

51 Compute the differential, or Jacobian matrix, and the Jacobian of the function $\underline{V} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $\underline{V}(x, y) = (x \cos y, x \sin y)$. State where \underline{V} defines an orientation preserving local diffeomorphism, and where it defines an orientation reversing local diffeomorphism.

52 Repeat question 51 for $\underline{V}(x, y) = (e^x \cos y, e^x \sin y)$.

53 Calculate the differential, or Jacobian matrix, and the Jacobian of the following transformations:

(a) $\underline{U}(u, v) = (x(u, v), y(u, v))$ where $x(u, v) = \frac{1}{2}(u + v)$ and $y(u, v) = \frac{1}{2}(u - v)$;

(b) $\underline{V}(r, \theta) = (x(r, \theta), y(r, \theta))$ where $x(r, \theta) = r \cos \theta$ and $y(r, \theta) = r \sin \theta$;

(c) $\underline{W}(r, \theta, \phi) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$.

54 Adapted from exam question 2009 (Section B) Q7:

(a) Let $\underline{V} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a vector field. Give the definition of \underline{V} being differentiable at a point \underline{a} .

(b) Let $\underline{V}(x)$ and $\underline{W}(x)$ be two differentiable vector fields in \mathbb{R}^2 . Give formulae for the two differentials $D\underline{V}_{\underline{x}}$ and $D\underline{W}_{\underline{x}}$.

(c) Use the chain rule to show that the differential of the composite map $\underline{U}(\underline{x}) := \underline{V}(\underline{W})$ satisfies

$$D\underline{U}_{\underline{x}} = D\underline{V}_{\underline{W}} D\underline{W}_{\underline{x}}.$$

55 Adapted from exam question 2018 (Section B) Q8:

(a) Given a vector field $\underline{u}(\underline{x}) = \underline{w}(\underline{v}(\underline{x}))$, use the chain rule to show that $D\underline{u}(\underline{x}) = D\underline{w}(\underline{v}) D\underline{v}(\underline{x})$, and hence $J(\underline{u}) = J(\underline{w})J(\underline{v})$.

(b) Let

$$\underline{v}(\underline{x}) = (v_1, v_2) = (\cos y, \sin x)$$

$$\underline{w}(\underline{x}) = (w_1, w_2) = (x^2 - y^2, 2xy),$$

and define $\underline{u}(\underline{x}) = \underline{w}(\underline{v}(\underline{x}))$. Use the result from part (a) to calculate $J(\underline{u})$. Verify your answer by direct substitution.