83 *For a simple closed curve C in the $(x, y)-$plane, show by Green's theorem that the area enclosed is $A = \frac{1}{2}\oint_C (xdy - ydx)$. (Note, 'simple' means that C does not cross itself, which means that it encloses a well-defined area A.)*

Solution: Area is given by $A = \int_S dA$, where S is the surface enclosed by the simple curve C in the (x, y) plane. Green's theorem states:

$$\oint_C Pdx + Qdy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

In this case, the RHS of the formula given in the question is matched if we take $P = -\frac{1}{2}y$ and $Q = \frac{1}{2}x$; it is then easy to check that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ and so the result is indeed equal to the area. (Note, many other choices of P and Q would also give the area.)

84 *Evaluate $\oint \mathbf{F} \cdot \mathbf{dx}$ around the circle $x^2 + y^2 + 2x = 0$, where $\mathbf{F} = y\mathbf{e_1} - x\mathbf{e_2}$, both directly and by using Green's theorem in the plane.*

Solution:

$$\oint \mathbf{F} \cdot \mathbf{dx} = \oint (F_1\, dx + F_2\, dy) = \oint (y\, dx - x\, dy)$$

Using Green's theorem in the plane this is

$$\iint_A \left(-\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y}\right) dx\, dy = -2 \iint_A dx\, dy = -2A$$

where A is the area inside the circle $x^2 + y^2 + 2x = 0$, or $(x+1)^2 + y^2 = 1$. Since this has radius 1, the area is π and so $\oint \mathbf{F} \cdot \mathbf{dx} = -2\pi$.

Alternatively, the circle can be parametrised as $\mathbf{x}\,(t) = (\cos(t) - 1)\,\mathbf{e_1} + \sin(t)\,\mathbf{e_2}$. Then $\frac{d\mathbf{x}\,(t)}{dt} = -\sin(t)\,\mathbf{e_1} + \cos(t)\,\mathbf{e_2}$ so

$$\begin{aligned}\oint \mathbf{F} \cdot \mathbf{dx} &= \int_0^{2\pi} (\sin(t)\,\mathbf{e_1} - (\cos(t) - 1)\,\mathbf{e_2}) \cdot (-\sin(t)\,\mathbf{e_1} + \cos(t)\,\mathbf{e_2})\, dt \\ &= \int_0^{2\pi} \left(-\sin^2(t) - \cos^2(t) + \cos(t)\right) dt = -2\pi\,.\end{aligned}$$

85 *Evaluate $\oint_C 2xdy - 3ydx$ around the square with vertices at $(x, y) = (0, 2)$, $(2, 0)$, $(-2, 0)$ and $(0, -2)$.*

Solution:

First, using Green's theorem in the plane:

$$\oint_C 2xdy - 3ydx = \iint_A \left(\frac{\partial(2x)}{\partial x} + \frac{\partial(3y)}{\partial y}\right) dx\, dy = 5 \iint_A dx\, dy = 5A$$

where A is the area of the square. Now $A = 8$ so the integral is equal to 40.

We could equivalently embed the whole problem in three dimensions and use Stokes' theorem. The required integral can be written as

$$\oint_C \mathbf{F} \cdot d\mathbf{x}$$

where $\mathbf{F} = -3y\mathbf{e_1} + 2x\mathbf{e_2}$ and C is the given contour, thought of as lying in the $z = 0$ plane. By Stokes, and noting that the unit normal to the planar surface spanning C is $\mathbf{e_3}$, this is equal to $\iint_A (\underline{\nabla} \times \mathbf{F}) \cdot \mathbf{e_3}\, dA$, and since $(\underline{\nabla} \times \mathbf{F})_3 = 5$ this reduces to the previous calculation.

Finally, it is possible (and a good exercise) to do the contour integral directly. The contour can be split into four straight lines as shown in Figure 1.

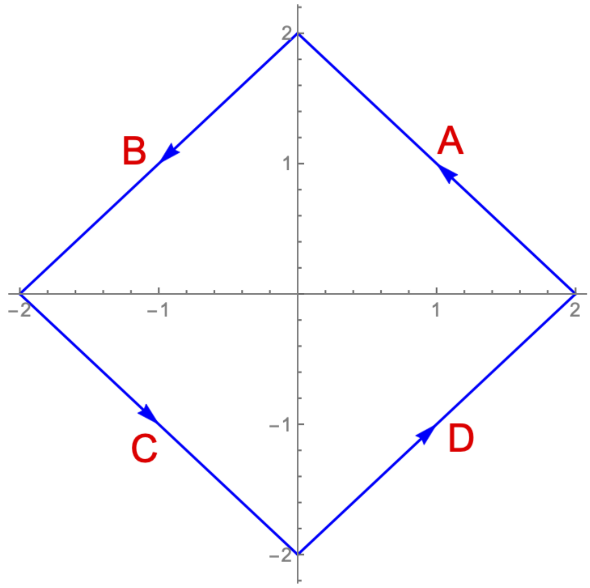

Figure 1: A square contour in the plane.

On **A**, write points as $(t, 2-t)$ with t running from 2 to 0; then $dx = dt$, $dy = -dt$, and

$$\int_{\mathbf{A}} -3y\,dx + 2x\,dy = \int_2^0 (-3(2-t) - 2t)\,dt = \left[-6t + \tfrac{1}{2}t^2\right]_2^0 = 12 - 2 = 10\,.$$

On **B**, write points as $(t, 2+t)$ with t running from 0 to -2; then $dx = dy = dt$ and

$$\int_C -3y\,dx + 2x\,dy = \int_0^{-2} (-3(2+t) + 2t)\,dt = \left[-6t - \tfrac{1}{2}t^2\right]_0^{-2} = 12 - 2 = 10\,.$$

On **C**, write points as $(t, -2-t)$ with t running from -2 to 0; then $dx = dt$, $dy = -dt$, and

$$\int_C -3y\,dx + 2x\,dy = \int_{-2}^0 (-3(-2-t) - 2t)\,dt = \left[6t + \tfrac{1}{2}t^2\right]_{-2}^0 = 12 - 2 = 10\,.$$

On **D**, write points as $(t, -2+t)$ with t running from 0 to 2; then $dx = dy = dt$ and

$$\int_C -3y\,dx + 2x\,dy = \int_0^2 (-3(-2+t) + 2t)\,dt = \left[6t - \tfrac{1}{2}t^2\right]_0^2 = 12 - 2 = 10\,.$$

Adding them all up gives 40, as before. Other parametrisations could have been used; for the particular ones taken here, you could also have written the parameter as x instead of t, but be extra careful with the signs! (And *check* that you agree with the calculation given here...)

86 *Let $\underline{v}$ be the radial vector field $\underline{v}(\underline{x}) = \underline{x}$.*

(a) *Compute $l_1 = \int_{C_1} \underline{v} \cdot d\underline{x}$, where C_1 is the straight-line contour from the origin to the point $(2,0,0)$.*

(b) *Compute $l_2 = \int_{C_2} \underline{v} \cdot d\underline{x}$, where C_2 is the semi-circular contour from the origin to the point $(2,0,0)$ defined by $0 \le x \le 2$, $y = +\sqrt{1-(x-1)^2}$, $z = 0$. [It may help to start by sketching C_2, and then to parameterize it as $\underline{x}(t) = (1-\cos t, \sin t, 0)$ with $0 \le t \le \pi$.]*

(c) *You should have found that $l_1 = l_2$. Explain this result using Stokes' theorem.*

Solution:

(a) Parameterizing C_1 by the distance x from the origin, at the point $\underline{x}(x) = x\,\underline{e}_1$ on C_1 we have $\underline{v} = x\,\underline{e}_x$ and $d\underline{x} = \underline{e}_1\,dx$, and so

$$l_1 = \int_0^2 x\,dx = 2\,.$$

(b) Parameterizing C_2 by the angle t, $\underline{v}(\underline{x}(t)) = (1-\cos t, \sin t, 0)$ while $d\underline{x}(t) = (\sin t, \cos t, 0)\,dt$, so $\underline{v} \cdot d\underline{x} = \sin t\,dt$ and

$$l_2 = \int_0^\pi \sin t\,dt = 2\,.$$

(c) As predicted in the question, $l_1 = l_2$. To explain this using Stokes' theorem, note first that if $\underline{v}(\underline{x}) = \underline{x}$ then $\underline{\nabla} \times \underline{v} = \underline{0}$ (see question 15(b)). Now let C_3 be the closed contour which runs from the origin to $(2,0,0)$ along C_1, then returns to the origin by taking C_2 in the reverse direction. Then $\int_{C_3} \underline{v} \cdot d\underline{x} = l_1 - l_2$. But by Stokes' theorem, $\int_{C_3} \underline{v} \cdot d\underline{x} = \int_{S_3} (\underline{\nabla} \times \underline{v}) \cdot d\underline{A} = 0$, where S_3 is a surface spanning C_3. Hence $l_1 - l_2 = 0$, or $l_1 = l_2$, as required.

87 *Integrate* $\operatorname{curl}\mathbf{F}$, *where* $\mathbf{F} = 3y\mathbf{e_1} - xz\mathbf{e_2} + yz^2\mathbf{e_3}$, *over the portion S of the surface $2z = x^2 + y^2$ below the plane $z = 2$, both directly and by using Stokes' theorem. Take the area elements of S to point outwards, so that their z components are negative.*

Solution: The geometrical setup is illustrated in Figure 2.

First proceed directly. Calculate $\underline{\nabla} \times \mathbf{F} = (z^2 + x)\mathbf{e_1} - (3+z)\mathbf{e_3}$. Now consider the surface of the paraboloid $2z = x^2 + y^2$, The surface is the zero level set of $f = x^2 + y^2 - 2z$, so a normal vector to it is $\underline{\nabla} f = 2x\mathbf{e_1} + 2y\mathbf{e_2} - 2\mathbf{e_3}$, and

$$d\mathbf{A} = -\frac{\underline{\nabla} f}{\mathbf{e_3} \cdot \underline{\nabla} f}\,dxdy = (x\mathbf{e_1} + y\mathbf{e_2} - \mathbf{e_3})\,dxdy.$$

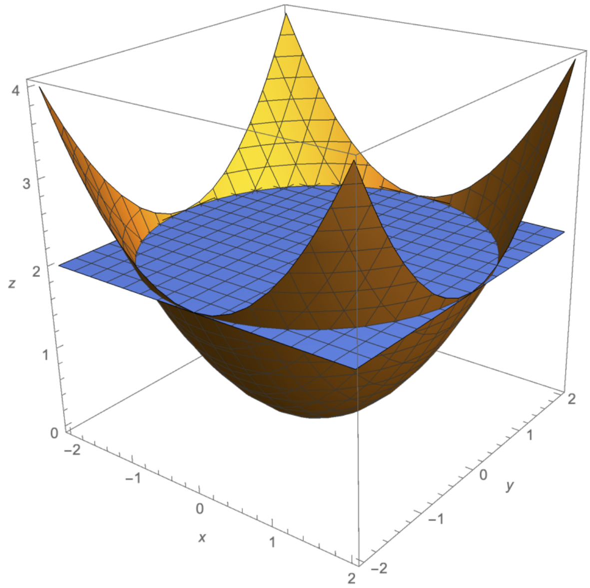

Figure 2: The intersection of the parabaloid $2z = x^2 + y^2$ and the plane $z = 2$

Note the minus sign – this comes from the choice to take the 'outward' normal to be pointing *out* of the paraboloid, which (as should be clear from the plot) means its z component is negative, leading to the minus sign in the formula for $d\mathbf{A}$.

Hence

$$\begin{aligned}\int_S \underline{\nabla} \times \mathbf{F} \cdot d\mathbf{A} &= \iint_D \big((z^2+x)\mathbf{e_1} - (3+z)\mathbf{e_3}\big) \cdot \big(x\mathbf{e_1} + y\mathbf{e_2} - \mathbf{e_3}\big)\, dxdy \\ &= \iint_D \big((z^2+x)x + 3 + z\big)\, dxdy \\ &= \iint_D \big((\tfrac{1}{4}(x^2+y^2)^2 + x)x + 3 + \tfrac{1}{2}(x^2+y^2)\big)\, dxdy\end{aligned}$$

where D is the projection of S onto the x,y plane (a circle of radius 2) and the fact that $z = \frac{1}{2}(x^2+y^2)$ was used in the last line to write the integral in terms of x and y alone.

To calculate this integral, change to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Then $dxdy = rdrd\theta$, and

$$\int_S (\underline{\nabla} \times \mathbf{F}) \cdot d\mathbf{A} = \int_0^{2\pi} d\theta \int_0^2 rdr\, \big(\tfrac{1}{4}r^5\cos\theta + r^2\cos^2\theta + 3 + \tfrac{1}{2}r^2\big)$$

giving $\int_S (\underline{\nabla} \times \mathbf{F}) \cdot d\mathbf{A} = \int_0^{2\pi} \left(\frac{2^5}{7}\cos\theta + 4\cos^2\theta + 8\right) d\theta = 20\pi$.

Alternatively, we can use Stokes' theorem to say that $\int_S (\underline{\nabla} \times \mathbf{F}) \cdot d\mathbf{A} = \oint_C \mathbf{F} \cdot d\mathbf{x}$ where C is the boundary of S, a circle of radius 2 described by

$$x^2 + y^2 = 4\,, \quad z = 2\,.$$

Note that since our normal is pointing *down* (i.e. in the negative-z direction), C should be traversed *clockwise* in the x,y plane. To get this right, write C in parametric form as $x = 2\cos t$, $y = -2\sin t$, $z = 2$ with t running from 0 to 2π. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{x} \;=\; \int_0^{2\pi}\big(-6\sin t\,\mathbf{e_1} - 4\cos t\,\mathbf{e_2} - 8\sin t\,\mathbf{e_3}\big)\cdot\big(-2\sin t\,\mathbf{e_1} - 2\cos t\,\mathbf{e_2}\big)\,dt$$
$$= \int_0^{2\pi}\big(12\sin^2 t + 8\cos^2 t\big)\,dt \;=\; 20\pi,$$

as before.

Note: It is also possible to integrate the curl of $\mathbf{F}$ over the flat disk $S' = \{x^2+y^2 \le 4, z = 2\}$ which also has C as its boundary. For this disk, picking the normal to again point downwards, $d\mathbf{A} = -\mathbf{e_3}\,dxdy$ everywhere, and so

$$\int_{S'} \underline{\nabla}\times\mathbf{F}\cdot d\mathbf{A} = \iint_{x^2+y^2\le 2}\big((4{+}x)\mathbf{e_1} - 5\mathbf{e_3}\big)\cdot\big(-\mathbf{e_3}\,dxdy\big)$$
$$= \iint_{x^2+y^2\le 2} 5\,dxdy \;=\; 20\pi\,.$$

Observe that this is the same answer as for the surface integral over S. This has to be the case since both are equal to the the line integral round C, but can you see a direct argument for it? (Hint: consider using the divergence theorem.)

88 *The paraboloid of equation $z = x^2 + y^2$ intersects the plane $z = y$ in a curve C. Calculate $\oint_C \mathbf{v}\cdot\mathbf{dx}$ for $\mathbf{v} = 2z\mathbf{e_1} + x\mathbf{e_2} + y\mathbf{e_3}$ using Stokes' theorem. Check your answer by evaluating the line integral directly.*

Solution: First, a picture of the situation is shown below in Figure 3.

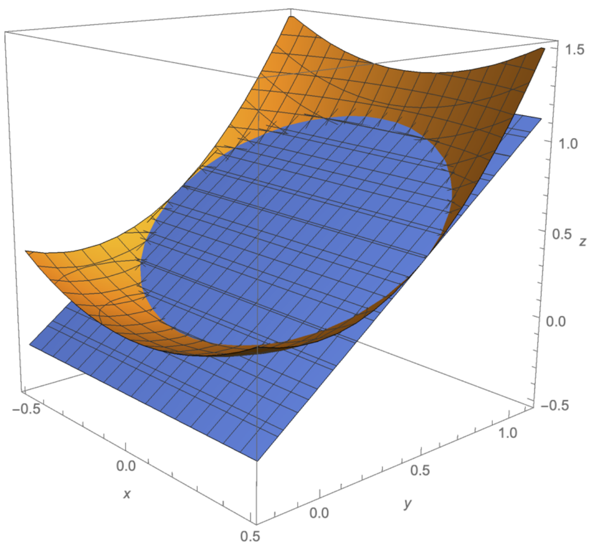

Figure 3: The intersection of the paraboloid $z = x^2 + y^2$ and the plane $z = y$.

On C, we have $z = x^2 + y^2$ and $z = y$. Hence $x^2 + y^2 = y$, or $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$. Stokes' theorem says that

$$\oint_C \mathbf{v} \cdot d\mathbf{x} = \int_S (\underline{\nabla} \times \mathbf{v}) \cdot d\mathbf{A}$$

where S is any surface bounded by C, and $\underline{\nabla} \times \mathbf{v} = \text{(calculate)} = \mathbf{e_1} + 2\mathbf{e_2} + \mathbf{e_3} = (1, 2, 1)$. There are then two ways to proceed.

1) (Harder) Take S to be the portion S of the paraboloid $z = x^2 + y^2$ that lies inside C. Using method 1 from lectures, we can parametrise the paraboloid by x and y, setting $\mathbf{x}\,(x, y) = x\,\mathbf{e_1} + y\,\mathbf{e_2} + (x^2 + y^2)\,\mathbf{e_3}$, so $\frac{\partial \mathbf{x}}{\partial x} = \mathbf{e_1} + 2x\,\mathbf{e_3}$ and $\frac{\partial \mathbf{x}}{\partial y} = \mathbf{e_2} + 2y\,\mathbf{e_3}$. Hence

$$d\mathbf{A} = (\mathbf{e_1} + 2x\,\mathbf{e_3}) \times (\mathbf{e_2} + 2y\,\mathbf{e_3})\,dxdy = (-2x\,\mathbf{e_1} - 2y\,\mathbf{e_2} + \mathbf{e_3})\,dxdy\,.$$

Equivalently, we can use method 2 and define the paraboloid implicitly as the level set $f(x, y, z) = 0$ where $f(x, y, z) = z - x^2 - y^2$, so $\underline{\nabla} f = -2x\,\mathbf{e_1} - 2y\,\mathbf{e_2} + \mathbf{e_3}$ and

$$d\mathbf{A} = \frac{\underline{\nabla} f}{\mathbf{e_3} \cdot \underline{\nabla} f}\,dxdy = (-2x\,\mathbf{e_1} - 2y\,\mathbf{e_2} + \mathbf{e_3})\,dxdy$$

which is the same as the formula found using method 1.

Either way, we have

$$(\underline{\nabla} \times \mathbf{v}) \cdot d\mathbf{A} = (\mathbf{e_1} + 2\mathbf{e_2} + \mathbf{e_3}) \cdot (-2x\,\mathbf{e_1} - 2y\,\mathbf{e_2} + \mathbf{e_3})\,dxdy = (-2x - 4y + 1)\,dxdy$$

and this must be integrated over D, the projection of S onto the x,y plane. Setting $Y = y - \frac{1}{2}$, $4y = 4Y + 2$ and the integral is

$$\iint_{x^2+Y^2 \le \frac{1}{4}} (-1 - 2x - 4Y)\,dxdY = -\frac{\pi}{4}\,.$$

(Note, the integrals of $-2x$ and $-4Y$ vanish by symmetry.)

2) (Easier) Take S to be the part of the plane $z = y$ encircled by C, a (tilted) flat disk. On this disk the unit normal is $\widehat{\mathbf{n}} = \frac{1}{\sqrt{2}}(-\mathbf{e_2} + \mathbf{e_3})$ and the area elements are $d\mathbf{A} = \widehat{\mathbf{n}}\,dx\,\sqrt{2}dy = (-\mathbf{e_2} + \mathbf{e_3})\,dxdy = (0, -1, 1)dxdy$. Then the area integral is

$$\iint_D (1, 2, 1) \cdot (0, -1, 1)\,dxdy = -\iint_D dxdy = -\frac{\pi}{4}$$

since the area of D, a disk of radius $\frac{1}{2}$, is $\frac{\pi}{4}$.

Note, the methods chosen here to evaluate the surface integrals are not compulsory. For example, the level set method could also be used for the second case.

As a check, we can do the line integral directly. Given the characterisation of C found above, it can be parametrised as $\mathbf{x}\,(t) = \frac{1}{2}\cos t\,\mathbf{e_1} + \frac{1}{2}(1 + \sin t)\,\mathbf{e_2} + \frac{1}{2}(1 + \sin t)\,\mathbf{e_3}$

with $0 \leq t \leq 2\pi$. Then $\frac{d\mathbf{x}}{dt} = -\frac{1}{2}\sin t\,\mathbf{e_1} + \frac{1}{2}\cos t\,\mathbf{e_2} + \frac{1}{2}\cos t\,\mathbf{e_3}$ while $\mathbf{v}(\mathbf{x}\,(t)) = (1+\sin t)\,\mathbf{e_1} + \frac{1}{2}\cos t\,\mathbf{e_2} + \frac{1}{2}(1+\sin t)\,\mathbf{e_3}$, and so

$$\oint_C \mathbf{v}\cdot d\mathbf{x}$$
$$= \tfrac{1}{4}\int_0^{2\pi}\big((2{+}2\sin t)\,\mathbf{e_1} + \cos t\,\mathbf{e_2} + (1{+}\sin t)\,\mathbf{e_3}\big)\cdot\big(-\sin t\,\mathbf{e_1} + \cos t\,\mathbf{e_2} + \cos t\,\mathbf{e_3}\big)dt$$
$$= \tfrac{1}{4}\int_0^{2\pi}\big(-2\sin t - 2\sin^2 t + \cos^2 t + \cos t + \cos t\sin t\big)dt \;=\; -\frac{\pi}{4}$$

as before.

89 *For an open surface S with boundary C, show that $2\int_S \mathbf{u}\cdot d\mathbf{A} = \oint_C (\mathbf{u}\times\mathbf{x}\,)\cdot d\mathbf{x}$, where $\mathbf{u}$ is a fixed vector.*

Solution: By Stokes' theorem, the RHS is equal to $\int_S \underline{\nabla}\times(\mathbf{u}\times\mathbf{x}\,)\cdot d\mathbf{A}$. In addition,

$$(\underline{\nabla}\times(\mathbf{u}\times\mathbf{x}\,))_i = \varepsilon_{ijk}\partial_j\varepsilon_{klm}u_l x_m = \varepsilon_{ijk}\varepsilon_{klm}u_l\delta_{jm} = (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})u_l\delta_{jm} = 2u_i$$

so $\underline{\nabla}\times(\mathbf{u}\times\mathbf{x}\,) = 2\mathbf{u}$, and, integrating, the LHS of the given formula is matched.

90 *Verify Stokes' theorem for the upper hemispherical surface S: $z = \sqrt{1-x^2-y^2}$, $z \geq 0$, with $\mathbf{F}$ equal to the radial vector field, i.e. $\mathbf{F}(x,y,z) = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$.*

Solution: The surface is bounded by the circle C: $x^2+y^2 = 1$, $z = 0$. Parametrising this by $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = 0$, $\frac{d\mathbf{x}}{dt} = -\sin t\,\mathbf{e_1} + \cos t\,\mathbf{e_2}$ and so
(a) $\oint_C \mathbf{F}\cdot d\mathbf{x} = \int_0^{2\pi}(\cos t\times(-\sin t) + \sin t\times\cos t)\,dt = 0$;
(b) $(\underline{\nabla}\times\mathbf{F})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}x_k = \varepsilon_{ijk}\delta_{jk} = 0$.
Hence $\oint_C \mathbf{F}\cdot d\mathbf{x} = 0 = \int_S(\underline{\nabla}\times\mathbf{F})\cdot d\mathbf{A}$, as required.

91 *Let $\mathbf{F} = ye^z\,\mathbf{e_1} + xe^z\,\mathbf{e_2} + xye^z\,\mathbf{e_3}$. Show that the integral of $\mathbf{F}$ around a regular closed curve C that is the boundary of a surface S is zero.*

Solution: This one is easy! Simply compute $\underline{\nabla}\times\mathbf{F} = \mathbf{0}$ (where some details of this calculation should be given) and then the result follows by Stokes' theorem.

92 *By applying Stokes' theorem to the vector field $\mathbf{G} = |\mathbf{x}\,|^2\mathbf{a}$ (with $\mathbf{a}$ a constant vector), or otherwise, show that $\int_S \mathbf{x}\times d\mathbf{A} = -\frac{1}{2}\oint_C |\mathbf{x}\,|^2\,d\mathbf{x}$, where S is the area bounded by the closed curve C.*

Solution: Note first that $(\underline{\nabla}\times\mathbf{G})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(x_l x_l a_k) = \varepsilon_{ijk}2x_l\delta_{jl}a_k = 2\varepsilon_{ijk}x_j a_k$. Using Stokes' theorem for $\mathbf{G}$, $\oint_C \mathbf{G}\cdot d\mathbf{x} = \int_S \underline{\nabla}\times\mathbf{G}\cdot d\mathbf{A}$, i.e.

$$\oint_C |\mathbf{x}\,|^2 a_k dx_k = \int_S 2\varepsilon_{ijk}x_j a_k dA_i \quad\text{or}\quad a_k\oint_C |\mathbf{x}\,|^2 dx_k = -a_k\int_S 2\varepsilon_{kji}x_j dA_i$$

(note the minus sign from the swap of indices on the second ε). Since this is true for all (constant) a_k, we deduce

$$\oint_C |\mathbf{x}\,|^2 dx_k = -\int_S 2\varepsilon_{kji}x_j dA_i$$

from which the result follows immediately.
Aside: Here we are dealing with the line integral of a scalar, which strictly speaking didn't get defined yet. For curve C parametrised by $\mathbf{x}\,(t)$, $t_0 \le t \le t_1$, and a scalar field $f(\mathbf{x}\,)$, it is simply the vector-valued integral

$$\int_C f(\mathbf{x}\,)d\mathbf{x} \;=\; \int_{t_0}^{t_1} f(\mathbf{x}\,(t))\tfrac{d\mathbf{x}\,(t)}{dt}\,dt\,.$$

(But beware, some authors use the same words for the scalar-valued quantity $\int_C f(\mathbf{x}\,)\,ds = \int_{t_0}^{t_1} f(\mathbf{x}\,(t))\,|\frac{d\mathbf{x}\,(t)}{dt}|\,dt$.)

93 *Exam question June 2002 (Section B): Evaluate the line integral $I = \int_C \mathbf{F}\cdot d\mathbf{x}$ where $\mathbf{F}(\mathbf{x}\,) = 2y\mathbf{e_1} + z\mathbf{e_2} + 3y\mathbf{e_3}$ and the path C is the intersection of the surface of equation $x^2+y^2+z^2 = 4z$ and the surface of equation $z = x+2$, taken in a clockwise direction to an observer at the origin. A picture of the path is required, as well as full justifications of the theoretical results you might use.*

Solution: The first surface is $x^2+y^2+(z-2)^2 = 4$, i.e. a sphere of radius 2 centred at $(0,0,2)$. The x and y coordinates of its intersection C with the plane $z = x+2$ therefore satisfy $2x^2+y^2 = 4$. Note, this is an ellipse with semi-major and semi-minor axes 2 and $\sqrt{2}$, as it is the projection of the circular curve C, lying in 3 dimensions, onto the x,y plane. A cutaway picture of the intersection is shown below in Figure 4

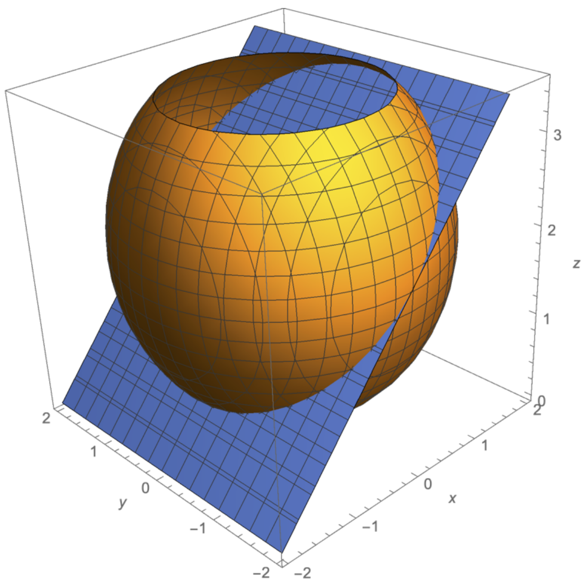

Figure 4: The in tersection of the sphere $x^2+y^2+(z-2)^2 = 4$ with the plane $z = x+2$.

To calculate the line integral directly, we can parametrise C by $x(t) = \sqrt{2}\cos t$, $y(t) = 2\sin t$, $z(t) = 2+\sqrt{2}\cos t$, so $\frac{d\mathbf{x}}{dt} = -\sqrt{2}\sin t\,\mathbf{e_1} + 2\cos t\,\mathbf{e_2} - \sqrt{2}\sin t\,\mathbf{e_3}$. (Note, clockwise round C when viewed from the origin is the same as anticlockwise when viewed from above, so this parametrisation goes round C in the correct direction.) Thus

$$\oint_C \mathbf{F}\cdot d\mathbf{x}$$
$$= \int_0^{2\pi} \mathbf{F}(\mathbf{x}(t))\cdot\frac{d\mathbf{x}(t)}{dt}\,dt$$
$$= \int_0^{2\pi} (4\sin t\,\mathbf{e_1} + (2+\sqrt{2}\cos t)\,\mathbf{e_2} + 6\sin t\,\mathbf{e_3})\cdot(-\sqrt{2}\sin t\,\mathbf{e_1} + 2\cos t\,\mathbf{e_2} - \sqrt{2}\sin t\,\mathbf{e_3})\,dt$$
$$= \int_0^{2\pi} (-4\sqrt{2}\sin^2 t + 4\cos t + 2\sqrt{2}\cos^2 t - 6\sqrt{2}\sin^2 t)\,dt$$
$$= -4\sqrt{2}\,\pi + 2\sqrt{2}\,\pi - 6\sqrt{2}\,\pi = -8\sqrt{2}\,\pi\,,$$

using $\int_0^{2\pi}\sin^2 t\,dt = \int_0^{2\pi}\cos^2 t\,dt = \pi$ and $\int_0^{2\pi}\cos t\,dt = 0$ to get to the last line.

Alternatively, we can use Stokes. The simplest surface spanning C is the disk in the plane $z = x+2$, which we'll call D. This is a level set of $f = z-x$, so $\underline{\nabla} f = (-1,0,1)$ while $\underline{\nabla}\times\mathbf{F} = (2,0,-2)$. Putting the pieces together, and letting E be the elliptical projection of D into the x,y plane,

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{x} &= \int_D \underline{\nabla}\times\mathbf{F}\cdot d\mathbf{A}\\ &= \iint_E \frac{(\underline{\nabla}\times\mathbf{F})\cdot\underline{\nabla} f}{\mathbf{e_3}\cdot\underline{\nabla} f}\,dxdy\\ &= \iint_E (-4)\,dxdy \;=\; -4\times(\text{area of } E) \;=\; -8\sqrt{2}\,\pi\,,\end{aligned}$$

since the area of an ellipse with semi-major and semi-minor axes a and b is equal to πab.

94 *Exam question 2012 (Section B) Q9(a)(ii): Use Stokes' theorem to calculate the line integral $\oint_C ydx+zdy+xdz$, where C is the intersection of the surfaces $x^2+y^2+z^2 = a^2$ and $x+y+z = 0$ and is orientated anticlockwise when viewed from above.*
Suggestion: Instead of one of the two standard methods for surface integrals, just use $d\mathbf{A} = \frac{\underline{\nabla} f}{|\underline{\nabla} f|}dA$ and think about how the surfaces intersect.

Solution: By Stokes, with $\mathbf{F} = (y,z,x)$ and $\underline{\nabla}\times\mathbf{F} = (-1,-1,-1)$,

$$\oint_C ydx+zdy+xdz = \int_S \underline{\nabla}\times\mathbf{F}\cdot d\mathbf{A}$$

Pick S to lie in the plane $f = x+y+z = 0$; then $\underline{\nabla} f/|\underline{\nabla} f| = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and hence

$$\int_S \underline{\nabla}\times\mathbf{F}\cdot d\mathbf{A} = \int_S (-\sqrt{3})\,dA = -\sqrt{3}\,\pi a^2$$

since the plane cuts through the centre of the sphere, so C is a great circle with radius a, enclosing a planar area πa^2.

95 *Consider the vector field $\mathbf{F} = y\,\mathbf{e_1} + (z-x)\,\mathbf{e_2} + (x^3+y)\,\mathbf{e_3}$. Evaluate $\int_S (\underline{\nabla}\times\mathbf{F})\cdot\mathbf{dA}$ explicitly, where*

(i) S is the disk $x^2 + y^2 \leq 4$, $z = 1$,

(ii) S is the surface of a paraboloid $x^2 + y^2 = 5 - z$ above the plane $z = 1$.

Verify that each of these agrees with Stokes' theorem by considering the integral of $\mathbf{F}$ *around each bounding contour.*

Solution: Calculating, $\underline{\nabla} \times \mathbf{F} = -3x^2\, \mathbf{e_2} - 2\, \mathbf{e_3}$. Then

(i) On the given disk, $d\mathbf{A} = \mathbf{e_3}\, dxdy$ so

$$\int_S \underline{\nabla} \times \mathbf{F} \cdot d\mathbf{A} = \iint_{x^2+y^2\leq 4} (-2)\, dxdy = -8\pi$$

as the disk has radius 2 and hence area 4π.

Now consider the line integral round the boundary C of the disk, parametrising it as $\mathbf{x}\,(t) = 2\cos t\, \mathbf{e_1} + 2\sin t\, \mathbf{e_2} + \mathbf{e_3}$, $0 \leq t \leq 2\pi$. Then $\frac{d\mathbf{x}}{dt} = -2\sin t\, \mathbf{e_1} + 2\cos t\, \mathbf{e_2}$, and

$$\oint_C \mathbf{F} \cdot d\mathbf{x} = \int_0^{2\pi} \big(-4\sin^2 t + 2(1 - 2\cos t)\cos t\big)\, dt = -8\pi\,.$$

(ii) On the paraboloid, $f \equiv x^2 + y^2 + z = 5$, so using method 2 and describing the surface S as a level set of f with a projection A onto the x,y plane, and computing $\underline{\nabla} f = 2x\, \mathbf{e_1} + 2y\, \mathbf{e_2} + \mathbf{e_3}$,

$$\begin{aligned}
\int_S \underline{\nabla} \times \mathbf{F} \cdot d\mathbf{A} &= \iint_A \frac{(\underline{\nabla} \times \mathbf{F}) \cdot \underline{\nabla} f}{\mathbf{e_3} \cdot \underline{\nabla} f}\, dxdy \\
&= \iint_A \big(-3x^2\, \mathbf{e_2} - 2\, \mathbf{e_3}\big) \cdot \big(2x\, \mathbf{e_1} + 2y\, \mathbf{e_2} + \mathbf{e_3}\big)\, dxdy \\
&= \iint_A \big(-6x^2 y - 2\big)\, dxdy\,.
\end{aligned}$$

Now A is a disk of radius 2 centred on the origin (see the plot below: on C, the boundary of S, we have $x^2 + y^2 = 5 - z$ and $z = 1$, implying $x^2 + y^2 = 4$), giving us

$$\iint_{x^2+y^2\leq 4} \big(-6x^2 y - 2\big)\, dxdy = \iint_{x^2+y^2\leq 4} \big(-2\big)\, dxdy = -8\pi$$

(the integral of $-6x^2 y$ vanishing by symmetry). Alternatively we can switch to polar coordinates and compute

$$\begin{aligned}
\int_0^{2\pi} d\theta \int_0^2 r dr\, \big(-6r^3 \cos^2\theta \sin\theta - 2\big) &= \int_0^2 r dr\, \big[2r^3 \cos^3\theta - 2\theta\big]_0^{2\pi} \\
&= \int_0^2 r dr\, (-4\pi) = -4\pi\, \big[\tfrac{1}{2} r^2\big]_0^2 = -8\pi\,.
\end{aligned}$$

Note that this surface has the same boundary as the disk in part (i), so the line integral calculation is the same as before.

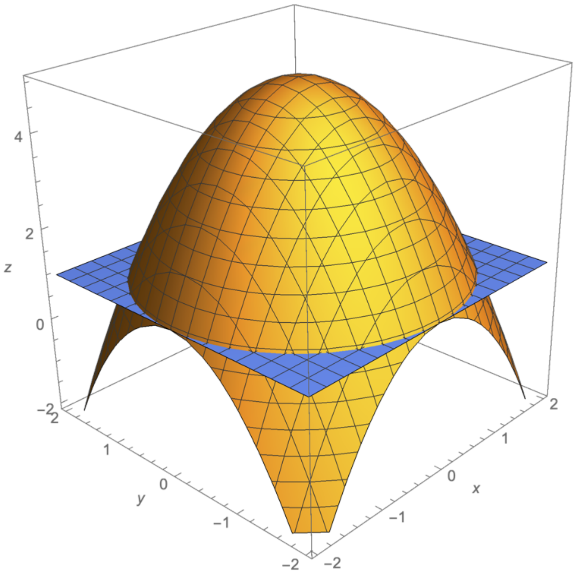

Figure 5: The intersection of the paraboloid $x^2 + y^2 = 5 - z$ and the plane $z = 1$.

96 *Evaluate the line integral*

$$\int_{P_1}^{P_2} yz\,dx + xz\,dy + (xy + z^2)\,dz$$

from P_1 with co-ordinates (1,0,0) to P_2 at (1,0,1) explicitly,

(a) along a straight line in the z–direction, and

(b) along a helical path parametrised by $\mathbf{x}(t) = \mathbf{e_1}\cos t + \mathbf{e_2}\sin t + \mathbf{e_3}\,t/2\pi$, where t varies along the path.

Compare these two results – does this suggest that there might be a general formula for the integral from $P_1 = (1,0,0)$ to any point P with coordinates (x,y,z)? Check your answer when $P = P_2$.

Solution: Call the straight path C_1. On this path $y = 0$ and $x = 1$, so $dx = dy = 0$ and

$$\int_{C_1} (yz\,dx + xz\,dy + (xy + z^2)\,dz) = \int_0^1 z^2\,dz = \tfrac{1}{3}\,.$$

Call the helical path C_2. On C_2 we have $x = \cos t$, $y = \sin t$ so $dx = -\sin t\,dt$ and $dy = \cos t\,dt$, while $z = \frac{t}{2\pi}$ so $dz = \frac{1}{2\pi}\,dt$. Hence

$$\begin{aligned}\int_{C_2} (yz\,dx + xz\,dy + (xy + z^2)\,dz) &= \tfrac{1}{2\pi}\int_0^{2\pi} \left(-t\sin^2 t + t\cos^2 t + \cos t\sin t + \tfrac{t^2}{4\pi^2}\right) dt \\ &= \tfrac{1}{2\pi}\int_0^{2\pi} \left(t\cos 2t + \tfrac{1}{2}\sin 2t + \tfrac{t^2}{4\pi^2}\right) dt \\ &= \tfrac{1}{2\pi}\int_0^{2\pi} \tfrac{t^2}{4\pi^2}\,dt = \tfrac{1}{8\pi^3}\left[\tfrac{1}{3}t^3\right]_0^{2\pi} = \tfrac{1}{3}\,.\end{aligned}$$

(If you are averse to integrating by parts to get from the second line to the third, note that $\int_0^{2\pi} t\cos 2t\,dt = \int_0^{2\pi}(2\pi - t)\cos 2t\,dt$ by symmetry, so both must be equal to half their sum, which is clearly zero.)

These two results are the same! This suggests (but does not prove) that we might be computing the line integral of a conservative field. To check, compute $\underline{\nabla} \times \mathbf{F} =$ (compute) $= \mathbf{0}$. This means we must have $\mathbf{F} = \underline{\nabla}\phi$ for some scalar field ϕ. To find ϕ, set $\mathbf{F} = \underline{\nabla}\phi$ and work component by component:

$$F_1 = \frac{\partial \phi}{\partial x} \;\Rightarrow\; yz = \frac{\partial \phi}{\partial x} \;\Rightarrow\; \phi(x,y,z) = xyz + f(y,z)\,.$$

Then

$$F_2 = \frac{\partial \phi}{\partial y} \;\Rightarrow\; xz = xz + \frac{\partial f}{\partial y} \;\Rightarrow\; \frac{\partial f}{\partial y} = 0 \;\Rightarrow\; f(y,z) = g(z)\,.$$

Finally

$$F_3 = \frac{\partial \phi}{\partial z} \;\Rightarrow\; xy + z^2 = xy + \frac{\partial g}{\partial z} \;\Rightarrow\; \frac{\partial g}{\partial z} = z^2 \;\Rightarrow\; g(z) = \tfrac{1}{3}z^3 + c\,.$$

Thus $\phi(x,y,z) = xyz + \frac{1}{3}z^3 + c$, and the line integral of $\mathbf{F}$ from $(1,0,0)$ to any point (x,y,z) is equal to $\phi(x,y,z) - \phi(1,0,0) = xyz + \frac{1}{3}z^3$. If $(x,y,z) = P_2 = (1,0,1)$ this is equal to $\frac{1}{3}$, agreeing with the previous results.

97 *State a necessary and sufficient condition for a vector field* $\mathbf{F}$ *to be expressible in the form* $\mathbf{F} = \underline{\nabla}\,\phi$ *in some simply-connected region. The scalar field* ϕ *is called a scalar potential, though sometimes the opposite sign is used.*

Determine whether the following vector fields are expressible in this form

(i) $(\mathbf{a}\cdot\mathbf{x}\,)\,\mathbf{x}\,$, *(ii)* $(\mathbf{a}\cdot\mathbf{x}\,)\,\mathbf{a}$, *(iii)* $(\mathbf{a}\cdot\mathbf{a})\,\mathbf{x}\,$, *(iv)* $\mathbf{a}\times\mathbf{x}\,$, *(v)* $\mathbf{a}\times(\mathbf{a}\times\mathbf{x}\,)$,

and find the vector fields, $\mathbf{F}$, *for which the corresponding potentials are*

(a) $\frac{1}{2}(\mathbf{a}\cdot\mathbf{x}\,)^2$, *(b)* $\frac{1}{2}a^2|\mathbf{x}\,|^2 - \frac{1}{2}(\mathbf{a}\cdot\mathbf{x}\,)^2$.

Here $\mathbf{a}$ *is a constant non-zero vector, and* $a^2 = |\mathbf{a}|^2$.

Solution: The necessary and sufficient condition is $\underline{\nabla} \times \mathbf{F} = \mathbf{0}$.

(i) $\mathbf{F} = (\mathbf{a}\cdot\mathbf{x}\,)\mathbf{x}$:

$$(\underline{\nabla} \times \mathbf{F})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(a_l x_l x_k) = \varepsilon_{ijk}a_l(\delta_{jl}x_k + x_l\delta_{jk}) = \varepsilon_{ijk}a_j x_k = (\mathbf{a}\times\mathbf{x}\,)_i \neq 0$$

so $\mathbf{F} \neq \underline{\nabla}\phi$.

(ii) $\mathbf{F} = (\mathbf{a}\cdot\mathbf{x}\,)\mathbf{a}$:

$$(\underline{\nabla} \times \mathbf{F})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(a_l x_l a_k) = \varepsilon_{ijk}a_l\delta_{jl}a_k = \varepsilon_{ijk}a_j a_k = 0$$

so $\mathbf{F} = \underline{\nabla}\phi$ for some ϕ.

(iii) $\mathbf{F} = (\mathbf{a}\cdot\mathbf{a})\mathbf{x}$:

$$(\underline{\nabla} \times \mathbf{F})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(a_l a_l x_k) = \varepsilon_{ijk} a_l a_l \delta_{jk} = \varepsilon_{ijj} a_l a_l = 0$$

so $\mathbf{F} = \underline{\nabla}\phi$ for some ϕ.

(iv) $\mathbf{F} = \mathbf{a}\times\mathbf{x}$:

$$(\underline{\nabla} \times \mathbf{F})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\varepsilon_{klm} a_l x_m) = \varepsilon_{ijk}\varepsilon_{klm} a_l \delta_{jm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_l \delta_{jm} = 2a_i \neq 0$$

so $\mathbf{F} \neq \underline{\nabla}\phi$.

(v) $\mathbf{F} = \mathbf{a}\times(\mathbf{a}\times\mathbf{x})$:

$$\begin{aligned}(\mathbf{a}\times(\mathbf{a}\times\mathbf{x}))_i &= \varepsilon_{ijk} a_j \varepsilon_{klm} a_l x_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_j a_l x_m = a_j a_i x_j - a_j a_j x_i \\ &= ((\mathbf{a}\cdot\mathbf{x})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{x})_i\end{aligned}$$

so $\mathbf{a}\times(\mathbf{a}\times\mathbf{x}) = (\mathbf{a}\cdot\mathbf{x})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{x}$, and $\underline{\nabla}\times\mathbf{F} = \mathbf{0}$ by (ii) and (iii); hence $\mathbf{F} = \underline{\nabla}\phi$ for some ϕ.

For the second part:

(a) $\phi = \frac{1}{2}(\mathbf{a}\cdot\mathbf{x})^2 = \frac{1}{2}(a_j x_j a_k x_k)$ so

$$F_i = \tfrac{1}{2} a_j a_k \frac{\partial}{\partial x_i}(x_j x_k) = \tfrac{1}{2} a_j a_k(\delta_{ij} x_k + x_j \delta_{ik}) = \tfrac{1}{2}(a_i a_k x_k + a_j a_i x_j) = ((\mathbf{a}\cdot\mathbf{x})\mathbf{a})_i\,.$$

Hence $\mathbf{F} = (\mathbf{a}\cdot\mathbf{x})\mathbf{a}$.

(b) $\phi = \frac{1}{2}a^2|\mathbf{x}|^2 - \frac{1}{2}(\mathbf{a}\cdot\mathbf{x})^2$: we have $(\underline{\nabla}(\frac{1}{2}a^2|\mathbf{x}|^2))_i = \frac{1}{2}a^2\frac{\partial}{\partial x_i}(x_j x_j) = a^2 x_i$ and combining this with the result from part (a), $\mathbf{F} = a^2\mathbf{x} - (\mathbf{a}\cdot\mathbf{x})\mathbf{a}$.

98 *Exam question June 2002 (Section B):*

(a) State the conditions for the line integral $\int_c \mathbf{F}\cdot d\mathbf{x}$ from $\mathbf{x}_0$ to $\mathbf{x}_1$ to be independent of the path connecting these two points.

(b) Determine the value of the line integral $\int_c \mathbf{F}\cdot d\mathbf{x}$ where

$$\mathbf{F} = (e^{-y} - ze^{-x})\,\mathbf{e_1} + (e^{-z} - xe^{-y})\,\mathbf{e_2} + (e^{-x} - ye^{-z})\,\mathbf{e_3},$$

and C is the path

$$x = \frac{1}{\ln 2}\ln(1+p), \quad y = \sin\frac{\pi p}{2}, \quad z = \frac{1-e^p}{1-e}\,,$$

with the parameter p in the range $0 \le p \le 1$, from $(0,0,0)$ to $(1,1,1)$.

Solution:

(a) The necessary and sufficient condition for the integral not to depend of the path is that $\mathbf{F} = \underline{\nabla}\phi$ for some ϕ, or, equivalently, that $\underline{\nabla}\times\mathbf{F} = \mathbf{0}$ (assuming that the region under consideration is simply connected).

(b) We have $\underline{\nabla} \times \mathbf{F} =$ (calculate) $= \mathbf{0}$. Hence we can swap the given C for any other path joining $(0,0,0)$ to $(1,1,1)$ without changing the answer. Simplest is to take the straight line $\mathbf{x}\,(t) = (t,t,t)$ with $0 \le t \le 1$, so $\frac{d\mathbf{x}}{dt} = (1,1,1)$. Thus

$$I = 3\int_0^1 \left(e^{-t} - te^{-t}\right)dt = 3\int_0^1 \frac{d}{dt}\left(te^{-t}\right)dt = 3\left[te^{-t}\right]_0^1 = \frac{3}{e}\,.$$

Alternatively, we can figure out $\phi(x,y,z)$. We need

$$\frac{\partial\phi}{\partial x} = F_1 = e^{-y} - ze^{-x} \Rightarrow \phi(x,y,z) = xe^{-y} + ze^{-x} + f(y,z)$$

and then

$$\frac{\partial\phi}{\partial y} = -xe^{-y} + \frac{\partial f}{\partial y} = F_2 = e^{-z} - xe^{-y} \Rightarrow f(y,z) = ye^{-z} + g(z)$$

and finally

$$\frac{\partial\phi}{\partial z} = e^{-x} - ye^{-z} + \frac{\partial g}{\partial z} = F_3 = e^{-x} - ye^{-x} \Rightarrow g(z) = \text{const} = A\,,\ \text{say.}$$

Hence $\phi(x,y,z) = xe^{-y} + ze^{-x} + ye^{-z} + A$ and

$$\int_C \mathbf{F}\cdot d\mathbf{x} = \phi(1,1,1) - \phi(0,0,0) = (e^{-1}+e^{-1}+e^{-1}+A) - (0+A) = \frac{3}{e}$$

as before.

99 *If $\mathbf{F} = x\,\mathbf{e_1} + y\,\mathbf{e_2}$, calculate $\int_S \mathbf{F}\cdot\mathbf{dA}$, where S is the part of the surface $z = 9 - x^2 - y^2$ that is above the x,y plane, by applying the divergence theorem to the volume bounded by the surface and the piece it cuts out of the x, y plane.*
Hint: what is $\mathbf{F}\cdot\mathbf{dA}$ on the x,y plane?

Solution: The normal vector to the x,y plane is $\mathbf{e_3}$, so $\mathbf{F}\cdot d\mathbf{A} = 0$ there. Hence the surface integral side of the divergence theorem, when applied to the volume V mentioned in the question, comes only from the integral over the part S of the paraboloid above the x,y plane, which is exactly what the question asks us to calculate. Hence the desired surface integral is equal to the volume integral of $\underline{\nabla}\cdot\mathbf{F} = 2$. Thus

$$\begin{aligned}\int_S \mathbf{F}\cdot d\mathbf{A} = \int_{\partial V} \mathbf{F}\cdot d\mathbf{A} &= \int_V \underline{\nabla}\cdot\mathbf{F}\,dV \quad \text{by the divergence theorem}\\ &= 2\int_0^9 \left(\iint_{x^2+y^2\le 9-z} dxdy\right)dz\\ &= 2\int_0^9 \pi(9-z)\,dz\\ &= 2\pi\left[9z - \tfrac{1}{2}z^2\right]_0^9 = 81\pi\,.\end{aligned}$$

100 *Evaluate each of the integrals below as* **either** *a volume integral* **or** *a surface integral, whichever is easier:*

(a) $\int_S \mathbf{x}\cdot\mathbf{dA}$ *over the whole surface of a cylinder bounded by* $x^2+y^2=R^2$, $z=0$ *and* $z=L$. *Note that* $\mathbf{x}$ *means* $x\,\mathbf{e_1}+y\,\mathbf{e_2}+z\,\mathbf{e_3}$.

(b) $\int_S \mathbf{F}\cdot\mathbf{dA}$, *where* $\mathbf{F}=x\cos^2 y\,\mathbf{e_1}+xz\,\mathbf{e_2}+z\sin^2 y\,\mathbf{e_3}$, *over the surface of a sphere with centre at the origin and radius* π.

(c) $\int_V \underline{\nabla}\cdot\mathbf{F}\,dV$, *where* $\mathbf{F}=\sqrt{x^2+y^2}(x\,\mathbf{e_1}+y\,\mathbf{e_2})$, *over the three-dimensional volume* $x^2+y^2\le R^2$, $0\le z\le L$.

Solution:

(a) $\int_S \mathbf{x}\cdot d\mathbf{A}=\int_V \underline{\nabla}\cdot\mathbf{x}\;dV=\int_V 3\,dV=3\times(\text{volume of }V)=3\pi R^2L$.

(b) Note, $\underline{\nabla}\cdot\mathbf{F}=\cos^2 y+\sin^2 y=1$, so $\int_S \mathbf{F}\cdot\mathbf{dA}=\int_V dV=\frac{4}{3}\pi^4$.

(c) This time, convert to a surface integral. Since $\mathbf{F}$ has no z component, the ends of the cylinder don't contribute. Furthermore, on the curved surface S of the cylinder, the unit normal is $\widehat{\mathbf{n}}=(x\mathbf{e_1}+y\mathbf{e_2})/\sqrt{x^2+y^2}$ (and is parallel to $\mathbf{F}$). Hence on this surface $\mathbf{F}\cdot d\mathbf{A}=x^2+y^2=R^2$ and $\int_V \underline{\nabla}\cdot\mathbf{F}\,dV=\int_{\partial V}\mathbf{F}\cdot d\mathbf{A}=\int_S \mathbf{F}\cdot d\mathbf{A}=R^2\int_S dA=R^2\times(\text{area of }S)=2\pi R^3L$.

101 *Suppose the vector field* $\mathbf{F}$ *is everywhere tangent to the closed surface* S*, which encloses the volume* V*. Prove that*

$$\int_V \underline{\nabla}\cdot\mathbf{F}\,dV=0\,.$$

Solution: By the divergence theorem, $\int_V \underline{\nabla}\cdot\mathbf{F}\,dV=\int_S \mathbf{F}\cdot d\mathbf{A}=\int_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dA=0$, since to be tangent to S, $\mathbf{F}$ must be orthogonal to $\widehat{\mathbf{n}}$, the unit normal to S.

102 *Exam question June 2003 (Section A): Evaluate* $\int_S \mathbf{B}(\mathbf{x})\cdot\mathbf{dA}$, *where*

$$\mathbf{B}(\mathbf{x})=(8x+\alpha\,y-z)\,\mathbf{e_1}+(x+2y+\beta\,z)\,\mathbf{e_2}+(\gamma\,x+y-z)\,\mathbf{e_3}$$

and S *is the surface of the sphere having centre at* (α,β,γ) *and radius* γ*, where* $\alpha,\beta\in\mathbb{R}$*, and* γ *is an arbitrary positive real number.*

Solution: Calculating, $\underline{\nabla}\cdot\mathbf{B}=8+2-1=9$, so by the divergence theorem

$$\int_S \mathbf{B}\cdot d\mathbf{A}=\int_V \underline{\nabla}\cdot\mathbf{B}\,dV=9\int_V dV=9\times(\text{volume of }V)=12\pi\gamma^3.$$

103 *Let* A *be the interior of the circle of unit radius centred on the origin. Evaluate* $\iint_A \exp(x^2+y^2)\,dx\,dy$ *by making a change of variables to polar co-ordinates.*

Solution:

$$\begin{aligned}\iint_A \exp(x^2+y^2)\,dx\,dy&=\int_0^{2\pi}\Big(\int_0^1 e^{r^2}r\,dr\Big)\,d\theta=\int_0^{2\pi}\left[\tfrac{1}{2}e^{r^2}\right]_0^1 d\theta\\&=\int_0^{2\pi}\big(\tfrac{1}{2}(e-1)\big)\,d\theta=(e-1)\pi\,.\end{aligned}$$

(Or, do the θ integral first.)

104 *Let A be the region $0 \le y \le x$ and $0 \le x \le 1$. Evaluate $\int_A (x+y)\, dx\, dy$ by making the change of variables $x = u+v$, $y = u-v$. Check your answer by evaluating the integral directly.*

Solution: First find the Jacobian:

$$J(\underline{U}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2\,.$$

Then

$$I = \int_A (x+y)\, dxdy = \int_U 2u\, |J|\, dudv = 4 \int_U u\, dudv$$

where U is the area of integration in the new coordinates, which is bounded by a triangle in the u,v plane with edges $y = x \Rightarrow v = 0$, $x = 1 \Rightarrow u+v = 1$, $y = 0 \Rightarrow u = v$. This leads to

$$I = 4\int_0^{1/2} \left(\int_v^{1-v} u\, du \right) dv = 4\int_0^{1/2} \left[\tfrac{1}{2}u^2\right]_v^{1-v} dv = 4\int_0^{1/2} \left(\tfrac{1}{2} - v\right) dv = \tfrac{1}{2}\,.$$

Alternatively the integral can be done directly:

$$I = \int_0^1 \int_y^1 (x+y)\, dxdy = \int_0^1 \left[\tfrac{1}{2}x^2 + xy\right]_y^1 dy = \int_0^1 (\tfrac{1}{2} + y - \tfrac{1}{2}y^2 - y^2)\, dy = \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{6} - \tfrac{1}{3} = \tfrac{1}{2}$$

which agrees!

105 *The expressions below are called Green's first and second identities. Derive them from the divergence theorem with $\mathbf{F} = f\underline{\nabla} g$ or $g\underline{\nabla} f$ as appropriate, where f and g are differentiable functions:*

$$\int_V (f\underline{\nabla}^2 g + \underline{\nabla} f \cdot \underline{\nabla} g)\, dV = \int_S (f\underline{\nabla} g) \cdot \mathbf{dA}\ ,$$

$$\int_V (f\underline{\nabla}^2 g - g\underline{\nabla}^2 f)\, dV = \int_S (f\underline{\nabla} g - g\underline{\nabla} f) \cdot \mathbf{dA}\ .$$

where the volume V is that enclosed by the closed surface S.

Solution: First apply the divergence theorem to $f\underline{\nabla} g$ as follows:

$$\begin{aligned} \int_S (f\underline{\nabla} g) \cdot d\mathbf{A} &= \int_V \underline{\nabla} \cdot (f\underline{\nabla} g)\, dV \\ &= \int_V (\underline{\nabla} f \cdot \underline{\nabla} g + f\underline{\nabla} \cdot \underline{\nabla} g)\, dV \\ &= \int_V \left(\underline{\nabla} f \cdot \underline{\nabla} g + f\underline{\nabla}^2 g\right) dV \quad \text{as required.} \end{aligned}$$

Swapping f and g, we also have

$$\int_S (g\underline{\nabla} f) \cdot d\mathbf{A} = \int_V \left(\underline{\nabla} g \cdot \underline{\nabla} f + g\underline{\nabla}^2 f\right) dV,$$

and subtracting this from the first result gives us the second identity.

106 *A gas holder has the form of a vertical cylinder of radius R and height H with hemispherical top also of radius R. The density, ρ, (i.e. the mass per unit volume) of the gas inside varies with height z above the base according to the relation $\rho = C\exp(-z)$, where C is a constant. Calculate the total mass of gas in the holder, taking care to define any coordinate system used and the range of the corresponding variables.*

Use this result to find the integral of the field

$$\mathbf{F} \;=\; B\,z\,e^{-y}\,\mathbf{e_1} \;+\; C\,y\,e^{-z}\,\mathbf{e_2}$$

over the curved surface of the gas holder.
Hint: it may help to note first what the integral over the base of the holder is.

Solution: For the first part we need the mass in the gas holder, which given that ρ is the density is equal to $M_{\text{total}} = \int_V \rho\, dV$. Split this integral into two parts, the first over the cylinder and the second over the hemispherical top. A convenient choice of coordinates is to put the origin at the centre of the circular base, which will therefore be in the plane $z = 0$. Then

$$\begin{aligned} M_{\text{cylinder}} &= \int_0^H \int_{x^2+y^2\le R^2} Ce^{-z}\, dxdydz \\ &= \int_0^H \pi R^2\, Ce^{-z}\, dz \\ &= \pi C R^2 \left[-e^{-z}\right]_0^H = \pi C R^2 \left(1 - e^{-H}\right)\,. \end{aligned}$$

For the top, we have $x^2 + y^2 + (z - H)^2 = R^2$, since it is centred at $(0, 0, H)$. Setting $w = z - H$ the required integral is

$$\begin{aligned} M_{\text{top}} &= \int_0^R \int_{x^2+y^2\le R^2-w^2} Ce^{-(w+H)}\, dxdydw \\ &= \int_0^R \pi(R^2 - w^2)\, Ce^{-w-H}\, dw \\ &= \pi C e^{-H} \int_0^R (R^2 - w^2)\, e^{-w}\, dw\,. \end{aligned}$$

Now $\int_0^R R^2 e^{-w}\, dw = R^2\left[-e^{-w}\right]_0^R = R^2(1 - e^{-R})$, while

$$\begin{aligned} \int_0^R w^2 e^{-w}\, dw &= \left[-w^2 e^{-w}\right]_0^R + \int_0^R 2we^{-w} dw \\ &= -R^2 e^{-R} + \left[-2we^{-w}\right]_0^R + \int_0^R 2e^{-w} dw \\ &= -R^2 e^{-R} - 2Re^{-R} + 2 - 2e^{-R}. \end{aligned}$$

Adding up the bits,

$$M_{\text{top}} = \pi C e^{-H}\big(R^2 - 2 + 2Re^{-R} + 2e^{-R}\big)$$

and so the total mass is

$$M_{\text{total}} = M_{\text{cylinder}} + M_{\text{top}} = \pi C R^2 + 2\pi C e^{-H}\big(Re^{-R} + e^{-R} - 1\big)\,.$$

(Not the most edifying calculation in the world!)

For the second part, we need the integral over the curved surface, which we will call S. This is the boundary ∂V of V less the flat bit on the bottom. However, as suggested by the hint, we can notice that $\mathbf{F}\cdot\widehat{\mathbf{n}} = 0$ on this flat bit (since $\widehat{\mathbf{n}} = -\mathbf{e_3}$ there). Hence

$$\int_S \mathbf{F}\cdot d\mathbf{A} = \int_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dA = \int_{\partial V}\mathbf{F}\cdot\widehat{\mathbf{n}}\,dA = \int_{\partial V}\mathbf{F}\cdot d\mathbf{A} = \int_V \underline{\nabla}\cdot\mathbf{F}\,dV$$

using the divergence theorem, and putting in rather more steps than strictly needed. Now $\underline{\nabla}\cdot\mathbf{F} = Ce^{-z}$, which is exactly the function ρ we took the trouble to integrate over V in the first part. Hence

$$\int_S \mathbf{F}\cdot d\mathbf{A} = \pi C R^2 + 2\pi C e^{-H}\big(Re^{-R} + e^{-R} - 1\big)\,.$$

107 *Exam question June 2001 (Section B): In electrostatic theory, Gauss' law states that the flux of an electrostatic (vector) field $\mathbf{E}(\mathbf{x})$ over some closed surface S is equal to the enclosed charge q divided by a constant ϵ_0 :*

$$\int_S \mathbf{E}(\mathbf{x})\cdot\mathbf{dA} \;=\; \frac{q}{\epsilon_0}\,.$$

If the electrostatic field is given by $\mathbf{E}(x,y) = \alpha x\mathbf{e_1} + \beta y\mathbf{e_2}$, use Gauss' law to find the total charge in the compact region bounded by the surface S consisting of S_1, the curved portion of the half-cylinder $z = (R^2 - y^2)^{1/2}$ of length H; S_2 and S_3 the two semi-circular plane end pieces; and S_4, the rectangular portion of the (x,y)-plane. (In equations, the relevant bounded region may be described by $z^2+y^2 \le R^2, z\ge 0, -\frac{H}{2}\le x \le \frac{H}{2}$). Express your result in terms of α, β, R and H.

Solution: Since the surface is closed we can use the divergence theorem to switch from a surface to a volume integral. First calculate $\underline{\nabla}\cdot\mathbf{E} = \alpha+\beta$; then, letting V denote the region enclosed by S,

$$\begin{aligned}\int_S \mathbf{E}\cdot d\mathbf{A} &= \int_V (\alpha+\beta)\,dV\\ &= (\alpha+\beta)\times\text{(volume of the half cylinder)}\\ &= \tfrac{1}{2}(\alpha+\beta)\pi R^2 H.\end{aligned}$$

Hence the total charge enclosed is $q = \frac{1}{2}\epsilon_0(\alpha+\beta)\pi R^2 H.$

108 *Exam question June 2002 (Section A): In electrostatic theory, Gauss' law states that the flux of an electrostatic field $\mathbf{E}(\mathbf{x})$ over a closed surface S is given by the ratio of the enclosed charge q and a constant ϵ_0. Calculate the electric charge enclosed in the ellipsoid of equation $x^2 + \frac{1}{2}y^2 + z^2 = 1$ in the presence of*

(a) *an electrostatic field* $\mathbf{E}(\mathbf{x}) = yz\mathbf{e_1} + xz\mathbf{e_2} + xy\mathbf{e_3}$,

(b) *an electrostatic field* $\mathbf{E}(\mathbf{x}) = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$.

Solution: In both parts, the best tactic to replace $\int_S \mathbf{E} \cdot d\mathbf{A}$ by $\int_V \underline{\nabla} \cdot \mathbf{E}\, dV$, quoting the divergence theorem.

(a) In this case, $\underline{\nabla} \cdot \mathbf{E} = 0$, so the enclosed electric charge is zero.

(b) This time, $\underline{\nabla} \cdot \mathbf{E} = 3$, so the enclosed electric charge is equal to 3 times the volume of the ellipsoid, multiplied by ϵ_0. The given ellipsoid is a unit sphere (which has volume $\frac{4}{3}\pi$), expanded by a factor of $\sqrt{2}$ in the y direction. Hence its volume is $\sqrt{2} \times \frac{4}{3}\pi$, and the charge enclosed is $4\sqrt{2}\,\pi\,\epsilon_0$. (Alternatively, one can grind through the volume integral explicitly.)