7 Second order differential equations

7.1 Linear constant coefficient homogeneous ODEs

The general form of a second order linear constant coefficient ODE is

$$\alpha_2 y'' + \alpha_1 y' + \alpha_0 y = \phi(x) \qquad (\dagger)$$

where $\alpha_2 \neq 0, \alpha_1, \alpha_0$ are constants (the constant coefficients) and $\phi(x)$ is an arbitrary function of x. The ODE is still second order and linear if these constants are replaced by functions of x, but then the ODE is not so easy to solve.

We first restrict to the case $\phi(x) = 0$ ie.

$$\alpha_2 y'' + \alpha_1 y' + \alpha_0 y = 0 \qquad (\dagger\dagger)$$

in which case the second order ODE is called **homogeneous**.

The first point to note is that if y_1 and y_2 are any two solutions of the homogeneous ODE $(\dagger\dagger)$ then so is any arbitrary linear combination
$y = Ay_1 + By_2$, where A and B are constants.
Proof: $y = Ay_1 + By_2$ so $y' = Ay_1' + By_2'$ and $y'' = Ay_1'' + By_2''$. Hence
$\alpha_2 y'' + \alpha_1 y' + \alpha_0 y = \alpha_2(Ay_1'' + By_2'') + \alpha_1(Ay_1' + By_2') + \alpha_0(Ay_1 + By_2) =$
$= A(\alpha_2 y_1'' + \alpha_1 y_1' + \alpha_0 y_1) + B(\alpha_2 y_2'' + \alpha_1 y_2' + \alpha_0 y_2) = 0 + 0 = 0.$
Note that the linearity of the ODE is crucial for this result.

The general solution of the ODE $(\dagger\dagger)$ contains two arbitrary constants (because the ODE is second order). If y_1 and y_2 are any two independent particular solutions then the general solution is given by $y = Ay_1 + By_2$, with A and B the two arbitrary constants. The task is therefore to find two independent particular solutions.

To find a particular solution, look for one in the form $y = e^{\lambda x}$.
Putting this into $(\dagger\dagger)$ yields the **characteristic** (or **auxiliary**) **equation**

$$\alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0 = 0. \qquad (Char)$$

There are 3 cases to consider, depending upon the type of roots of $(Char)$.

(i). Distinct real roots.
If $(Char)$ has real roots $\lambda_1 \neq \lambda_2$ then $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are the two required particular solutions and the general solution is

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}.$$

Eg. Solve $y'' - y' - 2y = 0$. Then $(Char)$ is $\lambda^2 - \lambda - 2 = 0 = (\lambda - 2)(\lambda + 1)$ with roots $\lambda = 2, -1$. The general solution is $y = Ae^{2x} + Be^{-x}$.

(ii). Repeated real root.
$(Char)$ has a repeated real root if $\alpha_1^2 = 4\alpha_0\alpha_2$ ie. when $\alpha_0 = \frac{\alpha_1^2}{4\alpha_2}$.
$(Char)$ then reduces to

$$\alpha_2\lambda^2 + \alpha_1\lambda + \frac{\alpha_1^2}{4\alpha_2} = 0 = \alpha_2\left(\lambda + \frac{\alpha_1}{2\alpha_2}\right)^2 \quad \text{with} \quad \lambda_1 = -\frac{\alpha_1}{2\alpha_2} \text{ the real double root.}$$

Thus $(Char)$ has produced only one solution $y_1 = e^{\lambda_1 x}$.
However, as we now show, $y = xe^{\lambda_1 x}$ is also a solution in this case.
The original ODE (††) now takes the form

$$\alpha_2 y'' + \alpha_1 y' + \frac{\alpha_1^2}{4\alpha_2} y = 0 = \alpha_2(y'' - 2\lambda_1 y' + \lambda_1^2 y).$$

With $y = xe^{\lambda_1 x}$ we have $y' = e^{\lambda_1 x} + \lambda_1 x e^{\lambda_1 x}$ and $y'' = 2\lambda_1 e^{\lambda_1 x} + \lambda_1^2 x e^{\lambda_1 x}$. Hence

$$y'' - 2\lambda_1 y' + \lambda_1^2 y = 2\lambda_1 e^{\lambda_1 x} + \lambda_1^2 x e^{\lambda_1 x} - 2\lambda_1(e^{\lambda_1 x} + \lambda_1 x e^{\lambda_1 x}) + \lambda_1^2 x e^{\lambda_1 x} = 0.$$

$y_1 = e^{\lambda_1 x}$ and $y_2 = xe^{\lambda_1 x}$ are the two required particular solutions and the general solution is

$$y = Ae^{\lambda_1 x} + Bxe^{\lambda_1 x}.$$

Eg. Solve $2y'' - 12y' + 18y = 0$. Then $(Char)$ is $2\lambda^2 - 12\lambda + 18 = 0 = 2(\lambda - 3)^2$ with double root $\lambda = 3$ The general solution is $y = Ae^{3x} + Bxe^{3x}$.

(iii). Complex roots.
If the roots of $(Char)$ are complex then they come in a complex conjugate pair ie. $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. We therefore have the two solutions $y_1 = e^{\lambda_1 x} = e^{\alpha x}e^{i\beta x}$ and $y_2 = e^{\lambda_2 x} = e^{\alpha x}e^{-i\beta x}$, but these are not the solutions we are looking for, since they are complex and we want real solutions. However, we have seen that we can take any linear combination of these solutions (even with complex coefficients) and it will still be a solution. In particular the combinations

$$Y_1 = \frac{1}{2}(y_1 + y_2) = \frac{1}{2}e^{\alpha x}(e^{i\beta x} + e^{-i\beta x}) = e^{\alpha x}\cos(\beta x)$$
$$Y_2 = \frac{1}{2i}(y_1 - y_2) = \frac{1}{2i}e^{\alpha x}(e^{i\beta x} - e^{-i\beta x}) = e^{\alpha x}\sin(\beta x)$$

are both real and provide the two particular solutions we want.
The general solution is an arbitrary real linear combination of Y_1 and Y_2 ie.

$$y = Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x) = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x))$$

with A and B real. Recall that α and β are both real and are the real and imaginary parts of the complex root of $(Char)$.

Eg. Solve $y'' - 6y' + 10y = 0$. Then $(Char)$ is $\lambda^2 - 6\lambda + 10 = 0$ with roots

$$\lambda = \frac{6 \pm \sqrt{36 - 40}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i = \alpha \pm i\beta.$$

Hence $\alpha = 3$ and $\beta = 1$ and the general solution is $y = e^{3x}(A\cos x + B\sin x)$.

7.2 The method of undetermined coefficients

We now return to the general inhomogeneous ODE

$$\alpha_2 y'' + \alpha_1 y' + \alpha_0 y = \phi(x) \qquad (\dagger)$$

where $\phi(x)$ is a function that is not identically zero.

The general solution to (†) is obtained as a sum of two parts

$$y = y_{CF} + y_{PI}$$

where y_{CF} is called the **complementary function** and is the general solution of the homogeneous version of the ODE ie. (††) which is obtained by removing the $\phi(x)$ term. This part of the solution contains the two arbitrary constants and we have already seen how to find this.
y_{PI} is called the **particular integral** and is any particular solution of (†). We haven't yet seen any methods to find this. However, before we look at a method, let us first show that the combination $y = y_{CF} + y_{PI}$ is indeed a solution of (†). We have that $y' = y'_{CF} + y'_{PI}$ and $y'' = y''_{CF} + y''_{PI}$ so put these into the left hand side of (†) to get

$$\alpha_2 y'' + \alpha_1 y' + \alpha_0 y = \alpha_2(y''_{CF} + y''_{PI}) + \alpha_1(y'_{CF} + y'_{PI}) + \alpha_0(y_{CF} + y_{PI})$$

$$= \underbrace{\alpha_2 y''_{CF} + \alpha_1 y'_{CF} + \alpha_0 y_{CF}}_{=0 \text{ by } (\dagger\dagger)} + \underbrace{\alpha_2 y''_{PI} + \alpha_1 y'_{PI} + \alpha_0 y_{PI}}_{=\phi(x) \text{ by } (\dagger)} = \phi(x).$$

To find a particular integral we shall use the **method of undetermined coefficients,** which can be applied if $\phi(x)$ takes certain forms such as polynomials, exponentials or trigonometric functions, including sums and products of these. The idea is to try an appropriate form of the solution that contains some unknown constant coefficients which are then determined by explicit calculation and the requirement that the try yields a solution. The table below lists the kind of terms that can be dealt with if they appear in $\phi(x)$ (they can come with any constant coefficients in front) together with the form to try for y_{PI}, containing undetermined constant coefficients a_i.

Term in $\phi(x)$	Form to try for y_{PI}
$e^{\gamma x}$	$a_1 e^{\gamma x}$
x^n	$a_0 + a_1 x + ... + a_n x^n$
$\cos(\gamma x)$	$a_1 \cos(\gamma x) + a_2 \sin(\gamma x)$
$\sin(\gamma x)$	$a_1 \cos(\gamma x) + a_2 \sin(\gamma x)$

The motivation for the above is that the form of the try when differentiated zero, once or twice, should have a similar functional form to $\phi(x)$, as there is then a chance that the left hand side of (†) could equal the right hand side. If $\phi(x)$ contains sums/products of the kind of terms above then try sums/products of the listed forms for the try.

Special case rule: If the suggested form of the try for y_{PI} is contained in y_{CF} then we know that this form wont work as the left hand side of (†) will then be identically zero and so cant equal $\phi(x)$. In this case first multiply the suggested form of the try by x to get the correct try. In some special cases this rule may need to be applied twice as x times the suggested try may also be contained in y_{CF}.

Eg. Solve $y'' - y' - 2y = 7 - 2x^2$.
From earlier we already know that $y_{CF} = Ae^{2x} + Be^{-x}$.
To find y_{PI} we try the form $y = a_0 + a_1 x + a_2 x^2$
to simplify the notation for the calculation we have dropped the $_{PI}$ subscript here.
Now $y' = a_1 + 2a_2 x$ and $y'' = 2a_2$, so putting these into the ODE gives $2a_2 - (a_1 + 2a_2 x) - 2(a_0 + a_1 x + a_2 x^2) = 7 - 2x^2$
$= 2a_2 - a_1 - 2a_0 + x(-2a_2 - 2a_1) - 2a_2 x^2$.
Comparing coefficients of x^2, x^1, x^0 gives $-2a_2 = -2,\ (a_2 = 1),\ -2a_2 - 2a_1 = 0,$
$(a_1 = -1),\ 2a_2 - a_1 - 2a_0 = 7,\ (a_0 = -2)$.
We have found $y_{PI} = -2 - x + x^2$ and the general solution is $y = y_{CF} + y_{PI}$ ie.
$y = Ae^{2x} + Be^{-x} - 2 - x + x^2$.

Eg. Solve $y'' + 4y' + 4y = 5e^{3x}$.
First find y_{CF} by solving $y'' + 4y' + 4y = 0$.
$(Char)$ is $\lambda^2 + 4\lambda + 4 = 0 = (\lambda + 2)^2$ with $\lambda = -2$ repeated,
hence $y_{CF} = e^{-2x}(A + Bx)$.
For y_{PI} try $y = a_1 e^{3x}$ so $y' = 3a_1 e^{3x}$ and $y'' = 9a_1 e^{3x}$. Put these into the ODE
$9a_1 e^{3x} + 12a_1 e^{3x} + 4a_1 e^{3x} = 5e^{3x} = 25a_1 e^{3x}$ so $a_1 = \frac{1}{5}$ giving $y_{PI} = \frac{1}{5}e^{3x}$.
The general solution is $y = y_{CF} + y_{PI}$ ie $y = e^{-2x}(A + Bx) + \frac{1}{5}e^{3x}$.

Eg. Solve $y'' - 2y' + 2y = 10\cos(2x)$.
First find y_{CF} by solving $y'' - 2y' + 2y = 0$.
$(Char)$ is $\lambda^2 - 2\lambda + 2 = 0$ with $\lambda = 1 \pm i$. Hence $y_{CF} = e^x(A\cos x + B\sin x)$.
For y_{PI} try $y = a_1\cos(2x) + a_2\sin(2x)$ so $y' = -2a_1\sin(2x) + 2a_2\cos(2x)$ and $y'' = -4a_1\cos(2x) - 4a_2\sin(2x)$. Put these into the ODE $y'' - 2y' + 2y = -4a_1\cos(2x) - 4a_2\sin(2x) - 2(-2a_1\sin(2x) + 2a_2\cos(2x)) + 2(a_1\cos(2x) + a_2\sin(2x)) = 10\cos(2x) = (-2a_1 - 4a_2)\cos(2x) + (-2a_2 + 4a_1)\sin(2x)$.
Comparing coefficients of $\sin(2x)$ and $\cos(2x)$ gives $-2a_2 + 4a_1 = 0,\ (a_2 = 2a_1),\ -2a_1 - 4a_2 = 10,\ -10a_1 = 10,\ (a_1 = -1,\ a_2 = -2)$.
Thus $y_{PI} = -\cos(2x) - 2\sin(2x)$.
The general solution is $y = e^x(A\cos x + B\sin x) - \cos(2x) - 2\sin(2x)$.

Eg. Solve $y'' - y' - 2y = 6e^{-x}$.
From earlier we already know that $y_{CF} = Ae^{2x} + Be^{-x}$.
To find y_{PI} we first think that we should try the form $y = a_1 e^{-x}$ but note that this is contained in y_{CF} so instead we try $y = a_1 x e^{-x}$. Then
$y' = a_1 e^{-x} - a_1 x e^{-x}$ and $y'' = -2a_1 e^{-x} + a_1 x e^{-x}$. Put into ODE
$-2a_1 e^{-x} + a_1 x e^{-x} - (a_1 e^{-x} - a_1 x e^{-x}) - 2a_1 x e^{-x} = 6e^{-x} = -3a_1 e^{-x}$ so $a_1 = -2$ and $y_{PI} = -2xe^{-x}$.
The general solution is $y = Ae^{2x} + Be^{-x} - 2xe^{-x}$.

Eg. Solve $y'' - y' - 2y = e^x(8\sin(3x) - 14\cos(3x))$.
From earlier we already know that $y_{CF} = Ae^{2x} + Be^{-x}$.
For y_{PI} try $y = e^x(a_1 \cos(3x) + a_2 \sin(3x))$.
It is an exercise to show that $a_1 = 1,\ a_2 = -1$.
The general solution is then $y = Ae^{2x} + Be^{-x} + e^x(\cos(3x) - \sin(3x))$.

Eg. Solve $y'' - y' - 2y = -5 + 9x - 2x^3 + 4\cos x + 2\sin x$.
From earlier we already know that $y_{CF} = Ae^{2x} + Be^{-x}$.
For y_{PI} try $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 \cos x + a_5 \sin x$.
It is an exercise to show that
$a_0 = 1,\ a_1 = 0,\ a_2 = -\frac{3}{2},\ a_3 = 1,\ a_4 = -1,\ a_5 = -1$.
The general solution is then $y = Ae^{2x} + Be^{-x} + 1 - \frac{3}{2}x^2 + x^3 - \cos x - \sin x$.

7.3 Initial and boundary value problems

The values of the two arbitrary constants in the general solution of a second order ODE are fixed by specifying two extra requirements on the solution and/or its derivative.

An **initial value problem** (IVP) is when we require $y(x_0) = y_0$ and $y'(x_0) = \delta$ for given constants x_0, y_0, δ. In this case the two constraints are given at the same value of the independent variable.

A **boundary value problem** (BVP) is when we require $y(x_0) = y_0$ and $y(x_1) = y_1$ for given constants $x_0 \neq x_1, y_0, y_1$. In this case the two constraints are given at two different values of the independent variable.

The way to solve an IVP or BVP is to first find the general solution of the ODE and then determine the values of the two constants in this solution so that the extra conditions are satisfied.

Eg. Solve the IVP $\quad y'' - y' - 2y = 7 - 2x^2, \quad y(0) = 5,\ y'(0) = 1.$
From earlier we already know that the general solution of the ODE is
$y(x) = Ae^{2x} + Be^{-x} - 2 - x + x^2.$
Therefore $y(0) = A + B - 2 = 5,$ so $\ A + B = 7.$
Now $y'(x) = 2Ae^{2x} - Be^{-x} - 1 + 2x,$ giving $y'(0) = 2A - B - 1 = 1,$ so $2A - B = 2.$
The solution of these two equations for A and B is $A = 3,\ B = 4.$
Therefore the solution of the IVP is $y = 3e^{2x} + 4e^{-x} - 2 - x + x^2.$

Eg. Solve the BVP $\quad 4y'' + y = 0, \quad y(0) = 1,\ y(\pi) = 2.$
$(Char)$ is $4\lambda^2 + 1 = 0$ with roots $\lambda = \pm\frac{i}{2}.$
The general solution is $y = A\cos(\frac{x}{2}) + B\sin(\frac{x}{2}).$
$y(0) = A = 1$ and $y(\pi) = B = 2.$
Hence the solution of the BVP is $y = \cos(\frac{x}{2}) + 2\sin(\frac{x}{2}).$

7.4 The method of variation of parameters

So far, the only method we have seen to construct a particular integral is the method of undetermined coefficients. As we have seen, this method only applies if the inhomogeneous term ϕ takes particular forms. In this section we consider a more general method to find a particular integral, known as the method of variation of parameters.

Defn. Given two differentiable functions, $y_1(x), y_2(x)$, we define the **Wronskian** to be

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

It is obvious that if y_1 and y_2 are linearly dependent then $W(y_1, y_2)$ is identically zero, that is, zero for all x. Therefore, if $W(y_1, y_2)$ is not identically zero then this implies that y_1 and y_2 are linearly independent.

The task at hand is to find a particular integral for the inhomogeneous ODE (†),

$$\alpha_2 y'' + \alpha_1 y' + \alpha_0 y = \phi.$$

The starting point is to consider the two linearly independent solutions y_1, y_2 of the homogeneous version of the ODE (††). In other words $y_{CF} = Ay_1 + By_2$.
The method of variation of parameters is to look for a particular integral by replacing the constants in y_{CF} by functions. Namely, a solution to (†) is sought in the form

$$y = u_1 y_1 + u_2 y_2$$

for functions $u_1(x), u_2(x)$. Given this form then

$$y' = u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2'.$$

The form we have chosen has two arbitrary functions and the ODE will only give one relation, hence we need to impose a second condition. This condition is chosen to be

$$u_1' y_1 + u_2' y_2 = 0$$

so that the derivative now simplifies to

$$y' = u_1 y_1' + u_2 y_2'.$$

Differentiating once more gives

$$y'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''.$$

Putting these expressions into (†) yields

$$\alpha_2(u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'') + \alpha_1(u_1 y_1' + u_2 y_2') + \alpha_0(u_1 y_1 + u_2 y_2) = \phi.$$

Using the fact that both y_1 and y_2 solve (††) simplifies the above equation to

$$\alpha_2(u_1'y_1' + u_2'y_2') = \phi.$$

We now have two equations for the two unknown functions u_1', u_2' and this gives

$$u_1' = -\frac{y_2\phi/\alpha_2}{W(y_1,y_2)}, \qquad u_2' = \frac{y_1\phi/\alpha_2}{W(y_1,y_2)},$$

which are solved by direct integration

$$u_1 = -\int \frac{y_2\phi/\alpha_2}{W(y_1,y_2)}\,dx, \qquad u_2 = \int \frac{y_1\phi/\alpha_2}{W(y_1,y_2)}\,dx.$$

As a simple first example, we will apply the method of variation of parameters to find a particular integral that could also be found using the method of undetermined coefficients.

Eg. Solve $y'' - y = e^{2x}$.
We have that $y_{CF} = Ae^x + Be^{-x}$ hence $y_1 = e^x$ and $y_2 = e^{-x}$.

$$W(y_1,y_2) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

Using the above formulae

$$u_1 = -\int \frac{e^{-x}e^{2x}}{-2}\,dx = \int \frac{1}{2}e^x\,dx = \frac{1}{2}e^x.$$

$$u_2 = \int \frac{e^x e^{2x}}{-2}\,dx = \int \frac{-1}{2}e^{3x}\,dx = -\frac{1}{6}e^{3x}.$$

$$y_{PI} = u_1y_1 + u_2y_2 = \frac{1}{2}e^xe^x - \frac{1}{6}e^{3x}e^{-x} = \frac{1}{3}e^{2x}.$$

$$y = y_{CF} + y_{PI} = Ae^x + Be^{-x} + \frac{1}{3}e^{2x}.$$

Note that if you don't want to remember the formulae you can simply remember that the key step in the method is to remove the u_1', u_2' terms from y'.
Here is the same problem solved via this slightly longer way of proceeding.

$y = u_1e^x + u_2e^{-x}$ hence $y' = [u_1'e^x + u_2'e^{-x}] + u_1e^x - u_2e^{-x}$

we set $[u_1'e^x + u_2'e^{-x}] = 0$ and differentiate y' again to get $y'' = u_1'e^x - u_2'e^{-x} + u_1e^x + u_2e^{-x}$.

Putting all the expressions into the ODE gives $u_1'e^x - u_2'e^{-x} = e^{2x}$.
Adding and subtracting the two equations $u_1'e^x + u_2'e^{-x} = 0$ and $u_1'e^x - u_2'e^{-x} = e^{2x}$ yields $u_1' = \frac{1}{2}e^x$ and $u_2' = -\frac{1}{2}e^{3x}$, which are the same expressions as found above using the formulae.

Here is an example that needs the method of variation of parameters as the method of undetermined coefficients is inapplicable.

Eg. Solve

$$y'' - 2y' + y = \frac{e^x}{x^2+1}.$$

CF: (CHAR) $\lambda^2 - 2\lambda + 1 = 0 = (\lambda - 1)^2,\ y_{CF} = (A + Bx)e^x.$

$y_1 = e^x,\ y_2 = xe^x$, with $W(y_1, y_2) = y_1 y_2' - y_2 y_1' = e^{2x}$

$$u_1 = -\int \frac{e^x}{x^2+1}\frac{xe^x}{e^{2x}}\,dx = -\int \frac{x}{x^2+1}\,dx = -\frac{1}{2}\log(x^2+1)$$

$$u_2 = \int \frac{e^x}{x^2+1}\frac{e^x}{e^{2x}}\,dx = \int \frac{1}{x^2+1}\,dx = \tan^{-1} x$$

$y_{PI} = -\frac{1}{2}e^x \log(x^2+1) + xe^x \tan^{-1} x$

$y = y_{CF} + y_{PI} = (A + Bx)e^x - \frac{1}{2}e^x \log(x^2+1) + xe^x \tan^{-1} x$

7.5 Systems of first order linear ODEs

A system of n coupled first order linear ODEs for n dependent variables can be written as a single n^{th} order linear ODE for a single dependent variable by eliminating the other dependent variables. An associated IVP is obtained if all the values of the dependent variables are specified at a single value of the independent variable.

Eg. Find the solution $y(x), z(x)$ of the pair of first order linear ODEs
$y' = -y + z + x^2, \quad z' = -8y + 5z + 8x^2 - 7x + 1,$
satisfying the initial condition $y(0) = 2,\ z(0) = 0.$

We solve by first finding a second order ODE for $y(x)$.
From the first equation $z = y' + y - x^2$ hence $z' = y'' + y' - 2x$.
Substituting these expressions into the second equation gives
$y'' + y' - 2x = -8y + 5(y' + y - x^2) + 8x^2 - 7x + 1$
ie. the second order ODE $\quad y'' - 4y' + 3y = 3x^2 - 5x + 1$
We can now solve this ODE using our earlier methods.
$(Char)$ is $\lambda^2 - 4\lambda + 3 = 0 = (\lambda - 3)(\lambda - 1)$ with roots $\lambda = 1, 3$. Hence $y_{CF} = Ae^x + Be^{3x}$
For y_{PI} try $y = a_0 + a_1 x + a_2 x^2$ to give
$y'' - 4y' + 3y = 2a_2 - 4(a_1 + 2a_2 x) + 3(a_0 + a_1 x + a_2 x^2) = 3a_2 x^2 + (3a_1 - 8a_2)x + 2a_2 - 4a_1 + 3a_0 = 3x^2 - 5x + 1$
Comparing coefficients gives the solution $a_2 = 1,\ a_1 = 1,\ a_0 = 1.$
The general solution for y is therefore $y = Ae^x + Be^{3x} + x^2 + x + 1$
To obtain z we use the earlier relation $z = y' + y - x^2 = 2Ae^x + 4Be^{3x} + 3x + 2.$
We now have the general solution for both y and z.
From the general solution $y(0) = A + B + 1 = 2$ and $z(0) = 2A + 4B + 2 = 0$ with solution $A = 3,\ B = -2.$
Hence the solution of the IVP is $\quad y = 3e^x - 2e^{3x} + x^2 + x + 1, \quad z = 6e^x - 8e^{3x} + 3x + 2.$