1 Maps between real vector spaces

1.1 General notation

- $\mathbb{R}$: the set of real numbers, which we think of as points on a line.
- $\mathbb{R}^n$: the set of ordered n-tuples $(x_1, x_2, \ldots, x_n)$ where each x_i is real ($x_i \in \mathbb{R}$). Such an n-tuple can be thought of as the cartesian coordinates of a point in n-dimensional space. $\mathbb{R}^n$ is a real vector space (see Linear Algebra I for the full set of axioms of a real vector space), and so we refer to elements of $\mathbb{R}^n$ as n-dimensional vectors. You may be used to using column vector notation for such vectors, but here we will use row vector notation.
- Recall that the standard basis vector e_i is defined as the vector with a 1 in the i^{th} position, and a 0 in all other positions, e.g. $x_2 = (0, 1, 0, \ldots, 0)$.

The standard basis vectors $\underline{e}_i$ are othonormal (orthogonal, and normalised to have length 1) with respect to the scalar (dot) product on R^n : $\underline{e}_i.\underline{e}_j = \left\{ \begin{array}{ll} 1, & i = j \\ 0, & i \neq j \end{array} \right\} \equiv \delta_{ij}.\delta_{ij}$ as just defined is called the Kronecker delta, and we will revisit this in our later section on index notation 4.

In terms of the standard basis of $\mathbb{R}^n$, $\{e_1, e_2, \ldots, e_n\}$, the position vector of a point in $\mathbb{R}^n$, $\underline{x}$ can be written as

$$\underline{x} = x_1\underline{e}_1 + x_2\underline{e}_2 + \cdots + x_n\underline{e}_n = \sum_{i=1}^{n} x_i\underline{e}_i = x_i\underline{e}_i.$$

Note: Index notation (as in the final expression in (1.1), $x_i\underline{e}_i$) is a very important convention: when we see an index (like i) repeated, we will assume that it is to be summed over from 1 to n and leave off the "$\sum_{i=1}^{n}$". This convention is known as the **Einstein Summation Convention** (ESC). Index notation is sufficiently important in vector calculus that we will have a whole section on it later in the course.

For low values of n, we will often write $x_1, x_2, \ldots$ as $x, y, \ldots$

$$\begin{array}{rll} \text{e.g.} \quad n = 2: & \underline{x} & = x\underline{e}_1 + y\underline{e}_2 \\ n = 3: & \underline{x} & = x\underline{e}_1 + y\underline{e}_2 + z\underline{e}_3. \end{array}$$

Important: **Don't forget the vector signs**, i.e. $\underline{x} \neq x$. xe_i has a quite different meaning to $\underline{x}.e_i$.

Given two vectors $\underline{u}, \underline{v} \in \mathbb{R}^n$

$$\underline{u} = u_1\underline{e}_1 + u_2\underline{e}_2 + \cdots + u_n\underline{e}_n = \sum_{i=1}^{n} u_i\underline{e}_i = u_i\underline{e}_i$$

$$\underline{v} = v_1\underline{e}_1 + v_2\underline{e}_2 + \cdots + v_n\underline{e}_n = \sum_{i=1}^{n} v_i\underline{e}_i = v_i\underline{e}_i$$

(note the use of the Einstein summation convention again here), the scalar (dot) product between these vectors is then

$$\underline{u}.\underline{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum_{i=1}^{n} u_iv_i = u_iv_i.$$

This can easily be proved by multiplying out $\underline{u}.\underline{v}$, and using the fact that the standard basis vectors are orthonormal.

The length or magnitude of $\underline{u}$ is found using the dot product between $\underline{u}$ and itself,

$$|\underline{u}| = \sqrt{\underline{u}.\underline{u}},$$

and if θ is the angle between $\underline{u}$ and $\underline{v}$, then

$$\underline{u}.\underline{v} = |\underline{u}|\,|\underline{v}|\cos(\theta).$$

$|\underline{x}|$ is sometimes denoted r (note that this is a scalar quantity, and not a vector), since it is the radial coordinate of $\underline{x}$ in spherical polars. Similarly $\underline{x}$ is sometimes denoted $\underline{r}$.

Draw vectors as arrows:

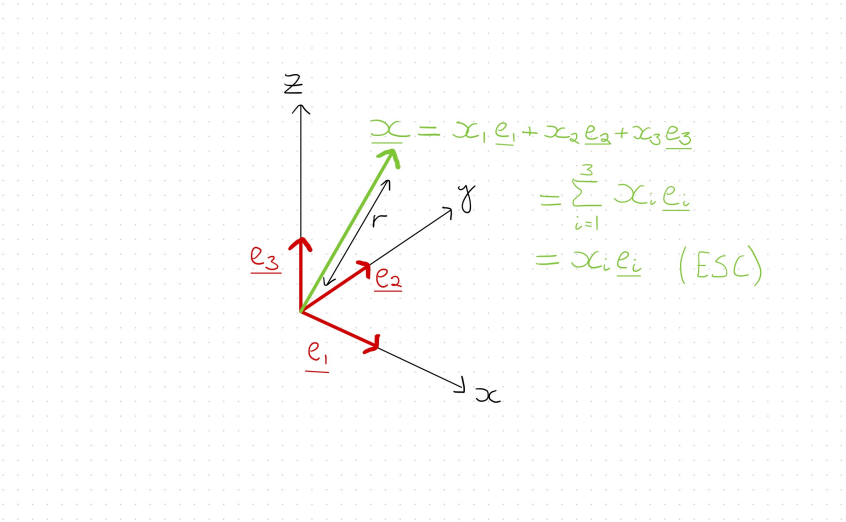

Figure 2: The vector $\underline{x}$ illustrated as the position vector of a point in $\mathbb{R}^3$. The length of $\underline{x}$, r, is shown, as are the standard basis vectors of $\mathbb{R}^3$. Using Einstein Summation Convention, we can write $\underline{x}$ in terms of the standard basis as $\underline{x} = x_i e_i$

1.2 Scalar fields, vector fields and curves

Now we can introduce the main objects of study in this course, scalar fields, vector fields and curves. These are all defined as maps from one real vector space to another, that is as maps from $\mathbb{R}^m \to \mathbb{R}^n$.

- **Scalar fields** are real-valued functions on $\mathbb{R}^n$, i.e. maps

$$\begin{aligned} \mathbb{R}^n &\to \mathbb{R} \\ \underline{x} &\mapsto f(\underline{x}) \end{aligned}$$

  e.g. for $n = 3$, we could have the function f defined as

$$f(\underline{x}) \quad = \quad \frac{xy}{\tan z}.$$

  Note that here the argument of the function is underlined to show that it's a vector quantity. $f(\underline{x})$ could also be written as $f(x, y, z)$.

  The functions of two variables that you studied in the Epiphany term of Calculus I were all examples of scalar fields.

- **Vector fields** are vector-valued functions on $\mathbb{R}^n$, i.e. maps

$$\begin{aligned} \mathbb{R}^n &\to \mathbb{R}^n \\ \underline{x} &\mapsto \underline{f}(\underline{x}). \end{aligned}$$

  Note that since the image of $\underline{x}$ is also a vector, we underline the function $\underline{f}$ to indicate this. In textbooks you may also see vector fields written in bold, rather than underlined.

  Example 1 *Given a constant vector $\underline{a} \in \mathbb{R}^n$, we might have the vector field $\underline{f}$ on $\mathbb{R}^n$, given by*

$$\underline{f}(\underline{x}) = (\underline{a}.\underline{x})\underline{x}.$$

  If we let $n = 2$ and take $\underline{a} = (1, 1)$, then the vector field near the origin looks as in Figure 3.

  Pay careful attention to the two different types of multiplication being used in this example. We take the scalar product *between $\underline{a}$ and $\underline{x}$, and we can then use* scalar multiplication *to multiply the vector $\underline{x}$ by the scalar $(\underline{a}.\underline{x})$.*

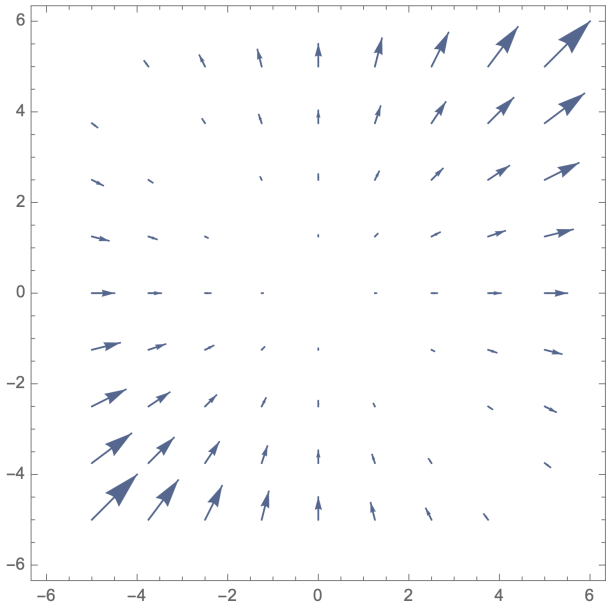

Figure 3: A plot of the vector field $\underline{f}(\underline{x}) = (a.x)x$ near the origin in 2 dimensions with $a = (1, 1)$. The vectors are drawn at a sample of points $\underline{x}$, as arrows.

Note: We can also write the formula "in components", i.e. using index notation - by giving a formula for the ith component of $\underline{f}$. So example 1 would be (using index notation and ESC):

$$f_i \quad = \quad (a_j x_j)x_i$$

- **Curves** in $\mathbb{R}^n$ are given parametrically by specifying $\underline{x}$ as a function $\underline{x}(t)$ of some parameter, t say. That is, a curve is a map

$$\mathbb{R} \to \mathbb{R}^n$$
$$t \mapsto \underline{x}(t).$$

Since the image of t is a vector quantity, we underline the function $\underline{x}(t)$ to indicate this. t itself is a scalar quantity however, and so is not underlined.

Example 2 *Given constant vectors $\underline{a}, \underline{b} \in \mathbb{R}^n$, the curve*

$$\text{e.g.} \quad \underline{x}(t) = \underline{a} + t\underline{b},$$

which can be written in components as

$$x_i(t) = a_i + tb_i,$$

is a straight line in $\mathbb{R}^n$, which goes through the point $\underline{a}$ and is parallel to $\underline{b}$.

If $\underline{x}(t)$ is differentiable, then $\frac{d\underline{x}}{dt}$ is tangent to the curve (if non-zero). (If you studied Dynamics I, then you've already come across this idea in that course. There, the *trajectory* of a particle $\underline{r}(t)$ was a curve in space, parameterised by time t, the *velocity* was the derivative of the trajectory with respect to time, $\frac{d\underline{r}(t)}{dt}$, and the *acceleration* was the second derivative $\frac{d^2\underline{r}(t)}{dt^2}$.)

Example 3 *A helix in $\mathbb{R}^3$ can be parameterised as*

$$\underline{x}(t) = \cos(t)\underline{e}_1 + \sin(t)\underline{e}_2 + t\underline{e}_3.$$

The tangent to the helix is therefore given by

$$\frac{d\underline{x}}{dt}(t) = -\sin(t)\underline{e}_1 + \cos(t)\underline{e}_2 + \underline{e}_3.$$

Note that the standard basis vectors are constant, and hence the components of the derivative of $\underline{x}(t)$ with respect to t, are just the derivatives of the components of $\underline{x}(t)$.

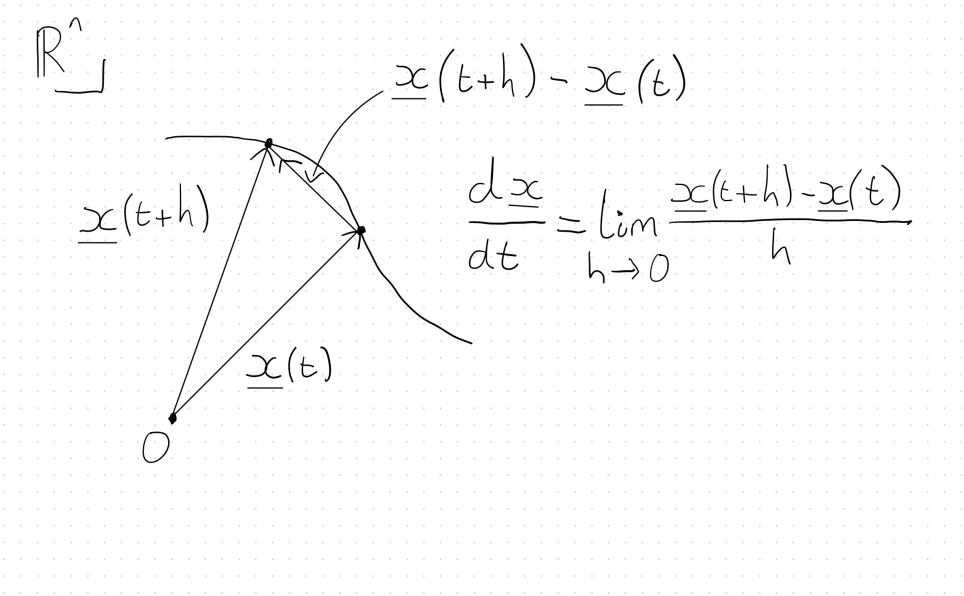

Figure 4: The tangent to a curve $\underline{x}(t)$ is found by differentiating the curve with respect to t.

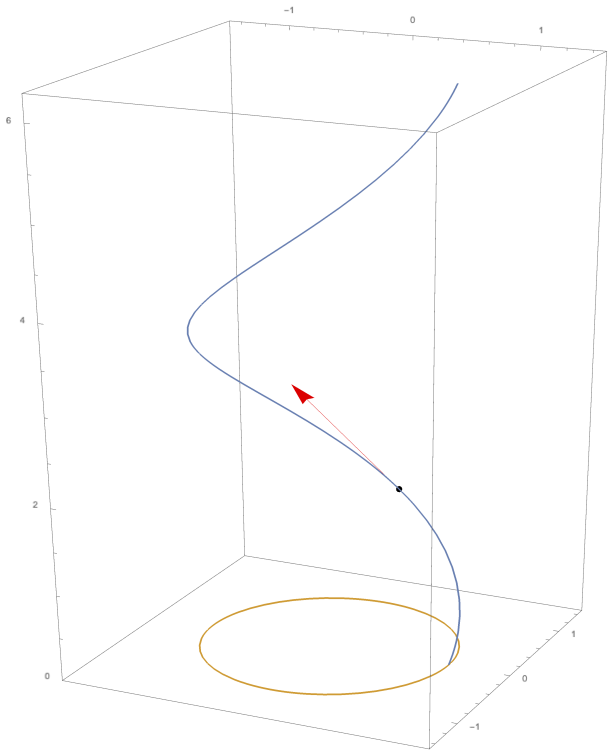

Figure 5: A helix in $\mathbb{R}^3$, with the tangent at a point shown.

Note: If a curve is parameterised in terms of the so-called *arc-length* s (we won't define this precisely here) along the curve from a fixed point on it, then $|\frac{d\underline{x}}{ds}| = 1$.

$$\begin{aligned} & & \text{with} \quad t = s, \quad h = \delta s, \\ & & |\underline{x}(s+\delta s) - \underline{x}(s)| \simeq \delta s \\ \Rightarrow \quad & \lim_{\delta s \to 0} \frac{|\underline{x}(s+\delta s) - \underline{x}(s)|}{\delta s} = 1 \\ & \Rightarrow \quad |\frac{d\underline{x}}{ds}| = 1 \qquad \text{a unit vector.} \end{aligned}$$

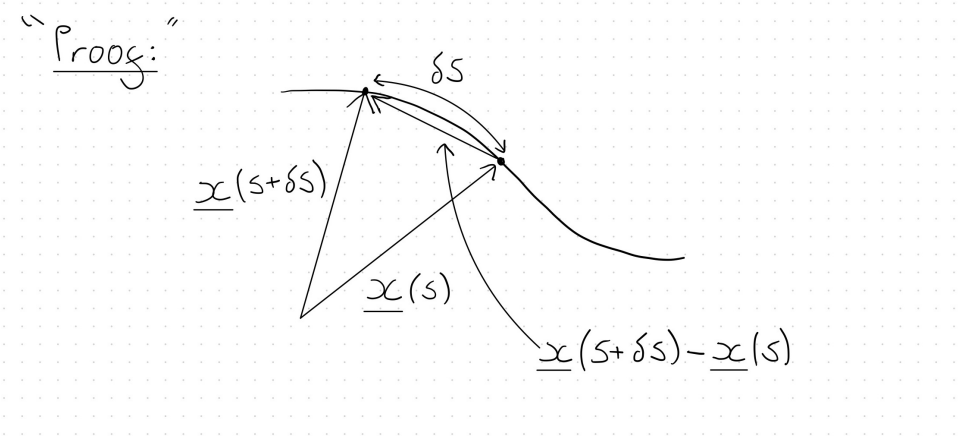

Figure 6: If the curve is parameterised by its arc-length (which we haven't rigorously defined), then for small δs, the length of the vector $\underline{x}(s+\delta s) - \underline{x}(s)$ becomes approximately the same as the length of the curve segment δs. In the limit that $\delta s \to 0$, these become equal, and hence the length of the derivative of the curve with respect to s (the tangent) becomes 1.

1.3 Partial derivatives and the chain rule

In Calculus I, given a function of two variables $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$, you learned that the partial derivatives

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ \frac{\partial f}{\partial y} &= \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}, \end{aligned}$$

tell us the rate of change of the function f as we move parallel to the x- and y- axes respectively. If we write x, y as x_1, x_2 & $f(x,y)$ as $f(\underline{x})$, with $\underline{x} = x_1\underline{e}_1 + x_2\underline{e}_2$, then we can re-express the partial derivatives using vector notation as

$$\begin{aligned} \frac{\partial f}{\partial x} \equiv \quad \frac{\partial f}{\partial x_1} &= \lim_{h \to 0} \frac{f(\underline{x} + h\underline{e}_1) - f(\underline{x})}{h} \\ \frac{\partial f}{\partial y} \equiv \quad \frac{\partial f}{\partial x_2} &= \lim_{h \to 0} \frac{f(\underline{x} + h\underline{e}_2) - f(\underline{x})}{h}. \end{aligned}$$

This now suggests the obvious generalisation to scalar fields in n dimensions. If we let $f(\underline{x}) : \mathbb{R}^n \to \mathbb{R}$, then the function f has n (1^{st} order) partial derivatives given by

$$\frac{\partial f(\underline{x})}{\partial x_a} = \lim_{h \to 0} \frac{f(\underline{x} + h\underline{e}_a) - f(\underline{x})}{h},$$

for $a = 1, 2, \ldots, n$. These partial derivatives tell us about the rate of change of the function as we move parallel to any of the n coordinate axes in n dimensions.

You also learned that the rate of change of a function $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ along a parametrically defined curve C given as $(x(t), y(t))$ can be found using the chain rule. Along this curve we have $F(t) \equiv f(x(t), y(t))$. Note that I don't write the function of t as $f(t)$, since f is a map from $\mathbb{R}^2$ and hence strictly speaking $f(t)$ is a different function, this time a map from $\mathbb{R}$. To avoid the confusion of two different functions with the same name, we will denote the restricted function as $F(t)$. The chain rule then tells us that

$$\frac{dF(t)}{dt} = \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y}.$$

To extend this to the n-dimensional case using vector notation, we first note that in the two-dimensional case, the curve C is given by

$$\underline{x}(t) = x_1(t)\underline{e}_1 + x_2(t)\underline{e}_2,$$

with $x_1(t) = x(t)$ and $x_2(t) = y(t)$. Similarly, $f(x,y)$ is $f(\underline{x})$, so

$$\frac{dF(t)}{dt} = \frac{df(\underline{x}(t))}{dt} = \frac{dx_1}{dt}\frac{\partial f}{\partial x_1} + \frac{dx_2}{dt}\frac{\partial f}{\partial x_2}.$$

We can now see how this should be generalised to the case of a scalar field in n dimensions. The curve C can be given parametrically as

$$\underline{x}(t) = x_1(t)\underline{e}_1 + \ldots + x_n(t)\underline{e}_n.$$

Our scalar field is given as $f(\underline{x}) : \mathbb{R}^n \to \mathbb{R}$, and the restriction of the scalar field to the curve C can then be written as $F(t) = f(\underline{x}(t))$. The chain rule then tells us that

$$\frac{dF(t)}{dt} = \frac{d}{dt}f(\underline{x}(t)) = \frac{dx_1}{dt}\frac{\partial f}{\partial x_1} + \cdots + \frac{dx_n}{dt}\frac{\partial f}{\partial x_n}.$$

Note: The chain rule holds for differentiable functions of two variables. You defined what this meant in the context of functions of two variables in Calculus I, and we shall revisit precisely what it means for a scalar function in n dimensions to be differentiable in section 5. For now we simply assume that our scalar fields are indeed differentiable.