83 *For a simple closed curve C in the $(x, y)-$plane, show by Green's theorem that the area enclosed is $A = \frac{1}{2} \oint_C (xdy - ydx)$. (Note, 'simple' means that C does not cross itself, which means that it encloses a well-defined area A.)*

84 *Evaluate $\oint \mathbf{F} \cdot \mathbf{dx}$ around the circle $x^2 + y^2 + 2x = 0$, where $\mathbf{F} = y\mathbf{e_1} - x\mathbf{e_2}$, both directly and by using Green's theorem in the plane.*

85 *Evaluate $\oint_C 2xdy - 3ydx$ around the square with vertices at $(x, y) = (0, 2)$, $(2, 0)$, $(-2, 0)$ and $(0, -2)$.*

86 *Let $\underline{v}$ be the radial vector field $\underline{v}(\underline{x}) = \underline{x}$.*

(a) Compute $l_1 = \int_{C_1} \underline{v} \cdot d\underline{x}$, where C_1 is the straight-line contour from the origin to the point $(2, 0, 0)$.

(b) Compute $l_2 = \int_{C_2} \underline{v} \cdot d\underline{x}$, where C_2 is the semi-circular contour from the origin to the point $(2, 0, 0)$ defined by $0 \le x \le 2$, $y = +\sqrt{1 - (x{-}1)^2}$, $z = 0$. [It may help to start by sketching C_2 , and then to parameterize it as $\underline{x}(t) = (1 - \cos t, \sin t, 0)$ with $0 \le t \le \pi$.]

(c) You should have found that $l_1 = l_2$. Explain this result using Stokes' theorem.

87 *Integrate $\operatorname{curl} \mathbf{F}$, where $\mathbf{F} = 3y\mathbf{e_1} - xz\mathbf{e_2} + yz^2\mathbf{e_3}$, over the portion S of the surface $2z = x^2 + y^2$ below the plane $z = 2$, both directly and by using Stokes' theorem. Take the area elements of S to point outwards, so that their z components are negative.*

88 *The paraboloid of equation $z = x^2 + y^2$ intersects the plane $z = y$ in a curve C. Calculate $\oint_C \mathbf{v} \cdot \mathbf{dx}$ for $\mathbf{v} = 2z\mathbf{e_1} + x\mathbf{e_2} + y\mathbf{e_3}$ using Stokes' theorem. Check your answer by evaluating the line integral directly.*

89 *For an open surface S with boundary C, show that $2 \int_S \mathbf{u} \cdot d\mathbf{A} = \oint_C (\mathbf{u} \times \mathbf{x}) \cdot d\mathbf{x}$, where $\mathbf{u}$ is a fixed vector.*

90 *Verify Stokes' theorem for the upper hemispherical surface S: $z = \sqrt{1 - x^2 - y^2}$, $z \ge 0$, with $\mathbf{F}$ equal to the radial vector field, i.e. $\mathbf{F}(x, y, z) = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$.*

91 *Let $\mathbf{F} = ye^z\, \mathbf{e_1} + xe^z\, \mathbf{e_2} + xye^z\, \mathbf{e_3}$. Show that the integral of $\mathbf{F}$ around a regular closed curve C that is the boundary of a surface S is zero.*

92 *By applying Stokes' theorem to the vector field $\mathbf{G} = |\mathbf{x}\,|^2\mathbf{a}$ (with $\mathbf{a}$ a constant vector), or otherwise, show that $\int_S \mathbf{x} \times d\mathbf{A} = -\frac{1}{2} \oint_C |\mathbf{x}\,|^2\, d\mathbf{x}$, where S is the area bounded by the closed curve C.*

93 *Exam question June 2002 (Section B): Evaluate the line integral $I = \int_C \mathbf{F} \cdot d\mathbf{x}$ where $\mathbf{F}(\mathbf{x}\,) = 2y\mathbf{e_1} + z\mathbf{e_2} + 3y\mathbf{e_3}$ and the path C is the intersection of the surface of equation $x^2 + y^2 + z^2 = 4z$ and the surface of equation $z = x + 2$, taken in a clockwise direction to an observer at the origin. A picture of the path is required, as well as full justifications of the theoretical results you might use.*

94 *Exam question 2012 (Section B) Q9(a)(ii): Use Stokes' theorem to calculate the line integral* $\oint_C y dx + z dy + x dz$, *where C is the intersection of the surfaces* $x^2+y^2+z^2 = a^2$ *and* $x + y + z = 0$ *and is orientated anticlockwise when viewed from above.*
Suggestion: Instead of one of the two standard methods for surface integrals, just use $d\mathbf{A} = \frac{\nabla f}{|\nabla f|} dA$ *and think about how the surfaces intersect.*

95 *Consider the vector field* $\mathbf{F} = y\,\mathbf{e_1} + (z-x)\,\mathbf{e_2} + (x^3+y)\,\mathbf{e_3}$. *Evaluate* $\int_S (\underline{\nabla} \times \mathbf{F}) \cdot \mathbf{dA}$ *explicitly, where*

(i) S *is the disk* $x^2 + y^2 \leq 4$, $z = 1$,

(ii) S *is the surface of a paraboloid* $x^2 + y^2 = 5 - z$ *above the plane* $z = 1$.

Verify that each of these agrees with Stokes' theorem by considering the integral of $\mathbf{F}$ *around each bounding contour.*

96 *Evaluate the line integral*

$$\int_{P_1}^{P_2} yz\,dx + \; xz\,dy + \; (xy + z^2)\,dz$$

from P_1 *with co-ordinates (1,0,0) to* P_2 *at (1,0,1) explicitly,*

(a) *along a straight line in the z–direction, and*

(b) *along a helical path parametrised by* $\mathbf{x}\,(t) = \mathbf{e_1}\,\cos t + \mathbf{e_2}\,\sin t + \mathbf{e_3}\,t/2\pi$, *where* t *varies along the path.*

Compare these two results – does this suggest that there might be a general formula for the integral from $P_1 = (1, 0, 0)$ *to any point* P *with coordinates* (x, y, z)*? Check your answer when* $P = P_2$.

97 *State a necessary and sufficient condition for a vector field* $\mathbf{F}$ *to be expressible in the form* $\mathbf{F} = \underline{\nabla}\,\phi$ *in some simply-connected region. The scalar field* ϕ *is called a scalar potential, though sometimes the opposite sign is used.*

Determine whether the following vector fields are expressible in this form

(i) $(\mathbf{a}\cdot\mathbf{x}\,)\,\mathbf{x}$, (ii) $(\mathbf{a}\cdot\mathbf{x}\,)\,\mathbf{a}$, (iii) $(\mathbf{a}\cdot\mathbf{a}\,)\,\mathbf{x}$, (iv) $\mathbf{a}\times\mathbf{x}$, (v) $\mathbf{a}\times(\mathbf{a}\times\mathbf{x}\,)$,

and find the vector fields, $\mathbf{F}$, *for which the corresponding potentials are*

(a) $\frac{1}{2}(\mathbf{a}\cdot\mathbf{x}\,)^2$, (b) $\frac{1}{2}a^2|\mathbf{x}\,|^2 - \frac{1}{2}(\mathbf{a}\cdot\mathbf{x}\,)^2$.

Here $\mathbf{a}$ *is a constant non-zero vector, and* $a^2 = |\mathbf{a}|^2$.

98 *Exam question June 2002 (Section B):*

(a) *State the conditions for the line integral* $\int_c \mathbf{F}\cdot d\mathbf{x}$ *from* $\mathbf{x}_0$ *to* $\mathbf{x}_1$ *to be independent of the path connecting these two points.*

(b) *Determine the value of the line integral* $\int_c \mathbf{F}\cdot d\mathbf{x}$ *where*

$$\mathbf{F} = (e^{-y} - ze^{-x})\,\mathbf{e_1} + (e^{-z} - xe^{-y})\,\mathbf{e_2} + (e^{-x} - ye^{-z})\,\mathbf{e_3},$$

and C is the path

$$x = \frac{1}{\ln 2}\ln(1+p), \quad y = \sin\frac{\pi p}{2}, \quad z = \frac{1-e^p}{1-e},$$

with the parameter p in the range $0 \le p \le 1$, from $(0,0,0)$ to $(1,1,1)$.

99 *If $\mathbf{F} = x\,\mathbf{e_1} + y\,\mathbf{e_2}$, calculate $\int_S \mathbf{F}\cdot\mathbf{dA}$, where S is the part of the surface $z = 9 - x^2 - y^2$ that is above the x,y plane, by applying the divergence theorem to the volume bounded by the surface and the piece it cuts out of the x, y plane.*
Hint: what is $\mathbf{F}\cdot\mathbf{dA}$ on the x,y plane?

100 *Evaluate each of the integrals below as* **either** *a volume integral* **or** *a surface integral, whichever is easier:*

(a) $\int_S \mathbf{x}\cdot\mathbf{dA}$ *over the whole surface of a cylinder bounded by $x^2+y^2 = R^2$, $z = 0$ and $z = L$. Note that $\mathbf{x}$ means $x\,\mathbf{e_1} + y\,\mathbf{e_2} + z\,\mathbf{e_3}$.*

(b) $\int_S \mathbf{F}\cdot\mathbf{dA}$, *where $\mathbf{F} = x\cos^2 y\,\mathbf{e_1} + xz\,\mathbf{e_2} + z\sin^2 y\,\mathbf{e_3}$, over the surface of a sphere with centre at the origin and radius π.*

(c) $\int_V \underline{\nabla}\cdot\mathbf{F}\,dV$, *where $\mathbf{F} = \sqrt{x^2+y^2}(x\,\mathbf{e_1} + y\,\mathbf{e_2})$, over the three-dimensional volume $x^2+y^2 \le R^2$, $0 \le z \le L$.*

101 *Suppose the vector field $\mathbf{F}$ is everywhere tangent to the closed surface S, which encloses the volume V. Prove that*

$$\int_V \underline{\nabla}\cdot\mathbf{F}\,dV = 0\,.$$

102 *Exam question June 2003 (Section A): Evaluate $\int_S \mathbf{B}(\mathbf{x})\cdot\mathbf{dA}$, where*

$$\mathbf{B}(\mathbf{x}) = (8x + \alpha\,y - z)\,\mathbf{e_1} + (x + 2y + \beta\,z)\,\mathbf{e_2} + (\gamma\,x + y - z)\,\mathbf{e_3}$$

and S is the surface of the sphere having centre at (α,β,γ) and radius γ, where $\alpha,\beta \in \mathbb{R}$, and γ is an arbitrary positive real number.

103 *Let A be the interior of the circle of unit radius centred on the origin. Evaluate $\iint_A \exp(x^2+y^2)\,dx\,dy$ by making a change of variables to polar co-ordinates.*

104 *Let A be the region $0 \le y \le x$ and $0 \le x \le 1$. Evaluate $\int_A (x+y)\,dx\,dy$ by making the change of variables $x = u+v$, $y = u-v$. Check your answer by evaluating the integral directly.*

105 *The expressions below are called Green's first and second identities. Derive them from the divergence theorem with $\mathbf{F} = f\underline{\nabla}g$ or $g\underline{\nabla}f$ as appropriate, where f and g are differentiable functions:*

$$\int_V (f\underline{\nabla}^2 g + \underline{\nabla}f\cdot\underline{\nabla}g)\,dV = \int_S (f\underline{\nabla}g)\cdot\mathbf{dA}\,,$$

$$\int_V (f\underline{\nabla}^2 g - g\underline{\nabla}^2 f)\,dV = \int_S (f\underline{\nabla}g - g\underline{\nabla}f)\cdot\mathbf{dA}\,.$$

where the volume V is that enclosed by the closed surface S.

106 *A gas holder has the form of a vertical cylinder of radius R and height H with hemispherical top also of radius R. The density, ρ, (i.e. the mass per unit volume) of the gas inside varies with height z above the base according to the relation $\rho = C\exp(-z)$, where C is a constant. Calculate the total mass of gas in the holder, taking care to define any coordinate system used and the range of the corresponding variables.*

Use this result to find the integral of the field

$$\mathbf{F} \;=\; B\,z\,e^{-y}\,\mathbf{e_1} \;+\; C\,y\,e^{-z}\,\mathbf{e_2}$$

over the curved surface of the gas holder.

Hint: it may help to note first what the integral over the base of the holder is.

107 *Exam question June 2001 (Section B): In electrostatic theory, Gauss' law states that the flux of an electrostatic (vector) field $\mathbf{E}(\mathbf{x})$ over some closed surface S is equal to the enclosed charge q divided by a constant ϵ_0:*

$$\int_S \mathbf{E}(\mathbf{x}) \cdot \mathbf{dA} \;=\; \frac{q}{\epsilon_0}\,.$$

If the electrostatic field is given by $\mathbf{E}(x,y) \;=\; \alpha x\mathbf{e_1} + \beta y\mathbf{e_2}$, use Gauss' law to find the total charge in the compact region bounded by the surface S consisting of S_1, the curved portion of the half-cylinder $z = (R^2 - y^2)^{1/2}$ of length H; S_2 and S_3 the two semi-circular plane end pieces; and S_4, the rectangular portion of the (x,y)-plane. (In equations, the relevant bounded region may be described by $z^2 + y^2 \le R^2, z \ge 0, -\frac{H}{2} \le x \le \frac{H}{2}$). Express your result in terms of α, β, R and H.

108 *Exam question June 2002 (Section A): In electrostatic theory, Gauss' law states that the flux of an electrostatic field $\mathbf{E}(\mathbf{x})$ over a closed surface S is given by the ratio of the enclosed charge q and a constant ϵ_0. Calculate the electric charge enclosed in the ellipsoid of equation $x^2 + \frac{1}{2}y^2 + z^2 = 1$ in the presence of*

(a) *an electrostatic field* $\mathbf{E}(\mathbf{x}) = yz\mathbf{e_1} + xz\mathbf{e_2} + xy\mathbf{e_3}\,,$

(b) *an electrostatic field* $\mathbf{E}(\mathbf{x}) = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}\,.$