6 First order differential equations

6.1 First order separable ODEs

A **differential equation** is an equation that relates an unknown function (the dependent variable) to one or more of its derivatives. The **order** of a differential equation is the order of the highest derivative that appears in the equation. A function that satisfies the differential equation is called a **solution** and the process of finding a solution is called **solving** the differential equation. If the dependent variable depends on only a single unknown variable (the independent variable) then the differential equation is called an **ordinary differential equation** (ODE). We shall deal only with ODEs.
Differential equations for functions with several independent variables and involving partial derivatives are called **partial differential equations** (PDEs). These are generally much harder to solve and wont be discussed in this course.

We shall usually discuss ODEs using y to denote the dependent variable and x for the independent variable, so solving the ODE involves finding $y(x)$.

The generic first order ODE may be written in the form

$$\frac{dy}{dx} = f(x,y) \qquad (\star)$$

Generically this will have a one-parameter family of solutions ie. the **general solution** will contain an arbitrary constant. Assigning a particular value to this constant gives a **particular solution**. A particular solution will be singled out by the assignment of an **initial value** ie. requiring $y(x_0) = y_0$, for given x_0 and y_0. In this case the ODE is called an **initial value problem** (IVP). To solve an IVP, first solve the ODE to find the general solution and then determine the value of the constant so that the IVP is also satisfied.

Depending upon the form of the function $f(x,y)$ there are various methods that can be used to solve certain kinds of ODEs. We shall discuss these in turn.

The ODE $(\star)$ is **separable** if $f(x,y) = X(x)Y(y)$.
It can be solved by direct integration as

$$\int \frac{1}{Y(y)}\,dy = \int X(x)\,dx.$$

Eg. Solve the IVP

$$\frac{dy}{dx} = xe^{y-x} \quad \text{with} \quad y(0) = 0.$$

$$y' = xe^{-x}e^{y}, \quad \int e^{-y}\,dy = \int xe^{-x}\,dx, \quad -e^{-y} = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + c$$

$$y = -\log(e^{-x}(1+x) - c). \quad y(0) = 0 = \log(1-c), \quad \text{so} \quad c = 0.$$

$$y = -\log(e^{-x}(1+x)) = x - \log(1+x).$$

Eg. Solve

$$y' = \frac{2x(1+y^2)}{(1+x^2)^2}.$$

$$\int \frac{1}{1+y^2}\,dy = \int \frac{2x}{(1+x^2)^2}\,dx, \quad \tan^{-1} y = -\frac{1}{1+x^2} - c, \quad y = -\tan\left(\frac{1}{1+x^2} + c\right).$$

Eg. Solve the IVP

$$y' = \frac{x^2 y - y}{1+y}, \quad y(3) = 1.$$

$$\int \frac{1+y}{y}\,dy = \int (x^2 - 1)\,dx, \quad y + \log|y| = \frac{x^3}{3} - x + c. \quad \text{Put } x = 3 \text{ and } y = 1$$

$$1 = 6 + c, \quad c = -5, \quad \text{hence} \quad y + \log|y| = \frac{x^3}{3} - x - 5.$$

In this case the final solution can only be written in implicit form ie. it cannot be written in an explicit form $y = \ldots$ where the right hand side is a function of x only.

6.2 First order homogeneous ODEs

The ODE $(\star)$ is **homogeneous** if $f(tx, ty) = f(x, y) \;\; \forall\, t \in \mathbb{R}$.

In this case the substitution $y = xv$ produces a separable ODE for $v(x)$.

Eg. Solve

$$y' = \frac{y^2 - x^2}{xy}.$$

$$f(tx, ty) = \frac{t^2y^2 - t^2x^2}{txty} = \frac{y^2 - x^2}{xy} = f(x, y) \quad \text{hence homogeneous.}$$

Put $y = xv$ then $y' = v + xv'$ and the ODE becomes

$$v + xv' = \frac{x^2v^2 - x^2}{x^2v} = v - \frac{1}{v}, \quad v' = -\frac{1}{xv}, \quad \int v\, dv = \int -\frac{1}{x}\, dx, \quad \frac{v^2}{2} = -\log|x| + \log c,$$

$$v = \pm\sqrt{2\log\left|\frac{c}{x}\right|}, \quad y = xv = \pm x\sqrt{2\log\left|\frac{c}{x}\right|}.$$

6.3 First order linear ODEs

The ODE $(\star)$ is **linear** if $f(x,y) = -p(x)y + q(x)$, in which case we can write the standard form of a linear first order ODE as

$$y' + py = q$$

where p and q can be any functions of x.
In this case the ODE can be solved in terms of the **integrating factor**

$$I(x) = e^{\int p(x)\,dx}.$$

The solution is given by

$$y = \frac{1}{I(x)} \int I(x)q(x)\,dx.$$

Proof: We need to show that y satisfies the ODE $y' + py = q$.

The first step is to observe that $I' = e^{\int p\,dx} p = Ip$.

With $y = \frac{1}{I}\int Iq\,dx$ we have

$$y' = -\frac{I'}{I^2}\int Iq\,dx + \frac{1}{I}Iq = -\frac{I'}{I}y + q = -\frac{Ip}{I}y + q = -py + q,$$

as required.

Eg. Solve the IVP

$$y' - \frac{2}{x}y = 3x^3, \quad y(-1) = 2.$$

In the above notation $p = -\frac{2}{x}$ and $q = 3x^3$.

$$I = \exp\left(\int p\,dx\right) = \exp\left(\int -\frac{2}{x}\,dx\right) = \exp\left(-2\log x\right) = \frac{1}{x^2}.$$

$$y = \frac{1}{I}\int Iq\,dx = x^2 \int \frac{1}{x^2}3x^3\,dx = x^2\int 3x\,dx = x^2\left(\frac{3}{2}x^2 + c\right) = \frac{3}{2}x^4 + cx^2.$$

Using $y(-1) = 2$ we have $2 = \frac{3}{2} + c$ hence $c = \frac{1}{2}$ giving $y = \frac{3}{2}x^4 + \frac{1}{2}x^2$.

6.4 First order exact ODEs

An alternative form in which to write a first order ODE is

$$M(x,y)\,dx + N(x,y)\,dy = 0. \qquad (\star\star)$$

Rearranging gives

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

so this agrees with the form $(\star)$ with $f(x,y) = \frac{M(x,y)}{N(x,y)}$.
Note that a given $f(x,y)$ does not correspond to a unique choice of $M(x,y)$ and $N(x,y)$ as only their ratio determines the ODE, so there is a freedom to multiply both M and N by the same arbitrary function.

For any function $g(x,y)$ the **total differential** dg is defined to be

$$dg = \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy.$$

You may think of dg as the small change in the function g that arises in moving from the point (x,y) to $(x+dx, y+dy)$ where dx and dy are both small. In particular, if dg is identically zero then g is a constant.

The ODE $(\star\star)$ is called **exact** if there exists a function $g(x,y)$ such that the left hand side of $(\star\star)$ is equal to the total derivative dg ie.

$$M = \frac{\partial g}{\partial x} \quad \text{and} \quad N = \frac{\partial g}{\partial y}.$$

In this case the ODE says that $dg = 0$ hence $g = constant$ and this yields the solution of the ODE.

The equality of the mixed partial derivatives $\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y}$ requires that an exact equation satisfies $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
This results in the following **test for exactness**

$$\text{The ODE} \quad M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{is exact iff} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Eg. Show that $(3e^{3x}y + e^x)\,dx + (e^{3x} + 1)\,dy = 0$ is an exact equation and hence solve it.
In this example $M = 3e^{3x}y + e^x$ and $N = e^{3x} + 1$.
$\frac{\partial M}{\partial y} = 3e^{3x} = \frac{\partial N}{\partial x}$ hence this equation is exact.

$$\text{From } M = \frac{\partial g}{\partial x} \text{ we have } \frac{\partial g}{\partial x} = 3e^{3x}y + e^x \quad \text{giving} \;\; g = e^{3x}y + e^x + \phi(y)$$

where the usual constant of integration is replaced by an arbitrary function of integration $\phi(y)$ because of the partial differentiation. To determine this function we use

$$\frac{\partial g}{\partial y} = N, \quad \frac{\partial g}{\partial y} = e^{3x} + 1 = e^{3x} + \phi', \quad \text{so } \phi' = 1, \text{ giving } \phi = y.$$

$$\text{Finally we have that} \quad g = e^{3x}y + e^x + y = \text{constant} = c.$$

In this case we can write the solution in explicit form $y = (c - e^x)/(e^{3x} + 1)$.
As an exercise check that this does indeed solve the starting ODE
$y' = -(3e^{3x}y + e^x)/(e^{3x} + 1)$. Note that this example is also a linear ODE so could also have been solved using an integrating factor.

Consider an ODE $\quad M\,dx + N\,dy = 0 \quad$ that is not exact ie. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.
By multiplying the ODE by a function $I(x, y)$ we can get an equivalent representation of the ODE as $\quad m\,dx + n\,dy = 0 \quad$ where $m = MI$ and $n = NI$. If this equation is exact ie. $\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$ then I is called an **integrating factor** for the original ODE.

Eg. Show that x is an integrating factor for $\quad (3xy - y^2)\,dx + x(x - y)\,dy = 0$.
$M = 3xy - y^2$ so $\frac{\partial M}{\partial y} = 3x - 2y$, but $N = x^2 - xy$ so $\frac{\partial N}{\partial x} = 2x - y$.
Therefore $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and the ODE is not exact.
However, with the integrating factor $I = x$ we get $m = MI = 3x^2y - y^2x$ and $n = NI = x^3 - x^2y$ so now $\frac{\partial m}{\partial y} = 3x^2 - 2yx = \frac{\partial n}{\partial x}$ which is exact.
We can now solve this exact equation using the above method.
$\frac{\partial g}{\partial x} = m = 3x^2y - y^2x$ giving $g = x^3y - \frac{1}{2}y^2x^2 + \phi(y)$.
$\frac{\partial g}{\partial y} = n = x^3 - x^2y = x^3 - x^2y + \phi'$ so $\phi' = 0$ and we can take $\phi = 0$.
This gives the final solution $g = \text{constant} = c$ ie. $x^3y - \frac{1}{2}y^2x^2 = c$.

6.5 Bernoulli equations

A **Bernoulli equation** is a nonlinear ODE of the form

$$y' + p(x)y = q(x)y^n$$

where $n \neq 0, 1,$ otherwise the equation is simply linear.

These ODEs can be solved by first performing the substitution $v = y^{1-n}$, which converts the equation to a linear ODE for $v(x)$.

Eg. Solve

$$y' - \frac{2y}{x} = -x^2y^2.$$

This is a Bernoulli equation with $n = 2$ hence put $v = \frac{1}{y},$ which gives $v' = -\frac{y'}{y^2}$.
Dividing the ODE by y^2 yields

$$\frac{y'}{y^2} - \frac{2}{xy} = -x^2$$

which in terms of v is

$$-v' - \frac{2v}{x} = -x^2$$

ie the linear equation

$$v' + \frac{2}{x}v = x^2.$$

We solve this with an integrating factor $I = \exp\left(\int \frac{2}{x}\,dx\right) = \exp(2\log x) = x^2$ as

$$v = \frac{1}{x^2}\int x^2x^2\,dx = \frac{1}{x^2}\left(\frac{1}{5}x^5 + \frac{c}{5}\right).$$

So finally

$$y = \frac{1}{v} = \frac{5x^2}{x^5 + c}.$$