41 *For which values of (x, y) are the following continuous:*

(a) $x/(x^2 + y^2 + 1)$,

Solution: *Use the theorem that if f and g are continuous at a point then so are $f + g$, fg and f/g if g does not vanish there. Taking $f = x$ and $g = y$ all the functions can be built up by repeated applications of this result, so all the functions will be continuous whenever their denominators are non-zero so this function is continuous everywhere since $x^2 + y^2 + 1 > 0$,*

(b) $x/(x^2 + y^2)$,

Solution: *everywhere except the origin,*

(c) $(x + y)/(x - y)$,

Solution: *everywhere except the line $y = x$,*

(d) $x^3/(y - x^2)$?

Solution: *everywhere except the parabola $y = x^2$.*

42 *Let the scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as*

$$f(\underline{x}) = \begin{cases} 1 + \dfrac{x^2}{y} & y \neq 0, \\ 1 & y = 0. \end{cases}$$

(a) *Show that, along any straight line through the origin, $\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = f(\underline{0})$.*

(b) *Is $f(\underline{x})$ continuous at $\underline{0}$? Explain your answer, with reference to the first part of this question.*

Solution:

(a) *On a straight line through the origin, we have either $y = mx$ for some fixed $m \in \mathbb{R}$, or $x = 0$.*

On the line $x = 0$, the limit as $\underline{x} \to \underline{0}$ becomes the limit $y \to 0$, and so we have

$$\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = \lim_{y\to 0} \left(1 + \frac{0}{y}\right) = 1.$$

On the line $y = mx$, the limit as $\underline{x} \to \underline{0}$ becomes the limit $x \to 0$, and so we have

$$\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = \lim_{x\to 0} \left(1 + \frac{x^2}{mx}\right) = \lim_{x\to 0} \left(1 + \frac{x}{m}\right) = 1.$$

Since $f(\underline{0}) = 1$, we have shown that $\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = f(\underline{0})$ along any straight line throughout the origin.

(b) *In order for f to be continuous at $\underline{0}$ we must have $\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = f(\underline{0})$ as a limit in $\mathbb{R}^2$, not just when we restrict to straight lines through the origin. Since we have $f(\underline{0}) = 1$, we would require $\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = 1$, which means that for all $\epsilon > 0$, we could find a $\delta > 0$ such that*

$$|f(x) - 1| = \left|1 + \frac{x^2}{y} - 1\right| = \left|\frac{x^2}{y}\right| < \epsilon$$

for all $\underline{x}$ such that $0 < |\underline{x}| < \delta$. However, for any δ, the set of points $0 < |\underline{x}| < \delta$ will contain points with y much less than x^2 (sometimes written $y << x^2$), including points where $\left|\frac{x^2}{y}\right| > \epsilon$ for any fixed ϵ.

An alternative way to think about this is to consider $\lim_{\underline{x}\to\underline{0}} f(\underline{x})$ along the curve $y = x^2$. On this curve, the limit as $\underline{x} \to \underline{0}$ becomes the limit $x \to 0$, and so we have

$$\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = \lim_{x\to 0}\left(1 + \frac{x^2}{x^2}\right) = \lim_{x\to 0}(2) = 2.$$

Since this limit is not equal to 1, ϵ we cannot hope to find a δ, such that $|f(x)-1| < \epsilon$ for all $0 < |\underline{x}| < \delta$, for every $\epsilon > 0$.

43 *Let the scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as*

$$f(\underline{x}) = \begin{cases} \dfrac{x^2y^2}{x^4+y^4} & \underline{x} \neq \underline{0}, \\ 0 & \underline{x} = \underline{0}. \end{cases}$$

(a) *Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist at $\underline{0}$, and find their values at this point.*

(b) *Show that $\lim_{\underline{x}\to\underline{0}} f(\underline{x})$ does not exist, and hence $f(\underline{x})$ is not continuous at the origin. Comment on this in relation to the previous part of this question.*

Solution:

(a) Using the definitions of the partial derivatives as limits, at the origin we have

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0/h^4 - 0}{h} = 0$$
$$\frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0 - 0/h^4}{h} = 0,$$

and so both partials exist with value 0.

(b) In order for $f(\underline{x})$ to be continuous at the origin, $\lim_{\underline{x}\to\underline{0}} f(\underline{x})$ must exist and be equal to $f(\underline{0}) = 0$. However, considering $f(\underline{x})$ along the line $y = x$, the limit $\underline{x} \to \underline{0}$ becomes the limit $x \to 0$, and we have

$$\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = \lim_{x\to 0}\frac{x^4}{x^4+x^4} = \frac{1}{2} \neq 0.$$

Given $\epsilon > 0$, we therefore cannot hope to find a $\delta > 0$ such that $|f(\underline{x}) - 0| < \epsilon$ for all $0 < |\underline{x}| < \delta$. Therefore $\lim_{\underline{x}\to\underline{0}} f(\underline{x})$ is not equal to 0, and $f(\underline{x})$ is not continuous at $\underline{0}$.

By considering the limit along the line $y = 0$ (or similarly along $x = 0$) we can see that the limit as $\underline{x} \to \underline{0}$ becomes the limit $x \to 0$ (or $y \to 0$), and so we have

$$\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = \lim_{x\to 0}\frac{0}{x^4} = 0,$$

and since the limit along this line is not equal to the limit along the line $y = x$, the limit $\lim_{\underline{x}\to\underline{0}} f(\underline{x})$ cannot exist.

The partial derivatives exist at the origin even though the function f is not continuous at the origin. This is because the partial derivatives depend on the behaviour of the function as we move parallel to one of the coordinate axes, whereas continuity depends on the behaviour of the function in all directions.

44 *Which of the following sets are open?.*

(a) $\{(x, y, z) : x > 0\}$,

Solution: *An open set has to contain an open ball centred on each of its points, so this is open because for any point we can construct a sphere centred on that point and lying entirely within the set, even points arbitrarily close to the plane $x = 0$. To make this really precise, suppose $\underline{a} = (a_1, a_2, a_3)$ is in the set. Then $a_1 > 0$ and if we take $\delta = a_1/2 > 0$, then $B_\delta(\underline{a})$ is entirely in the set.*

To see this, consider $(x, y, z) = \underline{x} \in B_\delta(\underline{a})$. Since $\underline{x} \in B_\delta(\underline{a})$, $a_1 - \delta < x < a_1 + \delta$, so $\frac{a_1}{2} < x$. But then $x = x - \frac{a_1}{2} + \frac{a_1}{2} = x - \frac{a_1}{2} + \delta > 0$. So $\underline{x}$ is in the set.

Since this works for all points $\underline{a}$ in the set, the set is open.

(b) $\{(x, y, z) : y \geq 0\}$,

Solution: *this is not open because it contains points for which $y = 0$. Let $\underline{x}$ be such a point, and consider a sphere of radius $\delta > 0$ around $\underline{x}$. Then $\underline{x} - \delta\underline{e}_2 = (x, -\delta, z) \in B_\delta(\underline{x})$, and this point does not lie in the original set.*

(c) $\{(x, y, z) : 1 > (x^2 + y^2)/z\}$,

Solution: *$1 > (x^2 + y^2)/z$ implies $z > x^2 + y^2$ or $z < 0$, and this is open because for any point we can construct a sphere centred on that point and lying entirely within the set, even points arbitrarily close to the paraboloid $z = x^2 + y^2$ or to $z = 0$. (Beware: you might forget that points with $z < 0$ are also in the set!)*

(d) $\{(x, y, z) : 1 \geq (x^2 + y^2)/z\}$*?*

Solution: *this is not open because it contains points for which $z = x^2 + y^2$, and any sphere with such a point as its centre contains points outside the set.*

45 *Prove that an open ball, as defined in lectures, is an open set.*

Solution: Let $B_\delta(\underline{a})$ be the open ball centred at $\underline{a}$ with radius δ, and let $\underline{x}$ be any point in $B_\delta(\underline{a})$. We need to show that there's a δ' such that $B_{\delta'}(\underline{x}) \subset B_\delta(\underline{a})$. Since $\underline{x} \in B_\delta(\underline{a})$, $|\underline{x} - \underline{a}| < \delta$. Let $\delta' = \delta - |\underline{x} - \underline{a}|$. By the remark just made, $\delta' > 0$, and if $\underline{x}'$ is any point in $B_{\delta'}(\underline{x})$ then $|\underline{x}' - \underline{x}| < \delta'$ which implies $|\underline{x}' - \underline{a}| = |\underline{x}' - \underline{x} + \underline{x} - \underline{a}| \leq |\underline{x}' - \underline{x}| + |\underline{x} - \underline{a}|$ (by the triangle inequality). Hence $|\underline{x}' - \underline{a}| < \delta' + |\underline{x} - \underline{a}| = \delta$, and so $\underline{x}' \in B_\delta(\underline{a})$, which is what we needed to prove. (This all becomes much clearer if you draw a picture – basically we are just setting δ' equal to the distance from $\underline{x}$ to the edge of $B_\delta(\underline{a})$.)

46 *Prove that the intersection of two open sets, as defined in lectures, is another open set. (Note that the empty set is an open set: since it contains no points, the statement that every point in it sits inside an open ball which is also in the set is vacuously true.) What about the intersection of a finite number of open sets? And what about the intersection of an infinite number of open sets?*

Solution: Let the two open sets be S_1 and S_2, and put $S = S_1 \cap S_2$. If $S = \emptyset$ then there's nothing more to be done since $\emptyset$ is open. If $S \neq \emptyset$ then let $\underline{a}$ be any point in S. Then $\underline{a} \in S_1$, and since S_1 is open we can find a value of $\delta_1 > 0$ such that $B_{\delta_1}(\underline{a}) \subset S_1$; and similarly $\underline{a} \in S_2$, and we can find a value of $\delta_2 > 0$ such that $B_{\delta_2}(\underline{a}) \subset S_2$. Now set $\delta = \min(\delta_1, \delta_2)$; then $B_\delta(\underline{a})$ is a subset of both S_1 and S_2, which means that it is a subset of S. Since this works for *any* point $\underline{a} \in S$, this proves that S is open.

For the intersection S of a finite number of open sets S_1, S_2, $\ldots S_n$, the argument is much the same: either S is the empty set, or else for any point $\underline{a}$ in S we can take $\delta = \min(\delta_1, \delta_2, \ldots \delta_n)$ and show that $B_\delta(\underline{a})$ is in S.

However this doesn't necessarily work for an infinite intersection, as $\min(\delta_1, \delta_2, \ldots)$ might be zero if there are infinitely many δ_is. For example, if for $n = 1, 2, \ldots$ S_n is the open ball centred on the origin with radius $1/n$, then the interection of all the S_ns is the set containing the single point $\underline{0}$, which is *not* an open set.

47 *Exam question June 2014 (Section A):*

(a) *Give the definition of the open ball $B_\delta(\mathbf{a})$ with centre $\mathbf{a} \in \mathbb{R}^n$ and radius $\delta > 0$, and define what it means for a subset S of $\mathbb{R}^n$ to be open.*

(b) *Which of the following subsets of $\mathbb{R}^2$ are open? In each case, justify your answer in terms of the definition you gave in part (a).*

(i) $S_1 = \{(x, y) : x > 2\}$,

(ii) $S_2 = \{(x, y) : x > 2, y = 2\}$,

(iii) $S_3 = \{(x, y) : x > 2, y > 2\}$.

Solution:

(a) $B_\delta(\mathbf{a}) = \{\underline{x} \in \mathbb{R}^n : |\underline{x} - \underline{a}| < \delta\}$; a subset S of $\mathbb{R}^n$ is *open* if for each point $\underline{a} \in S$ there is an open ball $B_\delta(\underline{a})$ which is also in S (where δ might depend on $\underline{a}$).

(b) open, not open, open. (NB: 'not open' is not the same as 'closed'!) In each case some justification should be given. (Sketch for part (i): if $\underline{a} = (a_1, a_2) \in S_1$ then $a_1 - 2 > 0$. Let $\delta = a_1 - 2 > 0$, and consider $\underline{x} = (x, y) \in B_\delta(\underline{a})$. We want to show $\underline{x} \in S_1$, which means we need to show $x > 2$. We have $x - 2 = x - a_1 + a_1 - 2 = x - a_1 + \delta$. But $|x - a_1| < \delta \implies -\delta < x - a_1 < \delta$, so $x - 2 = x - a_1 + \delta > 0$, and hence $B_\delta(\underline{a}) \subset S_1$.)

48 *Exam question (last part) June 2014 (Section B): Determine the points of $\mathbb{R}^2$ at which the function $f(x, y) = |xy + x + y + 1|$ is*
(a) continuously differentiable; (b) differentiable.

Hint: *first factorise f.*

Solution: $f(x, y) = |xy + x + y + 1| = |(x + 1)(y + 1)|$ so $f(x, y) = (x + 1)(y + 1)$ for $(x + 1)(y + 1) > 0$ and $f(x, y) = -(x + 1)(y + 1)$ for $(x + 1)(y + 1) < 0$. Hence away from the lines $x = -1$ or $y = -1$, f is a polynomial in x and y and therefore has continuous partial derivatives, and hence is both continuously differentiable, and, by the theorem from lectures, differentiable in this region. It remains to consider the two lines

$x = -1$ and $y = -1$.
On the line $x = -1$, we have

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(-1+h, y) - f(-1, y)}{h} = \lim_{h \to 0} \frac{|h(y+1)|}{h} = \lim_{h \to 0} \frac{|h|}{h} |y+1| \, .$$

Hence for $y \neq -1$ the limit doesn't exist as $|h|/h = \pm 1$; while for $y = -1$ the limit exists and is zero. Also on the line $x = -1$ we have

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(-1, y+h) - f(-1, y)}{h} = 0 \, .$$

Similarly on the line $y = -1$, $\frac{\partial f}{\partial x} = 0$ while $\frac{\partial f}{\partial y}$ does not exist except at $x = -1$.
Since one or other partial derivative does not exist on the lines $x = -1$ and $y = -1$ away from $(x, y) = (-1, -1)$, f is neither continuously differentiable nor differentiable on these lines away from $(-1, -1)$; it is also not continuously differentiable at $(-1, -1)$ since the partial derivatives do not exist on these lines away from that point.
Finally we must ask whether f is differentiable at $\underline{a} = (-1, -1)$. Both partials of f are zero at that point, so *if* f were differentiable there, the remainder term would have to be $R(\underline{h}) = f(\underline{a} + \underline{h}) - f(\underline{a})$. Consider $\underline{h} = (h_1, h_2)$: then

$$\frac{R(\underline{h})}{|\underline{h}|} = \frac{f(-1 + h_1, -1 + h_2) - f(-1, -1)}{|\underline{h}|} = \frac{|h_1| h_2|}{|\underline{h}|} \, .$$

Since $|h_1| \leq |\underline{h}|$ and $|h_2| \leq |\underline{h}|$, we have $R(\underline{h})/|\underline{h}| \leq |\underline{h}|$ and since $|\underline{h}| \to 0$ as $\underline{h} \to 0$, the same must be true of $R(\underline{h})/|\underline{h}|$. Hence f *is* differentiable at $(-1, 1)$.
Conclusion: (a) f is continuously differentiable at all points in $\mathbb{R}^2$ away from the lines $x = -1$ or $y = -1$; (b) f is differentiable at all points in (a) and also at $(-1, -1)$.

49 *Determine the points of* $\mathbb{R}^2$ *at which the function* $f(x, y) = |x^2 - y^2|$ *is*
(a) continuously differentiable; (b) differentiable.

Solution: $f(x, y) = |x^2 - y^2|$ is $x^2 - y^2$ for $x^2 - y^2 = (x - y)(x + y) > 0$ and $y^2 - x^2$ for $x^2 - y^2 = (x - y)(x + y) < 0$. For the first case $(x - y)(x + y) > 0$ for $x - y > 0$ and $x + y > 0$ or $x - y < 0$ and $x + y < 0$, call this part of $\mathbb{R}^2$ region 1. For the second case $(x - y)(x + y) < 0$ for $x - y > 0$ and $x + y < 0$ or $x - y < 0$ and $x + y > 0$, call this part of $\mathbb{R}^2$ region 2. Within both regions the function is a polynomial in x and y and so has continuous partial derivatives, hence the function is continuously differentiable there. In neighbourhoods of points on the line $y = \pm x$ the function is no longer a polynomial and we have to be more careful so use the definition of the partial derivative: for $y = x = a$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{|(a + h)^2 - a^2|}{h} = \lim_{h \to 0} \frac{|2ah + h^2|}{h}$$

If $a \neq 0$ we can neglect the h^2 term in comparison to the ah piece so we get

$$\lim_{h \to 0} \frac{|2ah|}{h} = \lim_{h \to 0} \frac{|2a| \, |h|}{h}$$

which does not exist because $\frac{|h|}{h}$ is ± 1 depending on the sign of h. However if $a = 0$ then the limit is

$$\lim_{h\to 0} \frac{|h^2|}{h} = 0 \,.$$

Similar arguments show that $\frac{\partial f}{\partial y}$ also does not exist on this line except at the origin, and also that neither partial derivative exists on the line $y = -x$, except at the origin. Since the p.d.s do not exist on the lines $y = \pm x$ away from $\mathbf{0}$ the function cannot be differentiable there. It might be differentiable at $\mathbf{0}$ but cannot be continuously differentiable there, because the p.d.s are not continuous at $\mathbf{0}$ (since they do not exist on the lines). Now investigate the differentiablity of f at the origin. Consider

$$\frac{R}{|\mathbf{h}|} = \frac{f(\mathbf{h}) - f(\mathbf{0}) - \mathbf{h}\cdot\nabla f}{|\mathbf{h}|} = \frac{|h_1^2 - h_2^2|}{|\mathbf{h}|}$$

Now

$$\frac{|h_1^2 - h_2^2|}{|\mathbf{h}|} \le \frac{h_1^2}{|\mathbf{h}|} + \frac{h_2^2}{|\mathbf{h}|} \le |\mathbf{h}| + |\mathbf{h}|$$

we have that $|R/|\mathbf{h}|| < \epsilon$ whenever $|\mathbf{h}| < \delta$ by taking $\delta = \epsilon/2$ so $\lim_{\mathbf{h}\to 0} R/|\mathbf{h}| = 0$ and f is differentiable at the origin.

50 *Define* $f : \mathbb{R}^2 \to \mathbb{R}$ *by* $f(\mathbf{0}) = 0$ *whilst for* $\mathbf{x} \neq \mathbf{0}$*:*

$$f(\mathbf{x}) = \frac{x^3}{x^2 + y^2} \,.$$

Calculate the partial derivatives of f *with respect to* x *and* y *at* $\mathbf{x} = \mathbf{0}$ *using their definitions as limits. Defining* $R(\mathbf{h})$ *at the origin by* $R(\mathbf{h}) = f(\mathbf{h}) - f(\mathbf{0}) - \mathbf{h}\cdot\nabla f$ *as usual, show that* $R(\mathbf{h})/|\mathbf{h}|$ *does not tend to zero as* $\mathbf{h}$ *tends to* $\mathbf{0}$*, so that* f *is not differentiable at the origin.*

On the line through the origin, $\mathbf{x} = \mathbf{b}t$*, (with* $\mathbf{b}$ *a constant vector),* f *becomes a function of the single variable* t*,* $f(\mathbf{b}t)$*. Write* $\mathbf{b} = \mathbf{e_1}b_1 + \mathbf{e_2}b_2$ *and use this to write* $f(\mathbf{b}t)$ *explicitly as a function of* t*. Show that this function is differentiable at the origin, i.e.* df/dt *exists at* $t = 0$ *despite* $f(\mathbf{x})$ *not being differentiable at* $\mathbf{0}$*.*

Solution:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(h\mathbf{e_1}) - f(\mathbf{0})}{h} = \lim_{h\to 0} \frac{h - 0}{h} = 1$$

and

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(h\mathbf{e_2}) - f(\mathbf{0})}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = 0 \,.$$

With the standard definition of R,

$$\frac{R}{|\mathbf{h}|} = \frac{f(\mathbf{h}) - f(\mathbf{0}) - \mathbf{h}\cdot\nabla f}{|\mathbf{h}|} = \frac{h_1^3}{(h_1^2 + h_2^2)^{3/2}} - 0 - \frac{h_1}{(h_1^2 + h_2^2)^{1/2}} = \frac{-h_1 h_2^2}{(h_1^2 + h_2^2)^{3/2}}$$

now consider how this behaves as the origin is approached along the line $h_2 = mh_1$ then

$$\frac{R}{|\mathbf{h}|} = \frac{-m^2 h_1^3}{(1 + m^2)^{3/2} h_1^3} = \frac{-m^2}{(1 + m^2)^{3/2}}$$

which remains constant as the origin is approached. Since this is not zero f cannot be differentiable at the origin.

$$f(\mathbf{b}t) = \frac{(b_1 t)^3}{(b_1 t)^2 + (b_2 t)^2} = t\frac{b_1^3}{b_1^2 + b_2^2}$$

which is t multiplied by a constant (this also holds at $\mathbf{0}$ since f is defined to vanish there), and so is differentiable at the origin.

51 *Exam Question June 2022 (Part B)*

(a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar field on $\mathbb{R}^n$. Define what it means for the limit of $f(\underline{x})$ as $\underline{x}$ tends to $\underline{a}$ to be L.

(b) Define what it means for f to be continuous at $\underline{a}$.

(c) Let f be a scalar field on $\mathbb{R}^2$ given by

$$f(x,y) = \begin{cases} xy(x^2-y^2)/(x^2+y^2) & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that f is continuous on $\mathbb{R}^2$, stating any results that you use.

(d) Is f differentiable at the origin?

(e) Show that, at the origin,

$$\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}.$$

Solution:

(a) The limit of $f(\underline{x})$ as $\underline{x}$ tends to $\underline{a}$ is L, or $\lim_{\underline{x}\to\underline{a}} f(\underline{x}) = L$, if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(\underline{x}) - L| < \epsilon \quad \forall x$ such that $0 < |\underline{x} - \underline{a}| < \delta$.

(b) f is continuous at $\underline{a}$ if $f(\underline{a})$ exists and $\lim_{\underline{x}\to\underline{a}} f(\underline{x}) = f(\underline{a})$.

(c) For any $\underline{a} \neq \underline{0}$, f is continuous at $\underline{a}$ by Theorem 5.3. To see that f is continuous at $\underline{0}$, we first note that $f(\underline{0}) = 0$, and so we must have $\lim_{\underline{x}\to\underline{a}} f(\underline{x}) = 0$. Letting $\delta = \sqrt{\epsilon}$, then if $|\underline{x}| < \delta = \sqrt{\epsilon}$

$$|f(\underline{x}) - 0| \leq \frac{|\underline{x}||\underline{x}||x^2 - y^2|}{|\underline{x}|^2} \leq |x^2 + y^2| < \delta^2 = \epsilon.$$

Therefore $\lim_{\underline{x}\to\underline{a}} f(\underline{x}) = 0 = f(\underline{a})$, and so f is also continuous at $\underline{0}$. We have therefore shown that f is continuous on $\mathbb{R}^2$.

(d) We first calculate the partial derivatives at the origin, using their limit definitions.

$$\frac{\partial f}{\partial x}(\underline{0}) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0/h^2 - 0}{h} = 0,$$

and similarly

$$\frac{\partial f}{\partial y}(\underline{0}) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0 - 0/h^2}{h} = 0,$$

and therefore $\underline{\nabla} f(\underline{0}) = \underline{0}$.
By definition, f is differentiable at $\underline{0}$, if $f(\underline{x}+\underline{h}) - f(\underline{x}) = \underline{h}\cdot\underline{\nabla} f + R(\underline{h})$, with $\lim_{\underline{h}\to\underline{0}} R(\underline{h})/|\underline{h}| = 0$. At $\underline{x}=\underline{0}$, with $\underline{h}=(h_1,h_2)$ we therefore have

$$R(\underline{h}) = \frac{h_1h_2(h_1^2-h_2^2)}{h_1^2+h_2^2} - 0 - 0,$$

and so

$$0 \le \left|\frac{R(\underline{h})}{|\underline{h}|}\right| = \frac{|h_1||h_2||h_1^2-h_2^2|}{|\underline{h}|^3} \le \frac{|\underline{h}||\underline{h}||h_1^2-h_2^2|}{|\underline{h}|^3} \le \frac{|h_1^2+h_2^2|}{|\underline{h}|} = \frac{|\underline{h}|^2}{|\underline{h}|} = |\underline{h}|.$$

By the squeezing theorem we therefore have that $\lim_{\underline{h}\to\underline{0}}\left|\frac{R(\underline{h})}{|\underline{h}|}\right| = 0$. Since $-\left|\frac{R(\underline{h})}{|\underline{h}|}\right| \le \frac{R(\underline{h})}{|\underline{h}|} \le \left|\frac{R(\underline{h})}{|\underline{h}|}\right|$, we therefore also have $\lim_{\underline{h}\to\underline{0}}\frac{R(\underline{h})}{|\underline{h}|} = 0$ by another application of the squeezing theorem, and hence f is differentiable at $\underline{0}$.

(e) We will first need to calculate $f_x(0,y)$ for $x\neq 0$, and $f_y(x,0)$ for $y\neq 0$.
For $y\neq 0$, we have

$$f_x(0,y) = \lim_{h\to 0}\frac{f(0+h,y)-f(0,y)}{h} = \lim_{h\to 0}\frac{hy(h^2-y^2)}{h(h^2+y^2)} = \lim_{h\to 0}\frac{y(h^2-y^2)}{(h^2+y^2)} = -y,$$

and similarly for $x\neq 0$ we have

$$f_y(x,0) = \lim_{h\to 0}\frac{f(x,0+h)-f(x,0)}{h} = \lim_{h\to 0}\frac{xh(x^2-h^2)}{h(x^2+h^2)} = \lim_{h\to 0}\frac{x(x^2-h^2)}{(x^2+h^2)} = x.$$

Therefore

$$\frac{\partial^2 f}{\partial x\partial y}(\underline{0}) = \lim_{h\to 0}\frac{f_y(0+h,0)-f_y(0,0)}{h} = \lim_{h\to 0}\frac{h-0}{h} = 1,$$

and similarly

$$\frac{\partial^2 f}{\partial y\partial x}(\underline{0}) = \lim_{h\to 0}\frac{f_x(0,0+h)-f_x(0,0)}{h} = \lim_{h\to 0}\frac{-h-0}{h} = -1,$$

and so $\frac{\partial^2 f}{\partial x\partial y} \neq \frac{\partial^2 f}{\partial y\partial x}$ at the origin as required.

52 *If $y = 1 + xy^5$ show that y may be written in the form $y = f(x)$ in a neighbourhood of $(0,1)$ and find the gradient of the graph of f at the point $(0,1)$.*

Solution: $y = 1 + xy^5$ is a level curve of $H = y - 1 - xy^5$ with $H = 0$ and passes through $(0,1)$. H is polynomial in x and y and so has continuous derivatives and is therefore differentiable. The implicit function theorem then implies that y may be written in the form $y = f(x)$ in a neighbourhood of $(0,1)$ if $\frac{\partial H}{\partial y}\neq 0$ there. Now $\frac{\partial H}{\partial y} = 1 - 5xy^4 = 1$ at $(0,1)$ so the condition is satisfied. The implicit function theorem also gives the gradient of the function as

$$f'(0) = -\frac{\partial H}{\partial x}\Big/\frac{\partial H}{\partial y},$$

now $\frac{\partial H}{\partial x} = -y^5 = -1$ at $(0,1)$ so $f'(0) = 1$.

53 *Show that the equation $xy^3 - y^2 - 3x^2 + 1 = 0$ can be written in the form $y = f(x)$ in a neighbourhood of the point $(0, 1)$, and in the form $y = g(x)$ in a neighbourhood of the point $(0, -1)$. Is it true that $f(x)$ and $g(x)$ are equivalent as functions of x? What are the critical values of the curve $H(x, y) = xy^3 - y^2 - 3x^2 + 1$, and what are the regular values of this curve?*

Solution: $xy^3 - y^2 - 3x^2 + 1 = 0$ is a level curve of $H = xy^3 - y^2 - 3x^2 + 1$ with $H = 0$ and passes through $(0, \pm 1)$. H is polynomial in x and y and so has continuous derivatives and is therefore differentiable. $\frac{\partial H}{\partial y} = 3xy^2 - 2y$, so at $(0, \pm 1)$, $\frac{\partial H}{\partial y} \neq 0$. The implicit function theorem then implies that y may be written in the form $y = f(x)$ in a neighbourhood of the point $(0, 1)$, and in the form $y = g(x)$ in a neighbourhood of the point $(0, -1)$. Note that at $x = 0$ we must have $f(0) = 1$ but $g(0) = -1$ so $f(x)$ and $g(x)$ cannot be equivalent as functions of x. In fact solving $xy^3 - y^2 - 3x^2 + 1 = 0$ for y gives three solutions, since it is a cubic equation and one passes through $(0, 1)$ another through $(0, -1)$.

To find the critical values of $H(x, y)$, we must consider values of c for which the curve $H(x, y) = c$ contains critical points, i.e. points where $\underline{\nabla} H = 0$. We have $\frac{\partial H}{\partial x} = y^3 - 6x$, so $\underline{\nabla} H = 0$ if we have $y^3 - 6x = 0$ and $3xy^2 - 2y = 0$. If $y \neq 0$, we can rearrange these equations in terms of x and set them equal to give $\frac{y^3}{6} = \frac{2}{3y}$, with real solutions $y = \pm\sqrt{2}$. Substituting this back to find x, we have $\underline{\nabla} H = 0$ at the points $(\pm\frac{\sqrt{2}}{3}, \pm\sqrt{2})$, so these are critical points of $H(x, y)$. If $y = 0$, then we have $\underline{\nabla} H = 0$ if $x = 0$, so there is also a critical point at the origin. Critical values of $H(x, y)$ are values c where $H(x, y) = c$ contains critical points of H, so we need to find which values of c the critical points correspond to. Since $H(\pm\frac{\sqrt{2}}{3}, \pm\sqrt{2}) = -\frac{1}{3}$, we have one critical value at $c = -\frac{1}{3}$, and since $H(\underline{0}) = 1$, we have another critical value at $c = 1$. The regular values are therefore $\mathbb{R} - \{-\frac{1}{3}, 1\}$.

54 *Determine whether or not the equation $x^2 + y + \sin(xy) = 0$ can be written in the form $y = f(x)$ or in the form $x = g(y)$ in some small open disc about the origin for some suitable continuously differentiable functions f, g.*

Solution: $x^2 + y + \sin(xy) = 0$ is a level curve of $H = x^2 + y + \sin(xy)$ with $H = 0$ and passes through the origin. H has continuous partial derivatives and so is differentiable. $\frac{\partial H}{\partial y} = 1 + x\cos(xy)$ is 1 at the origin and $\frac{\partial H}{\partial x} = 2x + y\cos(xy)$ vanishes at the origin, so by the implicit function theorem y may be written in the form $y = f(x)$ in a neighbourhood of the origin with differentiable f, but x cannot be written in the form $x = g(y)$.

55 *Exam question May 2015 (Section B, lightly edited):*
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the scalar function $f(x, y) = e^{xy} - x + y$.

(a) *Find the vector equations of the tangent and normal lines to the curve $f(x, y) = 0$ at the points $(1, 0)$ and $(0, -1)$.*

(b) *Use the implicit function theorem for functions of two variables to determine whether or not we can guarantee the curve $f(x, y) = 2$ can be written in the form $y = g(x)$ for some differentiable function $g(x)$ in the neighbourhoods of the points (i)*

$(0, 1)$; *(ii)* $(-1, 0)$.
Determine also whether we can guarantee the curve can be written as $x = h(y)$ *for some differentiable function* $h(y)$*, in the neighbourhoods of the same two points.*

(c) *Does the function* $f(x, y)$ *have any critical points? Justify your answer. (You can quote without proof that* $|xe^{-x^2}| < 1$ *for all* $x \in \mathbb{R}$.*)*

Solution:

(a) We have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{e}_1 + \frac{\partial f}{\partial y}\mathbf{e}_1 \,, \qquad \frac{\partial f}{\partial x} = ye^{xy} - 1 \,, \quad \frac{\partial f}{\partial y} = xe^{xy} + 1 \,,$$

and so $\nabla f(1, 0) = -\mathbf{e}_1 + 2\mathbf{e}_2$ and $\nabla f(0, -1) = -2\mathbf{e}_1 + \mathbf{e}_2$.
Hence the equations of the normal lines at $(1, 0)$ and $(0, -1)$ can be written as

$$\mathbf{x} = \mathbf{e}_1 + \lambda(-\mathbf{e}_1 + 2\mathbf{e}_2) = (1 - \lambda)\mathbf{e}_1 + 2\lambda\mathbf{e}_2$$

and

$$\mathbf{x} = -\mathbf{e}_2 + \lambda(-2\mathbf{e}_1 + \mathbf{e}_2) = -2\lambda\mathbf{e}_1 + (\lambda - 1)\mathbf{e}_2$$

respectively. Vectors perpendicular to ∇f at $(1, 0)$ and $(0, -1)$ are (for example) $2\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_1 + 2\mathbf{e}_2$, so the two tangent lines can be written as

$$\mathbf{x} = \mathbf{e}_1 + \mu(2\mathbf{e}_1 + \mathbf{e}_2) = (1 + 2\mu)\mathbf{e}_1 + \mu\mathbf{e}_2$$

and

$$\mathbf{x} = -\mathbf{e}_2 + \mu(\mathbf{e}_1 + 2\mathbf{e}_2) = \mu\mathbf{e}_1 + (2\mu - 1)\mathbf{e}_2 \,.$$

(b) Reminder of implicit function theorem (not needed for the problem sheet, though it *did* feature in the exam question: if $f(x, y) : U \to \mathbb{R}$ is differentiable on U with U open in $\mathbb{R}^2$, and if (x_0, y_0) is a point on the level curve $f(x, y) = c$ at which $\frac{\partial f}{\partial y} \neq 0$, then a differentiable function $g(x)$ exists in a neighbourhood of $x = x_0$ such that (I) $f(x, g(x)) = c$ and (II) $\frac{dg}{dx} = -\frac{\partial f}{\partial x}/\frac{\partial f}{\partial y}$, with $g(x_0) = y_0$. Here, f is everywhere differentiable and we just need to check the partials. At $(0, 1)$, $\frac{\partial f}{\partial y} = 1$ so the curve *can* be written as $y = g(x)$; but at $(-1, 0)$, $\frac{\partial f}{\partial y} = 0$ so we *cannot guarantee* the curve can be written as $y = g(x)$. For the final part, swap x and y in the implicit function theorem and check the partial x derivatives: at $(0, 1)$, $\frac{\partial f}{\partial x} = 0$ so we *cannot guarantee* the curve can be written as $x = h(y)$; and at $(-1, 0)$, $\frac{\partial f}{\partial x} = -1$ so the curve *can* be written as $x = h(y)$.

(c) For a critical point we need $\nabla f = \mathbf{0}$, so $ye^{xy} = 1$ and $xe^{xy} = -1$ must hold. Adding implies $(x + y)e^{xy} = 0$, so $y = -x$. Then $xe^{-x^2} = -1$ is required, but this contradicts the inequality given in the question. Hence f has no critical points.

56 *Part of Exam Question May 2017 (Section B):*

(c) Consider the function

$$f(x,y) = (3x+y)e^{3xy}.$$

Determine whether or not the curve $f(x,y) = c$ can be written in the form $y = g(x)$, and if not, state clearly the points (x_0, y_0) and corresponding values of c where problems occur. You may assume that f is differentiable on $\mathbb{R}^2$.

(d) Using $f(x,y)$ as given in the previous part, determine whether or not the curve $(f(x,y) = c)$ can be written in the form $x = h(y)$, and if not, state clearly the points (x_0, y_0) where problems occur.

(e) Using $f(x,y)$ as in the previous parts of this question, are there any points where the curve $f(x,y) = c$ can neither be written as $y = g(x)$, nor as $x = h(y)$?

Solution:

(c) By the implicit function theorem, if (x_0, y_0) is a point on the curve $f(x,y) = c$ at which $\frac{\partial f}{\partial y} \neq 0$, then the curve $f(x,y) = c$ can be written in the form $y = g(x)$ in a neighbourhood of the point (x_0, y_0). We have

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}((3x+y)e^{3xy}) \\
&= e^{3xy} + (3x+y)3xe^{3xy} \\
&= (1 + 9x^2 + 3xy)e^{3xy}
\end{aligned}$$

$$\text{so} \qquad \frac{\partial f}{\partial y} = 0 \iff 1 + 9x^2 + 3xy = 0$$

$$\text{If } x \neq 0 \quad \Longrightarrow \quad y = -\frac{(1+9x^2)}{3x} = -\left(3x + \frac{1}{3x}\right) \qquad (\dagger).$$

So problems occur at point (x_0, y_0) with $y_0 = -(3x_0 + 1/3x_0)$.
Now we need to find which values of c these problems occur at, so we evaluate

$$\begin{aligned}
c = f(x_0, y_0) &= \left(3x_0 + \left(-\left(3x_0 + \frac{1}{3x_0}\right)\right)\right) e^{3x_0(-(3x_0+1/3x_0))} \\
&= -\frac{1}{3x_0} e^{-9x_0^2 - 1},
\end{aligned}$$

which for $x \in \mathbb{R} - \{0\}$ can give any value for c except $c = 0$.

(d) Similarly, for $x = h(y)$ need $\frac{\partial f}{\partial x} \neq 0$. Fast way is to notice that f is symmetric under the interchange $3x \leftrightarrow y$, which gives the result. The long way is

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}((3x+y)e^{3xy}) \\
&= 3e^{3xy} + (3x+y)3ye^{3xy} \\
&= (3 + 3y^2 + 9xy)e^{3xy}
\end{aligned}$$

$$\text{so} \qquad \frac{\partial f}{\partial y} = 0 \iff 3 + 3y^2 + 9xy = 0$$

$$\text{If } y \neq 0 \quad \Longrightarrow \quad x = -\frac{(3+3y^2)}{9y} = -\frac{1}{3}\left(y + \frac{1}{y}\right) \qquad (\ddagger).$$

So problems occur at point (x_0, y_0) with $x_0 = -\frac{1}{3}\left(y_0 + \frac{1}{y_0}\right)$.

(e) Points where we can neither write $y = g(x)$ nor $x = h(y)$ are the *critical points*, where $\underset{\sim}{\nabla} f = \underset{\sim}{0}$. At these points, both (†) and (‡) are satisfied simultaneously. If we substitute $3x_0 = -\left(y_0 + \frac{1}{y_0}\right)$ into (†), remembering that this requires $y_0 \neq 0$, gives

$$\begin{aligned} y_0 &= \left(y_0 + \frac{1}{y_0}\right) + 1/\left(y_0 + \frac{1}{y_0}\right) \\ \implies 0 &= \frac{1}{y_0} + 1/\left(y_0 + \frac{1}{y_0}\right) \\ \implies 0 &= \left(y_0 + \frac{1}{y_0}\right) + y_0 \\ \implies 0 &= 2y_0^2 + 1. \end{aligned}$$

Since $2y_0^2 + 1 > 0$, no points simultaneously satisfy (†) and (‡), and hence no points exist where f can neither be written as $y = g(x)$ nor as $x = h(y)$.

57 *For each of the following two surfaces, show that the surface can be parameterised as $\underset{\sim}{x}(x, y) = x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + g(x, y)\underset{\sim}{e}_3$, and show that the normal vector $\frac{\partial \underset{\sim}{x}}{\partial x} \times \frac{\partial \underset{\sim}{x}}{\partial y}$ can be written as*

$$\frac{\partial \underset{\sim}{x}}{\partial x} \times \frac{\partial \underset{\sim}{x}}{\partial y} = \frac{\underset{\sim}{\nabla} f}{\underset{\sim}{e}_3 \cdot \underset{\sim}{\nabla} f},$$

where the surface is a level set of a scalar field $f(\underset{\sim}{x})$ which you should specify (and which you may assume to be differentiable).

(a) *The upper hemisphere of radius $\sqrt{2}$, centred on the origin (where $z > 0$).*

(b) *The surface defined by $e^{x+y+z} = 1 - (xy)^2$.*

Solution:

(a) The upper hemisphere of radius $\sqrt{2}$, centred on the origin, is the set of points $\underset{\sim}{x}$ such that $x^2 + y^2 + z^2 = 2$ with $z > 0$. On this surface we can therefore explicitly write $z = g(x, y) = \sqrt{2 - x^2 - y^2}$, and the surface can therefore be parameterised as

$$\begin{aligned} \underset{\sim}{x}(x, y) &= x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + g(x, y)\underset{\sim}{e}_3 \\ &= x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + \sqrt{2 - x^2 - y^2}\,\underset{\sim}{e}_3. \end{aligned}$$

We therefore have

$$\begin{aligned} \frac{\partial \underset{\sim}{x}}{\partial x} &= \underset{\sim}{e}_1 - x\left(2 - x^2 - y^2\right)^{-1/2}\underset{\sim}{e}_3 \\ \frac{\partial \underset{\sim}{x}}{\partial y} &= \underset{\sim}{e}_2 - y\left(2 - x^2 - y^2\right)^{-1/2}\underset{\sim}{e}_3, \qquad \text{and so} \end{aligned}$$

$$\frac{\partial \underset{\sim}{x}}{\partial x} \times \frac{\partial \underset{\sim}{x}}{\partial y} = x\left(2 - x^2 - y^2\right)^{-1/2}\underset{\sim}{e}_1 + y\left(2 - x^2 - y^2\right)^{-1/2}\underset{\sim}{e}_2 + \underset{\sim}{e}_3.$$

The surface is a level set of the scalar field $f(\underline{x}) = x^2 + y^2 + z^2$, and so we have

$$\begin{aligned} \underline{\nabla}\, f &= 2x\underline{e}_1 + 2y\underline{e}_2 + 2z\underline{e}_3, \qquad \text{and so} \\ \frac{\underline{\nabla}\, f}{\underline{e}_3 \cdot \underline{\nabla}\, f} &= \frac{x}{z}\underline{e}_1 + \frac{y}{z}\underline{e}_2 + \underline{e}_3 \\ &= x\left(2 - x^2 - y^2\right)^{-1/2}\underline{e}_1 + y\left(2 - x^2 - y^2\right)^{-1/2}\underline{e}_2 + \underline{e}_3 \\ &= \frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y}. \end{aligned}$$

(b) The surface is the level set $h(\underline{x}) = 1$, of the scalar field $h(\underline{x}) = e^{x+y+z} + (xy)^2$, which we assume to be differentiable. By the IFT for surfaces, the surface may be written as $z = g(x, y)$ in a neighbourhood of any point (x_0, y_0, z_0) at which $\frac{\partial h}{\partial z} \neq 0$, and this function $g(x, y)$ satisfies

$$\begin{aligned} h(x, y, g(x, y)) &= 1 \\ \frac{\partial g}{\partial x} &= \frac{-\partial h/\partial x}{\partial h/\partial z} \qquad (\dagger) \\ \frac{\partial g}{\partial y} &= \frac{-\partial h/\partial y}{\partial h/\partial z} \qquad (\ddagger)\,, \end{aligned}$$

in a neighbourhood of (x_0, y_0, z_0).
We have

$$\begin{aligned} \frac{\partial h}{\partial x} &= e^{x+y+z} + 2xy^2 \\ \frac{\partial h}{\partial y} &= e^{x+y+z} + 2x^2y \\ \frac{\partial h}{\partial z} &= e^{x+y+z}, \end{aligned}$$

and so $\frac{\partial h}{\partial z} \neq 0$ at all points (x_0, y_0, z_0). So, by the IFT, such a function $g(x, y)$ exists in a neighbourhood of all points, and so the surface can be parameterised as $\underline{x}(x, y) = x\underline{e}_1 + y\underline{e}_2 + g(x, y)\underline{e}_3$.
We therefore have

$$\begin{aligned} \frac{\partial \underline{x}}{\partial x} &= \underline{e}_1 + \frac{\partial g}{\partial x}\underline{e}_3 \\ \frac{\partial \underline{x}}{\partial y} &= \underline{e}_2 + \frac{\partial g}{\partial y}\underline{e}_3, \qquad \text{and so} \\ \frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y} &= -\frac{\partial g}{\partial x}\underline{e}_1 - \frac{\partial g}{\partial y}\underline{e}_2 + \underline{e}_3, \end{aligned}$$

and so using $(\dagger)$ and $(\ddagger)$,

$$\begin{aligned} \frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y} &= \frac{\partial h/\partial x}{\partial h/\partial z}\underline{e}_1 + \frac{\partial h/\partial y}{\partial h/\partial z}\underline{e}_2 + \underline{e}_3 \\ &= \left(1 + 2xy^2e^{-(x+y+z)}\right)\underline{e}_1 + \left(1 + 2x^2ye^{-(x+y+z)}\right)\underline{e}_2 + \underline{e}_3 \\ &= \frac{\underline{\nabla}\, h}{\underline{e}_3 \cdot \underline{\nabla}\, h}. \end{aligned}$$

Since at all points on this surface we have $e^{x+y+z} = 1 - (xy)^2$, we can write this normal in terms of x and y only (as we did for the first part of this question), as

$$\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y} = \left(1 + 2xy^2(1 - x^2y^2)^{-1}\right) \underline{e}_1 + \left(1 + 2x^2y(1 - x^2y^2)^{-1}\right) \underline{e}_2 + \underline{e}_3.$$