

57 Let  $A$  be the region bounded by the positive  $x$ - and  $y$ -axes and the line  $3x + 4y = 10$ . Compute  $\iint_A (x^2 + y^2) dx dy$ , taking the integrals in both orders and checking that your answers agree.

58 In the following integrals sketch the integration regions and then evaluate the integrals. Next interchange the order of integrations and re-evaluate.

(a)  $\int_0^1 \left( \int_x^1 xy dy \right) dx,$

(b)  $\int_0^{\pi/2} \left( \int_0^{\cos \theta} \cos \theta dr \right) d\theta,$

(c)  $\int_0^1 \left( \int_1^{2-y} (x+y)^2 dx \right) dy.$

59 Exam question 2010 (Section A) Q4: Calculate the double integral

$$\iint_A (|x| + |y|) dx dy.$$

where  $A$  is the region defined by  $|x| + |y| \leq 1$ .

60 Exam question 2011 (Section A) Q4: Change the order of integration in the double integral

$$\int_0^2 \int_x^{2x} f(x, y) dy dx.$$

61 Exam question May 2017 (Section A): A solid cylinder  $C$  of radius 1 and height 1 is defined by

$$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, 0 \leq z \leq 1\}.$$

Show that the paraboloid  $P = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}$  cuts  $C$  into two pieces of equal volume.

62 Compute the iterated integral

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dy dx.$$

Now reverse the order of integrations and re-evaluate. Why doesn't your answer contradict Fubini's theorem? (Hints: for the first integral, with respect to  $y$ , it might help to aim at something that can be integrated by parts using  $\frac{d}{dy} \left( \frac{1}{x^2 + y^2} \right) = -\frac{2y}{(x^2 + y^2)^2}$ ; and in answering to the last part of the question, the fact that  $\int_0^1 \left| \frac{x^2 - y^2}{(x^2 + y^2)^2} \right| dy \geq \int_0^x \frac{x^2 - y^2}{(x^2 + y^2)^2} dy$  could be useful.)

63 Let  $B$  be the region bounded by the five planes  $x = 0$ ,  $y = 0$ ,  $z = 0$ ,  $x + y = 1$ , and  $z = x + y$ .

(a) Find the volume of  $B$ .

(b) Evaluate  $\int_B x dV$ .

(c) Evaluate  $\int_B y dV$ .

64 A function  $f(x, y)$  is defined by

$$f(x, y) = \begin{cases} 1 & \text{if } -1 < x-y < 0 \\ -1 & \text{if } 0 < x-y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute  $\int_0^\infty \left( \int_{-\infty}^\infty f(x, y) dx \right) dy$  and also  $\int_{-\infty}^\infty \left( \int_0^\infty f(x, y) dy \right) dx$  (for the second case it might help to draw a picture). Comment on your two answers – why does Fubini fail?

65 A function  $f(x, y)$  is defined by

$$f(x, y) = \begin{cases} 2^{2(n+1)} & \text{if } 2^{-(n+1)} < x < 2^{-n}, \quad 2^{-(n+1)} < y < 2^{-n} \\ -2^{2n+3} & \text{if } 2^{-(n+1)} < x < 2^{-n}, \quad 2^{-(n+2)} < y < 2^{-(n+1)} \\ 0 & \text{otherwise,} \end{cases}$$

for  $n \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$ .

Compute  $\int_0^1 \int_0^1 f(x, y) dx dy$ , and  $\int_0^1 \int_0^1 f(x, y) dy dx$ . Does this contradict Fubini's theorem?

66 Write the line integral

$$\int_C x dx + y dy + (xz - y) dz$$

in the form  $\int_C \mathbf{v} \cdot d\mathbf{x}$  for a suitable vector field  $\mathbf{v}(\mathbf{x})$ , and compute its value when  $C$  is the curve given by  $\mathbf{x}(t) = t^2 \mathbf{e}_1 + 2t \mathbf{e}_2 + 4t^3 \mathbf{e}_3$  with  $0 \leq t \leq 1$ .

67 Evaluate  $\int_\sigma \mathbf{F} \cdot d\mathbf{x}$ , where  $\mathbf{F} = y \mathbf{e}_1 + 2x \mathbf{e}_2 + y \mathbf{e}_3$  and the path  $\sigma$  is given by  $\mathbf{x}(t) = t \mathbf{e}_1 + t^2 \mathbf{e}_2 + t^3 \mathbf{e}_3$ ,  $0 \leq t \leq 1$ .

68 Let  $\underline{A}(\underline{x})$  be the vector field  $\underline{A}(x, y, z) = x \underline{e}_1 + y \underline{e}_2 + z \underline{e}_3$ .

(a) Compute the line integral  $\int_C \underline{A} \cdot d\underline{x}$  where  $C$  is the straight line from the origin to the point  $(1, 1, 1)$ .

(b) Show (by finding  $f$ ) that the vector field  $\underline{A}$  from part (a) is equal to  $\nabla f(\underline{x})$  for some scalar field  $f$ , and that your answer to part (a) is equal to  $f(1, 1, 1) - f(0, 0, 0)$ .

69 Show that the result from question 68 applies in general: if the vector field  $\underline{v}(\underline{x})$  in  $\mathbb{R}^n$  is the gradient of a scalar field  $f(\underline{x})$ , so that  $\underline{v} = \nabla f$ , and if  $C$  is a curve in  $\mathbb{R}^n$  running from  $\underline{x} = \underline{a}$  to  $\underline{x} = \underline{b}$ , then  $\int_C \underline{v} \cdot d\underline{x} = f(\underline{b}) - f(\underline{a})$ . (Hint: use the chain rule.)

70 Use the result from question 69 to evaluate  $\int_C 2xyz dx + x^2 z dy + x^2 y dz$ , where  $C$  is any regular curve connecting  $(1, 1, 1)$  to  $(1, 2, 4)$ .

71 Compute the surface integral,  $\int_S \mathbf{F} \cdot d\mathbf{A}$ , of the vector field  $\mathbf{F} = (3x^2, -2yx, 8)$  over the surface given by the plane  $z = 2x - y$  with  $0 \leq x \leq 2$ ,  $0 \leq y \leq 2$ ,

(a) using method 2 from lectures,

(b) using method 1 from lectures.

72 Let  $\mathbf{F}(x, y, z) = (z, x, y)$ , and  $S$  be the part of the surface of the sphere  $x^2 + y^2 + (z - 1)^2 = r^2$  above the plane  $z = 0$ . Assume that  $r > 1$  so that the boundary  $C$  of  $S$ , where  $S$  intersects the plane  $z = 0$ , is non-empty.

(a) Compute the line integral  $\oint_C \mathbf{F} \cdot d\mathbf{x}$ .

(b) By parameterising the surface  $S$  with spherical coordinates centred on the point  $(0, 0, 1)$ , compute the surface integral of the curl of  $\mathbf{F}$ ,  $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{A}$ .

73 Let  $\underline{A}(\underline{x})$  be the vector field  $\underline{A}(x, y, z) = z \underline{e}_1 + x \underline{e}_2 + y \underline{e}_3$ ,  $C$  be the circle in the  $x, y$ -plane of radius  $r$  centred on the origin, and  $S$  the disk in the  $x, y$ -plane whose boundary is  $C$ .

(a) Compute the line integral  $\oint_C \underline{A} \cdot d\underline{x}$ .

(b) Compute the surface integral of the curl of  $\underline{A}$ ,  $\int_S (\nabla \times \underline{A}) \cdot d\mathbf{A}$ .

74 Let  $\underline{A}(\underline{x})$  be the vector field  $\underline{A}(x, y, z) = z \underline{e}_1 + x \underline{e}_2 + y \underline{e}_3$ ,  $C$  be the boundary of a square  $abcd$  in the  $y, z$ -plane with vertices  $\underline{a} = \underline{0}$ ,  $\underline{b} = a \underline{e}_2$ ,  $\underline{c} = a \underline{e}_2 + a \underline{e}_3$  and  $\underline{d} = a \underline{e}_3$ , and  $S$  be the region of the  $y, z$ -plane bounded by  $C$ .

(a) Compute the line integral  $\oint_C \underline{A} \cdot d\underline{x}$ .

(b) Compute the surface integral of the curl of  $\underline{A}$ ,  $\int_S (\nabla \times \underline{A}) \cdot d\mathbf{A}$ .

75 Based on exam question May 2015 (Section A) Q5 (which didn't contain part (b)):

(a) Calculate  $\int_V (\nabla \cdot \underline{U}) dV$  where  $V$  is the solid cube with faces  $x = \pm 1$ ,  $y = \pm 1$  and  $z = \pm 1$  and

$$\underline{U}(x, y, z) = (xy^2, yx^2, z).$$

(b) Calculate  $\int_S \underline{U} \cdot d\underline{A}$ , where  $S$  is the surface of  $V$ .