7 *Compute the gradient,* $\underline{\nabla} f$*, for the following functions:*

(a) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$,

Solution: $\partial f/\partial x = x/\sqrt{x^2 + y^2 + z^2}$, $\partial f/\partial y = y/\sqrt{x^2 + y^2 + z^2}$, *and* $\partial f/\partial z = z/\sqrt{x^2 + y^2 + z^2}$, *so* $\underline{\nabla} f = (x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3})/\sqrt{x^2 + y^2 + z^2} = \mathbf{x}/|\mathbf{x}|$.

(b) $f(x, y, z) = xy + yz + xz$,

Solution: $\partial f/\partial x = y + z$, $\partial f/\partial y = x + z$, $\partial f/\partial z = y + x$, *so* $\underline{\nabla} f = (y + z)\mathbf{e_1} + (x + z)\mathbf{e_2} + (x + y)\mathbf{e_3}$.

(c) $f(x, y, z) = 1/(x^2 + y^2 + z^2)$.

Solution: $\partial f/\partial x = -2x/(x^2 + y^2 + z^2)^2$, $\partial f/\partial y = -2y/(x^2 + y^2 + z^2)^2$, *and* $\partial f/\partial z = -2z/(x^2 + y^2 + z^2)^2$ *so* $\underline{\nabla} f = -2(x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3})/(x^2 + y^2 + z^2)^2 = -2\mathbf{x}/|\mathbf{x}|^4$.

8 *Show that* $\underline{h}(s) = \left(s/\sqrt{2}, \cos(s/\sqrt{2}), \sin(s/\sqrt{2})\right)$ *is the arc-length parameterisation of a helix, that is that* $|\frac{dh}{ds}| = 1 \quad \forall s$.

Calculate the directional derivative of the scalar field $f(\underline{x}) = (\log(x^2 + y^2 + z^2))$ *along* $\underline{h}(s)$ *at* $s = \sqrt{2}\pi$.

Solution: To show that the curve is parameterised by arc-length, we need to show that $|\frac{dh}{ds}| = 1 \quad \forall s$. We have

$$\frac{d\underline{h}}{ds} = \left(1/\sqrt{2}, -\frac{1}{\sqrt{2}}\sin(s/\sqrt{2}), \frac{1}{\sqrt{2}}\cos(s/\sqrt{2})\right),$$

and therefore

$$\begin{aligned} \left|\frac{d\underline{h}}{ds}\right| &= \sqrt{\frac{1}{2} + \frac{1}{2}\sin^2(s/\sqrt{2}) + \frac{1}{2}\cos^2(s/\sqrt{2})} \\ &= \sqrt{\frac{1}{2} + \frac{1}{2}} \\ &= 1 \end{aligned}$$

The directional derivative of $f(\underline{x})$ along $\underline{h}(s)$ at $s = \sqrt{2}\pi$ is then given by

$$\frac{df(\underline{h})}{ds}(\sqrt{2}\pi) = \frac{d\underline{h}}{ds}(\sqrt{2}\pi).\underline{\nabla} f(\underline{h}(\sqrt{2}\pi)),$$

and so we need to calculate the gradient of f and evaluate this at $\underline{h}(\sqrt{2}\pi)$. We have

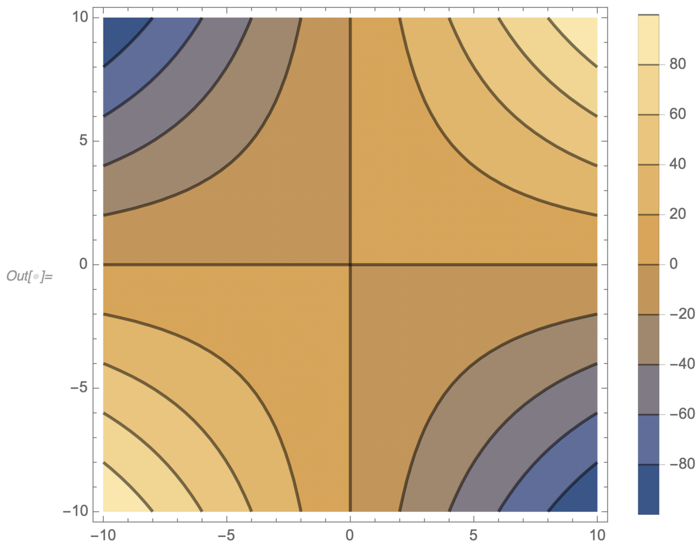
$$\begin{aligned} \underline{\nabla} f &= \left(\frac{2x}{x^2 + y^2 + z^2}, \frac{2y}{x^2 + y^2 + z^2}, \frac{2z}{x^2 + y^2 + z^2}\right) \\ &= \frac{2\underline{x}}{x^2 + y^2 + z^2} \\ \underline{h}(\sqrt{2}\pi) &= (\pi, \cos(\pi), \sin(\pi)) \\ &= (\pi, -1, 0), \end{aligned}$$

and so

$$\begin{aligned}
\frac{df(\underline{h})}{ds}(\sqrt{2}\pi) &= \frac{d\underline{h}}{ds}(\sqrt{2}\pi).\underline{\nabla} f(\underline{h}(\sqrt{2}\pi)) \\
&= \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\sin(\pi), \frac{1}{\sqrt{2}}\cos(\pi)\right).\underline{\nabla} f(\pi,-1,0) \\
&= \frac{2}{\pi^2+1}\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right).(\pi,-1,0) \\
&= \frac{\pi\sqrt{2}}{(\pi^2+1)}.
\end{aligned}$$

9 *Draw a sketch of the contour plot of the scalar field on $\mathbb{R}^2$ $f(\underline{x}) = xy$, as well as the gradient of f. What do you notice?*

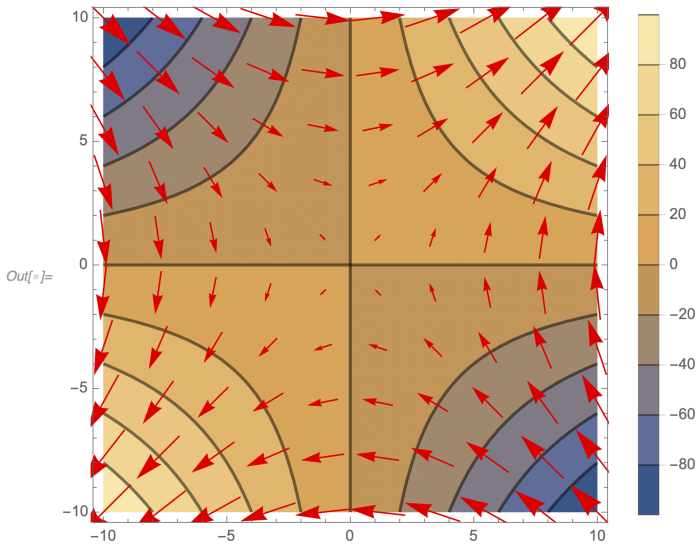
Solution: The level sets of f are of the form $xy = c$ for a constant $c \in \mathbb{R}$. Rearranging this as $y = c/x$, we can then plot a few of these level sets. This looks as follows:



The gradient of f is given as

$$\begin{aligned}
\text{grad} f = \underline{\nabla} f &= \left(\underline{e}_1\frac{\partial}{\partial x}, \underline{e}_2\frac{\partial}{\partial y}\right) f \\
&= (y, x)\,.
\end{aligned}$$

A plot of this overlaid on to top of the contour plot of f is as follows:



We see that the vectors of the vector field $\underline{\nabla} f$ are normal to the level sets of f, as we expect.

10 *Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be scalar fields on $\mathbb{R}^3$, $h : \mathbb{R} \to \mathbb{R}$ be a function on $\mathbb{R}$ and a be a constant in $\mathbb{R}$. Show (using the definition of $\underline{\nabla}$) that*

$$\underline{\nabla}(af(\underline{x})g(\underline{x}) + h(f(\underline{x}))) = a(\underline{\nabla} f)g + af\underline{\nabla} g + \underline{\nabla} f \frac{dh}{df}.$$

Solution: For this question, we are supposed to use only the definition of the gradient in $\mathbb{R}^3$, not the properties of the gradient. This is just a slog in keeping track of all the terms. We have

$$\begin{aligned}
\underline{\nabla}(af(\underline{x})g(\underline{x}) + h(f(\underline{x}))) &= \underline{e}_1 \frac{\partial}{\partial x}(af(\underline{x})g(\underline{x}) + h(f(\underline{x}))) + \underline{e}_2 \frac{\partial}{\partial y}(af(\underline{x})g(\underline{x}) + h(f(\underline{x}))) \\
&\quad + \underline{e}_3 \frac{\partial}{\partial z}(af(\underline{x})g(\underline{x}) + h(f(\underline{x}))) \\
&= \underline{e}_1 \left(a\frac{\partial}{\partial x} f(\underline{x})g(\underline{x}) + \frac{\partial}{\partial x} h(f(\underline{x})) \right) \qquad \text{by linearity} \\
&\quad + \underline{e}_2 \left(a\frac{\partial}{\partial y} f(\underline{x})g(\underline{x}) + \frac{\partial}{\partial y} h(f(\underline{x})) \right) \qquad \text{of partial} \\
&\quad + \underline{e}_3 \left(a\frac{\partial}{\partial z} f(\underline{x})g(\underline{x}) + \frac{\partial}{\partial z} h(f(\underline{x})) \right) \qquad \text{derivatives} \\
&= \underline{e}_1 \left(a\frac{\partial f(\underline{x})}{\partial x} g(\underline{x}) + af(\underline{x})\frac{\partial g(\underline{x})}{\partial x} + \frac{\partial h(f)}{\partial f}\frac{\partial f(\underline{x})}{\partial x} \right) \\
&\quad + \underline{e}_2 \left(a\frac{\partial f(\underline{x})}{\partial y} g(\underline{x}) + af(\underline{x})\frac{\partial g(\underline{x})}{\partial y} + \frac{\partial h(f)}{\partial f}\frac{\partial f(\underline{x})}{\partial y} \right) \\
&\quad + \underline{e}_3 \left(a\frac{\partial f(\underline{x})}{\partial z} g(\underline{x}) + af(\underline{x})\frac{\partial g(\underline{x})}{\partial z} + \frac{\partial h(f)}{\partial f}\frac{\partial f(\underline{x})}{\partial z} \right),
\end{aligned}$$

where we used the product rule and chain rule for the partial derivative in each component. We can now recollect the terms to give

$$\begin{aligned}\underline{\nabla}(af(\underline{x})g(\underline{x}) + h(f(\underline{x}))) &= a\left(\underline{e}_1\frac{\partial f(\underline{x})}{\partial x}g(\underline{x}) + \underline{e}_2\frac{\partial f(\underline{x})}{\partial y}g(\underline{x}) + \underline{e}_3\frac{\partial f(\underline{x})}{\partial z}g(\underline{x})\right)\\ &+ a\left(\underline{e}_1 f(\underline{x})\frac{\partial g(\underline{x})}{\partial x} + \underline{e}_2 f(\underline{x})\frac{\partial g(\underline{x})}{\partial y} + \underline{e}_3 f(\underline{x})\frac{\partial g(\underline{x})}{\partial z}\right)\\ &+ \left(\underline{e}_1\frac{\partial h(f)}{\partial f}\frac{\partial f(\underline{x})}{\partial x} + \underline{e}_2\frac{\partial h(f)}{\partial f}\frac{\partial f(\underline{x})}{\partial y} + \underline{e}_3\frac{\partial h(f)}{\partial f}\frac{\partial f(\underline{x})}{\partial z}\right)\\ &= a(\underline{\nabla} f)g + af\underline{\nabla} g + \underline{\nabla} f\frac{dh}{df}.\end{aligned}$$

11 *Exam question June 2001 (Section B): You are given the following family of scalar functions labelled by a real parameter λ : $\Phi_\lambda(x, y, z) = (y - \lambda)\cos x + zxy$.*

(a) *What are their derivatives in the direction $\mathbf{V} = \mathbf{e_1} + 2(\mathbf{e_2} + \mathbf{e_3})$?*

Solution: *$\nabla\Phi_\lambda = \mathbf{e_1}((\lambda - y)\sin x + zy) + \mathbf{e_2}(\cos x + zx) + \mathbf{e_3}xy$ and the directional derivative of Φ_λ in the direction of $\mathbf{V}$ is*

$$\frac{\mathbf{V}}{|\mathbf{V}|}\cdot\nabla\Phi_\lambda = \frac{\mathbf{e_1} + 2(\mathbf{e_2} + \mathbf{e_3})}{\sqrt{1+4+4}}\cdot\nabla\Phi_\lambda$$

$$= \frac{1}{3}\left((\lambda - y)\sin x + zy + 2\cos x + 2zx + 2xy\right)$$

(b) *Which member of the family has its gradient at the point $(\frac{\pi}{2}, 1, 1)$ equal to $\frac{\pi}{2}(\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3})$?*

Solution: *$\nabla\Phi_\lambda(\frac{\pi}{2}, 1, 1) = \mathbf{e_1}\lambda + \mathbf{e_2}\pi/2 + \mathbf{e_3}\pi/2$ so take $\lambda = \pi/2$.*

(c) *Calling this particular member of the family Φ_{λ_0}, in which direction is Φ_{λ_0} decreasing most rapidly when starting at the point $(\frac{\pi}{2}, 1, 1)$?*

Solution: *At this point Φ_{λ_0} decreases most rapidly in the direction of $-\nabla\Phi_{\lambda_0} = -\frac{\pi}{2}(\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3})$.*

12 *Exam question June 2002 (Section A): Give the unit vector normal to the surface of equation $x^2/a^2 + y^2/b^2 + z^2/c^2 = 4$ where a, b, c are three real constants.*
What is the unit vector normal to a sphere of radius 2 at the point $(x, y, z) = (\sqrt{2}, 0, \sqrt{2})$?

Solution: $\nabla f(\mathbf{x})$ is orthogonal to the level surface $f = const.$ at the point $\mathbf{x}$, so take $f = x^2/a^2 + y^2/b^2 + z^2/c^2$, then $\nabla f(\mathbf{x}) = \mathbf{e_1}2x/a^2 + \mathbf{e_2}2y/b^2 + \mathbf{e_3}2z/c^2$ is normal to the surface at $\mathbf{x}$. A unit vector normal to the surface is therefore $\mathbf{n} \equiv \nabla f(\mathbf{x})/|\nabla f(\mathbf{x})| = (\mathbf{e_1}x/a^2 + \mathbf{e_2}y/b^2 + \mathbf{e_3}z/c^2)/\sqrt{x^2/a^4 + y^2/b^4 + z^2/c^4}$
When $a = b = c = 1$ the ellipsoid in the first part of the question becomes a sphere of radius 2, so substituting this and $(x, y, z) = (\sqrt{2}, 0, \sqrt{2})$ into $\mathbf{n}$ gives $(\mathbf{e_1}\sqrt{2} + \mathbf{e_3}\sqrt{2})/2$, which is a unit vector along the radial direction at $(x, y, z) = (\sqrt{2}, 0, \sqrt{2})$, as it should be.

13 *Find the vector equations of tangent and normal lines in $\mathbb{R}^2$ to the following curves at the given points*

(a) $x^2 + 2y^2 = 3$ *at* $(1, 1)$,

Solution: *Set* $f(x, y) = x^2 + 2y^2$ *so the curve is the level set* $f = 3$. $\nabla f = 2x\mathbf{e_1} + 4y\mathbf{e_2}$ *is orthogonal to this. At* $(1, 1)$ $\nabla f = 2\mathbf{e_1} + 4\mathbf{e_2}$*. The line through* $(1, 1)$ *parallel to* $\mathbf{e_1}+2\mathbf{e_2}$ *has vector parametric equation* $\mathbf{x} = \mathbf{e_1}+\mathbf{e_2}+t(\mathbf{e_1}+2\mathbf{e_2})$*, this is the normal. The line through* $(1, 1)$ *orthogonal to* $\mathbf{e_1} + 2\mathbf{e_2}$*, i.e. parallel to* $2\mathbf{e_1} - \mathbf{e_2}$*, has vector parametric equation* $\mathbf{x} = \mathbf{e_1} + \mathbf{e_2} + t(2\mathbf{e_1} - \mathbf{e_2})$*, this is the tangent.*

(b) $xy = 1$ *at* $(2, 1/2)$,

Solution: *This time, set* $f(x, y) = xy$ *so the curve is the level set* $f = 1$. $\nabla f = y\underline{e}_1 + x\underline{e}_2$*, which is equal to* $1/2\,\underline{e}_1 + 2\,\underline{e}_2$ *at* $(2, 1/2)$*. The normal line can therefore be written in vector form as* $\underline{x} = 2\,\underline{e}_1 + 1/2\,\underline{e}_2 + t(1/2\,\underline{e}_1 + 2\,\underline{e}_2)$*. Picking a vector orthogonal to* ∇f*, say* $2\,\underline{e}_1 - 1/2\,\underline{e}_2$*, the tangent line can be written as* $\underline{x} = 2\,\underline{e}_1 + 1/2\,\underline{e}_2 + t(2\,\underline{e}_1 - 1/2\,\underline{e}_2)$*.*

(c) $x^2 - y^3 = 3$ *at* $(2, 1)$.

Solution: *Now* $f(x, y) = x^2 - y^3$*, the relevant level set is* $f = 3$*, and* $\nabla f = 2x\underline{e}_1 - 3y^2\underline{e}_2$*. At* $(2, 1)$ *this is* $4\,\underline{e}_1 - 3\,\underline{e}_2$ *and so an equation for the normal is* $\underline{x} = 2\,\underline{e}_1 + \underline{e}_2 + t(4\,\underline{e}_1 - 3\,\underline{e}_2)$*, and for the tangent,* $\underline{x} = 2\,\underline{e}_1 + \underline{e}_2 + t(3\,\underline{e}_1 + 4\,\underline{e}_2)$*.*

14 *Exam question June 2003 (Section A): Find the directional derivative of the function* $\phi(x, y, z) = xy^2z^3$ *at the point* $P = (1, 1, 1)$ *in the direction from P towards* $Q = (3, 1, -1)$*. Starting from P, in which direction is the directional derivative maximum and what is the value of this maximum?*

Solution: The directional derivative of ϕ at P in the direction from P towards $\mathbf{Q} = (3, 1, -1)$ is $\mathbf{n} \cdot \nabla\phi(\mathbf{P})$ where $\mathbf{n}$ is a unit vector in this direction, i.e. $\mathbf{n} = (\mathbf{Q}-\mathbf{P})/|\mathbf{Q}-\mathbf{P}|$. Now $\nabla\phi = \mathbf{e_1}y^2z^3 + \mathbf{e_2}2xyz^3 + \mathbf{e_3}3xy^2z^2$, so $\nabla\phi(\mathbf{P}) = \mathbf{e_1} + \mathbf{e_2}2 + \mathbf{e_3}3$, and $\mathbf{n} = (\mathbf{e_1}2 - \mathbf{e_3}2)/\sqrt{8} = (\mathbf{e_1} - \mathbf{e_3})/\sqrt{2}$ so the required directional derivative is $(\mathbf{e_1} + \mathbf{e_2}2 + \mathbf{e_3}3) \cdot (\mathbf{e_1} - \mathbf{e_3})/\sqrt{2}$ which equals $-\sqrt{2}$. The directional derivative is a maximum in the direction of $\mathbf{e_1} + \mathbf{e_2}2 + \mathbf{e_3}3$, i.e. parallel to $\mathbf{e_1} + \mathbf{e_2}2 + \mathbf{e_3}3$, and its value then is $|\nabla\phi| = \sqrt{1 + 4 + 9} = \sqrt{14}$.

15 *Exam question June 2002 (Section A): What is the derivative of the scalar function* $\phi(x, y, z) = x\cos z - y$ *in the direction* $\mathbf{V} = \mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}$*? What is the gradient at the point* $(x, y, z) = (0, 1, \pi/2)$*? In which direction is* ϕ *increasing the most when moving away from this point?*

Solution: $\nabla\phi(x, y, z) = \mathbf{e_1}\cos z - \mathbf{e_2} - \mathbf{e_3}x\sin z$, so the derivative in the direction of $\mathbf{V}$ is $|\mathbf{V}|^{-1}\mathbf{V} \cdot \nabla\phi(x, y, z) = \sqrt{3}^{-1}(\cos z - 1 - x\sin z)$. At $(x, y, z) = (0, 1, \pi/2)$ the gradient is $\nabla\phi(x, y, z) = -\mathbf{e_2}$. ϕ increases the most when moving in the direction of $\nabla\phi(x, y, z) = -\mathbf{e_2}$ away from this point.

16 *A marble is released from the point* $(1, 1, c - a - b)$ *on the elliptic paraboloid defined by* $z = c - ax^2 - by^2$*, where* a, b, c *are positive real numbers and the* z*-coordinate is vertical. In which direction in the* (x, y) *plane does the marble begin to roll?*

Solution: Here $z = f(x, y)$ is the height of the marble, and this decreases the fastest in the direction of $-\nabla f = 2ax\mathbf{e_1} + 2by\mathbf{e_2} = 2a\mathbf{e_1} + 2b\mathbf{e_2}$ at $(1, 1, c - a - b)$.

17 *In which direction does the function $f(x, y) = x^2 - y^2$ increase fastest at the points (a) $(1, 0)$, (b) $(-1, 0)$, (c) $(2, 1)$? Illustrate with a sketch.*

Solution: f increases the fastest in the direction of its gradient $\nabla f = \mathbf{e_1}2x - \mathbf{e_2}2y$. At (a) $(1, 0)$, $\nabla f = 2\mathbf{e_1}$, a unit vector in this direction is $\mathbf{e_1}$, (b) $(-1, 0)$, $\nabla f = -2\mathbf{e_1}$, a unit vector in this direction is $-\mathbf{e_1}$, (c) $(2, 1)$, $\nabla f = 4\mathbf{e_1} - 2\mathbf{e_2}$ a unit vector in this direction is $(2\mathbf{e_1} - \mathbf{e_2})/\sqrt{5}$.

18 *Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$.*

(a) In which direction is the directional derivative of f at $(1, 1)$ equal to zero?

Solution: *We have $f(x, y) = 1 - 2y^2/(x^2 + y^2) = 2x^2/(x^2 + y^2) - 1$ so*

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(1 - 2y^2/(x^2 + y^2)) = 4xy^2/(x^2 + y^2)^2$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2x^2/(x^2 + y^2) - 1) = -4x^2y/(x^2 + y^2)^2$$

So at $(1, 1)$ $\nabla f = \mathbf{e_1} - \mathbf{e_2}$. The directional derivative in the direction of the unit vector $\mathbf{n}$ is $\mathbf{n} \cdot \nabla f$, which vanishes when $\mathbf{n}$ and ∇f are perpendicular, i.e. when $\mathbf{n} = \pm(\mathbf{e_1} + \mathbf{e_2})/\sqrt{2}$.

(b) What about at an arbitrary point (x_0, y_0) in the first quadrant?

Solution: *At (x_0, y_0) $\nabla f = 4x_0y_0(y_0\mathbf{e_1} - x_0\mathbf{e_2})/(x_0^2 + y_0^2)^2$ which is perpendicular to $\mathbf{n} = \pm(x_0\mathbf{e_1} + y_0\mathbf{e_2})/\sqrt{x_0^2 + y_0^2}$*

(c) Describe the level curves of f and discuss them in the light of the result in (b).

Solution: *The level curves are orthogonal to ∇f, and so tangent to $\mathbf{n}$. They are thus straight lines through the origin.*