5 Double integrals

5.1 Rectangular regions

Recall that for a function $f(x)$ the definite integral $\int_a^b f(x)\,dx$ is the signed area under the curve $y = f(x)$ between $x = a$ and $x = b$. This interpretation of the definite integral can be generalized for a function of two variables to a volume under a surface.
Given a function $f(x, y)$ and a region D in the (x, y) plane, the double integral

$$\iint_D f(x, y)\,dxdy$$

is the signed volume of the region between the surface $z = f(x, y)$ and the region D in the $z = 0$ plane (see Figure 29).

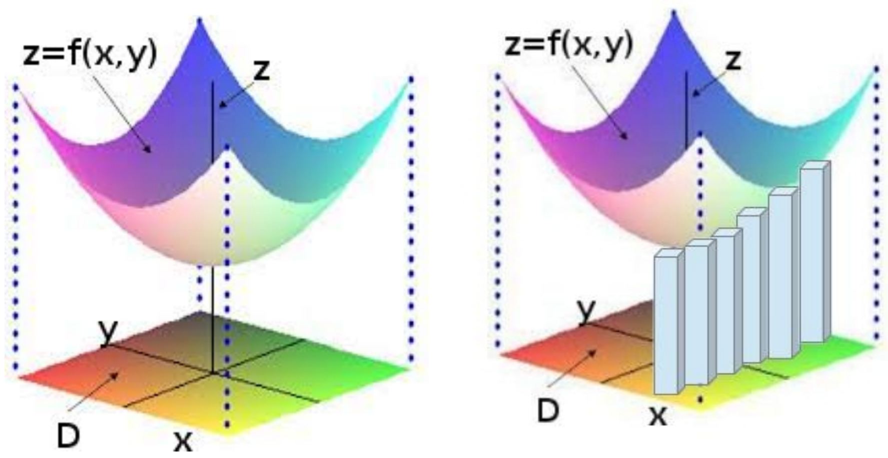

Figure 29: The double integral is the signed volume of the region between the surface $z = f(x, y)$ and the region D in the $z = 0$ plane. It is obtained as the limit of a Riemann sum of cuboid volumes.

In a similar manner to our earlier construction of the definite integral, in terms of the limit of a Riemann sum of areas of rectangles, we can define the double integral as the limit of a Riemann sum of volumes of cuboids.
For simplicity, consider the case of a rectangular region $D = [a_0, a_1] \times [b_0, b_1]$. We construct a subdivision S of D by defining the points of a two-dimensional lattice as $a_0 = x_0 < x_1 < \ldots < x_n = a_1$ and $b_0 = y_0 < y_1 < \ldots < y_m = b_1$. The edge lengths of the lattice are equal to $dx_i = x_i - x_{i-1}$ for $i = 1, \ldots, n$ and $dy_j = y_j - y_{j-1}$ for $j = 1, \ldots, m$. The norm of the subdivision $|S|$ is given by the maximum of all the dx_i and dy_j.
We introduce sample points $(p_i, q_j) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ inside each rectangle of the lattice. The area of the rectangle is $dx_i dy_j$ and we associate to this rectangle a cuboid of height $f(p_i, q_j)$.

The volume of all the cuboids is the Riemann sum

$$\mathcal{R} = \sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i, q_j) dx_i dy_j$$

and the double integral is the limit

$$\iint\limits_D f(x,y)\, dxdy = \lim_{|S| \to 0} \mathcal{R}.$$

In the special case that $f(x,y)$ is a constant, say $f(x,y) = c$, then $\iint\limits_D f(x,y)\, dxdy = c \times \mathrm{area}(D)$.

For a rectangular region $D = [a_0, a_1] \times [b_0, b_1]$ a practical way to compute the double integral is to use the following iterated integral result (valid providing f is continuous on D)

$$\iint\limits_D f(x,y)\, dxdy = \int_{a_0}^{a_1} \left(\int_{b_0}^{b_1} f(x,y)\, dy \right) dx = \int_{b_0}^{b_1} \left(\int_{a_0}^{a_1} f(x,y)\, dx \right) dy.$$

Eg. Given the region $D = [-2, 1] \times [0, 1]$, calculate $\iint\limits_D (x^2 + y^2)\, dxdy$.

$$\iint\limits_D (x^2 + y^2)\, dxdy = \int_{-2}^{1} \left(\int_{0}^{1} (x^2 + y^2)\, dy \right) dx = \int_{-2}^{1} \left(\left[yx^2 + \frac{1}{3} y^3 \right]_{y=0}^{y=1} \right) dx$$

$$= \int_{-2}^{1} (x^2 + \frac{1}{3})\, dx = \left[\frac{x^3}{3} + \frac{x}{3} \right]_{-2}^{1} = \frac{1}{3}(1 + 1 + 8 + 2) = 4.$$

As an exercise, check that the same result is obtained if the integration is performed in the other order.

5.2 Beyond rectangular regions

To compute double integrals beyond rectangular regions we need to consider the following definition.

Defn: A region D of the plane is called $\mathbf{y-simple}$ if every line that is parallel to the y-axis and intersects D, does so in a single line segment (or a single point if this is on the boundary of D).

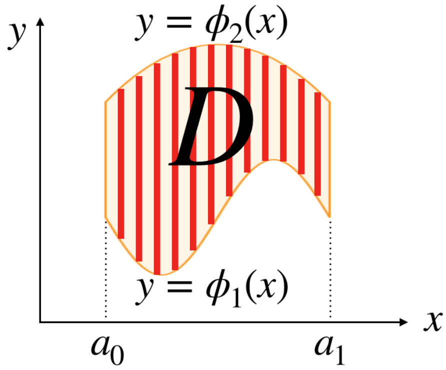

Figure 30: A y-simple region.

See Figure 30 for an example of a y-simple region. The boundaries of this region are given by the curves $y = \phi_1(x)$ and $y = \phi_2(x)$ for $a_0 \leq x \leq a_1$.

The double integral can be calculated as an iterated integral by integrating over y first

$$\iint_D f(x, y)\, dxdy = \int_{a_0}^{a_1} \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy \right) dx.$$

Eg. Sketch the region D that lies between the curves $y = x$ and $y = x^2$ for $0 \leq x \leq 1$ and calculate $\iint\limits_D 6xy\, dxdy$.

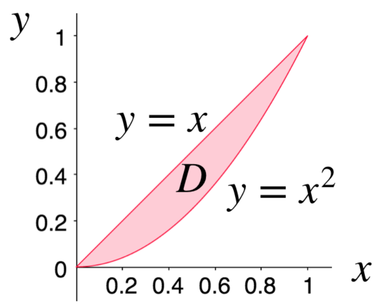

Figure 31: A sketch of the region D in the example.

As the region D is y-simple then

$$\iint\limits_D 6xy\, dxdy = \int_0^1 \left(\int_{x^2}^{x} 6xy\, dy \right) dx = \int_0^1 \Big[3xy^2 \Big]_{y=x^2}^{y=x} dx = 3 \int_0^1 (x^3 - x^5)\, dx$$

$$= 3 \left[\frac{x^4}{4} - \frac{x^6}{6} \right]_0^1 = 3 \left(\frac{1}{4} - \frac{1}{6} \right) = \frac{1}{4}.$$

There is a similar definition of an x-simple region.

Defn: A region D of the plane is called $\mathbf{x - simple}$ if every line that is parallel to the x-axis and intersects D, does so in a single line segment (or a single point if this is on the boundary of D).

Figure 32 shows an example of an x-simple region that is not y-simple.

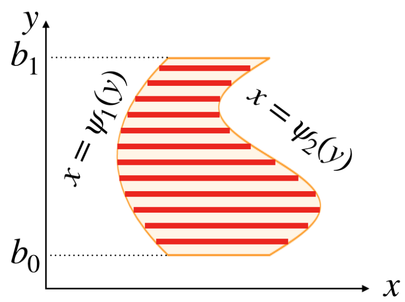

Figure 32: A region that is x-simple but is not y-simple.

The boundaries of this region are the curves $x = \psi_1(y)$ and $x = \psi_2(y)$ for $b_0 \leq y \leq b_1$. The double integral can be calculated as an iterated integral by integrating over x first

$$\iint_D f(x,y)\, dxdy = \int_{b_0}^{b_1} \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx \right) dy.$$

A region can be both x-simple and y-simple, in which case the double integral can be calculated as an iterated integral by integrating over x first or y first. The region in the previous example is x-simple, so we can recalculate this double integral using this property.

Same Eg. D is the region between the curves $y = x$ and $y = x^2$ for $0 \leq x \leq 1$. Use the fact that this region is x-simple to calculate $\iint\limits_D 6xy\, dxdy$.

The boundaries of D are given by the curves $x = y$ and $x = \sqrt{y}$ for $0 \leq y \leq 1$. Note from Figure 31 that moving along a line (that passes through D) parallel to the x-axis with x increasing, intersects the curve $x = y$ before it intersects the curve $x = \sqrt{y}$. Thus y is the lower limit in the x integration and $\sqrt{y}$ is the upper limit.

$$\iint_D 6xy\, dxdy = \int_0^1 \left(\int_y^{\sqrt{y}} 6xy\, dx \right) dy = \int_0^1 \left[3x^2 y \right]_{x=y}^{x=\sqrt{y}} dy = 3 \int_0^1 (y^2 - y^3)\, dy$$

$$= 3\left[\frac{y^3}{3} - \frac{y^4}{4} \right]_0^1 = 3\left(\frac{1}{3} - \frac{1}{4} \right) = \frac{1}{4}.$$

Sometimes a region may be both x-simple and y-simple but we can only perform the integration explicitly for one of the ordering choices ie. either integrating over x first or y first.

Eg. The boundaries of D are given by the curves $y = x$ and $y = \sqrt{x}$ for $0 \leq x \leq 1$.
Calculate $\iint\limits_D \frac{e^y}{y}\, dxdy$.
If we use the fact that D is y-simple then we have

$$\iint_D \frac{e^y}{y}\, dxdy = \int_0^1 \left(\int_x^{\sqrt{x}} \frac{e^y}{y}\, dy \right) dx,$$

but we can't do the integration. However, if we use the fact that D is x-simple and the bounding curves are $x = y$ and $x = y^2$ then

$$\iint_D \frac{e^y}{y}\, dxdy = \int_0^1 \left(\int_{y^2}^{y} \frac{e^y}{y}\, dx \right) dy = \int_0^1 \left[\frac{xe^y}{y} \right]_{x=y^2}^{x=y} dy = \int_0^1 (e^y - ye^y)\, dy$$

$$= [e^y]_0^1 - [ye^y]_0^1 + \int_0^1 e^y\, dy = e - 2.$$

5.3 Integration using polar coordinates

Sometimes it may be useful to describe the region of integration D in terms of polar coordinates r and θ, where $x = r\cos\theta$ and $y = r\sin\theta$. To convert a double integral into polar coordinates we need to know what to do with the area element $dA = dxdy$. Recall that this area element came from the definition of the integral in terms of a division of D into small rectangles. The area of a small rectangle with side lengths dx and dy is $dA = dxdy$ which is the area element in the infinitesimal limit.

We therefore need to know the area dA of a small region obtained by taking the point with polar cooridnates (r, θ) and extending r by dr and θ by $d\theta$, as shown in Figure 33. We see that the area is approximately a rectangle with area $dA \approx rd\theta dr$ and this approximation improves as the area decreases. In the infinitesimal limit that defines the integral the result is $dA = dxdy = rd\theta dr$.

We now know how to convert a double integral to polar coordinates, we replace $dxdy$ with $rd\theta dr$. Note the crucial factor of r here. We can then calculate the double integral as an iterated integral if the region D is θ-simple or r-simple (with the obvious definitions of these terms).

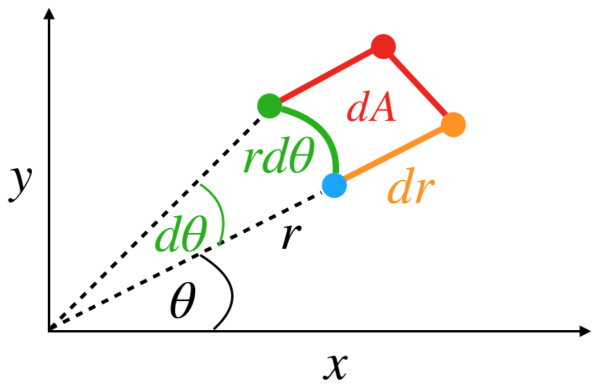

Figure 33: The area element in polar coordinates.

Eg. Let D be the region between the curves $x^2+y^2=1$ and $x^2+y^2=4$ satisfying $x \geq 0$ and $y \geq 0$. Calculate $\iint\limits_D xy\,dxdy$.

In polar coordinates the region D is given by $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$.
The area element is $dxdy = rd\theta dr$ and the integrand is $xy = r^2 \cos\theta \sin\theta$.

$$\iint\limits_D xy\,dxdy = \iint\limits_D r^2 \sin\theta \cos\theta\, r dr d\theta = \int_1^2 \left(\int_0^{\pi/2} r^3 \sin\theta \cos\theta\, d\theta \right) dr$$

$$= \int_1^2 \left(\int_0^{\pi/2} \frac{1}{2} \sin(2\theta)\, d\theta \right) r^3\, dr = \int_1^2 \left[-\frac{1}{4} \cos(2\theta) \right]_{\theta=0}^{\theta=\pi/2} r^3\, dr = \int_1^2 \frac{1}{2} r^3\, dr$$

$$= \left[\frac{1}{8} r^4 \right]_1^2 = \frac{15}{8}.$$

As an exercise, you can check that you obtain the same result if you perform the calculation without changing to polar coordinates.

5.4 Change of variables and the Jacobian

Using polar coordinates r, θ rather than Cartesian coordiantes x, y is a particular example of a change of variables. In general we might want to change variables from x, y to new variables u, v given by relations of the form $x = g(u, v)$ and $y = h(u, v)$ for some functions g and h. Not only do we need to know what the integration region D looks like in the new variables, but we also need the area element $dxdy$ in the new variables. The following definition plays the central role in this issue.

Defn. The **Jacobian** of the transformation from the variables x, y to u, v is

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

ie. it is the determinant of the 2×2 matrix of partial derivatives.

To obtain the area element $dxdy$ in terms of the new variables we first compute the Jacobian J and then use the result that

$$dxdy = |J| dudv$$

where $|J|$ is the absolute value of the Jacobian.

As an example, we can rederive the area element in polar coordinates. We have $x = r\cos\theta$ and $y = r\sin\theta$, so in this case r, θ are the new variables, playing the role of u, v in the general setting.

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

giving $dxdy = |J| drd\theta = rdrd\theta$ as before.

Eg. Let D be the square in the (x, y) plane with vertices $(0,0), (1,1), (2,0), (1,-1)$. Calculate $\iint\limits_D (x+y)\,dxdy$ by making the change of variables $x = u + v$ and $y = u - v$.

First we need to determine the region D in terms of the variables $[u, v]$. Using $u = (x+y)/2$ and $v = (x-y)/2$ we find that the vertices map to the $[u, v]$ coordinates $[0,0], [1,0], [1,1], [0,1]$. The edges of the square lie along the lines $y = x,\ y = -x,\ y = 2 - x,\ y = x - 2$ which map to the lines $v = 0,\ u = 0,\ u = 1,\ v = 1$. Therefore in the $[u, v]$ plane D is again a square, with the vertices given above.
The Jacobian of the transformation is

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$$

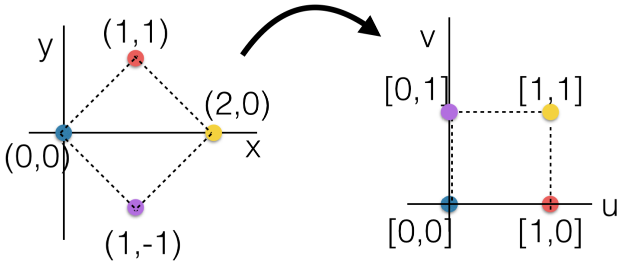

Figure 34: A square in the (x, y) plane is transformed into a square in the $[u, v]$ plane.

Thus $dxdy = |J|dudv = 2dudv$.
Finally, the integrand is $x + y = 2u$. Putting all this together we get

$$\iint_D (x+y)\,dxdy = \iint_D 4u\,dudv = \int_0^1 \left(\int_0^1 4u\,dv \right) du$$

$$= \int_0^1 \Big[4uv \Big]_{v=0}^{v=1} du = \int_0^1 4u\,du = \Big[2u^2 \Big]_0^1 = 2.$$

5.5 The Gaussian integral

An important integral that appears in a wide range of mathematical contexts is the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$$

where a is a positive constant.

We can derive this result using a trick that involves calculating a double integral, as follows. We define I to be the required integral

$$I = \int_{-\infty}^{\infty} e^{-ax^2}\,dx$$

and observe that

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-ax^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-ay^2}\,dy\right) = \iint\limits_{\mathbb{R}^2} e^{-a(x^2+y^2)}\,dxdy.$$

We now use polar coordinates to evaluate this double integral

$$I^2 = \iint\limits_{\mathbb{R}^2} e^{-ar^2} r\,drd\theta = \int_0^{\infty}\left(\int_0^{2\pi} re^{-ar^2}\,d\theta\right)dr = 2\pi\int_0^{\infty} re^{-ar^2}\,dr$$

$$= -2\pi\left[\frac{e^{-ar^2}}{2a}\right]_0^{\infty} = \frac{\pi}{a}.$$

Hence $I = \sqrt{\pi/a}$, which is the required result.