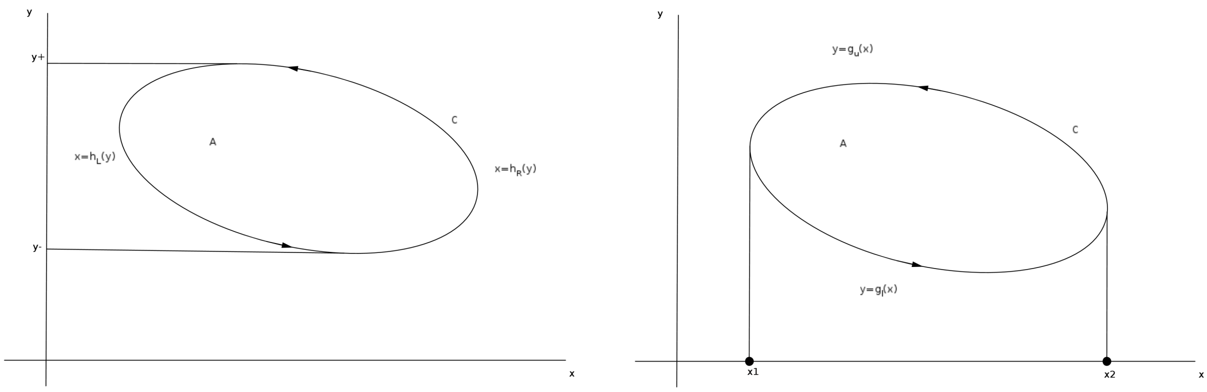
9 Proofs of the three big theorems (Non-Examinable)

In this chapter we return to the 'big three' theorems, and indicate how they can be proved. We'll begin with Green's theorem, and work up from there. Don't worry about memorising these proofs, but instead try to understand how they work. The chapter also includes a number of further examples which you may find useful in your revision.

9.1 Green's theorem in the plane

Recall that the 'coordinate' version of the theorem reads as follows. Suppose $P(x, y)$ and $Q(x, y)$, $(x, y) \in \mathbb{R}^2$, are continuously differentiable scalar fields in 2 dimensions, and suppose that C is a simple closed curve in the x-y plane, traversed anticlockwise and surrounding an area A. Then

$$\boxed{\oint_C (P(x,y)dx + Q(x,y)dy) = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy\,,}$$



(a) (b)

Figure 52: Area of integration, A, with boundary C. Plot (a) shows the curve split into left and right sections $x = h_L(y)$, $x = h_R(y)$, while plot (b) shows the curve split into upper and lower sections $y = g_U(x)$ and $y = g_L(x)$.

Proof of Green's theorem in the plane: We'll make the additional assumption that the area A is both horizontally and vertically simple, meaning that each point in A lies between exactly two boundary points (on C) to its left and right, and also between exactly two boundary points above and below, as illustrated in figure 52. We'll start with the right hand side of the theorem and look at the two terms, which we'll label as ① and ②, separately, remembering that we can interchange the order of integration by Fubini's Theorem, and splitting up the integration region and its bounding curve in the two ways shown in the figure. We have

$$\begin{aligned} RHS &= \int_A \frac{\partial Q}{\partial x}\, dx\, dy - \int_A \frac{\partial P}{\partial y}\, dx\, dy \\ &= \underbrace{\int_A \frac{\partial Q}{\partial x}\, dx\, dy}_{①} \underbrace{- \int_A \frac{\partial P}{\partial y}\, dy\, dx}_{②} \end{aligned}$$

where Fubini's theorem was used to swap the order of integration in the second term in going from the first to the second line. Now evaluating the two terms in turn,

$$
\begin{aligned}
① &= \int_{y-}^{y+} \int_{h_L(y)}^{h_R(y)} \frac{\partial Q}{\partial x} dx\, dy = \int_{y-}^{y+} \Big[Q(x,y) \Big]_{h_L(y)}^{h_R(y)} dy \\
&= \int_{y-}^{y+} \left(Q(h_R, y) - Q(h_L, y) \right) dy \qquad \text{(now split the integral)} \\
&= \int_{y-}^{y+} Q(h_R, y)\, dy + \int_{y+}^{y-} Q(h_L, y)\, dy \quad \text{(note change of limits in 2}^{\text{nd}} \text{ integral)} \\
&= \oint_C Q\, dy\,.
\end{aligned}
$$

We follow a similar argument for ②:

$$
\begin{aligned}
② &= \int_{x-}^{x+} \int_{g_L(x)}^{g_U(x)} -\frac{\partial P}{\partial y} dy dx = \int_{x-}^{x+} \Big[-P(x,y) \Big]_{g_L(x)}^{g_U(x)} dx \\
&= \int_{x-}^{x+} \left(-P(x, g_U) + P(x, g_L) \right) dx \qquad \text{(now split the integral)} \\
&= \int_{x+}^{x-} P(x, g_U)\, dx + \int_{x-}^{x+} P(x, g_L)\, dx \quad \text{(note change of limits in 1}^{\text{st}} \text{ integral)} \\
&= \oint_C P\, dx\,.
\end{aligned}
$$

The last steps in both calculations (① and ②) are made using the fact that the curve C can be split as in the diagram, i.e. $C = h_L(y) \cup h_R(y) = g_U(x) \cup g_L(x)$.

Finally we can bring ① and ② back together to see that ①+② = LHS, as required by the theorem. (To prove the theorem for more complicated regions, divide it up into subregions first, and then add the results, noting that line integrals on bits of boundaries shared by two subregions will be traversed in opposite directions, and hence cancel.)

9.2 Stokes' theorem

Stokes' theorem generalises Green's theorem in the plane, relating an integral over a surface S, now in $\mathbb{R}^3$, to the line integral over the boundary of S, C. This is most clearly seen by writing Green's theorem in vector form as

$$\oint_C \underline{F} \cdot d\underline{x} = \int_A (\nabla \times \underline{F}) \cdot \underline{e}_3\, dx\, dy\,,$$

where $\underline{e}_3$ is the unit vector in the z direction, $\underline{F}(x,y,z) = (P(x,y), Q(x,y), R)$, A is a region in the x-y plane, and C is the curve which bounds A, traversed anticlockwise when viewed from above.

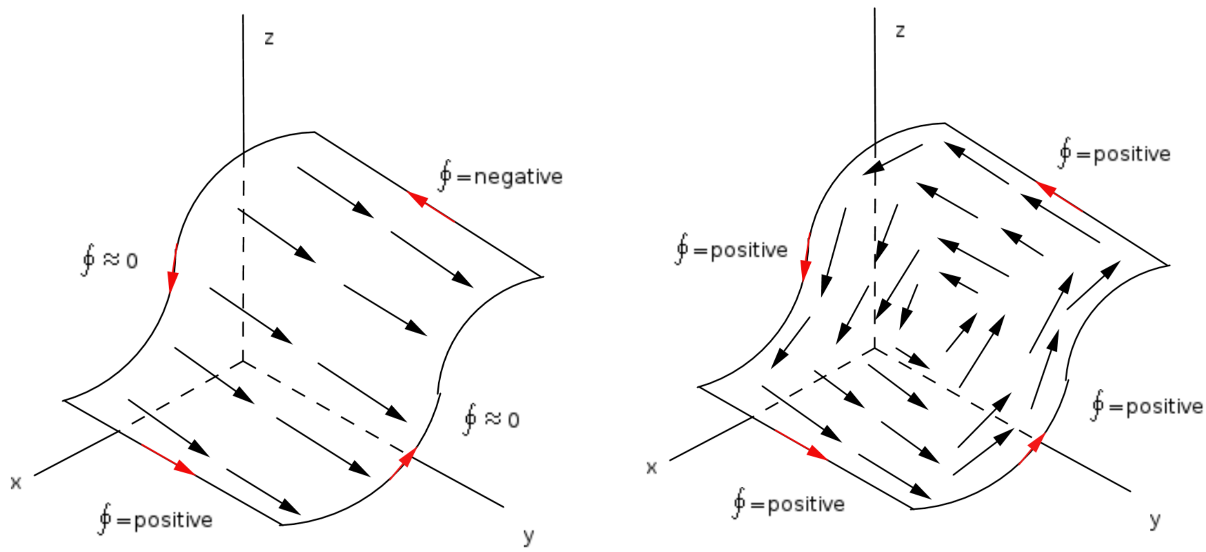
Stokes' looks much the same:

$$\boxed{\oint_C \underline{F} \cdot d\underline{x} = \int_S (\nabla \times \underline{F}) \cdot d\underline{A}}$$

where $\underline{F}(x,y,z)$ is now *any* continuously differentiable vector field in $\mathbb{R}^3$ and $S \subset \mathbb{R}^3$ is a smooth oriented surface which is bounded by the closed curve C. As before, $d\underline{A}$ is shorthand for $\widehat{\underline{n}}\, dA$, and $\widehat{\underline{n}}$ is a unit vector normal to the surface at the location of the area element dA, pointing up when viewed from the side of the surface from which C appears to be traversed anticlockwise.

Put simply, Stokes' theorem relates the microscopic circulation (curl), of some quantity $\underline{F}$ on a surface to the total circulation around the boundary of that surface. As a warm-up to the full proof, let's start by looking at this intuitively without specific vector fields or surfaces.

Figure 53 (adapted from http://www.youtube.com/watch?v=9iaYNaENVH4) shows two vector fields over a surface and its bounding curve. The line integral part of Stokes' theorem, $\oint_C \underline{F} \cdot d\underline{x}$, has been split into



(a) (b)

Figure 53: Sketches of two vector fields over a surface. The black arrows represent the vector field and the red arrows the direction of the line integral around the boundary of the surface. Sketch (a) has the line integral approximately zero while for sketch (b) it is positive.

four sections, one along each edge of the surface. The orientation of the line integral has been chosen to be positive and is shown with red arrows.

Looking at the bottom of sketch (a), we see that field lines and path element $d\underline{x}$ are aligned therefore the dot product will be positive, along left and right sides respectively the field lines and $d\underline{x}$ are perpendicular therefore the line integral here contributes nothing, finally along the top field lines and $d\underline{x}$ are in opposite directions therefore giving us a negative dot product. Adding these to get the overall line integral will give us zero (or close to) as the top and bottom sections cancel each other. This is in agreement with what we would expect from the surface integral part of Stokes' theorem, $\int_S (\nabla \times \underline{F}) \cdot \underline{dA}$, since the homogeneous field lines will not induce any rotation, so $\nabla \times \underline{F} = 0$.

Conversely for the field shown in sketch (b), at all 4 sections of the line integral the field lines are parallel to $d\underline{x}$ and so the dot product will produce a positive result, making the overall line integral positive. Again this agrees with the surface integral part of Stokes' theorem as the circular field lines will induce a rotation and so a positive curl, which in turn will cause the surface integral to be positive.

This is a very rough argument, far from a proof, but with any luck you can see how it might be that the surface integral of the curl of a vector field over S is related to the line integral of the vector field around the bounding curve C. Now for a more rigorous proof.

Proof of Stokes' theorem: The basic setup is a differentiable vector field, $\underline{F}(x, y, z)$, and a surface $S \subset \mathbb{R}^3$ bounded by a closed curve C, as shown in figure 54.

The surface S can be described parametrically by a continuously differentiable map $\underline{x}(u, v) : U \to \mathbb{R}^3$ with $\underline{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))$. If the boundary of the parameter domain $U \subset \mathbb{R}^2$ is a closed curve $\widetilde{C}$, given parametrically as $\underline{u}(t) = (u(t), v(t))$, $t_1 \le t \le t_2$, then the points of the boundary C of S are $\underline{x}(\underline{u}(t))$, again with $t_1 \le t \le t_2$.

Now consider the right hand side of Stokes' theorem, writing everything as a two-dimensional area integral over the region U in the u-v plane. Let's write it out in full first – it will help us to appreciate

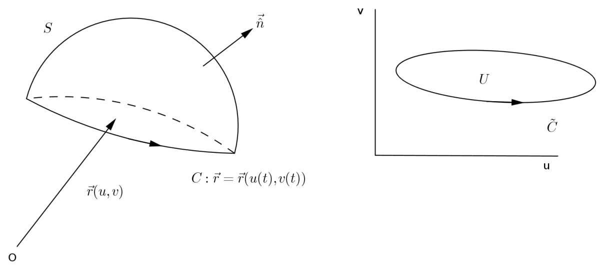


Figure 54: The surface $S \subset \mathbb{R}^3$ with its bounding curve C. Also shown is the parameter domain $U \subset \mathbb{R}^2$ with its bounding curve $\widetilde{C}$, in the parametrised coordinate system u, v. Under the mapping $(u,v) \mapsto \underline{x}(u,v)$, $\widetilde{C} \to C$.

the utility of index notation!

$$
\begin{aligned}
I &= \int_S (\nabla \times \underline{F}) \cdot d\underline{A} = \int_U (\nabla \times \underline{F}) \cdot (\frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v})\, du\, dv \\
&= \int_U \left[\left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \right) \left(\frac{\partial x_2}{\partial u} \frac{\partial x_3}{\partial v} - \frac{\partial x_2}{\partial v} \frac{\partial x_3}{\partial u} \right) + \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \right) \left(\frac{\partial x_3}{\partial u} \frac{\partial x_1}{\partial v} - \frac{\partial x_3}{\partial v} \frac{\partial x_1}{\partial u} \right) \right. \\
&\qquad \left. + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) \left(\frac{\partial x_1}{\partial u} \frac{\partial x_2}{\partial v} - \frac{\partial x_1}{\partial v} \frac{\partial x_2}{\partial u} \right) \right] du\, dv
\end{aligned}
$$

Now we'll rewrite the integrand using the epsilon and (Kronecker) delta symbols; the expression is much more concise, and can be manipulated as follows:

$$
\begin{aligned}
(\nabla \times \underline{F}) \cdot (\underline{x}_u \times \underline{x}_v) = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j} \epsilon_{ilm} \frac{\partial x_l}{\partial u} \frac{\partial x_m}{\partial v} &= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \frac{\partial F_k}{\partial x_j} \frac{\partial x_l}{\partial u} \frac{\partial x_m}{\partial v} \\
&= \frac{\partial F_k}{\partial x_j} \frac{\partial x_j}{\partial u} \frac{\partial x_k}{\partial v} - \frac{\partial F_k}{\partial x_j} \frac{\partial x_k}{\partial u} \frac{\partial x_j}{\partial v} \\
&= \left(\frac{\partial F_k}{\partial x_j} \frac{\partial x_j}{\partial u} \right) \frac{\partial x_k}{\partial v} - \left(\frac{\partial F_k}{\partial x_j} \frac{\partial x_j}{\partial v} \right) \frac{\partial x_k}{\partial u} \\
&= \frac{\partial F_k}{\partial u} \frac{\partial x_k}{\partial v} - \frac{\partial F_k}{\partial v} \frac{\partial x_k}{\partial u} \quad \text{(using the chain rule in reverse)} \\
&= \frac{\partial}{\partial u} \left(F_k \frac{\partial x_k}{\partial v} \right) - \cancel{F_k \frac{\partial^2 x_k}{\partial u \partial v}} - \frac{\partial}{\partial v} \left(F_k \frac{\partial x_k}{\partial u} \right) + \cancel{F_k \frac{\partial^2 x_k}{\partial v \partial u}} \\
&\qquad \text{(using the product rule in reverse)} \\
&= \frac{\partial}{\partial u} \left(F_k \frac{\partial x_k}{\partial v} \right) - \frac{\partial}{\partial v} \left(F_k \frac{\partial x_k}{\partial u} \right)
\end{aligned}
$$

Written in this way, the surface integral I has been expressed as an area integral, and as a bonus the integrand has turned out to be in just the right form to enable us to use Green's theorem in the plane. The 'plane' is now the u-v plane, with $P(u,v) = \left(F_k \frac{\partial x_k}{\partial u} \right)$ and $Q(u,v) = \left(F_k \frac{\partial x_k}{\partial v} \right)$. Hence,

$$
\begin{aligned}
I &= \int_U \left(\frac{\partial}{\partial u} \left(F_k \frac{\partial x_k}{\partial v} \right) - \frac{\partial}{\partial v} \left(F_k \frac{\partial x_k}{\partial u} \right) \right) du\, dv \quad \text{(now use Green's theorem)} \\
&= \oint_{\widetilde{C}} \left(F_k \frac{\partial x_k}{\partial u} du + F_k \frac{\partial x_k}{\partial v} dv \right) \qquad \text{(note this is a line integral in the } u,v \text{ plane)} \\
&= \int_{t_1}^{t_2} F_k \left(\frac{\partial x_k}{\partial u} \frac{du}{dt} + \frac{\partial x_k}{\partial v} \frac{dv}{dt} \right) dt \qquad \text{(now use chain rule in reverse on the term in brackets)} \\
&= \int_{t_1}^{t_2} F_k \frac{dx_k}{dt} dt = \oint_C \underline{F} \cdot d\underline{x}\,.
\end{aligned}
$$

Hence Stokes' theorem follows from Green's theorem in the plane, which was proved earlier.

9.3 The divergence theorem

The divergence theorem gives us a relationship between volume integrals and surface integrals, just as Stokes' theorem related surface integrals to line integrals. Recall its content: if $\underline{F}$ is a continuously differentiable vector field defined over a volume $V \subset \mathbb{R}^3$ with bounding surface S, then

$$\boxed{\int_S \underline{F} \cdot d\underline{A} = \int_V \nabla \cdot \underline{F}\, dV}$$

As before, $d\underline{A} = \widehat{\underline{n}}\, dA$, and now the unit normal $\widehat{\underline{n}}$ should be chosen to point *out* of the volume V.

Aside: It may not look like it, but the divergence theorem is actually a higher dimensional version of Green's theorem in the plane. This can be seen by rewriting Green's theorem as follows.

Recall Green's theorem:

$$I = \oint_C P dx + Q dy = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy\,, \tag{9.1}$$

where C is a closed curve which bounds an area A in the x-y plane. We can parametrise the curve with a parameter t as $\underline{x}(t) = (x(t), y(t))$; then the tangent at any point on the curve is given by $\frac{d\underline{x}}{dt}$.

Now we can write the line integral in terms of t:

$$I = \oint_C \left(P \frac{dx}{dt} + Q \frac{dy}{dt} \right) dt\,, \tag{9.2}$$

and (though this may seem a little odd right now) let us define the following 2-D vectors:

$$\begin{aligned} \underline{F} &= (Q, -P) \\ \underline{N} &= (\frac{dy}{dt}, -\frac{dx}{dt})\,, \end{aligned}$$

so that the integrand of equation (9.2) can be written as the following dot product:

$$I = \oint_C \underline{F} \cdot \underline{N}\, dt\,. \tag{9.3}$$

Now $\underline{N}$ is normal to the curve C, as can be checked by taking its dot product with the tangent $\frac{d\underline{x}}{dt}$:

$$\underline{N} \cdot \frac{d\underline{x}}{dt} = \frac{dy}{dt}\frac{dx}{dt} - \frac{dx}{dt}\frac{dy}{dt} = 0.$$

We may also write:

$$\underline{N} = \widehat{\underline{n}}\, |\underline{N}| = \widehat{\underline{n}} \sqrt{\left(\tfrac{dy}{dt}\right)^2 + \left(\tfrac{dx}{dt}\right)^2} = \widehat{\underline{n}} \frac{ds}{dt}\,,$$

where s is the arc length and $\widehat{\underline{n}}$ is the unit normal. Now we can write equation (9.3) as $I = \oint_C \underline{F} \cdot \widehat{\underline{n}} \frac{ds}{dt}\, dt = \oint_C \underline{F} \cdot \widehat{\underline{n}}\, ds$.

Finally with $\underline{F} = (Q, -P)$ we can rewrite the RHS of equation (9.1) as $\int_A \nabla \cdot \underline{F}\, dA$ and so we get Green's theorem in the following form:

$$\oint_C \underline{F} \cdot \widehat{\underline{n}}\, ds = \int_A \nabla \cdot \underline{F}\, dA$$

which is clearly a lower dimensional form of the divergence theorem

$$\int_S \underline{F} \cdot \widehat{\underline{n}}\, dA = \int_V \nabla \cdot \underline{F}\, dV.$$

This is another example of the analogies between the big theorems that were mentioned at the end of section 10.1. With this aside out of the way, we return to the proof of the full theorem.

Proof of the divergence theorem: Suppose that, in components, the vector field in the statement of the theorem is $\underline{F} = (F_1, F_2, F_3)$. Then we can write

$$\underline{F} = \underline{F}_1 + \underline{F}_2 + \underline{F}_3$$

where $\underline{F}_1 = F_1\,\underline{e}_1$, $\underline{F}_2 = F_2\,\underline{e}_2$ and $\underline{F}_3 = F_3\,\underline{e}_3$. We'll start by proving the theorem for $\underline{F}_3$. Suppose that V projects onto a region A in the x-y plane, and assume further that V is vertically simple, meaning that each point in the interior of V lies between exactly two points above and below it on the boundary of V, with all three of these points projecting onto the same point on the x-y plane in the interior of A. This means that S, the boundary of V, can be split into two halves as $S = S_L \cup S_U$ where S_L ('lower') contains all of the 'below' points on S, and S_U ('upper') contains all of the 'above' points, as illustrated in figure 55.

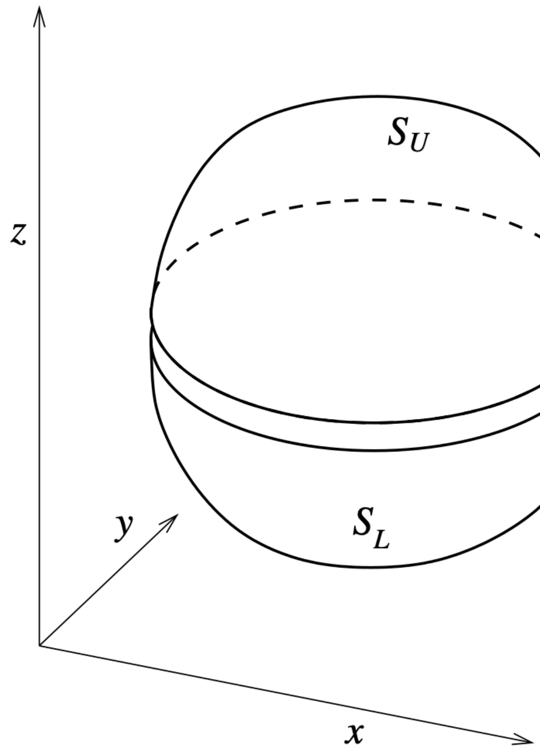


Figure 55: A vertically simple volume, and the corresponding split of its boundary S as $S = S_L \cup S_U$.

Much as in method 2 for doing surface integrals, we'll parametrise S_L and S_U using x and y, with the z coordinates of points on the two surfaces being given by two functions g and h, say, so that on S_L, $z = g(x, y)$, while on S_U, $z = h(x, y)$.

Now consider the volume integral side of the divergence theorem, a triple integral in x, y and z. Opting to do the z integral first, which by Fubini's theorem is allowed, we have

$$\begin{aligned}
\int_V \nabla\cdot\underline{F}_3\,dV &= \iint_A \left(\int_{g(x,y)}^{h(x,y)} \frac{\partial F_3}{\partial z}\,dz\right) dx\,dy \\
&= \iint_A \big[F_3(x,y,z)\big]_{z=g(x,y)}^{z=h(x,y)}\,dx\,dy \\
&= \iint_A \big(F_3(x,y,h(x,y,z)) - F_3(x,y,g(x,y))\big)\,dx\,dy = (*)\,.
\end{aligned}$$

On the other ('surface') side of the theorem, we can write

$$\int_S \underline{F}_3\cdot d\underline{A} = \int_{S_U} \underline{F}_3\cdot d\underline{A} - \int_{S_L} \underline{F}_3\cdot d\underline{A}$$

where both surface integrals on the RHS are to be taken with 'upwards' pointing (i.e. positive z component) normals. Note that all normals in the integral on the LHS point out of V, which on S_L means downwards. The minus sign in front of the second term on the RHS converts the 'upwards-normals' integral to the downwards-normals version, so as to match the LHS.

Now treat S_L as a parametrised surface with coordinates $\underline{x}(x,y) = (x, y, g(x,y))$. By method 1, we have

$$\begin{aligned}\int_{S_L} \underline{F}_3 \cdot d\underline{A} &= \iint_A \underline{F}_3(\underline{x}(x,y)) \cdot \left(\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y}\right) dx\, dy \\ &= \iint_A F_3(x,y,g(x,y))\, \underline{e}_3 \cdot ((1,0,g_x) \times (0,1,g_y))\, dx\, dy \\ &= \iint_A F_3(x,y,g(x,y))\, \underline{e}_3 \cdot (-g_x, -g_y, 1)\, dx\, dy = \iint_A F_3(x,y,g(x,y))\, dx\, dy\,.\end{aligned}$$

Likewise,

$$\int_{S_U} \underline{F}_3 \cdot d\underline{A} = \iint_A F_3(x,y,h(x,y))\, dx\, dy$$

and hence

$$\int_S \underline{F}_3 \cdot d\underline{A} = \iint_A \big(F_3(x,y,h(x,y)) - F_3(x,y,g(x,y))\big) dx\, dy = (*)$$

and we've proved that

$$\int_V \nabla \cdot \underline{F}_3\, dV = \int_S \underline{F}_3 \cdot d\underline{A}\,.$$

In the same way (but doing the x or y integrals first)

$$\int_V \nabla \cdot \underline{F}_1\, dV = \int_S \underline{F}_1 \cdot d\underline{A}$$

and

$$\int_V \nabla \cdot \underline{F}_2\, dV = \int_S \underline{F}_2 \cdot d\underline{A}\,.$$

Adding these up and using $\underline{F} = \underline{F}_1 + \underline{F}_2 + \underline{F}_3$,

$$\int_V \nabla \cdot \underline{F}\, dV = \int_S \underline{F} \cdot d\underline{A}\,,$$

as required. Notice that this is very similar to the way that Green's theorem in the plane was proved earlier. Given the aside that began this subsection, this might not come as a huge surprise.

9.4 Further examples

Bonus example 5. Verify Stokes' theorem for the vector field $\underline{F}(x,y,z) = (y,z,x)$ and the parabolic surface $z = 1 - (x^2 + y^2)$, $z \geq 0$, shown in figure 56.

We first need to find the bounding curve C, in order to calculate the line integral part of Stokes' theorem. It can be found, see figure, by locating the intersection of the surface and the $z = 0$ plane.

$$z = 0 = 1 - (x^2 + y^2) \quad \Rightarrow \quad x^2 + y^2 = 1, \quad \text{the unit circle.}$$

Next we are going to parametrise the curve and calculate $I_1 \equiv \oint_C \underline{F} \cdot d\underline{x}$. As seen earlier, a parametrisation for a unit circle in $\mathbb{R}^2$ is

$$x(t) = \cos t\,, \quad y(t) = \sin t\,,$$

and to situate it in the $z = 0$ plane we simply set $z(t) = 0$. Using this we can write $\underline{x}(t)$, $\underline{F}(\underline{x}(t))$ and $d\underline{x}(t)/dt$ in terms of the parameter t:

$$\begin{aligned}\underline{x}(t) &= (\cos t, \sin t, 0)\,; \\ \underline{F}(\underline{x}(t)) &= (\sin t, 0, \cos t)\,; \\ \frac{d\underline{x}(t)}{dt} &= (-\sin t, \cos t, 0)\,.\end{aligned}$$

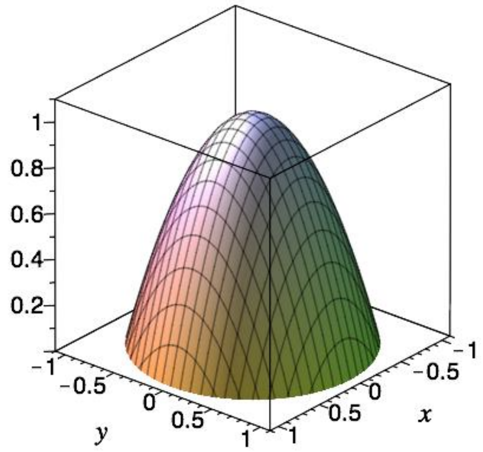


Figure 56: Parabolic surface $z = 1 - (x^2 + y^2)$.

So now we are ready to do the line integral:

$$\begin{aligned} I_1 = \oint_C \underline{F} \cdot d\underline{x} &= \int_0^{2\pi} (\sin t, 0, \cos t) \cdot (-\sin t, \cos t, 0)\, dt \\ &= \int_0^{2\pi} -\sin^2 t\, dt \\ &= \tfrac{1}{2} \int_0^{2\pi} (\cos(2t) - 1)\, dt \;=\; -\pi\,. \end{aligned}$$

Now let's tackle the surface integral: $I_2 \equiv \int_S (\nabla \times \underline{F}) \cdot d\underline{A}$. We will start in Cartesian coordinates and calculate the integrand using 'method 2', the level surface method. First off,

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (-1, -1, -1)\,.$$

Next we recall from section 9how to calculate the area element $d\underline{A} = \widehat{\underline{n}}\, dA$. Since we can express our surface as a level set of a scalar field i.e. $f(x, y, z) = z + x^2 + y^2 = 1$, and

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\underline{e}_1 + \frac{\partial f}{\partial y}\underline{e}_2 + \frac{\partial f}{\partial z}\underline{e}_3 = (2x, 2y, 1)\,,$$

our integrand is given by:

$$(\nabla \times \underline{F}) \cdot d\underline{A} = \frac{(-1, -1, -1) \cdot (2x, 2y, 1)}{\underline{e}_3 \cdot (2x, 2y, 1)} dx\, dy = (-2x - 2y - 1)\, dx\, dy\,,$$

which leaves us ready to compute the double integral:

$$I_2 = \int_{x^2+y^2 \le 1} (-2x - 2y - 1)\, dx\, dy\,.$$

Quickest at this stage is to spot that the integration region is symmetrical in both the x and y directions, so the integrals of $-2x$ and $-2y$ both vanish; this leaves us with

$$I_2 = \int_{x^2+y^2 \le 1} (-1)\, dx\, dy = -(\text{area of unit disk}) = -\pi\,.$$

Alternatively, we change to polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$), with the two parameters r and θ running between 0 and 1 and 0 and 2π respectively. (Projecting the surface integral onto the x,y plane so that our region for the double integral is the unit circle.) The change of variables replaces $dA = dx\,dy$ by $dA = rdrd\theta$; this comes from the Jacobian, as discussed in section 10.5. Hence

$$\begin{aligned} I_2 = \int_S (\nabla \times \underline{F}) \cdot d\underline{A} &= \int_0^{2\pi} \int_0^1 (-2r\cos\theta - 2r\sin\theta - 1)\, r\,dr\,d\theta \\ &= \int_0^{2\pi} \left(-\tfrac{2}{3}\cos\theta - \tfrac{2}{3}\sin\theta - \tfrac{1}{2}\right) d\theta \\ &= \left[-\tfrac{2}{3}\sin\theta + \tfrac{2}{3}\cos\theta - \tfrac{1}{2}\theta\right]_0^{2\pi} \;=\; -\pi\,. \end{aligned}$$

Either way, we have obtained the same result as before, and $\oint_C \underline{F} \cdot d\underline{x} = \int_S (\nabla \times \underline{F}) \cdot d\underline{A}$, as required.

Bonus example 6. Use the divergence theorem to compute

$$\int_S \underline{F} \cdot d\underline{A}$$

where $\underline{F} = (y^2 z, y^3, xz)$ and S is the surface of the cube $|x| \leq 1\,,\ |y| \leq 1\,,\ 0 \leq z \leq 2\,.$
Answer: $\nabla \cdot \underline{F} = 3y^2 + x$, so

$$\begin{aligned} \int_S \underline{F} \cdot d\underline{A} &= \int_V (3y^2 + x)\, dV \\ &= \int_{-1}^{1} \int_{-1}^{1} \int_0^2 (3y^2 + x)\, dz\, dx\, dy \\ &= 2 \int_{-1}^{1} \int_{-1}^{1} (3y^2 + x)\, dx\, dy \\ &= 2 \int_{-1}^{1} \left[3y^2 x + \frac{1}{2}x^2\right]_{-1}^{1} dx\, dy \\ &= 2 \int_{-1}^{1} 6y^2\, dy = \frac{12}{3}\left[y^3\right]_{-1}^{1} = 8\,. \end{aligned}$$

Bonus example 7. Verify the divergence theorem by calculating both left and right hand sides of

$$\int_S \underline{F} \cdot d\underline{A} = \int_V \nabla \cdot \underline{F}\, dV\,,$$

when $\underline{F} = (7x, 0, -z)$, S is the sphere $x^2 + y^2 + z^2 = 4$, and V is the volume inside it.

Let us start with the surface integral. Since the surface is a sphere, which does not sit in a single-valued fashion above any single plane, we use the parametric form: $\int_S \underline{F} \cdot d\underline{A} = \int_U \underline{F}(\underline{x}(u,v)) \cdot (\underline{x}_u \times \underline{x}_v)\, du\, dv$. As in section 9, the (radius 2) sphere can be parametrised as $\underline{x}(u,v) = (2\sin(u)\cos(v), 2\sin(u)\sin(v), 2\cos(u))$, with $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$, and so (as we calculated in example 7 above) $(\underline{x}_u \times \underline{x}_v) = 2\sin(u)\,\underline{x}$. Now we construct our integral in terms of the parameters u and v:

$$\begin{aligned}
\oint_U \underline{F}(\underline{x}(u,v)) \cdot (\underline{x}_u \times \underline{x}_v)\, du\, dv &= \int_0^\pi \int_0^{2\pi} (14\sin(u)\cos(v), 0, -2\cos(u)) \cdot (2\sin(u)\underline{x})\, dv\, du \\
&= \int_0^\pi \int_0^{2\pi} (56\sin^3(u)\cos^2(v) - 8\cos^2(u)\sin(u))\, dv\, du \\
&= \int_0^\pi \left[56\sin^3(u)\left(\frac{v}{2} + \frac{\sin(2v)}{4}\right) - 8\cos^2(u)\sin(u)v \right]_0^{2\pi} du \\
&= \int_0^\pi (56\pi\sin^3(u) - 16\pi\cos^2(u)\sin(u))\, du \\
&= \int_0^\pi (56\pi\sin(u)(1-\cos^2(u)) - 16\pi\cos^2(u)\sin(u))\, du \\
&= \int_0^\pi (-72\pi\sin(u)\cos^2(u) + 56\pi\sin(u))\, du \\
&= \left[-72\pi\frac{(-\cos^3(u))}{3} + 56\pi(-\cos(u)) \right]_0^\pi = 64\,\pi\,.
\end{aligned}$$

Next, the volume integral. The first step is to calculate the divergence of $\underline{F}$:

$$\nabla \cdot \underline{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (7x, 0, -z) = 6\,.$$

Hence the integral is

$$\int_V 6\, dV \;=\; 6\left(\frac{4}{3}\pi r^3\right) \;=\; 64\,\pi\,.$$

This was much simpler than the surface integral, and in fact we didn't have to integrate at all – we just used the formula for the volume of a sphere!

Bonus example 8. Evaluate the line integral $\oint_C \underline{F} \cdot d\underline{x}$ with $\underline{F} = (y, \frac{z}{2}, \frac{3y}{2})$ around the curve C given by the intersection of the sphere $x^2 + y^2 + z^2 = 6z$, and the plane $z = x + 3$. The curve is a circle lying in the plane $z = x + 3$. If we use Stokes' theorem then we need to think about how to set up the surface integral for a surface with C as its boundary. If we do this by using x, y as parameters then we will need the projection of C onto the x, y-plane which is given by: $x^2 + y^2 + (x+3)^2 = 6(x+3) = 2x^2 + y^2 = 9$, an ellipse.
The line integral can be done directly, or we can apply Stokes' theorem: $\oint_C \underline{F} \cdot d\underline{x} = \int_S (\nabla \times \underline{F}) \cdot d\underline{A}$. The easiest surface to consider that has C as its boundary is the flat disk S in the $z = x + 3$ plane (we could have used either parts of the sphere that have C as boundary but that would be more complcated).

Let's start by calculating the curl of F:

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z/2 & 3y/2 \end{vmatrix} = (1, 0, -1)$$

And now let's find $d\underline{A}$ using method 2, describing S as (part of) a level set and parametrising it using x and y:

$$d\underline{A} = \frac{\nabla f}{\mathbf{e}_3 \cdot \nabla f}\, dx\, dy$$

Taking $f(x,y,z) = z - x$ so the plane is $f(x,y,z) = 3$, $\nabla f = (-1, 0, 1)$, $\underline{e}_3 \cdot \nabla f = 1$, and our integrand is $(\nabla \times \underline{F}).d\underline{A} = -2\, dx\, dy$. The region of integration A is the interior of the ellipse $2x^2 + y^2 = 9$ in the x,y plane. Finally we can use a simple change of variables to convert the ellipse to a circle and make the area integration even simpler: if we set $\bar{x} = \sqrt{2}x$, $d\bar{x} = \sqrt{2}dx$, then our ellipse becomes the circle $\bar{x}^2 + y^2 = 9$, and we can apply the usual coordinate transformations as follows ($\bar{A}$ (a circle) is the transformation of

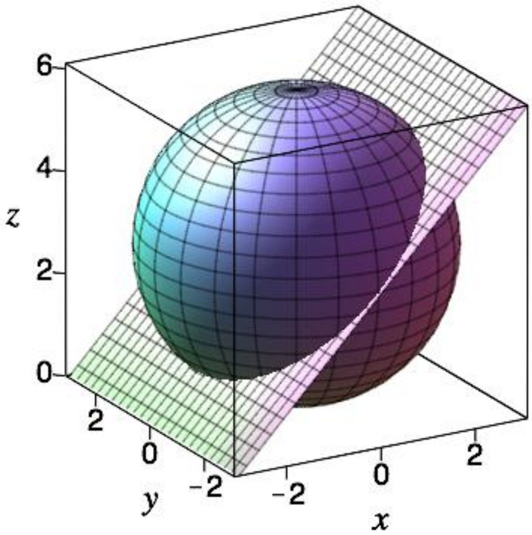


Figure 57: Intersection of sphere $x^2 + y^2 + z^2 - 6z = 0$ and plane $z = x + 3$.

region A (an ellipse)):

$$\begin{aligned}\int_S (\nabla \times \underline{F}) \cdot d\underline{A} &= -2 \int_A dx\, dy = -\sqrt{2} \int_{\bar{A}} d\bar{x}\, dy \\ &= -\sqrt{2} \int_0^{2\pi} \int_0^3 r\, dr\, d\theta \\ &= -\sqrt{2} \int_0^{2\pi} \tfrac{9}{2}\, d\theta \;=\; -9\sqrt{2}\pi\,.\end{aligned}$$

Alternatively, we can do the line integral directly around the ellipse $2x^2 + y^2 = 9$. This can be written in the standard form as $\frac{x^2}{(3/\sqrt{2})^2} + \frac{y^2}{3^2} = 1$ and using the parametrisation from the table in section 9.1, $x(t) = \frac{3}{\sqrt{2}} \cos(t)$, $y(t) = 3 \sin(t)$ and so $z(t) = x(t) + 3 = \frac{3}{\sqrt{2}} \cos(t) + 3$. We can then find the line element along the path to be $d\underline{x} = (\frac{-3}{\sqrt{2}} \sin(t), 3 \cos(t), \frac{-3}{\sqrt{2}} \sin(t))\, dt$, and so the integral is

$$\begin{aligned}\oint_C \underline{F} \cdot d\underline{x} &= \int_0^{2\pi} (3 \sin(t), \tfrac{3}{2\sqrt{2}} \cos(t) + \tfrac{3}{2}, \tfrac{9}{2} \sin(t)) \cdot (\tfrac{-3}{\sqrt{2}} \sin(t), 3 \cos(t), \tfrac{-3}{\sqrt{2}} \sin(t))\, dt \\ &= \int_0^{2\pi} \left(-\tfrac{9}{\sqrt{2}} \sin^2 t + \tfrac{9}{2\sqrt{2}} \cos^2 t + \tfrac{9}{2} \cos t - \tfrac{27}{2\sqrt{2}} \sin^2 t\right) dt \\ &= -\tfrac{9}{\sqrt{2}}\pi + \tfrac{9}{2\sqrt{2}}\pi - \tfrac{27}{2\sqrt{2}}\pi = -9\sqrt{2}\pi\,,\end{aligned}$$

as before.

Bonus example 9. The next example is a little more general: Suppose $\underline{B}(\underline{x})$ is defined everywhere in $\mathbb{R}^3$, and that S is a closed surface in $\mathbb{R}^3$. Show that $\int_S (\nabla \times \underline{B}) \cdot d\underline{A} = 0$, (i) using Stokes' theorem, and (ii) using the divergence theorem.

(i) Using Stokes' theorem, the proof takes a little work. First imagine the closed surface split into two surfaces S_+ and S_-, bounded by the curves C and $\bar{C}$ respectively, as shown in figure 58.

Now:

$$\int_S (\nabla \times \underline{B}) \cdot d\underline{A} = \int_{S_+} (\nabla \times \underline{B}) \cdot d\underline{A} + \int_{S_-} (\nabla \times \underline{B}) \cdot d\underline{A}\,.$$

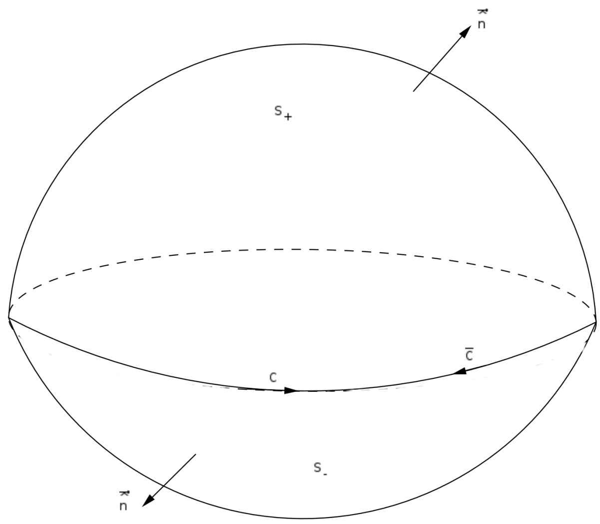


Figure 58: The closed surface is split into two open surfaces; S_+ is bounded (positively) by curve C and S_- is bounded by $\bar{C}$. The unit normal is oriented outwards as usual. Note that $\bar{C}$ is just C traversed backwards.

Next, apply Stokes' theorem:

$$\begin{aligned}\int_{S_+}(\nabla\times\underline{B})\cdot d\underline{A}+\int_{S_-}(\nabla\times\underline{B})\cdot d\underline{A}&=\oint_C\underline{B}\cdot d\underline{x}+\oint_{\bar{C}}\underline{B}\cdot d\underline{x}\\&=\oint_C\underline{B}\cdot d\underline{x}-\oint_C\underline{B}\cdot d\underline{x}=0\,.\end{aligned}$$

(ii) Using the divergence theorem, the result is a little easier to see:

$$\int_S(\nabla\times\underline{B})\cdot d\underline{A}=\int_V\nabla\cdot(\nabla\times\underline{B})\,dV=\int_V 0\,dV=0\,,$$

since the divergence of a curl is zero. Try to prove it using index notation!

Bonus example 10. One final quick example: show that the volume enclosed by a closed surface S is $\frac{1}{3}\int_S\underline{x}\cdot d\underline{A}$.

First off, remember that $\underline{x}=(x,y,z)$. Now let's show that $\frac{1}{3}\int_S\underline{x}\cdot d\underline{A}$ is indeed equal to volume by applying the divergence theorem:

$$\tfrac{1}{3}\int_S\underline{x}\cdot d\underline{A}=\tfrac{1}{3}\int_V\nabla\cdot\underline{x}\,dV=\int_V dV=V,\ \text{as required.}$$

One last request: if you spot any typos in these notes, or if you find that anything is particularly obscure, please let me know! (At `s.m.fearn@durham.ac.uk`)