8 Taylor series

8.1 Taylor's theorem

Taylor's theorem states that if $f(x)$ has $n+1$ continuous derivatives in an open interval I that contains the point $x=a$, then $\forall\ x \in I$

$$f(x) = \left(\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \right) + R_n(x)$$

$$\text{where} \quad R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt \quad \text{is called the } \textbf{remainder}.$$

Proof:
Fix $x \in I$ then $\int_a^x f'(t)\,dt = f(x) - f(a)$, so

$$f(x) = f(a) + \int_a^x f'(t)\,dt.$$

If we think of the integrand in the above as $f'(t) \cdot 1$ then we can perform an integration by parts where we choose the constant of integration to be $-x$ so that $\int 1\,dt = (t-x)$. This gives

$$\begin{aligned} f(x) &= f(a) + \Big[f'(t)(t-x) \Big]_a^x - \int_a^x f''(t)(t-x)\,dt \\ &= f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)\,dt. \end{aligned}$$

Note that this proves the theorem for $n=1$. To prove the theorem for $n=2$ we perform another integration by parts on this equation,

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)\,dt \\ &= f(a) + f'(a)(x-a) - \left[f''(t)\frac{(x-t)^2}{2} \right]_a^x + \int_a^x f'''(t)\frac{(x-t)^2}{2}\,dt \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \int_a^x f'''(t)\frac{(x-t)^2}{2}. \end{aligned}$$

This proves the theorem for $n=2$. Continuing in this way the theorem follows after performing an integration by parts a total of n times. More formally, this can be written in the form of a proof by induction.

8.2 Taylor polynomials

The combination $P_n(x) = f(x) - R_n(x)$ is a polynomial in x of degree n called **the n^{th} order Taylor polynomial of $f(x)$ about $x = a$**

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If the term 'about $x = a$' is not included when referring to the n^{th} order Taylor polynomial of $f(x)$, then by default this is taken to mean the choice $a = 0$, as this is the most common case.

The Taylor polynomial $P_n(x)$ is an approximation to the function $f(x)$. Generically, it is a good approximation if x is close to a and the approximation improves with increasing order n. The remainder provides an exact expression for the error in the approximation.

Eg. Calculate the n^{th} order Taylor polynomial of e^x.
$f(x) = e^x,\ f'(x) = e^x,\ \ldots, f^{(k)}(x) = e^x$ so $f^{(k)}(0) = 1$ giving

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Figure 35 shows the graph of e^x and its Taylor polynomials of order 0 to 3.

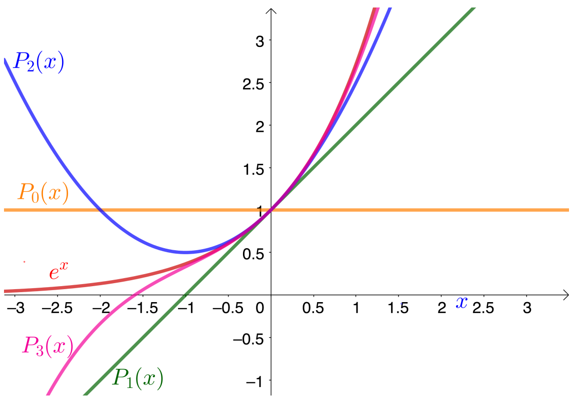

Figure 35: The function e^x (red) and its associated Taylor polynomials $P_0(x), P_1(x), P_2(x), P_3(x)$ of order 0 (orange), 1 (green), 2 (blue) and 3 (pink).

If $f(x)$ is infinitely differentiable on an open interval I that contains the point $x = a$ and in addition for each $x \in I$ we have that $\lim_{n\to\infty} R_n(x) = 0$, then we say that $f(x)$ can be expanded as a **Taylor series** about $x = a$ and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Infinite series and discussions/tests concerning their convergence are covered in the Analysis I course.

Note that the n^{th} order Taylor polynomial is obtained by taking just the first $(n+1)$ terms of the Taylor series (where we count terms even if they happen to be zero).

Eg. From our earlier calculation we have that the Taylor series of e^x is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

where $\cdots$ denotes an infinite number of terms with increasing powers of x.

Note that if we set $x = 1$ in the above Taylor series then we obtain a formula for Euler's number

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Eg. Calculate the Taylor series of $\sin x$.
$f(x) = \sin x,\ f(0) = 0, \quad f'(x) = \cos x,\ f'(0) = 1,$
$f''(x) = -\sin x,\ f''(0) = 0, \quad f'''(x) = -\cos x,\ f'''(0) = -1,$
$f^{(4)}(x) = \sin x$ so now the pattern repeats and we see that for all non-negative integers k we have that $f^{2k}(0) = 0$ and $f^{2k+1}(0) = (-1)^k$.
The Taylor series of $\sin x$ is therefore given by

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Observe that only odd powers of x appear in the Taylor series of $\sin x$ as this is an odd function.

A similar calculation (exercise) yields the Taylor series of $\cos x$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

in which only even powers of x appear, as $\cos x$ is an even function.

To calculate the Taylor series of $\sinh x$ observe that for all non-negative integer k we have that $f(x) = \sinh x$ satisfies

$$f^{(k)}(x) = \begin{cases} \sinh x & \text{if } k \text{ is even} \\ \cosh x & \text{if } k \text{ is odd} \end{cases}$$

thus $f^{(2k)}(0) = 0$ and $f^{(2k+1)}(0) = 1$ giving

$$\sinh x = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

A similar calculation (exercise) yields the Taylor series of $\cosh x$

$$\cosh x = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

These two results can also be obtained directly from the Taylor series of e^x by noting that $\cosh x$ and $\sinh x$ are the even and odd parts of e^x.

As $\log x$ is not defined at $x = 0$ it does not make sense to consider the Taylor series of $\log x$ about $x = 0$. Instead we consider the Taylor series of $\log x$ about $x = 1$.
$f(x) = \log x,\ f(1) = 0, \quad f'(x) = \frac{1}{x},\ f'(1) = 1,$
$f''(x) = -\frac{1}{x^2},\ f''(1) = -1, \quad f'''(x) = \frac{2}{x^3},\ f'''(1) = 2,$
$f^{(4)}(x) = -\frac{3{\cdot}2}{x^4},\ f^{(4)}(1) = -3{\cdot}2$ and in general for k a positive integer $f^{(k)}(x) = (-1)^{k-1}\frac{(k-1)!}{x^k},\ f^{(k)}(1) = (-1)^{k-1}(k-1)!$
Thus the Taylor series of $\log x$ about $x = 1$ is

$$\log x = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{k!}(x-1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k$$

This result is more often expressed in terms of the variable $X = x - 1$, when it becomes $\log(1+X) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} X^k$.
If we now rename the variable X as x we get

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

This result is not valid at $x = -1$, and this is to be expected because the logarithm is not defined when its argument is zero. A careful examination of the requirement $\lim_{n\to\infty} R_n(x) = 0$ reveals that x must satisfy $-1 < x \leq 1$ for the above Taylor series to be valid (ie. for the series to converge).

8.3 Lagrange form for the remainder

There is a more convenient expression for the remainder term in Taylor's theorem. The **Lagrange form for the remainder** is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}, \qquad \text{for some } \; c \in (a,x).$$

To prove this expression for the remainder we will first need to prove the following lemma:

Lemma
Let $h(t)$ be differentiable $n+1$ times on $[a,x]$ with $h^{(k)}(a) = 0$ for $0 \le k \le n$ and $h(x) = 0$.
Then $\exists\, c \in (a,x)$ s.t. $h^{(n+1)}(c) = 0$.

Proof:
$h(a) = 0 = h(x)$ so by Rolle's theorem $\exists\, c_1 \in (a,x)$ s.t. $h'(c_1) = 0$.
$h'(a) = 0 = h'(c_1)$ so by Rolle's theorem $\exists\, c_2 \in (a,c_1)$ s.t. $h''(c_1) = 0$.
Repeating this argument a total of n times we arrive at
$h^{(n)}(a) = 0 = h^{(n)}(c_n)$ so by Rolle's theorem $\exists\, c_{n+1} \in (a,c_n)$ s.t. $h^{(n+1)}(c_{n+1}) = 0$.
This proves the lemma with $c = c_{n+1} \in (a,x)$.

Proof of the Lagrange form of the remainder:
Consider the function

$$h(t) = (f(t) - P_n(t))(x-a)^{n+1} - (f(x) - P_n(x))(t-a)^{n+1}.$$

By construction $h(x) = 0$.
Also $\frac{d^k}{dt^k}(t-a)^{n+1}$ is zero when evaluated at $t = a$ for $0 \le k \le n$. Furthermore, by definition of the Taylor polynomial $P_n(t)$ we have that $f^{(k)}(a) = P_n^{(k)}(a)$ for $0 \le k \le n$.
Hence $h^{(n)}(a) = 0$ for $0 \le k \le n$.
$h(t)$ therefore satisfies the conditions of the lemma and we have that
$\exists\, c \in (a,x)$ s.t. $h^{(n+1)}(c) = 0$.
As $P_n(t)$ is a polynomial of degree n then $P_n^{(n+1)}(t) = 0$.
Also, $\frac{d^{n+1}}{dt^{n+1}}(t-a)^{n+1} = (n+1)!$ hence

$$0 = h^{(n+1)}(c) = (x-a)^{n+1} f^{(n+1)}(c) - (n+1)!(f(x) - P_n(x))$$

Rearranging this expression gives the required result

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

One use of the Lagrange form of the remainder is to provide an upper bound on the error of a Taylor polynomial approximation to a function.

Suppose that $|f^{(n+1)}(t)| \leq M, \quad \forall\, t$ in the closed interval between a and x.
Then $|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$ provides a bound on the error.

Eg. Show that the error in approximating e^x by its 6^{th} order Taylor polynomial is always less than 0.0006 throughout the interval $[0, 1]$.

In this example $f(x) = e^x$ and $n = 6$ so we first require an upper bound on $|f^{(7)}(t)| = |e^t|$ for $t \in [0, 1]$. As e^t is monotonic increasing and positive then $|e^t| \leq e^1 = e < 3$. Thus, in the above notation, we may take $M = 3$.
As $a = 0$ we now have that $|R_6(x)| < \frac{3|x|^7}{7!} \leq \frac{3}{7!}$ for $x \in [0, 1]$.
Evaluating $\frac{3}{7!} = \frac{1}{1680} < 0.0006$ and the required result has been shown.

8.4 Calculating limits using Taylor series

**** Only use this method in assignments/exam questions when told to do so ****

Defn: Let n be a positive integer.
We say that $f(x) = o(x^n)$ (as $x \to 0$) if $\lim_{x\to 0} \frac{f(x)}{x^n} = 0$.

In particular, if α is any non-zero constant then $\alpha x^m = o(x^n)$ iff $m > n$.

Egs. $3x^4 = o(x^3),\ 3x^4 = o(x^2),\ 3x^8 - x^6 = o(x^5),\ o(x^8) + o(x^5) = o(x^5)$.

If we know that a Taylor series is valid in a suitable open interval, then it may be useful in calculating certain limits.
It can be shown that the Taylor series we have already seen for $\sin x$ and $\cos x$ are valid for all $x \in \mathbb{R}$. Although we have not proved this result you may assume that it is true and use it in calculating limits. As examples, we shall now calculate the two important trigonometric limits that we saw earlier.

Eg. Use Taylor series to calculate $\lim_{x\to 0} \frac{\sin x}{x}$.
From the Taylor series of $\sin x$ we have that $\sin x = x - \frac{x^3}{3!} + o(x^4)$. Hence

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + o(x^4)}{x} = \lim_{x\to 0}(1 - \frac{x^2}{3!} + o(x^3)) = \lim_{x\to 0}(1 + o(x)) = 1.$$

Eg. Use Taylor series to calculate $\lim_{x\to 0} \frac{1-\cos x}{x}$.
From the Taylor series of $\cos x$ we have that $\cos x = 1 - \frac{x^2}{2} + o(x^3)$. Hence

$$\lim_{x\to 0} \frac{1 - \cos x}{x} = \lim_{x\to 0} \frac{1 - 1 + \frac{x^2}{2} + o(x^3)}{x} = \lim_{x\to 0} \frac{\frac{x^2}{2} + o(x^3)}{x} = \lim_{x\to 0} \left(\frac{x}{2} + o(x^2)\right) = 0.$$