2 Limits and continuity

2.1 Definition of a limit and continuity

Rough idea #1:
A function $f(x)$ has a **limit** L at $x = a$ if $f(x)$ is close to L whenever x is close to a.

Rough idea #2: Lets think about the function $f(x)$ near $x = a$.
If you approximate $f(x)$ by a constant L then you will make an error given by $|f(x) - L|$. Suppose $\varepsilon > 0$ is the εrror at which you become unhappy with your approximation, that is, you are happy as long as $|f(x) - L| < \varepsilon$. The important issue for a limit is whether I can keep you happy simply by making sure that x stays within a δistance $\delta > 0$ of a, that is by restricting to $0 < |x - a| < \delta$. Note that we dont even care about what happens exactly at $x = a$. If I can always find a δistance δ that keeps you happy, no matter how small you choose your εrror ε, then we say that $f(x)$ has a **limit** L as x tends to a.

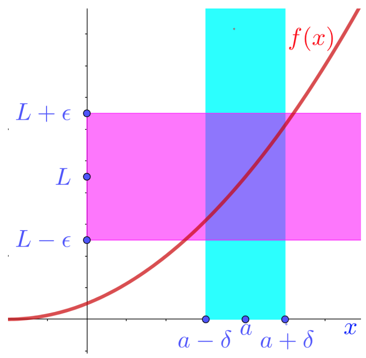

Figure 10: The idea of a limit.

Defn: $f(x)$ has a **limit** L as x tends to a if

$$\forall \ \varepsilon > 0 \ \exists \ \delta > 0 \ \text{s.t} \ |f(x) - L| < \varepsilon \ \text{ when } \ 0 < |x - a| < \delta.$$

We then write

$$\lim_{x \to a} f(x) = L \quad \text{or equivalently} \quad f(x) \to L \ \text{ as } \ x \to a.$$

If there is no such L then we say that *no limit exists*.

The limit does not require that $f(a)$ is equal to L or even that $f(a)$ exists. This is because of the inequality $0 < |x - a|$ in the definition.

There is a lot more of this ε and δ stuff in Analysis I, where proofs concerning limits are considered. In this course we shall be less formal and deal only with methods to calculate limits or see that no limit exists.

Rough idea #3:
If $f(a)$ exists then it would seem that this is a good candidate for the limit L. This naive view turns out to be correct only if the graph of $f(x)$ is a single unbroken curve with no holes or jumps, at least in a small region around $x = a$.

Defn. A function $f(x)$ is **continuous at the point** $x = a$ if the following three properties all hold:
- $f(a)$ exists.
- $\lim_{x \to a} f(x)$ exists.
- $\lim_{x \to a} f(x)$ is equal to $f(a)$.

Defn. A function $f(x)$ is **continuous on a subset** S of its domain if it is continuous at every point in S.

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Eg. $f(x) = x^2$ is continuous and $\lim_{x \to 2} x^2 = f(2) = 4$.

Eg. The following function has a discontinuity at $x = 0$

$$f(x) = \begin{cases} x^2 & \text{if } -1 \le x < 0 \\ 1 & \text{if } x = 0 \\ x^2 & \text{if } 0 < x \le 1 \end{cases}$$

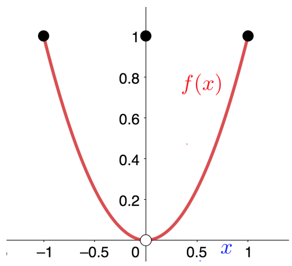

Figure 11: A graph illustrating the use of open and closed circles. The limit exists at $x = 0$.

As mentioned earlier, to indicate on a graph exactly which points are included we use a closed circle to denote an included point, so if it is at the end of a curve segment this denotes that this end is a closed interval. We use an open circle to denote an excluded point associated with an open interval. This notation is demonstrated in Figure 11 where the graph of the above function is presented. This function is not continuous at $x = 0$ but the limit exists at this point and

$\lim_{x \to 0} f(x) = 0 \neq 1 = f(0)$.

Eg. The following function also has a discontinuity at $x = 0$ (see Figure 12 for its graph)

$$f(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

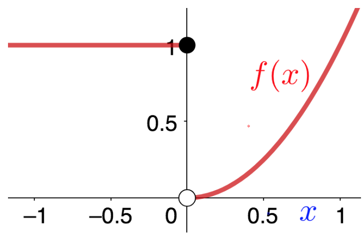

Figure 12: A graph illustrating the use of open and closed circles. No limit exists at $x = 0$.

In this case no limit exists at $x = 0$. This is clear because if we consider $x < 0$ then to get arbitrarily close to the function for x arbitrarily close to 0 would require the limit to be 1. However, if we consider $x > 0$ then to get arbitrarily close to the function for x arbitrarily close to 0 would require the limit to be 0. These two requirements are incompatible and so no limit exists at $x = 0$.

Eg. For the function $f(x) = \sin(1/x)$ no limit exists at $x = 0$ because as x approaches zero this function oscillates between 1 and -1 over smaller and smaller intervals. In particular, in any given interval $0 < x < \delta$ it is always possible to find values x_1 and x_2 s.t. $f(x_1) = 1$ and $f(x_2) = -1$, thus $f(x)$ cannot remain close to any given constant L.

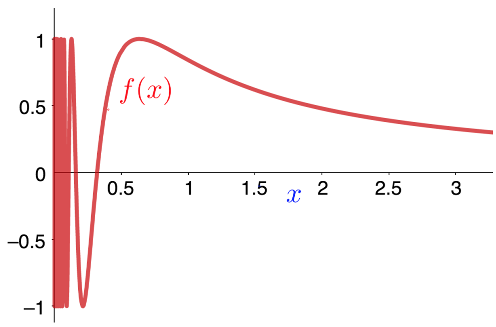

Figure 13: A graph of the function $\sin(1/x)$.

2.2 Facts about limits and continuity

There are some simple **facts about limits** that follow from the definition:

• The limit is unique.

• If $f(x) = g(x)$ (except possibly at $x = a$) on some open interval containing a then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

• If $f(x) \geq K$ on either an interval (a, b) or an interval (c, a)
and if $\lim_{x\to a} f(x) = L$ then $L \geq K$.
(A similar result holds by replacing all $\geq$ with $\leq$).

• If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then
(i) $\lim_{x\to a}(f(x) + g(x)) = L + M$
(ii) $\lim_{x\to a}(f(x)g(x)) = LM$
(iii) if $M \neq 0$ then $\lim_{x\to a}(f(x)/g(x)) = L/M$.
Note: In Durham the results (i),(ii),(iii) are sometimes known as the
Calculus Of Limits Theorem (COLT). However, this seems to be a rather grand name for these results and you will not find this name in any books.

There are also some simple **facts about continuous functions** that follow from the definition and the above results:

• If $f(x)$ and $g(x)$ are continuous functions then so are $f(x) + g(x)$, $f(x)g(x)$ and $f(x)/g(x)$.

• All polynomial, rational, trigonometric and hyperbolic functions are continuous.

• If $f(x)$ is continuous then so is $|f(x)|$.

• If $\lim_{x\to a} g(x) = L$ exists and $f(x)$ is continuous at $x = L$ then $\lim_{x\to a}(f \circ g)(x) = f(L)$.

Eg. All the following functions are continuous:
$2x^3 + x + 7, \quad 3x/(x - 1), \quad |(1 + x^2)/\sin x|, \quad \tan x.$

Note: Although $\tan x$ is continuous, it is not continuous on the interval $[0, \pi]$, because this interval includes the point $x = \pi/2$ which is not in the domain of $\tan x$.

Eg. $\lim_{x\to\pi/2} x^2 \sin x = (\pi/2)^2 \sin(\pi/2) = \pi^2/4$.

Eg. $\lim_{x\to 0} \frac{1}{x}$ does not exist. The value of $f(x)$ can be made greater than any finite constant by taking x sufficiently close to zero, so no limit exists.

Eg. Calculate $\lim_{x \to 3} \frac{2x^2-18}{x-3}$.

$$\frac{2x^2-18}{x-3} = \frac{2(x+3)(x-3)}{x-3} = 2(x+3) \text{ if } x \neq 3.$$

The value of the function at $x = 3$ is irrelevant in defining the limit as $x \to 3$ so

$$\lim_{x \to 3} \frac{2x^2-18}{x-3} = \lim_{x \to 3} 2(x+3) = 12.$$

Eg. Calculate $\lim_{x \to 25} \frac{\sqrt{x}-5}{x-25}$.

$$\frac{\sqrt{x}-5}{x-25} = \frac{(\sqrt{x}-5)(\sqrt{x}+5)}{(x-25)(\sqrt{x}+5)} = \frac{x-25}{(x-25)(\sqrt{x}+5)} = \frac{1}{\sqrt{x}+5} \text{ if } x \neq 25.$$

$$\lim_{x \to 25} \frac{\sqrt{x}-5}{x-25} = \lim_{x \to 25} \frac{1}{\sqrt{x}+5} = \frac{1}{10}.$$

2.3 The pinching theorem

The pinching (squeezing) theorem: If $g(x) \le f(x) \le h(x)$ for all $x \ne a$ in some open interval containing a and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} f(x) = L$.

Eg. Calculate $\lim_{x\to 0} x^2 \sin(\frac{1}{x})$.

As $-1 \le \sin(\frac{1}{x}) \le 1$ then $-x^2 \le x^2 \sin(\frac{1}{x}) \le x^2$. Also $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$. Hence by the pinching theorem $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$.

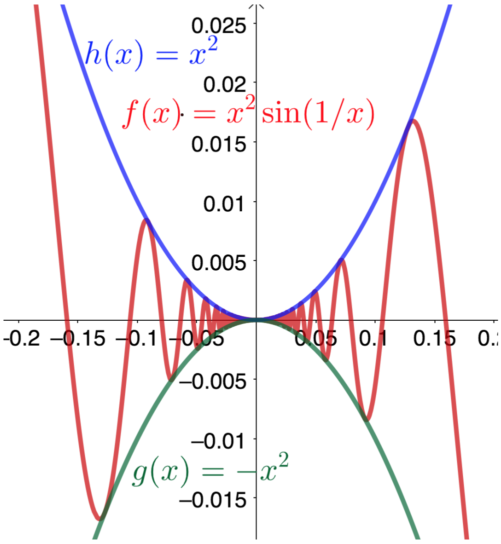

Figure 14: Graphs of $f(x) = x^2 \sin(1/x)$ and the bounding functions $g(x) = -x^2$ and $h(x) = x^2$.

2.4 Two trigonometric limits

Two important trigonometric limits that you may assume to be true and use are:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0}\frac{1-\cos x}{x} = 0.$$

The graphs in Figure 15 provide some evidence to support these results.

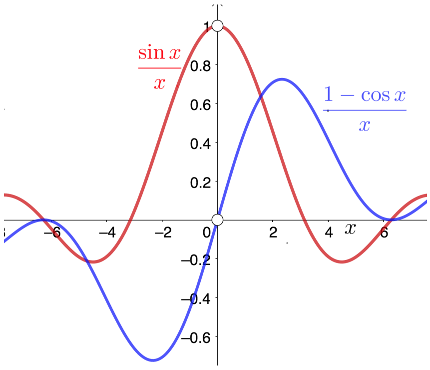

Figure 15: Graphs of $\frac{\sin x}{x}$ and $\frac{1-\cos x}{x}$.

The first of these limits can be proved by applying the pinching theorem but to do this we will need to prove the inequalities $\sin x < x < \tan x$ for $0 < x < \pi/2$.

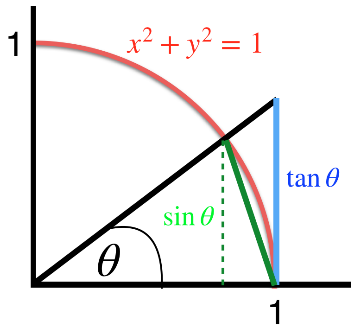

Figure 16: Some geometry to prove some inequalities

Consider the geometry illustrated in the above figure, where $0 < \theta < \pi/2$.
Let T_1 be the area of the triangle with two black sides and a solid green side.
Let S be the area of the sector with two black sides and a red arc of the unit circle.
Let T_2 be the area of the triangle with two black sides and a blue side.
Clearly we have the inequalities $T_1 < S < T_2$.
Now $T_1 = \frac{1}{2}\sin\theta$, $\quad S = \pi\frac{\theta}{2\pi} = \frac{1}{2}\theta$, $\quad T_2 = \frac{1}{2}\tan\theta$ hence $\sin\theta < \theta < \tan\theta$ as required.

Next consider upper and lower bounding functions for $\frac{x}{\sin x}$ where $x \in (0, \pi/2)$. Using the above inequalities for x we have

$$1 = \frac{\sin x}{\sin x} < \frac{x}{\sin x} < \frac{\tan x}{\sin x} = \frac{1}{\cos x}.$$

As all the combinations of functions in this inequality produce even functions then the result extends to $x \in (-\pi/2, 0) \cup (0, \pi/2)$. The limits of the bounding functions are $\lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0}(1/\cos x) = 1$ hence by the pinching theorem $\lim_{x\to 0}(x/\sin x) = 1$ and therefore $\lim_{x\to 0}((\sin x)/x) = 1$.

Now that we have proved the first trigonometric limit we can use it to prove the second,

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \frac{x}{(1+\cos x)} = 0.$$

Here are some more examples that use the first trigonometric limit.

Eg. Calculate $\lim_{x\to 0} \frac{\sin(2x)}{x}$.

$$\lim_{x\to 0} \frac{\sin(2x)}{x} = \lim_{x\to 0} \frac{2\sin(2x)}{2x} = \lim_{u\to 0} \frac{2\sin(u)}{u} = 2 \cdot 1 = 2.$$

The above used the change of variable $u = 2x$ and then the first of the two important trigonometric limits given above.

Eg. Calculate $\lim_{x\to 0} \frac{\tan x}{x}$.

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x\cos x} = \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{\cos x}\right) = 1 \cdot 1 = 1.$$

Eg. Calculate $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.
For $-\pi < x < \pi$

$$\frac{1-\cos x}{x^2} = \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x}$$

Hence by using the first important trigonometric limit we have that

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \left\{ \left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x} \right\} = (1)^2 \cdot \frac{1}{1+1} = \frac{1}{2}.$$

2.5 Classification of discontinuities

If a function $f(x)$ is not continuous at a point $x = a$ then we say it has a discontinuity at $x = a$. There are different types of discontinuity and these are best classified by considering how the function behaves on each side of the point $x = a$. This motivates the definition of the following one-sided limits.

Defn: $f(x)$ has a **right-sided limit** $L^+ = \lim_{x\to a^+} f(x)$ as x tends to a from above if

$$\forall \ \varepsilon > 0 \ \exists \ \delta > 0 \ \text{ s.t } \ |f(x) - L^+| < \varepsilon \ \text{ when } \ 0 < x - a < \delta.$$

Note that the difference between this definition and the definition of the limit is the removal of the modulus sign in $|x - a|$. This means that we only need to worry about points to the right of a when requiring the function to be close to L^+.

Similarly, we have the definition
Defn: $f(x)$ has a **left-sided limit** $L^- = \lim_{x\to a^-} f(x)$ as x tends to a from below if

$$\forall \ \varepsilon > 0 \ \exists \ \delta > 0 \ \text{ s.t } \ |f(x) - L^-| < \varepsilon \ \text{ when } \ 0 < a - x < \delta.$$

Here we only need to worry about points to the left of a when requiring the function to be close to L^-.

It should be obvious that $L = \lim_{x\to a} f(x)$ exists iff L^+ and L^- both exist and are equal. In which case $L = L^+ = L^-$.

There are 3 types of discontinuity:

(i) **Removable discontinuity**.
In this case L exists but $f(a) \neq L$.
The discontinuity can be removed to make the continuous function

$$g(x) = \begin{cases} f(x) & \text{if } \ x \neq a \\ L & \text{if } \ x = a. \end{cases}$$

Eg. The following function (Figure 17i) has a removable discontinuity at $x = 0$,

$$f(x) = \begin{cases} x^2 & \text{if } \ x \neq 0 \\ 1 & \text{if } \ x = 0. \end{cases}$$

Removing this discontinuity yields the continuous function $g(x) = x^2$.

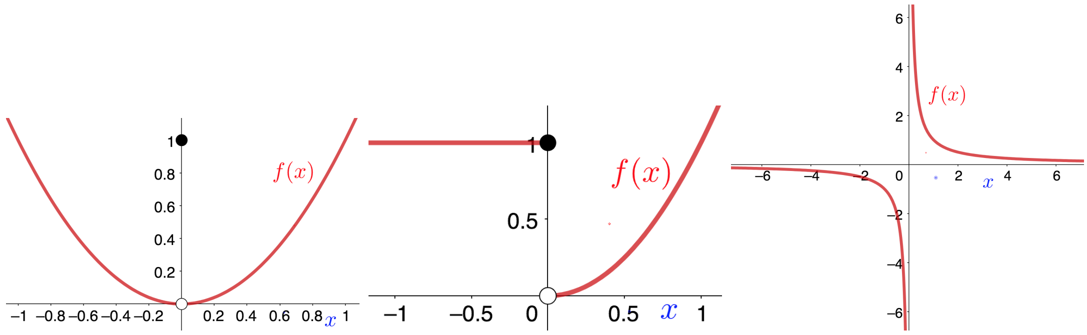

Figure 17: 3 types of discontinuities at $x = 0$: (i) removable; (ii) jump; (iii) infinite.

(ii) **Jump discontinuity.**
In this case both L^+ and L^- exist but $L^+ \neq L^-$.

Eg. The following function (Figure 17ii) has a jump discontinuity at $x = 0$,

$$f(x) = \begin{cases} 1 & \text{if} \ \ x \leq 0 \\ x^2 & \text{if} \ \ x > 0. \end{cases}$$

In this example $L^+ = 0 \neq 1 = L^-$.

Eg. The signum function

$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

has a jump discontinuity at $x = 0$. In this example $L^+ = 1 \neq -1 = L^-$.

(iii) **Infinite discontinuity.**
In this case at least one of L^+ or L^- does not exist.

Eg. The function $f(x) = 1/x$ (Figure 17iii) has an infinite discontinuity at $x = 0$.
In this example neither L^+ nor L^- exist.

2.6 Limits as $x \to \infty$.

So far we have only been concerned with the limit of a function $f(x)$ as x approaches a finite point a. However, it is also possible to define a limit as $x \to \infty$.

Rough idea #4:
A function $f(x)$ has a limit L as $x \to \infty$ if $f(x)$ can be kept arbitrarily close to L by making x sufficiently large.

Defn: $f(x)$ has a **limit** L **as** x **tends to** ∞ if

$$\forall \ \ \varepsilon > 0 \ \ \exists \ \ S > 0 \ \ \text{s.t} \ \ |f(x) - L| < \varepsilon \ \ \text{when} \ \ x > S.$$

We then write

$$\lim_{x\to\infty} f(x) = L \quad \text{or equivalently} \quad f(x) \to L \ \ \text{as} \ \ x \to \infty.$$

If there is no such L then we say that *no limit exists.*

The corresponding definition associated with $\lim_{x\to-\infty} f(x) = L$ should be obvious
Defn: $f(x)$ has a **limit** L **as** x **tends to** $-\infty$ if

$$\forall \ \ \varepsilon > 0 \ \ \exists \ \ S < 0 \ \ \text{s.t} \ \ |f(x) - L| < \varepsilon \ \ \text{when} \ \ x < S.$$

Eg. $\lim_{x\to\infty} \frac{1}{x} = 0$. This should be clear because $\frac{1}{x}$ can be made as close to zero as required by making x sufficiently large. An easy way to calculate limits as $x \to \infty$ is to first make the substitution $u = 1/x$ and then use the fact that if $\lim_{x\to\infty} f(x)$ exists then $\lim_{x\to\infty} f(x) = \lim_{u\to 0^+} f(1/u)$. This has transformed the limit into one that we are already familiar with.
Eg. $\lim_{x\to\infty} \frac{1}{x} = \lim_{u\to 0^+} u = 0$.

Eg. Calculate $\lim_{x\to\infty} \frac{x\cos(1/x)+2}{x}$. By using the substitution $u = 1/x$ we have that

$$\lim_{x\to\infty} \frac{x\cos(1/x)+2}{x} = \lim_{u\to 0^+} \frac{\frac{1}{u}\cos u + 2}{\frac{1}{u}} = \lim_{u\to 0^+} (\cos u + 2u) = \cos(0) + 0 = 1.$$

Our earlier discussion of a horizontal asymptote can now be made more precise, in that the graph of a function $f(x)$ will have a horizontal asymptote $y = L$ if $\lim_{x\to\infty} f(x) = L$.

$$\text{Eg.} \quad \lim_{x\to\infty} \frac{2x+3}{x+5} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{1+\frac{5}{x}} = \frac{2+0}{1+0} = 2.$$

Here we have used the result that $\lim_{x\to\infty} \frac{1}{x} = 0$, together with the earlier facts about limits. We see that calculating this limit shows that the graph of this rational function has the horizontal asymptote $y = 2$, reproducing our earlier observations about the horizontal asymptotes of rational functions.

2.7 The intermediate value theorem

The intermediate value theorem states that if $f(x)$ is continuous on $[a, b]$ and u is any number between $f(a)$ and $f(b)$ (ie. either $f(a) < u < f(b)$ or $f(b) < u < f(a)$) then $\exists$ (at least one) $c \in (a, b)$ s.t. $f(c) = u$.

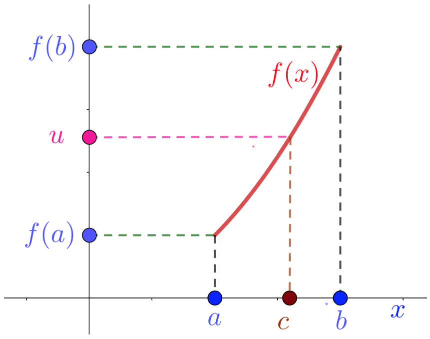

Figure 18: An illustration of the intermediate value theorem.

Eg. $f(x) = \sin x$ is continuous on $[0, \pi/2]$ and $f(0) = 0 < \frac{1}{2} < 1 = f(\pi/2)$ so by the intermediate value theorem there is (at least) one point x in $(0, \pi/2)$ s.t. $\sin x = \frac{1}{2}$.

It is important that $f(x)$ is continuous throughout the interval, otherwise the theorem does not apply.

Eg. $f(x) = \frac{\text{sgn}(x)}{1+x^2}$ has $f(-1) = -\frac{1}{2} < \frac{1}{5} < \frac{1}{2} = f(1)$ but there is no x in $(-1, 1)$ s.t. $f(x) = \frac{1}{5}$. The intermediate value theorem does not apply because $f(x)$ is not continuous at $x = 0$. There is a jump discontinuity at $x = 0$ with $\lim_{x\to 0^+} f(x) = 1 \neq -1 = \lim_{x\to 0^-} f(x)$.

An application of the intermediate value theorem is locating the zeros of a function. If the function $f(x)$ is continuous on $[a, b]$ and we know that either $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$ then by the intermediate value theorem the equation $f(x) = 0$ has at least one root between a and b.

Eg. The function $f(x) = x^2 - 2$ is continuous on $[1, 2]$ with $f(1) = -1 < 0$ and $f(2) = 2 > 0$, so there is at least one root of the equation $x^2 - 2 = 0$ in $(1, 2)$ (of course the root is $x = \sqrt{2} = 1.4142...$).

A repeated iterated application of this approach gives the *bisection method*, which can be used to locate the roots of a wide variety of equations to any desired accuracy.