Calculus I, Gradescope Assignment, Week 7

Q1. Find an expression for $\frac{dy}{dx}$ in terms of x and y given the relation $2y^{3/2} + xy - x = 0$. 2 marks

Solution:

Differentiating with respect to x gives $3\sqrt{y}y' + y + xy' - 1 = 0$ 1 mark
hence $y' = (1-y)/(3\sqrt{y} + x)$. 1 mark

Q2. Given that y is a differentiable function of x and satisfies the equation, $x^3 - 2y^2 + xy = 0$, calculate y' at the point $(x, y) = (1, 1)$. 3 marks

Solution:

Differentiating with respect to x gives $3x^2 - 4yy' + y + xy' = 0$ 1 mark
At $(x, y) = (1, 1)$ this becomes $3 - 4y' + 1 + y' = 0$ 1 mark
so $y' = 4/3$. 1 mark

Q3. Find all local maxima and minima, and hence the global extreme values, of
$f(x) = x^4 - 2x^2 + 1$ in the interval $[-2, 2]$. 7 marks

Solution:

Note that $f(x) = x^4 - 2x^2 + 1$ is twice differentiable at all points.
$f'(x) = 4x(x^2 - 1)$, hence $f'(x) = 0$ at $x = 0, \pm 1$, which are all in $[-2, 2]$. 1 mark
Also, $f''(x) = 4(3x^2 - 1)$.
$f''(0) = -4 < 0$, hence $x = 0$ is a local maximum with $f(0) = 1$. 1 mark
$f''(\pm 1) = 8 > 0$, hence $x = \pm 1$ are local minima with $f(\pm 1) = 0$. 1 mark
$f'(-2) = -24 < 0$, hence $x = -2$ is an endpoint maximum with $f(-2) = 9$. 1 mark
$f'(2) = 24 > 0$, hence $x = 2$ is an endpoint maximum with $f(2) = 9$. 1 mark
The global minimum is therefore 0 1 mark
and the global maximum is 9. 1 mark

Q4. Find the global extreme values of $f(x) = x|x^2 - 6| - \frac{3}{2}x^2 + 2$ in $[-2, 4]$. 10 marks

Solution:

Note that $f(x) = x|x^2 - 6| - \frac{3}{2}x^2 + 2$ is differentiable everywhere in $(-2, 4)$ except at $x = \sqrt{6}$. 1 mark
First restrict to $x \in (-2, \sqrt{6})$, then $f(x) = -x(x^2 - 6) - \frac{3}{2}x^2 + 2$ with
$f'(x) = -3x^2 + 6 - 3x = -3(x + 2)(x - 1)$. 1 mark
Hence $f'(x) = 0$ in $(-2, \sqrt{6})$ iff $x = 1$. $f(1) = \frac{11}{2}$. 1 mark
Now restrict to $x \in (\sqrt{6}, 4)$, then $f(x) = x(x^2 - 6) - \frac{3}{2}x^2 + 2$ with
$f'(x) = 3x^2 - 6 - 3x = 3(x - 2)(x + 1)$. 1 mark
Hence $f'(x) \neq 0$ in $(\sqrt{6}, 4)$. 1 mark

The only other possibilities for the global extreme values are $x = \sqrt{6}$ (where $f(x)$ is not differentiable) and the endpoints $x = -2, 4$. Checking these values gives

$f(\sqrt{6}) = -7$ 1 mark

$f(-2) = -8$ 1 mark

$f(4) = 18.$ 1 mark

The global maximum is $\max\{\frac{11}{2}, -7, -8, 18\} = 18.$ 1 mark

The global minimum is $\min\{\frac{11}{2}, -7, -8, 18\} = -8.$ 1 mark

Q5. Either find the global maximum or justify that it does not exist for $f(x) = x^4 - 2x^2$ in $[-\frac{1}{3}, \frac{4}{3}]$, 6 marks

Solution:

$f'(x) = 4x(x^2 - 1) = 0$ in $(-\frac{1}{3}, \frac{4}{3})$ iff $x = 0, 1.$ 1 mark

$f(0) = 0$ 1 mark

$f(1) = -1$ 1 mark

$f(-\frac{1}{3}) = -\frac{17}{81}$ 1 mark

$f(\frac{4}{3}) = -\frac{32}{81}$ 1 mark

Global maximum is 0. 1 mark

Q6. Either find the global maximum or justify that it does not exist for $f(x) = x^4 - 2x^2$ in $[-\frac{1}{3}, 2]$. 2 marks

Solution:

Same as in the previous question, $f'(x) = 4x(x^2 - 1) = 0$ in $(-\frac{1}{3}, 2)$ iff $x = 0, 1$.

$f(0) = 0,\ f(1) = -1,\ f(-\frac{1}{3}) = -\frac{17}{81}$

except that now $f(2) = 8$ 1 mark

hence the global maximum is 8. 1 mark

Q7. Either find the global maximum or justify that it does not exist for $f(x) = x^4 - 2x^2$ in $(0, 1]$. 4 marks

Solution:

$f'(x) = 4x(x^2 - 1) < 0$ in $(0, 1)$ 1 mark

so $f(x)$ is decreasing in this interval. 1 mark

As the left-hand point is not contained in the given interval $(0, 1]$ 1 mark

then there is no global maximum in $(0, 1]$. 1 mark