3.10 Summary: Differentiation

You should have a good understanding of the definition of the derivative, the general Leibniz rule, L'Hôpital's rule, local and global extreme values and how to find them, the Mean Value Theorem (MVT) and the equivalent Rolle's theorem, the inverse function rule and partial derivatives. Here are some key points:

- The *derivative* of $f(x)$ at $x = a$ is (if it exists) $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.
- The *Leibniz rule* is $(fg)' = f'g + fg'$ and extends in an obvious way to products of more than two functions, e.g. $(fgh)' = f'gh + fg'h + fgh'$.
- The *general Leibniz rule* gives an efficient way to calculate higher order derivatives of a product of two functions, specifically $D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} (D^k f)(D^{n-k} g)$.
- The *chain rule* tells us how to differentiate a function of a function, $(f \circ g)'(x) = f'(g(x))g'(x)$.
- It may be possible to evaluate an indeterminate limit $0/0$ or ∞/∞ by *L'Hôpital's rule* $\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(x)/g'(x)$.
- The *extreme value theorem* states that continuous functions on closed intervals always attain upper and lower bounds (*global maxima and minima*).
- A *monotonically increasing/decreasing* function $f(x)$ on an interval is a function which never decreases/increases as x increases on that interval. *Strictly monotonic* excludes the possibility that the function may be constant on some subinterval.
- A function $f(x)$ has a *local maximum/minimum* at $x = a$ if $f(a)$ is a global maximum/minimum on some open interval containing a.
- If f is differentiable and has a local max/min at $x = a$ then it has a *stationary point* at $x = a$, i.e. $f'(a) = 0$. You can test whether a stationary point is a local max or local min or neither with the *first derivative test*, or often with the *second derivative test*.
- All *extreme values* (maxima and minima) of a function on an interval will be at *critical points* (stationary points or points where the function is not differentiable) or at the *endpoints* of the interval. If you are looking for the global extreme values, just evaluate the function at all those points – no need to identify whether each point is a local max or min.
- If f is continuous on $[a, b]$ and differentiable on (a, b) then the *Mean Value Theorem (MVT)* says that $\exists c \in (a, b)$ s.t. $f'(c)$ equals the gradient of the straight line from $(a, f(a))$ to $(b, f(b))$. Rolle's theorem is the MVT in the case where $f(a) = f(b)$ so $f'(c) = 0$.
- If f is continuous on $[a, b]$ and differentiable on (a, b) with $f'(x) \neq 0$ on (a, b) then the *inverse function rule* tells that the derivative of f^{-1} exists on (a, b) and tells us how to relate it to the derivative of f. Letting $g = f^{-1}$ we have $g'(x) = 1/f'(g(x))$.

- If we have a function of more than one variable we can define *partial derivatives* with respect to one of the variables as the ordinary derivative when we treat all the other variables as constants. E.g. for $f(x, y)$ we have the partial derivative of f wrt. x at (x, y): $f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y)-f(x,y)}{h}$ etc.