9.6 Summary: Fourier Series

Fourier series attempt to represent periodic functions as linear combinations of sines and cosines. You should know how to calculate Fourier coefficients so you can construct Fourier series (or Fourier partial sums). You should understand the interpretation of Fourier series as (infinite-dimensional) vectors where the orthogonal basis functions are the sines and cosines, and the components are the Fourier coefficients. Here are some key points:

- If $f(x)$ is a function of period $2L$, its (full-range) *Fourier series* is the infinite series $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L))$ where the Fourier coefficients are given by $a_n = \frac{1}{L}\int_{-L}^{L} \cos(n\pi x/L) f(x) dx$ (including $n = 0$) and $b_n = \frac{1}{L}\int_{-L}^{L} \sin(n\pi x/L) f(x) dx$. This arises for the vector interpretation with scalar product of $\phi(x)$ and $\psi(x)$ being $\frac{1}{L}\int_{-L}^{L} \phi\psi dx$, noting that the sines and cosines are orthonormal using this scalar product.

- If we include only the first N sines and cosines (i.e. have $\sum_{n=1}^{N}$ rather than $\sum_{n=1}^{\infty}$) we have a *Fourier partial sum*. The Fourier series is then given by the limit as $N \to \infty$.

- An odd function has a *sine Fourier series* (all $a_n = 0$) while an even function has a *cosine Fourier series* (all $b_n = 0$).

- *Dirichlet's theorem* says that if $f(x)$ is a periodic function with a finite number of extreme values and jump discontinuities in each period, and with $|f(x)|$ integrable over a period then the Fourier series converges (to some real number) for all x. It converges to $f(x)$ at all points where f is continuous, and at jump discontinuities it converges to the midpoint of the jump.

- The *Gibbs phenomenon* is the fact that at a jump discontinuity the Fourier series overshoots at the upper end and undershoots at the lower end by approximately 9% of the jump. This is visible in the Fourier partial sums with the peaks becoming both narrower and closer to the location of the jump as $N \to \infty$, but not reducing in height.

- Given any function $f(x)$ on $(-L, L]$ we can define its *periodic extension* to be the function of period $2L$ which equals $f(x)$ on $(-L, L]$. We can also define the *odd periodic extension* of $f(x)$ on $(0, L)$ to be the periodic extension of $f_o(x) = \begin{cases} f(x) & \text{if} x \in (0, L) \\ 0 & \text{if} x \in \{0, L\} \\ -f(-x) & \text{if} x \in (-L, 0) \end{cases}$. Similarly the *even periodic extension* is the periodic extension of $f_e(x) = \begin{cases} f(x) & \text{if} x \in [0, L] \\ f(-x) & \text{if} x \in (-L, 0) \end{cases}$.
We call the Fourier series for these odd/even periodic extensions *half-range sine/cosine series*.

- *Parseval's theorem* says that $\frac{1}{L}\int_{-L}^{L} (f(x))^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$. It is the Fourier series equivalent of the fact that the magnitude squared of a vector is the sum of the squares of its components. Parseval's theorem can be used to evaluate infinite series if we can realise the series in terms of squares of Fourier coefficients for some function and we can calculate the integral of the function squared.

- We can also evaluate infinite series if we have a Fourier series which we know converges to a specific value at some point – e.g. at a point where the function is continuous and Dirichlet's theorem guarantees the Fourier series converges to the value of the function at that point.

- Using the orthonormal basis functions $e^{in\pi x/L}$ with scalar product of ϕ and ψ defined by $\frac{1}{2L}\int_{-L}^{L}\overline{\phi}\psi dx$, we can write the *Fourier series in complex form* $\sum_{-\infty}^{\infty} c_n e^{in\pi x/L}$. The Fourier coefficients are given by $c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\pi x/L} f(x)dx$. If f is a real-valued function then $c_{-n} = \overline{c}_n$. Note that the complex form of the Fourier series is just a different way to express the usual real Fourier series using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.