5.6 Summary: Double integration

You should have a good understanding of how to calculate double integrals and methods such as swapping the order of integration or changing variables. Here are some key points:

- The *double integral* of a function of two variables $f(x, y)$ over a region D is the definite integral $\int\int_D f(x, y)dxdy$ which can be defined as a limit of Riemann sums. This limit always exists if f is continuous on a closed region D. This has the interpretation as the (signed) volume between the surface $f(x, y)$ and the region D in the (x, y)-plane. If $f(x, y) = 1$ the integral gives the area of D.

- You should know how to calculate double integrals for x-*simple* and y-*simple* regions which are regions which can be divided into strips parallel to the x-axis or y-axis respectively with each strip being a single interval. You should know how to change the order of integration for regions which are both x-simple and y-simple – note that this process is trivial for rectangular regions but otherwise requires some care.

- Sometimes for a region which is both x-simple and y-simple we can only calculate the integral for one order of integration, sometimes both ways are possible but one way is easier.

- A change of variable from (x, y) to (u, v) can be performed but care must be taken so that the integration measure is correct, i.e. that the value of the integral doesn't depend on the choice of variables. This can be derived by ensuring that the double integral of 1 always gives the area of any region D. The result is $dxdy = |J|dudv$ where J is the *Jacobian* determinant $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$.

- A particular change of variables is from 2d Cartesian coordinates (x, y) to 2d polar coordinates (r, θ) where $dxdy = rdrd\theta$.

- Changes of variable may be motivated by the shape of the region D or by the form of the integrand but both factors will affect how easy or hard it is to calculate the integral.

- The 1d *Gaussian integral* $I = \int_{-\infty}^{\infty} e^{-ax^2}dx$ for $a > 0$ is an interesting example where we can square it to find a 2d Gaussian integral which is a double integral we can evaluate by changing to polar coordinates. This then gives the result for the original Gaussian integral $I = \sqrt{\pi/a}$ which we had no way to calculate as a single variable integral.