

19 Compute the divergence, $\nabla \cdot \mathbf{A}$, of the following vector fields:

(a) $\mathbf{A}(x, y, z) = yz\mathbf{e}_1 + xz\mathbf{e}_2 + xy\mathbf{e}_3,$

Solution:

$$\nabla \cdot \mathbf{A} = \frac{\partial(yz)}{\partial x} + \frac{\partial(xz)}{\partial y} + \frac{\partial(xy)}{\partial z} = 0 + 0 + 0 = 0.$$

(b) $\mathbf{A}(x, y, z) = (x^2 + y^2 + z^2)(3\mathbf{e}_1 + 4\mathbf{e}_2 + 5\mathbf{e}_3),$

Solution:

$$\nabla \cdot \mathbf{A} = \frac{\partial(3(x^2 + y^2 + z^2))}{\partial x} + \frac{\partial(4(x^2 + y^2 + z^2))}{\partial y} + \frac{\partial(5(x^2 + y^2 + z^2))}{\partial z} = 6x + 8y + 10z.$$

Alternatively write $\mathbf{A} = |\mathbf{x}|^2 \mathbf{B}$ with $\mathbf{B} = 3\mathbf{e}_1 + 4\mathbf{e}_2 + 5\mathbf{e}_3$ a constant vector and use $\nabla \cdot (|\mathbf{x}|^2 \mathbf{B}) = (\nabla(|\mathbf{x}|^2)) \cdot \mathbf{B} + |\mathbf{x}|^2 \nabla \cdot \mathbf{B}$ with

$$\begin{aligned} \nabla(|\mathbf{x}|^2) &= \mathbf{e}_1 \frac{\partial(x^2 + y^2 + z^2)}{\partial x} + \mathbf{e}_2 \frac{\partial(x^2 + y^2 + z^2)}{\partial y} + \mathbf{e}_3 \frac{\partial(x^2 + y^2 + z^2)}{\partial z} \\ &= \mathbf{e}_1 2x + \mathbf{e}_2 2y + \mathbf{e}_3 2z = 2\mathbf{x} \end{aligned}$$

and $\nabla \cdot \mathbf{B} = 0$.

(c) $\mathbf{A}(x, y, z) = (x + y)\mathbf{e}_1 + (y + z)\mathbf{e}_2 + (z + x)\mathbf{e}_3.$

Solution:

$$\nabla \cdot \mathbf{A} = \frac{\partial(x + y)}{\partial x} + \frac{\partial(y + z)}{\partial y} + \frac{\partial(z + x)}{\partial z} = 1 + 1 + 1 = 3.$$

20 Compute the curl, $\nabla \times \mathbf{A}$, of each of the vector fields, \mathbf{A} , in the previous question.

Solution:

(a) $\mathbf{A}(x, y, z) = yz\mathbf{e}_1 + xz\mathbf{e}_2 + xy\mathbf{e}_3,$

$$\begin{aligned} \nabla \times \mathbf{A} &= \mathbf{e}_1 \left(\frac{\partial(xy)}{\partial y} - \frac{\partial(xz)}{\partial z} \right) + \mathbf{e}_2 \left(\frac{\partial(yz)}{\partial z} - \frac{\partial(xy)}{\partial x} \right) + \mathbf{e}_3 \left(\frac{\partial(xz)}{\partial x} - \frac{\partial(yz)}{\partial y} \right) \\ &= \mathbf{e}_1(x - x) + \mathbf{e}_2(y - y) + \mathbf{e}_3(z - z) = \mathbf{0}. \end{aligned}$$

(b) $\mathbf{A}(x, y, z) = (x^2 + y^2 + z^2)(3\mathbf{e}_1 + 4\mathbf{e}_2 + 5\mathbf{e}_3),$

$$\begin{aligned} \nabla \times \mathbf{A} &= \mathbf{e}_1 \left(\frac{\partial(5(x^2 + y^2 + z^2))}{\partial y} - \frac{\partial(4(x^2 + y^2 + z^2))}{\partial z} \right) \\ &\quad + \mathbf{e}_2 \left(\frac{\partial(3(x^2 + y^2 + z^2))}{\partial z} - \frac{\partial(5(x^2 + y^2 + z^2))}{\partial x} \right) \\ &\quad + \mathbf{e}_3 \left(\frac{\partial(4(x^2 + y^2 + z^2))}{\partial x} - \frac{\partial(3(x^2 + y^2 + z^2))}{\partial y} \right) \\ &= \mathbf{e}_1(10y - 8z) + \mathbf{e}_2(6z - 10x) + \mathbf{e}_3(8x - 6y). \end{aligned}$$

Alternatively write $\mathbf{A} = |\mathbf{x}|^2 \mathbf{B}$ with $\mathbf{B} = 3\mathbf{e}_1 + 4\mathbf{e}_2 + 5\mathbf{e}_3$ a constant vector and use $\nabla \times (|\mathbf{x}|^2 \mathbf{B}) = (\nabla(|\mathbf{x}|^2)) \times \mathbf{B} + |\mathbf{x}|^2 \nabla \times \mathbf{B}$ with $\nabla(|\mathbf{x}|^2) = 2\mathbf{x}$ and $\nabla \times \mathbf{B} = 0$.

$$(c) \quad \mathbf{A}(x, y, z) = (x + y)\mathbf{e}_1 + (y + z)\mathbf{e}_2 + (z + x)\mathbf{e}_3.$$

$$\begin{aligned} \nabla \times \mathbf{A} &= \\ \mathbf{e}_1 \left(\frac{\partial(z+x)}{\partial y} - \frac{\partial(y+z)}{\partial z} \right) &+ \mathbf{e}_2 \left(\frac{\partial(x+y)}{\partial z} - \frac{\partial(z+x)}{\partial x} \right) + \mathbf{e}_3 \left(\frac{\partial(y+z)}{\partial x} - \frac{\partial(x+y)}{\partial y} \right) \\ &= \mathbf{e}_1(0 - 1) + \mathbf{e}_2(0 - 1) + \mathbf{e}_3(0 - 1) = -(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3). \end{aligned}$$

21 If $f(r)$ is a differentiable function of $r = |\mathbf{x}|$, show that

$$(a) \quad \text{grad } f(r) = f'(r) \mathbf{x} / r,$$

Solution:

$$\begin{aligned} \text{grad } f(r) &= \mathbf{e}_1 \frac{\partial f(r)}{\partial x} + \mathbf{e}_2 \frac{\partial f(r)}{\partial y} + \mathbf{e}_3 \frac{\partial f(r)}{\partial z} \\ &= \mathbf{e}_1 \frac{df}{dr} \frac{\partial r}{\partial x} + \mathbf{e}_2 \frac{df}{dr} \frac{\partial r}{\partial y} + \mathbf{e}_3 \frac{df}{dr} \frac{\partial r}{\partial z}. \end{aligned}$$

Now $r = \sqrt{x^2 + y^2 + z^2}$ so

$$\frac{\partial r}{\partial x} = \frac{1}{2} \frac{2x}{r} = \frac{x}{r}$$

and similarly

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

so

$$\text{grad } f(r) = \frac{df}{dr} \left(\mathbf{e}_1 \frac{x}{r} + \mathbf{e}_2 \frac{y}{r} + \mathbf{e}_3 \frac{z}{r} \right) = f'(r) \mathbf{x} / r.$$

$$(b) \quad \text{curl } [f(r)\mathbf{x}] = 0.$$

Solution: Using $\text{curl}(f\mathbf{V}) = (\nabla f) \times \mathbf{V} + f \text{curl } \mathbf{V}$, the result to part (a) and

$$\text{curl } \mathbf{x} = \mathbf{e}_1 \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) + \mathbf{e}_2 \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \right) + \mathbf{e}_3 \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) = \mathbf{0},$$

gives $\text{curl}(f\mathbf{x}) = (\nabla f) \times \mathbf{x} + f \text{curl } \mathbf{x} = f'(r) \frac{\mathbf{x}}{r} \times \mathbf{x} + \mathbf{0} = \mathbf{0}$ since $\mathbf{x} \times \mathbf{x} = \mathbf{0}$.

22 Let \mathbf{x} be the position vector in three dimensions, with $r = |\mathbf{x}|$, and let \mathbf{a} be a constant vector. Show that

$$(a) \quad \text{div } \mathbf{x} = 3,$$

Solution:

$$\nabla \cdot \mathbf{x} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3.$$

$$(b) \quad \text{curl } \mathbf{x} = 0,$$

Solution:

$$\text{curl } \mathbf{x} = \mathbf{e}_1 \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) + \mathbf{e}_2 \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \right) + \mathbf{e}_3 \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) = \mathbf{0}.$$

(c) $\text{grad } r = \mathbf{x}/r,$

Solution:

$$\frac{\partial r}{\partial x} = \frac{1}{2} \frac{2x}{r} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

so

$$\text{grad } r = \mathbf{e}_1 x/r + \mathbf{e}_2 y/r + \mathbf{e}_3 z/r = \mathbf{x}/r.$$

(d) $\text{div}(r^n \mathbf{x}) = (n+3)r^n,$

Solution: Use $\text{div}(f\mathbf{V}) = (\nabla f) \cdot \mathbf{V} + f \text{div } \mathbf{V}$ with $f(\mathbf{x}) = r^n$ so that using 21(a) $\nabla f = nr^{n-1} \mathbf{x}/r = nr^{n-2} \mathbf{x}$ and $\mathbf{V} = \mathbf{x}$ so

$$\text{div}(r^n \mathbf{x}) = nr^{n-2} \mathbf{x} \cdot \mathbf{x} + r^n \text{div } \mathbf{x} = nr^n + 3r^n = (n+3)r^n.$$

(e) $\text{grad}(\mathbf{a} \cdot \mathbf{x}) = \mathbf{a},$

Solution: $\mathbf{a} \cdot \mathbf{x} = a_1 x + a_2 y + a_3 z$ so

$$\frac{\partial(\mathbf{a} \cdot \mathbf{x})}{\partial x} = a_1, \quad \frac{\partial(\mathbf{a} \cdot \mathbf{x})}{\partial y} = a_2, \quad \frac{\partial(\mathbf{a} \cdot \mathbf{x})}{\partial z} = a_3$$

hence $\text{grad}(\mathbf{a} \cdot \mathbf{x}) = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 = \mathbf{a}.$

(f) $\text{div}(\mathbf{a} \times \mathbf{x}) = 0,$

Solution: $\mathbf{a} \times \mathbf{x} = \mathbf{e}_1(a_2 z - a_3 y) + \mathbf{e}_2(a_3 x - a_1 z) + \mathbf{e}_3(a_1 y - a_2 x)$ so

$$\text{div}(\mathbf{a} \times \mathbf{x}) = \frac{\partial(a_2 z - a_3 y)}{\partial x} + \frac{\partial(a_3 x - a_1 z)}{\partial y} + \frac{\partial(a_1 y - a_2 x)}{\partial z} = 0.$$

(g) $\text{curl}(\mathbf{a} \times \mathbf{x}) = 2\mathbf{a},$

Solution:

$$\begin{aligned} \text{curl}(\mathbf{a} \times \mathbf{x}) &= \mathbf{e}_1 \left(\frac{\partial(a_1 y - a_2 x)}{\partial y} - \frac{\partial(a_3 x - a_1 z)}{\partial z} \right) \\ &+ \mathbf{e}_2 \left(\frac{\partial(a_2 z - a_3 y)}{\partial z} - \frac{\partial(a_1 y - a_2 x)}{\partial x} \right) + \mathbf{e}_3 \left(\frac{\partial(a_3 x - a_1 z)}{\partial x} - \frac{\partial(a_2 z - a_3 y)}{\partial y} \right) \\ &= \mathbf{e}_1(a_1 + a_1) + \mathbf{e}_2(a_2 + a_2) + \mathbf{e}_3(a_3 + a_3) = 2\mathbf{a}. \end{aligned}$$

(h) $\text{curl}(r^2 \mathbf{a}) = 2(\mathbf{x} \times \mathbf{a}),$

Solution: Use $\text{curl}(f\mathbf{V}) = (\text{grad } f) \times \mathbf{V} + f \text{curl } \mathbf{V}$ with $f = r^2 = |\mathbf{x}|^2$ and $\nabla(|\mathbf{x}|^2) = 2\mathbf{x}$ so that $\text{curl}(r^2 \mathbf{a}) = 2\mathbf{x} \times \mathbf{a} + r^2 \text{curl } \mathbf{a} = 2\mathbf{x} \times \mathbf{a}.$

(i) $\nabla^2(1/r) = 0, \quad \text{if } r \neq 0,$

Solution: $\nabla^2(1/r) \equiv \nabla \cdot \nabla(1/r).$ From 21(a) with $f = 1/r$ we have $\nabla(1/r) = -\mathbf{x}/r^3$. Then from 22(d) $\nabla \cdot (\mathbf{x}/r^3) = (-3 + 3)/r^3 = 0$ so $\nabla^2(1/r) = 0$, but this is only valid for $r \neq 0$ since the calculation involves division by r .

(j) $\nabla^2(\log r) = 1/r^2, \quad \text{if } r \neq 0,$

Solution: $\nabla^2(\log r) \equiv \nabla \cdot \nabla(\log r)$. From 21 (a) with $f = \log r$, $\nabla(\log r) = \mathbf{x}/r^2$. From 22 (d) $\nabla \cdot (\mathbf{x}/r^2) = (-2+3)/r^3 = 1/r^2$ so $\nabla^2(\log r) = 1/r^2$, but as in part (i) this is only valid for $r \neq 0$ since the calculation involves division by r .

$$(k) \quad \operatorname{div}[(\mathbf{a} \cdot \mathbf{x})\mathbf{x}] = 4\mathbf{a} \cdot \mathbf{x},$$

Solution: Using $\operatorname{div} f\mathbf{V} = (\nabla f) \cdot \mathbf{V} + f \operatorname{div} \mathbf{V}$, with $f = \mathbf{a} \cdot \mathbf{x}$ so $\nabla f = \mathbf{a}$ from part (e) and $\mathbf{V} = \mathbf{x}$ so $\operatorname{div} \mathbf{x} = 3$ from part (a) hence $\operatorname{div}[(\mathbf{a} \cdot \mathbf{x})\mathbf{x}] = \mathbf{a} \cdot \mathbf{x} + \mathbf{a} \cdot \mathbf{x} 3 = 4\mathbf{a} \cdot \mathbf{x}$.

$$(l) \quad \operatorname{div}[\mathbf{x} \times (\mathbf{x} \times \mathbf{a})] = 2\mathbf{a} \cdot \mathbf{x},$$

Solution: Use $\mathbf{x} \times (\mathbf{x} \times \mathbf{a}) = \mathbf{x}(\mathbf{x} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{x} \cdot \mathbf{x})$ and then from part (k) use $\operatorname{div}(\mathbf{x}(\mathbf{x} \cdot \mathbf{a})) = 4\mathbf{x} \cdot \mathbf{a}$ and

$$\nabla \cdot (|\mathbf{x}|^2 \mathbf{a}) = (\nabla(|\mathbf{x}|^2)) \cdot \mathbf{a} + |\mathbf{x}|^2 \nabla \cdot \mathbf{a} = 2\mathbf{x} \cdot \mathbf{a}$$

so

$$\operatorname{div}[\mathbf{x} \times (\mathbf{x} \times \mathbf{a})] = \operatorname{div}(\mathbf{x}(\mathbf{x} \cdot \mathbf{a})) - \nabla \cdot (|\mathbf{x}|^2 \mathbf{a}) = 4\mathbf{x} \cdot \mathbf{a} - 2\mathbf{x} \cdot \mathbf{a}.$$

$$(m) \quad \operatorname{curl}(\mathbf{a} \times \mathbf{x}/r^3) = 3(\mathbf{a} \cdot \mathbf{x})\mathbf{x}/r^5 - \mathbf{a}/r^3,$$

Solution:

$$\operatorname{curl}(\mathbf{a} \times \mathbf{x}/r^3) = (\operatorname{curl}(\mathbf{a} \times \mathbf{x}))/r^3 + (\nabla r^{-3}) \times (\mathbf{a} \times \mathbf{x})$$

Using part (g) $\operatorname{curl}(\mathbf{a} \times \mathbf{x}) = 2\mathbf{a}$ together with $\mathbf{x} \times (\mathbf{a} \times \mathbf{x}) = \mathbf{a}(\mathbf{x} \cdot \mathbf{x}) - \mathbf{x}(\mathbf{a} \cdot \mathbf{x})$ and $\nabla r^{-3} = -3\mathbf{x}/r^5$ (from 21 (a)) gives

$$\operatorname{curl}(\mathbf{a} \times \mathbf{x}/r^3) = \frac{2\mathbf{a}}{r^3} - \frac{3}{r^5}(\mathbf{a}r^2 - \mathbf{x}(\mathbf{a} \cdot \mathbf{x})) = 3(\mathbf{a} \cdot \mathbf{x})\mathbf{x}/r^5 - \mathbf{a}/r^3.$$

$$(n) \quad \text{Exam question June 2002 (Section A): calculate the curl of } (\mathbf{a} \cdot \mathbf{x})\mathbf{x}.$$

Solution:

$$\operatorname{curl}((\mathbf{a} \cdot \mathbf{x})\mathbf{x}) = (\nabla(\mathbf{a} \cdot \mathbf{x})) \times \mathbf{x} + \mathbf{a} \cdot \mathbf{x} \operatorname{curl} \mathbf{x} = \mathbf{a} \times \mathbf{x}$$

using part parts (e) and (b).

23 If \mathbf{x} is the position vector, $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$, \mathbf{a} is a constant vector, $\mathbf{F} = (\mathbf{a} \cdot \mathbf{x})\mathbf{x}$ and $\mathbf{G} = r^2\mathbf{a}$, (with $r = |\mathbf{x}|$), show that

$$(a) \quad \operatorname{div} \mathbf{F} = 2 \operatorname{div} \mathbf{G} = 4\mathbf{a} \cdot \mathbf{x},$$

Solution: Using the product rule

$$\operatorname{div} \mathbf{F} = \nabla \cdot ((\mathbf{a} \cdot \mathbf{x})\mathbf{x}) = (\nabla(\mathbf{a} \cdot \mathbf{x})) \cdot \mathbf{x} + (\mathbf{a} \cdot \mathbf{x}) \nabla \cdot \mathbf{x}$$

As in 22 (e), $\mathbf{a} \cdot \mathbf{x} = a_1x + a_2y + a_3z$ so

$$\frac{\partial(\mathbf{a} \cdot \mathbf{x})}{\partial x} = a_1, \quad \frac{\partial(\mathbf{a} \cdot \mathbf{x})}{\partial y} = a_2, \quad \frac{\partial(\mathbf{a} \cdot \mathbf{x})}{\partial z} = a_3$$

hence $\text{grad}(\mathbf{a} \cdot \mathbf{x}) = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 = \mathbf{a}$ and as in 22 (a)

$$\nabla \cdot \mathbf{x} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3,$$

so

$$\text{div } \mathbf{F} = \nabla \cdot ((\mathbf{a} \cdot \mathbf{x}) \mathbf{x}) = \mathbf{a} \cdot \mathbf{x} + 3\mathbf{a} \cdot \mathbf{x} = 4\mathbf{a} \cdot \mathbf{x}.$$

Using the product rule

$$\text{div } \mathbf{G} = (\nabla(r^2)) \cdot \mathbf{a} + r^2 \nabla \cdot \mathbf{a}$$

and $\nabla(|\mathbf{x}|^2) = 2\mathbf{x}$, $\nabla \cdot \mathbf{a} = 0$ so that $\text{div } \mathbf{G} = 2\mathbf{a} \cdot \mathbf{x}$.

(b) $\text{curl } \mathbf{G} = -2 \text{curl } \mathbf{F} = 2\mathbf{x} \times \mathbf{a},$

Solution: Use $\text{curl}(f\mathbf{V}) = (\text{grad } f) \times \mathbf{V} + f \text{curl } \mathbf{V}$ with $f = r^2 = |\mathbf{x}|^2$ and $\nabla(|\mathbf{x}|^2) = 2\mathbf{x}$ so that

$$\text{curl } \mathbf{G} = \text{curl}(r^2 \mathbf{a}) = 2\mathbf{x} \times \mathbf{a} + r^2 \text{curl } \mathbf{a} = 2\mathbf{x} \times \mathbf{a}.$$

Now use this product rule with $f = \mathbf{a} \cdot \mathbf{x}$ and $\mathbf{V} = \mathbf{x}$ so that

$$\nabla \times ((\mathbf{a} \cdot \mathbf{x}) \mathbf{x}) = (\nabla(\mathbf{a} \cdot \mathbf{x})) \times \mathbf{x} + (\mathbf{a} \cdot \mathbf{x}) \nabla \times \mathbf{x}$$

As in 22 (b)

$$\text{curl } \mathbf{x} = \mathbf{e}_1 \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) + \mathbf{e}_2 \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \right) + \mathbf{e}_3 \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) = \mathbf{0},$$

so

$$\nabla \times ((\mathbf{a} \cdot \mathbf{x}) \mathbf{x}) = \mathbf{a} \times \mathbf{x} = -\mathbf{x} \times \mathbf{a}.$$

(c) $\text{div } \text{curl } \mathbf{F} = \text{div } \text{curl } \mathbf{G} = 0,$

Solution: $\text{div } \text{curl } \mathbf{F} = \text{div}(\mathbf{a} \times \mathbf{x}) = 0$ as in 22 (f):

$\mathbf{a} \times \mathbf{x} = \mathbf{e}_1(a_2z - a_3y) + \mathbf{e}_2(a_3x - a_1z) + \mathbf{e}_3(a_1y - a_2x)$ so

$$\text{div}(\mathbf{a} \times \mathbf{x}) = \frac{\partial(a_2z - a_3y)}{\partial x} + \frac{\partial(a_3x - a_1z)}{\partial y} + \frac{\partial(a_1y - a_2x)}{\partial z} = 0.$$

Since $\text{curl } \mathbf{G} = -2 \text{curl } \mathbf{F}$ this implies that $\text{div } \text{curl } \mathbf{G} = 0$.

(d) $\text{curl } \text{curl } \mathbf{G} = -2 \text{curl } \text{curl } \mathbf{F} = -4\mathbf{a}.$

Solution:

$$\text{curl } \text{curl } \mathbf{G} = 2 \text{curl}(\mathbf{x} \times \mathbf{a}) = -4\mathbf{a}$$

using $\mathbf{a} \times \mathbf{x} = -\mathbf{x} \times \mathbf{a}$ and 22 (g)

$$\begin{aligned} \text{curl}(\mathbf{a} \times \mathbf{x}) &= \mathbf{e}_1 \left(\frac{\partial(a_1y - a_2x)}{\partial y} - \frac{\partial(a_3x - a_1z)}{\partial z} \right) \\ &+ \mathbf{e}_2 \left(\frac{\partial(a_2z - a_3y)}{\partial z} - \frac{\partial(a_1y - a_2x)}{\partial x} \right) + \mathbf{e}_3 \left(\frac{\partial(a_3x - a_1z)}{\partial x} - \frac{\partial(a_2z - a_3y)}{\partial y} \right) \\ &= \mathbf{e}_1(a_1 + a_1) + \mathbf{e}_2(a_2 + a_2) + \mathbf{e}_3(a_3 + a_3) = 2\mathbf{a}. \end{aligned}$$

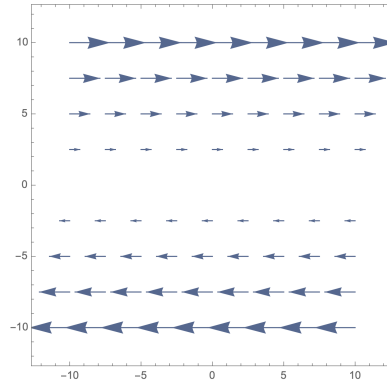
Since $\text{curl } \mathbf{G} = -2 \text{curl } \mathbf{F}$ this implies that

$$\text{curl } \text{curl } \mathbf{G} = -2 \text{curl } \text{curl } \mathbf{F} = -4\mathbf{a}.$$

24 Exam question June 2001 (Section A):

- (a) Give a representation of the vector function $\mathbf{A}(x, y) = y\mathbf{e}_1$ as a collection of arrows in the region of the (x, y) -plane bounded by $(x_1, y_1) = (-2, 2), (x_2, y_2) = (2, 2), (x_3, y_3) = (2, -2), (x_4, y_4) = (-2, -2)$.

Solution:



The vector field $\mathbf{A}(x, y)$

- (b) Calculate the curl of the vector field $\mathbf{A}(x, y) = (-y\mathbf{e}_1 + x\mathbf{e}_2)/(x^2 + y^2)$ defined everywhere in the (x, y) -plane except at the origin. (You can consider \mathbf{A} to be embedded in three dimensions, independent of z and with zero z component.)

Solution:

$$\begin{aligned}\nabla \times \mathbf{A} &= \mathbf{e}_1 \left(\frac{\partial 0}{\partial y} - \frac{\partial}{\partial z} \frac{x}{x^2 + y^2} \right) + \mathbf{e}_2 \left(\frac{\partial}{\partial z} \frac{-y}{x^2 + y^2} - \frac{\partial 0}{\partial x} \right) + \mathbf{e}_3 \left(\frac{\partial}{\partial x} \frac{x}{x^2 + y^2} + \frac{\partial}{\partial y} \frac{y}{x^2 + y^2} \right) \\ &= \mathbf{e}_3 \left(\frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} + \frac{1}{x^2 + y^2} - \frac{2y^2}{(x^2 + y^2)^2} \right) = \mathbf{0},\end{aligned}$$

- (c) Give the unit vector normal to the surface of equation $ax + by = cz$, where a, b, c , are three real constants.

Solution: $ax + by = cz$ is the equation of a plane, the general form of which would be $\mathbf{x} \cdot \mathbf{V} = k$ where \mathbf{V} is orthogonal to the plane. Here $\mathbf{V} = (a, b, -c)$, so a unit vector normal to the surface is $(a/\sqrt{a^2 + b^2 + c^2}, b/\sqrt{a^2 + b^2 + c^2}, -c/\sqrt{a^2 + b^2 + c^2})$. Alternatively: in general ∇f is normal to the level-surface $f = k$, here $f = ax + by - cz$ so $\nabla f = (a, b, -c)$ and again a unit vector normal to the surface is $\nabla f/|\nabla f| = (a/\sqrt{a^2 + b^2 + c^2}, b/\sqrt{a^2 + b^2 + c^2}, -c/\sqrt{a^2 + b^2 + c^2})$.

- (d) (Slightly modified from exam) Let \mathbf{x} be the position vector in 3-dimensions and \mathbf{a} be a constant vector. Use the result $\mathbf{x} \times (\mathbf{x} \times \mathbf{a}) = \mathbf{x}(\mathbf{x} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{x} \cdot \mathbf{x})$ to show that $\text{div} [\mathbf{x} \times (\mathbf{x} \times \mathbf{a})] = 2\mathbf{a} \cdot \mathbf{x}$.

Solution: Using $\mathbf{x} \times (\mathbf{x} \times \mathbf{a}) = \mathbf{x}(\mathbf{x} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{x} \cdot \mathbf{x})$, then $\text{div} [\mathbf{x} \times (\mathbf{x} \times \mathbf{a})] = \text{div} (\mathbf{x}(\mathbf{x} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{x} \cdot \mathbf{x})) = (\nabla \cdot \mathbf{x})(\mathbf{x} \cdot \mathbf{a}) + \mathbf{x} \cdot \nabla(\mathbf{x} \cdot \mathbf{a}) - \mathbf{a} \cdot \nabla(\mathbf{x} \cdot \mathbf{x})$. This is $3(\mathbf{x} \cdot \mathbf{a}) + (\mathbf{x} \cdot \mathbf{a}) - 2(\mathbf{x} \cdot \mathbf{a}) = 2(\mathbf{x} \cdot \mathbf{a})$