar{k}\right)$$

(1) Find a unit normal vector to the given surface $x^2y+2xz=4$ at the point $(2,-2,3)$

(2) Find a unit normal vector to the surface $x^2+y^2+2z^2=26$ at the point $(2,2,3)$

(3) If $f = x^3+y^3+3xyz+z^3$ find ∇f at $(1,2,3)$.

① Given Surface $\phi(x,y,z) = x^2y + 2xz - 4$ and $P(2,-2,3)$

To find Normal vector

$\frac{\partial \phi}{\partial x} = 2xy + 2z$ | $\frac{\partial \phi}{\partial y} = x^2$ | $\frac{\partial \phi}{\partial z} = 2x.$

By def, $\nabla \phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z} = \bar{i}(2xy+2z) + \bar{j}x^2 + \bar{k}\,2x.$

$\therefore$ Normal vector of ϕ at $p = (\nabla\phi)_{P(2,-2,3)}$

$= \bar{i}(2.2.-2 + 2.3) + \bar{j}(2^2) + \bar{k}\,2.2$

$= -2\bar{i} + 4\bar{j} + 4\bar{k}.$

unit normal vector $= \frac{\nabla\phi}{|\nabla\phi|} = \frac{-2\bar{i}+4\bar{j}+4\bar{k}}{\sqrt{(-2)^2+4^2+4^2}}$

$= \frac{-2\bar{i}+4\bar{j}+4\bar{k}}{\sqrt{36}} = \frac{-\bar{i}+2\bar{j}+2\bar{k}}{3}$

② Given Surface $\phi(x,y,z) = x^2 + y^2 + 2z^2 - 26$ and $P(2,2,3)$

To find Normal vector

$\frac{\partial \phi}{\partial x} = 2x$ | $\frac{\partial \phi}{\partial y} = 2y$ | $\frac{\partial \phi}{\partial z} = 4z$

By def: $\nabla\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z} = \bar{i}\,2x + \bar{j}\,2y + \bar{k}\,4z$

$\therefore$ Normal vector at $p = (\nabla\phi)_{P(2,2,3)}$

$= 4\bar{i} + 4\bar{j} + 12\bar{k}$

unit ~~Nor~~ normal vector at $p = \frac{\nabla\phi}{(\nabla\phi)_P} = \frac{4\bar{i}+4\bar{j}+12\bar{k}}{\sqrt{4^2+4^2+12^2}}$

$= \frac{4\bar{i}+4\bar{j}+12\bar{k}}{\sqrt{176}}$

(3) Given $f = x^3+y^3+z^3+3xyz$

and $P(1,2,3)$

To find grad f

$\frac{\partial f}{\partial x} = 3x^2+3yz$ | $\frac{\partial f}{\partial y} = 3y^2+3xz$ | $\frac{\partial f}{\partial z} = 3z^2+3xy$

By def. $\text{grad} f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$= \bar{i}(3x^2+3yz) + \bar{j}(3y^2+3xz) + \bar{k}(3z^2+3xy)$

$(\text{grad} f)_{P(1,2,3)} = \bar{i}(3+3.2.3) + \bar{j}(3.2^2+3.1.3)$
$+\bar{k}(3.3^2+3.1.2)$

$= 21\bar{i} + 21\bar{j} + 33\bar{k}$.

Topic (2) Angle between two vectors

Let θ ~~between~~ be angle between two vectors $\bar{a}$ and $\bar{b}$ then 'θ' is given by.

$$\boxed{\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}}$$

Ex:(1) find the angle between the surfaces $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at $(2,-1,2)$.

Sol: Given Surfaces $x^2+y^2+z^2-9=0$

$z - x^2 - y^2 + 3 = 0$.

Let $\phi_1(x,y,z) = x^2+y^2+z^2-9$

$\phi_2(x,y,z) = z - x^2 - y^2 + 3$

and $P(2,-1,2)$

Let θ be the angle between ϕ_1 and ϕ_2

then $\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}$

To find $\bar{a}$ $\quad \bar{a} = (\nabla\phi_1)_p$.

$$\frac{\partial\phi_1}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2-9) = 2x$$

$$\frac{\partial\phi_1}{\partial y} = 2y \ ; \ \frac{\partial\phi_1}{\partial z} = 2z$$

$$\therefore \nabla\phi_1 = \bar{i}\frac{\partial\phi_1}{\partial x} + \bar{j}\frac{\partial\phi_1}{\partial y} + \bar{k}\frac{\partial\phi_1}{\partial z} = \bar{i}(2x)+\bar{j}(2y)+\bar{k}(2z)$$

Now $\bar{a} = (\nabla\phi_1)_{P(2,-1,2)}$

$$= \bar{i}(2.2)+\bar{j}(2.-1)+\bar{k}(2.2)$$

$$\boxed{\bar{a} = 4\bar{i}-2\bar{j}+4\bar{k}}$$

To find $\bar{b}$ $\quad \bar{b} = (\nabla\phi_2)_p$

$$\frac{\partial\phi_2}{\partial x} = -2x \ \Big| \ \frac{\partial\phi_2}{\partial y} = -2y \ \Big| \ \frac{\partial\phi_2}{\partial z} = 1$$

$$\therefore \nabla\phi_2 = \bar{i}(-2x)+\bar{j}(-2y)+\bar{k}$$

$$\bar{b} = (\nabla\phi_2)_{P(2,-1,2)} = \bar{i}(-2.2)+\bar{j}(-2.-1)+\bar{k}$$

$$= -4\bar{i}+2\bar{j}+\bar{k}.$$

To find θ $\quad \cos\theta = \dfrac{(4\bar{i}-2\bar{j}+4\bar{k})\cdot(-4\bar{i}+2\bar{j}+\bar{k})}{\sqrt{4^2+(-2)^2+4^2}\ \sqrt{(-4)^2+2^2+1}}$

$$= \frac{-16-4+4}{\sqrt{36}\ \sqrt{21}} = \frac{-16}{6\sqrt{21}}$$

$$= \frac{-8}{3\sqrt{21}} \quad (\text{or}) \quad \theta = \cos^{-1}\left(\frac{-8}{3\sqrt{21}}\right)$$

Ex:- (2) Calculate the angle between the normals to the surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$

Sol: Given surface $xy = z^2$

$$xy - z^2 = 0$$

Let $\phi(x,y,z) = xy - z^2$

and $P(4,1,2)$; $Q(3,3,-3)$

Let θ be the angle between the normals to the surface, then $\boxed{\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}}$

To find $\bar{a}$, $\quad \bar{a} = (\nabla\phi)_P$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}(xy - z^2) = y \;\Big|\; \frac{\partial\phi}{\partial y} = x \;\Big|\; \frac{\partial\phi}{\partial z} = -2z.$$

$$\therefore \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}(y) + \bar{j}(x) + \bar{k}(-2z).$$

$$\bar{a} = (\nabla\phi)_{P(4,1,2)} = \bar{i} + 4\bar{j} - 4\bar{k}.$$

To find $\bar{b}$ $\quad \bar{b} = (\nabla\phi)_{Q(3,3,-3)}$

$$= 3\bar{i} + 3\bar{j} + 6\bar{k}$$

To find θ $\quad \cos\theta = \dfrac{(\bar{i}+4\bar{j}-4\bar{k})\cdot(3\bar{i}+3\bar{j}+6\bar{k})}{\sqrt{1+4^2+4^2}\,\sqrt{9+9+36}}$

$$= \frac{3+12-24}{\sqrt{33}\,\sqrt{54}} = \frac{-9}{\sqrt{3\times 11}\,\sqrt{9\times 6}}$$

$$= \frac{-9}{\cancel{\sqrt{}}\,9\sqrt{22}} = \frac{-1}{\sqrt{22}}$$

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Ex:- 3 Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ is orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1,-1,2)$.

Sol: Given Surfaces $ax^2 - byz - (a+2)x = 0$

and $4x^2y + z^3 - 4 = 0$.

Let $\phi_1 = ax^2 - byz - (a+2)x$

$\phi_2 = 4x^2y + z^3 - 4$ and $P(1,-1,2)$

Given The two Surfaces ϕ_1 and ϕ_2 are orthogonal

$\Rightarrow$ $\boxed{\theta = 90^\circ}$ and $\cos\theta = \cos 90 = 0$

To find $\bar{a}$ $\bar{a} = (\nabla\phi_1)_P$

$\frac{\partial\phi_1}{\partial x} = 2ax - a - 2$ | $\frac{\partial\phi_1}{\partial y} = -bz$ | $\frac{\partial\phi_1}{\partial z} = -by$

$\nabla\phi_1 = \bar{i}(2ax - a - 2) + \bar{j}(-bz) + \bar{k}(-by)$

$\bar{a} = (\nabla\phi_1)_{P(1,-1,2)} = \bar{i}(2a - a - 2) - 2b\bar{j} + b\bar{k}$

To find $\bar{b}$ $\bar{b} = (\nabla\phi_2)_P$

$\frac{\partial\phi_2}{\partial x} = 8xy$ | $\frac{\partial\phi_2}{\partial y} = 4x^2$ | $\frac{\partial\phi_2}{\partial z} = 3z^2$

$\Rightarrow \nabla\phi_2 = \bar{i}(8xy) + \bar{j}(4x^2) + \bar{k}(3z^2)$

$\therefore \bar{b} = -8\bar{i} + 4\bar{j} + 12\bar{k}$

To find a, b $\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} \Rightarrow \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} = 0$

$\Rightarrow \bar{a}\cdot\bar{b} = 0$

$\Rightarrow \left(\bar{i}(a-2) - 2b\bar{j} + b\bar{k}\right)\cdot\left(-8\bar{i} + 4\bar{j} + 12\bar{k}\right) = 0$

$-8a + 16 - 8b + 12b = 0 \Rightarrow -8a + 4b + 16 = 0$ — (1)

Substituting given point $P(1,-1,2)$ in first Surface $ax^2 - byz - (a+2)x = 0$

$$a + 2b - a - 2 = 0 \Rightarrow 2b - 2 = 0$$
$$\Rightarrow 2b = 2$$
$$\boxed{b = 1}$$

Substituting b value in eq'n ①

$$-8a + 4b + 16 = 0 \Rightarrow -8a + 4 + 16 = 0$$
$$\Rightarrow -8a + 20 = 0 \Rightarrow 8a = 20$$
$$\Rightarrow \boxed{a = \frac{20}{8} = \frac{5}{2}}$$

Directional derivative

Def :- The rate of change of $f(x,y)$ in the direction of a vector $\bar{a}$ is called the directional derivative. It is defined as

$$\lim_{h \to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h}.$$

① The directional derivative of a Scalar function $\phi(x,y,z)$ in the direction of a vector $\bar{a}$ is defined as $\dfrac{\nabla\phi \cdot \bar{a}}{|\bar{a}|}$

② The directional derivative will be maximum in the direction of $\nabla\phi$

③ Maximum value of directional derivative $= |\nabla\phi|$.

Ex ① Find the directional derivative of $\phi = xy + yz + zx$ at $(1,2,0)$ in the direction of $\bar{i} + 2\bar{j} + 2\bar{k}$

Sol: Given $\phi = xy + yz + zx$

$P(1,2,0)$

$\bar{a} = \bar{i} + 2\bar{j} + 2\bar{k}$

Directional derivative $= \dfrac{\nabla\phi \cdot \bar{a}}{|\bar{a}|}$

To find $\nabla\phi$

$\dfrac{\partial\phi}{\partial x} = y + z \quad \Big| \quad \dfrac{\partial\phi}{\partial y} = x + z \quad \Big| \quad \dfrac{\partial\phi}{\partial z} = \dfrac{\partial\phi}{\partial z} = y + x$

$\nabla\phi = \bar{i}(y+z) + \bar{j}(x+z) + \bar{k}(y+x)$

$(\nabla\phi)_{P(1,2,0)} = 2\bar{i} + \bar{j} + 3\bar{k}$

To find directional derivative

Directional derivative $= \dfrac{(2\bar{i} + \bar{j} + 3\bar{k}) \cdot (\bar{i} + 2\bar{j} + 2\bar{k})}{\sqrt{1 + 2^2 + 2^2}}$

$= \dfrac{2 + 2 + 6}{\sqrt{9}} = \dfrac{10}{3}$

Ex:- ② Find the directional derivative of $f(x,y,z) = 4e^{2x-y+z}$ at the point $A(1,1,-1)$ in the direction of the vector towards the point $B(-3,5,6)$

Sol: Given $f(x,y,z) = 4e^{2x-y+z}$

and $A(1,1,-1)$, $B(-3,5,6)$

Directional derivative $= \dfrac{\nabla f \cdot \bar{a}}{|\bar{a}|}$

To find ∇f

$\dfrac{\partial f}{\partial x} = 4 \, e^{2x-y+z} \times 2 = 8e^{2x-y+z}$

$$\frac{\partial f}{\partial y} = 4e^{2x-y+z} \times -1 = -4e^{2x-y+z}$$

$$\frac{\partial f}{\partial z} = 4e^{2x-y+z} \times 1 = 4e^{2x-y+z}.$$

By def., $\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$$= \bar{i}\left(+8e^{2x-y+z}\right) + \bar{j}\left(-4e^{2x-y+z}\right) + \bar{k}\left(4e^{2x-y}\right.$$

$$(\nabla f)_{A(1,1,-1)} = 8\bar{i} - 4\bar{j} + 4\bar{k}.$$

To find $\bar{a}$ Here $\bar{a} = \overline{AB}$

$$= \overline{OB} - \overline{OA}$$

$$= -3\bar{i} + 5\bar{j} + 6\bar{k} - (\bar{i} + \bar{j} - \bar{k})$$

$$= -4\bar{i} - 4\bar{j} + 7\bar{k}$$

To find directional derivative

$$\text{Directional derivative} = \frac{(8\bar{i} - 4\bar{j} + 4\bar{k})\cdot(-4\bar{i} - 4\bar{j} + 7\bar{k})}{\sqrt{(-4)^2 + (-4)^2 + 7^2}}$$

$$= \frac{-32 + 16 + 28}{\sqrt{81}}$$

$$= \frac{12}{9} = \frac{4}{3}.$$

Excercise

1. Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$

2. Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$. Also calculate the maximum value of directional derivative.

③ Find the directional derivative of xyz^2+xz at $(1,1,1)$ in the direction normal to the surface $3xy^2+y=z$ at $(0,1,1)$

Sol: Let $\phi(x,y,z) = xyz^2+xz$

and $P(1,1,1)$

$f(x,y,z) = 3xy^2+y-z$

$Q(0,1,1)$

Directional derivative $= \dfrac{\nabla\phi\cdot\bar{a}}{|\bar{a}|}$

step i) To find $\nabla\phi$:-

$$\frac{\partial\phi}{\partial x} = yz^2+z \quad \Big| \quad \frac{\partial\phi}{\partial y} = xz^2 \quad \Big| \quad \frac{\partial\phi}{\partial z} = 2xyz+x.$$

By def. $\nabla\phi = \bar{i}\dfrac{\partial\phi}{\partial x}+\bar{j}\dfrac{\partial\phi}{\partial y}+\bar{k}\dfrac{\partial\phi}{\partial z}$.

$$\nabla\phi = \bar{i}(yz^2+z)+\bar{j}(xz^2)+\bar{k}(2xyz+x)$$

$$\nabla\phi_{\text{at } P(1,1,1)} = 2\bar{i}+\bar{j}+3\bar{k}$$

step ii) To find $\bar{a}$: Given $\bar{a}$ = normal to the surface f at Q

$= [\nabla f]_Q$

$$\frac{\partial f}{\partial x} = 3y^2 \quad \Big| \quad \frac{\partial f}{\partial y} = 6xy+1 \quad \Big| \quad \frac{\partial f}{\partial z} = -1$$

$$\nabla f = \bar{i}(3y^2)+\bar{j}(6xy+1)-\bar{k}$$

$$\therefore \bar{a} = (\nabla f)_{Q(0,1,1)} = 3\bar{i}+\bar{j}-\bar{k}$$

Step iii) To find directional derivative.

$$\text{Directional derivative} = \frac{(2\bar{i}+\bar{j}+3\bar{k})\cdot(3\bar{i}+\bar{j}-\bar{k})}{\sqrt{9+1+1}}$$

$$= \frac{6+1-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

(4) Find the directional derivative of $\phi = xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t,\ y = t^2,\ z = t^3$ at the point $(1,1,1)$

Sol: Given $\phi = xy^2 + yz^2 + zx^2$

Curve: $x = t,\ y = t^2,\ z = t^3$

and $P(1,1,1)$

To find $\nabla\phi$

$$\frac{\partial\phi}{\partial x} = y^2 + 2xz \quad \bigg| \quad \frac{\partial\phi}{\partial y} = 2xy + z^2 \quad \bigg| \quad \frac{\partial\phi}{\partial z} = 2yz + x^2$$

By def. $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$$= \bar{i}(y^2 + 2xz) + \bar{j}(2xy + z^2) + \bar{k}(2yz + x^2)$$

$$(\nabla\phi)_{(1,1,1)} = 3\bar{i} + 3\bar{j} + 3\bar{k}$$

to find $\bar{a}$ Given $\bar{a}$ = tangent to the curve

$$= \frac{d\bar{r}}{dx} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$$

$$= 1\bar{i} + 2t\bar{j} + 3t^2\bar{k}$$

$$\bar{a}_{P(1,1,1)} = \bar{i} + 2\bar{j} + 3\bar{k} \quad \left[\because x = t \text{ at } P \text{ we have } t = 1\right]$$

To find directional derivative

$$\text{Directional derivative} = \frac{\nabla\phi \cdot \bar{a}}{|\bar{a}|}$$

$$= \frac{(3\bar{i} + 3\bar{j} + 3\bar{k})\cdot(\bar{i} + 2\bar{j} + 3\bar{k})}{\sqrt{14}}$$

$$= \frac{18}{\sqrt{14}}$$

(5) The temperature at a point (x, y, z) is given by $T(x,y,z) = x^2+y^2-z$ a mosquito located at $(1,1,2)$ desires to fly in such a direction that it will get warm as soon as possible. In what direction should it fly?

Sol. Given $T(x,y,z) = x^2+y^2-z$

$P(1,1,2)$

gradient of a function gives the direction of rapid increase.

$\therefore$ The direction of rapid increase in temperature

$$= (\nabla T)_P$$

To find ∇T.

$$\frac{\partial T}{\partial x} = 2x \quad \Big| \quad \frac{\partial T}{\partial y} = 2y \quad \Big| \quad \frac{\partial T}{\partial z} = -1$$

$$\therefore \nabla T = \bar{i}(2x) + \bar{j}(2y) - \bar{k}$$

$$\text{Now } (\nabla T)_{(1,1,2)} = 2\bar{i} + 2\bar{j} - \bar{k}.$$

(6) Find the greatest value of the directional derivative of the function $f = x^2yz^3$ at $(2,1,-1)$

Sol. Given $f = x^2yz^3$ & $P(2,1,-1)$

To find ∇f

$$\frac{\partial f}{\partial x} = 2xyz^3 \quad \Big| \quad \frac{\partial f}{\partial y} = x^2z^3 \quad \Big| \quad \frac{\partial f}{\partial z} = 3x^2yz^2$$

$$\nabla f = \bar{i}(2xyz^3) + \bar{j}(x^2z^3) + \bar{k}(3x^2yz^2)$$

$$(\nabla f)_P = -4\bar{i} - 4\bar{j} + 12\bar{k}$$

Greatest value of Directional derivative $= \sqrt{16+16+144}$

$$= \sqrt{176} = 4\sqrt{11}$$

① Show that $\nabla[f(r)] = \frac{f'(r)}{r}\bar{r}$ where $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

<u>Proof:</u> Given $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

$\Rightarrow |\bar{r}|$ or $r = \sqrt{x^2+y^2+z^2}$

$$r^2 = x^2+y^2+z^2$$

differentiating the above with respect to x, y, z partially.

$2r\frac{\partial r}{\partial x} = 2x.$ | $2r\frac{\partial r}{\partial y} = 2y$ | $\frac{\partial r}{\partial z} = \frac{z}{r}.$

$\Rightarrow \frac{\partial r}{\partial x} = \frac{2x}{2r} = \frac{x}{r}$ | $\frac{\partial r}{\partial y} = \frac{y}{r}$

Now consider L.H.S $= \nabla[f(r)]$

$$= \bar{i}\frac{\partial}{\partial x}(f(r)] + \bar{j}\frac{\partial}{\partial y}(f(r)) + \bar{k}\frac{\partial}{\partial z}[f(r)]$$

$$= \bar{i}f'(r)\frac{\partial r}{\partial x} + j\, f'(r)\frac{\partial r}{\partial y} + \bar{k}\, f'(r)\frac{\partial r}{\partial z}$$

$$= f'(r)\left[\bar{i}\frac{x}{r} + \bar{j}\frac{y}{r} + \bar{k}\frac{z}{r}\right]$$

$$= \frac{f'(r)}{r}\left[x\bar{i}+y\bar{j}+z\bar{k}\right] = \frac{f'(r)}{r}\bar{r}.$$

② Prove that $\nabla(r^n) = n r^{n-2}\bar{r}$

<u>Sol:</u> Let $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

$\Rightarrow r = \sqrt{x^2+y^2+z^2}$

$\Rightarrow r^2 = x^2+y^2+z^2$

differentiating above equation with respect to x, y, z partially

$\frac{\partial r}{\partial x} = \frac{x}{r}$ | $\frac{\partial r}{\partial y} = \frac{y}{r}$ | $\frac{\partial r}{\partial z} = \frac{z}{r}.$

Now consider L.H.S $= \nabla(r^n)$

$$= \bar{i}\frac{\partial}{\partial x}(r^n) + \bar{j}\frac{\partial}{\partial y}(r^n) + \bar{k}\frac{\partial}{\partial z}(r^n)$$

$$= \bar{i}\, n.r^{n-1}\frac{\partial r}{\partial x} + \bar{j}\, n r^{n-1}\frac{\partial r}{\partial y} + \bar{k}\, n r^{n-1}\frac{\partial r}{\partial z}$$

$$= n r^{n-1}\left[\bar{i}\frac{x}{r} + \bar{j}\frac{y}{r} + \bar{k}\frac{z}{r}\right]$$

$$= n r^{n-1}\frac{\bar{r}}{r}$$

$$= n.r^{n-2}\,\bar{r}.$$

Divergence of a vector :- Let $\bar{f}$ be any continuously differentiable vector point function. Then "$\bar{i}\cdot\frac{\partial \bar{f}}{\partial x} + \bar{j}\cdot\frac{\partial \bar{f}}{\partial y} + \bar{k}\cdot\frac{\partial \bar{f}}{\partial z}$" is called the divergence of $\bar{f}$ and is written as $\operatorname{div}\bar{f}$ (or) $\nabla\cdot\bar{f}$

i.e., $\operatorname{div}\bar{f} = \bar{i}\cdot\frac{\partial \bar{f}}{\partial x} + \bar{j}\cdot\frac{\partial \bar{f}}{\partial y} + \bar{k}\cdot\frac{\partial \bar{f}}{\partial z}$.

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\bar{f}.$$

Note ① :- If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

then $\operatorname{div}\bar{f} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(f_1\bar{i} + f_2\bar{j} + f_3\bar{k}\right)$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Note:-② **Solenoidal vector**

A vector point function $\bar{f}$ is said to be Solenoidal if $\operatorname{div}\bar{f} = 0$

Ex:- (1) $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ then find $\operatorname{div}\bar{f}$ at $(1,-1,1)$

Sol: Given $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$

and $P(1,-1,1)$

$$\operatorname{div}\bar{f} = \nabla\cdot\bar{f}$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) - \frac{\partial}{\partial z}(3yz^2)$$

$$= y^2 + 2x^2z - 6yz.$$

$$[\operatorname{div}\bar{f}]_{P(1,-1,1)} = (-1)^2 + 2 1^2 \cdot 1 - 6(-1)\cdot 1$$

$$= 1 + 2 + 6 = 9$$

Ex. (2) Find $\operatorname{div}\bar{f}$ where $\bar{f} = \operatorname{grad}(x^3+y^3+z^3-3xyz)$

Sol: let $\phi = x^3+y^3+z^3-3xyz.$

To find $\bar{f}$

given $\bar{f} = \operatorname{grad}\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = 3x^2 - 3yz$ | $\frac{\partial\phi}{\partial y} = 3y^2 - 3xz$ | $\frac{\partial\phi}{\partial z} = 3z^2 - 3xy$

$$\therefore \bar{f} = \bar{i}(3x^2-3yz) + \bar{j}(3y^2-3xz) + \bar{k}(3z^2-3xy).$$

To find $\operatorname{div}\bar{f}$

$$\operatorname{div}\bar{f} = \nabla\cdot\bar{f} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot$$

$$\bar{i}(3x^2-3yz) + \bar{j}(3y^2-3xz) + \bar{k}(3z^2-3xy)$$

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$$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$$

$$= 6x + 6y + 6z.$$

Ex:- (3) If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is Solenoidal, find p.

Sol: Given $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$

$$\operatorname{div}\bar{f} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left[(x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}\right]$$

$$= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+pz)$$

$$= 1+1+p = 2+p$$

$\because$ $\bar{f}$ is Solenoidal $\Rightarrow$ $\operatorname{div}\bar{f} = 0$

$$\Rightarrow 2+p=0 \Rightarrow \boxed{p=-2}$$

Ex:- (4) Show that $3y^4z^2\bar{i} + z^3x^2\bar{j} - 3x^2y^2\bar{k}$ is Solenoidal.

Sol: Let $\bar{f} = 3y^4z^2\bar{i} + z^3x^2\bar{j} - 3x^2y^2\bar{k}$

$$\operatorname{div}\bar{f} = \nabla\cdot\bar{f}$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(3y^4z^2\bar{i} + z^3x^2\bar{j} - 3x^2y^2\bar{k}\right)$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(z^3x^2) + \frac{\partial}{\partial z}(-3x^2y^2)$$

$$= 0+0+0 = 0$$

$\Rightarrow \operatorname{div}\bar{f} = 0$

$\therefore$ $\bar{f}$ is Solenoidal.

(5) Show that $\frac{\bar{r}}{r^3}$ is solenoidal.

Sol: Let $\bar{f} = \frac{\bar{r}}{r^3} = \frac{1}{r^3}(x\bar{i} + y\bar{j} + z\bar{k})$

where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2$

$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \;|\; \frac{\partial r}{\partial y} = \frac{y}{r} \;|\; \frac{\partial r}{\partial z} = \frac{z}{r}$.

Consider $= \operatorname{div}\bar{f} = \nabla\cdot\bar{f}$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{x}{r^3}\bar{i} + \frac{y}{r^3}\bar{j} + \frac{z}{r^3}\bar{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(y/r^3\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)$$

$$= \frac{r^3 - 3r^2 x\frac{\partial r}{\partial x}}{r^6} + \frac{r^3 - 3r^2 y\frac{\partial r}{\partial y}}{r^6} + \frac{r^3 - 3r^2 z\frac{\partial r}{\partial z}}{r^6}$$

$$= \frac{r^3}{r^6} - \frac{3.x.x.\cancel{r^2}}{r^{\cancel{6}5}.\cancel{r}} + \frac{1}{r^3} - \frac{3r^2.y.y}{r^6.r} + \frac{r^3}{r^6} - \frac{3r^2.z.z}{r^6.r}$$

$$= \frac{3}{r^3} - \frac{3x^2}{r^5} - \frac{3y^2}{r^5} - \frac{3z^2}{r^5}$$

$$= \frac{3}{r^3} - \frac{3}{r^5}(x^2+y+z^2) = \frac{3}{r^3} - \frac{3}{r^5}.r^2$$

$$= \frac{3}{r^3} - \frac{3}{r^3} = 0$$

$\therefore \frac{\bar{r}}{r^3}$ is Solenoidal.

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① Prove that $\operatorname{div}(r^n \bar{r}) = (n+3) r^n$. Hence show that $r^n \bar{r}$ is solenoidal for $n=-3$.

Sol: Let $\bar{f} = r^n \bar{r} = r^n (x i + y j + z \bar{k})$

$\bar{r} = x i + y j + z \bar{k}$

$r = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2$

then $\frac{\partial r}{\partial x} = \frac{x}{r} \; \Big| \; \frac{\partial r}{\partial y} = y/r \; \Big| \; \frac{\partial r}{\partial z} = \frac{z}{r}.$

$\operatorname{div}(r^n . \bar{r}) = \nabla \cdot r^n \bar{r}$

$$= \left(i\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z} \right) \cdot \left(r^n x\, i + r^n y \bar{j} + r^n z \bar{k} \right)$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= n . r^{n-1} \frac{\partial r}{\partial x} . x + r^n + n . r^{n-1} \frac{\partial r}{\partial y} . y + r^n$$
$$+ n . r^{n-1} \frac{\partial r}{\partial z} . z + r^n$$

$$= n . r^{n-1} \left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r} \right] + 3 r^n$$

$$= n . r^{n-1} \frac{r^2}{r} + 3 r^n = n . r^n + 3 r^n$$

$$= (n+3) r^n$$

if $r^n \bar{r}$ is solenoidal then $\operatorname{div}(r^n \bar{r}) = 0$

$\Rightarrow (n+3) r^n = 0$

$\Rightarrow n+3 = 0$

$\Rightarrow n = -3.$

Curl of a vector :-

Def: Let $\bar{f}$ be any continuously differentiable vector point function. Then the vector function defined by $\bar{i}\times\frac{\partial \bar{f}}{\partial x}+\bar{j}\times\frac{\partial \bar{f}}{\partial y}+\bar{k}\times\frac{\partial \bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or $\nabla\times\bar{f}$

$$\text{Curl}\,\bar{f} = \bar{i}\times\frac{\partial \bar{f}}{\partial x}+\bar{j}\times\frac{\partial \bar{f}}{\partial y}+\bar{k}\times\frac{\partial \bar{f}}{\partial z}.$$

$$= \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\times\bar{f}$$

Note: If $\bar{f} = f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$

then $\text{curl}\,\bar{f} = \nabla\times\bar{f}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Irrotational

(2) A vector $\bar{f}$ is said to be Irrotational, if $\text{curl}\,\bar{f}=\bar{0}$

(3) If $\bar{f}$ is irrotational then there exists a scalar function $\phi(x,y,z)$ such that $\bar{f} = \text{grad}\,\phi$. i.e., $\bar{f}=\bar{i}\frac{\partial \phi}{\partial x}+\bar{j}\frac{\partial \phi}{\partial y}+\bar{k}\frac{\partial \phi}{\partial z}$

This ϕ is called scalar potential of $\bar{f}$.

Ex :- (1) If $\bar{f} = xy^2\bar{i}+2x^2yz\bar{j}-3yz^2\bar{k}$ find curl $\bar{f}$ at the point $(1,-1,1)$

Sol :- Given $\bar{f} = xy^2\bar{i}+2x^2yz\bar{j}-3yz^2\bar{k}$

$P(1,-1,1)$

$$\operatorname{Curl}\bar{f} = \nabla \times \bar{f}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \bar{i}\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ 2x^2yz & -3yz^2 \end{vmatrix} - \bar{j}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ xy^2 & -3yz^2 \end{vmatrix} + \bar{k}\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ xy^2 & 2x^2yz \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right] - \bar{j}\left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$$

$$= \bar{i}\left[-3z^2 - 2x^2y\right] - \bar{j}(0-0) + \bar{k}\left[4xyz - 2xy\right]$$

$$= \bar{i}(-3z^2 - 2x^2y) + \bar{k}(4xyz - 2xy]$$

$$\operatorname{Curl}\bar{f}_{P(1,-1,1)} = \bar{i}(-3+2) + \bar{k}(-4+2)$$

$$= -\bar{i} - 2\bar{k}$$

Ex: (2) Find curl $\bar{f}$ for $\bar{f} = 2xz^2\,\bar{i} - yz\bar{j} + 3xz^3\bar{k}$

Sol: $\bar{f} = 2xz^2\,\bar{i} - yz\bar{j} + 3xz^3\,\bar{k}$

$$\operatorname{Curl}\bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= \bar{i}\left[0 + y\right] - \bar{j}\left[3z^3 - 4xz\right] + \bar{k}\left[0 - 0\right]$$

$$= y\,\bar{i} - (3z^3 - 4xz)\,\bar{j}.$$

Ex ③ Find $\bar{f}$ where $\bar{f} = \text{grad}\,(x^3+y^3+z^3-3xyz)$

Sol: Let $\phi = x^3+y^3+z^3-3xyz$.

To find $\bar{f}$: $\bar{f} = \text{grad}\,\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = 3x^2 - 3yz \;\Big|\; \frac{\partial\phi}{\partial y} = 3y^2-3xz \;\Big|\; \frac{\partial\phi}{\partial z} = 3z^2-3xy$

$\therefore \bar{f} = \bar{i}(3x^2-3yz) + \bar{j}(3y^2-3xz) + \bar{k}(3z^2-3xy)$

To find curl $\bar{f}$: curl $\bar{f} = \nabla\times\bar{f}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= \bar{i}[-3x+3x] - \bar{j}[-3y+3y] + \bar{k}[-3z+3z]$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k}$$

$$= \bar{0}$$

$\Rightarrow$ curl $\bar{f} = \bar{0}$

④ Find constants a, b and c if the vector

$\bar{f} = (2x+3y+az)\bar{i} + (bx+2y+3z)\bar{j} + (2x+cy+3z)\bar{k}$

is Irrotational.

Sol: Curl $\bar{f} = \nabla\times\bar{f}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix}$$

$= \bar{i}[c-3] - \bar{j}[2-a] + \bar{k}[b-3]$

$\because \bar{f}$ is Irrotational $\Rightarrow$ curl $\bar{f} = \bar{0}$

$\Rightarrow \bar{i}(c-3) - \bar{j}(2-a) + \bar{k}(b-3) = \bar{0} = 0\bar{i} + 0\bar{j} + 0\bar{k}$

$\Rightarrow$ $c-3=0$, $c=3$ | $2-a=0$, $a=2$ | $b-3=0$, $b=3$

(5) Show that the vector point function $(x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ is irrotational and hence find its scalar potential function.

Sol: Let $\bar{f} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$

$$\text{curl}\,\bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$= \bar{i}(-x+x) - \bar{j}(-y+y) + \bar{k}(-z+z)$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k} = \bar{0}.$$

$\therefore \bar{f}$ is Irrotational.

To find ϕ:-

By def. $\exists\, \phi \ni \bar{f} = \text{grad}\,\phi = \bar{i}\dfrac{\partial \phi}{\partial x} + \bar{j}\dfrac{\partial \phi}{\partial y} + \bar{k}\dfrac{\partial \phi}{\partial z}$

$$(x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k} = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z} \quad \text{(A)}$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = x^2 - yz$$

Integration on b.s.

$$\Rightarrow \int \frac{\partial \phi}{\partial x}\,dx = \int (x^2-yz)\,dx$$

$$\Rightarrow \phi = \frac{x^3}{3} - xyz + C_1(y,z) \quad \text{(I)}$$

from (A), $\frac{\partial \phi}{\partial y} = y^2 - xz$

Integrating with respect to y on b.s

$$\phi = y^3/3 - xyz + c_2(x, z) \quad \text{—— (II)}$$

from (A), $\frac{\partial \phi}{\partial z} = z^2 - xy$

Integrating on b.s w.r.to z

$$\int \frac{\partial \phi}{\partial z} dz = \int z^2 - xy \, dz$$

$$\Rightarrow \quad \phi = \frac{z^3}{3} - xyz + c_3(x, y) \quad \text{—— (III)}$$

from (I), (II) & (III)

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz$$ is the required scalar potential function.

(6) Prove that $\bar{f} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$ is irrotational and find scalar potential function ϕ such that $\bar{f} = \nabla \phi$.

Sol: Curl $\bar{f} = \nabla \times \bar{f}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \bar{i}(1-1) + \bar{j}(1-1) + \bar{k}(1-1)$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k} = \bar{0}$$

To find ϕ

By def $\bar{f} = \text{grad}\,\phi$

$$\Rightarrow (y+z)\bar{i} + (x+z)\bar{j} + (z+x)\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} \quad \text{(A)}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = y+z$$

On Integration with respect to x.

$$\phi = xy + xz + c_1(y,z) \quad \text{(1)}$$

from (A), $\frac{\partial\phi}{\partial y} = x+z$

On Integration w.r.to y

$$\Rightarrow \phi = xy + yz + c_2(x,z) \quad \text{(2)}$$

from (A), $\frac{\partial\phi}{\partial z} = y+x$.

On Integration w.r.to z.

$$\phi = yz + xz + c_3(x,y) \quad \text{(3)}$$

from (1), (2), & (3)

$\phi = xy + yz + zx$ is the required scalar potential function.

① Find constants a, b, c so that the vector
$\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$ is irrotational. Also find ϕ such that $\bar{A} = \nabla\phi$.

Sol: $\text{Curl}\,\bar{A} = \nabla\times\bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$

$= \bar{i}(c+1) - \bar{j}(4-a) + \bar{k}(b-2)$ —— ①

Since $\bar{A}$ is irrotational, then $\text{curl}\,\bar{A} = \bar{0} = 0\bar{i} + 0\bar{j} + 0\bar{k}$ ②

Equating ① & ② $c+1=0$, $c=-1$ | $4-a=0$, $4=a$ | $b-2=0$, $b=2$

To find ϕ $\bar{A} = \nabla\phi$

i.e., $(x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k}$

$= \bar{i}\dfrac{\partial\phi}{\partial x} + \bar{j}\dfrac{\partial\phi}{\partial y} + \bar{k}\dfrac{\partial\phi}{\partial z}$ —— (A).

$\Rightarrow \dfrac{\partial\phi}{\partial x} = x+2y+4z$

On Integration, $\phi = \dfrac{x^2}{2} + 2xy + 4xz + c_1(y,z)$ (I)

from (A), $\dfrac{\partial\phi}{\partial y} = 2x-3y-z$.

Integrating w.r.to y, $\phi = 2xy - \dfrac{3y^2}{2} - yz + c_2(x,z)$ (II)

from (A), $\dfrac{\partial\phi}{\partial z} = 4x-y+2z$

Integrating w.r.to z, $\phi = \cancel{2x} \; 4xz - yz + z^2 + c_3(x,y)$ (III)

By observing (I), (II), (III)

$\phi = \dfrac{x^2}{2} + 2xy + 4xz - \dfrac{3y^2}{2} - yz + z^2$

② Prove that $r^n\bar{r}$ is Irrotational.

Sol: Let $\bar{f} = r^n\bar{r} = r^n(x\bar{i} + y\bar{j} + z\bar{k})$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

and $r = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2$

differentiating with respect to x, y, z partially, we get $\frac{\partial r}{\partial x} = \frac{x}{r}$ | $\frac{\partial r}{\partial y} = y/r$ | $\frac{\partial r}{\partial z} = \frac{z}{r}$.

Now $\operatorname{curl} \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ r^n x & r^n y & r^n z \end{vmatrix}$

$$= \bar{i}\left[z \cdot n\, r^{n-1} \frac{\partial r}{\partial y} - y\, n \cdot r^{n-1} \frac{\partial r}{\partial z}\right] - \bar{j}\left[z\, n\, r^{n-1} \frac{\partial r}{\partial x} - x\, n \cdot r^{n-1} \frac{\partial r}{\partial z}\right] + \bar{k}\left[y\, n \cdot r^{n-1} \frac{\partial r}{\partial x} - x\, n\, r^{n-1} \frac{\partial r}{\partial y}\right]$$

$$= \bar{i}\left[\frac{yz\, \cancel{n} \cdot r^{n-1}}{\cancel{r}} - \frac{yz\, \cancel{n} r^{n-1}}{\cancel{r}}\right] - \bar{j}(0) + \bar{k}(0)$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k} = \bar{0}$$

$\therefore r^n \bar{r}$ is irrotational.

__Laplacian Operator__: Laplacian operator is denoted by ∇^2 and is given by $\nabla \cdot \nabla$

$$\nabla^2 = \nabla \cdot \nabla = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

1. Prove that $\operatorname{div}(\operatorname{grad} r^m) = m(m+1) r^{m-2}$

(or) Prove that $\nabla^2 r^m = m(m+1) r^{m-2}$

__Sol:__ Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\Rightarrow r = \sqrt{x^2 + y^2 + z^2} \Rightarrow r^2 = x^2 + y^2 + z^2$

differentiating with respect to x, y, z partially

$\frac{\partial r}{\partial x} = \frac{x}{r}$ | $\frac{\partial r}{\partial y} = \frac{y}{r}$ | $\frac{\partial r}{\partial z} = \frac{z}{r}$.

Now Consider $\nabla^2(r^m) = \frac{\partial^2}{\partial x^2}(r^m) + \frac{\partial^2}{\partial y^2}(r^m) + \frac{\partial^2}{\partial z^2}(r^m)$

$$= \sum \frac{\partial^2}{\partial x^2}(r^m)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(r^m)\right)$$

$$= \sum \frac{\partial}{\partial x}\left[m r^{m-1}\frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[m r^{m-1}\frac{x}{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[m\, r^{m-2} x\right]$$

$$= \sum m\left[(m-2) r^{m-3}\frac{\partial r}{\partial x} x + m r^{m-2}\right]$$

$$= \sum m(m-2) r^{m-3}\frac{x^2}{r} + m r^{m-2}$$

$$= \sum m(m-2) r^{m-4} x^2 + m r^{m-2}$$

$$= m(m-2) r^{m-4} x^2 + m r^{m-2} + m(m-2) r^{m-4} y^2 + m r^{m-2} + m(m-2) r^{m-4} z^2 + m r^{m-2}$$

$$= m(m-2) r^{m-4}(x^2+y^2+z^2) + 3m r^{m-2}$$

$$= m(m-2) r^{m-4} \cdot r^2 + 3m r^{m-2}$$

$$= (m^2-2m) r^{m-2} + 3m r^{m-2}$$

$$= (m^2-2m+3m) r^{m-2}$$

$$= (m^2+m) r^{m-2}$$

$$= m(m+1) r^{m-2}$$

Hence the proof.

Vector Identities

$$\text{grad}\,\phi \text{ (or) } \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\text{div}\,\bar{f} \text{ (or) } \nabla\cdot\bar{f} = \bar{i}\cdot\frac{\partial\bar{f}}{\partial x} + \bar{j}\cdot\frac{\partial\bar{f}}{\partial y} + \bar{k}\cdot\frac{\partial\bar{f}}{\partial z} = \sum \bar{i}\frac{\partial}{\partial x}\cdot\bar{f}$$

$$\text{Curl}\,\bar{f} \text{ (or) } \nabla\times\bar{f} = \sum \bar{i}\frac{\partial}{\partial x}\times\bar{f} = \bar{i}\times\frac{\partial\bar{f}}{\partial x} + \bar{j}\times\frac{\partial\bar{f}}{\partial y} + \bar{k}\times\frac{\partial\bar{f}}{\partial z}.$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\bar{a}\cdot(\bar{b}\times\bar{c}) = (\bar{a}\times\bar{b})\cdot\bar{c}$$

$$\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$$

$$\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$$

$$\bar{a}\times\bar{b} = -\bar{b}\times\bar{a}$$

① Prove that $\text{curl}(\text{grad}\,\phi) = \bar{0}$

Sol: By def., $\text{grad}\,\phi = \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$$\text{Curl}(\text{grad}\,\phi) = \nabla\times(\text{grad}\,\phi)$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right] - \bar{j}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right] + \bar{k}\left[\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right]$$

$$= 0\bar{i} - 0\bar{j} + 0\bar{k}$$

$$= \bar{0}$$

$$\text{Curl}[\text{grad}\,\phi] = \bar{0}$$

② Prove that $\operatorname{div}[\operatorname{curl}\bar{f}]=0$

Sol: Let $\bar{f}=f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$

$$\operatorname{Curl}\bar{f}=\nabla\times\bar{f}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ f_1 & f_2 & f_3\end{vmatrix}$$

$$=\bar{i}\left[\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right]-\bar{j}\left[\frac{\partial f_3}{\partial x}-\frac{\partial f_1}{\partial z}\right]+\bar{k}\left[\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right]$$

$$\operatorname{div}[\operatorname{Curl}\bar{f}]=\nabla\cdot\operatorname{Curl}\bar{f}$$

$$=\left[\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right]\cdot\left[\bar{i}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)-\bar{j}\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_1}{\partial z}\right)+\bar{k}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\right]$$

$$=\frac{\partial}{\partial x}\left[\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right]-\frac{\partial}{\partial y}\left[\frac{\partial f_3}{\partial x}-\frac{\partial f_1}{\partial z}\right]+\frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)$$

$$=\cancel{\frac{\partial^2 f_3}{\partial x\partial y}}-\cancel{\frac{\partial^2 f_2}{\partial x\partial z}}-\cancel{\frac{\partial^2 f_3}{\partial y\partial x}}+\cancel{\frac{\partial^2 f_1}{\partial y\partial z}}+\cancel{\frac{\partial^2 f_2}{\partial z\partial x}}-\cancel{\frac{\partial^2 f_1}{\partial z\partial y}}$$

$$=0$$

$\therefore \operatorname{div}[\operatorname{curl}\bar{f}]=0.$

③ Prove that $\operatorname{div}(\bar{a}\times\bar{b})=\bar{b}\cdot\operatorname{Curl}\bar{a}-\bar{a}\cdot\operatorname{Curl}\bar{b}$

Sol: $\operatorname{div}(\bar{a}\times\bar{b})=\nabla\cdot(\bar{a}\times\bar{b})$

$$=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\cdot(\bar{a}\times\bar{b})$$

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$$= \sum \bar{i}\frac{\partial}{\partial x}\cdot(\bar{a}\times\bar{b})$$

$$= \sum \bar{i}\cdot\frac{\partial}{\partial x}(\bar{a}\times\bar{b})$$

$$= \sum \bar{i}\cdot\left[\frac{\partial\bar{a}}{\partial x}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial x}\right]$$

$$= \sum \bar{i}\cdot\left[\frac{\partial\bar{a}}{\partial x}\times\bar{b} - \frac{\partial\bar{b}}{\partial x}\times\bar{a}\right]$$

$$= \sum \bar{i}\cdot\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right) - \sum \bar{i}\cdot\left(\frac{\partial\bar{b}}{\partial x}\times\bar{a}\right)$$

$\boxed{\text{Scalar triple product: } \bar{a}\cdot(\bar{b}\times\bar{c}) = (\bar{a}\times\bar{b})\cdot\bar{c}}$

$$= \sum\left(\bar{i}\times\frac{\partial\bar{a}}{\partial x}\right)\cdot\bar{b} - \sum\left(\bar{i}\times\frac{\partial\bar{b}}{\partial x}\right)\cdot\bar{a}$$

$$= \text{Curl}\,\bar{a}\cdot\bar{b} - \text{Curl}\,\bar{b}\cdot\bar{a}$$

$$= (\nabla\times\bar{a})\cdot\bar{b} - (\nabla\times\bar{b})\cdot\bar{a}$$

(4) $\text{Curl}(\bar{a}\times\bar{b}) = \bar{a}\,\text{div}\,\bar{b} - \bar{b}\,\text{div}\,\bar{a} + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$

(or)

$\nabla\times(\bar{a}\times\bar{b}) = \bar{a}(\nabla\cdot\bar{b}) - \bar{b}(\nabla\cdot\bar{a}) + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$

<u>Sol:</u> By def. $\text{Curl}(\bar{a}\times\bar{b}) = \nabla\times(\bar{a}\times\bar{b})$

$$= \sum \bar{i}\frac{\partial}{\partial x}\times(\bar{a}\times\bar{b})$$

$$= \sum \bar{i}\times\frac{\partial}{\partial x}(\bar{a}\times\bar{b})$$

$$= \sum \bar{i}\times\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$= \sum \bar{i}\times\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right) + \sum \bar{i}\times\left(\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$= \sum \bar{i}\times\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right) - \sum \bar{i}\times\left(\frac{\partial\bar{b}}{\partial x}\times\bar{a}\right)$$

$$= \sum \bar{i} \times \left(\frac{\partial \bar{a}}{\partial x} \times \bar{b}\right) - \sum \bar{i} \times \left(\frac{\partial \bar{b}}{\partial x} \times \bar{a}\right)$$

$$= \sum (\bar{i} \cdot \bar{b}) \frac{\partial \bar{a}}{\partial x} - \sum \left(\bar{i} \cdot \frac{\partial \bar{a}}{\partial x}\right) \bar{b} - \sum (\bar{i} \cdot \bar{a}) \frac{\partial \bar{b}}{\partial x}$$

$$+ \sum \left(\bar{i} \cdot \frac{\partial \bar{b}}{\partial x}\right) \bar{a}$$

$$\left[\because \bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{a} \cdot \bar{b}) \bar{c}\right]$$

$$= \sum (\bar{b} \cdot \bar{i}) \frac{\partial \bar{a}}{\partial x} - \sum \left(\bar{i} \cdot \frac{\partial \bar{a}}{\partial x}\right) \bar{b} - \sum (\bar{a} \cdot \bar{i}) \frac{\partial \bar{b}}{\partial x}$$

$$+ \sum \left(\bar{i} \cdot \frac{\partial \bar{b}}{\partial x}\right) \bar{a}$$

$$= (\bar{b} \cdot \nabla) \bar{a} - (\nabla \cdot \bar{a}) \bar{b} - (\bar{a} \cdot \nabla) \bar{b} + (\nabla \cdot \bar{b}) \bar{a}$$

(5) Prove that $\nabla \times (\nabla \times \bar{a}) = \nabla(\nabla \cdot \bar{a}) - \nabla^2 \bar{a}$

(or)

$\text{Curl}(\text{curl}\, \bar{a}) = \text{grad}(\text{div}\, \bar{a}) - \nabla^2 \bar{a}$

Proof: Consider $\nabla \times (\nabla \times \bar{a})$

$$= \nabla \times \left(\bar{i} \times \frac{\partial \bar{a}}{\partial x} + \bar{j} \times \frac{\partial \bar{a}}{\partial y} + \bar{k} \times \frac{\partial \bar{a}}{\partial z}\right)$$

$$= \sum \bar{i} \times \frac{\partial}{\partial x} \left(\bar{i} \times \frac{\partial \bar{a}}{\partial x} + \bar{j} \times \frac{\partial \bar{a}}{\partial y} + \bar{k} \times \frac{\partial \bar{a}}{\partial z}\right)$$

$$= \sum \bar{i} \times \left(\bar{i} \times \frac{\partial^2 \bar{a}}{\partial x^2} + \bar{j} \times \frac{\partial^2 \bar{a}}{\partial x \partial y} + \bar{k} \times \frac{\partial^2 \bar{a}}{\partial x \partial z}\right)$$

$$= \sum \bar{i} \times \left(\bar{i} \times \frac{\partial^2 \bar{a}}{\partial x^2}\right) + \sum \bar{i} \times \left(\bar{j} \times \frac{\partial^2 \bar{a}}{\partial x \partial y}\right)$$

$$+ \sum \bar{i} \times \left(\bar{k} \times \frac{\partial^2 \bar{a}}{\partial x \partial z}\right)$$

$$\left[\text{Now } \bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{a} \cdot \bar{b}) \bar{c}\right]$$

$$= \sum\left(\bar{i}\cdot\frac{\partial^2\bar{a}}{\partial x^2}\right)i - \sum(\bar{i}\cdot\bar{i})\frac{\partial^2\bar{a}}{\partial x^2} + \sum\left(\bar{i}\cdot\frac{\partial^2\bar{a}}{\partial x\partial y}\right)\bar{j}$$

$$\cancel{-\sum(\bar{i}\cdot j)\frac{\partial^2\bar{a}}{\partial x\partial y}} + \sum\left(\bar{i}\cdot\frac{\partial^2\bar{a}}{\partial x\partial z}\right)\bar{k}$$

$$\cancel{-\sum(\bar{i}\cdot\bar{k})\frac{\partial^2\bar{a}}{\partial x\partial z}}$$

$$\left[\because \bar{i}\cdot\bar{j} = 0 = \bar{i}\cdot\bar{k} \;\&\; [\bar{i}\cdot\bar{i} = 1]\right]$$

$$= \sum i\frac{\partial}{\partial x}\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right) + \sum j\frac{\partial}{\partial y}\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right) + \sum k\frac{\partial}{\partial z}\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right)$$

$$-\sum\frac{\partial^2\bar{a}}{\partial x^2}$$

$$= \sum\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right) - \nabla^2\bar{a}$$

$$= \sum \nabla\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right) - \nabla^2\bar{a}$$

$$= \nabla(\nabla\cdot\bar{a}) - \nabla^2\bar{a}$$

Hence the proof.