21 *Compute the divergence, $\nabla \cdot \mathbf{A}$, of the following vector fields:*

(a) $\mathbf{A}(x,y,z) \;=\; yz\mathbf{e_1} + xz\mathbf{e_2} + xy\mathbf{e_3}$,

Solution:

$$\nabla \cdot \mathbf{A} = \frac{\partial(yz)}{\partial x} + \frac{\partial(xz)}{\partial y} + \frac{\partial(xy)}{\partial z} = 0 + 0 + 0 = 0\,.$$

(b) $\mathbf{A}(x,y,z) \;=\; (x^2+y^2+z^2)\,(3\mathbf{e_1} + 4\mathbf{e_2} + 5\mathbf{e_3})$,

Solution:

$$\nabla\cdot\mathbf{A} = \frac{\partial(3(x^2+y^2+z^2))}{\partial x} + \frac{\partial(4(x^2+y^2+z^2))}{\partial y} + \frac{\partial(5(x^2+y^2+z^2))}{\partial z} = 6x+8y+10z\,.$$

Alternatively write $\mathbf{A} = |\mathbf{x}|^2\mathbf{B}$ *with* $\mathbf{B} = 3\mathbf{e_1} + 4\mathbf{e_2} + 5\mathbf{e_3}$ *a constant vector and use* $\nabla\cdot(|\mathbf{x}|^2\mathbf{B}) = (\nabla(|\mathbf{x}|^2))\cdot\mathbf{B} + |\mathbf{x}|^2\nabla\cdot\mathbf{B}$ *with*

$$\begin{aligned}\nabla(|\mathbf{x}|^2)) &= \mathbf{e_1}\frac{\partial(x^2+y^2+z^2)}{\partial x} + \mathbf{e_2}\frac{\partial(x^2+y^2+z^2)}{\partial y} + \mathbf{e_3}\frac{\partial(x^2+y^2+z^2)}{\partial z}\\ &= \mathbf{e_1}2x + \mathbf{e_2}2y + \mathbf{e_3}2z = 2\mathbf{x}\end{aligned}$$

and $\nabla\cdot\mathbf{B} = 0$.

(c) $\mathbf{A}(x,y,z) \;=\; (x+y)\mathbf{e_1} + (y+z)\mathbf{e_2} + (z+x)\mathbf{e_3}$.

Solution:

$$\nabla \cdot \mathbf{A} = \frac{\partial(x+y)}{\partial x} + \frac{\partial(y+z)}{\partial y} + \frac{\partial(z+x)}{\partial z} = 1 + 1 + 1 = 3\,.$$

22 *Compute the curl, $\nabla\times\mathbf{A}$, of each of the vector fields, $\mathbf{A}$, in the previous question.*

Solution:

(a) $\mathbf{A}(x,y,z) \;=\; yz\mathbf{e_1} + xz\mathbf{e_2} + xy\mathbf{e_3}$,

$$\begin{aligned}\nabla\times\mathbf{A} &= \mathbf{e_1}\left(\frac{\partial(xy)}{\partial y} - \frac{\partial(xz)}{\partial z}\right) + \mathbf{e_2}\left(\frac{\partial(yz)}{\partial z} - \frac{\partial(xy)}{\partial x}\right) + \mathbf{e_3}\left(\frac{\partial(xz)}{\partial x} - \frac{\partial(yz)}{\partial y}\right)\\ &= \mathbf{e_1}(x-x) + \mathbf{e_2}(y-y) + \mathbf{e_3}(z-z) = \mathbf{0}\,.\end{aligned}$$

(b) $\mathbf{A}(x,y,z) \;=\; (x^2+y^2+z^2)\,(3\mathbf{e_1} + 4\mathbf{e_2} + 5\mathbf{e_3})$,

$$\begin{aligned}\nabla\times\mathbf{A} &= \mathbf{e_1}\left(\frac{\partial(5(x^2+y^2+z^2))}{\partial y} - \frac{\partial(4(x^2+y^2+z^2))}{\partial z}\right)\\ &+\mathbf{e_2}\left(\frac{\partial(3(x^2+y^2+z^2))}{\partial z} - \frac{\partial(5(x^2+y^2+z^2))}{\partial x}\right)\\ &+\mathbf{e_3}\left(\frac{\partial(4(x^2+y^2+z^2))}{\partial x} - \frac{\partial(3(x^2+y^2+z^2))}{\partial y}\right)\\ &= \mathbf{e_1}(10y-8z) + \mathbf{e_2}(6z-10x) + \mathbf{e_3}(8x-6y)\,.\end{aligned}$$

Alternatively write $\mathbf{A} = |\mathbf{x}|^2\mathbf{B}$ with $\mathbf{B} = 3\mathbf{e_1}+4\mathbf{e_2}+5\mathbf{e_3}$ a constant vector and use $\nabla\times(|\mathbf{x}|^2\mathbf{B}) = (\nabla(|\mathbf{x}|^2))\times\mathbf{B} + |\mathbf{x}|^2\nabla\times\mathbf{B}$ with $\nabla(|\mathbf{x}|^2)) = 2\mathbf{x}$ and $\nabla\times\mathbf{B} = 0$.

(c) $\mathbf{A}(x,y,z) \;=\; (x+y)\mathbf{e_1} + (y+z)\mathbf{e_2} + (z+x)\mathbf{e_3}.$

$$\nabla \times \mathbf{A} =$$
$$\mathbf{e_1}\left(\frac{\partial(z+x)}{\partial y} - \frac{\partial(y+z)}{\partial z}\right) + \mathbf{e_2}\left(\frac{\partial(x+y)}{\partial z} - \frac{\partial(z+x)}{\partial x}\right) + \mathbf{e_3}\left(\frac{\partial(y+z)}{\partial x} - \frac{\partial(x+y)}{\partial y}\right)$$
$$= \mathbf{e_1}(0-1) + \mathbf{e_2}(0-1) + \mathbf{e_3}(0-1) = -(\mathbf{e_1}+\mathbf{e_2}+\mathbf{e_3})\,.$$

23 *If $f(r)$ is a differentiable function of $r = |\mathbf{x}\,|$, for $\mathbf{x} \in \mathbb{R}^n,\; r \neq 0$, show that*

(a) $\operatorname{grad} f(r) \;=\; f'(r)\,\mathbf{x}\,/r,$

Solution:

$$\operatorname{grad} f(r) = \mathbf{e_1}\frac{\partial f(r)}{\partial x} + \mathbf{e_2}\frac{\partial f(r)}{\partial y} + \mathbf{e_3}\frac{\partial f(r)}{\partial z}$$
$$= \mathbf{e_1}\frac{df}{dr}\frac{\partial r}{\partial x} + \mathbf{e_2}\frac{df}{dr}\frac{\partial r}{\partial y} + \mathbf{e_3}\frac{df}{dr}\frac{\partial r}{\partial z}\,.$$

Now $r = \sqrt{x^2+y^2+z^2}$ so

$$\frac{\partial r}{\partial x} = \frac{1}{2}\frac{2x}{r} = \frac{x}{r}$$

and similarly

$$\frac{\partial r}{\partial y} = \frac{y}{r}\,, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

so

$$\operatorname{grad} f(r) = \frac{df}{dr}\left(\mathbf{e_1}\frac{x}{r} + \mathbf{e_2}\frac{y}{r} + \mathbf{e_3}\frac{z}{r}\right) = f'(r)\mathbf{x}\,/r\,.$$

(b) $\operatorname{curl}[f(r)\mathbf{x}] \;=\; 0$*, where now we let $n = 3$.*

Solution: *Using $\operatorname{curl}(f\mathbf{V}) = (\nabla f) \times \mathbf{V} + f\operatorname{curl}\mathbf{V}$, the result to part (a) and*

$$\operatorname{curl}\mathbf{x} \;= \mathbf{e_1}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \mathbf{e_2}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \mathbf{e_3}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = \mathbf{0}\,,$$

gives $\operatorname{curl}(f\mathbf{x}) = (\nabla f) \times \mathbf{x} + f\operatorname{curl}\mathbf{x} \;= f'(r)\frac{\mathbf{x}}{r} \times \mathbf{x} + \mathbf{0} = \mathbf{0}$ *since* $\mathbf{x} \times \mathbf{x} \;= \mathbf{0}$.

24 *Let $\mathbf{x}$ be the position vector in three dimensions, with $r = |\mathbf{x}\,|$, and let $\mathbf{a}$ be a constant vector. Show that*

(a) $\operatorname{div}\mathbf{x} \;=\; 3,$

Solution:

$$\nabla \cdot \mathbf{x} \;= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3\,.$$

(b) $\operatorname{curl}\mathbf{x} \;= 0,$

Solution:

$$\operatorname{curl}\mathbf{x} \;= \mathbf{e_1}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \mathbf{e_2}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \mathbf{e_3}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = \mathbf{0}\,.$$

(c) $\quad \operatorname{grad} r \;=\; \mathbf{x}/r$,

Solution:

$$\frac{\partial r}{\partial x} = \frac{1}{2}\frac{2x}{r} = \frac{x}{r}\,, \quad \frac{\partial r}{\partial y} = \frac{y}{r}\,, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

so

$$\operatorname{grad} r \;=\; \mathbf{e_1}x/r + \mathbf{e_2}y/r + \mathbf{e_3}z/r \;=\; \mathbf{x}/r\,.$$

(d) $\quad \operatorname{div}(r^n\mathbf{x}) \;=\; (n+3)\,r^n$,

Solution: *Use* $\operatorname{div}(f\mathbf{V}) \;=\; (\nabla f)\cdot\mathbf{V} + f\operatorname{div}\mathbf{V}$ *with* $f(\mathbf{x}) \;=\; r^n$ *so that using 21 (a)* $\nabla f = nr^{n-1}\mathbf{x}/r = nr^{n-2}\mathbf{x}$ *and* $\mathbf{V} = \mathbf{x}$ *so*

$$\operatorname{div}(r^n\mathbf{x}) \;=nr^{n-2}\mathbf{x}\cdot\mathbf{x} + r^n\operatorname{div}\mathbf{x} = nr^n + 3r^n = (n+3)r^n\,.$$

(e) $\quad \operatorname{grad}(\mathbf{a}\cdot\mathbf{x}) \;=\; \mathbf{a}$,

Solution: $\mathbf{a}\cdot\mathbf{x} \;= a_1x + a_2y + a_3z$ *so*

$$\frac{\partial(\mathbf{a}\cdot\mathbf{x})}{\partial x} = a_1, \quad \frac{\partial(\mathbf{a}\cdot\mathbf{x})}{\partial y} = a_2, \quad \frac{\partial(\mathbf{a}\cdot\mathbf{x})}{\partial z} = a_3$$

hence $\operatorname{grad}(\mathbf{a}\cdot\mathbf{x}) \;=\; a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3} \;=\; \mathbf{a}\,.$

(f) $\quad \operatorname{div}(\mathbf{a}\times\mathbf{x}) \;= 0$,

Solution: $\mathbf{a}\times\mathbf{x} \;= \mathbf{e_1}(a_2z - a_3y) + \mathbf{e_2}(a_3x - a_1z) + \mathbf{e_3}(a_1y - a_2x)$ *so*

$$\operatorname{div}(\mathbf{a}\times\mathbf{x}) \;=\; \frac{\partial(a_2z - a_3y)}{\partial x} + \frac{\partial(a_3x - a_1z)}{\partial y} + \frac{\partial(a_1y - a_2x)}{\partial z} = 0\,.$$

(g) $\quad \operatorname{curl}(\mathbf{a}\times\mathbf{x}) \;=\; 2\,\mathbf{a}$,

Solution:

$$\operatorname{curl}(\mathbf{a}\times\mathbf{x}) = \mathbf{e_1}\left(\frac{\partial(a_1y - a_2x)}{\partial y} - \frac{\partial(a_3x - a_1z)}{\partial z}\right)$$

$$+\mathbf{e_2}\left(\frac{\partial(a_2z - a_3y)}{\partial z} - \frac{\partial(a_1y - a_2x)}{\partial x}\right) + \mathbf{e_3}\left(\frac{\partial(a_3x - a_1z)}{\partial x} - \frac{\partial(a_2z - a_3y)}{\partial y}\right)$$

$$= \mathbf{e_1}(a_1 + a_1) + \mathbf{e_2}(a_2 + a_2) + \mathbf{e_3}(a_3 + a_3) = 2\mathbf{a}\,.$$

(h) $\quad \operatorname{curl}(r^2\mathbf{a}) \;=\; 2\,(\mathbf{x}\times\mathbf{a})$,

Solution: *Use* $\operatorname{curl}(f\mathbf{V}) \;=\; (\operatorname{grad} f)\times\mathbf{V} + f\operatorname{curl}\mathbf{V}$ *with* $f \;=\; r^2 \;=\; |\mathbf{x}|^2$ *and* $\nabla(|\mathbf{x}|^2) = 2\mathbf{x}$ *so that* $\operatorname{curl}(r^2\mathbf{a}) \;= 2\mathbf{x}\times\mathbf{a} + r^2\operatorname{curl}\mathbf{a} = 2\mathbf{x}\times\mathbf{a}\,.$

(i) $\quad \nabla^2(1/r) \;=\; 0 \;\;, \quad \text{if } r \neq 0$,

Solution: $\nabla^2(1/r) \;\equiv\; \nabla\cdot\nabla(1/r)$*. From 21 (a) with* $f \;= 1/r$ *we have* $\nabla(1/r) = -\mathbf{x}/r^3$*. Then from 22 (d)* $\nabla\cdot(\mathbf{x}/r^3) \;=\; (-3+3)/r^3 \;=\; 0$ *so* $\nabla^2(1/r) \;=\; 0$*, but this is only valid for* $r \neq 0$ *since the calculation involves division by* r*.*

(j) $\quad \nabla^2(\log r) \;=\; 1/r^2\,, \quad \text{if } r \neq 0$,

Solution: $\nabla^2(\log r) \equiv \nabla \cdot \nabla(\log r)$. *From 21 (a) with* $f = \log r$, $\nabla(\log r) = \mathbf{x}/r^2$. *From 22 (d)* $\nabla \cdot (\mathbf{x}/r^2) = (-2+3)/r^3 = 1/r^2$ *so* $\nabla^2(\log r) = 1/r^2$*, but as in part (i) this is only valid for* $r \neq 0$ *since the calculation involves division by* r.

(k) $\quad \operatorname{div}[(\mathbf{a}\cdot\mathbf{x})\mathbf{x}] = 4\,\mathbf{a}\cdot\mathbf{x}$,

Solution: *Using* $\operatorname{div} f\mathbf{V} = (\nabla f)\cdot\mathbf{V} + f\operatorname{div}\mathbf{V}$*, with* $f = \mathbf{a}\cdot\mathbf{x}$ *so* $\nabla f = \mathbf{a}$ *from part (e) and* $\mathbf{V} = \mathbf{x}$ *so* $\operatorname{div}\mathbf{x} = 3$ *from part (a) hence* $\operatorname{div}[(\mathbf{a}\cdot\mathbf{x})\mathbf{x}] = \mathbf{a}\cdot\mathbf{x} + \mathbf{a}\cdot\mathbf{x}\,3 = 4\mathbf{a}\cdot\mathbf{x}$.

(l) $\quad \operatorname{div}[\mathbf{x}\times(\mathbf{x}\times\mathbf{a})] = 2\,\mathbf{a}\cdot\mathbf{x}$,

Solution: *Use* $\mathbf{x}\times(\mathbf{x}\times\mathbf{a}) = \mathbf{x}(\mathbf{x}\cdot\mathbf{a}) - \mathbf{a}(\mathbf{x}\cdot\mathbf{x})$ *and then from part (k) use* $\operatorname{div}(\mathbf{x}(\mathbf{x}\cdot\mathbf{a})) = 4\mathbf{x}\cdot\mathbf{a}$ *and*

$$\nabla\cdot(|\mathbf{x}|^2\mathbf{a}) = (\nabla(|\mathbf{x}|^2))\cdot\mathbf{a} + |\mathbf{x}|^2\nabla\cdot\mathbf{a} = 2\mathbf{x}\cdot\mathbf{a}$$

so

$$\operatorname{div}[\mathbf{x}\times(\mathbf{x}\times\mathbf{a})] = \operatorname{div}(\mathbf{x}(\mathbf{x}\cdot\mathbf{a})) - \nabla\cdot(|\mathbf{x}|^2\mathbf{a}) = 4\mathbf{x}\cdot\mathbf{a} - 2\mathbf{x}\cdot\mathbf{a}\,.$$

(m) $\quad \operatorname{curl}(\mathbf{a}\times\mathbf{x}/r^3) = 3(\mathbf{a}\cdot\mathbf{x})\mathbf{x}/r^5 - \mathbf{a}/r^3$,

Solution:

$$\operatorname{curl}\left(\mathbf{a}\times\mathbf{x}/r^3\right) = \left(\operatorname{curl}(\mathbf{a}\times\mathbf{x})\right)/r^3 + (\nabla r^{-3})\times(\mathbf{a}\times\mathbf{x})$$

Using part (g) $\operatorname{curl}(\mathbf{a}\times\mathbf{x}) = 2\mathbf{a}$ *together with* $\mathbf{x}\times(\mathbf{a}\times\mathbf{x}) = \mathbf{a}(\mathbf{x}\cdot\mathbf{x}) - \mathbf{x}(\mathbf{a}\cdot\mathbf{x})$ *and* $\nabla r^{-3} = -3\mathbf{x}/r^5$ *(from 21 (a)) gives*

$$\operatorname{curl}\left(\mathbf{a}\times\mathbf{x}/r^3\right) = \frac{2\mathbf{a}}{r^3} - \frac{3}{r^5}(\mathbf{a}r^2 - \mathbf{x}(\mathbf{a}\cdot\mathbf{x})) = 3(\mathbf{a}\cdot\mathbf{x})\mathbf{x}/r^5 - \mathbf{a}/r^3\,.$$

(n) $\quad$ *Exam question June 2002 (Section A): calculate the curl of* $(\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}$.

Solution:

$$\operatorname{curl}((\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}) = (\nabla(\mathbf{a}\cdot\mathbf{x}))\times\mathbf{x} + \mathbf{a}\cdot\mathbf{x}\operatorname{curl}\mathbf{x} = \mathbf{a}\times\mathbf{x}$$

using part parts (e) and (b).

25 *If* $\mathbf{x}$ *is the position vector,* $\mathbf{x} = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$, $\mathbf{a}$ *is a constant vector,* $\mathbf{F} = (\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}$ *and* $\mathbf{G} = r^2\mathbf{a}$*, (with* $r = |\mathbf{x}|$*), show that*

(a) $\quad \operatorname{div}\mathbf{F} = 2\operatorname{div}\mathbf{G} = 4\,\mathbf{a}\cdot\mathbf{x}$,

Solution: *Using the product rule*

$$\operatorname{div}\mathbf{F} = \nabla\cdot((\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}) = (\nabla(\mathbf{a}\cdot\mathbf{x}))\cdot\mathbf{x} + (\mathbf{a}\cdot\mathbf{x})\,\nabla\cdot\mathbf{x}$$

As in 24 (e), $\mathbf{a}\cdot\mathbf{x} = a_1x + a_2y + a_3z$ *so*

$$\frac{\partial(\mathbf{a}\cdot\mathbf{x})}{\partial x} = a_1, \quad \frac{\partial(\mathbf{a}\cdot\mathbf{x})}{\partial y} = a_2, \quad \frac{\partial(\mathbf{a}\cdot\mathbf{x})}{\partial z} = a_3$$

hence $\operatorname{grad}(\mathbf{a}\cdot\mathbf{x}) \;=\; a_1\mathbf{e_1}+a_2\mathbf{e_2}+a_3\mathbf{e_3} \;=\; \mathbf{a}$ *and as in 24 (a)*

$$\nabla\cdot\mathbf{x} = \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z} = 1+1+1 = 3\,,$$

so

$$\operatorname{div}\mathbf{F} = \nabla\cdot((\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}) = \mathbf{a}\cdot\mathbf{x} + 3\mathbf{a}\cdot\mathbf{x} = 4\mathbf{a}\cdot\mathbf{x}\;.$$

Using the product rule

$$\operatorname{div}\mathbf{G} = (\nabla(r^2))\cdot\mathbf{a} + r^2\nabla\cdot\mathbf{a}$$

and $\nabla(|\mathbf{x}\,|^2) = 2\mathbf{x}$, $\nabla\cdot\mathbf{a} = 0$ *so that* $\operatorname{div}\mathbf{G} = 2\,\mathbf{a}\cdot\mathbf{x}$.

(b) $\operatorname{curl}\mathbf{G} = -2\operatorname{curl}\mathbf{F} = 2\,\mathbf{x}\times\mathbf{a}$,

Solution: *Use* $\operatorname{curl}(f\mathbf{V}) = (\operatorname{grad} f)\times\mathbf{V} + f\operatorname{curl}\mathbf{V}$ *with* $f = r^2 = |\mathbf{x}\,|^2$ *and* $\nabla(|\mathbf{x}\,|^2) = 2\mathbf{x}$ *so that*

$$\operatorname{curl}\mathbf{G} = \operatorname{curl}(r^2\mathbf{a}) = 2\mathbf{x}\times\mathbf{a} + r^2\operatorname{curl}\mathbf{a} = 2\mathbf{x}\times\mathbf{a}\,.$$

Now use this product rule with $f = \mathbf{a}\cdot\mathbf{x}$ *and* $\mathbf{V} = \mathbf{x}$ *so that*

$$\nabla\times((\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}) = (\nabla(\mathbf{a}\cdot\mathbf{x}))\times\mathbf{x} + (\mathbf{a}\cdot\mathbf{x})\nabla\times\mathbf{x}$$

As in 24 (b)

$$\operatorname{curl}\mathbf{x} = \mathbf{e_1}\left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right) + \mathbf{e_2}\left(\frac{\partial x}{\partial z}-\frac{\partial z}{\partial x}\right) + \mathbf{e_3}\left(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right) = \mathbf{0}\,,$$

so

$$\nabla\times((\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}) = \mathbf{a}\times\mathbf{x} = -\mathbf{x}\times\mathbf{a}\,.$$

(c) $\operatorname{div}\operatorname{curl}\mathbf{F} = \operatorname{div}\operatorname{curl}\mathbf{G} = 0$,

Solution: $\operatorname{div}\operatorname{curl}\mathbf{F} = \operatorname{div}(\mathbf{a}\times\mathbf{x}) = 0$ *as in 24 (f):*
$\mathbf{a}\times\mathbf{x} = \mathbf{e_1}(a_2z - a_3y) + \mathbf{e_2}(a_3x - a_1z) + \mathbf{e_3}(a_1y - a_2x)$ *so*

$$\operatorname{div}(\mathbf{a}\times\mathbf{x}) = \frac{\partial(a_2z-a_3y)}{\partial x} + \frac{\partial(a_3x-a_1z)}{\partial y} + \frac{\partial(a_1y-a_2x)}{\partial z} = 0\,.$$

Since $\operatorname{curl}\mathbf{G} = -2\operatorname{curl}\mathbf{F}$ *this implies that* $\operatorname{div}\operatorname{curl}\mathbf{G} = 0$.

(d) $\operatorname{curl}\operatorname{curl}\mathbf{G} = -2\operatorname{curl}\operatorname{curl}\mathbf{F} = -4\,\mathbf{a}$.

Solution:

$$\operatorname{curl}\operatorname{curl}\mathbf{G} == 2\operatorname{curl}(\mathbf{x}\times\mathbf{a}) = -4\mathbf{a}$$

using $\mathbf{a}\times\mathbf{x} = -\mathbf{x}\times\mathbf{a}$ *and 22 (g)*

$$\operatorname{curl}(\mathbf{a}\times\mathbf{x}) = \mathbf{e_1}\left(\frac{\partial(a_1y-a_2x)}{\partial y} - \frac{\partial(a_3x-a_1z)}{\partial z}\right)$$

$$+\mathbf{e_2}\left(\frac{\partial(a_2z-a_3y)}{\partial z} - \frac{\partial(a_1y-a_2x)}{\partial x}\right) + \mathbf{e_3}\left(\frac{\partial(a_3x-a_1z)}{\partial x} - \frac{\partial(a_2z-a_3y)}{\partial y}\right)$$

$$= \mathbf{e_1}(a_1+a_1) + \mathbf{e_2}(a_2+a_2) + \mathbf{e_3}(a_3+a_3) = 2\mathbf{a}\,.$$

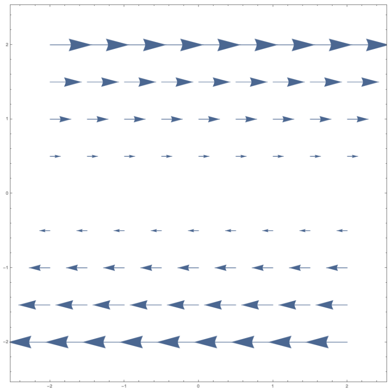
Since $\operatorname{curl}\mathbf{G} = -2\operatorname{curl}\mathbf{F}$ *this implies that*

$$\operatorname{curl}\operatorname{curl}\mathbf{G} = -2\operatorname{curl}\operatorname{curl}\mathbf{F} = -4\,\mathbf{a}\,.$$

26 *Exam question June 2001 (Section A):*

(a) *Give a representation of the vector function $\mathbf{A}(x, y) = y\mathbf{e_1}$ as a collection of arrows in the region of the (x, y)-plane bounded by $(x_1, y_1) = (-2, 2), (x_2, y_2) = (2, 2), (x_3, y_3) = (2, -2), (x_4, y_4) = (-2, -2)$.*

Solution:



The vector field $\mathbf{A}(x, y)$

(b) *Calculate the curl of the vector field $\mathbf{A}(x, y) = (-y\mathbf{e_1} + x\mathbf{e_2})/(x^2 + y^2)$ defined everywhere in the (x, y)-plane except at the origin. (You can consider $\mathbf{A}$ to be embedded in three dimensions, independent of z and with zero z component.)*

Solution:

$$\nabla\times\mathbf{A} = \mathbf{e_1}\left(\frac{\partial 0}{\partial y} - \frac{\partial}{\partial z}\frac{x}{x^2+y^2}\right) + \mathbf{e_2}\left(\frac{\partial}{\partial z}\frac{-y}{x^2+y^2} - \frac{\partial 0}{\partial x}\right) + \mathbf{e_3}\left(\frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2}\right)$$

$$= \mathbf{e_3}\left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right) = \underline{0}\,,$$

(c) *Give the unit vector normal to the surface of equation $ax + by = cz$, where a, b, c, are three real constants.*

Solution: *$ax+by = cz$ is the equation of a plane, the general form of which would be $\mathbf{x}\cdot\mathbf{V} = k$ where $\mathbf{V}$ is orthogonal to the plane. Here $\mathbf{V} = (a, b, -c)$, so a unit vector normal to the surface is $(a/\sqrt{a^2+b^2+c^2}, b/\sqrt{a^2+b^2+c^2}, -c/\sqrt{a^2+b^2+c^2})$. Alternatively: in general ∇f is normal to the level-surface $f = k$, here $f = ax + by - cz$ so $\nabla f = (a, b, -c)$ and again a unit vector normal to the surface is $\nabla f/|\nabla f| = (a/\sqrt{a^2+b^2+c^2}, b/\sqrt{a^2+b^2+c^2}, -c/\sqrt{a^2+b^2+c^2})$.*

(d) *(Slightly modified from exam) Let $\mathbf{x}$ be the position vector in 3-dimensions and $\mathbf{a}$ be a constant vector. Use the result $\mathbf{x}\times(\mathbf{x}\times\mathbf{a}) = \mathbf{x}(\mathbf{x}\cdot\mathbf{a}) - \mathbf{a}(\mathbf{x}\cdot\mathbf{x})$ to show that* $\operatorname{div}[\mathbf{x}\times(\mathbf{x}\times\mathbf{a})] = 2\mathbf{a}\cdot\mathbf{x}$.

Solution: *Using $\mathbf{x}\times(\mathbf{x}\times\mathbf{a}) = \mathbf{x}(\mathbf{x}\cdot\mathbf{a}) - \mathbf{a}(\mathbf{x}\cdot\mathbf{x})$, then* $\operatorname{div}[\mathbf{x}\times(\mathbf{x}\times\mathbf{a})] = \operatorname{div}(\mathbf{x}(\mathbf{x}\cdot\mathbf{a}) - \mathbf{a}(\mathbf{x}\cdot\mathbf{x})) = (\nabla\cdot\mathbf{x})(\mathbf{x}\cdot\mathbf{a}) + \mathbf{x}\cdot\nabla(\mathbf{x}\cdot\mathbf{a}) - \mathbf{a}\cdot\nabla(\mathbf{x}\cdot\mathbf{x})$. *This is* $3(\mathbf{x}\cdot\mathbf{a}) + (\mathbf{x}\cdot\mathbf{a}) - 2(\mathbf{x}\cdot\mathbf{a}) = 2(\mathbf{x}\cdot\mathbf{a})$

27 *Let $\underline{v}: \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Prove the vector identity*

$$\underline{v}\times(\underline{\nabla}\times\underline{v}) = \underline{\nabla}(|\underline{v}|^2/2) - (\underline{v}\cdot\underline{\nabla})\underline{v}.$$

Solution: We compute the curl of $\underline{v}$ as

$$\underline{\nabla} \times \underline{v} = \underline{e}_1(\partial_2 v_3 - \partial_3 v_2) + \underline{e}_2(\partial_3 v_1 - \partial_1 v_3) + \underline{e}_3(\partial_1 v_2 - \partial_2 v_1),$$

where we have used the shorthand $\partial_i \equiv \frac{\partial}{\partial x_i}$. The left hand side of the identity can therefore be written as

$$\begin{aligned}
\underline{v} \times (\underline{\nabla} \times \underline{v}) = \;& \underline{e}_1(v_2\,[\partial_1 v_2 - \partial_2 v_1] - v_3\,[\partial_3 v_1 - \partial_1 v_3]) \\
& + \underline{e}_2(v_3\,[\partial_2 v_3 - \partial_3 v_2] - v_1\,[\partial_1 v_2 - \partial_2 v_1]) \\
& + \underline{e}_3(v_1\,[\partial_3 v_1 - \partial_1 v_3] - v_2\,[\partial_2 v_3 - \partial_3 v_2]) \\
= \;& \underline{e}_1(v_2\partial_1 v_2 + v_3\partial_1 v_3 + v_1\partial_1 v_1 - v_1\partial_1 v_1 - v_2\partial_2 v_1 - v_3\partial_3 v_1) \\
& + \underline{e}_2(v_1\partial_2 v_1 + v_3\partial_2 v_3 + v_2\partial_2 v_2 - v_2\partial_2 v_2 - v_1\partial_1 v_2 - v_3\partial_3 v_2) \\
& + \underline{e}_3(v_1\partial_3 v_1 + v_2\partial_3 v_2 + v_3\partial_3 v_3 - v_3\partial_3 v_3 - v_1\partial_1 v_3 - v_2\partial_2 v_3) \\
= \;& \underline{e}_1(\partial_1|\underline{v}|^2/2 - (\underline{v}\cdot\underline{\nabla})v_1) + \underline{e}_2(\partial_2|\underline{v}|^2/2 - (\underline{v}\cdot\underline{\nabla})v_2) \\
& + \underline{e}_3(\partial_3|\underline{v}|^2/2 - (\underline{v}\cdot\underline{\nabla})v_3) \\
= \;& \underline{\nabla}(|\underline{v}|^2/2) - (\underline{v}\cdot\underline{\nabla})\underline{v},
\end{aligned}$$

proving the given identity.