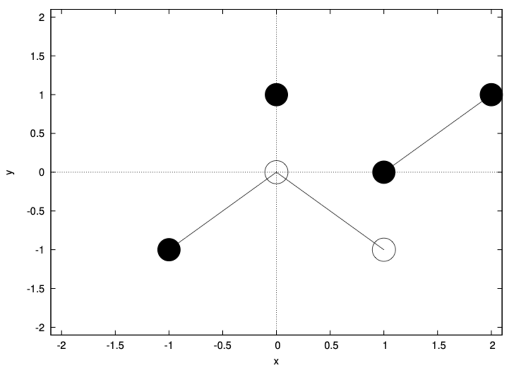
Calculus I, Chapter 2 Problems

Limits

Q1. Consider the given graph of the function $f(x)$. Are the following statements true or false?



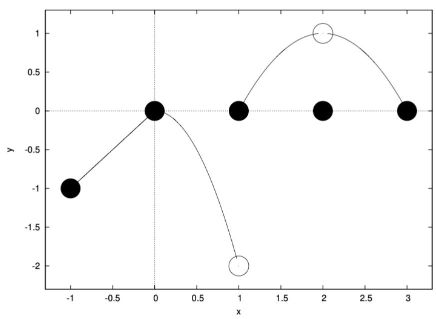
$f(x)$ for Q1

(a) $\lim_{x\to 0} f(x)$ exists, (b) $\lim_{x\to 0} f(x) = 0$, (c) $\lim_{x\to 0} f(x) = 1$
(d) $\lim_{x\to 1} f(x) = 1$, (e) $\lim_{x\to 1} f(x) = 0$, (f) $\lim_{x\to a} f(x)$ exists $\forall\ a \in (-1, 1)$.

Solution.

(a) true, (b) true, (c) false, (d) false, (e) false, (f) true.

Q2. Consider the given graph of the function $f(x)$. Are the following statements true or false?



$f(x)$ for Q2

(a) $\lim_{x\to 2} f(x)$ does not exist, (b) $\lim_{x\to 2} f(x) = 1$, (c) $\lim_{x\to 1} f(x)$ does not exist,
(d) $\lim_{x\to a} f(x)$ exists $\forall\ a \in (-1, 1)$ (e) $\lim_{x\to a} f(x)$ exists $\forall\ a \in (1, 3)$.

Solution.

(a) false, (b) true, (c) true, (d) true, (e) true.

Q3. If $f(x) > 0\ \forall\ x \neq a$ and $\lim_{x\to a} f(x) = L$, can we conclude that $L > 0$? Justify your answer.

Solution.

No. An example is provided by $f(x) = x^2$ with $a = 0$ so that $L = 0$ which is not positive.

Q4. Justify whether the following statement is true or false.

If $\lim_{x\to a} f(x)$ exists then so does $\lim_{x\to a} \sqrt{f(x)}$.

Solution. *False. An example is provided by $f(x) = -1$, with $a = 0$.*

Here $\lim_{x\to 0} f(x)$ exists (and is equal to -1) but $\sqrt{f(x)}$ is not a real function.

Q5. Calculate the following limits

(a) $\lim_{x\to 0}(2-x)$, (b) $\lim_{x\to -1} \frac{3x^2}{2x-1}$, (c) $\lim_{x\to \pi/2} x\sin x$, (d) $\lim_{x\to\pi} \frac{\cos x}{1-\pi}$.

Solution. *(a)* 2, *(b)* -1, *(c)* $\pi/2$, *(d)* $\frac{1}{\pi-1}$.

Q6. Calculate the following limits

(a) $\lim_{x\to 1} \frac{x^4-1}{x^3-1}$, (b) $\lim_{x\to 2} \frac{x^3-8}{x^4-16}$, (c) $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2}$, (d) $\lim_{x\to 4} \frac{4x-x^2}{2-\sqrt{x}}$.

Solution.

(a) $\lim_{x\to 1} \frac{x^4-1}{x^3-1} = \lim_{x\to 1} \frac{(x^2+1)(x+1)(x-1)}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{(x^2+1)(x+1)}{x^2+x+1} = 4/3.$

(b) $\lim_{x\to 2} \frac{x^3-8}{x^4-16} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x+2)(x-2)(x^2+4)} = \lim_{x\to 2} \frac{x^2+2x+4}{(x+2)(x^2+4)} = 3/8.$

(c) $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2} = \lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}+2)} = \lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{x+3-4} = 4.$

(d) $\lim_{x\to 4} \frac{4x-x^2}{2-\sqrt{x}} = \lim_{x\to 4} \frac{x(4-x)(2+\sqrt{x})}{(2-\sqrt{x})(2+\sqrt{x})} = \lim_{x\to 4}(x(2+\sqrt{x})) = 16.$

Q7. Calculate the limit as $x \to 0$ of the following

(a) $\frac{1-\cos x}{x^2}$, (b) $\frac{x^2}{1-\cos 2x}$, (c) $\frac{x^2}{1-\cos 4x}$.

Solution.

(a) $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \lim_{x\to 0} \left\{ \left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x} \right\} = 1/2.$

(b) $\lim_{x\to 0} \frac{x^2}{1-\cos 2x} = \lim_{x\to 0} \frac{x^2(1+\cos 2x)}{(1-\cos 2x)(1+\cos 2x)} = \lim_{x\to 0} \frac{(2x)^2(1+\cos 2x)}{4\sin^2 2x} = 1/2.$

(c) $\lim_{x\to 0} \frac{x^2}{1-\cos 4x} = \lim_{x\to 0} \frac{x^2(1+\cos 4x)}{(1-\cos 4x)(1+\cos 4x)} = \lim_{x\to 0} \frac{(4x)^2(1+\cos 4x)}{16\sin^2 4x} = 1/8.$

Q8. Does $\lim_{x\to 0} \frac{\sin(x+|x|)}{x}$ exist?

If the limit exists then find it.

Solution. *For* $x > 0$, $\frac{\sin(x+|x|)}{x} = \frac{\sin 2x}{x}$. *Hence* $\lim_{x\to 0^+} \frac{\sin(x+|x|)}{x} = \lim_{x\to 0^+} \frac{\sin 2x}{x} = \lim_{x\to 0^+} \frac{2\sin 2x}{2x} =$
For $x < 0$, $\frac{\sin(x+|x|)}{x} = 0$. *Hence* $\lim_{x\to 0^-} \frac{\sin(x+|x|)}{x} = 0$.

The left-sided and right-sided limits exist but are not equal, hence the limit does not exist.

Q9. In each case either evaluate the limit or state that no limit exists

(a) $\lim_{x\to 3} \frac{x^2+x+12}{x-3}$, (b) $\lim_{x\to 3} \frac{x^2+x-12}{x-3}$, (c) $\lim_{x\to 3} \frac{(x^2+x-12)^2}{x-3}$, (d) $\lim_{x\to 3} \frac{(x^2+x-12)}{(x-3)^2}$,

(e) $\lim_{h\to 0} \frac{1-1/h^2}{1+1/h^2}$, (f) $\lim_{h\to 0} \frac{1+1/h}{1+1/h^2}$.

Solution.

(a) no limit exists,

(b) $\lim_{x\to 3} \frac{x^2+x-12}{x-3} = \lim_{x\to 3} \frac{(x+4)(x-3)}{x-3} = 7$.

(c) $\lim_{x\to 3} \frac{(x^2+x-12)^2}{x-3} = \lim_{x\to 3} \frac{(x+4)^2(x-3)^2}{x-3} = \lim_{x\to 3} (x+4)^2(x-3) = 0$.

(d) $\frac{(x^2+x-12)}{(x-3)^2} = \frac{(x+4)(x-3)}{(x-3)^2} = \frac{x+4}{x-3}$ *hence no limit exists.*

(e) $\lim_{h\to 0} \frac{1-1/h^2}{1+1/h^2} = \lim_{h\to 0} \frac{h^2-1}{h^2+1} = -1$.

(f) $\lim_{h\to 0} \frac{1+1/h}{1+1/h^2} = \lim_{h\to 0} \frac{h^2+h}{h^2+1} = 0$.

Q10. Calculate the limit as $x \to \infty$ of the following

(a) $\frac{6x+7}{1-2x}$, (b) $\frac{x^2}{x^2+\sin^2 x}$.

Solution.

(a) $\lim_{x\to\infty} \frac{6x+7}{1-2x} = \lim_{x\to\infty} \frac{6+\frac{7}{x}}{\frac{1}{x}-2} = -3$.

(b) First note that $0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$.

As $\lim_{x\to\infty} \frac{1}{x^2} = 0$ *then by the pinching theorem* $\lim_{x\to\infty} \frac{\sin^2 x}{x^2} = 0$.

Thus $\lim_{x\to\infty} \frac{x^2}{x^2+\sin^2 x} = \lim_{x\to\infty} \frac{1}{1+\frac{\sin^2 x}{x^2}} = 1$.

Q11. Calculate the following limits

(a) $\lim_{x\to\infty} \frac{\cos(1/x)}{1+(1/x)}$, (b) $\lim_{x\to\infty} \left(\frac{1}{x}\right)^{1/x}$,

(c) $\lim_{x\to\infty} (3+\frac{2}{x})\cos(1/x)$, (d) $\lim_{x\to\infty} \{(\frac{3}{x^2} - \cos(1/x))(1+\sin(1/x))\}$.

Solution.

Set $u = 1/x$ *in each case*

(a) $\lim_{x\to\infty} \frac{\cos(1/x)}{1+(1/x)} = \lim_{u\to 0^+} \frac{\cos u}{1+u} = 1$.

(b) $\lim_{x\to\infty} \left(\frac{1}{x}\right)^{1/x} = \lim_{u\to 0^+} u^u = \lim_{u\to 0^+} \exp(\log u^u) = \lim_{u\to 0^+} \exp(u \log u) = e^0 = 1$.

(c) $\lim_{x\to\infty} (3+\frac{2}{x})\cos(1/x) = \lim_{u\to 0^+} (3+2u)\cos u = 3$.

(d) $\lim_{x\to\infty} (\frac{3}{x^2} - \cos(1/x))(1+\sin(1/x)) = \lim_{u\to 0^+} (3u^2 - \cos u)(1+\sin u) = -1$.

Q12. For each of the following statements, either give a proof that it is true or a counter example to show that it is false:

(a) If $g(x) > 0 \; \forall \; x > 0$ and $\lim_{x\to\infty}(f(x) - g(x)) = 0$ then $\lim_{x\to\infty}(f(x)/g(x)) = 1$.

(b) If $g(x) > 0 \; \forall \; x > 0$ and $\lim_{x\to\infty}(f(x)/g(x)) = 1$ then $\lim_{x\to\infty}(f(x) - g(x)) = 0$.

Solution.

(a) False. A counter example is provided by $f(x) = 1/x$ *and* $g(x) = 2/x$.

(b) False. A counter example is provided by $f(x) = x$ *and* $g(x) = x + 1$.

Q13. In each case either evaluate the limit or state that no limit exists

(a) $\lim_{u\to-5} \frac{u^2}{5-u}$, (b) $\lim_{y\to 0}(2y-8)^{1/3}$, (c) $\lim_{x\to 0} \frac{(x-2)(1-\cos 3x)}{2x}$, (d) $\lim_{t\to 5} \frac{t-5}{t^2-25}$,

(e) $\lim_{x\to-2} \frac{x+2}{\sqrt{x^2+5}-3}$, (f) $\lim_{x\to\infty} \frac{-3x^4+x^2+1}{-5x^4-1}$, (g) $\lim_{t\to 0} \frac{5t^3+8t^2}{3t^2-16t^4}$, (h) $\lim_{x\to 3} \frac{\tan(2(x-3))}{x-3}$,

(i) $\lim_{x\to-3} \frac{x+3}{x^2+4x+3}$, (j) $\lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2}$, (k) $\lim_{t\to 1} \frac{t^2+t-2}{t^2-1}$, (l) $\lim_{t\to-\infty} \frac{t^3+1}{t^2+1}$.

Solution.

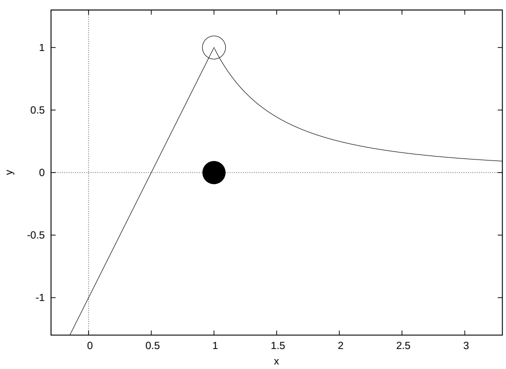
(a) $5/2$, *(b)* -2, *(c)* 0, *(d)* $1/10$, *(e)* $-3/2$, *(f)* $3/5$, *(g)* $8/3$, *(h)* 2, *(i)* $-1/2$, *(j)* $1/2$, *(k)* $3/2$, *(l) no limit exists.*

Continuous Functions

Q14. Sketch the graph of the function $f(x)$ and classify any discontinuities, where

$$f(x) = \begin{cases} 2x-1 & \text{if } x < 1 \\ 0 & \text{if } x = 1 \\ 1/x^2 & \text{if } x > 1 \end{cases}$$

Solution.

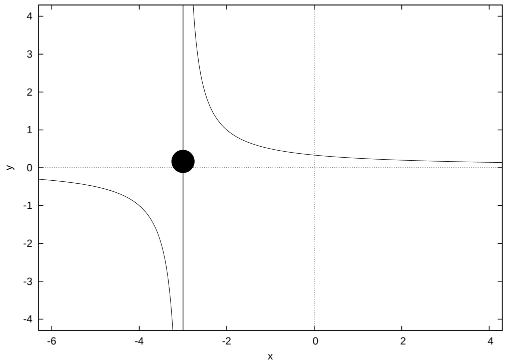


Solution for Q14. $f(x)$ with a removable discontinuity at $x = 1$.

Q15. Sketch the graph of the function $f(x)$ and classify any discontinuities, where

$$f(x) = \begin{cases} (x-3)/(x^2-9) & \text{if } x \neq \pm 3 \\ 1/6 & \text{if } x = \pm 3 \end{cases}$$

Solution.

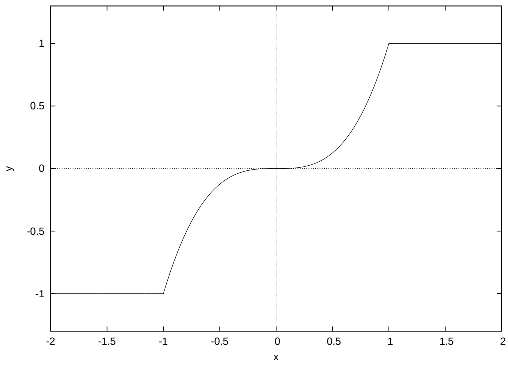


Solution for Q15. $f(x)$ with an infinite discontinuity at $x = -3$.

Q16. Sketch the graph of the function $f(x)$ and classify any discontinuities, where

$$f(x) = \begin{cases} -1 & \text{if } x < -1 \\ x^3 & \text{if } -1 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

Solution.



Solution for Q16. $f(x)$ with no discontinuities.