51 *Compute the differential, or Jacobian matrix, and the Jacobian of the function* $\underline{V} : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $\underline{V}(x, y) = (x \cos y, x \sin y)$. *State where* $\underline{V}$ *defines an orientation preserving local diffeomorphism, and where it defines an orientation reversing local diffeomorphism.*

52 *Repeat question 51 for* $\underline{V}(x, y) = (e^x \cos y, e^x \sin y)$.

53 *Calculate the differential, or Jacobian matrix, and the Jacobian of the following transformations:*

(a) $\underline{U}(u, v) = (x(u, v), y(u, v))$ *where* $x(u, v) = \frac{1}{2}(u + v)$ *and* $y(u, v) = \frac{1}{2}(u - v)$;

(b) $\underline{V}(r, \theta) = (x(r, \theta), y(r, \theta))$ *where* $x(r, \theta) = r \cos\theta$ *and* $y(r, \theta) = r \sin\theta$;

(c) $\underline{W}(r, \theta, \phi) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$.

54 *Adapted from exam question 2009 (Section B) Q7:*

(a) *Let* $\underline{V} : \mathbb{R}^n \to \mathbb{R}^n$ *be a vector field. Give the definition of* $\underline{V}$ *being differentiable at a point* $\underline{a}$.

(b) *Let* $\underline{V}(x)$ *and* $\underline{W}(x)$ *be two differentiable vector fields in* $\mathbb{R}^2$. *Give formulae for the two differentials* $D\underline{V}_{\underline{x}}$ *and* $D\underline{W}_{\underline{x}}$.

(c) *Use the chain rule to show that the differential of the composite map* $\underline{U}(\underline{x}) := \underline{V}(\underline{W})$ *satisfies*

$$D\underline{U}_{\underline{x}} = D\underline{V}_{\underline{W}} D\underline{W}_{\underline{x}}\,.$$

55 *Adapted from exam question 2018 (Section B) Q8:*

(a) *Given a vector field* $\underline{u}(\underline{x}) = \underline{w}(\underline{v}(\underline{x}))$, *use the chain rule to show that* $D\underline{u}(\underline{x}) = D\underline{w}(\underline{v})D\underline{v}(\underline{x})$, *and hence* $J(\underline{u}) = J(\underline{w})J(\underline{v})$.

(b) *Let*

$$\underline{v}(\underline{x}) = (v_1, v_2) = (\cos y, \sin x)$$
$$\underline{w}(\underline{x}) = (w_1, w_2) = (x^2 + y^3, x^2 y),$$

and define $\underline{u}(\underline{x}) = \underline{w}(\underline{v}(\underline{x}))$. *Use the result from part* (a) *to calculate* $J(\underline{u})$. *Verify your answer by direct substitution.*

56 (a) *Let* $\underline{v} : \mathbb{R}^n \to \mathbb{R}^n$ *be a vector field. Give the definition of* $\underline{V}$ *being differentiable on an open set* $U \subseteq \mathbb{R}^n$.

(b) *For* $\underline{x} = x\underline{e}_1 + y\underline{e}_2$, *let* $\underline{v} : \mathbb{R}^2 \to \mathbb{R}^2$ *be given by*

$$\underline{v}(\underline{x}) = (x^2 + y^2, x + y).$$

Using the definition of differentiability, show that $\underline{v}$ *is differentiable on* $\mathbb{R}^2$.

(c) *Draw a diagram to show where* $\underline{v}$ *defines an orientation preserving local diffeomorphism (on* $U \subseteq \mathbb{R}^2$*), and where* $\underline{v}$ *defines an orientation reversing local diffeomorphism (on* $V \subseteq \mathbb{R}^2$*).*