1. For the function $f(x,y) = \cos(x+y)\,\exp(x-y)$ calculate $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x^2$, $\partial^2 f/\partial y^2$, $\partial^2 f/\partial x\partial y$, $\partial^2 f/\partial y\partial x$. Use your results to show that $\partial^2 f/\partial x^2 = -\partial^2 f/\partial y^2$ and $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$.

   Solution $\partial f/\partial x = -\sin(x+y)\,\exp(x-y) + \cos(x+y)\,\exp(x-y)$, $\partial f/\partial y = -\sin(x+y)\,\exp(x-y) - \cos(x+y)\,\exp(x-y)$, $\partial^2 f/\partial x^2 = -2\sin(x+y)\,\exp(x-y)$, $\partial^2 f/\partial y^2 = 2\sin(x+y)\,\exp(x-y)$, $\partial^2 f/\partial x\partial y = -2\cos(x+y)\,\exp(x-y)$, $\partial^2 f/\partial y\partial x = -2\cos(x+y)\,\exp(x-y)$. By inspection $\partial^2 f/\partial x^2 = -\partial^2 f/\partial y^2$ and $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$.

2. Let $F(t)$ be the value of the function $f(x,y,z) = \cos(xy)\,z$ restricted to the helix $x = \cos(t), y = \sin(t)\,, z = t$ which is parametrised by t and $-\infty < t < \infty$. Calculate dF/dt as a function of t (i) directly by substituting the equations of the helix into $f(x,y,z)$ to calculate $F(t)$ as a function of t and then differentiating, and (ii) using the chain rule. Note how similar these approaches are.

   Solution (i) Substitution gives $F = \cos(\cos(t)\sin(t))\,t$. Differentiating: $dF/dt = \sin(\cos(t)\sin(t))\,\sin^2(t)\,t - \sin(\cos(t)\sin(t))\,\cos^2(t)\,t + \cos(\cos(t)\sin(t))$. (ii) Using the chain rule:

$$\begin{aligned} dF/dt &= (\partial f/\partial x)(dx/dt) + (\partial f/\partial y)(dy/dt) + (\partial f/\partial z)(dz/dt) \\ &= \sin(xy)\,yz\,\sin(t) - \sin(xy)\,xz\,\cos(t) + \cos(xy) \\ &= \sin(\cos(t)\sin(t))\,\sin^2(t)\,t - \sin(\cos(t)\sin(t))\,\cos^2(t)\,t + \cos(\cos(t)\sin(t)). \end{aligned}$$

3. If $\mathbf{a} = \sin 2t\,\mathbf{e_1} + e^t\,\mathbf{e_2} - (t^3 - 5t)\,\mathbf{e_3}$, find
   (a) $d\mathbf{a}/dt$, (b) $||d\mathbf{a}/dt||$, (c) $d^2\mathbf{a}/dt^2$, (d) $||d^2\mathbf{a}/dt^2||$, all at $t = 0$.

   Solution (a) $d\mathbf{a}/dt = 2\cos(2t)\,\mathbf{e_1} + e^t\,\mathbf{e_2} - (3t^2 - 5)\,\mathbf{e_3}$. At $t = 0$ this is $2\,\mathbf{e_1} + \mathbf{e_2} + 5\,\mathbf{e_3}$. (b) which has length $||d\mathbf{a}/dt|| = \sqrt{2^2 + 1^2 + 5^2} = \sqrt{30}$, (c) $d^2\mathbf{a}/dt^2 = -4\sin(2t)\,\mathbf{e_1} + e^t\,\mathbf{e_2} - 6t\,\mathbf{e_3}$, at $t = 0$ this is $\mathbf{e_2}$, (d) so $||d^2\mathbf{a}/dt^2|| = 1$ at $t = 0$.

4. Find a unit vector tangent to the space curve $x = t^3, y = t, z = t^2$ at $t = 2$.

   Solution If $\mathbf{x}(t) = t^3\mathbf{e_1} + t\mathbf{e_2} + t^2\mathbf{e_3}$ then $d\mathbf{x}(t)/dt = 3t^2\mathbf{e_1} + \mathbf{e_2} + 2t\mathbf{e_3}$ is parallel to the tangent. At $t = 2$ this is $12\mathbf{e_1} + \mathbf{e_2} + 4\mathbf{e_3}$ which has length $\sqrt{161}$ so a unit vector in this direction is $(12\mathbf{e_1} + \mathbf{e_2} + 4\mathbf{e_3})/\sqrt{161}$.

5. Use the chain rule to calculate df/dt when $f(\mathbf{x}) = \exp(-||\mathbf{x}||^2)$ is restricted to the curves:

   (a) $\mathbf{x} = \mathbf{e_1}\log t + \mathbf{e_2} t\log t + \mathbf{e_3} t$,

   (b) $(x,y,z) = (\cosh t, \sinh t, 0)$.

Solution The chain rule to use here is

$$\frac{df}{dt} = \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} + \frac{dz}{dt}\frac{\partial f}{\partial z}\,.$$

Since $f(\mathbf{x}) = \exp(-||\mathbf{x}||^2) = \exp(-(x^2+y^2+z^2))$ we have

$$\frac{\partial f}{\partial x} = -2x\,f\,, \quad \frac{\partial f}{\partial y} = -2y\,f\,, \quad \frac{\partial f}{\partial z} = -2z\,f\,.$$

(a) $\mathbf{x} = \mathbf{e_1}\log t + \mathbf{e_2} t\log t + \mathbf{e_3} t$, so $d\mathbf{x}/dt = \mathbf{e_1}/t + \mathbf{e_2}(\log t + 1) + \mathbf{e_3}$ and

$$\frac{df}{dt} = (1/t)(-2x)\,f + (\log t + 1)(-2y)\,f + (-2z)\,f$$
$$= -2((\log t)/t + (\log t+1)t\log t + t)\ \exp -((\log t)^2 + (t\log t)^2 + t^2)\,.$$

(b) $(x,y,z) = (\cosh t, \sinh t, 0)$, so $(dx/dt, dy/dt, dz/dt) = (\sinh t, \cosh t, 0)$ giving

$$\frac{df}{dt} = (\sinh t)(-2x)\,f + (\cosh t)(-2y)\,f + 0.(-2z)\,f$$
$$= -4\sinh t\cosh t\ \exp(-\cosh^2 t - \sinh^2 t) = -4\sinh t\cosh t\ \exp(-\cosh 2t)\,.$$

6. Show that the curve C, given as points $\underline{x}(s) = \left(\sin(s/\sqrt{2}), \cos(s/\sqrt{2}), s/\sqrt{2}\right)$, is the arc-length parameterisation of a helix, that is that $\left|\frac{dx}{ds}\right| = 1 \quad \forall s$.

Solution To show that the curve is parameterised by arc-length, we need to show that $|\frac{d\underline{x}}{ds}| = 1 \quad \forall s$. We have

$$\frac{d\underline{x}}{ds} = \left(\frac{1}{\sqrt{2}}\cos(s/\sqrt{2}), -\frac{1}{\sqrt{2}}\sin(s/\sqrt{2}), 1/\sqrt{2}\right),$$

and therefore

$$\left|\frac{d\underline{x}}{ds}\right| = \frac{1}{2}\cos^2(s/\sqrt{2}) + \frac{1}{2}\sin^2(s/\sqrt{2}) + \sqrt{\frac{1}{2}}$$
$$= \sqrt{\frac{1}{2}+\frac{1}{2}}$$
$$= 1.$$

7. Describe the curve $\gamma : \underline{x}(t) = (2t+1, t-3, 6-2t)$. Find the arc-length parameterisation of γ, that is, re-parameterise the curve in terms of a parameter s, such that $\left|\frac{d\underline{x}(s)}{ds}\right| = 1\,\forall s$.

Solution We can write $\underline{x}(t) = (1,-3,6) + t(2,1,-2)$, which we recognise as a parameterisation of a straight line through $\underline{a} = (1,-3,6)$ parallel to $\underline{b} = (2,1,-2)$.

Before trying to re-parameterise γ, we should check whether the curve is *already* parameterised in terms of arc-length. We have

$$\frac{d\underline{x}}{dt} = (2, 1, -2),$$

and so

$$\left|\frac{d\underline{x}}{dt}\right| = \sqrt{4+1+4} = 3.$$

Since this is not equal to 1 $\forall t$, t does not give the arc length parameterisation of γ.

Note that we cannot just multiply $\underline{x}(t)$ by $1/3$, as although the resulting curve would have tangent vectors of unit length, this would be a straight line through $\underline{a}/3$ rather than through $\underline{a}$, and so this is a parameterisation of a different curve. However, we can see that if we let $t = s/3$, then we will find that the resulting tangent vector $\frac{d\underline{x}}{ds}$ is unit length:

$$\begin{aligned}
\underline{x}(s) &= \left(2\frac{s}{3} + 1, \frac{s}{3} - 3, 6 - 2\frac{s}{3}\right), \\
\frac{d\underline{x}}{ds} &= \left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right), \\
\left|\frac{d\underline{x}}{ds}\right| &= \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1.
\end{aligned}$$

We can see that since the only changes we have made are to the parameter in the curve, this is still a parameterisation of a curve through $\underline{a}$ parallel to $\underline{b}$. This is therefore the arc-length parameterisation of γ that was required. If you choose to study Differential Geometry III, you will learn the general method for parameterising a curve in terms of its arc-length.

8. **Harder:** Let $\mathbf{t}$ denote the unit tangent vector to a space curve $\mathbf{a} = \mathbf{a}(s)$ in $\mathbb{R}^3$, where $\mathbf{a}(s)$ is assumed differentiable, and where s measures the arclength from some fixed point on the curve. Define the unit vector $\mathbf{n} = \frac{1}{\kappa}\frac{d\mathbf{t}}{ds}$, where κ is a scalar. Also define $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ as the *unit binormal vector* to the space curve.

   By considering the derivative of the product $\mathbf{t}.\mathbf{t}$, show that the the 3 vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ form an orthonormal basis of $\mathbb{R}^3$.

   Hence, prove that
   $$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{n}, \quad \text{and} \quad \frac{d\mathbf{n}}{ds} = \tau\mathbf{b} - \kappa\mathbf{t},$$
   where τ is some real constant.

   These formulae are of fundamental importance in differential geometry. They involve the curvature κ and the torsion τ. The reciprocals of these are the radius of curvature ($\rho = \frac{1}{\kappa}$) and the radius of torsion ($\sigma = \frac{1}{\tau}$).

Solution $\mathbf{t}$ is a unit tangent, and hence we have $\mathbf{t}.\mathbf{t} = 1$. Differentiating this expression with respect to s using the product rule gives $2\mathbf{t}.\frac{d\mathbf{t}}{ds} = 0$, and hence $\frac{d\mathbf{t}}{ds}$ is orthogonal to $\mathbf{t}$. Since $\mathbf{n} = \frac{1}{\kappa}\frac{d\mathbf{t}}{ds}$, we have $\mathbf{t}$ and $\mathbf{n}$ orthogonal, and since we define $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, $\mathbf{b}$ is orthogonal to the plane containing $\mathbf{t}$ and $\mathbf{n}$. We have therefore shown that these three vectors form an orthogonal basis of $\mathbb{R}^3$, and since both $\mathbf{t}$ and $\mathbf{n}$ are orthogonal vectors, then so is their vector product, and hence these vectors form an orthonormal basis of $\mathbb{R}^3$.

Since these vectors are all mutually orthogonal, alongside the relation $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, we also have $\mathbf{t} = \mathbf{n} \times \mathbf{b}$ and $\mathbf{n} = \mathbf{b} \times \mathbf{t}$, as can easily be seen using the right-hand rule.

Now the derivative $\frac{d\mathbf{b}}{ds}$ is given by

$$\begin{aligned}\frac{d\mathbf{b}}{ds} = \frac{d}{ds}(\mathbf{t} \times \mathbf{n}) &= \left(\frac{d\mathbf{t}}{ds}\right) \times \mathbf{n} + \mathbf{t} \times \left(\frac{d\mathbf{n}}{ds}\right) \\ &= (\kappa\mathbf{n}) \times \mathbf{n} + \mathbf{t} \times \left(\frac{d\mathbf{n}}{ds}\right) \\ &= \mathbf{t} \times \left(\frac{d\mathbf{n}}{ds}\right),\end{aligned}$$

since the cross product of $\mathbf{n}$ with itself is 0.

Since $\mathbf{n}$ is also a unit vector, we have $\mathbf{n}.\mathbf{n} = 1$, and as before differentiating this with respect to s gives $2\mathbf{n}.\frac{d\mathbf{n}}{ds} = 0$. $\frac{d\mathbf{n}}{ds}$ is therefore orthogonal to $\mathbf{n}$, and since $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ form a basis of $\mathbb{R}^3$, we therefore have that

$$\frac{d\mathbf{n}}{ds} = \sigma\mathbf{t} + \tau\mathbf{b}.$$

Substituting this into our equation for the derivative of $\mathbf{b}$ gives

$$\begin{aligned}\frac{d\mathbf{b}}{ds} &= \mathbf{t} \times (\sigma\mathbf{t} + \tau\mathbf{b}) \\ &= \tau\mathbf{t} \times \mathbf{b} \\ &= -\tau\mathbf{n},\end{aligned}$$

which is the first of the equations we were asked to show.

Finally, since $\mathbf{n} = \mathbf{b} \times \mathbf{t}$, we have

$$\begin{aligned}\frac{d\mathbf{n}}{ds} &= \frac{d\mathbf{b}}{ds} \times \mathbf{t} + \mathbf{b} \times \frac{d\mathbf{t}}{ds}, \\ &= (-\tau\mathbf{n}) \times \mathbf{t} + \mathbf{b} \times (\kappa\mathbf{n}) \quad = \tau\mathbf{b} - \kappa\mathbf{t},\end{aligned}$$

which is the second equation we were asked to show.

These are the Serret-Frenet formulas; see e.g. G.E.Hay, Vector and Tensor Analysis.

Bonus 1. If $f(x, y) = F(r, \theta)$ with $x = r\cos\theta$ and $y = r\sin\theta$, use the chain rule to compute $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2$ in terms of partial r- and θ- derivatives of F, and hence find the general rotationally-symmetric solution to $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = 0$ in two dimensions which is non-singular away from the origin.
Suggestion: Begin by writing $\partial r/\partial x$, $\partial r/\partial y$, $\partial\theta/\partial x$ and $\partial\theta/\partial y$ as functions of r and θ.

Bonus Solution Following the suggestion, and borrowing some of the calculations from section 0.2 of lectures, we have

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta\,; \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta\,,$$

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2} = -\frac{\sin\theta}{r}\,; \qquad \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}\,,$$

and so by the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\,,$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\,.$$

Hence

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right),$$

or, a little more explicitly

$$\frac{\partial^2 f}{\partial x^2} = \left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)F\,.$$

Now for a key point – each derivative in this formula acts on *everything* to its right. So when expanding the brackets on the RHS you'll get terms like

$$-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\cos\theta\,\frac{\partial}{\partial r}F = -\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\,\frac{\partial F}{\partial r}\right) = -\frac{\sin\theta}{r}\left(-\sin\theta\,\frac{\partial F}{\partial r} + \cos\theta\,\frac{\partial^2 F}{\partial\theta\partial r}\right)$$

where the first equality simply emphasises the fact that the $\partial/\partial\theta$ acts on everything to its right, while the second comes from using the product rule for derivatives. With practice you should be able to do these calculations in 'operator' notation, not bothering to write f and F explicitly but remembering that they are there implicitly, so the full calculation of $\partial^2/\partial x^2$ would look like

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\\
&= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta}\\
&\quad + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}\\
&= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}
\end{aligned}$$

(Using the equality of the mixed second partial derivatives to gather the terms together.) A similar calculation (exercise!) gives

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta\frac{\partial^2}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Adding the two, many terms cancel and/or simplify to leave the final result

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\,.$$

For the final part: for a rotationally-symmetric solution, F will be independent of θ and so the equation (which is called Laplace's equation) simplifies to

$$0 = \frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right)$$

(Since F only depends on one variable, there's no need to use the partial derivative symbol.) This equation is easily solved: $F(r) = C\log r + D$ with C and D two constants, and this is the general rotationally-symmetric solution to Laplace's equation in two dimensions.

9. Compute the gradient, $\underline{\nabla} f$, for the following functions:

   (a) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$,

   Solution $\partial f/\partial x = x/\sqrt{x^2+y^2+z^2}$, $\partial f/\partial y = y/\sqrt{x^2+y^2+z^2}$, and $\partial f/\partial z = z/\sqrt{x^2+y^2+z^2}$, so $\underline{\nabla} f = (x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3})/\sqrt{x^2+y^2+z^2} = \mathbf{x}\,/|\mathbf{x}\,|$.

   (b) $f(x,y,z) = xy + yz + xz$,

   Solution $\partial f/\partial x = y+z$, $\partial f/\partial y = x+z$, $\partial f/\partial z = y+x$, so $\underline{\nabla} f = (y+z)\mathbf{e_1} + (x+z)\mathbf{e_2} + (x+y)\mathbf{e_3}$.

   (c) $f(x,y,z) = 1/(x^2+y^2+z^2)$.

   Solution $\partial f/\partial x = -2x/(x^2+y^2+z^2)^2$, $\partial f/\partial y = -2y/(x^2+y^2+z^2)^2$, and $\partial f/\partial z = -2z/(x^2+y^2+z^2)^2$ so $\underline{\nabla} f = -2(x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3})/(x^2+y^2+z^2)^2 = -2\mathbf{x}\,/|\mathbf{x}\,|^4$.

10. Show that $\underline{h}(s) = \left(s/\sqrt{2}, \cos(s/\sqrt{2}), \sin(s/\sqrt{2})\right)$ is the arc-length parameterisation of a helix, then calculate the directional derivative of the scalar field $f(\underline{x}) = (\log(x^2+y^2+z^2))$ along $\underline{h}(s)$ at $s = \sqrt{2}\pi$.

    Solution To show that the curve is parameterised by arc-length, we need to show that $|\frac{dh}{ds}| = 1 \quad \forall s$. We have

    $$\frac{dh}{ds} = \left(1/\sqrt{2}, -\frac{1}{\sqrt{2}}\sin(s/\sqrt{2}), \frac{1}{\sqrt{2}}\cos(s/\sqrt{2})\right),$$

    and therefore

    $$\begin{aligned}\left|\frac{dh}{ds}\right| &= \sqrt{\frac{1}{2} + \frac{1}{2}\sin^2(s/\sqrt{2}) + \frac{1}{2}\cos^2(s/\sqrt{2})} \\ &= \sqrt{\frac{1}{2} + \frac{1}{2}} \\ &= 1\end{aligned}$$

The directional derivative of $f(\underline{x})$ along $\underline{h}(s)$ at $s = \sqrt{2}\pi$ is then given by

$$\frac{df(\underline{h})}{ds}(\sqrt{2}\pi) = \frac{d\underline{h}}{ds}(\sqrt{2}\pi).\underline{\nabla} f(\underline{h}(\sqrt{2}\pi)),$$

and so we need to calculate the gradient of f and evaluate this at $\underline{h}(\sqrt{2}\pi)$. We have

$$\begin{aligned}\underline{\nabla} f &= \left(\frac{2x}{x^2+y^2+z^2}, \frac{2y}{x^2+y^2+z^2}, \frac{2z}{x^2+y^2+z^2}\right)\\ &= \frac{2\underline{x}}{x^2+y^2+z^2}\\ \underline{h}(\sqrt{2}\pi) &= (\pi, \cos(\pi), \sin(\pi))\\ &= (\pi, -1, 0)\,,\end{aligned}$$

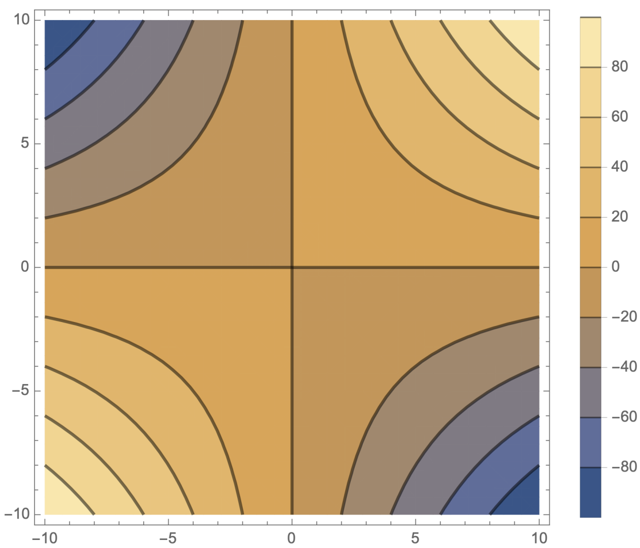
and so

$$\begin{aligned}\frac{df(\underline{h})}{ds}(\sqrt{2}\pi) &= \frac{d\underline{h}}{ds}(\sqrt{2}\pi).\underline{\nabla} f(\underline{h}(\sqrt{2}\pi))\\ &= \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\sin(\pi), \frac{1}{\sqrt{2}}\cos(\pi)\right).\underline{\nabla} f(\pi,-1,0)\\ &= \frac{2}{\pi^2+1}\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right).(\pi,-1,0)\\ &= \frac{\pi\sqrt{2}}{(\pi^2+1)}.\end{aligned}$$

11. Draw a sketch of the contour plot of the scalar field on $\mathbb{R}^2$ $f(\underline{x}) = xy$, as well as the gradient of f. What do you notice?

Solution The level sets of f are of the form $xy = c$ for a constant $c \in \mathbb{R}$. Rearranging this as $y = c/x$, we can then plot a few of these level sets. This looks as follows:

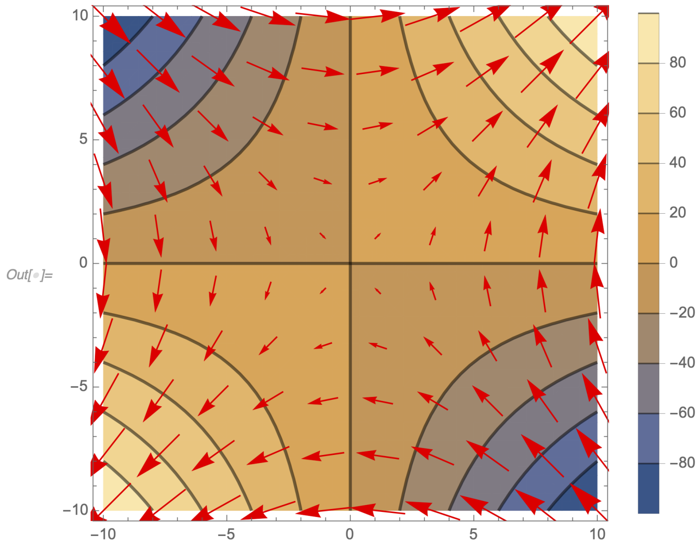
Out[∘]=



The gradient of f is given as

$$\begin{aligned} \operatorname{grad} f = \underline{\nabla} f &= \left(\underline{e}_1 \frac{\partial}{\partial x}, \underline{e}_2 \frac{\partial}{\partial y} \right) f \\ &= (y, x) \, . \end{aligned}$$

A plot of this overlaid on to top of the contour plot of f is as follows:



We see that the vectors of the vector field $\underline{\nabla} f$ are normal to the level sets of f, as we expect.

12. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be scalar fields on $\mathbb{R}^3$, $h : \mathbb{R} \to \mathbb{R}$ be a function on $\mathbb{R}$ and a be a constant in $\mathbb{R}$. Show (using the definition of $\underline{\nabla}$) that

$$\underline{\nabla}(af(\underline{x})g(\underline{x}) + h(f(\underline{x}))) = a(\underline{\nabla} f)g + af\underline{\nabla} g + \underline{\nabla} f \frac{dh}{df}.$$

Solution For this question, we are supposed to use only the definition of the gradient in $\mathbb{R}^3$, not the properties of the gradient. This is just a slog in keeping track of all the terms.

We have

$$\begin{aligned}
\underline{\nabla}(af(\underline{x})g(\underline{x}) + h(f(\underline{x}))) &= \underline{e}_1 \frac{\partial}{\partial x}(af(\underline{x})g(\underline{x}) + h(f(\underline{x}))) + \underline{e}_2 \frac{\partial}{\partial y}(af(\underline{x})g(\underline{x}) + h(f(\underline{x}))) \\
&\quad + \underline{e}_3 \frac{\partial}{\partial z}(af(\underline{x})g(\underline{x}) + h(f(\underline{x}))) \\
&= \underline{e}_1 \left(a\frac{\partial}{\partial x} f(\underline{x})g(\underline{x}) + \frac{\partial}{\partial x} h(f(\underline{x}))\right) \qquad \text{by linearity} \\
&\quad + \underline{e}_2 \left(a\frac{\partial}{\partial y} f(\underline{x})g(\underline{x}) + \frac{\partial}{\partial y} h(f(\underline{x}))\right) \qquad \text{of partial} \\
&\quad + \underline{e}_3 \left(a\frac{\partial}{\partial z} f(\underline{x})g(\underline{x}) + \frac{\partial}{\partial z} h(f(\underline{x}))\right) \qquad \text{derivatives} \\
&= \underline{e}_1 \left(a\frac{\partial f(\underline{x})}{\partial x} g(\underline{x}) + af(\underline{x})\frac{\partial g(\underline{x})}{\partial x} + \frac{\partial h(f)}{\partial f}\frac{\partial f(\underline{x})}{\partial x}\right) \\
&\quad + \underline{e}_2 \left(a\frac{\partial f(\underline{x})}{\partial y} g(\underline{x}) + af(\underline{x})\frac{\partial g(\underline{x})}{\partial y} + \frac{\partial h(f)}{\partial f}\frac{\partial f(\underline{x})}{\partial y}\right) \\
&\quad + \underline{e}_3 \left(a\frac{\partial f(\underline{x})}{\partial z} g(\underline{x}) + af(\underline{x})\frac{\partial g(\underline{x})}{\partial z} + \frac{\partial h(f)}{\partial f}\frac{\partial f(\underline{x})}{\partial z}\right),
\end{aligned}$$

where we used the product rule and chain rule for the partial derivative in each component. We can now recollect the terms to give

$$\begin{aligned}
\underline{\nabla}(af(\underline{x})g(\underline{x}) + h(f(\underline{x}))) &= a\left(\underline{e}_1 \frac{\partial f(\underline{x})}{\partial x} g(\underline{x}) + \underline{e}_2 \frac{\partial f(\underline{x})}{\partial y} g(\underline{x}) + \underline{e}_3 \frac{\partial f(\underline{x})}{\partial z} g(\underline{x})\right) \\
&\quad + a\left(\underline{e}_1 f(\underline{x})\frac{\partial g(\underline{x})}{\partial x} + \underline{e}_2 f(\underline{x})\frac{\partial g(\underline{x})}{\partial y} + \underline{e}_3 f(\underline{x})\frac{\partial g(\underline{x})}{\partial z}\right) \\
&\quad + \left(\underline{e}_1 \frac{\partial h(f)}{\partial f}\frac{\partial f(\underline{x})}{\partial x} + \underline{e}_2 \frac{\partial h(f)}{\partial f}\frac{\partial f(\underline{x})}{\partial y} + \underline{e}_3 \frac{\partial h(f)}{\partial f}\frac{\partial f(\underline{x})}{\partial z}\right) \\
&= a(\underline{\nabla} f)g + af\underline{\nabla} g + \underline{\nabla} f \frac{dh}{df}.
\end{aligned}$$

13. Exam question June 2001 (Section B): You are given the following family of scalar functions labelled by a real parameter λ : $\Phi_\lambda(x, y, z) = (y - \lambda)\mathrm{cos}x + zxy$.

    (a) What are their derivatives in the direction $\mathbf{V} = \mathbf{e_1} + 2(\mathbf{e_2} + \mathbf{e_3})$?

    Solution $\nabla\Phi_\lambda = \mathbf{e_1}((\lambda - y)\sin x + zy) + \mathbf{e_2}(\cos x + zx) + \mathbf{e_3}xy$ and the directional derivative of Φ_λ in the direction of $\mathbf{V}$ is

$$\frac{\mathbf{V}}{|\mathbf{V}|} \cdot \nabla\Phi_\lambda = \frac{\mathbf{e_1} + 2(\mathbf{e_2} + \mathbf{e_3})}{\sqrt{1+4+4}} \cdot \nabla\Phi_\lambda$$

$$= \frac{1}{3}\left((\lambda - y)\sin x + zy + 2\cos x + 2zx + 2xy\right)$$

(b) Which member of the family has its gradient at the point $(\frac{\pi}{2}, 1, 1)$ equal to $\frac{\pi}{2}(\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3})$?

Solution $\nabla\Phi_\lambda(\frac{\pi}{2}, 1, 1) = \mathbf{e_1}\lambda + \mathbf{e_2}\pi/2 + \mathbf{e_3}\pi/2$ so take $\lambda = \pi/2$.

(c) Calling this particular member of the family Φ_{λ_0}, in which direction is Φ_{λ_0} decreasing most rapidly when starting at the point $(\frac{\pi}{2}, 1, 1)$?

Solution At this point Φ_{λ_0} *decreases* most rapidly in the direction of $-\nabla\Phi_{\lambda_0} = -\frac{\pi}{2}(\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3})$.

14. Exam question June 2002 (Section A): Give the unit vector normal to the surface of equation $x^2/a^2 + y^2/b^2 + z^2/c^2 = 4$ where a, b, c are three real constants.
What is the unit vector normal to a sphere of radius 2 at the point $(x, y, z) = (\sqrt{2}, 0, \sqrt{2})$?

Solution $\nabla f(\mathbf{x})$ is orthogonal to the level surface $f = const.$ at the point $\mathbf{x}$, so take $f = x^2/a^2 + y^2/b^2 + z^2/c^2$, then $\nabla f(\mathbf{x}) = \mathbf{e_1}2x/a^2 + \mathbf{e_2}2y/b^2 + \mathbf{e_3}2z/c^2$ is normal to the surface at $\mathbf{x}$. A unit vector normal to the surface is therefore $\mathbf{n} \equiv \nabla f(\mathbf{x})/|\nabla f(\mathbf{x})| = (\mathbf{e_1}x/a^2 + \mathbf{e_2}y/b^2 + \mathbf{e_3}z/c^2)/\sqrt{x^2/a^4 + y^2/b^4 + z^2/c^4}$
When $a = b = c = 1$ the ellipsoid in the first part of the question becomes a sphere of radius 2, so substituting this and $(x, y, z) = (\sqrt{2}, 0, \sqrt{2})$ into $\mathbf{n}$ gives $(\mathbf{e_1}\sqrt{2} + \mathbf{e_3}\sqrt{2})/2$, which is a unit vector along the radial direction at $(x, y, z) = (\sqrt{2}, 0, \sqrt{2})$, as it should be.

15. Find the vector equations of tangent and normal lines in $\mathbb{R}^2$ to the following curves at the given points

(a) $x^2 + 2y^2 = 3$ at $(1, 1)$,

Solution Set $f(x, y) = x^2 + 2y^2$ so the curve is the level set $f = 3$. $\nabla f = 2x\mathbf{e_1} + 4y\mathbf{e_2}$ is orthogonal to this. At $(1, 1)$ $\nabla f = 2\mathbf{e_1} + 4\mathbf{e_2}$. The line through $(1, 1)$ parallel to $\mathbf{e_1} + 2\mathbf{e_2}$ has vector parametric equation $\mathbf{x} = \mathbf{e_1} + \mathbf{e_2} + t(\mathbf{e_1} + 2\mathbf{e_2})$, this is the normal. The line through $(1, 1)$ orthogonal to $\mathbf{e_1} + 2\mathbf{e_2}$, i.e. parallel to $2\mathbf{e_1} - \mathbf{e_2}$, has vector parametric equation $\mathbf{x} = \mathbf{e_1} + \mathbf{e_2} + t(2\mathbf{e_1} - \mathbf{e_2})$, this is the tangent.

(b) $xy = 1$ at $(2, 1/2)$,

Solution This time, set $f(x, y) = xy$ so the curve is the level set $f = 1$. $\nabla f = y\underline{e}_1 + x\underline{e}_2$, which is equal to $1/2\,\underline{e}_1 + 2\,\underline{e}_2$ at $(2, 1/2)$. The normal line can therefore be written in vector form as $\underline{x} = 2\,\underline{e}_1 + 1/2\,\underline{e}_2 + t(1/2\,\underline{e}_1 + 2\,\underline{e}_2)$. Picking a vector orthogonal to ∇f, say $2\,\underline{e}_1 - 1/2\,\underline{e}_2$, the tangent line can be written as $\underline{x} = 2\,\underline{e}_1 + 1/2\,\underline{e}_2 + t(2\,\underline{e}_1 - 1/2\,\underline{e}_2)$.

(c) $x^2 - y^3 = 3$ at $(2, 1)$.

Solution Now $f(x, y) = x^2 - y^3$, the relevant level set is $f = 3$, and $\nabla f = 2x\underline{e}_1 - 3y^2\underline{e}_2$. At $(2, 1)$ this is $4\,\underline{e}_1 - 3\,\underline{e}_2$ and so an equation for the normal is $\underline{x} = 2\,\underline{e}_1 + \underline{e}_2 + t(4\,\underline{e}_1 - 3\,\underline{e}_2)$, and for the tangent, $\underline{x} = 2\,\underline{e}_1 + \underline{e}_2 + t(3\,\underline{e}_1 + 4\,\underline{e}_2)$.

16. Exam question June 2003 (Section A): Find the directional derivative of the function $\phi(x,y,z) = xy^2z^3$ at the point $P = (1,1,1)$ in the direction from P towards $Q = (3,1,-1)$. Starting from P, in which direction is the directional derivative maximum and what is the value of this maximum?

    Solution The directional derivative of ϕ at P in the direction from P towards $\mathbf{Q} = (3,1,-1)$ is $\mathbf{n}\cdot\nabla\phi(\mathbf{P})$ where $\mathbf{n}$ is a unit vector in this direction, i.e. $\mathbf{n} = (\mathbf{Q}-\mathbf{P})/|\mathbf{Q}-\mathbf{P}|$. Now $\nabla\phi = \mathbf{e_1}y^2z^3 + \mathbf{e_2}2xyz^3 + \mathbf{e_3}3xy^2z^2$, so $\nabla\phi(\mathbf{P}) = \mathbf{e_1} + \mathbf{e_2}2 + \mathbf{e_3}3$, and $\mathbf{n} = (\mathbf{e_1}2 - \mathbf{e_3}2)/\sqrt{8} = (\mathbf{e_1} - \mathbf{e_3})/\sqrt{2}$ so the required directional derivative is $(\mathbf{e_1} + \mathbf{e_2}2 + \mathbf{e_3}3)\cdot(\mathbf{e_1} - \mathbf{e_3})/\sqrt{2}$ which equals $-\sqrt{2}$. The directional derivative is a maximum in the direction of $\mathbf{e_1} + \mathbf{e_2}2 + \mathbf{e_3}3$, i.e. parallel to $\mathbf{e_1} + \mathbf{e_2}2 + \mathbf{e_3}3$, and its value then is $|\nabla\phi| = \sqrt{1+4+9} = \sqrt{14}$.

17. Exam question June 2002 (Section A): What is the derivative of the scalar function $\phi(x,y,z) = x\cos z - y$ in the direction $\mathbf{V} = \mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}$? What is the gradient at the point $(x,y,z) = (0,1,\pi/2)$? In which direction is ϕ increasing the most when moving away from this point?

    Solution $\nabla\phi(x,y,z) = \mathbf{e_1}\cos z - \mathbf{e_2} - \mathbf{e_3}x\sin z$, so the derivative in the direction of $\mathbf{V}$ is $|\mathbf{V}|^{-1}\mathbf{V}\cdot\nabla\phi(x,y,z) = \sqrt{3}^{-1}(\cos z - 1 - x\sin z)$. At $(x,y,z) = (0,1,\pi/2)$ the gradient is $\nabla\phi(x,y,z) = -\mathbf{e_2}$. ϕ increases the most when moving in the direction of $\nabla\phi(x,y,z) = -\mathbf{e_2}$ away from this point.

18. A marble is released from the point $(1,1,c-a-b)$ on the elliptic paraboloid defined by $z = c - ax^2 - by^2$, where a, b, c are positive real numbers and the z-coordinate is vertical. In which direction in the (x,y) plane does the marble begin to roll?

    Solution Here $z = f(x,y)$ is the height of the marble, and this decreases the fastest in the direction of $-\nabla f = 2ax\mathbf{e_1} + 2by\mathbf{e_2} = 2a\mathbf{e_1} + 2b\mathbf{e_2}$ at $(1,1,c-a-b)$.

19. In which direction does the function $f(x,y) = x^2 - y^2$ increase fastest at the points (a) $(1,0)$, (b) $(-1,0)$, (c) $(2,1)$? Illustrate with a sketch.

    Solution f increases the fastest in the direction of its gradient $\nabla f = \mathbf{e_1}2x - \mathbf{e_2}2y$. At (a) $(1,0)$, $\nabla f = 2\mathbf{e_1}$, a unit vector in this direction is $\mathbf{e_1}$, (b) $(-1,0)$, $\nabla f = -2\mathbf{e_1}$, a unit vector in this direction is $-\mathbf{e_1}$, (c) $(2,1)$, $\nabla f = 4\mathbf{e_1} - 2\mathbf{e_2}$ a unit vector in this direction is $(2\mathbf{e_1} - \mathbf{e_2})/\sqrt{5}$.

20. Let $f(x,y) = (x^2-y^2)/(x^2+y^2)$.

    (a) In which direction is the directional derivative of f at $(1,1)$ equal to zero?

    Solution We have $f(x,y) = 1 - 2y^2/(x^2+y^2) = 2x^2/(x^2+y^2) - 1$ so

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(1 - 2y^2/(x^2+y^2)) = 4xy^2/(x^2+y^2)^2$$

and
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2x^2/(x^2+y^2)-1) = -4x^2y/(x^2+y^2)^2$$
So at $(1,1)$ $\nabla f = \mathbf{e_1} - \mathbf{e_2}$. The directional derivative in the direction of the unit vector $\mathbf{n}$ is $\mathbf{n}\cdot\nabla f$, which vanishes when $\mathbf{n}$ and ∇f are perpendicular, i.e. when $\mathbf{n} = \pm(\mathbf{e_1}+\mathbf{e_2})/\sqrt{2}$.

(b) What about at an arbitrary point (x_0, y_0) in the first quadrant?

Solution At (x_0, y_0) $\nabla f = 4x_0y_0(y_0\mathbf{e_1} - x_0\mathbf{e_2})/(x_0^2+y_0^2)^2$ which is perpendicular to $\mathbf{n} = \pm(x_0\mathbf{e_1} + y_0\mathbf{e_2})/\sqrt{x_0^2+y_0^2}$

(c) Describe the level curves of f and discuss them in the light of the result in (b).

Solution The level curves are orthogonal to ∇f, and so tangent to $\mathbf{n}$. They are thus straight lines through the origin.

21. Compute the divergence, $\nabla\cdot\mathbf{A}$, of the following vector fields:

(a) $$\mathbf{A}(x,y,z) = yz\mathbf{e_1} + xz\mathbf{e_2} + xy\mathbf{e_3},$$

Solution
$$\nabla\cdot\mathbf{A} = \frac{\partial(yz)}{\partial x} + \frac{\partial(xz)}{\partial y} + \frac{\partial(xy)}{\partial z} = 0+0+0 = 0\,.$$

(b) $$\mathbf{A}(x,y,z) = (x^2+y^2+z^2)\,(3\mathbf{e_1} + 4\mathbf{e_2} + 5\mathbf{e_3}),$$

Solution
$$\nabla\cdot\mathbf{A} = \frac{\partial(3(x^2+y^2+z^2))}{\partial x} + \frac{\partial(4(x^2+y^2+z^2))}{\partial y} + \frac{\partial(5(x^2+y^2+z^2))}{\partial z} = 6x+8y+10z\,.$$

Alternatively write $\mathbf{A} = |\mathbf{x}\,|^2\mathbf{B}$ with $\mathbf{B} = 3\mathbf{e_1} + 4\mathbf{e_2} + 5\mathbf{e_3}$ a constant vector and use $\nabla\cdot(|\mathbf{x}\,|^2\mathbf{B}) = (\nabla(|\mathbf{x}\,|^2))\cdot\mathbf{B} + |\mathbf{x}\,|^2\nabla\cdot\mathbf{B}$ with
$$\nabla(|\mathbf{x}\,|^2)) = \mathbf{e_1}\frac{\partial(x^2+y^2+z^2)}{\partial x} + \mathbf{e_2}\frac{\partial(x^2+y^2+z^2)}{\partial y} + \mathbf{e_3}\frac{\partial(x^2+y^2+z^2)}{\partial z}$$
$$= \mathbf{e_1}2x + \mathbf{e_2}2y + \mathbf{e_3}2z = 2\mathbf{x}$$
and $\nabla\cdot\mathbf{B} = 0$.

(c) $$\mathbf{A}(x,y,z) = (x+y)\mathbf{e_1} + (y+z)\mathbf{e_2} + (z+x)\mathbf{e_3}.$$

Solution
$$\nabla\cdot\mathbf{A} = \frac{\partial(x+y)}{\partial x} + \frac{\partial(y+z)}{\partial y} + \frac{\partial(z+x)}{\partial z} = 1+1+1 = 3\,.$$

22. Compute the curl, $\nabla\times\mathbf{A}$, of each of the vector fields, $\mathbf{A}$, in the previous question.

Solution

(a) $$\mathbf{A}(x,y,z) \;=\; yz\mathbf{e_1} + xz\mathbf{e_2} + xy\mathbf{e_3},$$

$$\nabla\times\mathbf{A} = \mathbf{e_1}\left(\frac{\partial(xy)}{\partial y} - \frac{\partial(xz)}{\partial z}\right) + \mathbf{e_2}\left(\frac{\partial(yz)}{\partial z} - \frac{\partial(xy)}{\partial x}\right) + \mathbf{e_3}\left(\frac{\partial(xz)}{\partial x} - \frac{\partial(yz)}{\partial y}\right)$$
$$= \mathbf{e_1}(x-x) + \mathbf{e_2}(y-y) + \mathbf{e_3}(z-z) = \mathbf{0}\,.$$

(b) $$\mathbf{A}(x,y,z) \;=\; (x^2+y^2+z^2)\,(3\mathbf{e_1} + 4\mathbf{e_2} + 5\mathbf{e_3}),$$

$$\begin{aligned}\nabla\times\mathbf{A} = \mathbf{e_1}&\left(\frac{\partial(5(x^2+y^2+z^2))}{\partial y} - \frac{\partial(4(x^2+y^2+z^2))}{\partial z}\right)\\ +\mathbf{e_2}&\left(\frac{\partial(3(x^2+y^2+z^2))}{\partial z} - \frac{\partial(5(x^2+y^2+z^2))}{\partial x}\right)\\ +\mathbf{e_3}&\left(\frac{\partial(4(x^2+y^2+z^2))}{\partial x} - \frac{\partial(3(x^2+y^2+z^2))}{\partial y}\right)\\ &= \mathbf{e_1}(10y-8z) + \mathbf{e_2}(6z-10x) + \mathbf{e_3}(8x-6y)\,.\end{aligned}$$

Alternatively write $\mathbf{A} = |\mathbf{x}\,|^2\mathbf{B}$ with $\mathbf{B} = 3\mathbf{e_1}+4\mathbf{e_2}+5\mathbf{e_3}$ a constant vector and use $\nabla\times(|\mathbf{x}\,|^2\mathbf{B}) = (\nabla(|\mathbf{x}\,|^2))\times\mathbf{B} + |\mathbf{x}\,|^2\nabla\times\mathbf{B}$ with $\nabla(|\mathbf{x}\,|^2)) = 2\mathbf{x}$ and $\nabla\times\mathbf{B} = 0$.

(c) $$\mathbf{A}(x,y,z) \;=\; (x+y)\mathbf{e_1} + (y+z)\mathbf{e_2} + (z+x)\mathbf{e_3}.$$

$$\nabla\times\mathbf{A} =$$
$$\mathbf{e_1}\left(\frac{\partial(z+x)}{\partial y} - \frac{\partial(y+z)}{\partial z}\right) + \mathbf{e_2}\left(\frac{\partial(x+y)}{\partial z} - \frac{\partial(z+x)}{\partial x}\right) + \mathbf{e_3}\left(\frac{\partial(y+z)}{\partial x} - \frac{\partial(x+y)}{\partial y}\right)$$
$$= \mathbf{e_1}(0-1) + \mathbf{e_2}(0-1) + \mathbf{e_3}(0-1) = -(\mathbf{e_1}+\mathbf{e_2}+\mathbf{e_3})\,.$$

23. If $f(r)$ is a differentiable function of $r = |\mathbf{x}\,|$, for $\mathbf{x} \in \mathbb{R}^n,\; r\neq 0$, show that

(a) $$\operatorname{grad} f(r) \;=\; f'(r)\,\mathbf{x}\,/r,$$

Solution

$$\operatorname{grad} f(r) = \mathbf{e_1}\frac{\partial f(r)}{\partial x} + \mathbf{e_2}\frac{\partial f(r)}{\partial y} + \mathbf{e_3}\frac{\partial f(r)}{\partial z}$$
$$= \mathbf{e_1}\frac{df}{dr}\frac{\partial r}{\partial x} + \mathbf{e_2}\frac{df}{dr}\frac{\partial r}{\partial y} + \mathbf{e_3}\frac{df}{dr}\frac{\partial r}{\partial z}\,.$$

Now $r = \sqrt{x^2+y^2+z^2}$ so

$$\frac{\partial r}{\partial x} = \frac{1}{2}\frac{2x}{r} = \frac{x}{r}$$

and similarly

$$\frac{\partial r}{\partial y} = \frac{y}{r}\,, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

so

$$\operatorname{grad} f(r) = \frac{df}{dr}\left(\mathbf{e_1}\frac{x}{r} + \mathbf{e_2}\frac{y}{r} + \mathbf{e_3}\frac{z}{r}\right) = f'(r)\mathbf{x}/r\,.$$

(b) $\operatorname{curl}[f(r)\mathbf{x}] = 0$, where now we let $n = 3$.

Solution Using $\operatorname{curl}(f\mathbf{V}) = (\nabla f) \times \mathbf{V} + f\operatorname{curl}\mathbf{V}$, the result to part (a) and

$$\operatorname{curl}\mathbf{x} = \mathbf{e_1}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \mathbf{e_2}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \mathbf{e_3}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = \mathbf{0}\,,$$

gives $\operatorname{curl}(f\mathbf{x}) = (\nabla f) \times \mathbf{x} + f\operatorname{curl}\mathbf{x} = f'(r)\frac{\mathbf{x}}{r} \times \mathbf{x} + \mathbf{0} = \mathbf{0}$ since $\mathbf{x} \times \mathbf{x} = \mathbf{0}$.

24. Let $\mathbf{x}$ be the position vector in three dimensions, with $r = |\mathbf{x}|$, and let $\mathbf{a}$ be a constant vector. Show that

(a) $\operatorname{div}\mathbf{x} = 3$,

Solution

$$\nabla\cdot\mathbf{x} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3\,.$$

(b) $\operatorname{curl}\mathbf{x} = 0$,

Solution

$$\operatorname{curl}\mathbf{x} = \mathbf{e_1}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \mathbf{e_2}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \mathbf{e_3}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = \mathbf{0}\,.$$

(c) $\operatorname{grad} r = \mathbf{x}/r$,

Solution

$$\frac{\partial r}{\partial x} = \frac{1}{2}\frac{2x}{r} = \frac{x}{r}\,, \quad \frac{\partial r}{\partial y} = \frac{y}{r}\,, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

so

$$\operatorname{grad} r = \mathbf{e_1}x/r + \mathbf{e_2}y/r + \mathbf{e_3}z/r = \mathbf{x}/r\,.$$

(d) $\operatorname{div}(r^n\mathbf{x}) = (n+3)\,r^n$,

Solution Use $\operatorname{div}(f\mathbf{V}) = (\nabla f)\cdot\mathbf{V} + f\operatorname{div}\mathbf{V}$ with $f(\mathbf{x}) = r^n$ so that using 21 (a) $\nabla f = nr^{n-1}\mathbf{x}/r = nr^{n-2}\mathbf{x}$ and $\mathbf{V} = \mathbf{x}$ so

$$\operatorname{div}(r^n\mathbf{x}) = nr^{n-2}\mathbf{x}\cdot\mathbf{x} + r^n\operatorname{div}\mathbf{x} = nr^n + 3r^n = (n+3)r^n\,.$$

(e) $\operatorname{grad}(\mathbf{a}\cdot\mathbf{x}) = \mathbf{a}$,

Solution $\mathbf{a}\cdot\mathbf{x} = a_1x + a_2y + a_3z$ so

$$\frac{\partial(\mathbf{a}\cdot\mathbf{x})}{\partial x} = a_1, \quad \frac{\partial(\mathbf{a}\cdot\mathbf{x})}{\partial y} = a_2, \quad \frac{\partial(\mathbf{a}\cdot\mathbf{x})}{\partial z} = a_3$$

hence $\operatorname{grad}(\mathbf{a}\cdot\mathbf{x}) = a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3} = \mathbf{a}\,.$

(f) $\operatorname{div}(\mathbf{a} \times \mathbf{x}) = 0$,

Solution $\mathbf{a} \times \mathbf{x} = \mathbf{e_1}(a_2 z - a_3 y) + \mathbf{e_2}(a_3 x - a_1 z) + \mathbf{e_3}(a_1 y - a_2 x)$ so

$$\operatorname{div}(\mathbf{a} \times \mathbf{x}) = \frac{\partial(a_2 z - a_3 y)}{\partial x} + \frac{\partial(a_3 x - a_1 z)}{\partial y} + \frac{\partial(a_1 y - a_2 x)}{\partial z} = 0\,.$$

(g) $\operatorname{curl}(\mathbf{a} \times \mathbf{x}) = 2\,\mathbf{a}$,

Solution

$$\operatorname{curl}(\mathbf{a} \times \mathbf{x}) = \mathbf{e_1}\left(\frac{\partial(a_1 y - a_2 x)}{\partial y} - \frac{\partial(a_3 x - a_1 z)}{\partial z}\right)$$
$$+\mathbf{e_2}\left(\frac{\partial(a_2 z - a_3 y)}{\partial z} - \frac{\partial(a_1 y - a_2 x)}{\partial x}\right) + \mathbf{e_3}\left(\frac{\partial(a_3 x - a_1 z)}{\partial x} - \frac{\partial(a_2 z - a_3 y)}{\partial y}\right)$$
$$= \mathbf{e_1}(a_1 + a_1) + \mathbf{e_2}(a_2 + a_2) + \mathbf{e_3}(a_3 + a_3) = 2\mathbf{a}\,.$$

(h) $\operatorname{curl}(r^2 \mathbf{a}) = 2\,(\mathbf{x} \times \mathbf{a})$,

Solution Use $\operatorname{curl}(f\mathbf{V}) = (\operatorname{grad} f) \times \mathbf{V} + f \operatorname{curl} \mathbf{V}$ with $f = r^2 = |\mathbf{x}|^2$ and $\nabla(|\mathbf{x}|^2) = 2\mathbf{x}$ so that $\operatorname{curl}(r^2\mathbf{a}) = 2\mathbf{x} \times \mathbf{a} + r^2 \operatorname{curl} \mathbf{a} = 2\mathbf{x} \times \mathbf{a}$.

(i) $\nabla^2(1/r) = 0$, if $r \neq 0$,

Solution $\nabla^2(1/r) \equiv \nabla \cdot \nabla(1/r)$. From 21 (a) with $f = 1/r$ we have $\nabla(1/r) = -\mathbf{x}/r^3$. Then from 22 (d) $\nabla \cdot (\mathbf{x}/r^3) = (-3+3)/r^3 = 0$ so $\nabla^2(1/r) = 0$, but this is only valid for $r \neq 0$ since the calculation involves division by r.

(j) $\nabla^2(\log r) = 1/r^2$, if $r \neq 0$,

Solution $\nabla^2(\log r) \equiv \nabla \cdot \nabla(\log r)$. From 21 (a) with $f = \log r$, $\nabla(\log r) = \mathbf{x}/r^2$. From 22 (d) $\nabla \cdot (\mathbf{x}/r^2) = (-2+3)/r^3 = 1/r^2$ so $\nabla^2(\log r) = 1/r^2$, but as in part (i) this is only valid for $r \neq 0$ since the calculation involves division by r.

(k) $\operatorname{div}[(\mathbf{a} \cdot \mathbf{x})\mathbf{x}] = 4\,\mathbf{a} \cdot \mathbf{x}$,

Solution Using $\operatorname{div} f\mathbf{V} = (\nabla f) \cdot \mathbf{V} + f \operatorname{div} \mathbf{V}$, with $f = \mathbf{a} \cdot \mathbf{x}$ so $\nabla f = \mathbf{a}$ from part (e) and $\mathbf{V} = \mathbf{x}$ so $\operatorname{div} \mathbf{x} = 3$ from part (a) hence $\operatorname{div}[(\mathbf{a} \cdot \mathbf{x})\mathbf{x}] = \mathbf{a} \cdot \mathbf{x} + \mathbf{a} \cdot \mathbf{x}\, 3 = 4\mathbf{a} \cdot \mathbf{x}$.

(l) $\operatorname{div}[\mathbf{x} \times (\mathbf{x} \times \mathbf{a})] = 2\,\mathbf{a} \cdot \mathbf{x}$,

Solution Use $\mathbf{x} \times (\mathbf{x} \times \mathbf{a}) = \mathbf{x}(\mathbf{x} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{x} \cdot \mathbf{x})$ and then from part (k) use $\operatorname{div}(\mathbf{x}(\mathbf{x} \cdot \mathbf{a})) = 4\mathbf{x} \cdot \mathbf{a}$ and

$$\nabla \cdot (|\mathbf{x}|^2 \mathbf{a}) = (\nabla(|\mathbf{x}|^2)) \cdot \mathbf{a} + |\mathbf{x}|^2 \nabla \cdot \mathbf{a} = 2\mathbf{x} \cdot \mathbf{a}$$

so

$$\operatorname{div}[\mathbf{x} \times (\mathbf{x} \times \mathbf{a})] = \operatorname{div}(\mathbf{x}(\mathbf{x} \cdot \mathbf{a})) - \nabla \cdot (|\mathbf{x}|^2 \mathbf{a}) = 4\mathbf{x} \cdot \mathbf{a} - 2\mathbf{x} \cdot \mathbf{a}\,.$$

(m) $\quad \text{curl}\,(\mathbf{a}\times\mathbf{x}/r^3) \;=\; 3\,(\mathbf{a}\cdot\mathbf{x})\mathbf{x}/r^5 \;-\; \mathbf{a}/r^3,$

Solution

$$\text{curl}\,\left(\mathbf{a}\times\mathbf{x}/r^3\right) = (\text{curl}\,(\mathbf{a}\times\mathbf{x}))\,/r^3 + (\nabla r^{-3})\times(\mathbf{a}\times\mathbf{x})$$

Using part (g) curl $(\mathbf{a}\times\mathbf{x}) = 2\mathbf{a}$ together with $\mathbf{x}\times(\mathbf{a}\times\mathbf{x}) = \mathbf{a}(\mathbf{x}\cdot\mathbf{x}) - \mathbf{x}\,(\mathbf{a}\cdot\mathbf{x})$ and $\nabla r^{-3} = -3\mathbf{x}/r^5$ (from 21 (a)) gives

$$\text{curl}\,\left(\mathbf{a}\times\mathbf{x}/r^3\right) \;=\; \frac{2\mathbf{a}}{r^3} - \frac{3}{r^5}(\mathbf{a}r^2 - \mathbf{x}\,(\mathbf{a}\cdot\mathbf{x})) = \;3\,(\mathbf{a}\cdot\mathbf{x})\mathbf{x}/r^5 \;-\; \mathbf{a}/r^3\,.$$

(n) Exam question June 2002 (Section A): calculate the curl of $(\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}$.

Solution

$$\text{curl}\,((\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}) = (\nabla(\mathbf{a}\cdot\mathbf{x}))\times\mathbf{x} + \mathbf{a}\cdot\mathbf{x}\,\text{curl}\,\mathbf{x} = \mathbf{a}\times\mathbf{x}$$

using part parts (e) and (b).

25. If $\mathbf{x}$ is the position vector, $\mathbf{x} = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$, $\mathbf{a}$ is a constant vector, $\mathbf{F} = (\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}$ and $\mathbf{G} = r^2\mathbf{a}$, (with $r = |\mathbf{x}|$), show that

(a) $\quad \text{div}\,\mathbf{F} = 2\;\text{div}\,\mathbf{G} = 4\,\mathbf{a}\cdot\mathbf{x}\,,$

Solution Using the product rule

$$\text{div}\,\mathbf{F} = \nabla\cdot((\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}) = (\nabla(\mathbf{a}\cdot\mathbf{x}))\cdot\mathbf{x} + (\mathbf{a}\cdot\mathbf{x})\,\nabla\cdot\mathbf{x}$$

As in 24 (e), $\mathbf{a}\cdot\mathbf{x} = a_1x + a_2y + a_3z$ so

$$\frac{\partial(\mathbf{a}\cdot\mathbf{x})}{\partial x} = a_1, \quad \frac{\partial(\mathbf{a}\cdot\mathbf{x})}{\partial y} = a_2, \quad \frac{\partial(\mathbf{a}\cdot\mathbf{x})}{\partial z} = a_3$$

hence $\text{grad}\,(\mathbf{a}\cdot\mathbf{x}) \;=\; a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3} \;=\; \mathbf{a}$ and as in 24 (a)

$$\nabla\cdot\mathbf{x} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3\,,$$

so

$$\text{div}\,\mathbf{F} = \nabla\cdot((\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}) = \mathbf{a}\cdot\mathbf{x} + 3\mathbf{a}\cdot\mathbf{x} = 4\mathbf{a}\cdot\mathbf{x}\,.$$

Using the product rule

$$\text{div}\,\mathbf{G} = (\nabla(r^2))\cdot\mathbf{a} + r^2\nabla\cdot\mathbf{a}$$

and $\nabla(|\mathbf{x}|^2) = 2\mathbf{x}$, $\nabla\cdot\mathbf{a} = 0$ so that $\text{div}\,\mathbf{G} = 2\,\mathbf{a}\cdot\mathbf{x}$.

(b) $\quad \text{curl}\,\mathbf{G} = -2\;\text{curl}\,\mathbf{F} = 2\,\mathbf{x}\times\mathbf{a},$

Solution Use $\mathrm{curl}\,(f\mathbf{V}) = (\mathrm{grad}\, f) \times \mathbf{V} + f\mathrm{curl}\,\mathbf{V}$ with $f = r^2 = |\mathbf{x}\,|^2$ and $\nabla(|\mathbf{x}\,|^2) = 2\mathbf{x}$ so that

$$\mathrm{curl}\,\mathbf{G} = \mathrm{curl}\,(r^2\mathbf{a}) = 2\mathbf{x} \times \mathbf{a} + r^2\mathrm{curl}\,\mathbf{a} = 2\mathbf{x} \times \mathbf{a}\,.$$

Now use this product rule with $f = \mathbf{a}\cdot\mathbf{x}$ and $\mathbf{V} = \mathbf{x}$ so that

$$\nabla \times ((\mathbf{a}\cdot\mathbf{x}\,)\,\mathbf{x}\,) = (\nabla(\mathbf{a}\cdot\mathbf{x}\,)) \times \mathbf{x} + (\mathbf{a}\cdot\mathbf{x}\,)\nabla\times\mathbf{x}$$

As in 24 (b)

$$\mathrm{curl}\,\mathbf{x} = \mathbf{e_1}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \mathbf{e_2}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \mathbf{e_3}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = \mathbf{0}\,,$$

so

$$\nabla \times ((\mathbf{a}\cdot\mathbf{x}\,)\,\mathbf{x}\,) = \mathbf{a}\times\mathbf{x} = -\mathbf{x} \times \mathbf{a}\,.$$

(c) $\mathrm{div}\,\mathrm{curl}\,\mathbf{F} = \mathrm{div}\,\mathrm{curl}\,\mathbf{G} = 0,$

Solution $\mathrm{div}\,\mathrm{curl}\,\mathbf{F} = \mathrm{div}\,(\mathbf{a}\times\mathbf{x}\,) = 0$ as in 24 (f):
$\mathbf{a}\times\mathbf{x} = \mathbf{e_1}(a_2z - a_3y) + \mathbf{e_2}(a_3x - a_1z) + \mathbf{e_3}(a_1y - a_2x)$ so

$$\mathrm{div}\,(\mathbf{a}\times\mathbf{x}\,) = \frac{\partial(a_2z - a_3y)}{\partial x} + \frac{\partial(a_3x - a_1z)}{\partial y} + \frac{\partial(a_1y - a_2x)}{\partial z} = 0\,.$$

Since $\mathrm{curl}\,\mathbf{G} = -2\ \mathrm{curl}\,\mathbf{F}$ this implies that $\mathrm{div}\,\mathrm{curl}\,\mathbf{G} = 0\,.$

(d) $\mathrm{curl}\,\mathrm{curl}\,\mathbf{G} = -2\ \mathrm{curl}\,\mathrm{curl}\,\mathbf{F} = -4\,\mathbf{a}.$

Solution

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{G} == 2\mathrm{curl}\,(\mathbf{x} \times \mathbf{a}) = -4\mathbf{a}$$

using $\mathbf{a}\times\mathbf{x} = -\mathbf{x} \times \mathbf{a}$ and 22 (g)

$$\mathrm{curl}\,(\mathbf{a}\times\mathbf{x}\,) = \mathbf{e_1}\left(\frac{\partial(a_1y - a_2x)}{\partial y} - \frac{\partial(a_3x - a_1z)}{\partial z}\right)$$

$$+\mathbf{e_2}\left(\frac{\partial(a_2z - a_3y)}{\partial z} - \frac{\partial(a_1y - a_2x)}{\partial x}\right) + \mathbf{e_3}\left(\frac{\partial(a_3x - a_1z)}{\partial x} - \frac{\partial(a_2z - a_3y)}{\partial y}\right)$$

$$= \mathbf{e_1}(a_1 + a_1) + \mathbf{e_2}(a_2 + a_2) + \mathbf{e_3}(a_3 + a_3) = 2\mathbf{a}\,.$$

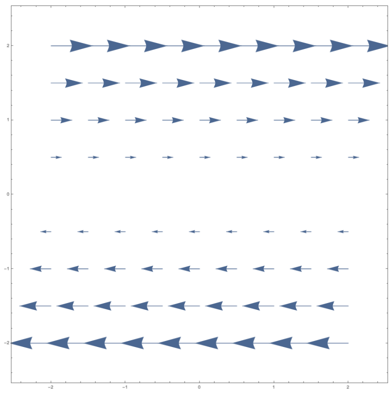
Since $\mathrm{curl}\,\mathbf{G} = -2\ \mathrm{curl}\,\mathbf{F}$ this implies that

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{G} = -2\ \mathrm{curl}\,\mathrm{curl}\,\mathbf{F} = -4\,\mathbf{a}\,.$$

26. Exam question June 2001 (Section A):

(a) Give a representation of the vector function $\mathbf{A}(x, y) = y\mathbf{e_1}$ as a collection of arrows in the region of the (x, y)-plane bounded by $(x_1, y_1) = (-2, 2), (x_2, y_2) = (2, 2), (x_3, y_3) = (2, -2), (x_4, y_4) = (-2, -2)$.

Solution



The vector field $\mathbf{A}(x, y)$

(b) Calculate the curl of the vector field $\mathbf{A}(x, y) = (-y\mathbf{e_1} + x\mathbf{e_2})/(x^2 + y^2)$ defined everywhere in the (x, y)-plane except at the origin. (You can consider $\mathbf{A}$ to be embedded in three dimensions, independent of z and with zero z component.)

Solution

$$\nabla\times\mathbf{A} = \mathbf{e_1}\left(\frac{\partial 0}{\partial y} - \frac{\partial}{\partial z}\frac{x}{x^2+y^2}\right)+\mathbf{e_2}\left(\frac{\partial}{\partial z}\frac{-y}{x^2+y^2} - \frac{\partial 0}{\partial x}\right)+\mathbf{e_3}\left(\frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2}\right)$$

$$= \mathbf{e_3}\left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right) = \underline{0}\,,$$

(c) Give the unit vector normal to the surface of equation $ax + by = cz$, where a, b, c, are three real constants.

Solution $ax+by = cz$ is the equation of a plane, the general form of which would be $\mathbf{x}\cdot\mathbf{V} = k$ where $\mathbf{V}$ is orthogonal to the plane. Here $\mathbf{V} = (a, b, -c)$, so a unit vector normal to the surface is $(a/\sqrt{a^2+b^2+c^2}, b/\sqrt{a^2+b^2+c^2}, -c/\sqrt{a^2+b^2+c^2})$. Alternatively: in general ∇f is normal to the level-surface $f = k$, here $f = ax + by - cz$ so $\nabla f = (a, b, -c)$ and again a unit vector normal to the surface is $\nabla f/|\nabla f| = (a/\sqrt{a^2+b^2+c^2}, b/\sqrt{a^2+b^2+c^2}, -c/\sqrt{a^2+b^2+c^2})$.

(d) (Slightly modified from exam) Let $\mathbf{x}$ be the position vector in 3-dimensions and $\mathbf{a}$ be a constant vector. Use the result $\mathbf{x}\times(\mathbf{x}\times\mathbf{a}) = \mathbf{x}\,(\mathbf{x}\cdot\mathbf{a}) - \mathbf{a}(\mathbf{x}\cdot\mathbf{x})$ to show that $\mathrm{div}\,[\,\mathbf{x}\times(\mathbf{x}\times\mathbf{a})] = 2\mathbf{a}\cdot\mathbf{x}$.

Solution Using $\mathbf{x}\times(\mathbf{x}\times\mathbf{a}) = \mathbf{x}\,(\mathbf{x}\cdot\mathbf{a}) - \mathbf{a}(\mathbf{x}\cdot\mathbf{x})$, then $\mathrm{div}\,[\,\mathbf{x}\times(\mathbf{x}\times\mathbf{a})] = \mathrm{div}\,(\mathbf{x}\,(\mathbf{x}\cdot\mathbf{a}) - \mathbf{a}(\mathbf{x}\cdot\mathbf{x})) = (\nabla\cdot\mathbf{x})(\mathbf{x}\cdot\mathbf{a}) + \mathbf{x}\cdot\nabla(\mathbf{x}\cdot\mathbf{a}) - \mathbf{a}\cdot\nabla(\mathbf{x}\cdot\mathbf{x})$. This is $3(\mathbf{x}\cdot\mathbf{a}) + (\mathbf{x}\cdot\mathbf{a}) - 2(\mathbf{x}\cdot\mathbf{a}) = 2(\mathbf{x}\cdot\mathbf{a})$

27. Let $\underline{v} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Prove the vector identity

$$\underline{v}\times(\underline{\nabla}\times\underline{v}) = \underline{\nabla}(|\underline{v}|^2/2) - (\underline{v}\cdot\underline{\nabla})\underline{v}.$$

Solution We compute the curl of $\underline{v}$ as

$$\underline{\nabla} \times \underline{v} = \underline{e}_1(\partial_2 v_3 - \partial_3 v_2) + \underline{e}_2(\partial_3 v_1 - \partial_1 v_3) + \underline{e}_3(\partial_1 v_2 - \partial_2 v_1),$$

where we have used the shorthand $\partial_i \equiv \frac{\partial}{\partial x_i}$. The left hand side of the identity can therefore be written as

$$\begin{aligned}
\underline{v} \times (\underline{\nabla} \times \underline{v}) = \;& \underline{e}_1(v_2\,[\partial_1 v_2 - \partial_2 v_1] - v_3\,[\partial_3 v_1 - \partial_1 v_3]) \\
&+ \underline{e}_2(v_3\,[\partial_2 v_3 - \partial_3 v_2] - v_1\,[\partial_1 v_2 - \partial_2 v_1]) \\
&+ \underline{e}_3(v_1\,[\partial_3 v_1 - \partial_1 v_3] - v_2\,[\partial_2 v_3 - \partial_3 v_2]) \\
= \;& \underline{e}_1(v_2\partial_1 v_2 + v_3\partial_1 v_3 + v_1\partial_1 v_1 - v_1\partial_1 v_1 - v_2\partial_2 v_1 - v_3\partial_3 v_1) \\
&+ \underline{e}_2(v_1\partial_2 v_1 + v_3\partial_2 v_3 + v_2\partial_2 v_2 - v_2\partial_2 v_2 - v_1\partial_1 v_2 - v_3\partial_3 v_2) \\
&+ \underline{e}_3(v_1\partial_3 v_1 + v_2\partial_3 v_2 + v_3\partial_3 v_3 - v_3\partial_3 v_3 - v_1\partial_1 v_3 - v_2\partial_2 v_3) \\
= \;& \underline{e}_1(\partial_1|\underline{v}|^2/2 - (\underline{v}\cdot\underline{\nabla})v_1) + \underline{e}_2(\partial_2|\underline{v}|^2/2 - (\underline{v}\cdot\underline{\nabla})v_2) \\
&+ \underline{e}_3(\partial_3|\underline{v}|^2/2 - (\underline{v}\cdot\underline{\nabla})v_3) \\
= \;& \underline{\nabla}(|\underline{v}|^2/2) - (\underline{v}\cdot\underline{\nabla})\underline{v},
\end{aligned}$$

proving the given identity.

28. Let $\mathbf{x}$ be the position vector in three dimensions, with $r = |\mathbf{x}\,|$, and let $\mathbf{a}$ be a constant vector. Using index notation, show that

(a) $\operatorname{div}\mathbf{x} \;=\; 3,$

Solution

$$\operatorname{div}\mathbf{x} \;=\; \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3\,.$$

(b) $\operatorname{curl}\mathbf{x} \;= 0,$

Solution

$$(\operatorname{curl}\mathbf{x}\,)_i = \varepsilon_{ijk}\frac{\partial x_k}{\partial x_j} = \varepsilon_{ijk}\delta_{jk} = 0\,,$$

using the fact that ε_{ijk} is antisymmetric under swaps of j and k while δ_{jk} is symmetric.

(c) $\operatorname{grad} r \;=\; \mathbf{x}\,/r,$

Solution

$$\begin{aligned}
(\operatorname{grad} r)_i = \frac{\partial}{\partial x_i} r = \;& \frac{\partial}{\partial x_i}(x_j x_j)^{1/2} \\
=\;& \frac{1}{2}(x_j x_j)^{-1/2}\,\frac{\partial}{\partial x_i} x_k x_k \\
=\;& \frac{1}{2}(x_j x_j)^{-1/2}\,(2\delta_{ik}x_k) \\
=\;& (x_j x_j)^{-1/2}\,x_i \\
=\;& x_i/r\,.
\end{aligned}$$

(d) $\operatorname{div}(r^n \mathbf{x}) \;=\; (n+3)\, r^n$,

Solution

$$\begin{aligned}\operatorname{div}(r^n \underline{x}) &= \frac{\partial}{\partial x_i}(r^n x_i) = (\frac{\partial}{\partial x_i} r^n) x_i + r^n (\frac{\partial}{\partial x_i} x_i) = n r^{n-1} (\frac{\partial}{\partial x_i} r) x_i + r^n \delta_{ii} \\ &= n r^{n-1} (x_i / r)\, x_i + 3 r^n = n r^n + 3 r^n = (n+3) r^n .\end{aligned}$$

(e) $\operatorname{grad}(\mathbf{a}\cdot\mathbf{x}) \;=\; \mathbf{a}$,

Solution

$$(\operatorname{grad}(\underline{a}\cdot\underline{x}))_i = \frac{\partial}{\partial x_i}(a_j x_j) = a_j \frac{\partial}{\partial x_i} x_j = a_j \delta_{ij} = a_i \,.$$

(f) $\operatorname{div}(\mathbf{a}\times\mathbf{x}) \;=0$,

Solution

$$\operatorname{div}(\underline{a}\times\underline{x}) = \frac{\partial}{\partial x_i}(\varepsilon_{ijk} a_j x_k) = \varepsilon_{ijk} a_j \delta_{ik} = 0\,,$$

using the fact that ε_{ijk} is antisymmetric under swaps of i and k while δ_{ik} is symmetric.

(g) $\operatorname{curl}(\mathbf{a}\times\mathbf{x}) \;=\; 2\,\mathbf{a}$,

Solution

$$\begin{aligned}(\operatorname{curl}(\underline{a}\times\underline{x}))_i &= \varepsilon_{ijk} \frac{\partial}{\partial x_j}(\varepsilon_{klm} a_l x_m) = \varepsilon_{ijk}\varepsilon_{klm} a_l \frac{\partial}{\partial x_j} x_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_l \frac{\partial}{\partial x_j} x_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_l \delta_{jm} \\ &= a_i \delta_{jj} - \delta_{ij} a_j = 3a_i - a_i = 2a_i \,.\end{aligned}$$

(h) $\operatorname{curl}(r^2\mathbf{a}) \;=\; 2\,(\mathbf{x}\times\mathbf{a})$,

Solution

$$\begin{aligned}(\operatorname{curl}(r^2\mathbf{a}))_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}(r^2 \mathbf{a}_k) = \varepsilon_{ijk}\Big(\frac{\partial}{\partial x_j} r^2\Big)\mathbf{a}_k \\ &= \varepsilon_{ijk} 2r \frac{\partial r}{\partial x_j}\,\mathbf{a}_k = \varepsilon_{ijk} 2r \frac{x_j}{r}\,\mathbf{a}_k = (2\,(\mathbf{x}\times\mathbf{a}))_i \,.\end{aligned}$$

(i) $\nabla^2(1/r) \;=\; 0$ if $r \neq 0$: using $\frac{\partial}{\partial x_i} r = x_i/r$ from part (c),

Solution

$$\begin{aligned}
\nabla^2(1/r) &= \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}(r^{-1}) = \frac{\partial}{\partial x_i}\big(-r^{-2}\frac{\partial}{\partial x_i}r\big) = \frac{\partial}{\partial x_i}\big(-r^{-3}x_i\big) \\
&= -\big(\frac{\partial}{\partial x_i}r^{-3}\big)x_i - r^{-3}\big(\frac{\partial}{\partial x_i}x_i\big) = 3r^{-4}\big(\frac{\partial}{\partial x_i}r\big)x_i - r^{-3}\delta_{ii} \\
&= 3r^{-5}x_ix_i - 3r^{-3} = (3-3)r^{-3} = 0\,.
\end{aligned}$$

Notice that this is only valid for $r \neq 0$ since the calculation involves division by r.

(j) $\nabla^2(\log r) \;=\; 1/r^2$ if $r \neq 0:$

Solution as in part (i), and using $\frac{\partial}{\partial x_i}r = x_i/r$ again,

$$\begin{aligned}
\nabla^2(\log r) &= \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}(\log r) = \frac{\partial}{\partial x_i}\big(r^{-1}\frac{\partial}{\partial x_i}r\big) = \frac{\partial}{\partial x_i}\big(r^{-2}x_i\big) \\
&= \big(\frac{\partial}{\partial x_i}r^{-2}\big)x_i + r^{-2}\big(\frac{\partial}{\partial x_i}x_i\big) = -2r^{-3}\big(\frac{\partial}{\partial x_i}r\big)x_i + r^{-2}\delta_{ii} \\
&= -2r^{-4}x_ix_i + 3r^{-2} = (-2+3)r^{-2} = 1/r^2.
\end{aligned}$$

Again, this is only valid for $r \neq 0$ since the calculation involves division by r.

(k) $\operatorname{div}[(\mathbf{a}\cdot\mathbf{x})\mathbf{x}] \;=\; 4\,\mathbf{a}\cdot\mathbf{x}\,,$

Solution

$$\begin{aligned}
\operatorname{div}[(\mathbf{a}\cdot\mathbf{x})\mathbf{x}] &= \frac{\partial}{\partial x_i}\big(a_jx_jx_i\big) = a_j\frac{\partial}{\partial x_i}\big(x_jx_i\big) = a_j\Big(\big(\frac{\partial}{\partial x_i}x_j\big)x_i + x_j\big(\frac{\partial}{\partial x_i}x_i\big)\Big) \\
&= a_j\big(\delta_{ij}x_i + x_j\delta_{ii}\big) = a_j\big(x_j + 3x_j\big) = 4a_jx_j = 4\underline{a}\cdot\underline{x}\,.
\end{aligned}$$

Using $\operatorname{div} f\mathbf{V} = (\nabla f)\cdot\mathbf{V} + f\operatorname{div}\mathbf{V}$, with $f = \mathbf{a}\cdot\mathbf{x}$ so $\nabla f = \mathbf{a}$ from part (e) and $\mathbf{V} = \mathbf{x}$ so $\operatorname{div}\mathbf{x} = 3$ from part (a) hence $\operatorname{div}[(\mathbf{a}\cdot\mathbf{x})\mathbf{x}] = \mathbf{a}\cdot\mathbf{x} + \mathbf{a}\cdot\mathbf{x}\,3 = 4\mathbf{a}\cdot\mathbf{x}\,.$

(l) $\operatorname{div}[\mathbf{x}\times(\mathbf{x}\times\mathbf{a})] \;=\; 2\,\mathbf{a}\cdot\mathbf{x}\,,$

Solution

$$\begin{aligned}
\operatorname{div}[\mathbf{x}\times(\mathbf{x}\times\mathbf{a})] &= \frac{\partial}{\partial x_i}\left(\varepsilon_{ijk}x_j\varepsilon_{klm}x_la_m\right) = \varepsilon_{ijk}\varepsilon_{klm}\frac{\partial}{\partial x_i}\left(x_jx_l\right)a_m \\
&= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\frac{\partial}{\partial x_i}\left(x_jx_l\right)a_m \\
&= \frac{\partial}{\partial x_i}\left(x_jx_i\right)a_j - \frac{\partial}{\partial x_i}\left(x_jx_j\right)a_i \\
&= \delta_{ij}x_ia_j + x_j\delta_{ii}a_j - \delta_{ij}x_ja_i - x_j\delta_{ij}a_i \\
&= x_ja_j + 3x_ja_j - x_ja_j - x_ja_j = 2x_ja_j = 2\underline{a}\cdot\underline{x}\,.
\end{aligned}$$

(m) curl $(\mathbf{a} \times \mathbf{x}/r^3) = 3(\mathbf{a}\cdot\mathbf{x})\mathbf{x}/r^5 - \mathbf{a}/r^3$,

Solution

$$
\begin{aligned}
\big(\text{curl}\ (\mathbf{a}\times\mathbf{x}/r^3)\big)_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\varepsilon_{klm}a_l x_m/r^3\right) = \varepsilon_{ijk}\varepsilon_{klm}a_l\frac{\partial}{\partial x_j}\left(x_m/r^3\right)\\
&= \varepsilon_{ijk}\varepsilon_{klm}a_l\Big(\delta_{jm}r^{-3} + x_m\frac{\partial}{\partial x_j}r^{-3}\Big)\\
&= \varepsilon_{ijk}\varepsilon_{klm}a_l\Big(\delta_{jm}r^{-3} - 3x_m r^{-4}\frac{\partial}{\partial x_j}r\Big)\\
&= \varepsilon_{ijk}\varepsilon_{klm}a_l\Big(\delta_{jm}r^{-3} - 3x_m r^{-5}x_j\Big)\\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,a_l\Big(\delta_{jm}r^{-3} - 3x_m r^{-5}x_j\Big)\\
&= a_i\delta_{jj}r^{-3} - a_j\delta_{ij}r^{-3} - 3a_i x_j r^{-5}x_j + 3a_j x_i r^{-5}x_j\\
&= 3a_i r^{-3} - a_i r^{-3} - 3a_i r^{-3} + 3x_i a_j x_j r^{-5}\\
&= -a_i r^{-3} + 3x_i a_j x_j r^{-5} = \big(-\underline{a}/r^3 + 3\underline{x}(\underline{a}\cdot\underline{x})/r^5\big)_i\ .
\end{aligned}
$$

Remark: in this question, and also in some of the earlier ones, there is some freedom in the order in which you, for example, use the product rule on derivatives, use the very useful formula, and simplify δs involving summed indices. In such cases any order is fine – the final answer will be the same.

(n) Exam question June 2002 (Section A): calculate the curl of $(\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}$.

Solution

$$
\begin{aligned}
\big(\text{curl}\,((\mathbf{a}\cdot\mathbf{x})\,\mathbf{x})\big)_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\big(a_l x_l x_k\big)\\
&= \varepsilon_{ijk}a_l\big(\delta_{jl}x_k + x_l\delta_{jk}\big)\\
&= \varepsilon_{ijk}a_j x_k + \varepsilon_{ijj}a_l x_l = \varepsilon_{ijk}a_j x_k = \big(\underline{a}\times\underline{x}\big)_i\,,
\end{aligned}
$$

using the fact that $\varepsilon_{ijj} = 0$, by the antisymmetry of the Levi-Civita symbol. Note: in the first line $\underline{a}\cdot\underline{x}$ was written using l as the repeated index rather than j or k, since j and k were already being used to write the curl.

General remark: Even though some of these index notation calculations look a little long, they are (almost) all self-contained, while in the version of this question on the Topic 3 sheet, there was much more quoting of results from other parts.

29. The vector $\underline{a}$ has components $(a_r) = (1,1,1)$ and the vector $\underline{b}$ has components $(b_r) = (2,3,4)$. In the following expressions state which indices are free and which are dummy, and give the numerical values of the expressions for each value that the free variable takes (e.g. for $a_r - b_r$ the free variable is r and it takes the values 1, 2, 3 so $a_1 - b_1 = -1$, $a_2 - b_2 = -2$, $a_3 - b_3 = -3$)

    (a) $a_r + b_r$,

Solution r is a free index so we need to specify 3 pieces of information. $a_r + b_r$ is the r-th component of the vector $\underline{a} + \underline{b}$ so we can give the information in the form $(a_r + b_r) = (3, 4, 5)$ or as $a_1 + b_1 = 3$, $a_2 + b_2 = 4$, $a_3 + b_3 = 5$.

(b) $a_r\, b_r$,

Solution r is a dummy index and $a_r\, b_r = a_1\, b_1 + a_2\, b_2 + a_3\, b_3 = 9$, which is the scalar product of $\underline{a}$ and $\underline{b}$.

(c) $a_r\, b_s\, a_r$,

Solution r is a dummy index, and s is free so to specify $a_r\, b_s\, a_r$ we need to specify 3 pieces of information, these are the components of the vector $\underline{b}\,\underline{a} \cdot \underline{a}$. Now $a_r\, a_r = a_1^2 + a_2^2 + a_3^2 = 3$ so $(a_r\, b_s\, a_r) = (3b_s) = (6, 9, 12)$

(d) $a_r\, b_s\, a_r\, b_s - a_r\, b_r\, a_s\, b_s$.

Solution r and s are both dummy indices. $a_r\, b_s\, a_r\, b_s - a_r\, b_r\, a_s\, b_s = (a_r\, a_r)(b_s\, b_s) - (a_r\, b_r)(a_s\, b_s)$ now $a_r\, a_r = 3$ and $b_s\, b_s = 4+9+16 = 29$ so $a_r\, b_s\, a_r\, b_s - a_r\, b_r\, a_s\, b_s = 3 \times 29 - 9^2 = 6$.

30. If δ_{rs} is the three-dimensional Kronecker delta, evaluate

(a) $\delta_{rs}\, \delta_{sr}\, \delta_{pq}\, \delta_{pq}$,

Solution $\delta_{rs}\, \delta_{sr}\, \delta_{pq}\, \delta_{pq} = \delta_{rr}\, \delta_{pq}\, \delta_{pq} = 3\, \delta_{pp} = 3 \times 3 = 9$.

(b) $\delta_{rs}\, \delta_{sk}\, \delta_{kl}\, \delta_{lr}$,

Solution $\delta_{rs}\, \delta_{sk}\, \delta_{kl}\, \delta_{lr} = \delta_{rk}\, \delta_{kl}\, \delta_{lr} = \delta_{rl}\, \delta_{lr} = \delta_{rr} = 3$.

(c) $\delta_{rs}\, \delta_{qr}\, \delta_{pq}\, \delta_{sp}$.

Solution $\delta_{rs}\, \delta_{qr}\, \delta_{pq}\, \delta_{sp} = \delta_{qs}\, \delta_{pq}\, \delta_{sp} = \delta_{ps}\, \delta_{sp} = \delta_{ss} = 3$.

31. If δ_{rs} is the three-dimensional Kronecker delta, simplify

(a) $(\delta_{rp}\, \delta_{sq} - \delta_{rq}\, \delta_{sp})\, a_p\, b_q$,

Solution $(\delta_{rp}\, \delta_{sq} - \delta_{rq}\, \delta_{sp})\, a_p\, b_q = \delta_{rp}\, \delta_{sq}\, a_p\, b_q - \delta_{rq}\, \delta_{sp}\, a_p\, b_q = \ a_r\, b_s - \ a_s\, b_r$,

(b) $(\delta_{rp}\, \delta_{sq} - \delta_{rq}\, \delta_{sp})\, \delta_{pq}$.

Solution $(\delta_{rp}\, \delta_{sq} - \delta_{rq}\, \delta_{sp})\, \delta_{pq} = \delta_{rp}\, \delta_{sq}\, \delta_{pq} - \delta_{rq}\, \delta_{sp}\, \delta_{pq} = \delta_{rs} - \delta_{rs} = 0$.

32. If δ_{rs} is the Kronecker delta in n dimensions, calculate

(a) δ_{rr},

Solution $\delta_{rr} = 1 + 1 + .. + 1$ (n terms), so $\delta_{rr} = n$,

(b) $\delta_{rs}\, \delta_{rs}$,

Solution $\delta_{rs}\, \delta_{rs} = \delta_{rr} = n$,

(c) $\delta_{rs}\,\delta_{st}\,\delta_{tr}$.

Solution $\delta_{rs}\,\delta_{st}\,\delta_{tr} = \delta_{rs}\,\delta_{sr} = \delta_{rr} = n$.

33. Starting from $\varepsilon_{ijk}\,\varepsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$ simplify as much as possible:

(a) $\varepsilon_{ijk}\varepsilon_{ijp}$,

Solution First use the cyclic symmetry of ε_{ijk} to rewrite the expression with the repeated indices next to each other: $\varepsilon_{ijk}\varepsilon_{ijp} = \varepsilon_{jki}\varepsilon_{ijp}$. Then, relabelling in the starting formula,

$$\varepsilon_{jki}\varepsilon_{ijp} = \delta_{jj}\,\delta_{kp} - \delta_{jp}\,\delta_{kj} = 3\,\delta_{kp} - \delta_{kp} = 2\,\delta_{kp}$$

(b) $\varepsilon_{ijk}\varepsilon_{ijk}$.

Solution Substituting $p = k$ into part (a),

$$\varepsilon_{ijk}\varepsilon_{ijk} = 2\,\delta_{kk} = 6$$

34. Calculate ε_{ijj} .

Solution ε_{ijk} vanishes whenever two indices take the same value, so ε_{ijj} also vanishes, being a sum of terms in each of which the last two indices take the same value.

35. Show, using index notation, that

(a) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = 0$,

Solution The i-th component of $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is

$$\begin{aligned}\varepsilon_{ijk}\,a_j\,(\varepsilon_{krs}\,b_r\,c_s) &= (\varepsilon_{ijk}\,\varepsilon_{rsk})\,a_j\,b_r\,c_s = (\delta_{ir}\,\delta_{js} - \delta_{is}\,\delta_{jr})\,a_j\,b_r\,c_s\\ &= \delta_{ir}\,\delta_{js}\,a_j\,b_r\,c_s - \delta_{is}\,\delta_{jr}\,a_j\,b_r\,c_s = a_s\,b_i\,c_s - a_r\,b_r\,c_i\end{aligned}$$

so the i-th component of $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b})$ is

$$\begin{aligned}&(a_s\,b_i\,c_s - a_r\,b_r\,c_i) + (b_s\,c_i\,a_s - b_r\,c_r\,a_i) + (c_s\,a_i\,b_s - c_r\,a_r\,b_i)\\ &\quad = a_i\,(c_s\,b_s - b_r\,c_r) + b_i\,(a_s\,c_s - c_r\,a_r) + c_i\,(b_s\,a_s - a_r\,b_r) = 0\end{aligned}$$

since the names of dummy indices can be changed.

(b) $$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= [\mathbf{d}\cdot(\mathbf{a}\times\mathbf{b})]\,\mathbf{c} - [\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})]\,\mathbf{d}\\ &= [\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})]\,\mathbf{b} - [\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})]\,\mathbf{a},\end{aligned}$$

Solution The i-th component of $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ is

$$\begin{aligned}\varepsilon_{ijk}\;(\varepsilon_{jrs}\,a_r\,b_s)\;(\varepsilon_{kpq}\,c_p\,d_q) &= \varepsilon_{ijk}\,\varepsilon_{jrs}\,\varepsilon_{kpq}\,a_r\,b_s\,c_p\,d_q\\ &= (\varepsilon_{jki}\,\varepsilon_{jrs})\,\varepsilon_{kpq}\,a_r\,b_s\,c_p\,d_q\\ &= \delta_{kr}\,\delta_{is}\,\varepsilon_{kpq}\,a_r\,b_s\,c_p\,d_q - \delta_{ks}\,\delta_{ir}\,\varepsilon_{kpq}\,a_r\,b_s\,c_p\,d_q\\ &= \varepsilon_{rpq}\,a_r\,b_i\,c_p\,d_q - \varepsilon_{spq}\,a_i\,b_s\,c_p\,d_q\end{aligned}$$

which is the same as the i-th component of $[\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})]\,\mathbf{b}\;-\;[\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})]\,\mathbf{a}$, because this is

$$[\,a_r\,(\varepsilon_{rpq}\,c_p\,d_q)]\;b_i\;-\;[\,b_s\,(\varepsilon_{spq}\,c_p\,d_q)]\;a_i$$

We can also expand the i-th component of $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ as

$$\varepsilon_{ijk}\;(\varepsilon_{jrs}\,a_r\,b_s)\;(\varepsilon_{kpq}\,c_p\,d_q) = \varepsilon_{ijk}\,\varepsilon_{jrs}\,\varepsilon_{kpq}\,a_r\,b_s\,c_p\,d_q$$

$$\begin{aligned}(\varepsilon_{ijk}\,\varepsilon_{pqk})\,\varepsilon_{jrs}\,a_r\,b_s\,c_p\,d_q &= \delta_{ip}\,\delta_{jq}\,\varepsilon_{jrs}\,a_r\,b_s\,c_p\,d_q - \delta_{iq}\,\delta_{jp}\,\varepsilon_{jrs}\,a_r\,b_s\,c_p\,d_q\\ &= \varepsilon_{qrs}\,a_r\,b_s\,c_i\,d_q\varepsilon_{prs}\,a_r\,b_s\,c_p\,d_i\end{aligned}$$

which is the same as the i-th component of $[\mathbf{d}\cdot(\mathbf{a}\times\mathbf{b})]\,\mathbf{c}\;-\;[\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})]\,\mathbf{d}$, because this is

$$[\,d_q\,(\varepsilon_{qrs}\,a_r\,b_s)]\;c_i\;-\;[\,c_p\,(\varepsilon_{prs}\,a_r\,b_s\,)]\;d_i\,.$$

(c) $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) \;=\; (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) \;-\; (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})\,,$

Solution

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= (\mathbf{a}\times\mathbf{b})_i(\mathbf{c}\times\mathbf{d})_i\\ &= \epsilon_{ijk}a_jb_k\epsilon_{ilm}c_ld_m\\ &= \epsilon_{ijk}\epsilon_{ilm}a_jb_kc_ld_m\\ &= (\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})a_jb_kc_ld_m\\ &= a_jc_jb_kd_k - a_jd_jb_kc_k\\ &= (\mathbf{a}\cdot\mathbf{c})(\mathbf{d}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}).\end{aligned}$$

(d) $\mathbf{a}\times[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})] \;=\; a^2\,(\mathbf{b}\times\mathbf{a})\,,$

Solution Note that the left-hand side of this expression is a vector, so we will write down the ith component of this vector, and check that this gives the ith component of the right-hand side of the expression. If this holds for all i, the expression is then true as a statement about vectors.

$$\begin{aligned}(\mathbf{a}\times[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})])_i &= \epsilon_{ijk}a_j\epsilon_{klm}a_l\epsilon_{mno}a_nm_o\\ &= \epsilon_{ijk}(\delta_{kn}\delta_{lo}-\delta_{ko}\delta_{ln})a_ja_la_nb_o\\ &= \epsilon_{ijk}a_ja_ka_lb_l - \epsilon_{ijk}a_jb_ka_la_l\\ &= (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\times\mathbf{a})_i - (\mathbf{a}\cdot\mathbf{a})(\mathbf{a}\times\mathbf{b})_i.\end{aligned}$$

Now note that $\mathbf{a}\times\mathbf{a}=0$ for all vectors $\mathbf{a}$, and recall that $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$, so we have

$$(\mathbf{a}\times[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})]) = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\times\mathbf{a}),$$

and letting $a=|\mathbf{a}|$, we have the required result.

(e) $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})+(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})=0\,.$

Solution Using part (a), we have

$$\begin{aligned}&(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})+(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})\\ &\quad=(\mathbf{a}\cdot\mathbf{c})(\mathbf{d}\cdot\mathbf{b})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})+(\mathbf{b}\cdot\mathbf{a})(\mathbf{c}\cdot\mathbf{d})\\ &\qquad-(\mathbf{b}\cdot\mathbf{d})(\mathbf{c}\cdot\mathbf{a})+(\mathbf{c}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{d})-(\mathbf{c}\cdot\mathbf{d})(\mathbf{a}\cdot\mathbf{b})\\ &\quad=0.\end{aligned}$$

36. Exam question June 2002 (Section A): Evaluate $\varepsilon_{ijk}\varepsilon_{ijl}x_kx_l$.

Solution

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{ijl}x_kx_l&=(\delta_{jj}\delta_{kl}-\delta_{jl}\delta_{kj})x_kx_l\\ &=(3\delta_{kl}-\delta_{kl})x_kx_l=2\delta_{kl}x_kx_l=2x_lx_l=2|\mathbf{x}|^2.\end{aligned}$$

37. Exam question June 2001 (Section A): Evaluate $\varepsilon_{ijk}\partial_i\partial_j(x_lx_l)^{1/2}$ away from the origin.

Solution The answer is zero, because the mixed second partial derivatives are symmetric in i and j while the ε_{ijk} is antisymmetic. For full marks you should say that the mixed partial derivatives are symmetric in i and j because they are continuous functions at all points away from the origin (which are the only points which need to be considered).

38. Exam question June 2003 (Section A): Calculate $\partial_i\left(\varepsilon_{ijk}\,\varepsilon_{jkl}\,x_l\right)$. (Hint: use the connection between $\partial_ix_j=\frac{\partial x_j}{\partial x_i}$ and the Kronecker delta.)

Solution Use $\partial_ix_j=\delta_{ij}$ in $\partial_i\left(\varepsilon_{ijk}\,\varepsilon_{jkl}\,x_l\right)=\varepsilon_{ijk}\,\varepsilon_{jkl}\,\partial_i\,x_l=\varepsilon_{ijk}\,\varepsilon_{jkl}\,\delta_{il}=\varepsilon_{ijk}\,\varepsilon_{jki}=\varepsilon_{ijk}\,\varepsilon_{ijk}=\delta_{ii}\delta_{jj}-\delta_{ij}\delta_{ji}=3\times3-\delta_{ii}=9-3=6$.

39. The functions f, g are scalars, while $\mathbf{A}$ and $\mathbf{B}$ are vector functions with components A_i and B_i respectively. Verify the following identities using index notation:

(a) $\operatorname{grad}(fg)\;=\;f\operatorname{grad}g\;+\;g\operatorname{grad}f,$

Solution The i-th component of the LHS is $\partial_i(fg)=f\partial_ig+g\partial_if$ by the product-rule. This is the i-th component of the RHS, which proves the identity.

(b) $\operatorname{grad}(\mathbf{A}\cdot\mathbf{B})\;=\;\mathbf{A}\times\operatorname{curl}\mathbf{B}+\mathbf{B}\times\operatorname{curl}\mathbf{A}\;+\;(\mathbf{B}\cdot\nabla)\mathbf{A}\;+\;(\mathbf{A}\cdot\nabla)\mathbf{B},$

Solution This time it's best to start on the RHS (general message: it's always a good idea to start with the most complicated side and then simplify). We have

$$\begin{aligned}(\mathbf{A}\times\operatorname{curl}\mathbf{B}+\mathbf{B}\times\operatorname{curl}\mathbf{A})_i&=\varepsilon_{ijk}A_j\varepsilon_{klm}\partial_lB_m+\varepsilon_{ijk}B_j\varepsilon_{klm}\partial_lA_m\\ &=\varepsilon_{ijk}\varepsilon_{klm}\left(A_j\partial_lB_m+B_j\partial_lA_m\right)\\ &=\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)\left(A_j\partial_lB_m+B_j\partial_lA_m\right)\\ &=A_j\partial_iB_j+B_j\partial_iA_j-A_j\partial_jB_i-B_j\partial_jA_i\end{aligned}$$

The last two terms in the final line are exactly cancelled by the i-th component of $(\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B}$, while the first two are reasonably-easily seen to be the i-th component of $\mathrm{grad}\,(\mathbf{A}\cdot\mathbf{B})$, proving the identity.

(c) $\mathrm{div}\,(f\mathbf{A}) \;=\; f\,\mathrm{div}\,\mathbf{A} \;+\; (\mathrm{grad} f)\cdot\mathbf{A},$

Solution The LHS is a scalar equal to $\partial_i(fA_i) = f\partial_i A_i + (\partial_i f)A_i$ by the product rule for derivatives. This equals the RHS, which proves the identity.

(d) $\mathrm{curl}\,(f\mathbf{A}) \;=\; f\,\mathrm{curl}\,\mathbf{A} \;+\; (\mathrm{grad}\,f)\times\mathbf{A},$

Solution The LHS is a vector whose i-th component is

$$\begin{aligned}\varepsilon_{ijk}\partial_j(fA_k) &= \varepsilon_{ijk}\left(f\partial_j A_k + (\partial_j f)A_k\right)\\ &= f\varepsilon_{ijk}\partial_j A_k + \varepsilon_{ijk}(\partial_j f)A_k\end{aligned}$$

This is the i-th component of the RHS, which proves the identity.

(e) $\mathrm{div}\,(\mathbf{A}\times\mathbf{B}) \;=\; \mathbf{B}\cdot\mathrm{curl}\,\mathbf{A} \;-\; \mathbf{A}\cdot\mathrm{curl}\,\mathbf{B},$

Solution The LHS is a scalar equal to $\partial_i(\varepsilon_{ijk}A_jB_k) = \varepsilon_{ijk}\partial_i(A_jB_k)$ since ε_{ijk} is just a number, independent of $\mathbf{x}$. Using the product-rule this becomes

$$\varepsilon_{ijk}(\partial_i A_j)B_k + \varepsilon_{ijk}A_j(\partial_i B_k) = B_k\left(\varepsilon_{kij}(\partial_i A_j)\right) - A_j(\varepsilon_{jik}\partial_i B_k)$$

This equals the RHS, which proves the identity.

(f) $\mathrm{curl}\,(\mathbf{A}\times\mathbf{B}) \;=\; (\mathrm{div}\,\mathbf{B})\mathbf{A} \;-\; (\mathrm{div}\,\mathbf{A})\mathbf{B} \;+\; (\mathbf{B}\cdot\nabla)\mathbf{A} \;-\; (\mathbf{A}\cdot\nabla)\mathbf{B},$

Solution The LHS is a vector whose i-th component is $\varepsilon_{ijk}\partial_j(\varepsilon_{krs}A_rB_s) = \varepsilon_{ijk}\varepsilon_{rsk}\partial_j(A_rB_s)$ since $\varepsilon_{krs} = \varepsilon_{rsk}$ is just a number, independent of $\mathbf{x}$. Using the product-rule, and the very useful formula, this becomes

$$\begin{aligned}&(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr})\left((\partial_j A_r)B_s + A_r(\partial_j B_s)\right)\\ &= \delta_{ir}\delta_{js}(\partial_j A_r)B_s + \delta_{ir}\delta_{js}A_r(\partial_j B_s) - \delta_{is}\delta_{jr}(\partial_j A_r)B_s - \delta_{is}\delta_{jr}A_r(\partial_j B_s)\\ &= (\partial_s A_i)B_s + A_i(\partial_s B_s) - (\partial_r A_r)B_i - A_r(\partial_r B_i) = A_i(\partial_s B_s) - (\partial_r A_r)B_i + (B_s\partial_s)A_i - (A_r\partial_r)B_i\end{aligned}$$

This is the i-th component of the RHS, which proves the identity.

(g) $\mathrm{div}\,\mathrm{curl}\,\mathbf{A} \;=\; 0,$

Solution The LHS is a scalar equal to $\partial_i(\varepsilon_{ijk}\partial_j A_k) = \varepsilon_{ijk}\partial_i\partial_j A_k$. On the assumption that $\partial_i\partial_j A_k$ is continuous then $\partial_i\partial_j A_k = \partial_j\partial_i A_k$ so $\varepsilon_{ijk}\partial_i\partial_j A_k = \varepsilon_{ijk}\partial_j\partial_i A_k = \varepsilon_{jik}\partial_i\partial_j A_k$ by re-naming dummy variables, but this is $-\varepsilon_{ijk}\partial_i\partial_j A_k$ by skew symmetry of ε_{ijk}, i.e. $\varepsilon_{ijk}\partial_i\partial_j A_k = -\varepsilon_{ijk}\partial_i\partial_j A_k$ which implies $\varepsilon_{ijk}\partial_i\partial_j A_k = 0$.

(h) $\mathrm{curl}\;\mathrm{curl}\,\mathbf{A} \;=\; \mathrm{grad}\,\mathrm{div}\,\mathbf{A} - \nabla^2\mathbf{A}.$

Solution The LHS is a vector whose i-th component is

$$\varepsilon_{ijk}\partial_j(\varepsilon_{krs}\partial_r A_s) = \varepsilon_{ijk}\varepsilon_{rsk}\partial_j(\partial_r A_s)$$

since $\varepsilon_{krs} = \varepsilon_{rsk}$ is just a number, independent of $\mathbf{x}$. Using the very useful formula, this becomes

$$(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr})\partial_j\partial_r A_s = \delta_{ir}\delta_{js}\partial_j\partial_r A_s - \delta_{is}\delta_{jr}\partial_j\partial_r A_s = \partial_s\partial_i A_s - \partial_r\partial_r A_i$$

On the assumption that $\partial_j\partial_r A_s$ is continuous we can write this as $\partial_i\partial_s A_s - \partial_r\partial_r A_i$. This is the i-th component of the RHS, which proves the identity.

40. What is the divergence of the vector function $\mathbf{A}(\mathbf{x}\,) = r\,\mathbf{x} + \nabla\,r$ where $\mathbf{x}$ is the position vector in 3 dimensions and $r = |\mathbf{x}\,|$? What is the corresponding result in n dimensions?

    Solution Using components, $\nabla\cdot\mathbf{A} = \partial_i\,(rx_i + \partial_i(r)) = \partial_i(r)x_i + r\partial_i x_i + \partial_i(\partial_i(r))\,.$ Using $\partial_i(r) = \partial_i(x_jx_j)^{1/2} = \frac{1}{2}(x_jx_j)^{-1/2}2x_i = x_i/r$ (as in 28(c)) this is

    $$\begin{aligned} x_ix_i/r + r\delta_{ii} + \partial_i(x_i/r) &= r + 3r + (\partial_i x_i)/r - x_ir^{-2}\partial_i(r) \\ &= 4r + \delta_{ii}/r - x_ir^{-2}x_i/r \\ &= 4r + 3/r - 1/r = 4r + 2/r\,. \end{aligned}$$

    In n dimensions $\delta_{ii} = n$ which changes the result to $(n+1)r + (n-1)/r$.

41. For which values of (x, y) are the following continuous:

    (a) $x/(x^2 + y^2 + 1)$,

    Solution Use the theorem that if f and g are continuous at a point then so are $f + g$, fg and f/g if g does not vanish there. Taking $f = x$ and $g = y$ all the functions can be built up by repeated applications of this result, so all the functions will be continuous whenever their denominators are non-zero so this function is continuous everywhere since $x^2 + y^2 + 1 > 0$,

    (b) $x/(x^2 + y^2)$,

    Solution everywhere except the origin,

    (c) $(x + y)/(x - y)$,

    Solution everywhere except the line $y = x$,

    (d) $x^3/(y - x^2)$?

    Solution everywhere except the parabola $y = x^2$.

42. Let the scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

    $$f(\underline{x}) = \begin{cases} 1 + \dfrac{x^2}{y} & y \neq 0, \\ \quad 1 & y = 0. \end{cases}$$

(a) Show that, along any straight line through the origin, $\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = f(\underline{0})$.

(b) Is $f(\underline{x})$ continuous at $\underline{0}$? Explain your answer, with reference to the first part of this question.

Solution

(a) On a straight line through the origin, we have either $y = mx$ for some fixed $m \in \mathbb{R}$, or $x = 0$.

On the line $x = 0$, the limit as $\underline{x} \to \underline{0}$ becomes the limit $y \to 0$, and so we have

$$\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = \lim_{y\to 0}\left(1 + \frac{0}{y}\right) = 1.$$

On the line $y = mx$, the limit as $\underline{x} \to \underline{0}$ becomes the limit $x \to 0$, and so we have

$$\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = \lim_{x\to 0}\left(1 + \frac{x^2}{mx}\right) = \lim_{x\to 0}\left(1 + \frac{x}{m}\right) = 1.$$

Since $f(\underline{0}) = 1$, we have shown that $\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = f(\underline{0})$ along any straight line throughout the origin.

(b) In order for f to be continuous at $\underline{0}$ we must have $\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = f(\underline{0})$ as a limit in $\mathbb{R}^2$, not just when we restrict to straight lines through the origin. Since we have $f(\underline{0}) = 1$, we would require $\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = 1$, which means that for all $\epsilon > 0$, we could find a $\delta > 0$ such that

$$|f(x) - 1| = \left|1 + \frac{x^2}{y} - 1\right| = \left|\frac{x^2}{y}\right| < \epsilon$$

for all $\underline{x}$ such that $0 < |\underline{x}| < \delta$. However, for any δ, the set of points $0 < |\underline{x}| < \delta$ will contain points with y much less than x^2 (sometimes written $y << x^2$), including points where $\left|\frac{x^2}{y}\right| > \epsilon$ for any fixed ϵ.

An alternative way to think about this is to consider $\lim_{\underline{x}\to\underline{0}} f(\underline{x})$ along the curve $y = x^2$. On this curve, the limit as $\underline{x} \to \underline{0}$ becomes the limit $x \to 0$, and so we have

$$\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = \lim_{x\to 0}\left(1 + \frac{x^2}{x^2}\right) = \lim_{x\to 0}(2) = 2.$$

Since this limit is not equal to 1, ϵ we cannot hope to find a δ, such that $|f(x) - 1| < \epsilon$ for all $0 < |\underline{x}| < \delta$, for every $\epsilon > 0$.

43. Let the scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(\underline{x}) = \begin{cases} \dfrac{x^2y^2}{x^4 + y^4} & \underline{x} \neq \underline{0}, \\ 0 & \underline{x} = \underline{0}. \end{cases}$$

(a) Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist at $\underline{0}$, and find their values at this point.

(b) Show that $\lim_{\underline{x}\to\underline{0}} f(\underline{x})$ does not exist, and hence $f(\underline{x})$ is not continuous at the origin. Comment on this in relation to the previous part of this question.

Solution

(a) Using the definitions of the partial derivatives as limits, at the origin we have

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0/h^4 - 0}{h} = 0$$
$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0 - 0/h^4}{h} = 0,$$

and so both partials exist with value 0.

(b) In order for $f(\underline{x})$ to be continuous at the origin, $\lim_{\underline{x}\to\underline{0}} f(\underline{x})$ must exist and be equal to $f(\underline{0}) = 0$. However, considering $f(\underline{x})$ along the line $y = x$, the limit $\underline{x} \to \underline{0}$ becomes the limit $x \to 0$, and we have

$$\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2} \neq 0.$$

Given $\epsilon > 0$, we therefore cannot hope to find a $\delta > 0$ such that $|f(\underline{x}) - 0| < \epsilon$ for all $0 < |\underline{x}| < \delta$. Therefore $\lim_{\underline{x}\to\underline{0}} f(\underline{x})$ is not equal to 0, and $f(\underline{x})$ is not continuous at $\underline{0}$.

By considering the limit along the line $y = 0$ (or similarly along $x = 0$) we can see that the limit as $\underline{x} \to \underline{0}$ becomes the limit $x \to 0$ (or $y \to 0$), and so we have

$$\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = \lim_{x\to 0} \frac{0}{x^4} = 0,$$

and since the limit along this line is not equal to the limit along the line $y = x$, the limit $\lim_{\underline{x}\to\underline{0}} f(\underline{x})$ cannot exist.

The partial derivatives exist at the origin even though the function f is not continuous at the origin. This is because the partial derivatives depend on the behaviour of the function as we move parallel to one of the coordinate axes, whereas continuity depends on the behaviour of the function in all directions.

44. Which of the following sets are open?.

(a) $\{(x, y, z) : x > 0\}$,

Solution An open set has to contain an open ball centred on each of its points, so this is open because for any point we can construct a sphere centred on that point and lying entirely within the set, even points arbitrarily close to the plane $x = 0$. To make this really precise, suppose $\underline{a} = (a_1, a_2, a_3)$ is in the set. Then $a_1 > 0$ and if we take $\delta = a_1/2 > 0$, then $B_\delta(\underline{a})$ is entirely in the set.

To see this, consider $(x, y, z) = \underline{x} \in B_\delta(\underline{a})$. Since $\underline{x} \in B_\delta(\underline{a})$, $a_1 - \delta < x < a_1 + \delta$, so $\frac{a_1}{2} < x$. But then $x = x - \frac{a_1}{2} + \frac{a_1}{2} = x - \frac{a_1}{2} + \delta > 0$. So $\underline{x}$ is in the set.

Since this works for all points $\underline{a}$ in the set, the set is open.

(b) $\{(x, y, z) : y \geq 0\}$,

Solution this is not open because it contains points for which $y = 0$. Let $\underline{x}$ be such a point, and consider a sphere of radius $\delta > 0$ around $\underline{x}$. Then $\underline{x} - \delta \underline{e}_2 = (x, -\delta, z) \in B_\delta(\underline{x})$, and this point does not lie in the original set.

(c) $\{(x, y, z) : 1 > (x^2 + y^2)/z\}$,

Solution $1 > (x^2 + y^2)/z$ implies $z > x^2 + y^2$ or $z < 0$, and this is open because for any point we can construct a sphere centred on that point and lying entirely within the set, even points arbitrarily close to the paraboloid $z = x^2 + y^2$ or to $z = 0$. (Beware: you might forget that points with $z < 0$ are also in the set!)

(d) $\{(x, y, z) : 1 \geq (x^2 + y^2)/z\}$?

Solution this is not open because it contains points for which $z = x^2 + y^2$, and any sphere with such a point as its centre contains points outside the set.

45. Prove that an open ball, as defined in lectures, is an open set.

Solution Let $B_\delta(\underline{a})$ be the open ball centred at $\underline{a}$ with radius δ, and let $\underline{x}$ be any point in $B_\delta(\underline{a})$. We need to show that there's a δ' such that $B_{\delta'}(\underline{x}) \subset B_\delta(\underline{a})$. Since $\underline{x} \in B_\delta(\underline{a})$, $|\underline{x} - \underline{a}| < \delta$. Let $\delta' = \delta - |\underline{x} - \underline{a}|$. By the remark just made, $\delta' > 0$, and if $\underline{x}'$ is any point in $B_{\delta'}(\underline{x})$ then $|\underline{x}' - \underline{x}| < \delta'$ which implies $|\underline{x}' - \underline{a}| = |\underline{x}' - \underline{x} + \underline{x} - \underline{a}| \leq |\underline{x}' - \underline{x}| + |\underline{x} - \underline{a}|$ (by the triangle inequality). Hence $|\underline{x}' - \underline{a}| < \delta' + |\underline{x} - \underline{a}| = \delta$, and so $\underline{x}' \in B_\delta(\underline{a})$, which is what we needed to prove. (This all becomes much clearer if you draw a picture – basically we are just setting δ' equal to the distance from $\underline{x}$ to the edge of $B_\delta(\underline{a})$.)

46. Prove that the intersection of two open sets, as defined in lectures, is another open set. (Note that the empty set is an open set: since it contains no points, the statement that every point in it sits inside an open ball which is also in the set is *vacuously* true.) What about the intersection of a finite number of open sets? And what about the intersection of an *infinite* number of open sets?

Solution Let the two open sets be S_1 and S_2, and put $S = S_1 \cap S_2$. If $S = \emptyset$ then there's nothing more to be done since $\emptyset$ is open. If $S \neq \emptyset$ then let $\underline{a}$ be any point in S. Then $\underline{a} \in S_1$, and since S_1 is open we can find a value of $\delta_1 > 0$ such that $B_{\delta_1}(\underline{a}) \subset S_1$; and similarly $\underline{a} \in S_2$, and we can find a value of $\delta_2 > 0$ such that $B_{\delta_2}(\underline{a}) \subset S_2$. Now set $\delta = \min(\delta_1, \delta_2)$; then $B_\delta(\underline{a})$ is a subset of both S_1 and S_2, which means that it is a subset of S. Since this works for *any* point $\underline{a} \in S$, this proves that S is open.

For the intersection S of a finite number of open sets S_1, S_2, $\ldots S_n$, the argument is much the same: either S is the empty set, or else for any point $\underline{a}$ in S we can take $\delta = \min(\delta_1, \delta_2, \ldots \delta_n)$ and show that $B_\delta(\underline{a})$ is in S.

However this doesn't necessarily work for an infinite intersection, as $\min(\delta_1, \delta_2, \ldots)$ might be zero if there are infinitely many δ_is. For example, if for $n = 1, 2, \ldots$ S_n is the open ball centred on the origin with radius $1/n$, then the interection of all the S_ns is the set containing the single point $\underline{0}$, which is *not* an open set.

47. Exam question June 2014 (Section A):

(a) Give the definition of the open ball $B_\delta(\mathbf{a})$ with centre $\mathbf{a} \in \mathbb{R}^n$ and radius $\delta > 0$, and define what it means for a subset S of $\mathbb{R}^n$ to be open.

(b) Which of the following subsets of $\mathbb{R}^2$ are open? In each case, justify your answer in terms of the definition you gave in part (a).

(i) $S_1 = \{(x, y) : x > 2\}$,

(ii) $S_2 = \{(x, y) : x > 2, y = 2\}$,

(iii) $S_3 = \{(x, y) : x > 2, y > 2\}$.

Solution

(a) $B_\delta(\mathbf{a}) = \{\underline{x} \in \mathbb{R}^n : |\underline{x} - \underline{a}| < \delta\}$; a subset S of $\mathbb{R}^n$ is *open* if for each point $\underline{a} \in S$ there is an open ball $B_\delta(\underline{a})$ which is also in S (where δ might depend on $\underline{a}$).

(b) open, not open, open. (NB: 'not open' is not the same as 'closed'!) In each case some justification should be given. (Sketch for part (i): if $\underline{a} = (a_1, a_2) \in S_1$ then $a_1 - 2 > 0$. Let $\delta = a_1 - 2 > 0$, and consider $\underline{x} = (x, y) \in B_\delta(\underline{a})$. We want to show $\underline{x} \in S_1$, which means we need to show $x > 2$. We have $x - 2 = x - a_1 + a_1 - 2 = x - a_1 + \delta$. But $|x - a_1| < \delta \implies -\delta < x - a_1 < \delta$, so $x - 2 = x - a_1 + \delta > 0$, and hence $B_\delta(\underline{a}) \subset S_1$.)

48. Exam question (last part) June 2014 (Section B): Determine the points of $\mathbb{R}^2$ at which the function $f(x, y) = |xy + x + y + 1|$ is
(a) continuously differentiable; (b) differentiable.

Hint: first factorise f.

Solution $f(x, y) = |xy + x + y + 1| = |(x + 1)(y + 1)|$ so $f(x, y) = (x + 1)(y + 1)$ for $(x + 1)(y + 1) > 0$ and $f(x, y) = -(x + 1)(y + 1)$ for $(x + 1)(y + 1) < 0$. Hence away from the lines $x = -1$ or $y = -1$, f is a polynomial in x and y and therefore has continuous partial derivatives, and hence is both continuously differentiable, and, by the theorem from lectures, differentiable in this region. It remains to consider the two lines $x = -1$ and $y = -1$.
On the line $x = -1$, we have

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(-1 + h, y) - f(-1, y)}{h} = \lim_{h \to 0} \frac{|h(y + 1)|}{h} = \lim_{h \to 0} \frac{|h|}{h} |y + 1| .$$

Hence for $y \neq -1$ the limit doesn't exist as $|h|/h = \pm 1$; while for $y = -1$ the limit exists and is zero. Also on the line $x = -1$ we have

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(-1, y + h) - f(-1, y)}{h} = 0 .$$

Similarly on the line $y = -1$, $\frac{\partial f}{\partial x} = 0$ while $\frac{\partial f}{\partial y}$ does not exist except at $x = -1$.
Since one or other partial derivative does not exist on the lines $x = -1$ and $y = -1$ away from $(x, y) = (-1, -1)$, f is neither continuously differentiable nor differentiable

on these lines away from $(-1,-1)$; it is also not continuously differentiable at $(-1,-1)$ since the partial derivatives do not exist on these lines away from that point.
Finally we must ask whether f is differentiable at $\underline{a} = (-1,-1)$. Both partials of f are zero at that point, so *if* f were differentiable there, the remainder term would have to be $R(\underline{h}) = f(\underline{a}+\underline{h}) - f(\underline{a})$. Consider $\underline{h} = (h_1, h_2)$: then

$$\frac{R(\underline{h})}{|\underline{h}|} = \frac{f(-1+h_1,-1+h_2) - f(-1,-1)}{|\underline{h}|} = \frac{|h_1||h_2|}{|\underline{h}|}\,.$$

Since $|h_1| \le |\underline{h}|$ and $|h_2| \le |\underline{h}|$, we have $R(\underline{h})/|\underline{h}| \le |\underline{h}|$ and since $|\underline{h}| \to 0$ as $\underline{h} \to 0$, the same must be true of $R(\underline{h})/|\underline{h}|$. Hence f *is* differentiable at $(-1,1)$.
Conclusion: (a) f is continuously differentiable at all points in $\mathbb{R}^2$ away from the lines $x=-1$ or $y=-1$; (b) f is differentiable at all points in (a) and also at $(-1,-1)$.

49. Determine the points of $\mathbb{R}^2$ at which the function $f(x,y) = |x^2 - y^2|$ is
(a) continuously differentiable; (b) differentiable.

Solution $f(x,y) = |x^2-y^2|$ is x^2-y^2 for $x^2-y^2 = (x-y)(x+y) > 0$ and y^2-x^2 for $x^2-y^2 = (x-y)(x+y) < 0$. For the first case $(x-y)(x+y) > 0$ for $x-y>0$ and $x+y>0$ or $x-y<0$ and $x+y<0$, call this part of $\mathbb{R}^2$ region 1. For the second case $(x-y)(x+y)<0$ for $x-y>0$ and $x+y<0$ or $x-y<0$ and $x+y>0$, call this part of $\mathbb{R}^2$ region 2. Within both regions the function is a polynomial in x and y and so has continuous partial derivatives, hence the function is continuously differentiable there. In neighbourhoods of points on the line $y=\pm x$ the function is no longer a polynomial and we have to be more careful so use the definition of the partial derivative: for $y=x=a$

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{|(a+h)^2 - a^2|}{h} = \lim_{h\to 0}\frac{|2ah+h^2|}{h}$$

If $a \neq 0$ we can neglect the h^2 term in comparison to the ah piece so we get

$$\lim_{h\to 0}\frac{|2ah|}{h} = \lim_{h\to 0}\frac{|2a|\,|h|}{h}$$

which does not exist because $\frac{|h|}{h}$ is ± 1 depending on the sign of h. However if $a=0$ then the limit is

$$\lim_{h\to 0}\frac{|h^2|}{h} = 0\,.$$

Similar arguments show that $\frac{\partial f}{\partial y}$ also does not exist on this line except at the origin, and also that neither partial derivative exists on the line $y=-x$, except at the origin. Since the p.d.s do not exist on the lines $y=\pm x$ away from $\mathbf{0}$ the function cannot be differentiable there. It might be differentiable at $\mathbf{0}$ but cannot be continuously differentiable there, because the p.d.s are not continuous at $\mathbf{0}$ (since they do not exist on the lines). Now investigate the differentiablity of f at the origin. Consider

$$\frac{R}{|\mathbf{h}|} = \frac{f(\mathbf{h}) - f(\mathbf{0}) - \mathbf{h}\cdot\nabla f}{|\mathbf{h}|} = \frac{|h_1^2 - h_2^2|}{|\mathbf{h}|}$$

Now
$$\frac{|h_1^2 - h_2^2|}{|\mathbf{h}|} \leq \frac{h_1^2}{|\mathbf{h}|} + \frac{h_2^2}{|\mathbf{h}|} \leq |\mathbf{h}| + |\mathbf{h}|$$
we have that $|R/|\mathbf{h}|| < \epsilon$ whenever $|\mathbf{h}| < \delta$ by taking $\delta = \epsilon/2$ so $\lim_{\mathbf{h}\to 0} R/|\mathbf{h}| = 0$ and f is differentiable at the origin.

50. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(\mathbf{0}) = 0$ whilst for $\mathbf{x} \neq \mathbf{0}$:
$$f(\mathbf{x}) = \frac{x^3}{x^2 + y^2}\,.$$
Calculate the partial derivatives of f with respect to x and y at $\mathbf{x} = \mathbf{0}$ *using their definitions as limits*. Defining $R(\mathbf{h})$ at the origin by $R(\mathbf{h}) = f(\mathbf{h}) - f(\mathbf{0}) - \mathbf{h}\cdot\nabla f$ as usual, show that $R(\mathbf{h})/|\mathbf{h}|$ does not tend to zero as $\mathbf{h}$ tends to $\mathbf{0}$, so that f is not differentiable at the origin.
On the line through the origin, $\mathbf{x} = \mathbf{b}t$, (with $\mathbf{b}$ a constant vector), f becomes a function of the single variable t, $f(\mathbf{b}t)$. Write $\mathbf{b} = \mathbf{e_1}b_1 + \mathbf{e_2}b_2$ and use this to write $f(\mathbf{b}t)$ explicitly as a function of t. Show that this function is differentiable at the origin, i.e. df/dt exists at $t = 0$ despite $f(\mathbf{x})$ not being differentiable at $\mathbf{0}$.

Solution
$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(h\mathbf{e_1}) - f(\mathbf{0})}{h} = \lim_{h\to 0}\frac{h-0}{h} = 1$$
and
$$\frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(h\mathbf{e_2}) - f(\mathbf{0})}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0\,.$$
With the standard definition of R,
$$\frac{R}{|\mathbf{h}|} = \frac{f(\mathbf{h}) - f(\mathbf{0}) - \mathbf{h}\cdot\nabla f}{|\mathbf{h}|} = \frac{h_1^3}{(h_1^2+h_2^2)^{3/2}} - 0 - \frac{h_1}{(h_1^2+h_2^2)^{1/2}} = \frac{-h_1h_2^2}{(h_1^2+h_2^2)^{3/2}}$$
now consider how this behaves as the origin is approached along the line $h_2 = mh_1$ then
$$\frac{R}{|\mathbf{h}|} = \frac{-m^2h_1^3}{(1+m^2)^{3/2}h_1^3} = \frac{-m^2}{(1+m^2)^{3/2}}$$
which remains constant as the origin is approached. Since this is not zero f cannot be differentiable at the origin.
$$f(\mathbf{b}t) = \frac{(b_1t)^3}{(b_1t)^2 + (b_2t)^2} = t\frac{b_1^3}{b_1^2 + b_2^2}$$
which is t multiplied by a constant (this also holds at $\mathbf{0}$ since f is defined to vanish there), and so is differentiable at the origin.

51. Exam Question June 2022 (Part B)

(a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar field on $\mathbb{R}^n$. Define what it means for the limit of $f(\underline{x})$ as $\underline{x}$ tends to $\underline{a}$ to be L.

(b) Define what it means for f to be continuous at $\underline{a}$.

(c) Let f be a scalar field on $\mathbb{R}^2$ given by

$$f(x,y) = \begin{cases} xy(x^2-y^2)/(x^2+y^2) & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that f is continuous on $\mathbb{R}^2$, stating any results that you use.

(d) Is f differentiable at the origin?

(e) Show that, at the origin,

$$\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}.$$

Solution

(a) The limit of $f(\underline{x})$ as $\underline{x}$ tends to $\underline{a}$ is L, or $\lim_{\underline{x}\to\underline{a}} f(\underline{x}) = L$, if $\forall\epsilon > 0$, $\exists\delta > 0$ such that $|f(\underline{x}) - L| < \epsilon \quad \forall x$ such that $0 < |\underline{x} - \underline{a}| < \delta$.

(b) f is continuous at $\underline{a}$ if $f(\underline{a})$ exists and $\lim_{\underline{x}\to\underline{a}} f(\underline{x}) = f(\underline{a})$.

(c) For any $\underline{a} \neq \underline{0}$, f is continuous at $\underline{a}$ by Theorem 5.3. To see that f is continuous at $\underline{0}$, we first note that $f(\underline{0}) = 0$, and so we must have $\lim_{\underline{x}\to\underline{a}} f(\underline{x}) = 0$. Letting $\delta = \sqrt{\epsilon}$, then if $|\underline{x}| < \delta = \sqrt{\epsilon}$

$$|f(\underline{x}) - 0| \leq \frac{|\underline{x}||\underline{x}||x^2-y^2|}{|\underline{x}|^2} \leq |x^2+y^2| < \delta^2 = \epsilon.$$

Therefore $\lim_{\underline{x}\to\underline{a}} f(\underline{x}) = 0 = f(\underline{a})$, and so f is also continuous at $\underline{0}$. We have therefore shown that f is continuous on $\mathbb{R}^2$.

(d) We first calculate the partial derivatives at the origin, using their limit definitions.

$$\frac{\partial f}{\partial x}(\underline{0}) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0/h^2 - 0}{h} = 0,$$

and similarly

$$\frac{\partial f}{\partial y}(\underline{0}) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0 - 0/h^2}{h} = 0,$$

and therefore $\underline{\nabla} f(\underline{0}) = \underline{0}$.

By definition, f is differentiable at $\underline{0}$, if $f(\underline{x}+\underline{h}) - f(\underline{x}) = \underline{h}\cdot\underline{\nabla} f + R(\underline{h})$, with $\lim_{\underline{h}\to\underline{0}} R(\underline{h})/|\underline{h}| = 0$. At $\underline{x} = \underline{0}$, with $\underline{h} = (h_1, h_2)$ we therefore have

$$R(\underline{h}) = \frac{h_1 h_2(h_1^2 - h_2^2)}{h_1^2 + h_2^2} - 0 - 0,$$

and so

$$0 \leq \left|\frac{R(\underline{h})}{|\underline{h}|}\right| = \frac{|h_1||h_2||h_1^2-h_2^2|}{|\underline{h}|^3} \leq \frac{|\underline{h}||\underline{h}||h_1^2-h_2^2|}{|\underline{h}|^3} \leq \frac{|h_1^2+h_2^2|}{|\underline{h}|} = \frac{|\underline{h}|^2}{|\underline{h}|} = |\underline{h}|.$$

By the squeezing theorem we therefore have that $\lim_{\underline{h}\to\underline{0}} \left|\frac{R(\underline{h})}{|\underline{h}|}\right| = 0$. Since $-\left|\frac{R(\underline{h})}{|\underline{h}|}\right| \leq \frac{R(\underline{h})}{|\underline{h}|} \leq \left|\frac{R(\underline{h})}{|\underline{h}|}\right|$, we therefore also have $\lim_{\underline{h}\to\underline{0}} \frac{R(\underline{h})}{|\underline{h}|} = 0$ by another application of the squeezing theorem, and hence f is differentiable at $\underline{0}$.

(e) We will first need to calculate $f_x(0,y)$ for $x \neq 0$, and $f_y(x,0)$ for $y \neq 0$.
For $y \neq 0$, we have

$$f_x(0,y) = \lim_{h\to 0} \frac{f(0+h,y)-f(0,y)}{h} = \lim_{h\to 0} \frac{hy(h^2-y^2)}{h(h^2+y^2)} = \lim_{h\to 0} \frac{y(h^2-y^2)}{(h^2+y^2)} = -y,$$

and similarly for $x \neq 0$ we have

$$f_y(x,0) = \lim_{h\to 0} \frac{f(x,0+h)-f(x,0)}{h} = \lim_{h\to 0} \frac{xh(x^2-h^2)}{h(x^2+h^2)} = \lim_{h\to 0} \frac{x(x^2-h^2)}{(x^2+h^2)} = x.$$

Therefore

$$\frac{\partial^2 f}{\partial x \partial y}(\underline{0}) = \lim_{h\to 0} \frac{f_y(0+h,0)-f_y(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1,$$

and similarly

$$\frac{\partial^2 f}{\partial y \partial x}(\underline{0}) = \lim_{h\to 0} \frac{f_x(0,0+h)-f_x(0,0)}{h} = \lim_{h\to 0} \frac{-h-0}{h} = -1,$$

and so $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$ at the origin as required.

52. If $y = 1 + xy^5$ show that y may be written in the form $y = f(x)$ in a neighbourhood of $(0,1)$ and find the gradient of the graph of f at the point $(0,1)$.

Solution $y = 1+xy^5$ is a level curve of $H = y-1-xy^5$ with $H = 0$ and passes through $(0,1)$. H is polynomial in x and y and so has continuous derivatives and is therefore differentiable. The implicit function theorem then implies that y may be written in the form $y = f(x)$ in a neighbourhood of $(0,1)$ if $\frac{\partial H}{\partial y} \neq 0$ there. Now $\frac{\partial H}{\partial y} = 1-5xy^4 = 1$ at $(0,1)$ so the condition is satisfied. The implicit function theorem also gives the gradient of the function as

$$f'(0) = -\frac{\partial H}{\partial x} \Big/ \frac{\partial H}{\partial y},$$

now $\frac{\partial H}{\partial x} = -y^5 = -1$ at $(0,1)$ so $f'(0) = 1$.

53. Show that the equation $xy^3 - y^2 - 3x^2 + 1 = 0$ can be written in the form $y = f(x)$ in a neighbourhood of the point $(0,1)$, and in the form $y = g(x)$ in a neighbourhood of the point $(0,-1)$. Is it true that $f(x)$ and $g(x)$ are equivalent as functions of x? What are the critical values of the curve $H(x,y) = xy^3 - y^2 - 3x^2 + 1$, and what are the regular values of this curve?

Solution $xy^3 - y^2 - 3x^2 + 1 = 0$ is a level curve of $H = xy^3 - y^2 - 3x^2 + 1$ with $H = 0$ and passes through $(0,\pm 1)$. H is polynomial in x and y and so has continuous derivatives and is therefore differentiable. $\frac{\partial H}{\partial y} = 3xy^2 - 2y$, so at $(0,\pm 1)$, $\frac{\partial H}{\partial y} \neq 0$. The implicit function theorem then implies that y may be written in the form $y = f(x)$ in a neighbourhood of the point $(0,1)$, and in the form $y = g(x)$ in a neighbourhood of the point $(0,-1)$. Note that at $x=0$ we must have $f(0)=1$ but $g(0)=-1$ so $f(x)$ and $g(x)$ cannot be equivalent as functions of x. In fact solving $xy^3-y^2-3x^2+1=0$ for y gives three solutions, since it is a cubic equation and one passes through $(0,1)$ another through $(0,-1)$.

To find the critical values of $H(x,y)$, we must consider values of c for which the curve $H(x,y)=c$ contains critical points, i.e. points where $\underline{\nabla} H=0$. We have $\frac{\partial H}{\partial x}=y^3-6x$, so $\underline{\nabla} H=0$ if we have $y^3-6x=0$ and $3xy^2-2y=0$. If $y\neq 0$, we can rearrange these equations in terms of x and set them equal to give $\frac{y^3}{6}=\frac{2}{3y}$, with real solutions $y=\pm\sqrt{2}$. Substituting this back to find x, we have $\underline{\nabla} H=0$ at the points $(\pm\frac{\sqrt{2}}{3},\pm\sqrt{2})$, so these are critical points of $H(x,y)$. If $y=0$, then we have $\underline{\nabla} H=0$ if $x=0$, so there is also a critical point at the origin. Critical values of $H(x,y)$ are values c where $H(x,y)=c$ contains critical points of H, so we need to find which values of c the critical points correspond to. Since $H(\pm\frac{\sqrt{2}}{3},\pm\sqrt{2})=-\frac{1}{3}$, we have one critical value at $c=-\frac{1}{3}$, and since $H(\underline{0})=1$, we have another critical value at $c=1$. The regular values are therefore $\mathbb{R}-\{-\frac{1}{3},1\}$.

54. Determine whether or not the equation $x^2+y+\sin(xy)=0$ can be written in the form $y=f(x)$ or in the form $x=g(y)$ in some small open disc about the origin for some suitable continuously differentiable functions f,g.

Solution $x^2+y+\sin(xy)=0$ is a level curve of $H=x^2+y+\sin(xy)$ with $H=0$ and passes through the origin. H has continuous partial derivatives and so is differentiable. $\frac{\partial H}{\partial y}=1+x\cos(xy)$ is 1 at the origin and $\frac{\partial H}{\partial x}=2x+y\cos(xy)$ vanishes at the origin, so by the implicit function theorem y may be written in the form $y=f(x)$ in a neighbourhood of the origin with differentiable f, but x cannot be written in the form $x=g(y)$.

55. Exam question May 2015 (Section B, lightly edited):
Let $f:\mathbb{R}^2\to\mathbb{R}$ be the scalar function $f(x,y)=e^{xy}-x+y$.
   (a) Find the vector equations of the tangent and normal lines to the curve $f(x,y)=0$ at the points $(1,0)$ and $(0,-1)$.
   (b) Use the implicit function theorem for functions of two variables to determine whether or not we can guarantee the curve $f(x,y)=2$ can be written in the form $y=g(x)$ for some differentiable function $g(x)$ in the neighbourhoods of the points (i) $(0,1)$; (ii) $(-1,0)$.
   Determine also whether we can guarantee the curve can be written as $x=h(y)$ for some differentiable function $h(y)$, in the neighbourhoods of the same two points.
   (c) Does the function $f(x,y)$ have any critical points? Justify your answer. (You can quote without proof that $|xe^{-x^2}|<1$ for all $x\in\mathbb{R}$.)

Solution

(a) We have

$$\nabla f=\frac{\partial f}{\partial x}\mathbf{e}_1+\frac{\partial f}{\partial y}\mathbf{e}_1\,,\qquad \frac{\partial f}{\partial x}=ye^{xy}-1\,,\quad \frac{\partial f}{\partial y}=xe^{xy}+1\,,$$

and so $\nabla f(1,0) = -\mathbf{e}_1 + 2\mathbf{e}_2$ and $\nabla f(0,-1) = -2\mathbf{e}_1 + \mathbf{e}_2$.
Hence the equations of the normal lines at $(1,0)$ and $(0,-1)$ can be written as

$$\mathbf{x} = \mathbf{e}_1 + \lambda(-\mathbf{e}_1 + 2\mathbf{e}_2) = (1-\lambda)\mathbf{e}_1 + 2\lambda\mathbf{e}_2$$

and

$$\mathbf{x} = -\mathbf{e}_2 + \lambda(-2\mathbf{e}_1 + \mathbf{e}_2) = -2\lambda\mathbf{e}_1 + (\lambda-1)\mathbf{e}_2$$

respectively. Vectors perpendicular to ∇f at $(1,0)$ and $(0,-1)$ are (for example) $2\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_1 + 2\mathbf{e}_2$, so the two tangent lines can be written as

$$\mathbf{x} = \mathbf{e}_1 + \mu(2\mathbf{e}_1 + \mathbf{e}_2) = (1+2\mu)\mathbf{e}_1 + \mu\mathbf{e}_2$$

and

$$\mathbf{x} = -\mathbf{e}_2 + \mu(\mathbf{e}_1 + 2\mathbf{e}_2) = \mu\mathbf{e}_1 + (2\mu-1)\mathbf{e}_2\,.$$

(b) Reminder of implicit function theorem (not needed for the problem sheet, though it *did* feature in the exam question: if $f(x,y) : U \to \mathbb{R}$ is differentiable on U with U open in $\mathbb{R}^2$, and if (x_0, y_0) is a point on the level curve $f(x,y) = c$ at which $\frac{\partial f}{\partial y} \neq 0$, then a differentiable function $g(x)$ exists in a neighbourhood of $x = x_0$ such that (I) $f(x, g(x)) = c$ and (II) $\frac{dg}{dx} = -\frac{\partial f}{\partial x} / \frac{\partial f}{\partial y}$, with $g(x_0) = y_0$. Here, f is everywhere differentiable and we just need to check the partials. At $(0,1)$, $\frac{\partial f}{\partial y} = 1$ so the curve *can* be written as $y = g(x)$; but at $(-1,0)$, $\frac{\partial f}{\partial y} = 0$ so we *cannot guarantee* the curve can be written as $y = g(x)$. For the final part, swap x and y in the implicit function theorem and check the partial x derivatives: at $(0,1)$, $\frac{\partial f}{\partial x} = 0$ so we *cannot guarantee* the curve can be written as $x = h(y)$; and at $(-1,0)$, $\frac{\partial f}{\partial x} = -1$ so the curve *can* be written as $x = h(y)$.

(c) For a critical point we need $\nabla f = \mathbf{0}$, so $ye^{xy} = 1$ and $xe^{xy} = -1$ must hold. Adding implies $(x+y)e^{xy} = 0$, so $y = -x$. Then $xe^{-x^2} = -1$ is required, but this contradicts the inequality given in the question. Hence f has no critical points.

56. Part of Exam Question May 2017 (Section B):

(c) Consider the function
$$f(x,y) = (3x+y)e^{3xy}.$$
Determine whether or not the curve $f(x,y) = c$ can be written in the form $y = g(x)$, and if not, state clearly the points (x_0, y_0) and corresponding values of c where problems occur. You may assume that f is differentiable on $\mathbb{R}^2$.

(d) Using $f(x,y)$ as given in the previous part, determine whether or not the curve $(f(x,y) = c)$ can be written in the form $x = h(y)$, and if not, state clearly the points (x_0, y_0) where problems occur.

(e) Using $f(x,y)$ as in the previous parts of this question, are there any points where the curve $f(x,y) = c$ can neither be written as $y = g(x)$, nor as $x = h(y)$?

Solution

(c) By the implicit function theorem, if (x_0, y_0) is a point on the curve $f(x, y) = c$ at which $\frac{\partial f}{\partial y} \neq 0$, then the curve $f(x, y) = c$ can be written in the form $y = g(x)$ in a neighbourhood of the point (x_0, y_0). We have

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}((3x + y)e^{3xy}) \\ &= e^{3xy} + (3x + y)3xe^{3xy} \\ &= (1 + 9x^2 + 3xy)e^{3xy}\end{aligned}$$

$$\text{so} \qquad \frac{\partial f}{\partial y} = 0 \iff 1 + 9x^2 + 3xy = 0$$

$$\text{If } x \neq 0 \quad \Longrightarrow \quad y = -\frac{(1 + 9x^2)}{3x} = -\left(3x + \frac{1}{3x}\right) \qquad (\dagger).$$

So problems occur at point (x_0, y_0) with $y_0 = -(3x_0 + 1/3x_0)$.
Now we need to find which values of c these problems occur at, so we evaluate

$$\begin{aligned}c = f(x_0, y_0) &= \left(3x_0 + \left(-\left(3x_0 + \frac{1}{3x_0}\right)\right)\right) e^{3x_0(-(3x_0+1/3x_0))} \\ &= -\frac{1}{3x_0}e^{-9x_0^2-1},\end{aligned}$$

which for $x \in \mathbb{R} - \{0\}$ can give any value for c except $c = 0$.

(d) Similarly, for $x = h(y)$ need $\frac{\partial f}{\partial x} \neq 0$. Fast way is to notice that f is symmetric under the interchange $3x \leftrightarrow y$, which gives the result. The long way is

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}((3x + y)e^{3xy}) \\ &= 3e^{3xy} + (3x + y)3ye^{3xy} \\ &= (3 + 3y^2 + 9xy)e^{3xy}\end{aligned}$$

$$\text{so} \qquad \frac{\partial f}{\partial y} = 0 \iff 3 + 3y^2 + 9xy = 0$$

$$\text{If } y \neq 0 \quad \Longrightarrow \quad x = -\frac{(3 + 3y^2)}{9y} = -\frac{1}{3}\left(y + \frac{1}{y}\right) \qquad (\ddagger).$$

So problems occur at point (x_0, y_0) with $x_0 = -\frac{1}{3}\left(y_0 + \frac{1}{y_0}\right)$.

(e) Points where we can neither write $y = g(x)$ nor $x = h(y)$ are the *critical points*, where $\underline{\nabla} f = \underline{0}$. At these points, both ($\dagger$) and ($\ddagger$) are satisfied simultaneously. If we substitute $3x_0 = -\left(y_0 + \frac{1}{y_0}\right)$ into ($\dagger$), remembering that this requires $y_0 \neq 0$, gives

$$\begin{aligned}y_0 &= \left(y_0 + \frac{1}{y_0}\right) + 1/\left(y_0 + \frac{1}{y_0}\right) \\ \Longrightarrow 0 &= \frac{1}{y_0} + 1/\left(y_0 + \frac{1}{y_0}\right) \\ \Longrightarrow 0 &= \left(y_0 + \frac{1}{y_0}\right) + y_0 \\ \Longrightarrow 0 &= 2y_0^2 + 1.\end{aligned}$$

Since $2y_0^2 + 1 > 0$, no points simultaneously satisfy (†) and (‡), and hence no points exist where f can neither be written as $y = g(x)$ nor as $x = h(y)$.

57. For each of the following two surfaces, show that the surface can be parameterised as $\underline{x}(x,y) = x\underline{e}_1 + y\underline{e}_2 + g(x,y)\underline{e}_3$, and show that the normal vector $\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y}$ can be written as
$$\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y} = \frac{\underline{\nabla} f}{\underline{e}_3 \cdot \underline{\nabla} f},$$
where the surface is a level set of a scalar field $f(\underline{x})$ which you should specify (and which you may assume to be differentiable).

(a) The upper hemisphere of radius two, centred on the origin (where $z > 0$).

(b) The surface defined by $e^{x+y+z} = 1 - (xy)^2$.

Solution

(a) The upper hemisphere of radius two, centred on the origin, is the set of points $\underline{x}$ such that $x^2 + y^2 + z^2 = 2$ with $z > 0$. On this surface we can therefore explicitly write $z = g(x,y) = \sqrt{2 - x^2 - y^2}$, and the surface can therefore be parameterised as
$$\begin{aligned}\underline{x}(x,y) &= x\underline{e}_1 + y\underline{e}_2 + g(x,y)\underline{e}_3 \\ &= x\underline{e}_1 + y\underline{e}_2 + \sqrt{2 - x^2 - y^2}\,\underline{e}_3.\end{aligned}$$
We therefore have
$$\begin{aligned}\frac{\partial \underline{x}}{\partial x} &= \underline{e}_1 - x\left(2 - x^2 - y^2\right)^{-1/2}\underline{e}_3 \\ \frac{\partial \underline{x}}{\partial y} &= \underline{e}_2 - y\left(2 - x^2 - y^2\right)^{-1/2}\underline{e}_3, \qquad \text{and so} \\ \frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y} &= x\left(2 - x^2 - y^2\right)^{-1/2}\underline{e}_1 + y\left(2 - x^2 - y^2\right)^{-1/2}\underline{e}_2 + \underline{e}_3.\end{aligned}$$
The surface is a level set of the scalar field $f(\underline{x}) = x^2 + y^2 + z^2$, and so we have
$$\begin{aligned}\underline{\nabla} f &= 2x\underline{e}_1 + 2y\underline{e}_2 + 2z\underline{e}_3, \qquad \text{and so} \\ \frac{\underline{\nabla} f}{\underline{e}_3 \cdot \underline{\nabla} f} &= \frac{x}{z}\underline{e}_1 + \frac{y}{z}\underline{e}_2 + \underline{e}_3 \\ &= x\left(2 - x^2 - y^2\right)^{-1/2}\underline{e}_1 + y\left(2 - x^2 - y^2\right)^{-1/2}\underline{e}_2 + \underline{e}_3 \\ &= \frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y}.\end{aligned}$$

(b) The surface is the level set $h(\underline{x}) = 1$, of the scalar field $h(\underline{x}) = e^{x+y+z} + (xy)^2$, which we assume to be differentiable. By the IFT for surfaces, the surface may be written as $z = g(x, y)$ in a neighbourhood of any point (x_0, y_0, z_0) at which $\frac{\partial h}{\partial z} \neq 0$, and this function $g(x, y)$ satisfies

$$\begin{aligned} h(x, y, g(x, y)) &= 1 \\ \frac{\partial g}{\partial x} &= \frac{-\partial h/\partial x}{\partial h/\partial z} \qquad (\dagger) \\ \frac{\partial g}{\partial y} &= \frac{-\partial h/\partial y}{\partial h/\partial z} \qquad (\ddagger)\,, \end{aligned}$$

in a neighbourhood of (x_0, y_0, z_0).

We have

$$\begin{aligned} \frac{\partial h}{\partial x} &= e^{x+y+z} + 2xy^2 \\ \frac{\partial h}{\partial y} &= e^{x+y+z} + 2x^2y \\ \frac{\partial h}{\partial z} &= e^{x+y+z}, \end{aligned}$$

and so $\frac{\partial h}{\partial z} \neq 0$ at all points (x_0, y_0, z_0). So, by the IFT, such a function $g(x, y)$ exists in a neighbourhood of all points, and so the surface can be parameterised as $\underline{x}(x, y) = x\underline{e}_1 + y\underline{e}_2 + g(x, y)\underline{e}_3$.

We therefore have

$$\begin{aligned} \frac{\partial \underline{x}}{\partial x} &= \underline{e}_1 + \frac{\partial g}{\partial x}\underline{e}_3 \\ \frac{\partial \underline{x}}{\partial y} &= \underline{e}_2 + \frac{\partial g}{\partial y}\underline{e}_3, \qquad \text{and so} \\ \frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y} &= -\frac{\partial g}{\partial x}\underline{e}_1 - \frac{\partial g}{\partial y}\underline{e}_2 + \underline{e}_3, \end{aligned}$$

and so using $(\dagger)$ and $(\ddagger)$,

$$\begin{aligned} \frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y} &= \frac{\partial h/\partial x}{\partial h/\partial z}\underline{e}_1 + \frac{\partial h/\partial y}{\partial h/\partial z}\underline{e}_2 + \underline{e}_3 \\ &= \left(1 + 2xy^2 e^{-(x+y+z)}\right)\underline{e}_1 + \left(1 + 2x^2 y e^{-(x+y+z)}\right)\underline{e}_2 + \underline{e}_3 \\ &= \frac{\underline{\nabla}\, h}{\underline{e}_3 \cdot \underline{\nabla}\, h}. \end{aligned}$$

Since at all points on this surface we have $e^{x+y+z} = 1 - (xy)^2$, we can write this normal in terms of x and y only (as we did for the first part of this question), as

$$\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y} = \left(1 + 2xy^2(1 - x^2y^2)^{-1}\right)\underline{e}_1 + \left(1 + 2x^2y(1 - x^2y^2)^{-1}\right)\underline{e}_2 + \underline{e}_3.$$

58. Compute the differential, or Jacobian matrix, and the Jacobian of the function $\underline{V} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\underline{V}(x, y) = (x\cos y, x\sin y)$. State where $\underline{V}$ defines an orientation preserving local diffeomorphism, and where it defines an orientation reversing local diffeomorphism.

Solution $\underline{V}(x, y) = (x\cos y, x\sin y)$ has Jacobian matrix

$$D\underline{V}_{(x,y)} = \begin{pmatrix} \cos y\,, & -x\sin y \\ \sin y\,, & x\cos y \end{pmatrix}$$

and so the Jacobian is

$$J(\underline{V}) = \begin{vmatrix} \cos y\,, & -x\sin y \\ \sin y\,, & x\cos y \end{vmatrix} = x\,.$$

Since $J(\underline{V}) > 0$ for $x > 0$, $\underline{V}$ is an orientation preserving local diffeomorphism on $R = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. $\underline{V}$ is an orientation reversing local diffeomorphism on $L = \{(x, y) \in \mathbb{R}^2 : x < 0\}$.

59. Repeat question 58 for $\underline{V}(x, y) = (e^x \cos y, e^x \sin y)\,.$

Solution $\underline{V}(x, y) = (e^x\cos y, e^x\sin y)$ has Jacobian matrix

$$D\underline{V}_{(x,y)} = \begin{pmatrix} e^x\cos y\,, & -e^x\sin y \\ e^x\sin y\,, & e^x\cos y \end{pmatrix}$$

and so the Jacobian is

$$J(\underline{V}) = \begin{vmatrix} e^x\cos y\,, & -e^x\sin y \\ e^x\sin y\,, & e^x\cos y \end{vmatrix} = e^{2x}\,.$$

Since $J(\underline{V}) > 0$ for all $x \in \mathbb{R}^2$, $\underline{V}$ defines an orientation preserving local diffeomorphism on all of $\mathbb{R}^2$ (to $\underline{V}(\mathbb{R}^2) = \mathbb{R}^2 - \{\underline{0}\}$).

60. Calculate the differential, or Jacobian matrix, and the Jacobian of the following transformations:

(a) $\underline{U}(u, v) = (x(u, v), y(u, v))$ where $x(u, v) = \frac{1}{2}(u + v)$ and $y(u, v) = \frac{1}{2}(u - v)\,;$

(b) $\underline{V}(r, \theta) = (x(r, \theta), y(r, \theta))$ where $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta\,;$

(c) $\underline{W}(r, \theta, \phi) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\,.$

Solution

(a)

$$D\underline{U} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\;; \quad J(\underline{U}) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}\,.$$

(b)

$$D\underline{V} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\;; \quad J(\underline{V}) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\,.$$

(c)

$$D\underline{W} = \begin{pmatrix} \frac{\partial W_1}{\partial r} & \frac{\partial W_1}{\partial \theta} & \frac{\partial W_1}{\partial \phi} \\ \frac{\partial W_2}{\partial r} & \frac{\partial W_2}{\partial \theta} & \frac{\partial W_2}{\partial \phi} \\ \frac{\partial W_3}{\partial r} & \frac{\partial W_3}{\partial \theta} & \frac{\partial W_3}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix};$$

$$J(\underline{W}) = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta\,.$$

61. Adapted from exam question 2009 (Section B) Q7:

(a) Let $\underline{V} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. Give the definition of $\underline{V}$ being differentiable at a point $\underline{a}$.

(b) Let $\underline{V}(x)$ and $\underline{W}(x)$ be two differentiable vector fields in $\mathbb{R}^2$. Give formulae for the two differentials $D\underline{V}_{\underline{x}}$ and $D\underline{W}_{\underline{x}}$.

(c) Use the chain rule to show that the differential of the composite map $\underline{U}(\underline{x}) := \underline{V}(\underline{W})$ satisfies

$$D\underline{U}_{\underline{x}} = D\underline{V}_{\underline{W}} D\underline{W}_{\underline{x}}\,.$$

Solution

(a) $\underline{V}$ is differentiable at a point if it can be well enough linearly approximated near that point. In particular, $\underline{V}$ is differentiable at $\underline{a}$ if there exists a linear function $\underline{L}(\underline{h}) : \mathbb{R}^n \to \mathbb{R}^n$ such that

i) $\underline{V}(\underline{a}+\underline{h}) - \underline{V}(\underline{a}) = \underline{L}(\underline{h}) + \underline{R}(\underline{h})$

ii) $\lim_{\underline{h}\to\underline{0}} \frac{R(\underline{h})}{|\underline{h}|} = 0$.

(b)

$$D\underline{V}_{\underline{x}} = \begin{pmatrix} \frac{\partial V_1}{\partial x} & \frac{\partial V_1}{\partial y} \\ \frac{\partial V_2}{\partial x} & \frac{\partial V_2}{\partial y} \end{pmatrix}$$

and

$$D\underline{W}_{\underline{x}} = \begin{pmatrix} \frac{\partial W_1}{\partial x} & \frac{\partial W_1}{\partial y} \\ \frac{\partial W_2}{\partial x} & \frac{\partial W_2}{\partial y} \end{pmatrix}.$$

(c)

$$\begin{aligned} D(\underline{U})_{\underline{x}} &= \begin{pmatrix} \frac{\partial V_1(\underline{W})}{\partial x} & \frac{\partial V_1(\underline{W})}{\partial y} \\ \frac{\partial V_2(\underline{W})}{\partial x} & \frac{\partial V_2(\underline{W})}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial V_1}{\partial W_1}\frac{\partial W_1}{\partial x} + \frac{\partial V_1}{\partial W_2}\frac{\partial W_2}{\partial x} & \frac{\partial V_1}{\partial W_1}\frac{\partial W_1}{\partial y} + \frac{\partial V_1}{\partial W_2}\frac{\partial W_2}{\partial y} \\ \frac{\partial V_2}{\partial W_1}\frac{\partial W_1}{\partial x} + \frac{\partial V_2}{\partial W_2}\frac{\partial W_2}{\partial x} & \frac{\partial V_2}{\partial W_1}\frac{\partial W_1}{\partial y} + \frac{\partial V_2}{\partial W_2}\frac{\partial W_2}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial V_1}{\partial W_1} & \frac{\partial V_1}{\partial W_2} \\ \frac{\partial V_2}{\partial W_1} & \frac{\partial V_2}{\partial W_2} \end{pmatrix} \begin{pmatrix} \frac{\partial W_1}{\partial x} & \frac{\partial W_1}{\partial y} \\ \frac{\partial W_2}{\partial x} & \frac{\partial W_2}{\partial y} \end{pmatrix}, \end{aligned}$$

as required.

62. Adapted from exam question 2018 (Section B) Q8:

(a) Given a vector field $\underline{u}(\underline{x}) = \underline{w}(\underline{v}(\underline{x}))$, use the chain rule to show that $D\underline{u}(\underline{x}) = D\underline{w}(\underline{v})D\underline{v}(\underline{x})$, and hence $J(\underline{u}) = J(\underline{w})J(\underline{v})$.

(b) Let

$$\begin{aligned}\underline{v}(\underline{x}) &= (v_1, v_2) = (\cos y, \sin x)\\ \underline{w}(\underline{x}) &= (w_1, w_2) = (x^2 + y^3, x^2 y),\end{aligned}$$

and define $\underline{u}(\underline{x}) = \underline{w}(\underline{v}(\underline{x}))$. Use the result from part (a) to calculate $J(\underline{u})$. Verify your answer by direct substitution.

Solution

(a) This follows from problem 61 part (c), but explicitly we have

$$\begin{aligned}D\underline{u}(\underline{x}) &= \begin{pmatrix} \frac{\partial w_1(\underline{v})}{\partial x} & \frac{\partial w_1(\underline{v})}{\partial y} \\ \frac{\partial w_2(\underline{v})}{\partial x} & \frac{\partial w_2(\underline{v})}{\partial y} \end{pmatrix}\\ &= \begin{pmatrix} \frac{\partial w_1}{\partial v_1}\frac{\partial v_1}{\partial x} + \frac{\partial w_1}{\partial v_2}\frac{\partial v_2}{\partial x} & \frac{\partial w_1}{\partial v_1}\frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial v_2}\frac{\partial v_2}{\partial y} \\ \frac{\partial w_2}{\partial v_1}\frac{\partial v_1}{\partial x} + \frac{\partial w_2}{\partial v_2}\frac{\partial v_2}{\partial x} & \frac{\partial w_2}{\partial v_1}\frac{\partial v_1}{\partial y} + \frac{\partial w_2}{\partial v_2}\frac{\partial v_2}{\partial y} \end{pmatrix}\\ &= \begin{pmatrix} \frac{\partial w_1}{\partial v_1} & \frac{\partial w_1}{\partial v_2} \\ \frac{\partial w_2}{\partial v_1} & \frac{\partial w_2}{\partial v_2} \end{pmatrix}\begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} \end{pmatrix}\\ &= D\underline{w}(\underline{v})D\underline{v}(\underline{x}).\end{aligned}$$

Now by taking determinants of both sides we have

$$\begin{aligned}|D\underline{u}(\underline{x})| &= |D\underline{w}(\underline{v})D\underline{v}(\underline{x})|\\ J(\underline{u}) &= |D\underline{w}(\underline{v})|\,|D\underline{v}(\underline{x})|\\ &= J(\underline{w})J(\underline{v}),\end{aligned}$$

as required.

(b) We have

$$D\underline{v}(\underline{x}) = \begin{pmatrix} 0 & -\sin y \\ \cos x & 0 \end{pmatrix},$$

so

$$J(\underline{v}) = \cos x \sin y,$$

and

$$D\underline{w}(\underline{v}) = \begin{pmatrix} 2v_1 & 3v_2^2 \\ 2v_1 v_2 & v_1^2 \end{pmatrix},$$

and so

$$J(\underline{w}) = 2v_1(v_1^2 - 3v_2^3).$$

We therefore have

$$\begin{aligned} J(\underline{u}) &= J(\underline{w})J(\underline{v}) \\ &= 2v_1(v_1^2 - 3v_2^3)(\cos x \sin y) \\ &= 2\cos y(\cos^2 y - 3\sin^3 x)(\cos x \sin y) \\ &= \sin 2y \cos x(\cos^2 y - 3\sin^3 x). \end{aligned}$$

Checking by direct substitution gives

$$\underline{u}(\underline{x}) = \underline{w}(\underline{v}(\underline{x})) = (\cos^2 y + \sin^3 x, \cos^2 y \sin x),$$

and hence

$$D\underline{u}(\underline{x}) = \begin{pmatrix} 3\cos x \sin^2 x & -2\cos y \sin y \\ \cos x \cos^2 y & -2\sin y \cos y \sin x \end{pmatrix},$$

and therefore

$$\begin{aligned} J(\underline{u}) &= -6\sin^3 x \cos x \sin y \cos y + 2\cos x \cos^3 y \sin y \\ &= 2\cos x \sin y \cos y(\cos^2 y - 3\sin^3 x) \\ &= \sin 2y \cos x(\cos^2 y - 3\sin^3 x), \end{aligned}$$

which agrees with the previous calculation.

63. (a) Let $\underline{v} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. Give the definition of $\underline{V}$ being differentiable on an open set $U \subseteq \mathbb{R}^n$.

(b) For $\underline{x} = x\underline{e}_1 + y\underline{e}_2$, let $\underline{v} : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\underline{v}(\underline{x}) = (x^2 + y^2, x + y).$$

Using the definition of differentiability, show that $\underline{v}$ is differentiable on $\mathbb{R}^2$.

(c) Draw a diagram to show where $\underline{v}$ defines an orientation preserving local diffeomorphism (on $U \subseteq \mathbb{R}^2$), and where $\underline{v}$ defines an orientation reversing local diffeomorphism (on $V \subseteq \mathbb{R}^2$).

Solution

(a) A vector field $\underline{v} : U \to \mathbb{R}^n$, with $U \subseteq \mathbb{R}^n$ open is differentiable at a point $\underline{a} \in U$ if $\exists$ a linear function $\underline{L} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\underline{v}(\underline{a} + \underline{h}) - \underline{v}(\underline{a}) = \underline{L}(\underline{h}) + \underline{R}(\underline{h})$$

$$\text{with} \qquad \lim_{\underline{h} \to \underline{0}} \frac{\underline{R}(\underline{h})}{|\underline{h}|} = \underline{0}.$$

A vector field is differentiable on an open set $U \subseteq \mathbb{R}^n$ if it is differentiable at every point $\underline{a} \in U$.

(b) We know from the lecture notes, that when $\underline{L}(\underline{h})$ exists, it is given by $D\underline{v}_{\underline{a}} \cdot \underline{h}$. In this case we have

$$D\underline{v}_{\underline{a}} = \begin{pmatrix} 2a_1 & 2y_1 \\ 1 & 1 \end{pmatrix},$$

and so, for any $\underline{a} \in \mathbb{R}^2$,

$$\begin{aligned}
\underline{R}(\underline{h}) &= \underline{v}(\underline{a}+\underline{h}) - \underline{v}(\underline{a}) - D\underline{v}_{\underline{a}} \cdot h \\
&= \left((a_1+h_1)^2 + (a_2+h_2)^2, a_1+h_1+a_2+h_2\right)^t \\
&\quad - \left(a_1^2+a_2^2, a_1+a_2\right)^t - (2a_1h_1+2a_2h_2, h_1+h_2)^t \\
&= (h_1^2+h_2^2, 0)^t.
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{\underline{h}\to\underline{0}} \frac{\underline{R}(\underline{h})}{|\underline{h}|} &= \lim_{\underline{h}\to\underline{0}} \left(\frac{h_1^2+h_2^2}{|\underline{h}|}, \frac{0}{|\underline{h}|}\right) \\
&= \lim_{\underline{h}\to\underline{0}} \left(\frac{|\underline{h}|^2}{|\underline{h}|}, 0\right) \\
&= \lim_{\underline{h}\to\underline{0}} (|\underline{h}|, 0) = \underline{0},
\end{aligned}$$

and hence $\underline{v}$ is differentiable at any $\underline{a} \in \mathbb{R}^2$, and therefore $\underline{v}$ is differentiable on $\mathbb{R}^2$ as required.

(c) Since, by the previous part of this question, $\underline{v}$ is differentiable on all of $\mathbb{R}^2$, we can apply the inverse function theorem. We have that $J(\underline{v}) = |D\underline{v}_{\underline{x}}| = 2(x-y)$, and so $\underline{v}$ defines a local diffeomorphism around all points where $x \neq y$. This local diffeomorphism is orientation preserving when $x > y$, and orientation reversing when $x < y$. A diagram indicating this is given below.

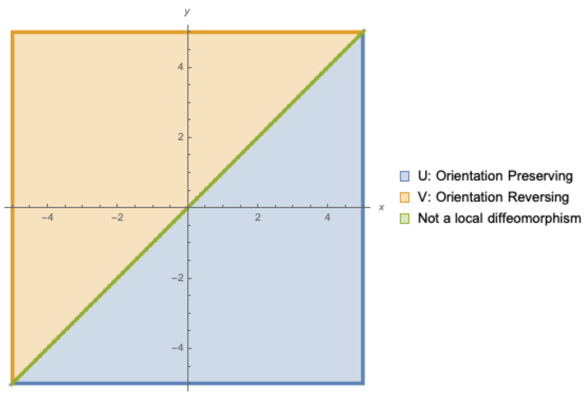


Figure 1: The vector field $\underline{v}$ defines an orientation preserving local diffeomorphism around all points in the region where $x > y$, it defines an orientation reversing local diffeomorphism around all points in the region where $y > x$, and it does not define a local diffeomorphism in open regions around points on the line $x = y$.

64. Let A be the region bounded by the positive x- and y-axes and the line $3x + 4y = 10$. Compute $\iint_A (x^2 + y^2)\, dx\, dy$, taking the integrals in both orders and checking that your answers agree.

Solution This can be integrated either $dx\, dy$ or $dy\, dx$, finding the limits of integration via the intersection of the line with the x and y axes respectively:

$$
\begin{aligned}
I &= \int_0^{5/2} \left(\int_0^{(10-4y)/3} (x^2 + y^2)\, dx \right) dy \\
&= \int_0^{5/2} \left[\tfrac{1}{3}x^3 + y^2 x \right]_0^{(10-4y)/3} dy \\
&= \int_0^{5/2} \left(\tfrac{1000}{81} - \tfrac{400}{27}y + \tfrac{250}{27}y^2 - \tfrac{172}{81}y^3 \right) dy \;=\; \frac{15625}{1296}\,.
\end{aligned}
$$

or

$$
\begin{aligned}
I &= \int_0^{10/3} \left(\int_0^{(10-3x)/4} (x^2 + y^2)\, dy \right) dx \\
&= \int_0^{10/3} \left[yx^2 + \tfrac{1}{3}y^3 \right]_0^{(10-3x)/4} dx \\
&= \int_0^{10/3} \left(\tfrac{125}{24} - \tfrac{75}{16}x + \tfrac{125}{32}x^2 - \tfrac{57}{64}x^3 \right) dx \;=\; \frac{15625}{1296}\,.
\end{aligned}
$$

65. In the following integrals sketch the integration regions and then evaluate the integrals. Next interchange the order of integrations and re-evaluate.

(a) $\int_0^1 (\int_x^1 xy\, dy)\, dx$,

(b) $\int_0^{\pi/2} (\int_0^{\cos\theta} \cos\theta\, dr)\, d\theta$,

(c) $\int_0^1 (\int_1^{2-y} (x+y)^2\, dx)\, dy$.

Solution (In all cases, sketches should also be included!)

(a)

$$
I = \int_0^1 (\int_x^1 xy\, dy)\, dx = \int_0^1 [\tfrac{1}{2}xy^2]_x^1\, dx = \int_0^1 (\tfrac{1}{2}x - \tfrac{1}{2}x^3)\, dx = [\tfrac{1}{4}x^2 - \tfrac{1}{8}x^4]_0^1 = \tfrac{1}{8}\,,
$$

or, interchanging the order of integrations,

$$
I = \int_0^1 (\int_0^y xy\, dx)\, dy = \int_0^1 [\tfrac{1}{2}x^2 y]_0^y\, dx = \int_0^1 (\tfrac{1}{2}y^3)\, dx = [\tfrac{1}{8}y^4]_0^1 = \tfrac{1}{8}\,.
$$

(b)

$$I = \int_0^{\pi/2} (\int_0^{\cos\theta} \cos\theta \, dr) \, d\theta = \int_0^{\pi/2} [r\cos\theta]_0^{\cos\theta} \, d\theta = \int_0^{\pi/2} \cos^2\theta \, d\theta = \pi/4 \, .$$

or, interchanging the order of integrations,

$$I = \int_0^1 (\int_0^{\cos^{-1} r} \cos\theta \, d\theta) \, dr = \int_0^1 [\sin\theta]_0^{\cos^{-1} r} \, dr = \int_0^1 \sqrt{1-r^2} \, dr \, .$$

Here $\sin(\cos^{-1} r)$ is the sine of the angle whose cosine is r, and so is given using the formula $\sin = \sqrt{1-\cos^2}$. Now substitute $r = \sin\varphi$ to get

$$I = \int_0^{\pi/2} \cos^2\varphi \, d\varphi = \pi/4 \, .$$

(c)

$$\begin{aligned} I &= \int_0^1 (\int_1^{2-y} (x+y)^2 \, dx) \, dy = \int_0^1 \left[\tfrac{1}{3}(x+y)^3\right]_1^{2-y} dy \\ &= \int_0^1 \left(\tfrac{8}{3} - \tfrac{1}{3}(1+y)^3\right) dy = \left[\tfrac{8}{3}y - \tfrac{1}{12}(1+y)^4\right]_0^1 dy = \tfrac{8}{3} - \tfrac{16}{12} + \tfrac{1}{12} = \tfrac{17}{12} \, . \end{aligned}$$

or, interchanging the order of integrations,

$$\begin{aligned} I &= \int_1^2 (\int_0^{2-x} (x+y)^2 \, dy) \, dx = \int_1^2 \left[\tfrac{1}{3}(x+y)^3\right]_0^{2-x} dx \\ &= \int_1^2 \left(\tfrac{8}{3} - \tfrac{1}{3}x^3\right) dx = \left[\tfrac{8}{3}x - \tfrac{1}{12}x^4\right]_1^2 dx = \tfrac{16}{3} - \tfrac{16}{12} - \tfrac{8}{3} + \tfrac{1}{12} = \tfrac{17}{12} \, . \end{aligned}$$

66. Exam question 2010 (Section A) Q4: Calculate the double integral

$$\iint_A (|x| + |y|) \, dx \, dy.$$

where A is the region defined by $|x| + |y| \leq 1$.

Solution The integration region is shown in Figure 2.

Due to the symmetry of the region and of the function $f(x,y) = |x| + |y|$, the integral is

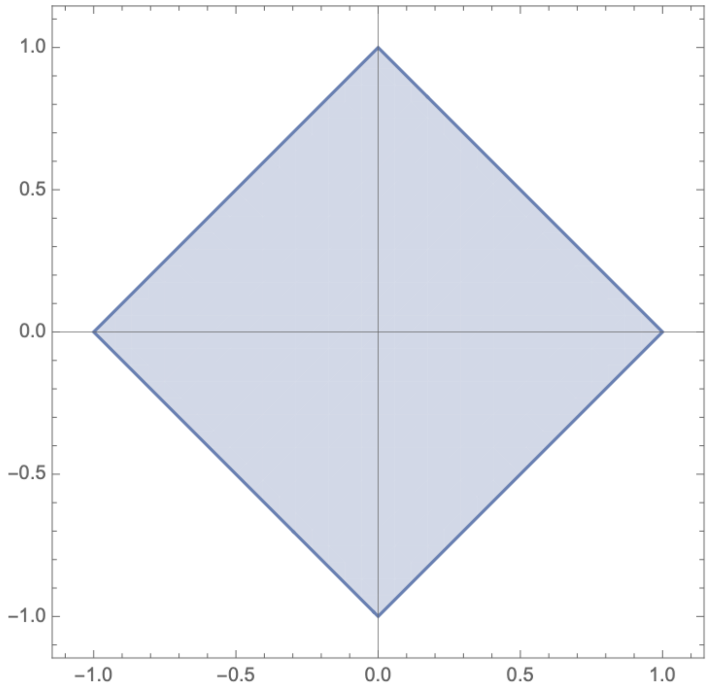


Figure 2: The region $|x| + |y| \leq 1$.

simply $4\times$ the integral over one quadrant. Therefore

$$\begin{aligned}
\iint_A (|x| + |y|)\, dx\, dy &= 4 \iint_{A \text{ in first quadrant}} (|x| + |y|)\, dx\, dy \\
&= 4 \int_0^1 \int_0^{1-y} (x + y)\, dx\, dy \\
&= 4 \int_0^1 \left[\tfrac{1}{2}x^2 + xy\right]_0^{1-y} dy \\
&= 4 \int_0^1 \left(\tfrac{1}{2}(1-y)^2 + (1-y)y\right) dy \\
&= 4 \int_0^1 \left(\tfrac{1}{2} - \tfrac{1}{2}y^2\right) dy \\
&= 2\Big[y - \tfrac{1}{3}y^3\Big]_0^1 = \frac{4}{3}\,.
\end{aligned}$$

67. Exam question 2011 (Section A) Q4: Change the order of integration in the double integral

$$\int_0^2 \int_x^{2x} f(x, y)\, dy\, dx\,.$$

Solution The area of integration is shown below in Figure 3. When we swap the order of integration, and perform x first, we require 2 integrals: one to accommodate part of triangle below the line $y = 2$, and one for the part above.

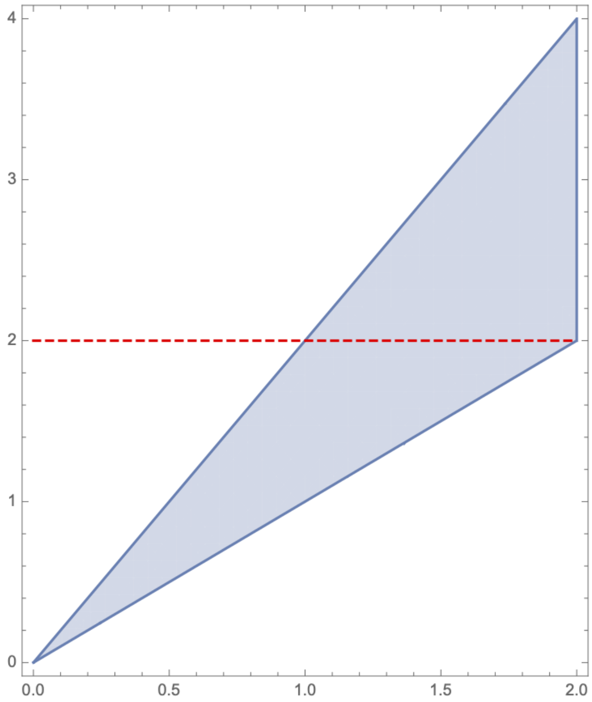


Figure 3: The required region. of integration

This is

$$\int_0^2 \int_{y/2}^{y} f(x,y)\,dx\,dy + \int_2^4 \int_{y/2}^{2} f(x,y)\,dx\,dy$$

68. Exam question May 2017 (Section A): A solid cylinder C of radius 1 and height 1 is defined by

$$C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1,\ 0 \leq z \leq 1\}\,.$$

Show that the paraboloid $P = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2\}$ cuts C into two pieces of equal volume.

Solution First, calculate the volume above P but inside C. Opting to do the z integral last, this is

$$V_{\text{above}} = \int_0^1 \left(\int_{x^2+y^2 \leq z} dxdy \right) dz$$

Recognising the integral in brackets as the area of a circle radius $\sqrt{z}$, we have

$$V_{\text{above}} = \int_0^1 (\pi z)\,dz = \left[\frac{\pi}{2} z^2\right]_0^1 = \frac{\pi}{2}\,.$$

(Other orders of integration would also be OK)

69. Compute the iterated integral

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy\, dx\,.$$

Now reverse the order of integrations and re-evaluate. Why doesn't your answer contradict Fubini's theorem? (Hints: for the first integral, with respect to y, it might help to aim at something that can be integrated by parts using $\frac{d}{dy}\left(\frac{1}{x^2+y^2}\right) = -\frac{2y}{(x^2+y^2)^2}$; and in answering to the last part of the question, the fact that $\int_0^1 \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| dy \geq \int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\, dy$ could be useful.)

Solution For the inner, y, integral one can either have a flash of inspriration, or else use the hint as follows:

$$\begin{aligned}
\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy &= \int_0^1 \left(\frac{x^2 + y^2}{(x^2 + y^2)^2} - \frac{2y^2}{(x^2 + y^2)^2}\right) dy \\
&= \int_0^1 \frac{1}{x^2 + y^2}\, dy - \int_0^1 \frac{2y^2}{(x^2 + y^2)^2}\, dy \\
&= \int_0^1 \frac{1}{x^2 + y^2}\, dy + \int_0^1 y \frac{d}{dy}\left(\frac{1}{x^2 + y^2}\right) dy\,.
\end{aligned}$$

The last term can be integrated by parts:

$$\int_0^1 y \frac{d}{dy}\left(\frac{1}{x^2 + y^2}\right) dy = \left[y\, \frac{1}{x^2 + y^2}\right]_0^1 - \int_0^1 \frac{dy}{dy}\, \frac{1}{x^2 + y^2}\, dy = \frac{1}{x^2 + 1} - \int_0^1 \frac{1}{x^2 + y^2}\, dy\,.$$

Adding everything up and cancelling the left-over integrals,

$$\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy = \frac{1}{1 + x^2}$$

and so the iterated integral is

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy\, dx = \int_0^1 \frac{1}{1 + x^2}\, dx = [\arctan(x)]_0^1 = \frac{\pi}{4}\,.$$

For the oppositely-ordered iterated integral the calculation is very similar, but take care with the signs! We have

$$\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dx = -\frac{1}{1 + y^2}$$

and so

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dx\, dy = -\int_0^1 \frac{1}{1 + y^2}\, dy = -\left[\arctan(x)\right]_0^1 = -\frac{\pi}{4}\,.$$

As hinted in the question, this is not what Fubini would predict. But the conditions for Fubini require the iterated integral of the modulus of the integrand to be finite, and this isn't true: taking the hint from the question,

$$\int_0^1 \int_0^1 \left| \frac{x^2 - y^2}{(x^2+y^2)^2} \right| dy\, dx \geq \int_0^1 \int_0^x \frac{x^2 - y^2}{(x^2+y^2)^2}\, dy\, dx$$
$$= \int_0^1 \left[\frac{y}{x^2+y^2} \right]_0^x dx = \int_0^1 \frac{1}{2x}\, dx = \infty\,.$$

(Strictly speaking one should regulate the potentially improper x integrals on the right-hand sides of this calculation, and thereby keep everything well-defined at all stages, by running x from $\delta > 0$ to 1 and then letting $\delta \to 0$ at the end, but the conclusion would be the same.) The 'volume' under the second surface is infinite, which is why the

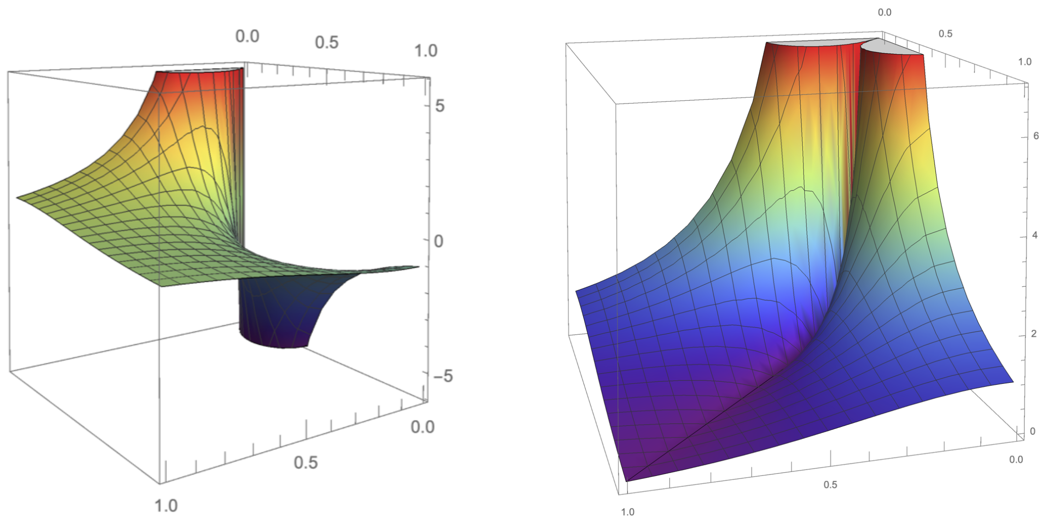


Figure 4: Plots of $\frac{x^2-y^2}{(x^2+y^2)^2}$ and of $\left|\frac{x^2-y^2}{(x^2+y^2)^2}\right|$.

conditions of Fubini's theorem are not met in this case.

70. Let B be the region bounded by the five planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$, and $z = x + y$.

(a) Find the volume of B.

(b) Evaluate $\int_B x\, dV$.

(c) Evaluate $\int_B y\, dV$.

Solution

(a) Volume of B:

$$\begin{aligned}\int_B dV = \int_0^1\int_0^{1-x}\int_0^{x+y} dzdydx &= \int_0^1\int_0^{1-x}(x+y)dydx\\ &= \int_0^1 \left[xy+\tfrac{1}{2}y^2\right]_0^{1-x} dx\\ &= \int_0^1 \left(x(1-x)+\tfrac{1}{2}(1-x)^2\right) dx\\ &= \int_0^1 \left(x-x^2+\tfrac{1}{2}-x+\tfrac{1}{2}x^2\right) dx\\ &= \left[\tfrac{1}{2}x-\tfrac{1}{6}x^3\right]_0^1 = \frac{1}{3}\,.\end{aligned}$$

(b)

$$\begin{aligned}\int_B x\,dV = \int_0^1\int_0^{1-x}\int_0^{x+y} x\,dzdydx &= \int_0^1 x\int_0^{1-x}(x+y)dydx\\ &= \int_0^1 \left(x^2-x^3+\tfrac{1}{2}x-x^2+\tfrac{1}{2}x^3\right) dx\\ &= \left[\tfrac{1}{4}x^2-\tfrac{1}{8}x^4\right]_0^1 = \frac{1}{8}\,.\end{aligned}$$

(c) Evaluate $\int_B y\,dV$.

$$\begin{aligned}\int_B y\,dV = \int_0^1\int_0^{1-x}\int_0^{x+y} y\,dzdydx &= \int_0^1\int_0^{1-x} y(x+y)dydx\\ &= \int_0^1 \left[\tfrac{1}{2}xy^2+\tfrac{1}{3}y^3\right]_0^{1-x} dx\\ &= \int_0^1 \left(\tfrac{1}{2}x(1-x)^2+\tfrac{1}{3}(1-x)^3\right) dx\\ &= \int_0^1 \tfrac{1}{6}\left(2-3x+x^3\right) dx\\ &= \tfrac{1}{6}\left[2x-\tfrac{3}{2}x^2+\tfrac{1}{4}x^4\right]_0^1 = \frac{1}{8}\,.\end{aligned}$$

71. A function $f(x,y)$ is defined by

$$f(x,y)=\begin{cases} 1 & \text{if } -1<x-y<0\\ -1 & \text{if } 0<x-y<1\\ 0 & \text{otherwise}\,.\end{cases}$$

Compute $\int_0^\infty\left(\int_{-\infty}^\infty f(x,y)\,dx\right)dy$ and also $\int_{-\infty}^\infty\left(\int_0^\infty f(x,y)\,dy\right)dx$ (for the second case it might help to draw a picture). Comment on your two answers – does this contradict Fubini's theorem?

Solution The first way round is easy! For any $y \in \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} f(x,y)\,dx = \int_{y-1}^{y} 1\,dx + \int_{y}^{y+1} -1\,dx = 1 - 1 = 0$$

and so

$$\int_{0}^{\infty} \left(\int_{-\infty}^{\infty} f(x,y)\,dx \right) dy = \int_{0}^{\infty} 0\,dy = 0\,.$$

For the opposite order of integration, what enters into the y integral depends on the value of x, as follows:

$$\int_{0}^{\infty} f(x,y)\,dy = \begin{cases} 0 & \text{if } x < -1 \\ \int_0^{x+1} 1\,dy = x{+}1 & \text{if } -1 < x < 0 \\ \int_0^{x} -1\,dy + \int_x^{x+1} 1\,dy = -x{+}1 & \text{if } 0 < x < 1 \\ \int_{x-1}^{x} -1\,dy + \int_x^{x+1} 1\,dy = 0 & \text{if } 1 < x\,. \end{cases}$$

Hence, and only including the non-zero bits,

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\int_{0}^{\infty} f(x,y)\,dy \right) dx &= \int_{-1}^{0} (x{+}1)\,dx + \int_{0}^{1} (-x{+}1)\,dx \\ &= \left[\tfrac{1}{2}x^2{+}x\right]_{-1}^{0} + \left[-\tfrac{1}{2}x^2{+}x\right]_{0}^{1} = (-\tfrac{1}{2}{+}1) + (-\tfrac{1}{2}{+}1) = 1\,. \end{aligned}$$

The two answers differ! This doesn't violate Fubini's theorem because if we were to replace $f(x,y)$ by $|f(x,y)|$ in the original double integral, the result would be infinite.

72. A function $f(x,y)$ is defined by

$$f(x,y) = \begin{cases} 2^{2(n+1)} & \text{if } 2^{-(n+1)} < x < 2^{-n}, \quad 2^{-(n+1)} < y < 2^{-n} \\ -2^{2n+3} & \text{if } 2^{-(n+1)} < x < 2^{-n}, \quad 2^{-(n+2)} < y < 2^{-(n+1)} \\ 0 & \text{otherwise}\,, \end{cases}$$

for $n \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$.

Compute $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$, and $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$. Does this contradict Fubini's theorem?

Solution Let's start by sketching the function $f(x,y)$ for $0 \leq x \leq 1$, $0 \leq y \leq 1$.

Let's first compute $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$. From the picture we can see that the integrand is

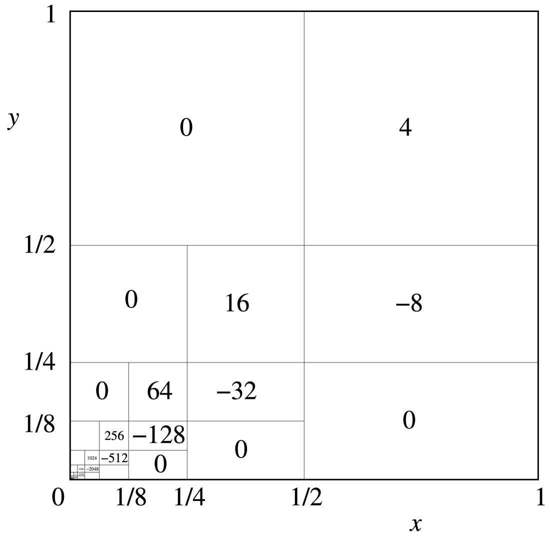


Figure 5: The function $f(x,y)$ for $0 \le x \le 1$, $0 \le y \le 1$.

dependent on both the x and y values that we're integrating over. Specifically, we have

$$\begin{aligned}
\int_0^1 \int_0^1 f(x,y)\,dy\,dx &= \int_{1/2}^1 \int_0^1 f(x,y)dy\,dx + \int_{1/4}^{1/2} \int_0^1 f(x,y)dy\,dx + \dots \\
&= \sum_{n=0}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} \int_0^1 f(x,y)dy\,dx \\
&= \sum_{n=0}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} \left(\int_{2^{-(n+1)}}^{2^{-n}} 2^{2(n+1)}dy + \int_{2^{-(n+2)}}^{2^{-(n+1)}} (-2^{2n+3})dy \right) dx \\
&= \sum_{n=0}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} \left(2^{-(n+1)}2^{2(n+1)} - 2^{-(n+2)}2^{2n+3}\right) \\
&= \sum_{n=0}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} 0 \\
&= 0,
\end{aligned}$$

where to go from the third line to the fourth we note that the integrands are simply constants.

Essentially, looking at the picture in vertical 'strips', we see that there are two rectangles where the function is non-zero. The upper rectangle is twice as long in the vertical direction but the integrand is half that of the integrand from the lower rectangle, and with opposite sign. Hence the two contributions cancel along each vertical slice of the integral.

The calculation for $\int_0^1\int_0^1 f(x,y)\,dx\,dy$ is similar.

$$\begin{aligned}
\int_0^1\int_0^1 f(x,y)\,dy\,dx &= \int_{1/2}^1\int_0^1 f(x,y)\,dx\,dy + \int_{1/4}^{1/2}\int_0^1 f(x,y)\,dx\,dy + \dots \\
&= \int_{1/2}^1\int_0^1 f(x,y)\,dx\,dy + \sum_{n=1}^{\infty}\int_{2^{-(n+1)}}^{2^{-n}}\int_0^1 f(x,y)\,dx\,dy \\
&= \int_{1/2}^1\int_{1/2}^1 4\,dx\,dy + \sum_{n=1}^{\infty}\int_{2^{-(n+1)}}^{2^{-n}}\left(\int_{2^{-(n+1)}}^{2^{-n}} 2^{2(n+1)}\,dx \right.\\
&\quad \left. - \int_{2^{-n}}^{2^{-n+1}} 2^{2n+1}\,dx\right)dy \\
&= 1 + \sum_{n=1}^{\infty}\int_{2^{-(n+1)}}^{2^{-n}} \left(2^{-(n+1)}2^{2(n+1)} - 2^{-n}2^{2n+1}\right)dy \\
&= 1 + \sum_{n=1}^{\infty}\int_{2^{-(n+1)}}^{2^{-n}} 0\,dy \\
&= 1.
\end{aligned}$$

The two answers differ. This doesn't contradict Fubini's theorem, as the double integral is not absolutely convergent.

73. Write the line integral

$$\int_C xdx + ydy + (xz-y)dz$$

in the form $\int_C \mathbf{v}\cdot d\mathbf{x}$ for a suitable vector field $\mathbf{v}(\mathbf{x})$, and compute its value when C is the curve given by $\mathbf{x}(t) = t^2\,\mathbf{e_1} + 2t\,\mathbf{e_2} + 4t^3\,\mathbf{e_3}$ with $0 \le t \le 1$.

Solution The integral to be evaluated is

$$I = \int_C xdx + ydy + (xz-y)dz = \int_C \mathbf{v}\cdot d\mathbf{x} = \int_0^1 \mathbf{v}(\mathbf{x}(t))\cdot\frac{d\mathbf{x}}{dt}\,dt\,.$$

where $\mathbf{v}(\mathbf{x}) = x\,\mathbf{e_1} + y\,\mathbf{e_2} + (xz-y)\,\mathbf{e_3}$. Following the recipe from lectures, we compute $\frac{d\mathbf{x}}{dt} = 2t\,\mathbf{e_1} + 2\,\mathbf{e_2} + 12t^2\,\mathbf{e_3}$ while $\mathbf{v}(\mathbf{x}(t)) = t^2\mathbf{e_1} + 2t\mathbf{e_2} + (4t^5-2t)\mathbf{e_3}$, so

$$I = \int_0^1 \left(2t^3 + 4t + 48t^7 - 24t^3\right)dt = \left[\tfrac{1}{2}t^4 + 2t^2 + 6t^8 - 6t^5\right]_0^1 = \tfrac{5}{2}\,.$$

Alteratively and a little more informally we can simply note that

$$x = t^2 \Rightarrow dx = 2t\,dt\,,\quad y = 2t \Rightarrow dy = 2\,dt\,,\quad z = 4t^3 \Rightarrow dz = 12t^2\,dt\,,$$

so $xdx = 2t^3\,dt$, $ydy = 4t$, and $(xz-y)dz = (4t^5-2t)\times 12t^2\,dt$, after which the calculation runs as before.

74. Evaluate $\int_\sigma \mathbf{F}\cdot d\mathbf{x}$, where $\mathbf{F} = y\,\mathbf{e_1} + 2x\,\mathbf{e_2} + y\,\mathbf{e_3}$ and the path σ is given by $\mathbf{x}\,(t) = t\,\mathbf{e_1} + t^2\,\mathbf{e_2} + t^3\,\mathbf{e_3}$, $0 \le t \le 1$.

Solution We have $\frac{d\mathbf{x}}{dt} = \mathbf{e_1} + 2t\,\mathbf{e_2} + 3t^2\,\mathbf{e_3}$, and $\mathbf{F}(\mathbf{x}\,(t)) = t^2\mathbf{e_1} + 2t\mathbf{e_2} + t^2\mathbf{e_3}$. Hence

$$\begin{aligned}\int_\sigma \mathbf{F}\cdot\mathbf{dx} &= \int_0^1 (t^2\mathbf{e_1} + 2t\mathbf{e_2} + t^2\mathbf{e_3})\cdot(\mathbf{e_1} + 2t\mathbf{e_2} + 3t^2\mathbf{e_3})\,dt \\ &= \int_0^1 (t^2 + 4t^2 + 3t^4)\,dt = \left[\tfrac{5}{3}t^3 + \tfrac{3}{5}t^5\right]_0^1 = \tfrac{34}{15}\,.\end{aligned}$$

75. Let $\underline{A}(\underline{x})$ be the vector field $\underline{A}(x,y,z) = x\,\underline{e}_1 + y\,\underline{e}_2 + z\,\underline{e}_3$.
   (a) Compute the line integral $\int_C \underline{A}\cdot d\underline{x}$ where C is the straight line from the origin to the point $(1,1,1)$.
   (b) Show (by finding f) that the vector field $\underline{A}$ from part (a) is equal to $\nabla f(\underline{x})$ for some scalar field f, and that your answer to part (a) is equal to $f(1,1,1) - f(0,0,0)$.

Solution

(a) Parametrise the curve as $\underline{x}(t) = (t,t,t)$ with t running from 0 to 1 (other choices would also be fine). Then $\frac{d\underline{x}(t)}{dt} = (1,1,1)$ and $\underline{A}(\underline{x}(t)) = (t,t,t)$ so $\frac{d\underline{x}(t)}{dt}\cdot\underline{A}(\underline{x}(t)) = 3t$. Hence

$$\int_C \underline{A}\cdot d\underline{x} = \int_0^1 \frac{d\underline{x}}{dt}\cdot\underline{A}(\underline{x}(t))\,dt = \int_0^1 3t\,dt = 3/2.$$

(b) If $\underline{A} = \nabla f$, $f_x = x$ so $f(x,y,z) = \frac{1}{2}x^2 + g(y,z)$; then $f_y = g_y = y$ so $g(y,z) = \frac{1}{2}y^2 + h(z)$; and finally $f_z = h_z = z$ so $h(z) = \frac{1}{2}z^2 + c$ where c is a constant. Hence $f(\underline{x}) = \frac{1}{2}\underline{x}^2 + c$, and it is easy to check that this works. Then $f(1,1,1) - f(0,0,0) = \frac{1}{2}(1^2 + 1^2 + 1^2) + c - c = 3/2$, which is indeed the same as the answer to part (a). (Aside: note that the procedure used here to find f does not always work: try it for $\underline{A}(x,y,z) = (z,x,y)$.)

76. Show that the result from question 75 applies in general: if the vector field $\underline{v}(\underline{x})$ in $\mathbb{R}^n$ is the gradient of a scalar field $f(\underline{x})$, so that $\underline{v} = \nabla f$, and if C is a curve in $\mathbb{R}^n$ running from $\underline{x} = \underline{a}$ to $\underline{x} = \underline{b}$, then $\int_C \underline{v}\cdot d\underline{x} = f(\underline{b}) - f(\underline{a})$. (Hint: use the chain rule.)

Solution Let the curve C be parametrised by $\underline{x}(t)$ with t running from t_1 to t_2 so that $\underline{x}(t_1) = \underline{a}$ and $\underline{x}(t_2) = \underline{b}$. Then by the definition of the line integral,

$$\int_C \underline{v}\cdot d\underline{x} = \int_{t_1}^{t_2} \underline{v}(\underline{x}(t))\cdot\frac{d\underline{x}(t)}{dt}\,dt = \int_{t_1}^{t_2} \nabla f(\underline{x}(t))\cdot\frac{d\underline{x}(t)}{dt}\,dt\,.$$

But by the chain rule (see **2.2** in the Michaelmas Summary), if $F(t) = f(\underline{x}(t))$, then $\frac{dF}{dt} = \nabla f\cdot\frac{d\underline{x}}{dt}$. Hence

$$\int_C \underline{v}\cdot d\underline{x} = \int_{t_1}^{t_2} \frac{dF}{dt}\,dt = \big[F(t)\big]_{t_1}^{t_2} = F(t_2) - F(t_1) = f(\underline{b}) - f(\underline{a})$$

since $F(t_1) = f(\underline{x}(t_1)) = f(\underline{a})$ and $F(t_2) = f(\underline{x}(t_2)) = f(\underline{b})$.

77. Use the result from question 76 to evaluate $\int_C 2xyzdx + x^2zdy + x^2ydz$, where C is any regular curve connecting (1,1,1) to (1,2,4).

Solution We look for ϕ such that $\phi_x = 2xyz, \Rightarrow \phi = x^2yz + f(y,z)$, then $\phi_y = x^2z \Rightarrow f_y = 0 \Rightarrow f(y,z) = g(z)$, then $\phi_z = x^2y \Rightarrow g =$ const. So with $\phi(x,y,z) = x^2yz$ we have $(2xyz, x^2z, x^2y) = \nabla\phi$, and (using the result of question 79)

$$\int_C 2xyzdx + x^2zdy + x^2ydz = \int_C \nabla(x^2yz)\cdot d\mathbf{x} = \left[x^2yz\right]_{(1,1,1)}^{(1,2,4)} = 8 - 1 = 7\,.$$

Note, as we will see when discussing Stokes' theorem, a necessary condition for this tactic to work is the fact that $\nabla\times(2xyz, x^2z, x^2y) = (x^2{-}x^2)\,\mathbf{e_1} + (2xy{-}2xy)\,\mathbf{e_2} + (2xz{-}2xz)\,\mathbf{e_3} = \underline{0}$.

78. Compute the surface integral, $\int_S \mathbf{F}\cdot d\mathbf{A}$, of the vector field $\mathbf{F} = (3x^2, -2yx, 8)$ over the surface given by the plane $z = 2x - y$ with $0 \le x \le 2$, $0 \le y \le 2$,

(a) using method 2 from lectures,

(b) using method 1 from lectures.

Solution

(a) The surface as the (zero) level set of the function $f = z - 2x + y$, which we can use to find the vector normal to the surface:

$$\mathbf{N} = \nabla f = (-2, 1, 1)\,.$$

As in method 2 from lectures, since we have given the surface as a level set we can now write

$$\begin{aligned}\int_S \mathbf{F}\cdot d\mathbf{A} &= \int_0^2\int_0^2 \frac{\mathbf{F}\cdot\nabla f}{\mathbf{e_3}\cdot\nabla f} dx\,dy \\ &= \int_0^2\int_0^2 (-6x^2 - 2yx + 8)\,dx\,dy \\ &= \int_0^2 \left[-2x^3 - yx^2 + 8x\right]_0^2 dy \\ &= \int_0^2 (-4y)\,dy \;=\; -8\,.\end{aligned}$$

(b) Using method 1 from lectures, the surface can be parameterised in terms of x and y as points $\underline{x}(x,y) = (x, y, 2x - y)$. We therefore have

$$\begin{aligned}\frac{\partial\underline{x}}{\partial x} &= (1, 0, 2) \\ \frac{\partial\underline{x}}{\partial y} &= (0, 1, -1)\end{aligned}$$

and so

$$\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y} = (-2, 1, 1). \tag{1}$$

The required integral is then

$$\begin{aligned}
\int_S \mathbf{F} \cdot d\mathbf{A} &= \int_{x,y} \mathbf{F}(\mathbf{x}(x,y)) \cdot \left(\frac{\partial \mathbf{x}}{\partial x} \times \frac{\partial \mathbf{x}}{\partial y} \right) dx\, dy \\
&= \int_0^2 \int_0^2 (-6x^2 - 2yx + 8)\, dx\, dy \\
&= -8,
\end{aligned}$$

in agreement with the answer from method 2.

The question was a little remiss in not specifying that upward-pointing (ie positive z component) normals should be taken when evaluating the surface integral. With downward-pointing normals, the answer would have been $+8$, and this answer is also valid.

79. Let $\mathbf{F}(x, y, z) = (z, x, y)$, and S be the part of the surface of the sphere $x^2 + y^2 + (z-1)^2 = r^2$ above the plane $z = 0$. Assume that $r > 1$ so that the boundary C of S, where S intersects the plane $z = 0$, is non-empty.

(a) Compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{x}$.

(b) By parameterising the surface S with spherical coordinates centred on the point $(0, 0, 1)$, compute the surface integral of the curl of $\mathbf{F}$, $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{A}$.

Solution

(a) The curve to be integrated over is the intersection of the sphere and the plane $z = 0$ in the picture below, which shows the situation for $r = 2$:
On C we have $z = 0$ and also $x^2 + y^2 + (z-1)^2 = r^2$, so $x^2 + y^2 = r^2 - 1$, and C is a circle in the x,y plane of radius $\rho = \sqrt{r^2 - 1}$. Parametrise it as $\mathbf{x}\,(t) = (\rho \cos t, \rho \sin t, 0)$ with $0 \le t \le 2\pi$, so $\mathbf{F}(\mathbf{x}\,(t)) = (0, \rho \cos t, \rho \sin t)$, $\frac{d\mathbf{x}\,(t)}{dt} = (-\rho \sin t, \rho \cos t, 0)$, and

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{x} &= \int_0^{2\pi} (-\rho \sin t, \rho \cos t, 0) \cdot (\rho \cos t, \rho \sin t, 0)\, dt \\
&= \int_0^{2\pi} \rho^2 \cos^2 t\, dt = \pi \rho^2 = \pi(r^2 - 1)\,.
\end{aligned}$$

(b) For the surface integral, follow the suggestion and set

$$\mathbf{x}\,(\theta, \phi) = (x(\theta, \phi), y(\theta, \phi), z(\theta, \phi))$$

with (compare with example 43 in your notes)

$$\begin{aligned}
x(\theta, \phi) &= r \sin\theta\, \cos\phi\,, \\
y(\theta, \phi) &= r \sin\theta\, \sin\phi\,, \\
z(\theta, \phi) &= 1 + r \cos\theta\,.
\end{aligned}$$

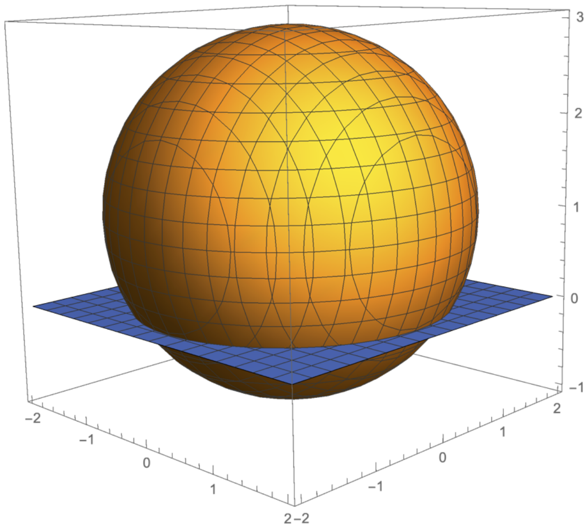


Figure 6: The intersection of the sphere $x^2 + y^2 + (z-1)^2 = r^2$ and the plane $z = 0$, shown for $r = 2$.

To figure out the parameter domain U, note that $z(\theta, \phi)$ must be positive and hence $0 \le \theta \le \cos^{-1}(-1/r)$, while $0 \le \phi \le 2\pi$. Furthermore $\nabla \times \mathbf{F} = (1, 1, 1)$, and

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial \theta} \times \frac{\partial \mathbf{x}}{\partial \phi} &= (r\cos\theta\cos\phi, r\cos\theta\sin\phi, -r\sin\theta) \times (-r\sin\theta\sin\phi, r\sin\theta\cos\phi, 0) \\ &= r^2(\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \sin\theta\cos\theta)\,.\end{aligned}$$

Hence

$$\begin{aligned}\int_S \nabla \times \mathbf{F} \cdot d\mathbf{A} &= \int_U \mathbf{F}(\mathbf{x}\,(u,v)) \cdot \left(\frac{\partial \mathbf{x}}{\partial \theta} \times \frac{\partial \mathbf{x}}{\partial \phi}\right) d\theta\, d\phi \\ &= r^2 \int_0^{\cos^{-1}(-1/r)} \int_0^{2\pi} \big((1,1,1)\cdot(\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \sin\theta\cos\theta)\big) d\phi\, d\theta \\ &= r^2 \int_0^{\cos^{-1}(-1/r)} \int_0^{2\pi} \big(\sin^2\theta\cos\phi + \sin^2\theta\sin\phi + \sin\theta\cos\theta\big) d\phi\, d\theta \\ &= r^2 \int_0^{\cos^{-1}(-1/r)} \big(2\pi\sin\theta\cos\theta\big)\, d\theta \quad \text{(the first two } \phi \text{ integrals vanishing)} \\ &= \pi r^2 \int_0^{\cos^{-1}(-1/r)} \sin 2\theta\, d\theta \\ &= \pi r^2 \Big[-\tfrac{1}{2}\cos 2\theta\Big]_0^{\cos^{-1}(-1/r)} \\ &= -\tfrac{1}{2}\pi r^2 \Big[2\cos^2\theta - 1\Big]_0^{\cos^{-1}(-1/r)} \\ &= -\tfrac{1}{2}\pi r^2 \Big(\frac{2}{r^2} - 2\Big) = \pi(r^2 - 1)\,.\end{aligned}$$

Remarks: Note that the answers to parts (a) and (b) are equal. We will see in Topic 8 that this result is predicted by Stokes' theorem. In this case it wouldn't have been appropriate to use method 2 for the surface integral (defining the surface implicitly and parametrising it by x and y) since for some parts of the surface there are two values of z for given values of x and y.

80. Let $\underline{A}(\underline{x})$ be the vector field $\underline{A}(x,y,z) = z\,\underline{e}_1 + x\,\underline{e}_2 + y\,\underline{e}_3$, C be the circle in the x,y-plane of radius r centred on the origin, and S the disk in the x,y-plane whose boundary is C.

    (a) Compute the line integral $\oint_C \underline{A}\cdot d\underline{x}$.

    (b) Compute the surface integral of the curl of $\underline{A}$, $\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{A}$.

Solution

(a) Parameterising C as $\underline{x}(t) = r\cos t\,\underline{e}_1 + r\sin t\,\underline{e}_2$, for $0\le t\le 2\pi$, we have $\underline{A}(\underline{x}(t)) = r\cos t\,\underline{e}_2 + r\sin t\,\underline{e}_3$ and $\frac{d\underline{x}(t)}{dt} = -r\sin t\,\underline{e}_1 + r\cos t\,\underline{e}_2$. Taking the dot product, $\underline{A}(\underline{x}(t))\cdot\frac{d\underline{x}(t)}{dt} = r^2\cos^2 t$ and so

$$\oint_C \underline{A}\cdot d\underline{x} = \int_0^{2\pi} r^2\cos^2 t\,dt = \frac{r^2}{2}\int_0^{2\pi}(\cos(2t)+1) = \frac{r^2}{2}\left[\frac{1}{2}\sin(2t)+t\right]_0^{2\pi} = \pi r^2.$$

(b) Parameterising the disk S as $\underline{x}(u,v) = (u\cos(v), u\sin(v), 0)$, for $0\le u\le r$, $0\le v\le 2\pi$, we have $\underline{x}_u = (\cos(v),\sin(v),0)$ and $\underline{x}_v = (-u\sin(v), u\cos(v), 0)$, and hence $\underline{x}_u\times\underline{x}_v = u\underline{e}_3$.
$\underline{\nabla}\times\underline{A} = (1,1,1)$, and so

$$\begin{aligned}\int_S(\nabla\times\mathbf{A})\cdot d\mathbf{A} &= \int_0^{2\pi}\int_0^r (1,1,1)\cdot(0,0,u)\,du\,dv\\ &= \int_0^{2\pi}\int_0^r u\,du\,dv\\ &= \pi r^2\end{aligned}$$

Remarks: Note that the answers to parts (a) and (b) are equal. We will see in Topic 8 that this result is predicted by Stokes' theorem.

81. Let $\underline{A}(\underline{x})$ be the vector field $\underline{A}(x,y,z) = z\,\underline{e}_1 + x\,\underline{e}_2 + y\,\underline{e}_3$, C be the a by a square $\underline{abcd}$ in the y,z-plane with vertices $\underline{a} = \underline{0}$, $\underline{b} = a\,\underline{e}_2$, $\underline{c} = a\,\underline{e}_2 + a\,\underline{e}_3$ and $\underline{d} = a\,\underline{e}_3$, and S be the region of the y,z-plane bounded by C.

    (a) Compute the line integral $\oint_C \underline{A}\cdot d\underline{x}$.

    (b) Compute the surface integral of the curl of $\underline{A}$, $\int_S(\nabla\times\mathbf{A})\cdot d\mathbf{A}$.

Solution

(a) Splitting C into four straight lines $C_a = \underline{ab}$, $C_b = \underline{bc}$, $C_c = \underline{cd}$ and $C_d = \underline{da}$, we have
$$\oint_{C_2} \underline{A} \cdot d\underline{x} = \int_{C_a} \underline{A} \cdot d\underline{x} + \int_{C_b} \underline{A} \cdot d\underline{x} + \int_{C_c} \underline{A} \cdot d\underline{x} + \int_{C_d} \underline{A} \cdot d\underline{x}$$
and the task is to evaluate the line integrals along these four straight lines. Taking them in turn:

- C_a runs from $\underline{0}$ to $a\,\underline{e}_2$, and can be parametrized as $\underline{x}(t) = t\,\underline{e}_2$ with t running from 0 to a. Then $\underline{A}(\underline{x}(t)) = t\,\underline{e}_3$ and $\frac{d\underline{x}(t)}{dt} = \underline{e}_2$ so $\underline{A}(\underline{x}(t)) \cdot \frac{d\underline{x}(t)}{dt} = 0$ and hence $\int_{C_a} \underline{A} \cdot d\underline{x} = 0$.
- C_b runs from $a\,\underline{e}_2$ to $a\,\underline{e}_2 + a\,\underline{e}_3$, and can be parametrized as $\underline{x}(t) = a\,\underline{e}_2 + t\,\underline{e}_3$ with t running from 0 to a. Then $\underline{A}(\underline{x}(t)) = t\,\underline{e}_1 + a\,\underline{e}_3$ and $\frac{d\underline{x}(t)}{dt} = \underline{e}_3$ so $\underline{A}(\underline{x}(t)) \cdot \frac{d\underline{x}(t)}{dt} = a$ and hence $\int_{C_b} \underline{A} \cdot d\underline{x} = \int_0^a a\,dt = [at]_0^a = a^2$.
- C_c runs from $a\,\underline{e}_2 + a\,\underline{e}_3$ to $a\,\underline{e}_3$, and can be parametrized as $\underline{x}(t) = t\,\underline{e}_2 + a\,\underline{e}_3$ with t running from a to 0. Then $\underline{A}(\underline{x}(t)) = a\,\underline{e}_1 + t\,\underline{e}_3$ and $\frac{d\underline{x}(t)}{dt} = \underline{e}_2$ so $\underline{A}(\underline{x}(t)) \cdot \frac{d\underline{x}(t)}{dt} = 0$ and hence $\int_{C_c} \underline{A} \cdot d\underline{x} = 0$.
- C_d runs from $a\,\underline{e}_3$ to $\underline{0}$, and can be parametrized as $\underline{x}(t) = t\,\underline{e}_3$ with t running from a to 0. Then $\underline{A}(\underline{x}(t)) = t\,\underline{e}_1$ and $\frac{d\underline{x}(t)}{dt} = \underline{e}_3$ so $\underline{A}(\underline{x}(t)) \cdot \frac{d\underline{x}(t)}{dt} = 0$ and hence $\int_{C_d} \underline{A} \cdot d\underline{x} = 0$.

Adding the four bits together gives the final result: $\oint_{C_2} \underline{A} \cdot d\underline{x} = a^2$.

(b) Using method 2, the surface S can be given as part of the level set $f(x, y, z) = x = 0$ for $0 \le y \le a\, 0 \le z \le a$. The surface S therefore has (right-pointing) normal given by $\underline{\nabla} f = \underline{e}_1$. Since we therefore have $\partial_x f \neq 0$ over the surface S, we can compute this integral by 'projecting' onto the y, z-plane using method 2.

We also have $\underline{\nabla} \times \underline{A} = (1, 1, 1)$, and hence
$$\begin{aligned}\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{A} &= \int_0^a \int_0^a \frac{(\nabla \times \underline{A}) \cdot \underline{\nabla} f}{\underline{e}_1 \cdot \underline{\nabla} f}\, dy\, dz \\ &= \int_0^a \int_0^a 1\, dy\, dz \\ &= a^2\end{aligned}$$

Remarks: Note that the answers to parts (a) and (b) are equal. We will see in Topic 8 that this result is predicted by Stokes' theorem.

82. Based on exam question May 2015 (Section A) Q5 (which didn't contain part (b)):

(a) Calculate $\int_V (\underline{\nabla} \cdot \underline{U})\, dV$ where V is the solid cube with faces $x = \pm 1$, $y = \pm 1$ and $z = \pm 1$ and
$$\underline{U}(x, y, z) = (\, x\, y^2, y\, x^2, z\,)\,.$$

(b) Calculate $\int_S \underline{U} \cdot d\underline{A}$, where S is the surface of V.

Solution

(a) We have $\nabla \cdot \mathbf{U} = y^2 + x^2 + 1$ so (changing the orders of integrations as convenient)

$$\begin{aligned}\int_V (\underline{\nabla} \cdot \underline{U})\, dV &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (y^2 + x^2 + 1)\, dxdydz \\ &= 4\int_{-1}^{1} y^2 dy + 4\int_{-1}^{1} x^2 dx + 8 \\ &= 4\left[\tfrac{1}{3}y^3\right]_{-1}^{1} + 4\left[\tfrac{1}{3}x^3\right]_{-1}^{1} + 8 \\ &= \tfrac{16}{3} + 8 = \tfrac{40}{3}\,.\end{aligned}$$

(b) The surface integral is the sum of six terms, one for each face of the cube:
• On the faces $x = \pm 1$, $\widehat{\mathbf{n}} = \pm \mathbf{e_1}$ and $\mathbf{U} = (\pm y^2, y, z)$, so $\mathbf{U} \cdot \widehat{\mathbf{n}} = y^2$ and each contributes $I_1 = \int_{-1}^{1}\int_{-1}^{1} y^2 dydz = \frac{4}{3}$.
• On the faces $y = \pm 1$, $\widehat{\mathbf{n}} = \pm \mathbf{e_2}$ and $\mathbf{U} = (x, \pm x^2, z)$, so $\mathbf{U} \cdot \widehat{\mathbf{n}} = x^2$ and each contributes $I_2 = \int_{-1}^{1}\int_{-1}^{1} x^2 dxdz = \frac{4}{3}$.
• On the faces $z = \pm 1$, $\widehat{\mathbf{n}} = \pm \mathbf{e_3}$ and $\mathbf{U} = (xy^2, yx^2, \pm 1)$, so $\mathbf{U} \cdot \widehat{\mathbf{n}} = 1$ and each contributes $I_3 = \int_{-1}^{1}\int_{-1}^{1} dxdy = 4$.
The total is $2I_1 + 2I_2 + 2I_3 = \frac{8}{3} + \frac{8}{3} + 8 = \frac{40}{3}$ as before.
Aside: the calculation is quite close to the one in part (b) – this is related to how the general proof of the divergence theorem works.

83. For a simple closed curve C in the $(x, y)-$plane, show by Green's theorem that the area enclosed is $A = \frac{1}{2}\oint_C (xdy - ydx)$. (Note, 'simple' means that C does not cross itself, which means that it encloses a well-defined area A.)

Solution Area is given by $A = \int_S dA$, where S is the surface enclosed by the simple curve C in the (x, y) plane. Green's theorem states:

$$\oint_C Pdx + Qdy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

In this case, the RHS of the formula given in the question is matched if we take $P = -\frac{1}{2}y$ and $Q = \frac{1}{2}x$; it is then easy to check that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ and so the result is indeed equal to the area. (Note, many other choices of P and Q would also give the area.)

84. Evaluate $\oint \mathbf{F} \cdot \mathbf{dx}$ around the circle $x^2 + y^2 + 2x = 0$, where $\mathbf{F} = y\mathbf{e_1} - x\mathbf{e_2}$, both directly and by using Green's theorem in the plane.

Solution

$$\oint \mathbf{F} \cdot \mathbf{dx} = \oint (F_1\, dx + F_2\, dy) = \oint (y\, dx - x\, dy)$$

Using Green's theorem in the plane this is

$$\iint_A \left(-\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y}\right) dx\, dy = -2\iint_A dx\, dy = -2A$$

where A is the area inside the circle $x^2+y^2+2x=0$, or $(x+1)^2+y^2=1$. Since this has radius 1, the area is π and so $\oint \mathbf{F}\cdot\mathbf{dx} = -2\pi$.

Alternatively, the circle can be parametrised as $\mathbf{x}(t)=(\cos(t)-1)\,\mathbf{e_1}+\sin(t)\,\mathbf{e_2}$. Then $\frac{d\mathbf{x}(t)}{dt}=-\sin(t)\,\mathbf{e_1}+\cos(t)\,\mathbf{e_2}$ so

$$\oint \mathbf{F}\cdot\mathbf{dx} = \int_0^{2\pi}(\sin(t)\,\mathbf{e_1}-(\cos(t)-1)\,\mathbf{e_2})\cdot(-\sin(t)\,\mathbf{e_1}+\cos(t)\,\mathbf{e_2})\,dt$$
$$= \int_0^{2\pi}\big(-\sin^2(t)-\cos^2(t)+\cos(t)\big)\,dt = -2\pi\,.$$

85. Evaluate $\oint_C 2xdy-3ydx$ around the square with vertices at $(x,y)=(0,2)$, $(2,0)$, $(-2,0)$ and $(0,-2)$.

Solution

First, using Green's theorem in the plane:

$$\oint_C 2xdy-3ydx = \iint_A\left(\frac{\partial(2x)}{\partial x}+\frac{\partial(3y)}{\partial y}\right)dx\,dy = 5\iint_A dx\,dy = 5A$$

where A is the area of the square. Now $A=8$ so the integral is equal to 40.

We could equivalently embed the whole problem in three dimensions and use Stokes' theorem. The required integral can be written as

$$\oint_C \mathbf{F}\cdot d\mathbf{x}$$

where $\mathbf{F}=-3y\mathbf{e_1}+2x\mathbf{e_2}$ and C is the given contour, thought of as lying in the $z=0$ plane. By Stokes, and noting that the unit normal to the planar surface spanning C is $\mathbf{e_3}$, this is equal to $\iint_A(\underline{\nabla}\times\mathbf{F})\cdot\mathbf{e_3}\,dA$, and since $(\underline{\nabla}\times\mathbf{F})_3=5$ this reduces to the previous calculation.

Finally, it is possible (and a good exercise) to do the contour integral directly. The contour can be split into four straight lines as shown in Figure 7.

On **A**, write points as $(t,2-t)$ with t running from 2 to 0; then $dx=dt$, $dy=-dt$, and

$$\int_{\mathbf{A}}-3y\,dx+2x\,dy=\int_2^0(-3(2-t)-2t)\,dt=\left[-6t+\tfrac{1}{2}t^2\right]_2^0=12-2=10\,.$$

On **B**, write points as $(t,2+t)$ with t running from 0 to -2; then $dx=dy=dt$ and

$$\int_C-3y\,dx+2x\,dy=\int_0^{-2}(-3(2+t)+2t)\,dt=\left[-6t-\tfrac{1}{2}t^2\right]_0^{-2}=12-2=10\,.$$

On **C**, write points as $(t,-2-t)$ with t running from -2 to 0; then $dx=dt$, $dy=-dt$, and

$$\int_C-3y\,dx+2x\,dy=\int_{-2}^0(-3(-2-t)-2t)\,dt=\left[6t+\tfrac{1}{2}t^2\right]_{-2}^0=12-2=10\,.$$

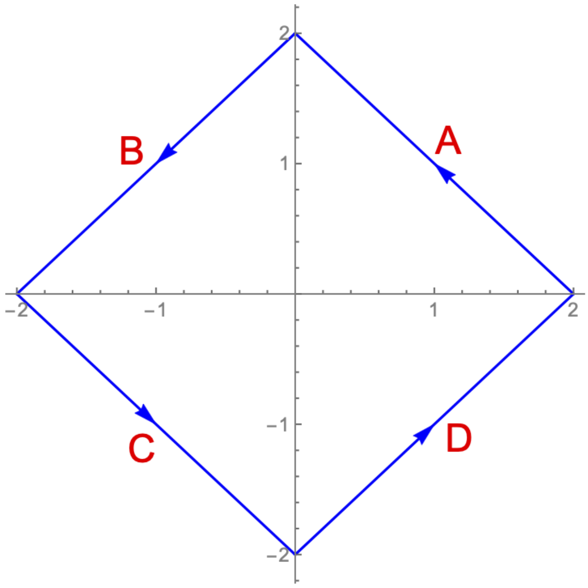


Figure 7: A square contour in the plane.

On **D**, write points as $(t, -2+t)$ with t running from 0 to 2; then $dx = dy = dt$ and

$$\int_C -3y\,dx + 2x\,dy = \int_0^2 (-3(-2+t) + 2t)\,dt = \left[6t - \tfrac{1}{2}t^2\right]_0^2 = 12 - 2 = 10\,.$$

Adding them all up gives 40, as before. Other parametrisations could have been used; for the particular ones taken here, you could also have written the parameter as x instead of t, but be extra careful with the signs! (And *check* that you agree with the calculation given here…)

86. Let $\underline{v}$ be the radial vector field $\underline{v}(\underline{x}) = \underline{x}$.
    (a) Compute $l_1 = \int_{C_1} \underline{v} \cdot d\underline{x}$, where C_1 is the straight-line contour from the origin to the point $(2, 0, 0)$.
    (b) Compute $l_2 = \int_{C_2} \underline{v} \cdot d\underline{x}$, where C_2 is the semi-circular contour from the origin to the point $(2, 0, 0)$ defined by $0 \le x \le 2$, $y = +\sqrt{1-(x-1)^2}$, $z = 0$. [It may help to start by sketching C_2 , and then to parameterize it as $\underline{x}(t) = (1 - \cos t, \sin t, 0)$ with $0 \le t \le \pi$.]
    (c) You should have found that $l_1 = l_2$. Explain this result using Stokes' theorem.

Solution

(a) Parameterizing C_1 by the distance x from the origin, at the point $\underline{x}(x) = x\,\underline{e}_1$ on C_1 we have $\underline{v} = x\,\underline{e}_x$ and $d\underline{x} = \underline{e}_1\,dx$, and so

$$l_1 = \int_0^2 x\,dx = 2\,.$$

(b) Parameterizing C_2 by the angle t, $\underline{v}(\underline{x}(t)) = (1 - \cos t, \sin t, 0)$ while $d\underline{x}(t) = (\sin t, \cos t, 0)\, dt$, so $\underline{v} \cdot d\underline{x} = \sin t\, dt$ and

$$l_2 = \int_0^{\pi} \sin t\, dt = 2\,.$$

(c) As predicted in the question, $l_1 = l_2$. To explain this using Stokes' theorem, note first that if $\underline{v}(\underline{x}) = \underline{x}$ then $\underline{\nabla} \times \underline{v} = \underline{0}$ (see question 15(b)). Now let C_3 be the closed contour which runs from the origin to $(2, 0, 0)$ along C_1, then returns to the origin by taking C_2 in the reverse direction. Then $\int_{C_3} \underline{v} \cdot d\underline{x} = l_1 - l_2$. But by Stokes' theorem, $\int_{C_3} \underline{v} \cdot d\underline{x} = \int_{S_3} (\underline{\nabla} \times \underline{v}) \cdot d\underline{A} = 0$, where S_3 is a surface spanning C_3. Hence $l_1 - l_2 = 0$, or $l_1 = l_2$, as required.

87. Integrate $\operatorname{curl} \mathbf{F}$, where $\mathbf{F} = 3y\mathbf{e_1} - xz\mathbf{e_2} + yz^2\mathbf{e_3}$, over the portion S of the surface $2z = x^2 + y^2$ below the plane $z = 2$, both directly and by using Stokes' theorem. Take the area elements of S to point outwards, so that their z components are negative.

Solution The geometrical setup is illustrated in Figure 8.

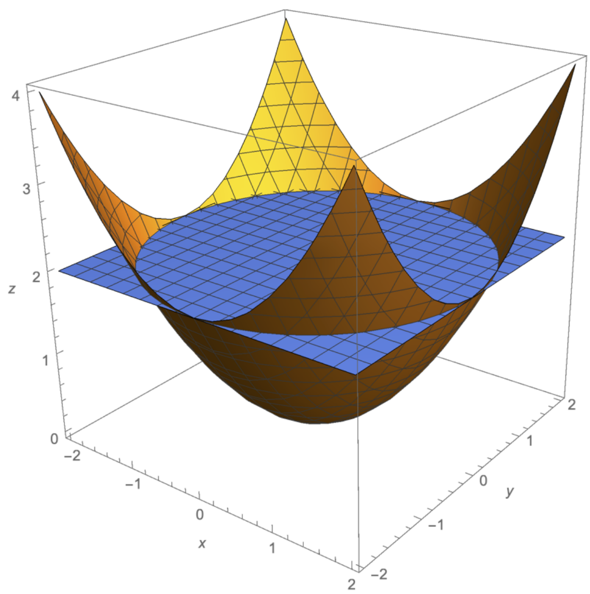


Figure 8: The intersection of the parabaloid $2z = x^2 + y^2$ and the plane $z = 2$

First proceed directly. Calculate $\underline{\nabla} \times \mathbf{F} = (z^2 + x)\mathbf{e_1} - (3 + z)\mathbf{e_3}\,.$ Now consider the surface of the paraboloid $2z = x^2 + y^2$, The surface is the zero level set of $f = x^2 + y^2 - 2z$, so a normal vector to it is $\underline{\nabla} f = 2x\mathbf{e_1} + 2y\mathbf{e_2} - 2\mathbf{e_3}$, and

$$d\mathbf{A} = -\frac{\underline{\nabla} f}{\mathbf{e_3} \cdot \underline{\nabla} f}\, dxdy = (x\mathbf{e_1} + y\mathbf{e_2} - \mathbf{e_3})\, dxdy.$$

Note the minus sign – this comes from the choice to take the 'outward' normal to be pointing *out* of the paraboloid, which (as should be clear from the plot) means its z component is negative, leading to the minus sign in the formula for $d\mathbf{A}$.

Hence

$$\begin{aligned}\int_S \underline{\nabla} \times \mathbf{F} \cdot d\mathbf{A} &= \iint_D \big((z^2+x)\mathbf{e_1} - (3+z)\mathbf{e_3}\big) \cdot \big(x\mathbf{e_1} + y\mathbf{e_2} - \mathbf{e_3}\big)\, dxdy \\ &= \iint_D \big((z^2+x)x + 3 + z\big)\, dxdy \\ &= \iint_D \big((\tfrac{1}{4}(x^2+y^2)^2 + x)x + 3 + \tfrac{1}{2}(x^2+y^2)\big)\, dxdy\end{aligned}$$

where D is the projection of S onto the x,y plane (a circle of radius 2) and the fact that $z = \frac{1}{2}(x^2+y^2)$ was used in the last line to write the integral in terms of x and y alone.

To calculate this integral, change to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Then $dxdy = rdrd\theta$, and

$$\int_S (\underline{\nabla} \times \mathbf{F}) \cdot d\mathbf{A} = \int_0^{2\pi} d\theta \int_0^2 rdr \left(\tfrac{1}{4}r^5\cos\theta + r^2\cos^2\theta + 3 + \tfrac{1}{2}r^2\right)$$

giving $\int_S (\underline{\nabla} \times \mathbf{F}) \cdot d\mathbf{A} = \int_0^{2\pi} \left(\frac{2^5}{7}\cos\theta + 4\cos^2\theta + 8\right) d\theta = 20\pi$.

Alternatively, we can use Stokes' theorem to say that $\int_S (\underline{\nabla} \times \mathbf{F}) \cdot d\mathbf{A} = \oint_C \mathbf{F} \cdot d\mathbf{x}$ where C is the boundary of S, a circle of radius 2 described by

$$x^2 + y^2 = 4\,, \quad z = 2\,.$$

Note that since our normal is pointing *down* (i.e. in the negative-z direction), C should be traversed *clockwise* in the x,y plane. To get this right, write C in parametric form as $x = 2\cos t$, $y = -2\sin t$, $z = 2$ with t running from 0 to 2π. Then

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{x} &= \int_0^{2\pi} \big(-6\sin t\,\mathbf{e_1} - 4\cos t\,\mathbf{e_2} - 8\sin t\,\mathbf{e_3}\big) \cdot \big(-2\sin t\,\mathbf{e_1} - 2\cos t\,\mathbf{e_2}\big)\, dt \\ &= \int_0^{2\pi} \big(12\sin^2 t + 8\cos^2 t\big)\, dt \;=\; 20\pi,\end{aligned}$$

as before.

Note: It is also possible to integrate the curl of $\mathbf{F}$ over the flat disk $S' = \{x^2 + y^2 \leq 4, z = 2\}$ which also has C as its boundary. For this disk, picking the normal to again point downwards, $d\mathbf{A} = -\mathbf{e_3}\, dxdy$ everywhere, and so

$$\begin{aligned}\int_{S'} \underline{\nabla} \times \mathbf{F} \cdot d\mathbf{A} &= \iint_{x^2+y^2\leq 2} \big((4+x)\mathbf{e_1} - 5\mathbf{e_3}\big) \cdot \big(-\mathbf{e_3}\, dxdy\big) \\ &= \iint_{x^2+y^2\leq 2} 5\, dxdy \;=\; 20\pi\,.\end{aligned}$$

Observe that this is the same answer as for the surface integral over S. This has to be the case since both are equal to the the line integral round C, but can you see a direct argument for it? (Hint: consider using the divergence theorem.)

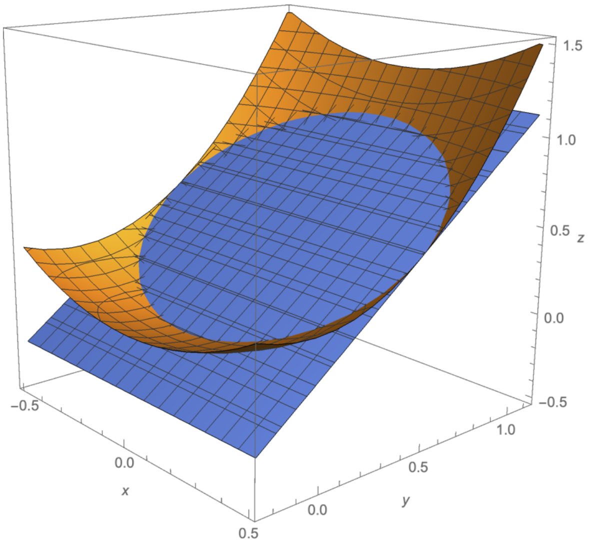


Figure 9: The intersection of the parabaloid $z = x^2 + y^2$ and the plane $z = y$.

88. The paraboloid of equation $z = x^2+y^2$ intersects the plane $z = y$ in a curve C. Calculate $\oint_C \mathbf{v} \cdot \mathbf{dx}$ for $\mathbf{v} = 2z\mathbf{e_1} + x\mathbf{e_2} + y\mathbf{e_3}$ using Stokes' theorem. Check your answer by evaluating the line integral directly.

Solution First, a picture of the situation is shown below in Figure 9.

On C, we have $z = x^2 + y^2$ and $z = y$. Hence $x^2 + y^2 = y$, or $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$. Stokes' theorem says that

$$\oint_C \mathbf{v} \cdot d\mathbf{x} = \int_S (\underline{\nabla} \times \mathbf{v}) \cdot d\mathbf{A}$$

where S is any surface bounded by C, and $\underline{\nabla} \times \mathbf{v} =$ (calculate) $= \mathbf{e_1} + 2\mathbf{e_2} + \mathbf{e_3} = (1, 2, 1)$. There are then two ways to proceed.

1) (Harder) Take S to be the portion S of the paraboloid $z = x^2 + y^2$ that lies inside C. Using method 1 from lectures, we can parametrise the paraboloid by x and y, setting $\mathbf{x}\,(x, y) = x\,\mathbf{e_1} + y\,\mathbf{e_2} + (x^2 + y^2)\,\mathbf{e_3}$, so $\frac{\partial \mathbf{x}}{\partial x} = \mathbf{e_1} + 2x\,\mathbf{e_3}$ and $\frac{\partial \mathbf{x}}{\partial y} = \mathbf{e_2} + 2y\,\mathbf{e_3}$. Hence

$$d\mathbf{A} = (\mathbf{e_1} + 2x\,\mathbf{e_3}) \times (\mathbf{e_2} + 2y\,\mathbf{e_3})\,dxdy = (-2x\,\mathbf{e_1} - 2y\,\mathbf{e_2} + \mathbf{e_3})\,dxdy\,.$$

Equivalently, we can use method 2 and define the paraboloid implicitly as the level set $f(x, y, z) = 0$ where $f(x, y, z) = z - x^2 - y^2$, so $\underline{\nabla} f = -2x\,\mathbf{e_1} - 2y\,\mathbf{e_2} + \mathbf{e_3}$ and

$$d\mathbf{A} = \frac{\underline{\nabla} f}{\mathbf{e_3} \cdot \underline{\nabla} f}\,dxdy = (-2x\,\mathbf{e_1} - 2y\,\mathbf{e_2} + \mathbf{e_3})\,dxdy$$

which is the same as the formula found using method 1.

Either way, we have

$$(\underline{\nabla} \times \mathbf{v}) \cdot d\mathbf{A} = (\mathbf{e_1} + 2\mathbf{e_2} + \mathbf{e_3}) \cdot (-2x\,\mathbf{e_1} - 2y\,\mathbf{e_2} + \mathbf{e_3})\,dxdy = (-2x - 4y + 1)\,dxdy$$

and this must be integrated over D, the projection of S onto the x,y plane. Setting $Y = y - \frac{1}{2}$, $4y = 4Y + 2$ and the integral is

$$\iint_{x^2+Y^2 \le \frac{1}{4}} (-1 - 2x - 4Y)\, dxdY = -\frac{\pi}{4}\,.$$

(Note, the integrals of $-2x$ and $-4Y$ vanish by symmetry.)

2) (Easier) Take S to be the part of the plane $z = y$ encircled by C, a (tilted) flat disk. On this disk the unit normal is $\widehat{\mathbf{n}} = \frac{1}{\sqrt{2}}(-\mathbf{e_2} + \mathbf{e_3})$ and the area elements are $d\mathbf{A} = \widehat{\mathbf{n}}\, dx\, \sqrt{2}dy = (-\mathbf{e_2} + \mathbf{e_3})\, dxdy = (0, -1, 1)dxdy$. Then the area integral is

$$\iint_D (1, 2, 1) \cdot (0, -1, 1)\, dxdy = -\iint_D dxdy = -\frac{\pi}{4}$$

since the area of D, a disk of radius $\frac{1}{2}$, is $\frac{\pi}{4}$.

Note, the methods chosen here to evaluate the surface integrals are not compulsory. For example, the level set method could also be used for the second case.

As a check, we can do the line integral directly. Given the characterisation of C found above, it can be parametrised as $\mathbf{x}\,(t) = \frac{1}{2}\cos t\, \mathbf{e_1} + \frac{1}{2}(1 + \sin t)\, \mathbf{e_2} + \frac{1}{2}(1 + \sin t)\, \mathbf{e_3}$ with $0 \le t \le 2\pi$. Then $\frac{d\mathbf{x}}{dt} = -\frac{1}{2}\sin t\, \mathbf{e_1} + \frac{1}{2}\cos t\, \mathbf{e_2} + \frac{1}{2}\cos t\, \mathbf{e_3}$ while $\mathbf{v}(\mathbf{x}\,(t)) = (1 + \sin t)\, \mathbf{e_1} + \frac{1}{2}\cos t\, \mathbf{e_2} + \frac{1}{2}(1 + \sin t)\, \mathbf{e_3}$, and so

$$\begin{aligned}
&\oint_C \mathbf{v} \cdot d\mathbf{x} \\
&= \tfrac{1}{4}\int_0^{2\pi} \big((2{+}2\sin t)\, \mathbf{e_1} + \cos t\, \mathbf{e_2} + (1{+}\sin t)\, \mathbf{e_3}\big) \cdot \big(-\sin t\, \mathbf{e_1} + \cos t\, \mathbf{e_2} + \cos t\, \mathbf{e_3}\big) dt \\
&= \tfrac{1}{4}\int_0^{2\pi} \big(-2\sin t - 2\sin^2 t + \cos^2 t + \cos t + \cos t \sin t\big) dt \;=\; -\frac{\pi}{4}
\end{aligned}$$

as before.

89. For an open surface S with boundary C, show that $2\int_S \mathbf{u} \cdot d\mathbf{A} = \oint_C (\mathbf{u} \times \mathbf{x}\,) \cdot d\mathbf{x}$, where $\mathbf{u}$ is a fixed vector.

Solution By Stokes' theorem, the RHS is equal to $\int_S \underline{\nabla} \times (\mathbf{u} \times \mathbf{x}\,) \cdot d\mathbf{A}$. In addition,

$$(\underline{\nabla} \times (\mathbf{u} \times \mathbf{x}\,))_i = \varepsilon_{ijk}\partial_j \varepsilon_{klm} u_l x_m = \varepsilon_{ijk}\varepsilon_{klm} u_l \delta_{jm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) u_l \delta_{jm} = 2u_i$$

so $\underline{\nabla} \times (\mathbf{u} \times \mathbf{x}\,) = 2\mathbf{u}$, and, integrating, the LHS of the given formula is matched.

90. Verify Stokes' theorem for the upper hemispherical surface S: $z = \sqrt{1 - x^2 - y^2}$, $z \ge 0$, with $\mathbf{F}$ equal to the radial vector field, i.e. $\mathbf{F}(x, y, z) = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$.

Solution The surface is bounded by the circle C: $x^2 + y^2 = 1$, $z = 0$. Parametrising this by $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = 0$, $\frac{d\mathbf{x}}{dt} = -\sin t\, \mathbf{e_1} + \cos t\, \mathbf{e_2}$ and so
(a) $\oint_C \mathbf{F} \cdot d\mathbf{x} = \int_0^{2\pi} (\cos t \times (-\sin t) + \sin t \times \cos t)\, dt = 0$;
(b) $(\underline{\nabla} \times \mathbf{F})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j} x_k = \varepsilon_{ijk}\delta_{jk} = 0$.
Hence $\oint_C \mathbf{F} \cdot d\mathbf{x} = 0 = \int_S (\underline{\nabla} \times \mathbf{F}) \cdot d\mathbf{A}$, as required.

91. Let $\mathbf{F} = ye^z\,\mathbf{e_1} + xe^z\,\mathbf{e_2} + xye^z\,\mathbf{e_3}$. Show that the integral of $\mathbf{F}$ around a regular closed curve C that is the boundary of a surface S is zero.

Solution This one is easy! Simply compute $\underline{\nabla} \times \mathbf{F} = \mathbf{0}$ (where some details of this calculation should be given) and then the result follows by Stokes' theorem.

92. By applying Stokes' theorem to the vector field $\mathbf{G} = |\mathbf{x}\,|^2\mathbf{a}$ (with $\mathbf{a}$ a constant vector), or otherwise, show that $\int_S \mathbf{x} \times d\mathbf{A} = -\frac{1}{2}\oint_C |\mathbf{x}\,|^2\,d\mathbf{x}$, where S is the area bounded by the closed curve C.

Solution Note first that $(\underline{\nabla} \times \mathbf{G})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(x_l x_l a_k) = \varepsilon_{ijk}2x_l\delta_{jl}a_k = 2\varepsilon_{ijk}x_j a_k$. Using Stokes' theorem for $\mathbf{G}$, $\oint_C \mathbf{G}\cdot d\mathbf{x} = \int_S \underline{\nabla} \times \mathbf{G}\cdot d\mathbf{A}$, i.e.

$$\oint_C |\mathbf{x}\,|^2 a_k dx_k = \int_S 2\varepsilon_{ijk}x_j a_k dA_i \quad \text{or} \quad a_k \oint_C |\mathbf{x}\,|^2 dx_k = -a_k \int_S 2\varepsilon_{kji}x_j dA_i$$

(note the minus sign from the swap of indices on the second ε). Since this is true for all (constant) a_k, we deduce

$$\oint_C |\mathbf{x}\,|^2 dx_k = -\int_S 2\varepsilon_{kji}x_j dA_i$$

from which the result follows immediately.
Aside: Here we are dealing with the line integral of a scalar, which strictly speaking didn't get defined yet. For curve C parametrised by $\mathbf{x}\,(t)$, $t_0 \le t \le t_1$, and a scalar field $f(\mathbf{x}\,)$, it is simply the vector-valued integral

$$\int_C f(\mathbf{x}\,)d\mathbf{x} = \int_{t_0}^{t_1} f(\mathbf{x}\,(t))\frac{d\mathbf{x}\,(t)}{dt}\,dt\,.$$

(But beware, some authors use the same words for the scalar-valued quantity $\int_C f(\mathbf{x}\,)\,ds = \int_{t_0}^{t_1} f(\mathbf{x}\,(t))\,|\frac{d\mathbf{x}\,(t)}{dt}|\,dt$.)

93. Exam question June 2002 (Section B): Evaluate the line integral $I = \int_C \mathbf{F}\cdot d\mathbf{x}$ where $\mathbf{F}(\mathbf{x}\,) = 2y\mathbf{e_1} + z\mathbf{e_2} + 3y\mathbf{e_3}$ and the path C is the intersection of the surface of equation $x^2+y^2+z^2 = 4z$ and the surface of equation $z = x+2$, taken in a clockwise direction to an observer at the origin. A picture of the path is required, as well as full justifications of the theoretical results you might use.

Solution The first surface is $x^2+y^2+(z-2)^2 = 4$, i.e. a sphere of radius 2 centred at $(0,0,2)$. The x and y coordinates of its intersection C with the plane $z = x+2$ therefore satisfy $2x^2+y^2 = 4$. Note, this is an ellipse with semi-major and semi-minor axes 2 and $\sqrt{2}$, as it is the projection of the circular curve C, lying in 3 dimensions, onto the x,y plane. A cutaway picture of the intersection is shown below in Figure 10

To calculate the line integral directly, we can parametrise C by $x(t) = \sqrt{2}\cos t$, $y(t) = 2\sin t$, $z(t) = 2+\sqrt{2}\cos t$, so $\frac{d\mathbf{x}}{dt} = -\sqrt{2}\sin t\,\mathbf{e_1} + 2\cos t\,\mathbf{e_2} - \sqrt{2}\sin t\,\mathbf{e_3}$. (Note,

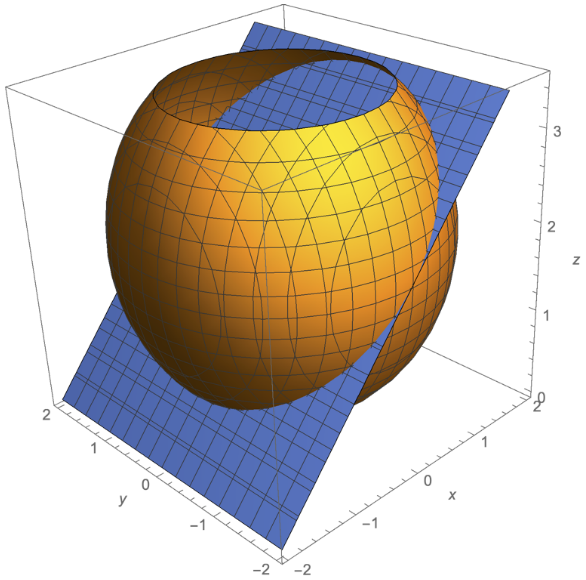


Figure 10: The in tersection of the sphere $x^2 + y^2 + (z-2)^2 = 4$ with the plane $z = x + 2$.

clockwise round C when viewed from the origin is the same as anticlockwise when viewed from above, so this parametrisation goes round C in the correct direction.) Thus

$$\begin{aligned}
&\oint_C \mathbf{F} \cdot d\mathbf{x} \\
&= \int_0^{2\pi} \mathbf{F}(\mathbf{x}(t)) \cdot \frac{d\mathbf{x}(t)}{dt}\, dt \\
&= \int_0^{2\pi} (4 \sin t\, \mathbf{e_1} + (2+\sqrt{2}\cos t)\, \mathbf{e_2} + 6 \sin t\, \mathbf{e_3}) \cdot (-\sqrt{2} \sin t\, \mathbf{e_1} + 2 \cos t\, \mathbf{e_2} - \sqrt{2} \sin t\, \mathbf{e_3})\, dt \\
&= \int_0^{2\pi} (-4\sqrt{2} \sin^2 t + 4 \cos t + 2\sqrt{2} \cos^2 t - 6\sqrt{2} \sin^2 t)\, dt \\
&= -4\sqrt{2}\,\pi + 2\sqrt{2}\,\pi - 6\sqrt{2}\,\pi = -8\sqrt{2}\,\pi\,,
\end{aligned}$$

using $\int_0^{2\pi} \sin^2 t\, dt = \int_0^{2\pi} \cos^2 t\, dt = \pi$ and $\int_0^{2\pi} \cos t\, dt = 0$ to get to the last line.

Alternatively, we can use Stokes. The simplest surface spanning C is the disk in the plane $z = x+2$, which we'll call D. This is a level set of $f = z - x$, so $\underline{\nabla} f = (-1, 0, 1)$ while $\underline{\nabla} \times \mathbf{F} = (2, 0, -2)$. Putting the pieces together, and letting E be the elliptical projection of D into the x,y plane,

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{x} &= \int_D \underline{\nabla} \times \mathbf{F} \cdot d\mathbf{A} \\
&= \iint_E \frac{(\underline{\nabla} \times \mathbf{F}) \cdot \underline{\nabla} f}{\mathbf{e_3} \cdot \underline{\nabla} f}\, dxdy \\
&= \iint_E (-4)\, dxdy \;=\; -4 \times (\text{area of } E) \;=\; -8\sqrt{2}\,\pi\,,
\end{aligned}$$

since the area of an ellipse with semi-major and semi-minor axes a and b is equal to πab.

94. Exam question 2012 (Section B) Q9(a)(ii): Use Stokes' theorem to calculate the line integral $\oint_C ydx+zdy+xdz$, where C is the intersection of the surfaces $x^2+y^2+z^2=a^2$ and $x+y+z=0$ and is orientated anticlockwise when viewed from above.
Suggestion: Instead of one of the two standard methods for surface integrals, just use $d\mathbf{A}=\frac{\nabla f}{|\nabla f|}dA$ and think about how the surfaces intersect.

Solution By Stokes, with $\mathbf{F}=(y,z,x)$ and $\underline{\nabla}\times\mathbf{F}=(-1,-1,-1)$,

$$\oint_C ydx+zdy+xdz=\int_S \underline{\nabla}\times\mathbf{F}\cdot d\mathbf{A}$$

Pick S to lie in the plane $f=x+y+z=0$; then $\underline{\nabla} f/|\underline{\nabla} f|=(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3})$ and hence

$$\int_S \underline{\nabla}\times\mathbf{F}\cdot d\mathbf{A}=\int_S(-\sqrt{3})\,dA=-\sqrt{3}\,\pi a^2$$

since the plane cuts through the centre of the sphere, so C is a great circle with radius a, enclosing a planar area πa^2.

95. Consider the vector field $\mathbf{F}=y\,\mathbf{e_1}+(z-x)\,\mathbf{e_2}+(x^3+y)\,\mathbf{e_3}$. Evaluate $\int_S(\underline{\nabla}\times\mathbf{F})\cdot\mathbf{dA}$ explicitly, where

(i) S is the disk $x^2+y^2\leq 4$, $z=1$,

(ii) S is the surface of a paraboloid $x^2+y^2=5-z$ above the plane $z=1$.

Verify that each of these agrees with Stokes' theorem by considering the integral of $\mathbf{F}$ around each bounding contour.

Solution Calculating, $\underline{\nabla}\times\mathbf{F}=-3x^2\,\mathbf{e_2}-2\,\mathbf{e_3}$. Then

(i) On the given disk, $d\mathbf{A}=\mathbf{e_3}\,dxdy$ so

$$\int_S \underline{\nabla}\times\mathbf{F}\cdot d\mathbf{A}=\iint_{x^2+y^2\leq 4}(-2)\,dxdy=-8\pi$$

as the disk has radius 2 and hence area 4π.
Now consider the line integral round the boundary C of the disk, parametrising it as $\mathbf{x}\,(t)=2\cos t\,\mathbf{e_1}+2\sin t\,\mathbf{e_2}+\mathbf{e_3}$, $0\leq t\leq 2\pi$. Then $\frac{d\mathbf{x}}{dt}=-2\sin t\,\mathbf{e_1}+2\cos t\,\mathbf{e_2}$, and

$$\oint_C \mathbf{F}\cdot d\mathbf{x}=\int_0^{2\pi}\left(-4\sin^2 t+2(1-2\cos t)\cos t\right)dt=-8\pi\,.$$

(ii) On the paraboloid, $f\equiv x^2+y^2+z=5$, so using method 2 and describing the surface S as a level set of f with a projection A onto the x,y plane, and computing

$\underline{\nabla} f = 2x\,\mathbf{e_1} + 2y\,\mathbf{e_2} + \mathbf{e_3}$,

$$\begin{aligned}\int_S \underline{\nabla} \times \mathbf{F} \cdot d\mathbf{A} &= \iint_A \frac{(\underline{\nabla} \times \mathbf{F}) \cdot \underline{\nabla} f}{\mathbf{e_3} \cdot \underline{\nabla} f}\,dxdy \\ &= \iint_A \big(-3x^2\,\mathbf{e_2} - 2\,\mathbf{e_3}\big) \cdot \big(2x\,\mathbf{e_1} + 2y\,\mathbf{e_2} + \mathbf{e_3}\big)\,dxdy \\ &= \iint_A \big(-6x^2 y - 2\big)\,dxdy\,.\end{aligned}$$

Now A is a disk of radius 2 centred on the origin (see the plot below: on C, the boundary of S, we have $x^2 + y^2 = 5 - z$ and $z = 1$, implying $x^2 + y^2 = 4$), giving us

$$\iint_{x^2+y^2\le 4} \big(-6x^2 y - 2\big)\,dxdy = \iint_{x^2+y^2\le 4} \big(-2\big)\,dxdy = -8\pi$$

(the integral of $-6x^2 y$ vanishing by symmetry). Alternatively we can switch to polar coordinates and compute

$$\begin{aligned}\int_0^{2\pi} d\theta \int_0^2 r dr\,\big(-6r^3\cos^2\theta\sin\theta - 2\big) &= \int_0^2 r dr\,\big[2r^3\cos^3\theta - 2\theta\big]_0^{2\pi} \\ &= \int_0^2 r dr\,(-4\pi) = -4\pi\,\big[\tfrac{1}{2}r^2\big]_0^2 = -8\pi\,.\end{aligned}$$

Note that this surface has the same boundary as the disk in part (i), so the line integral calculation is the same as before.

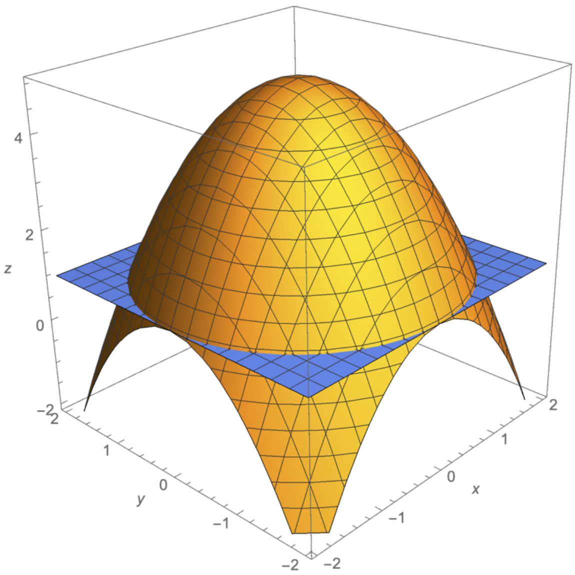


Figure 11: The intersection of the parabaloid $x^2 + y^2 = 5 - z$ and the plane $z = 1$.

96. Evaluate the line integral

$$\int_{P_1}^{P_2} yz\,dx + xz\,dy + (xy + z^2)\,dz$$

from P_1 with co-ordinates (1,0,0) to P_2 at (1,0,1) explicitly,

(a) along a straight line in the z–direction, and

(b) along a helical path parametrised by $\mathbf{x}(t) = \mathbf{e_1}\cos t + \mathbf{e_2}\sin t + \mathbf{e_3}\,t/2\pi$, where t varies along the path.

Compare these two results – does this suggest that there might be a general formula for the integral from $P_1 = (1,0,0)$ to any point P with coordinates (x,y,z)? Check your answer when $P = P_2$.

Solution Call the straight path C_1. On this path $y = 0$ and $x = 1$, so $dx = dy = 0$ and

$$\int_{C_1} (yz\,dx + xz\,dy + (xy + z^2)\,dz) = \int_0^1 z^2\,dz = \tfrac{1}{3}\,.$$

Call the helical path C_2. On C_2 we have $x = \cos t$, $y = \sin t$ so $dx = -\sin t\,dt$ and $dy = \cos t\,dt$, while $z = \frac{t}{2\pi}$ so $dz = \frac{1}{2\pi}\,dt$. Hence

$$\begin{aligned}\int_{C_2} (yz\,dx + xz\,dy + (xy + z^2)\,dz) &= \tfrac{1}{2\pi}\int_0^{2\pi}\left(-t\sin^2 t + t\cos^2 t + \cos t\sin t + \tfrac{t^2}{4\pi^2}\right)dt\\ &= \tfrac{1}{2\pi}\int_0^{2\pi}\left(t\cos 2t + \tfrac{1}{2}\sin 2t + \tfrac{t^2}{4\pi^2}\right)dt\\ &= \tfrac{1}{2\pi}\int_0^{2\pi}\tfrac{t^2}{4\pi^2}\,dt = \tfrac{1}{8\pi^3}\left[\tfrac{1}{3}t^3\right]_0^{2\pi} = \tfrac{1}{3}\,.\end{aligned}$$

(If you are averse to integrating by parts to get from the second line to the third, note that $\int_0^{2\pi} t\cos 2t\,dt = \int_0^{2\pi}(2\pi - t)\cos 2t\,dt$ by symmetry, so both must be equal to half their sum, which is clearly zero.)

These two results are the same! This suggests (but does not prove) that we might be computing the line integral of a conservative field. To check, compute $\underline{\nabla}\times\mathbf{F} = \text{(compute)} = \mathbf{0}$. This means we must have $\mathbf{F} = \underline{\nabla}\phi$ for some scalar field ϕ. To find ϕ, set $\mathbf{F} = \underline{\nabla}\phi$ and work component by component:

$$F_1 = \frac{\partial\phi}{\partial x} \;\Rightarrow\; yz = \frac{\partial\phi}{\partial x} \;\Rightarrow\; \phi(x,y,z) = xyz + f(y,z)\,.$$

Then

$$F_2 = \frac{\partial\phi}{\partial y} \;\Rightarrow\; xz = xz + \frac{\partial f}{\partial y} \;\Rightarrow\; \frac{\partial f}{\partial y} = 0 \;\Rightarrow\; f(y,z) = g(z)\,.$$

Finally

$$F_3 = \frac{\partial\phi}{\partial z} \;\Rightarrow\; xy + z^2 = xy + \frac{\partial g}{\partial z} \;\Rightarrow\; \frac{\partial g}{\partial z} = z^2 \;\Rightarrow\; g(z) = \tfrac{1}{3}z^3 + c\,.$$

Thus $\phi(x,y,z) = xyz + \frac{1}{3}z^3 + c$, and the line integral of $\mathbf{F}$ from $(1,0,0)$ to any point (x,y,z) is equal to $\phi(x,y,z) - \phi(1,0,0) = xyz + \frac{1}{3}z^3$. If $(x,y,z) = P_2 = (1,0,1)$ this is equal to $\frac{1}{3}$, agreeing with the previous results.

97. State a necessary and sufficient condition for a vector field $\mathbf{F}$ to be expressible in the form $\mathbf{F} = \underline{\nabla}\,\phi$ in some simply-connected region. The scalar field ϕ is called a *scalar potential*, though sometimes the opposite sign is used.

Determine whether the following vector fields are expressible in this form

(i) $(\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}$, (ii) $(\mathbf{a}\cdot\mathbf{x})\,\mathbf{a}$, (iii) $(\mathbf{a}\cdot\mathbf{a})\,\mathbf{x}$, (iv) $\mathbf{a}\times\mathbf{x}$, (v) $\mathbf{a}\times(\mathbf{a}\times\mathbf{x})$,

and find the vector fields, $\mathbf{F}$, for which the corresponding potentials are

(a) $\frac{1}{2}(\mathbf{a}\cdot\mathbf{x})^2$, (b) $\frac{1}{2}a^2|\mathbf{x}|^2 - \frac{1}{2}(\mathbf{a}\cdot\mathbf{x})^2$.

Here $\mathbf{a}$ is a constant non-zero vector, and $a^2 = |\mathbf{a}|^2$.

Solution The necessary and sufficient condition is $\underline{\nabla} \times \mathbf{F} = \mathbf{0}$.

(i) $\mathbf{F} = (\mathbf{a}\cdot\mathbf{x})\mathbf{x}$:

$$(\underline{\nabla} \times \mathbf{F})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(a_l x_l x_k) = \varepsilon_{ijk}a_l(\delta_{jl}x_k + x_l\delta_{jk}) = \varepsilon_{ijk}a_j x_k = (\mathbf{a}\times\mathbf{x})_i \neq 0$$

so $\mathbf{F} \neq \underline{\nabla}\phi$.

(ii) $\mathbf{F} = (\mathbf{a}\cdot\mathbf{x})\mathbf{a}$:

$$(\underline{\nabla} \times \mathbf{F})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(a_l x_l a_k) = \varepsilon_{ijk}a_l\delta_{jl}a_k = \varepsilon_{ijk}a_j a_k = 0$$

so $\mathbf{F} = \underline{\nabla}\phi$ for some ϕ.

(iii) $\mathbf{F} = (\mathbf{a}\cdot\mathbf{a})\mathbf{x}$:

$$(\underline{\nabla} \times \mathbf{F})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(a_l a_l x_k) = \varepsilon_{ijk}a_l a_l\delta_{jk} = \varepsilon_{ijj}a_l a_l = 0$$

so $\mathbf{F} = \underline{\nabla}\phi$ for some ϕ.

(iv) $\mathbf{F} = \mathbf{a}\times\mathbf{x}$:

$$(\underline{\nabla} \times \mathbf{F})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\varepsilon_{klm}a_l x_m) = \varepsilon_{ijk}\varepsilon_{klm}a_l\delta_{jm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})a_l\delta_{jm} = 2a_i \neq 0$$

so $\mathbf{F} \neq \underline{\nabla}\phi$.

(v) $\mathbf{F} = \mathbf{a}\times(\mathbf{a}\times\mathbf{x})$:

$$\begin{aligned}(\mathbf{a}\times(\mathbf{a}\times\mathbf{x}))_i &= \varepsilon_{ijk}a_j\varepsilon_{klm}a_l x_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})a_j a_l x_m = a_j a_i x_j - a_j a_j x_i \\ &= ((\mathbf{a}\cdot\mathbf{x})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{x})_i\end{aligned}$$

so $\mathbf{a}\times(\mathbf{a}\times\mathbf{x}) = (\mathbf{a}\cdot\mathbf{x})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{x}$, and $\underline{\nabla}\times\mathbf{F} = \mathbf{0}$ by (ii) and (iii); hence $\mathbf{F} = \underline{\nabla}\phi$ for some ϕ.

For the second part:

(a) $\phi = \frac{1}{2}(\mathbf{a}\cdot\mathbf{x})^2 = \frac{1}{2}(a_j x_j a_k x_k)$ so

$$F_i = \tfrac{1}{2}a_j a_k \frac{\partial}{\partial x_i}(x_j x_k) = \tfrac{1}{2}a_j a_k(\delta_{ij}x_k + x_j\delta_{ik}) = \tfrac{1}{2}(a_i a_k x_k + a_j a_i x_j) = ((\mathbf{a}\cdot\mathbf{x})\mathbf{a})_i\,.$$

Hence $\mathbf{F} = (\mathbf{a}\cdot\mathbf{x})\mathbf{a}$.

(b) $\phi = \frac{1}{2}a^2|\mathbf{x}|^2 - \frac{1}{2}(\mathbf{a}\cdot\mathbf{x})^2$: we have $(\underline{\nabla}(\frac{1}{2}a^2|\mathbf{x}|^2))_i = \frac{1}{2}a^2\frac{\partial}{\partial x_i}(x_j x_j) = a^2 x_i$ and combining this with the result from part (a), $\mathbf{F} = a^2\mathbf{x} - (\mathbf{a}\cdot\mathbf{x})\mathbf{a}$.

98. Exam question June 2002 (Section B):

(a) State the conditions for the line integral $\int_c \mathbf{F}\cdot d\mathbf{x}$ from $\mathbf{x}_0$ to $\mathbf{x}_1$ to be independent of the path connecting these two points.

(b) Determine the value of the line integral $\int_c \mathbf{F}\cdot d\mathbf{x}$ where

$$\mathbf{F} = (e^{-y} - ze^{-x})\,\mathbf{e_1} + (e^{-z} - xe^{-y})\,\mathbf{e_2} + (e^{-x} - ye^{-z})\,\mathbf{e_3},$$

and C is the path

$$x = \frac{1}{\ln 2}\ln(1+p), \quad y = \sin\frac{\pi p}{2}, \quad z = \frac{1-e^p}{1-e},$$

with the parameter p in the range $0 \le p \le 1$, from $(0,0,0)$ to $(1,1,1)$.

Solution

(a) The necessary and sufficient condition for the integral not to depend of the path is that $\mathbf{F} = \underline{\nabla}\phi$ for some ϕ, or, equivalently, that $\underline{\nabla}\times\mathbf{F} = \mathbf{0}$ (assuming that the region under consideration is simply connected).

(b) We have $\underline{\nabla}\times\mathbf{F} =$ (calculate) $= \mathbf{0}$. Hence we can swap the given C for any other path joining $(0,0,0)$ to $(1,1,1)$ without changing the answer. Simplest is to take the straight line $\mathbf{x}(t) = (t,t,t)$ with $0 \le t \le 1$, so $\frac{d\mathbf{x}}{dt} = (1,1,1)$. Thus

$$I = 3\int_0^1 \left(e^{-t} - te^{-t}\right)dt = 3\int_0^1 \frac{d}{dt}\left(te^{-t}\right)dt = 3\left[te^{-t}\right]_0^1 = \frac{3}{e}\,.$$

Alternatively, we can figure out $\phi(x,y,z)$. We need

$$\frac{\partial\phi}{\partial x} = F_1 = e^{-y} - ze^{-x} \Rightarrow \phi(x,y,z) = xe^{-y} + ze^{-x} + f(y,z)$$

and then

$$\frac{\partial\phi}{\partial y} = -xe^{-y} + \frac{\partial f}{\partial y} = F_2 = e^{-z} - xe^{-y} \Rightarrow f(y,z) = ye^{-z} + g(z)$$

and finally

$$\frac{\partial\phi}{\partial z} = e^{-x} - ye^{-z} + \frac{\partial g}{\partial z} = F_3 = e^{-x} - ye^{-x} \Rightarrow g(z) = \text{const} = A\,,\ \text{say.}$$

Hence $\phi(x,y,z) = xe^{-y} + ze^{-x} + ye^{-z} + A$ and

$$\int_C \mathbf{F}\cdot d\mathbf{x} = \phi(1,1,1) - \phi(0,0,0) = (e^{-1}+e^{-1}+e^{-1}+A) - (0+A) = \frac{3}{e}$$

as before.

99. If $\mathbf{F} = x\,\mathbf{e_1} + y\,\mathbf{e_2}$, calculate $\int_S \mathbf{F}\cdot\mathbf{dA}$, where S is the part of the surface $z = 9 - x^2 - y^2$ that is above the x,y plane, by applying the divergence theorem to the volume bounded by the surface and the piece it cuts out of the x, y plane.
Hint: what is $\mathbf{F}\cdot\mathbf{dA}$ *on* the x,y plane?

Solution The normal vector to the x,y plane is $\mathbf{e_3}$, so $\mathbf{F}\cdot d\mathbf{A} = 0$ there. Hence the surface integral side of the divergence theorem, when applied to the volume V mentioned in the question, comes only from the integral over the part S of the paraboloid above the x,y plane, which is exactly what the question asks us to calculate. Hence the desired surface integral is equal to the volume integral of $\underline{\nabla}\cdot\mathbf{F} = 2$. Thus

$$\begin{aligned}\int_S \mathbf{F}\cdot d\mathbf{A} = \int_{\partial V} \mathbf{F}\cdot d\mathbf{A} &= \int_V \underline{\nabla}\cdot\mathbf{F}\,dV \quad \text{by the divergence theorem}\\ &= 2\int_0^9 \Big(\iint_{x^2+y^2\le 9-z} dxdy\Big)\,dz\\ &= 2\int_0^9 \pi(9-z)\,dz\\ &= 2\pi\left[9z - \tfrac{1}{2}z^2\right]_0^9 = 81\pi\,.\end{aligned}$$

100. Evaluate each of the integrals below as **either** a volume integral **or** a surface integral, whichever is easier:

(a) $\int_S \mathbf{x}\cdot\mathbf{dA}$ over the whole surface of a cylinder bounded by $x^2 + y^2 = R^2$, $z = 0$ and $z = L$. Note that $\mathbf{x}$ means $x\,\mathbf{e_1} + y\,\mathbf{e_2} + z\,\mathbf{e_3}$.

(b) $\int_S \mathbf{F}\cdot\mathbf{dA}$, where $\mathbf{F} = x\cos^2 y\,\mathbf{e_1} + xz\,\mathbf{e_2} + z\sin^2 y\,\mathbf{e_3}$, over the surface of a sphere with centre at the origin and radius π.

(c) $\int_V \underline{\nabla}\cdot\mathbf{F}\,dV$, where $\mathbf{F} = \sqrt{x^2+y^2}(x\,\mathbf{e_1} + y\,\mathbf{e_2})$, over the three-dimensional volume $x^2 + y^2 \le R^2$, $0 \le z \le L$.

Solution

(a) $\int_S \mathbf{x}\cdot d\mathbf{A} = \int_V \underline{\nabla}\cdot\mathbf{x}\,dV = \int_V 3\,dV = 3\times(\text{volume of } V) = 3\pi R^2 L\,.$

(b) Note, $\underline{\nabla}\cdot\mathbf{F} = \cos^2 y + \sin^2 y = 1$, so $\int_S \mathbf{F}\cdot\mathbf{dA} = \int_V dV = \frac{4}{3}\pi^4$.

(c) This time, convert to a surface integral. Since $\mathbf{F}$ has no z component, the ends of the cylinder don't contribute. Furthermore, on the curved surface S of the cylinder, the unit normal is $\widehat{\mathbf{n}} = (x\mathbf{e_1} + y\mathbf{e_2})/\sqrt{x^2+y^2}$ (and is parallel to $\mathbf{F}$). Hence on this surface $\mathbf{F}\cdot d\mathbf{A} = x^2 + y^2 = R^2$ and $\int_V \underline{\nabla}\cdot\mathbf{F}\,dV = \int_{\partial V}\mathbf{F}\cdot d\mathbf{A} = \int_S \mathbf{F}\cdot d\mathbf{A} = R^2\int_S dA = R^2\times(\text{area of } S) = 2\pi R^3 L\,.$

101. Suppose the vector field $\mathbf{F}$ is everywhere tangent to the closed surface S, which encloses the volume V. Prove that

$$\int_V \underline{\nabla}\cdot\mathbf{F}\,dV = 0\,.$$

Solution By the divergence theorem, $\int_V \underline{\nabla} \cdot \mathbf{F}\, dV = \int_S \mathbf{F} \cdot d\mathbf{A} = \int_S \mathbf{F} \cdot \widehat{\mathbf{n}}\, dA = 0$, since to be tangent to S, $\mathbf{F}$ must be orthogonal to $\widehat{\mathbf{n}}$, the unit normal to S.

102. Exam question June 2003 (Section A): Evaluate $\int_S \mathbf{B}(\mathbf{x}) \cdot \mathbf{dA}$, where

$$\mathbf{B}(\mathbf{x}) = (8x + \alpha\, y - z)\, \mathbf{e_1} + (x + 2y + \beta\, z)\, \mathbf{e_2} + (\gamma\, x + y - z)\, \mathbf{e_3}$$

and S is the surface of the sphere having centre at (α, β, γ) and radius γ, where $\alpha, \beta \in \mathbb{R}$, and γ is an arbitrary positive real number.

Solution Calculating, $\underline{\nabla} \cdot \mathbf{B} = 8 + 2 - 1 = 9$, so by the divergence theorem

$$\int_S \mathbf{B} \cdot d\mathbf{A} = \int_V \underline{\nabla} \cdot \mathbf{B}\, dV = 9 \int_V dV = 9 \times (\text{volume of } V) = 12\pi\gamma^3.$$

103. Let A be the interior of the circle of unit radius centred on the origin. Evaluate $\iint_A \exp(x^2 + y^2)\, dx\, dy$ by making a change of variables to polar co-ordinates.

Solution

$$\begin{aligned}\iint_A \exp(x^2 + y^2)\, dx\, dy &= \int_0^{2\pi} (\int_0^1 e^{r^2} r\, dr)\, d\theta = \int_0^{2\pi} \left[\tfrac{1}{2} e^{r^2}\right]_0^1 d\theta \\ &= \int_0^{2\pi} \left(\tfrac{1}{2}(e-1)\right) d\theta = (e-1)\pi\,.\end{aligned}$$

(Or, do the θ integral first.)

104. Let A be the region $0 \le y \le x$ and $0 \le x \le 1$. Evaluate $\int_A (x + y)\, dx\, dy$ by making the change of variables $x = u + v$, $y = u - v$. Check your answer by evaluating the integral directly.

Solution First find the Jacobian:

$$J(\underline{U}) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2\,.$$

Then

$$I = \int_A (x + y)\, dxdy = \int_U 2u\, |J|\, dudv = 4 \int_U u\, dudv$$

where U is the area of integration in the new coordinates, which is bounded by a triangle in the u,v plane with edges $y = x \Rightarrow v = 0$, $x = 1 \Rightarrow u + v = 1$, $y = 0 \Rightarrow u = v$. This leads to

$$I = 4 \int_0^{1/2} \left(\int_v^{1-v} u\, du \right) dv = 4 \int_0^{1/2} \left[\tfrac{1}{2} u^2\right]_v^{1-v} dv = 4 \int_0^{1/2} \left(\tfrac{1}{2} - v\right) dv = \tfrac{1}{2}\,.$$

Alternatively the integral can be done directly:

$$I = \int_0^1 \int_y^1 (x+y)\, dxdy = \int_0^1 \left[\tfrac{1}{2}x^2 + xy\right]_y^1 dy = \int_0^1 (\tfrac{1}{2} + y - \tfrac{1}{2}y^2 - y^2)\, dy = \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{6} - \tfrac{1}{3} = \tfrac{1}{2}$$

which agrees!

105. The expressions below are called Green's first and second identities. Derive them from the divergence theorem with $\mathbf{F} = f\underline{\nabla} g$ or $g\underline{\nabla} f$ as appropriate, where f and g are differentiable functions:

$$\int_V (f\underline{\nabla}^2 g + \underline{\nabla} f \cdot \underline{\nabla} g)\, dV = \int_S (f\underline{\nabla} g) \cdot \mathbf{dA}\ ,$$

$$\int_V (f\underline{\nabla}^2 g - g\underline{\nabla}^2 f)\, dV = \int_S (f\underline{\nabla} g - g\underline{\nabla} f) \cdot \mathbf{dA}\ .$$

where the volume V is that enclosed by the closed surface S.

Solution First apply the divergence theorem to $f\underline{\nabla} g$ as follows:

$$\begin{aligned}
\int_S (f\underline{\nabla} g) \cdot d\mathbf{A} &= \int_V \underline{\nabla} \cdot (f\underline{\nabla} g)\, dV \\
&= \int_V (\underline{\nabla} f \cdot \underline{\nabla} g + f\underline{\nabla} \cdot \underline{\nabla} g)\, dV \\
&= \int_V \left(\underline{\nabla} f \cdot \underline{\nabla} g + f\underline{\nabla}^2 g\right) dV \quad \text{as required.}
\end{aligned}$$

Swapping f and g, we also have

$$\int_S (g\underline{\nabla} f) \cdot d\mathbf{A} = \int_V \left(\underline{\nabla} g \cdot \underline{\nabla} f + g\underline{\nabla}^2 f\right) dV,$$

and subtracting this from the first result gives us the second identity.

106. A gas holder has the form of a vertical cylinder of radius R and height H with hemispherical top also of radius R. The density, ρ, (i.e. the mass per unit volume) of the gas inside varies with height z above the base according to the relation $\rho = C\exp(-z)$, where C is a constant. Calculate the total mass of gas in the holder, taking care to define any coordinate system used and the range of the corresponding variables.

Use this result to find the integral of the field

$$\mathbf{F} \;=\; B\, z\, e^{-y}\, \mathbf{e_1} \;+\; C\, y\, e^{-z}\, \mathbf{e_2}$$

over the curved surface of the gas holder.

Hint: it may help to note first what the integral over the base of the holder is.

Solution For the first part we need the mass in the gas holder, which given that ρ is the density is equal to $M_{\text{total}} = \int_V \rho\, dV$. Split this integral into two parts, the first over the cylinder and the second over the hemispherical top. A convenient choice of coordinates is to put the origin at the centre of the circular base, which will therefore be in the plane $z = 0$. Then

$$\begin{aligned}
M_{\text{cylinder}} &= \int_0^H \int_{x^2+y^2\le R^2} Ce^{-z}\, dxdydz \\
&= \int_0^H \pi R^2\, Ce^{-z}\, dz \\
&= \pi C R^2 \left[-e^{-z}\right]_0^H = \pi C R^2 \left(1 - e^{-H}\right)\ .
\end{aligned}$$

For the top, we have $x^2 + y^2 + (z - H)^2 = R^2$, since it is centred at $(0, 0, H)$. Setting $w = z - H$ the required integral is

$$\begin{aligned} M_{\text{top}} &= \int_0^R \int_{x^2+y^2 \le R^2 - w^2} C e^{-(w+H)}\, dxdydw \\ &= \int_0^R \pi(R^2 - w^2)\, C e^{-w-H}\, dw \\ &= \pi C e^{-H} \int_0^R (R^2 - w^2)\, e^{-w}\, dw\,. \end{aligned}$$

Now $\int_0^R R^2 e^{-w}\, dw = R^2 \left[-e^{-w}\right]_0^R = R^2(1 - e^{-R})$, while

$$\begin{aligned} \int_0^R w^2 e^{-w}\, dw = \left[-w^2 e^{-w}\right]_0^R + \int_0^R 2we^{-w} dw \\ &= -R^2 e^{-R} + \left[-2we^{-w}\right]_0^R + \int_0^R 2e^{-w} dw \\ &= -R^2 e^{-R} - 2Re^{-R} + 2 - 2e^{-R}. \end{aligned}$$

Adding up the bits,

$$M_{\text{top}} = \pi C e^{-H}\big(R^2 - 2 + 2Re^{-R} + 2e^{-R}\big)$$

and so the total mass is

$$M_{\text{total}} = M_{\text{cylinder}} + M_{\text{top}} = \pi C R^2 + 2\pi C e^{-H}\big(Re^{-R} + e^{-R} - 1\big)\,.$$

(Not the most edifying calculation in the world!)

For the second part, we need the integral over the curved surface, which we will call S. This is the boundary ∂V of V less the flat bit on the bottom. However, as suggested by the hint, we can notice that $\mathbf{F} \cdot \widehat{\mathbf{n}} = 0$ on this flat bit (since $\widehat{\mathbf{n}} = -\mathbf{e_3}$ there). Hence

$$\int_S \mathbf{F} \cdot d\mathbf{A} = \int_S \mathbf{F} \cdot \widehat{\mathbf{n}}\, dA = \int_{\partial V} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dA = \int_{\partial V} \mathbf{F} \cdot d\mathbf{A} = \int_V \underline{\nabla} \cdot \mathbf{F}\, dV$$

using the divergence theorem, and putting in rather more steps than strictly needed. Now $\underline{\nabla} \cdot \mathbf{F} = Ce^{-z}$, which is exactly the function ρ we took the trouble to integrate over V in the first part. Hence

$$\int_S \mathbf{F} \cdot d\mathbf{A} = \pi C R^2 + 2\pi C e^{-H}\big(Re^{-R} + e^{-R} - 1\big)\,.$$

107. Exam question June 2001 (Section B): In electrostatic theory, Gauss' law states that the flux of an electrostatic (vector) field $\mathbf{E}(\mathbf{x})$ over some closed surface S is equal to the enclosed charge q divided by a constant ϵ_0 :

$$\int_S \mathbf{E}(\mathbf{x}) \cdot \mathbf{dA} = \frac{q}{\epsilon_0}\,.$$

If the electrostatic field is given by $\mathbf{E}(x, y) = \alpha x\mathbf{e_1} + \beta y\mathbf{e_2}$, use Gauss' law to find the total charge in the compact region bounded by the surface S consisting of S_1, the curved portion of the half-cylinder $z = (R^2 - y^2)^{1/2}$ of length H; S_2 and S_3 the two semi-circular plane end pieces; and S_4, the rectangular portion of the (x, y)-plane. (In equations, the relevant bounded region may be described by $z^2 + y^2 \le R^2, z \ge 0, -\frac{H}{2} \le x \le \frac{H}{2}$). Express your result in terms of α, β, R and H.

Solution Since the surface is closed we can use the divergence theorem to switch from a surface to a volume integral. First calculate $\underline{\nabla} \cdot \mathbf{E} = \alpha + \beta$; then, letting V denote the region enclosed by S,

$$\begin{aligned}\int_S \mathbf{E} \cdot d\mathbf{A} &= \int_V (\alpha + \beta)\, dV \\ &= (\alpha + \beta) \times \text{(volume of the half cylinder)} \\ &= \tfrac{1}{2}(\alpha + \beta)\pi R^2 H.\end{aligned}$$

Hence the total charge enclosed is $q = \frac{1}{2}\epsilon_0(\alpha + \beta)\pi R^2 H$.

108. Exam question June 2002 (Section A): In electrostatic theory, Gauss' law states that the flux of an electrostatic field $\mathbf{E}(\mathbf{x})$ over a closed surface S is given by the ratio of the enclosed charge q and a constant ϵ_0. Calculate the electric charge enclosed in the ellipsoid of equation $x^2 + \frac{1}{2}y^2 + z^2 = 1$ in the presence of

(a) an electrostatic field $\mathbf{E}(\mathbf{x}) = yz\mathbf{e_1} + xz\mathbf{e_2} + xy\mathbf{e_3}$,

(b) an electrostatic field $\mathbf{E}(\mathbf{x}) = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$.

Solution In both parts, the best tactic to replace $\int_S \mathbf{E} \cdot d\mathbf{A}$ by $\int_V \underline{\nabla} \cdot \mathbf{E}\, dV$, quoting the divergence theorem.

(a) In this case, $\underline{\nabla} \cdot \mathbf{E} = 0$, so the enclosed electric charge is zero.

(b) This time, $\underline{\nabla} \cdot \mathbf{E} = 3$, so the enclosed electric charge is equal to 3 times the volume of the ellipsoid, multiplied by ϵ_0. The given ellipsoid is a unit sphere (which has volume $\frac{4}{3}\pi$), expanded by a factor of $\sqrt{2}$ in the y direction. Hence its volume is $\sqrt{2} \times \frac{4}{3}\pi$, and the charge enclosed is $4\sqrt{2}\,\pi\,\epsilon_0$. (Alternatively, one can grind through the volume integral explicitly.)