

25 Let \mathbf{x} be the position vector in three dimensions, with $r = |\mathbf{x}|$, and let \mathbf{a} be a constant vector. Using index notation, show that

(a) $\operatorname{div} \mathbf{x} = 3$,

Solution:

$$\operatorname{div} \mathbf{x} = \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3.$$

(b) $\operatorname{curl} \mathbf{x} = 0$,

Solution:

$$(\operatorname{curl} \mathbf{x})_i = \varepsilon_{ijk} \frac{\partial x_k}{\partial x_j} = \varepsilon_{ijk} \delta_{jk} = 0,$$

using the fact that ε_{ijk} is antisymmetric under swaps of j and k while δ_{jk} is symmetric.

(c) $\operatorname{grad} r = \mathbf{x}/r$,

Solution:

$$\begin{aligned} (\operatorname{grad} r)_i &= \frac{\partial}{\partial x_i} r = \frac{\partial}{\partial x_i} (x_j x_j)^{1/2} \\ &= \frac{1}{2} (x_j x_j)^{-1/2} \frac{\partial}{\partial x_i} x_k x_k \\ &= \frac{1}{2} (x_j x_j)^{-1/2} (2\delta_{ik} x_k) \\ &= (x_j x_j)^{-1/2} x_i \\ &= x_i / r. \end{aligned}$$

(d) $\operatorname{div} (r^n \mathbf{x}) = (n+3)r^n$,

Solution:

$$\begin{aligned} \operatorname{div} (r^n \underline{x}) &= \frac{\partial}{\partial x_i} (r^n x_i) = \left(\frac{\partial}{\partial x_i} r^n \right) x_i + r^n \left(\frac{\partial}{\partial x_i} x_i \right) = nr^{n-1} \left(\frac{\partial}{\partial x_i} r \right) x_i + r^n \delta_{ii} \\ &= nr^{n-1} (x_i / r) x_i + 3r^n = nr^n + 3r^n = (n+3)r^n. \end{aligned}$$

(e) $\operatorname{grad} (\mathbf{a} \cdot \mathbf{x}) = \mathbf{a}$,

Solution:

$$(\operatorname{grad} (\underline{a} \cdot \underline{x}))_i = \frac{\partial}{\partial x_i} (a_j x_j) = a_j \frac{\partial}{\partial x_i} x_j = a_j \delta_{ij} = a_i.$$

(f) $\operatorname{div} (\mathbf{a} \times \mathbf{x}) = 0$,

Solution:

$$\operatorname{div} (\underline{a} \times \underline{x}) = \frac{\partial}{\partial x_i} (\varepsilon_{ijk} a_j x_k) = \varepsilon_{ijk} a_j \delta_{ik} = 0,$$

using the fact that ε_{ijk} is antisymmetric under swaps of i and k while δ_{ik} is symmetric.

(g) $\operatorname{curl} (\mathbf{a} \times \mathbf{x}) = 2\mathbf{a}$,

Solution:

$$\begin{aligned}
(\operatorname{curl}(\underline{a} \times \underline{x}))_i &= \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\varepsilon_{klm} a_l x_m) = \varepsilon_{ijk} \varepsilon_{klm} a_l \frac{\partial}{\partial x_j} x_m \\
&= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) a_l \frac{\partial}{\partial x_j} x_m = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) a_l \delta_{jm} \\
&= a_i \delta_{jj} - \delta_{ij} a_j = 3a_i - a_i = 2a_i.
\end{aligned}$$

$$(h) \quad \operatorname{curl}(r^2 \mathbf{a}) = 2(\mathbf{x} \times \mathbf{a}),$$

Solution:

$$\begin{aligned}
(\operatorname{curl}(r^2 \mathbf{a}))_i &= \varepsilon_{ijk} \frac{\partial}{\partial x_j} (r^2 a_k) = \varepsilon_{ijk} \left(\frac{\partial}{\partial x_j} r^2 \right) a_k \\
&= \varepsilon_{ijk} 2r \frac{\partial r}{\partial x_j} a_k = \varepsilon_{ijk} 2r \frac{x_j}{r} a_k = (2(\mathbf{x} \times \mathbf{a}))_i.
\end{aligned}$$

$$(i) \quad \nabla^2(1/r) = 0 \text{ if } r \neq 0: \text{ using } \frac{\partial}{\partial x_i} r = x_i/r \text{ from part (c),}$$

Solution:

$$\begin{aligned}
\nabla^2(1/r) &= \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} (r^{-1}) = \frac{\partial}{\partial x_i} \left(-r^{-2} \frac{\partial}{\partial x_i} r \right) = \frac{\partial}{\partial x_i} (-r^{-3} x_i) \\
&= -\left(\frac{\partial}{\partial x_i} r^{-3} \right) x_i - r^{-3} \left(\frac{\partial}{\partial x_i} x_i \right) = 3r^{-4} \left(\frac{\partial}{\partial x_i} r \right) x_i - r^{-3} \delta_{ii} \\
&= 3r^{-5} x_i x_i - 3r^{-3} = (3 - 3)r^{-3} = 0.
\end{aligned}$$

Notice that this is only valid for $r \neq 0$ since the calculation involves division by r .

$$(j) \quad \nabla^2(\log r) = 1/r^2 \text{ if } r \neq 0:$$

Solution: as in part (i), and using $\frac{\partial}{\partial x_i} r = x_i/r$ again,

$$\begin{aligned}
\nabla^2(\log r) &= \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} (\log r) = \frac{\partial}{\partial x_i} \left(r^{-1} \frac{\partial}{\partial x_i} r \right) = \frac{\partial}{\partial x_i} (r^{-2} x_i) \\
&= \left(\frac{\partial}{\partial x_i} r^{-2} \right) x_i + r^{-2} \left(\frac{\partial}{\partial x_i} x_i \right) = -2r^{-3} \left(\frac{\partial}{\partial x_i} r \right) x_i + r^{-2} \delta_{ii} \\
&= -2r^{-4} x_i x_i + 3r^{-2} = (-2 + 3)r^{-2} = 1/r^2.
\end{aligned}$$

Again, this is only valid for $r \neq 0$ since the calculation involves division by r .

$$(k) \quad \operatorname{div}[(\mathbf{a} \cdot \mathbf{x})\mathbf{x}] = 4\mathbf{a} \cdot \mathbf{x},$$

Solution:

$$\begin{aligned}
\operatorname{div}[(\mathbf{a} \cdot \mathbf{x})\mathbf{x}] &= \frac{\partial}{\partial x_i} (a_j x_j x_i) = a_j \frac{\partial}{\partial x_i} (x_j x_i) = a_j \left(\left(\frac{\partial}{\partial x_i} x_j \right) x_i + x_j \left(\frac{\partial}{\partial x_i} x_i \right) \right) \\
&= a_j (\delta_{ij} x_i + x_j \delta_{ii}) = a_j (x_j + 3x_j) = 4a_j x_j = 4\mathbf{a} \cdot \mathbf{x}.
\end{aligned}$$

Using $\operatorname{div} f\mathbf{V} = (\nabla f) \cdot \mathbf{V} + f \operatorname{div} \mathbf{V}$, with $f = \mathbf{a} \cdot \mathbf{x}$ so $\nabla f = \mathbf{a}$ from part (e) and $\mathbf{V} = \mathbf{x}$ so $\operatorname{div} \mathbf{x} = 3$ from part (a) hence $\operatorname{div}[(\mathbf{a} \cdot \mathbf{x})\mathbf{x}] = \mathbf{a} \cdot \mathbf{x} + \mathbf{a} \cdot \mathbf{x} 3 = 4\mathbf{a} \cdot \mathbf{x}$.

$$(l) \quad \operatorname{div} [\mathbf{x} \times (\mathbf{x} \times \mathbf{a})] = 2 \mathbf{a} \cdot \mathbf{x},$$

Solution:

$$\begin{aligned} \operatorname{div} [\mathbf{x} \times (\mathbf{x} \times \mathbf{a})] &= \frac{\partial}{\partial x_i} (\varepsilon_{ijk} x_j \varepsilon_{klm} x_l a_m) = \varepsilon_{ijk} \varepsilon_{klm} \frac{\partial}{\partial x_i} (x_j x_l) a_m \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \frac{\partial}{\partial x_i} (x_j x_l) a_m \\ &= \frac{\partial}{\partial x_i} (x_j x_i) a_j - \frac{\partial}{\partial x_i} (x_j x_j) a_i \\ &= \delta_{ij} x_i a_j + x_j \delta_{ii} a_j - \delta_{ij} x_j a_i - x_j \delta_{ij} a_i \\ &= x_j a_j + 3x_j a_j - x_j a_j - x_j a_j = 2x_j a_j = 2 \underline{a} \cdot \underline{x}. \end{aligned}$$

$$(m) \quad \operatorname{curl} (\mathbf{a} \times \mathbf{x} / r^3) = 3 (\mathbf{a} \cdot \mathbf{x}) \mathbf{x} / r^5 - \mathbf{a} / r^3,$$

Solution:

$$\begin{aligned} (\operatorname{curl} (\mathbf{a} \times \mathbf{x} / r^3))_i &= \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\varepsilon_{klm} a_l x_m / r^3) = \varepsilon_{ijk} \varepsilon_{klm} a_l \frac{\partial}{\partial x_j} (x_m / r^3) \\ &= \varepsilon_{ijk} \varepsilon_{klm} a_l \left(\delta_{jm} r^{-3} + x_m \frac{\partial}{\partial x_j} r^{-3} \right) \\ &= \varepsilon_{ijk} \varepsilon_{klm} a_l \left(\delta_{jm} r^{-3} - 3x_m r^{-4} \frac{\partial}{\partial x_j} r \right) \\ &= \varepsilon_{ijk} \varepsilon_{klm} a_l \left(\delta_{jm} r^{-3} - 3x_m r^{-5} x_j \right) \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) a_l \left(\delta_{jm} r^{-3} - 3x_m r^{-5} x_j \right) \\ &= a_i \delta_{jj} r^{-3} - a_j \delta_{ij} r^{-3} - 3a_i x_j r^{-5} x_j + 3a_j x_i r^{-5} x_j \\ &= 3a_i r^{-3} - a_i r^{-3} - 3a_i r^{-3} + 3x_i a_j x_j r^{-5} \\ &= -a_i r^{-3} + 3x_i a_j x_j r^{-5} = (-\underline{a} / r^3 + 3 \underline{x} (\underline{a} \cdot \underline{x}) / r^5)_i. \end{aligned}$$

Remark: in this question, and also in some of the earlier ones, there is some freedom in the order in which you, for example, use the product rule on derivatives, use the very useful formula, and simplify δ s involving summed indices. In such cases any order is fine – the final answer will be the same.

$$(n) \quad \text{Exam question June 2002 (Section A): calculate the curl of } (\mathbf{a} \cdot \mathbf{x}) \mathbf{x}.$$

Solution:

$$\begin{aligned} (\operatorname{curl} ((\mathbf{a} \cdot \mathbf{x}) \mathbf{x}))_i &= \varepsilon_{ijk} \frac{\partial}{\partial x_j} (a_l x_l x_k) \\ &= \varepsilon_{ijk} a_l (\delta_{jl} x_k + x_l \delta_{jk}) \\ &= \varepsilon_{ijk} a_j x_k + \varepsilon_{ijj} a_l x_l = \varepsilon_{ijk} a_j x_k = (\underline{a} \times \underline{x})_i, \end{aligned}$$

using the fact that $\varepsilon_{ijj} = 0$, by the antisymmetry of the Levi-Civita symbol. Note: in the first line $\underline{a} \cdot \underline{x}$ was written using l as the repeated index rather than j or k , since j and k were already being used to write the curl.

General remark: Even though some of these index notation calculations look a little long, they are (almost) all self-contained, while in the version of this question on the Topic 3 sheet, there was much more quoting of results from other parts.

- 26 The vector \underline{a} has components $(a_r) = (1, 1, 1)$ and the vector \underline{b} has components $(b_r) = (2, 3, 4)$. In the following expressions state which indices are free and which are dummy, and give the numerical values of the expressions for each value that the free variable takes (e.g. for $a_r - b_r$ the free variable is r and it takes the values 1, 2, 3 so $a_1 - b_1 = -1$, $a_2 - b_2 = -2$, $a_3 - b_3 = -3$)

(a) $a_r + b_r$,

Solution: r is a free index so we need to specify 3 pieces of information. $a_r + b_r$ is the r -th component of the vector $\underline{a} + \underline{b}$ so we can give the information in the form $(a_r + b_r) = (3, 4, 5)$ or as $a_1 + b_1 = 3$, $a_2 + b_2 = 4$, $a_3 + b_3 = 5$.

(b) $a_r b_r$,

Solution: r is a dummy index and $a_r b_r = a_1 b_1 + a_2 b_2 + a_3 b_3 = 9$, which is the scalar product of \underline{a} and \underline{b} .

(c) $a_r b_s a_r$,

Solution: r is a dummy index, and s is free so to specify $a_r b_s a_r$ we need to specify 3 pieces of information, these are the components of the vector $\underline{b} \cdot \underline{a}$. Now $a_r a_r = a_1^2 + a_2^2 + a_3^2 = 3$ so $(a_r b_s a_r) = (3b_s) = (6, 9, 12)$

(d) $a_r b_s a_r b_s - a_r b_r a_s b_s$.

Solution: r and s are both dummy indices. $a_r b_s a_r b_s - a_r b_r a_s b_s = (a_r a_r)(b_s b_s) - (a_r b_r)(a_s b_s)$ now $a_r a_r = 3$ and $b_s b_s = 4 + 9 + 16 = 29$ so $a_r b_s a_r b_s - a_r b_r a_s b_s = 3 \times 29 - 9^2 = 6$.

- 27 If δ_{rs} is the three-dimensional Kronecker delta, evaluate

(a) $\delta_{rs} \delta_{sr} \delta_{pq} \delta_{pq}$,

Solution: $\delta_{rs} \delta_{sr} \delta_{pq} \delta_{pq} = \delta_{rr} \delta_{pq} \delta_{pq} = 3 \delta_{pp} = 3 \times 3 = 9$.

(b) $\delta_{rs} \delta_{sk} \delta_{kl} \delta_{lr}$,

Solution: $\delta_{rs} \delta_{sk} \delta_{kl} \delta_{lr} = \delta_{rk} \delta_{kl} \delta_{lr} = \delta_{rl} \delta_{lr} = \delta_{rr} = 3$.

(c) $\delta_{rs} \delta_{qr} \delta_{pq} \delta_{sp}$.

Solution: $\delta_{rs} \delta_{qr} \delta_{pq} \delta_{sp} = \delta_{qs} \delta_{pq} \delta_{sp} = \delta_{ps} \delta_{sp} = \delta_{ss} = 3$.

- 28 If δ_{rs} is the three-dimensional Kronecker delta, simplify

(a) $(\delta_{rp} \delta_{sq} - \delta_{rq} \delta_{sp}) a_p b_q$,

Solution: $(\delta_{rp} \delta_{sq} - \delta_{rq} \delta_{sp}) a_p b_q = \delta_{rp} \delta_{sq} a_p b_q - \delta_{rq} \delta_{sp} a_p b_q = a_r b_s - a_s b_r$,

(b) $(\delta_{rp} \delta_{sq} - \delta_{rq} \delta_{sp}) \delta_{pq}$.

Solution: $(\delta_{rp} \delta_{sq} - \delta_{rq} \delta_{sp}) \delta_{pq} = \delta_{rp} \delta_{sq} \delta_{pq} - \delta_{rq} \delta_{sp} \delta_{pq} = \delta_{rs} - \delta_{rs} = 0$.

- 29 If δ_{rs} is the Kronecker delta in n dimensions, calculate

(a) δ_{rr} ,

Solution: $\delta_{rr} = 1 + 1 + \dots + 1$ (n terms), so $\delta_{rr} = n$,

(b) $\delta_{rs} \delta_{rs}$,

Solution: $\delta_{rs} \delta_{rs} = \delta_{rr} = n$,

(c) $\delta_{rs} \delta_{st} \delta_{tr}$.

Solution: $\delta_{rs} \delta_{st} \delta_{tr} = \delta_{rs} \delta_{sr} = \delta_{rr} = n$.

30 Starting from $\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$ simplify as much as possible:

(a) $\varepsilon_{ijk} \varepsilon_{ijp}$,

Solution: First use the cyclic symmetry of ε_{ijk} to rewrite the expression with the repeated indices next to each other: $\varepsilon_{ijk} \varepsilon_{ijp} = \varepsilon_{jki} \varepsilon_{ijp}$. Then, relabelling in the starting formula,

$$\varepsilon_{jki} \varepsilon_{ijp} = \delta_{jj} \delta_{kp} - \delta_{jp} \delta_{kj} = 3 \delta_{kp} - \delta_{kp} = 2 \delta_{kp}$$

(b) $\varepsilon_{ijk} \varepsilon_{ijk}$.

Solution: Substituting $p = k$ into part (a),

$$\varepsilon_{ijk} \varepsilon_{ijk} = 2 \delta_{kk} = 6$$

31 Calculate ε_{ijj} .

Solution: ε_{ijk} vanishes whenever two indices take the same value, so ε_{ijj} also vanishes, being a sum of terms in each of which the last two indices take the same value.

32 Show, using index notation, that

(a) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$,

Solution: The i -th component of $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is

$$\begin{aligned} \varepsilon_{ijk} a_j (\varepsilon_{krs} b_r c_s) &= (\varepsilon_{ijk} \varepsilon_{rsk}) a_j b_r c_s = (\delta_{ir} \delta_{js} - \delta_{is} \delta_{jr}) a_j b_r c_s \\ &= \delta_{ir} \delta_{js} a_j b_r c_s - \delta_{is} \delta_{jr} a_j b_r c_s = a_s b_i c_s - a_r b_r c_i \end{aligned}$$

so the i -th component of $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ is

$$\begin{aligned} (a_s b_i c_s - a_r b_r c_i) &+ (b_s c_i a_s - b_r c_r a_i) + (c_s a_i b_s - c_r a_r b_i) \\ &= a_i (c_s b_s - b_r c_r) + b_i (a_s c_s - c_r a_r) + c_i (b_s a_s - a_r b_r) = 0 \end{aligned}$$

since the names of dummy indices can be changed.

(b) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{d} \cdot (\mathbf{a} \times \mathbf{b})] \mathbf{c} - [\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})] \mathbf{d}$
 $= [\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})] \mathbf{b} - [\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})] \mathbf{a}$,

Solution: The i -th component of $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$ is

$$\begin{aligned} \varepsilon_{ijk} (\varepsilon_{jrs} a_r b_s) (\varepsilon_{kpq} c_p d_q) &= \varepsilon_{ijk} \varepsilon_{jrs} \varepsilon_{kpq} a_r b_s c_p d_q \\ &= (\varepsilon_{jki} \varepsilon_{jrs}) \varepsilon_{kpq} a_r b_s c_p d_q \\ &= \delta_{kr} \delta_{is} \varepsilon_{kpq} a_r b_s c_p d_q - \delta_{ks} \delta_{ir} \varepsilon_{kpq} a_r b_s c_p d_q \\ &= \varepsilon_{rpq} a_r b_i c_p d_q - \varepsilon_{spq} a_i b_s c_p d_q \end{aligned}$$

which is the same as the i -th component of $[\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})] \mathbf{b} - [\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})] \mathbf{a}$, because this is

$$[a_r (\varepsilon_{rpq} c_p d_q)] b_i - [b_s (\varepsilon_{spq} c_p d_q)] a_i$$

We can also expand the i -th component of $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$ as

$$\begin{aligned} \varepsilon_{ijk} (\varepsilon_{jrs} a_r b_s) (\varepsilon_{kpq} c_p d_q) &= \varepsilon_{ijk} \varepsilon_{jrs} \varepsilon_{kpq} a_r b_s c_p d_q \\ (\varepsilon_{ijk} \varepsilon_{pqk}) \varepsilon_{jrs} a_r b_s c_p d_q &= \delta_{ip} \delta_{jq} \varepsilon_{jrs} a_r b_s c_p d_q - \delta_{iq} \delta_{jp} \varepsilon_{jrs} a_r b_s c_p d_q \\ &= \varepsilon_{qrs} a_r b_s c_i d_q \varepsilon_{prs} a_r b_s c_p d_i \end{aligned}$$

which is the same as the i -th component of $[\mathbf{d} \cdot (\mathbf{a} \times \mathbf{b})] \mathbf{c} - [\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})] \mathbf{d}$, because this is

$$[d_q (\varepsilon_{qrs} a_r b_s)] c_i - [c_p (\varepsilon_{prs} a_r b_s)] d_i.$$

$$(c) \quad (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}),$$

Solution:

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= (\mathbf{a} \times \mathbf{b})_i (\mathbf{c} \times \mathbf{d})_i \\ &= \varepsilon_{ijk} a_j b_k \varepsilon_{ilm} c_l d_m \\ &= \varepsilon_{ijk} \varepsilon_{ilm} a_j b_k c_l d_m \\ &= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) a_j b_k c_l d_m \\ &= a_j c_j b_k d_k - a_j d_j b_k c_k \\ &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{d} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}). \end{aligned}$$

$$(d) \quad \mathbf{a} \times [\mathbf{a} \times (\mathbf{a} \times \mathbf{b})] = a^2 (\mathbf{b} \times \mathbf{a}),$$

Solution: Note that the left-hand side of this expression is a vector, so we will write down the i th component of this vector, and check that this gives the i th component of the right-hand side of the expression. If this holds for all i , the expression is then true as a statement about vectors.

$$\begin{aligned} (\mathbf{a} \times [\mathbf{a} \times (\mathbf{a} \times \mathbf{b})])_i &= \varepsilon_{ijk} a_j \varepsilon_{klm} a_l \varepsilon_{mno} a_n b_o \\ &= \varepsilon_{ijk} (\delta_{kn} \delta_{lo} - \delta_{ko} \delta_{ln}) a_j a_l a_n b_o \\ &= \varepsilon_{ijk} a_j a_k a_l b_l - \varepsilon_{ijk} a_j b_k a_l a_l \\ &= (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \times \mathbf{a})_i - (\mathbf{a} \cdot \mathbf{a})(\mathbf{a} \times \mathbf{b})_i. \end{aligned}$$

Now note that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for all vectors \mathbf{a} , and recall that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, so we have

$$(\mathbf{a} \times [\mathbf{a} \times (\mathbf{a} \times \mathbf{b})]) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \times \mathbf{a}),$$

and letting $a = |\mathbf{a}|$, we have the required result.

$$(e) \quad (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{d}) + (\mathbf{c} \times \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{d}) = 0.$$

Solution: Using part (a), we have

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{d}) + (\mathbf{c} \times \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{d}) \\ &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{d} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) + (\mathbf{b} \cdot \mathbf{a})(\mathbf{c} \cdot \mathbf{d}) \\ &\quad - (\mathbf{b} \cdot \mathbf{d})(\mathbf{c} \cdot \mathbf{a}) + (\mathbf{c} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{d}) - (\mathbf{c} \cdot \mathbf{d})(\mathbf{a} \cdot \mathbf{b}) \\ &= 0. \end{aligned}$$

33 Exam question June 2002 (Section A): Evaluate $\varepsilon_{ijk}\varepsilon_{ijl}x_kx_l$.

Solution:

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{ijl}x_kx_l &= (\delta_{jj}\delta_{kl} - \delta_{jl}\delta_{kj})x_kx_l \\ &= (3\delta_{kl} - \delta_{kl})x_kx_l = 2\delta_{kl}x_kx_l = 2x_lx_l = 2|\mathbf{x}|^2.\end{aligned}$$

34 Exam question June 2001 (Section A): Evaluate $\varepsilon_{ijk}\partial_i\partial_j(x_lx_l)^{1/2}$ away from the origin.

Solution: The answer is zero, because the mixed second partial derivatives are symmetric in i and j while the ε_{ijk} is antisymmetric. For full marks you should say that the mixed partial derivatives are symmetric in i and j because they are continuous functions at all points away from the origin (which are the only points which need to be considered).

35 Exam question June 2003 (Section A): Calculate $\partial_i(\varepsilon_{ijk}\varepsilon_{jkl}x_l)$. (Hint: use the connection between $\partial_ix_j = \frac{\partial x_j}{\partial x_i}$ and the Kronecker delta.)

Solution: Use $\partial_ix_j = \delta_{ij}$ in $\partial_i(\varepsilon_{ijk}\varepsilon_{jkl}x_l) = \varepsilon_{ijk}\varepsilon_{jkl}\partial_ix_l = \varepsilon_{ijk}\varepsilon_{jkl}\delta_{il} = \varepsilon_{ijk}\varepsilon_{jki} = \varepsilon_{ijk}\varepsilon_{ijk} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji} = 3 \times 3 - \delta_{ii} = 9 - 3 = 6$.

36 The functions f, g are scalars, while \mathbf{A} and \mathbf{B} are vector functions with components A_i and B_i respectively. Verify the following identities using index notation:

(a) $\text{grad}(fg) = f \text{grad } g + g \text{grad } f,$

Solution: The i -th component of the LHS is $\partial_i(fg) = f\partial_i g + g\partial_i f$ by the product-rule. This is the i -th component of the RHS, which proves the identity.

(b) $\text{grad}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times \text{curl } \mathbf{B} + \mathbf{B} \times \text{curl } \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B},$

Solution: This time it's best to start on the RHS (general message: it's always a good idea to start with the most complicated side and then simplify). We have

$$\begin{aligned}(\mathbf{A} \times \text{curl } \mathbf{B} + \mathbf{B} \times \text{curl } \mathbf{A})_i &= \varepsilon_{ijk}A_j\varepsilon_{klm}\partial_l B_m + \varepsilon_{ijk}B_j\varepsilon_{klm}\partial_l A_m \\ &= \varepsilon_{ijk}\varepsilon_{klm}(A_j\partial_l B_m + B_j\partial_l A_m) \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(A_j\partial_l B_m + B_j\partial_l A_m) \\ &= A_j\partial_i B_j + B_j\partial_i A_j - A_j\partial_j B_i - B_j\partial_j A_i\end{aligned}$$

The last two terms in the final line are exactly cancelled by the i -th component of $(\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B}$, while the first two are reasonably-easily seen to be the i -th component of $\text{grad}(\mathbf{A} \cdot \mathbf{B})$, proving the identity.

(c) $\text{div}(f\mathbf{A}) = f \text{div } \mathbf{A} + (\text{grad } f) \cdot \mathbf{A},$

Solution: The LHS is a scalar equal to $\partial_i(fA_i) = f\partial_i A_i + (\partial_i f)A_i$ by the product rule for derivatives. This equals the RHS, which proves the identity.

(d) $\text{curl}(f\mathbf{A}) = f \text{curl } \mathbf{A} + (\text{grad } f) \times \mathbf{A},$

Solution: The LHS is a vector whose i -th component is

$$\begin{aligned}\varepsilon_{ijk}\partial_j(fA_k) &= \varepsilon_{ijk}(f\partial_j A_k + (\partial_j f)A_k) \\ &= f\varepsilon_{ijk}\partial_j A_k + \varepsilon_{ijk}(\partial_j f)A_k\end{aligned}$$

This is the i -th component of the RHS, which proves the identity.

(e) $\operatorname{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \operatorname{curl} \mathbf{A} - \mathbf{A} \cdot \operatorname{curl} \mathbf{B},$

Solution: The LHS is a scalar equal to $\partial_i(\varepsilon_{ijk}A_jB_k) = \varepsilon_{ijk}\partial_i(A_jB_k)$ since ε_{ijk} is just a number, independent of \mathbf{x} . Using the product-rule this becomes

$$\varepsilon_{ijk}(\partial_iA_j)B_k + \varepsilon_{ijk}A_j(\partial_iB_k) = B_k(\varepsilon_{kij}(\partial_iA_j)) - A_j(\varepsilon_{jik}\partial_iB_k)$$

This equals the RHS, which proves the identity.

(f) $\operatorname{curl}(\mathbf{A} \times \mathbf{B}) = (\operatorname{div} \mathbf{B})\mathbf{A} - (\operatorname{div} \mathbf{A})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B},$

Solution: The LHS is a vector whose i -th component is $\varepsilon_{ijk}\partial_j(\varepsilon_{krs}A_rB_s) = \varepsilon_{ijk}\varepsilon_{rsk}\partial_j(A_rB_s)$ since $\varepsilon_{krs} = \varepsilon_{rsk}$ is just a number, independent of \mathbf{x} . Using the product-rule, and the very useful formula, this becomes

$$\begin{aligned} & (\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr})((\partial_jA_r)B_s + A_r(\partial_jB_s)) \\ &= \delta_{ir}\delta_{js}(\partial_jA_r)B_s + \delta_{ir}\delta_{js}A_r(\partial_jB_s) - \delta_{is}\delta_{jr}(\partial_jA_r)B_s - \delta_{is}\delta_{jr}A_r(\partial_jB_s) \\ &= (\partial_sA_i)B_s + A_i(\partial_sB_s) - (\partial_rA_r)B_i - A_r(\partial_rB_i) = A_i(\partial_sB_s) - (\partial_rA_r)B_i + (B_s\partial_s)A_i - (A_r\partial_r)B_i \end{aligned}$$

This is the i -th component of the RHS, which proves the identity.

(g) $\operatorname{div} \operatorname{curl} \mathbf{A} = 0,$

Solution: The LHS is a scalar equal to $\partial_i(\varepsilon_{ijk}\partial_jA_k) = \varepsilon_{ijk}\partial_i\partial_jA_k$. On the assumption that $\partial_i\partial_jA_k$ is continuous then $\partial_i\partial_jA_k = \partial_j\partial_iA_k$ so $\varepsilon_{ijk}\partial_i\partial_jA_k = \varepsilon_{ijk}\partial_j\partial_iA_k = \varepsilon_{jik}\partial_i\partial_jA_k$ by re-naming dummy variables, but this is $-\varepsilon_{ijk}\partial_i\partial_jA_k$ by skew symmetry of ε_{ijk} , i.e. $\varepsilon_{ijk}\partial_i\partial_jA_k = -\varepsilon_{ijk}\partial_i\partial_jA_k$ which implies $\varepsilon_{ijk}\partial_i\partial_jA_k = 0$.

(h) $\operatorname{curl} \operatorname{curl} \mathbf{A} = \operatorname{grad} \operatorname{div} \mathbf{A} - \nabla^2 \mathbf{A}.$

Solution: The LHS is a vector whose i -th component is

$$\varepsilon_{ijk}\partial_j(\varepsilon_{krs}\partial_rA_s) = \varepsilon_{ijk}\varepsilon_{rsk}\partial_j(\partial_rA_s)$$

since $\varepsilon_{krs} = \varepsilon_{rsk}$ is just a number, independent of \mathbf{x} . Using the very useful formula, this becomes

$$(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr})\partial_j\partial_rA_s = \delta_{ir}\delta_{js}\partial_j\partial_rA_s - \delta_{is}\delta_{jr}\partial_j\partial_rA_s = \partial_s\partial_iA_s - \partial_r\partial_rA_i$$

On the assumption that $\partial_j\partial_rA_s$ is continuous we can write this as $\partial_i\partial_sA_s - \partial_r\partial_rA_i$. This is the i -th component of the RHS, which proves the identity.

37 What is the divergence of the vector function $\mathbf{A}(\mathbf{x}) = r\mathbf{x} + \nabla r$ where \mathbf{x} is the position vector in 3 dimensions and $r = |\mathbf{x}|$? What is the corresponding result in n dimensions?

Solution: Using components, $\nabla \cdot \mathbf{A} = \partial_i(r x_i + \partial_i(r)) = \partial_i(r)x_i + r\partial_i x_i + \partial_i(\partial_i(r))$. Using $\partial_i(r) = \partial_i(x_j x_j)^{1/2} = \frac{1}{2}(x_j x_j)^{-1/2} 2x_i = x_i/r$ (as in 25(c)) this is

$$\begin{aligned} x_i x_i / r + r \delta_{ii} + \partial_i(x_i / r) &= r + 3r + (\partial_i x_i) / r - x_i r^{-2} \partial_i(r) \\ &= 4r + \delta_{ii} / r - x_i r^{-2} x_i / r \\ &= 4r + 3/r - 1/r = 4r + 2/r. \end{aligned}$$

In n dimensions $\delta_{ii} = n$ which changes the result to $(n+1)r + (n-1)/r$.