28 *Let* $\mathbf{x}$ *be the position vector in three dimensions, with* $r = |\mathbf{x}|$, *and let* $\mathbf{a}$ *be a constant vector. Using index notation, show that*

(a) $\operatorname{div}\mathbf{x} = 3$,

(b) $\operatorname{curl}\mathbf{x} = 0$,

(c) $\operatorname{grad} r = \mathbf{x}/r$,

(d) $\operatorname{div}(r^n\mathbf{x}) = (n+3)\, r^n$,

(e) $\operatorname{grad}(\mathbf{a}\cdot\mathbf{x}) = \mathbf{a}$,

(f) $\operatorname{div}(\mathbf{a}\times\mathbf{x}) = 0$,

(g) $\operatorname{curl}(\mathbf{a}\times\mathbf{x}) = 2\,\mathbf{a}$,

(h) $\operatorname{curl}(r^2\mathbf{a}) = 2\,(\mathbf{x}\times\mathbf{a})$,

(i) $\nabla^2(1/r) = 0$ *if* $r \neq 0$*: using* $\frac{\partial}{\partial x_i} r = x_i/r$ *from part (c),*

(j) $\nabla^2(\log r) = 1/r^2$ *if* $r \neq 0$*:*

(k) $\operatorname{div}[(\mathbf{a}\cdot\mathbf{x})\mathbf{x}] = 4\,\mathbf{a}\cdot\mathbf{x}$,

(l) $\operatorname{div}[\mathbf{x}\times(\mathbf{x}\times\mathbf{a})] = 2\,\mathbf{a}\cdot\mathbf{x}$,

(m) $\operatorname{curl}(\mathbf{a}\times\mathbf{x}/r^3) = 3\,(\mathbf{a}\cdot\mathbf{x})\mathbf{x}/r^5 - \mathbf{a}/r^3$,

(n) *Exam question June 2002 (Section A): calculate the curl of* $(\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}$.

29 *The vector* $\underline{a}$ *has components* $(a_r) = (1,1,1)$ *and the vector* $\underline{b}$ *has components* $(b_r) = (2,3,4)$. *In the following expressions state which indices are free and which are dummy, and give the numerical values of the expressions for each value that the free variable takes (e.g. for* $a_r - b_r$ *the free variable is* r *and it takes the values* $1,\ 2,\ 3$ *so* $a_1 - b_1 = -1$, $a_2 - b_2 = -2$, $a_3 - b_3 = -3$*)*

(a) $a_r + b_r$,

(b) $a_r\, b_r$,

(c) $a_r\, b_s\, a_r$,

(d) $a_r\, b_s\, a_r\, b_s - a_r\, b_r\, a_s\, b_s$.

30 *If* δ_{rs} *is the three-dimensional Kronecker delta, evaluate*

(a) $\delta_{rs}\,\delta_{sr}\,\delta_{pq}\,\delta_{pq}$,

(b) $\delta_{rs}\,\delta_{sk}\,\delta_{kl}\,\delta_{lr}$,

(c) $\delta_{rs}\,\delta_{qr}\,\delta_{pq}\,\delta_{sp}$.

31 *If* δ_{rs} *is the three-dimensional Kronecker delta, simplify*

(a) $(\delta_{rp}\,\delta_{sq} - \delta_{rq}\,\delta_{sp})\, a_p\, b_q$,

(b) $(\delta_{rp}\,\delta_{sq} - \delta_{rq}\,\delta_{sp})\, \delta_{pq}$.

32 *If* δ_{rs} *is the Kronecker delta in* n *dimensions, calculate*

(a) δ_{rr},

(b) $\delta_{rs}\,\delta_{rs}$,

(c) $\delta_{rs}\,\delta_{st}\,\delta_{tr}$.

33 *Starting from* $\varepsilon_{ijk}\,\varepsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$ *simplify as much as possible:*

(a) $\varepsilon_{ijk}\varepsilon_{ijp}$,

(b) $\varepsilon_{ijk}\varepsilon_{ijk}$.

34 *Calculate* ε_{ijj}.

35 *Show, using index notation, that*

(a) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = 0$,

(b) $$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [\mathbf{d}\cdot(\mathbf{a}\times\mathbf{b})]\,\mathbf{c} - [\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})]\,\mathbf{d} = [\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})]\,\mathbf{b} - [\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})]\,\mathbf{a},$$

(c) $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$,

(d) $\mathbf{a}\times[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})] = a^2\,(\mathbf{b}\times\mathbf{a})$,

(e) $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) + (\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d}) + (\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d}) = 0$.

36 *Exam question June 2002 (Section A): Evaluate* $\varepsilon_{ijk}\varepsilon_{ijl}x_kx_l$.

37 *Exam question June 2001 (Section A): Evaluate* $\varepsilon_{ijk}\partial_i\partial_j(x_lx_l)^{1/2}$ *away from the origin.*

38 *Exam question June 2003 (Section A): Calculate* $\partial_i\,(\varepsilon_{ijk}\,\varepsilon_{jkl}\,x_l)$. *(Hint: use the connection between* $\partial_i x_j = \frac{\partial x_j}{\partial x_i}$ *and the Kronecker delta.)*

39 *The functions* f, g *are scalars, while* $\mathbf{A}$ *and* $\mathbf{B}$ *are vector functions with components* A_i *and* B_i *respectively. Verify the following identities using index notation:*

(a) $\mathrm{grad}\,(fg) = f\,\mathrm{grad}\,g + g\,\mathrm{grad}\,f$,

(b) $\mathrm{grad}\,(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times\mathrm{curl}\,\mathbf{B} + \mathbf{B}\times\mathrm{curl}\,\mathbf{A} + (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B}$,

(c) $\mathrm{div}\,(f\mathbf{A}) = f\,\mathrm{div}\,\mathbf{A} + (\mathrm{grad} f)\cdot\mathbf{A}$,

(d) $\mathrm{curl}\,(f\mathbf{A}) = f\,\mathrm{curl}\,\mathbf{A} + (\mathrm{grad}\,f)\times\mathbf{A}$,

(e) $\mathrm{div}\,(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\mathrm{curl}\,\mathbf{A} - \mathbf{A}\cdot\mathrm{curl}\,\mathbf{B}$,

(f) $\mathrm{curl}\,(\mathbf{A}\times\mathbf{B}) = (\mathrm{div}\,\mathbf{B})\mathbf{A} - (\mathrm{div}\,\mathbf{A})\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$,

(g) $\mathrm{div}\,\mathrm{curl}\,\mathbf{A} = 0$,

(h) $\mathrm{curl}\;\mathrm{curl}\,\mathbf{A} = \mathrm{grad}\,\mathrm{div}\,\mathbf{A} - \nabla^2\mathbf{A}$.

40 *What is the divergence of the vector function* $\mathbf{A}(\mathbf{x}) = r\,\mathbf{x} + \nabla\,r$ *where* $\mathbf{x}$ *is the position vector in 3 dimensions and* $r = |\mathbf{x}|$*? What is the corresponding result in* n *dimensions?*