Calculus I: Background Material
Michaelmas Term

Paul Sutcliffe

Contents

These brief notes review some background material that is assumed knowledge at the start of the course. Please have a glance through this material to check that you are familiar with all the ideas.

1 Notation

Symbol	Denotes
$\forall$	for every
$\exists$	there exists
s.t.	such that
iff	if and only if

The set of real numbers is denoted by $\mathbb{R}$.
Open interval, $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.
Closed interval, $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.
Half-open interval, $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, etc.
Semi-infinite interval, $(a, \infty) = \{x \in \mathbb{R} : x > a\}$, etc.
$\mathbb{R}$ is the same as the interval $(-\infty, \infty)$.
For two sets A and B then $A \cup B$ is the union, $A \cap B$ is the intersection and $A \backslash B$ is the set difference, namely, the set of all elements in A that are not in B.

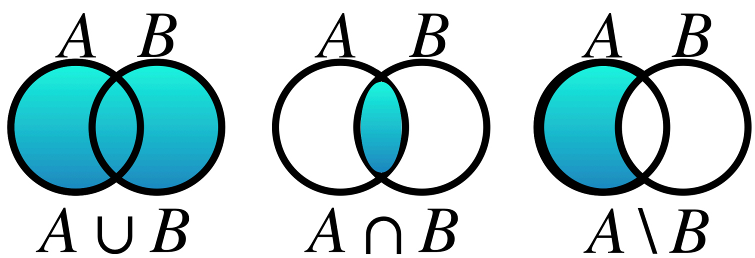

Figure 1: Set theory notation.

2 Graphs and transformations

The graph of a function $f : \mathbb{R} \mapsto \mathbb{R}$ can be shifted, reflected and rescaled by applying some simple transformations to the function and its argument, as follows:

• **translation**:
$f(x) + a$ shifts the graph of $f(x)$ up by an amount a.
$f(x + b)$ shifts the graph of $f(x)$ to the left by an amount b.

• **reflection**:
$-f(x)$ reflects the graph of $f(x)$ about the x-axis.
$f(-x)$ reflects the graph of $f(x)$ about the y-axis.

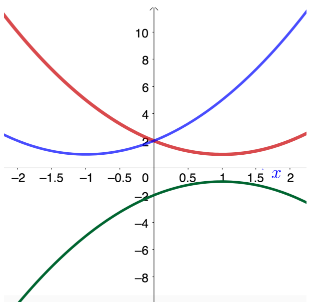

Figure 2: Graph of $f(x) = x^2 - 2x + 2$ (red), together with $-f(x)$ (green), and $f(-x)$ (blue).

$|f(x)|$ reflects the portions of the graph of $f(x)$ that are below the x-axis about the x-axis.

• **scaling**:
$\lambda f(x)$ is a vertical stretch of the graph of $f(x)$ if $\lambda > 1$ and a vertical contraction if $0 < \lambda < 1$.
$f(\mu x)$ is a horizontal contraction of the graph of $f(x)$ if $\mu > 1$ and a horizontal stretch if $0 < \mu < 1$.

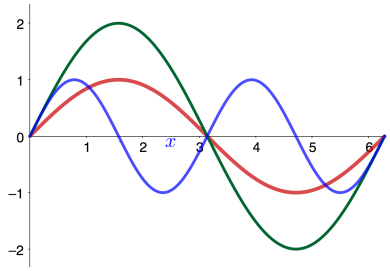

Figure 3: Graph of $f(x) = \sin x$ (red), together with $2f(x)$ (green), and $f(2x)$ (blue).

Combining these transformations in turn is a way to create graphs from the graph of a simpler starting function. As an example, the figure shows how to obtain the graph of the function $f(x) = 2 + 2|(x-1)^2 - 1|$ over $[-1, 3]$ by starting from the function $g(x) = x^2$.

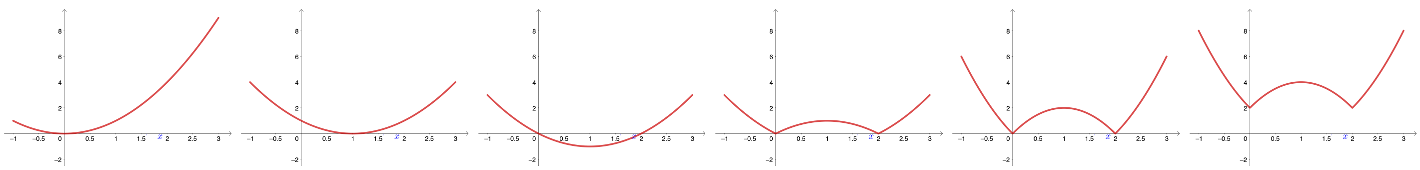

Figure 4: Graphs of x^2, $(x-1)^2$, $(x-1)^2-1$, $|(x-1)^2-1|$, $2|(x-1)^2-1|$, $2+2|(x-1)^2-1|$.

Here are some tips for graphing the reciprocal $1/f(x)$ from the graph of $f(x)$.
If $f(x)$ is increasing (decreasing) then $1/f(x)$ is decreasing (increasing).
The graphs of $f(x)$ and $1/f(x)$ intersect iff $f(x) = \pm 1$.

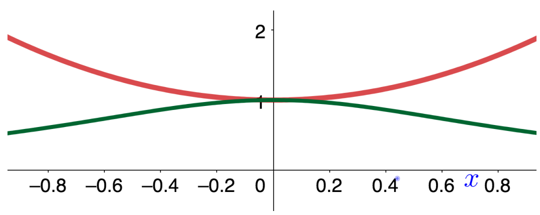

Figure 5: Graph of $f(x) = 1 + x^2$ (red), together with $1/f(x)$ (green).

3 Rational functions

A **rational function** is a ratio of two polynomials $f(x) = p(x)/q(x)$.
The **simplified form** of a rational function is obtained by removing any common factors in the numerator and denominator.

$$\text{Eg.} \qquad f(x) = \frac{2x-4}{x^2-4} = \frac{2(x-2)}{(x+2)(x-2)} = \frac{2}{x+2} \quad \text{if} \;\; x \neq 2.$$

Note that we had to exclude the point $x = 2$ in the last equality to avoid a division by zero.
If, in simplified form, $q(a) = 0$ then $x = a$ is a **vertical asymptote** of the graph of f, which is a vertical line given by the equation $x = a$.

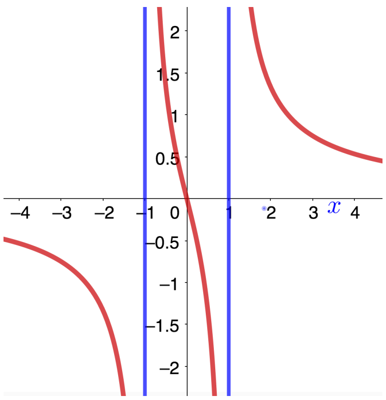

Figure 6: Graph of $f(x) = 2x/(x^2 - 1)$ and the vertical asymptotes $x = \pm 1$.

There can also be **horizontal asymptotes** though a correct treatment requires the concept of a limit, which we will study later in the course. Roughly, a horizontal asymptote involves considering the behaviour of the function when $|x|$ is large. For rational functions a horizontal asymptote exists only if the degree of the numerator is not larger than the degree of the denominator. In this case we can write

$$f(x) = \frac{p_0 + p_1 x + \ldots + p_n x^n}{q_0 + q_1 x + \ldots + q_n x^n}$$

and the horizontal asymptote is the line $y = p_n/q_n$. In the example of Figure 6 we have $n = 2$ and $p_2 = 0, \; q_2 = 1$, so the horizontal asymptote is the line $y = 0$, which the graph approaches when $|x|$ is large.

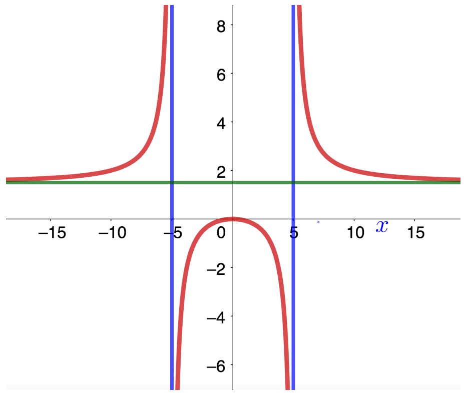

Figure 7: Graph of $f(x) = 3x^2/(2x^2 - 50)$, showing the vertical asymptotes $x = \pm 5$ and the horizontal asymptote $y = 3/2$.

A rational function is called a **proper** rational function if the degree of the numerator is less than the degree of the denominator. A rational function can always be written as the sum of a polynomial and a proper rational function by performing long division.

Eg. $f(x) = (1 + x^6)/(x^3 - x^2)$ is not a proper rational function.

$$
\begin{array}{r|l}
 & \quad\quad x^3 + x^2 + x + 1 \\ \hline
x^3 - x^2 & x^6 + 1 \\
 & \underline{x^6 - x^5} \\
 & \quad x^5 + 1 \\
 & \quad \underline{x^5 - x^4} \\
 & \quad\quad x^4 + 1 \\
 & \quad\quad \underline{x^4 - x^3} \\
 & \quad\quad\quad x^3 + 1 \\
 & \quad\quad\quad \underline{x^3 - x^2} \\
 & \quad\quad\quad\quad x^2 + 1
\end{array}
$$

This yields the required decomposition

$$\frac{1 + x^6}{x^3 - x^2} = x^3 + x^2 + x + 1 + \frac{x^2 + 1}{x^3 - x^2}.$$

4 Exponential and logarithm functions

An **exponential function** is any function of the form $f(x) = b^x$ where b (called the **base**) is a positive real number not equal to 1.

For all values of the base the graph passes through the point $(0, 1)$. However, there are two types of behaviour:

• If $b > 1$ the graph is increasing and has a horizontal asymptote $y = 0$ along the negative x-axis.

• If $0 < b < 1$ the graph is decreasing and has a horizontal asymptote $y = 0$ along the positive x-axis.

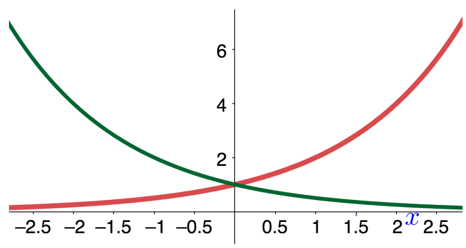

Figure 8: Graph of $f(x) = 2^x$ (red) and $g(x) = (\frac{1}{2})^x = \frac{1}{2^x} = 1/f(x)$ (green).

Two important properties of exponential functions are

$$b^x b^y = b^{x+y} \quad \text{and} \quad (b^x)^y = b^{xy}.$$

There is a special value of the base (called the natural base) $b = e = 2.718...$, where e is called Euler's number and is transcendental (like π) ie. it is not the root of a polynomial equation with rational coefficients. One way to calculate e is via the sum

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

The function $f(x) = e^x$ is usually referred to as the exponential function.

An exponential function $f(x) = b^x$ admits an inverse $f^{-1}(x)$. To determine this write $y = f^{-1}(x)$ so that $f(y) = x$ ie. $b^y = x$. This is the defining relation of a **logarithm function** $g(x) = \log_b x = f^{-1}(x)$ which is simply the name given to the inverse of an exponential function $f(x) = b^x$. Explicitly, $y = \log_b x$ iff $b^y = x$.

Notation: I shall use the notation $\log x$ as the shorthand for the logarithm $\log_e x$, which is known as the natural logarithm.

Warning: This notation is not universal as some people use $\ln x$ to denote the natural logarithm and instead use $\log x$ to denote $\log_{10} x$.

A logarithm grows very slowly for x large and positive (more slowly than a linear function) and has a vertical asymptote at $x = 0$ (see Figure 9).

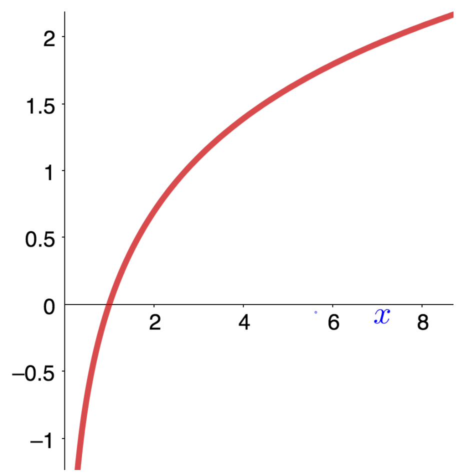

Figure 9: The natural logarithm $\log x$ and the vertical asymptote at $x = 0$.

Some important properties of logarithms are:

- $\log_b 1 = 0$ from $b^0 = 1$.
- $\log_b b^x = x$ from the definition as the inverse of an exponential.
- $\log_b(\alpha\beta) = \log_b \alpha + \log_b \beta$ from $b^\alpha b^\beta = b^{\alpha+\beta}$.
- $\log_b(\alpha/\beta) = \log_b \alpha - \log_b \beta$ from $b^\alpha / b^\beta = b^{\alpha-\beta}$.
- $\log_b(x^r) = r \log_b x$ from $(a^x)^r = a^{xr}$.

There is an equivalent definition of the logarithm as an integral. For $t \in (0, \infty)$ we can define

$$\log t = \int_1^t \frac{1}{x}\, dx.$$

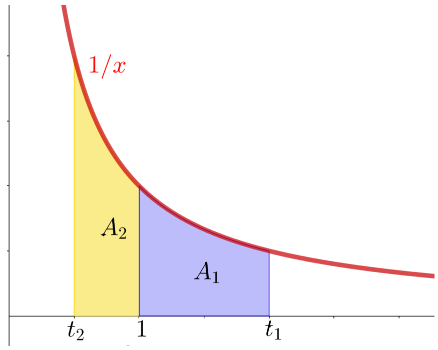

Figure 10: The logarithm function in terms of areas under the curve $y = \frac{1}{x}$.

In Figure 10 we illustrate the logarithm function in terms of areas under the curve $y = \frac{1}{x}$. For $t_1 > 1$ then $\log t_1$ is the area A_1 and for $0 < t_2 < 1$ then $\log t_2$ is minus the area A_2.

The indefinite integral $\int \frac{1}{x}\, dx = \log x$ is only valid for $x > 0$. However, we can extend this relation for the indefinite integral to all $x \neq 0$ as $\int \frac{1}{x}\, dx = \log |x|$. It is easy to prove this result by considering the derivative of $\log(-x)$ for $x < 0$.

If n is a positive integer then x^n has the obvious definition of x multiplied by itself n times. Also, $x^{-n} = 1/x^n$.

We define $x^{\frac{1}{n}}$ as the inverse function of y^n in the interval $(0, \infty)$, which has $x^{\frac{1}{n}} > 0$ and $(x^{\frac{1}{n}})^n = x \quad \forall\, x \in (0, \infty)$.

If q is an integer then we define $x^{\frac{q}{n}} = (x^{\frac{1}{n}})^q$ for $x > 0$. From this definition we have that

$$x^{\frac{q}{n}} = \exp(\log(x^{\frac{1}{n}})^q) = \exp(q \log(x^{\frac{1}{n}})) = \exp(\frac{q}{n} n \log(x^{\frac{1}{n}})) = \exp(\frac{q}{n} \log(x^{\frac{1}{n}})^n)$$

$$= \exp(\frac{q}{n} \log x).$$

Thus for any rational number $r = q/n$ we have that for $x > 0$

$$x^r = \exp(r \log x).$$

If a is irrational (eg. $a = \sqrt{2}$) then we take this formula as a definition of x^a,

$$x^a = \exp(a \log x).$$

5 Trigonometric functions

Defn. A function $f(x)$ is **periodic** if $\exists\ p > 0$ s.t. $f(x+p) = f(x)\ \forall\ x$.
The smallest such value of p is called the **period** of f.

The simplest periodic functions are the trigonometric functions $\sin x$ and $\cos x$ with period 2π. The function $\tan x = \sin x / \cos x$ has period π.with vertical asymptotes at $x = (\frac{1}{2} + n)\pi$.

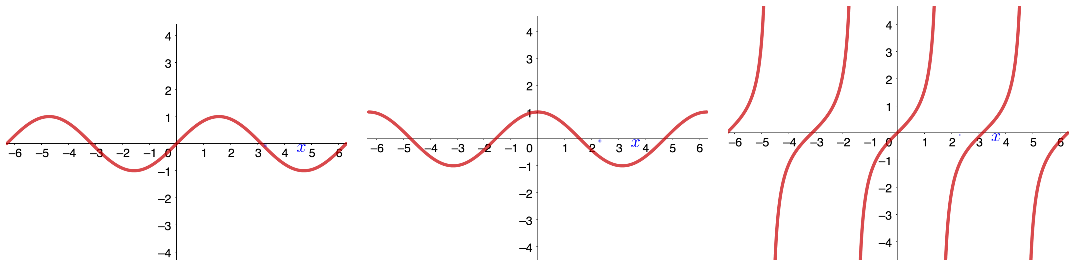

Figure 11: The trigonometric functions $\sin x$, $\cos x$ and $\tan x$.

It is expected that you know the values of the trigonometric functions at the particular values of the argument shown in the table, plus those related by the symmetries apparent in the plots shown in Figure 11.

radians degrees	0 0	$\pi/6$ 30	$\pi/4$ 45	$\pi/3$ 60	$\pi/2$ 90
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

It is also expected that you know the important trigonometric identities:

- $\sin^2 x + \cos^2 x = 1$.
- addition formulae, $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$, & $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
- double angle formulae, $\sin 2x = 2 \sin x \cos x$, & $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- half-angle formulae, $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$, & $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

The reciprocals of the trigonometric functions have their own notation:
$\csc x = 1/\sin x$, $\sec x = 1/\cos x$, $\cot x = 1/\tan x = \cos x/\sin x$.

The above formulae lead to related formulae for these functions eg. $\sec^2 x = 1 + \tan^2 x$.

6 Hyperbolic functions

Hyperbolic functions are defined in terms of the exponential function via

$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \quad \cosh x = \frac{1}{2}(e^x + e^{-x}), \quad \tanh x = \sinh x / \cosh x.$$

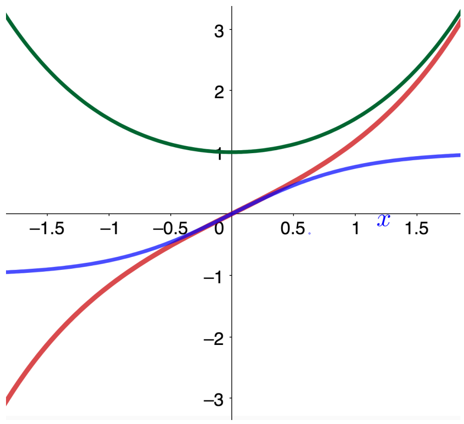

Figure 12: Graphs of the functions $\sinh x$ (red), $\cosh x$ (green), and $\tanh x$ (blue).

From the properties of the exponential function it is easy to show that some basic properties include

- $\sinh 0 = 0$ and $\cosh 0 = 1$.

- If $x > 0$ then $0 < \sinh x < \cosh x$.

- Addition formulae, $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$, & $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$.

- Double argument formulae, $\sinh(2x) = 2 \sinh x \cosh x$, & $\cosh(2x) = \sinh^2 x + \cosh^2 x$.

- $\cosh^2 x - \sinh^2 x = 1$.

The reciprocals of hyperbolic functions have their own notation:
$\text{cosech}\, x = 1/\sinh x$, $\text{sech}\, x = 1/\cosh x$, $\coth x = 1/\tanh x$.

The above formulae lead to related formulae for these functions eg. $\text{sech}^2 x = 1 - \tanh^2 x$.

7 Rules of differentiation

You are expected to know the following rules of differentiation.
If $f(x)$ and $g(x)$ are differentiable at x then so are the following:

• **sum rule:**
$f(x) + g(x)$ with derivative $f'(x) + g'(x)$.

• **product rule:**
$f(x)g(x)$ with derivative $f'(x)g(x) + f(x)g'(x)$.

and providing $g(x) \neq 0$ then so are the following:

• **reciprocal rule:**
$\frac{1}{g(x)}$ with derivative $-\frac{g'(x)}{(g(x))^2}$.

• **quotient rule:**
$\frac{f(x)}{g(x)}$ with derivative $\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$.

Eg. Differentiate $u(x) = (3x^2 - 1)/(x^4 + 3x + 1)$

$$u'(x) = \frac{(3x^2-1)'(x^4+3x+1) - (3x^2-1)(x^4+3x+1)'}{(x^4+3x+1)^2}$$

$$= \frac{6x(x^4+3x+1) - (3x^2-1)(4x^3+3)}{(x^4+3x+1)^2} = \frac{6x^5+18x^2+6x-(12x^5+9x^2-4x^3-3)}{(x^4+3x+1)^2}$$

$$= \frac{-6x^5+4x^3+9x^2+6x+3}{(x^4+3x+1)^2}.$$

8 Standard integrals

A **closed form expression** is a function obtained from elementary functions by a finite sequence of elementary operations.

$$\text{Eg.} \quad \frac{e^{x^2} + \sin(1+x^2)}{x^4 + e^{\cos x}} \quad \text{is a closed form expression.}$$

A basic problem in integration is given a closed form expression $f(x)$ can we find a closed form expression for its indefinite integral $F(x) = \int f(x)\,dx$?

In general this is not possible, for example, $\int e^{-x^2}\,dx$ does not have a closed form expression.

The game is to try and reduce $\int f(x)\,dx$ to some standard integrals by using a number of rules of integration eg. integration by parts, substitution,...

Some standard integrals that you are expected to know include

$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + c, \quad \text{for constant } a \neq -1$$

$$\int \frac{1}{x}\,dx = \log|x| + c$$

$$\int e^x\,dx = e^x + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \mathrm{cosec}^2 x\,dx = -\cot x + c$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$$

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1} x + c$$

9 Integration by parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Eg.

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2e^{3x} - \int \frac{2}{3}xe^{3x}\,dx = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \int \frac{2}{9}e^{3x}\,dx$$
$$= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} \; + c.$$

Eg. $\displaystyle\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x \; + c.$

Eg. $\displaystyle\int \log x\,dx = \int (\log x)1\,dx = (\log x)x - \int \frac{1}{x}x\,dx = x\log x - x + c = x(\log x - 1) + c.$

Eg. $\displaystyle\int e^x\cos x\,dx = e^x\sin x - \int e^x\sin x\,dx = e^x\sin x + e^x\cos x - \int e^x\cos x\,dx.$

Hence $\displaystyle\int e^x\cos x\,dx = \frac{1}{2}e^x(\sin x + \cos x) \; + c.$

10 Integration by substitution

If $F(x) = \int f(x)\,dx$ then $F(u(x)) = \int f(u(x))\,u'(x)\,dx$.
In more compact notation, by making the substitution from x to $u(x)$ we have

$$\int f(x)\,dx = \int f(u)\,du, \qquad \text{where} \quad du = u'(x)\,dx.$$

Eg. Calculate $\int \frac{x}{2+3x^4}\,dx$.
Make the substitution $u = \sqrt{\frac{3}{2}}x^2$ then $du = \sqrt{6}x\,dx$ and

$$\int \frac{x}{2+3x^4}\,dx = \int \left(\frac{1}{\sqrt{6}}\right)\frac{1}{2+2u^2}\,du = \int \left(\frac{1}{2\sqrt{6}}\right)\frac{1}{1+u^2}\,du = \frac{1}{2\sqrt{6}}\tan^{-1}u \; + c$$
$$= \frac{1}{2\sqrt{6}}\tan^{-1}(\sqrt{\frac{3}{2}}x^2) \; + c.$$

Eg. Calculate $\int \tan x\,dx$.
Put $u = \cos x$ then $du = -\sin x\,dx$ and

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int -\frac{1}{u}\,du = -\log|u| \; + c = -\log|\cos x| \; + c.$$

Eg. Calculate $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$.
Put $u = \sin x$ then $du = \cos x\,dx$.
When $x = 0$ then $u = 0$ and when $x = \pi/2$ then $u = 1$. Therefore

$$\int_0^{\frac{\pi}{2}} \cos^3 x\,dx = \int_0^{\frac{\pi}{2}} \cos x(1 - \sin^2 x)\,dx = \int_0^1 (1 - u^2)\,du = \left[u - \frac{1}{3}u^3\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Eg. Calculate $\int \frac{1}{(x^2+4)^2}\,dx$.
Put $x = 2\tan u$ then $dx = 2\sec^2 u\,du$ and

$$\int \frac{1}{(x^2+4)^2}\,dx = \int \frac{2\sec^2 u\,du}{(4\tan^2 u + 4)^2} = \int \frac{(\tan^2 u + 1)\,du}{8(\tan^2 u + 1)^2} = \int \frac{du}{8(\tan^2 u + 1)}$$

$$= \int \frac{1}{8}\cos^2 u\,du = \int \frac{1}{16}(1 + \cos 2u)\,du = \frac{1}{16}(u + \frac{1}{2}\sin 2u) = \frac{1}{16}(u + \sin u \cos u)$$

$$= \frac{1}{16}\left(u + \frac{\tan u}{\sec^2 u}\right) = \frac{1}{16}\left(u + \frac{\tan u}{1 + \tan^2 u}\right) = \frac{1}{16}\left(\tan^{-1}\frac{x}{2} + \frac{\frac{x}{2}}{1 + \frac{x^2}{4}}\right)$$

$$\text{Thus} \quad \int \frac{1}{(x^2+4)^2}\,dx = \frac{1}{16}\left(\tan^{-1}\frac{x}{2} + \frac{2x}{4 + x^2}\right) + c.$$

For a general function $f(x)$ the formula

$$\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c$$

is a simple application of integration by substitution.
Simply put $u = f(x)$ so that $du = f'(x)\,dx$. Then

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{1}{u}\,du = \log|u| + c = \log|f(x)| + c.$$

$$\text{Eg.} \quad \int \frac{x^3 + 1}{x^4 + 4x + 7}\,dx = \frac{1}{4}\log|x^4 + 4x + 7| + c.$$

11 Integration of rational functions

Given a rational function, first of all decompose it into a polynomial and a proper rational function. As the integration of a polynomial is trivial we can now restrict attention to the integration of proper rational functions.

Every proper rational function can be written as a sum of **partial fractions** which take the form

$$\frac{A}{(x-\alpha)^k} \quad \text{and} \quad \frac{Bx+C}{(x^2+\beta x+\gamma)^l}$$

where the quadratic that appears in the above has no real roots.

The number and type of partial fractions that appear depends on the factors in the denominator of the rational function.

Each factor in the denominator of the form $(x-\alpha)^k$ generates a partial fraction of the form

$$\frac{A_1}{(x-\alpha)} + \frac{A_2}{(x-\alpha)^2} + \ldots + \frac{A_k}{(x-\alpha)^k}.$$

Each factor in the denominator of the form $(x^2+\beta x+\gamma)^l$ generates a partial fraction of the form

$$\frac{B_1x+C_1}{(x^2+\beta x+\gamma)} + \frac{B_2x+C_2}{(x^2+\beta x+\gamma)^2} + \ldots + \frac{B_lx+C_l}{(x^2+\beta x+\gamma)^l}.$$

The coefficients A_k, B_l, C_l can be found directly by comparison with the rational function either by the substitution of specific values of x or by comparing coefficients of x. The integration is then completed by integrating the partial fraction.

Eg. Calculate $\int \frac{2x}{x^2-x-2}\,dx$.

$$\frac{2x}{x^2-x-2} = \frac{2x}{(x-2)(x+1)} = \frac{a}{x-2} + \frac{b}{x+1}.$$

To determine a and b multiply the above by $(x-2)(x+1)$ to give
$2x = a(x+1) + b(x-2)$. Putting $x=2$ yields $4=3a$ and putting $x=-1$ yields $-2=-3b$ hence

$$\frac{2x}{x^2-x-2} = \frac{\frac{4}{3}}{x-2} + \frac{\frac{2}{3}}{x+1}.$$

Integrating this expression gives the required integral

$$\int \frac{2x}{x^2-x-2}\,dx = \int \frac{\frac{4}{3}}{x-2} + \frac{\frac{2}{3}}{x+1} = \frac{4}{3}\log|x-2| + \frac{2}{3}\log|x+1| + c.$$

Eg. Calculate $\int \frac{2x^2+3}{x^3-2x^2+x}\,dx$.

$$\frac{2x^2+3}{x^3-2x^2+x} = \frac{2x^2+3}{x(x-1)^2} = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{(x-1)^2}$$

As $2x^2+3 = a(x-1)^2 + bx(x-1) + cx$ then putting $x=0$ gives $a=3$, while $x=1$ gives $5=c$. Comparing coefficients of x^2 gives $2=a+b$ thus $b=-1$.

$$\int \frac{2x^2+3}{x^3-2x^2+x}\,dx = \int \frac{3}{x} - \frac{1}{x-1} + \frac{5}{(x-1)^2}\,dx = 3\log|x| - \log|x-1| - \frac{5}{x-1} + C.$$

Eg. Calculate $\int \frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}\,dx$.

$$\frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2} = \frac{a}{x+1} + \frac{bx+c}{x^2+4} + \frac{dx+f}{(x^2+4)^2}$$

It is an exercise to show that $a=2,\ b=1,\ c=0,\ d=0,\ f=-1$.

$$\int \frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}\,dx = \int \frac{2}{x+1} + \frac{x}{x^2+4} - \frac{1}{(x^2+4)^2}\,dx$$

$$= 2\log|1+x| + \frac{1}{2}\log(x^2+4) - \frac{1}{8}\left(\frac{x}{x^2+4}\right) - \frac{1}{16}\tan^{-1}\frac{x}{2} + C.$$