Calculus I
Michaelmas Term

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Contents

1 Functions

1.1 Functions, domain and range

Defn: A **function** f is a correspondence between two sets D (called the *domain*) and C (called the *codomain*), that assigns to each element of D one and only one element of C. The notation to indicate the domain and codomain is $f : D \mapsto C$.

For $x \in D$ we write $f(x)$ to denote the assigned element in C, and call this the value of f at x, or the image of x under f, where x is called the argument of the function. Extending the above notation we write

$$f : D \mapsto C : x \mapsto f(x)$$

ie. "f is a function from D to C that associates x in D to $f(x)$ in C".

The set of all images is called the **range** of f and our notation for the domain and range is

$$\mathrm{Dom}\, f \qquad \text{and} \qquad \mathrm{Ran}\, f = \{f(x) : x \in \mathrm{Dom}\, f\}.$$

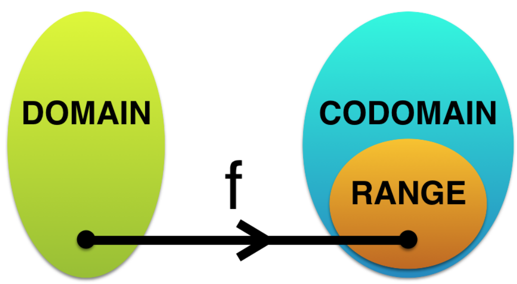


Figure 1: Illustration of the domain, codomain and range of a function.

In (almost all of) this term of the module we shall only deal with real-valued functions of a real variable, meaning that our functions assign real numbers to real numbers ie. both the domain and codomain are subsets of $\mathbb{R}$.

There are various ways of representing a function, but perhaps the most familiar is through an explicit expression.

Eg. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2, \ \forall x \in \mathbb{R}$.
In this case $\mathrm{Dom}\, f = \mathbb{R}$ and $\mathrm{Ran}\, f = [0, \infty)$.
We say that f *maps* the real line onto $[0, \infty)$.

Eg. $f(x) = \sqrt{2x + 4}, \ x \in [0, 6]$. Here $\mathrm{Dom}\, f = [0, 6]$ and $\mathrm{Ran}\, f = [2, 4]$.

If the domain of a function f is not explicitly given, then it is taken to be the maximal set of real numbers x for which $f(x)$ is a real number.

Eg. $f(x) = \sqrt{2x+4}$. Here $\operatorname{Dom} f = [-2, \infty)$ and $\operatorname{Ran} f = [0, \infty)$.

Eg. $f(x) = 1/(1-x)$. Here $\operatorname{Dom} f = \mathbb{R}\backslash\{1\} = (-\infty, 1) \cup (1, \infty)$ and $\operatorname{Ran} f = \mathbb{R}\backslash\{0\}$.

In addition to explicit expressions, a function f may be represented by other means, for example, as a set of ordered pairs (x, y), where $x \in \operatorname{Dom} f$ and $y \in \operatorname{Ran} f$.

Note: $y(x)$ is another common notation for a function. The element x in the domain is called the **independent variable** and the element y in the range is called the **dependent variable**. This notation is often used if the function is defined by an equation in two variables, including differential equations (see later).

The function $f(x) = x^2$ is defined by the equation $y = x^2$, as we have simply written $y = f(x)$. However, not all equations will define a function.

Eg. The equation $y^2 = x$ does *not* define a non-trivial function $y(x)$, because for $x < 0$ there are no real solutions for y, and for $x > 0$ there are two solutions for y, whereas a function must assign only one element of the range for each element of the domain.

1.2 The graph of a function

Defn: The **graph** of a function f is the set of all points (x, y) in the xy-plane with $x \in \text{Dom}\, f$ and $y = f(x)$, ie.

$$\text{graph f} = \{(x, y) : x \in \text{Dom}\, f \text{ and } \ y = f(x)\}.$$

The graph of a function over the interval $[a, b]$ is the portion of the graph where the argument is restricted to this interval.

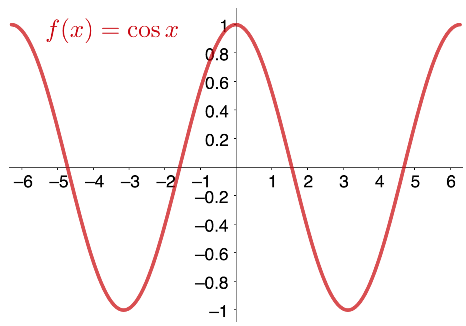


Figure 2: The graph of $f(x) = \cos x$ over $[-2\pi, 2\pi]$.

Note: If you are asked to graph a function, but no interval is given, then try to choose an appropriate interval that includes all the interesting behaviour eg. turning points.

To indicate on a graph exactly which points are included we use a closed circle to denote an included point, so if it is at the end of a curve segment this denotes that this end is a closed interval. We use an open circle to denote an excluded point associated with an open interval.

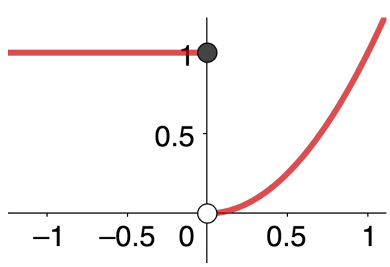


Figure 3: A graph illustating the use of open (white) and closed (black) circles.

The graph of a function is a curve in the plane, but not every curve is the graph of a function. The following simple test determines whether a curve is the graph of a function.

The vertical line test.
If any vertical line intersects the curve more than once then the curve is not the graph of a function, otherwise it is.

The proof is obvious, given the defining property of a function that only one element in the range is associated with an element in the domain.

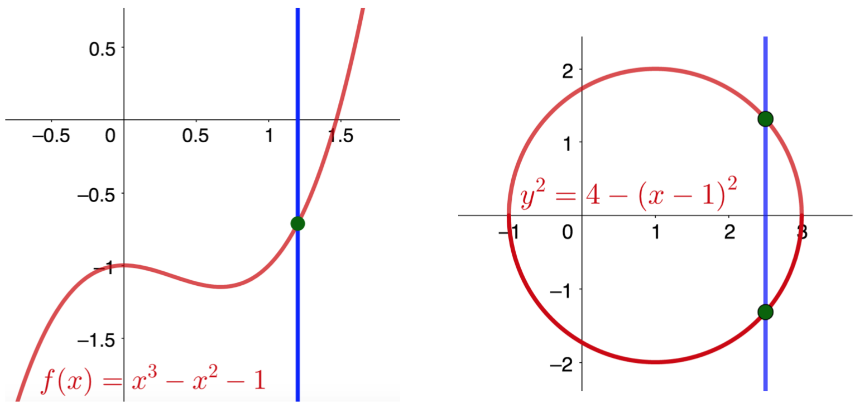


Figure 4: The vertical line test applied to a cubic curve and a circle.

In the figure the first curve is the graph of a function: in fact the cubic function $f(x) = x^3 - x^2 - 1$. The second curve is not the graph of a function as the vertical line shown intersects the curve twice. In fact the curve is given by the equation $y^2 = 4 - (x-1)^2$ ie. the circle of radius 2 with centre $(x, y) = (1, 0)$. Note that even when a curve is not the graph of a function, some sections of the curve may be graphs of functions. For the above circle example the section given by $y \geq 0$ is the graph of the function $f(x) = \sqrt{4 - (x-1)^2}$ with Dom $f = [-1, 3]$.

1.3 Even and odd functions

Defn. A function f is **even** if $f(x) = f(-x)\ \forall \pm x \in$ Dom f.
Defn. A function f is **odd** if $f(x) = -f(-x)\ \forall \pm x \in$ Dom f.

The graph of an even function is symmetric under a reflection about the y-axis.
The graph of an odd function is symmetric under a rotation by $180°$ about the origin (equivalent to a reflection in the y-axis followed by a reflection in the x-axis).

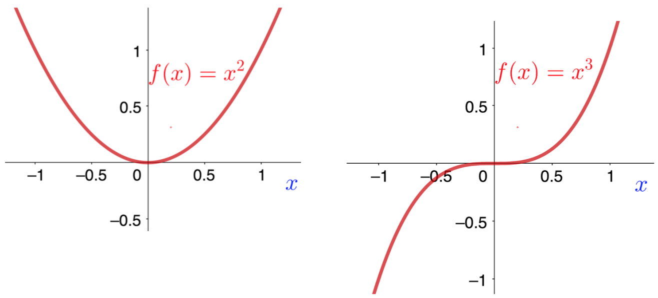


Figure 5: Graphs of the even function x^2 and the odd function x^3.

All functions $f : \mathbb{R} \mapsto \mathbb{R}$ can be written as the sum of an even and an odd function

$$f(x) = f_{even}(x) + f_{odd}(x),$$

$$\text{where} \quad f_{even}(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x) \quad \text{and} \quad f_{odd}(x) = \frac{1}{2}f(x) - \frac{1}{2}f(-x).$$

This decomposition is often useful, as is the ability to spot an even or odd function, as it can simplify some calculations, as we shall see in later sections.

Eg. $f(x) = (1 + x)\sin x$ with $f_{even}(x) = x\sin x$ and $f_{odd}(x) = \sin x$.

Eg. $f(x) = e^x$ with $f_{even}(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$ and $f_{odd}(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh x$.

1.4 Piecewise functions

Some functions are defined **piecewise** ie. different expressions are given for different intervals in the domain.

Eg. The absolute value (or modulus) function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

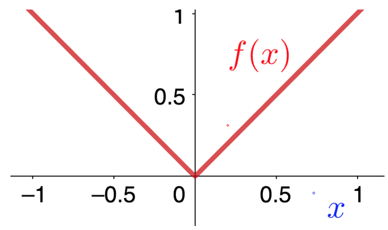


Figure 6: The graph of $|x|$ over $[-1, 1]$.

Eg.

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ x - 1 & \text{if } 1 \leq x \leq 2. \end{cases}$$

In this example Dom $f = [0, 2]$ but the function has a discontinuity at $x = 1$ (more about this later).

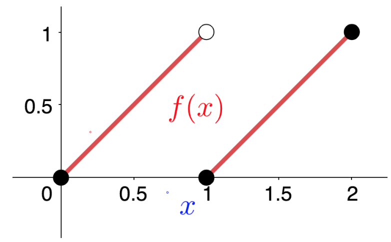


Figure 7: The graph of a piecewise function.

A **step function** is a piecewise function which is constant on each piece. An example is the Heaviside step function $H(x)$ defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Note that with this definition Dom $H = \mathbb{R} \backslash \{0\}$. It is sometimes convenient to extend the domain to $\mathbb{R}$ by defining the value of $H(0)$ (some obvious candidates are $0, \frac{1}{2}, 1$).

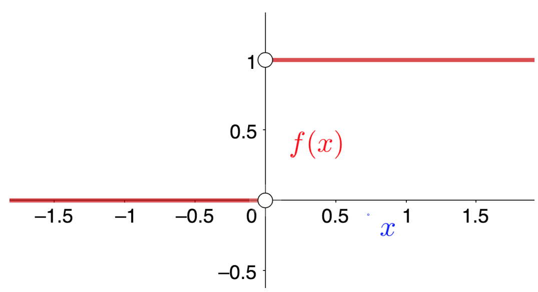


Figure 8: The Heaviside step function.

1.5 Operations with functions

Given two functions f and g we can define the following:

- the **sum** is $(f+g)(x) = f(x) + g(x)$, with domain $\text{Dom}\, f \cap \text{Dom}\, g$.

- the **difference** is $(f-g)(x) = f(x) - g(x)$, with domain $\text{Dom}\, f \cap \text{Dom}\, g$.

- the **product** is $(fg)(x) = f(x)g(x)$, with domain $\text{Dom}\, f \cap \text{Dom}\, g$.

- the **ratio** is $(f/g)(x) = f(x)/g(x)$, with domain $(\text{Dom}\, f \cap \text{Dom}\, g)\backslash\{x : g(x) = 0\}$.

- the **composition** is $(f \circ g)(x) = f(g(x))$, with domain $\{x \in \text{Dom}\, g : g(x) \in \text{Dom}\, f\}$.

Note that $f \circ g$ and $g \circ f$ are usually different functions.
Eg. $f(x) = \sin x,\ g(x) = x^2$ then $(f \circ g)(x) = \sin(x^2)$ but $(g \circ f)(x) = \sin^2 x$.

Note that the sum and difference, along with **scalar multiplication** $(cf)(x) = c \times f(x)$ for any constant c are special case of **linear combinations** of functions. The most general linear combination of f and g is $(af + bg)(x) = a \times f(x) + b \times g(x)$ for some constants a and b, with domain $\text{Dom}\, f \cap \text{Dom}\, g$..

1.6 Inverse functions

Defn. A function $f : D \mapsto C$ is **surjective** (or onto) if Ran $f = C$,
ie. if $\forall\ y \in C\ \exists\ x \in D$ s.t. $f(x) = y$.

Eg. $f : \mathbb{R} \mapsto \mathbb{R}$ given by $f(x) = 2x + 1$ is surjective.

Eg. $f : \mathbb{R} \mapsto \mathbb{R}$ given by $f(x) = x^2$ is not surjective, since any negative number in the codomain is not the image of an element in the domain.

Defn. A function $f : D \mapsto C$ is **injective** (or one-to-one) if $\forall\ x_1, x_2 \in D$ with $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

Eg. $f(x) = 2x + 1$ is injective since $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Eg. $f(x) = x^2$ is not injective since $f(x) = f(-x)$ and $x \neq -x$ if $x \neq 0$.

The following simple test can be applied to see if a function is injective from its graph.

The horizontal line test.
If no horizontal line intersects the graph of f more than once then f is injective, otherwise it is not.

Eg. The function $f(x) = x^3 - 3x$ is surjective as Ran $f = \mathbb{R}$ but it is not injective as the horizontal line $y = 1$ intersects the graph of f at 3 points (see the figure).

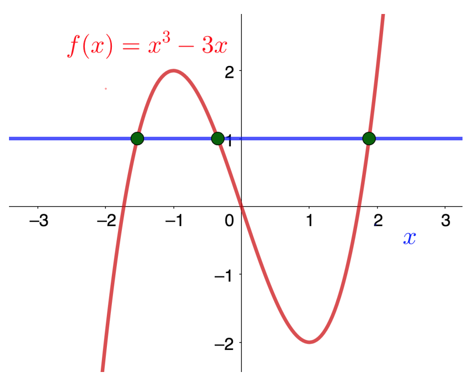


Figure 9: Graph of $f(x) = x^3 - 3x$ and the horizontal line $y = 1$.

Defn. A function $f : D \mapsto C$ is **bijective** if it is both surjective and injective.

Eg. From above we have seen that the function $f(x) = 2x + 1$ is bijective.

Theorem of inverse functions.
A bijective function f admits a unique inverse, denoted f^{-1}, such that

$$(f^{-1} \circ f)(x) = x = (f \circ f^{-1})(x).$$

It is clear from the definition that Dom $f^{-1} =$ Ran f and Ran $f^{-1} =$ Dom f.

As the inverse undoes the effect of the function an equivalent definition is

$$f(x) = y \quad \text{iff} \quad f^{-1}(y) = x.$$

This second definition is the same as the fact that the graph of the inverse function f^{-1} is given by reflecting the graph of f in the line $y = x$. However, note that in general the curve given by such a reflection is not the graph of a function – in particular it will not be single-valued if f is not injective.

Eg. We have seen that $f(x) = 2x + 1$ is bijective, so lets find its inverse. To simplify notation first write $y = f^{-1}(x)$. Using the property $x = f(f^{-1}(x))$, we have $x = f(y) = 2y + 1$ and hence $y = \frac{1}{2}(x - 1) = f^{-1}(x)$.

Note that given an injective function which is not surjective, we can make a bijective function by taking the codomain equal to the range (this automatically makes a function surjective), and then an inverse exists. This is essentially a harmless modification of the function since the function does not really care what its codomain is, the only requirement is that the range is a subset of the codomain.

Eg. We have seen that $f(x) = x^2$ is not an injective function if Dom $f = \mathbb{R}$, but it is injective if we take Dom $f = [0, \infty)$. With this choice we can take the codomain equal to Ran $f = [0, \infty)$ and f is now a bijective function. The inverse is $f^{-1}(x) = \sqrt{x}$ with Dom $f^{-1} =$ Ran $f = [0, \infty)$.

Note that here we made the function injective by restricting its domain. While we can always do this, in general doing so loses information about the function since the original non-injective function would have given values for other inputs values. However, in special cases such as even functions or periodic functions, we can restrict the domain and have a simple description of how to generate the original function (say with domain $\mathbb{R}$) from the function with the restricted domain. We used this above to find an inverse for $f(x) = x^2$ and this is also how we define an inverse for $f(x) = \sin x$ etc.

Warning: Don't confuse the notation for the inverse with the same notation for the reciprocal.

1.7 Summary: Functions

You should have a good and precise mathematical understanding of functions and various definitions, particularly focussing on real-valued functions of a single real variable. Here are some key points:

- A function f is a mapping from its *domain*, Dom f, to a *codomain*. Its *range*, Ran f, is the image of Dom f, i.e. the set of all values that $f(x)$ can actually take for $x \in$ Dom f.
- Functions are single-valued, i.e. $f(x)$ must have a unique value for all $x \in$ Dom f. This can be checked graphically by the *vertical line test*.
- Functions can be *even* – symmetric under reflection in the y-axis.
- Functions can be *odd* – symmetric under rotation by $180°$ around the origin, or equivalently by two reflections, in the x- and y-axes (in either order).
- Typical functions are neither even nor odd, but can be uniquely written as a sum of an even and an odd function, i.e. $f(x) = f_{even}(x) + f_{odd}(x)$. You should know how to define f_{even} and f_{odd} in terms of f.
- We can define *piecewise* functions using different expressions for different parts of the domain. At boundaries of intervals we use filled or empty circles to indicate that a point is or is not included.
- You should be familiar with the operations on functions: *linear combinations*, *product*, *ratio* and *composition*, and know what the domain of the resulting functions is in terms of the domains of the original functions.
- You should know the conditions for the *inverse* of a function to exist. You should know the definitions of *surjective*, *injective* and *bijective*.

2 Limits and continuity

2.1 Definition of a limit and continuity

Rough idea #1:
A function $f(x)$ has a **limit** L at $x = a$ if $f(x)$ is close to L whenever x is close to a.

Rough idea #2: Lets think about the function $f(x)$ near $x = a$.
If you approximate $f(x)$ by a constant L then you will make an error given by $|f(x)-L|$. Suppose $\varepsilon > 0$ is the εrror at which you become unhappy with your approximation, that is, you are happy as long as $|f(x)-L| < \varepsilon$. The important issue for a limit is whether I can keep you happy simply by making sure that x stays within a δistance $\delta > 0$ of a, that is by restricting to $0 < |x-a| < \delta$. Note that we dont even care about what happens exactly at $x = a$. If I can always find a δistance δ that keeps you happy, no matter how small you choose your εrror ε, then we say that $f(x)$ has a **limit** L as x tends to a.

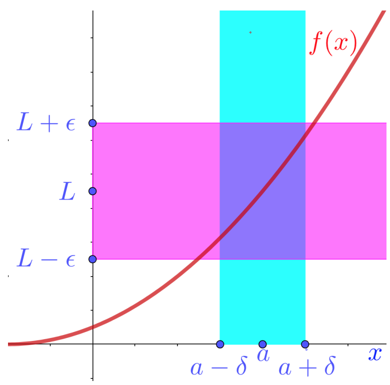


Figure 10: The idea of a limit.

Defn: $f(x)$ has a **limit** L as x tends to a if

$$\forall \quad \varepsilon > 0 \quad \exists \quad \delta > 0 \quad \text{s.t} \quad |f(x) - L| < \varepsilon \quad \text{when} \quad 0 < |x - a| < \delta.$$

We then write

$$\lim_{x \to a} f(x) = L \quad \text{or equivalently} \quad f(x) \to L \quad \text{as} \quad x \to a.$$

If there is no such L then we say that *no limit exists*.

The limit does not require that $f(a)$ is equal to L or even that $f(a)$ exists. This is because of the inequality $0 < |x - a|$ in the definition.

There is a lot more of this ε and δ stuff in Analysis I, where proofs concerning limits are considered. In this course we shall be less formal and deal only with methods to calculate limits or see that no limit exists.

Rough idea #3:
If $f(a)$ exists then it would seem that this is a good candidate for the limit L. This naive view turns out to be correct only if the graph of $f(x)$ is a single unbroken curve with no holes or jumps, at least in a small region around $x = a$.

Defn. A function $f(x)$ is **continuous at the point** $x = a$ if the following three properties all hold:
• $f(a)$ exists.
• $\lim_{x\to a} f(x)$ exists.
• $\lim_{x\to a} f(x)$ is equal to $f(a)$.

Defn. A function $f(x)$ is **continuous on a subset** S of its domain if it is continuous at every point in S.

Defn. A function $f(x)$ is **continuous** if it is continuous at every point in its domain.

Eg. $f(x) = x^2$ is continuous and $\lim_{x\to 2} x^2 = f(2) = 4$.

Eg. The following function has a discontinuity at $x = 0$

$$f(x) = \begin{cases} x^2 & \text{if } -1 \leq x < 0 \\ 1 & \text{if } x = 0 \\ x^2 & \text{if } 0 < x \leq 1 \end{cases}$$

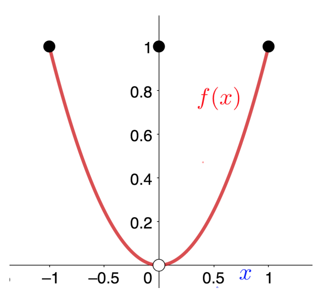


Figure 11: A graph illustrating the use of open and closed circles. The limit exists at $x = 0$.

As mentioned earlier, to indicate on a graph exactly which points are included we use a closed circle to denote an included point, so if it is at the end of a curve segment this denotes that this end is a closed interval. We use an open circle to denote an excluded point associated with an open interval. This notation is demonstrated in Figure 11 where the graph of the above function is presented. This function is not continuous at $x = 0$ but the limit exists at this point and

$\lim_{x\to 0} f(x) = 0 \neq 1 = f(0)$.

Eg. The following function also has a discontinuity at $x = 0$ (see Figure 12 for its graph)

$$f(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

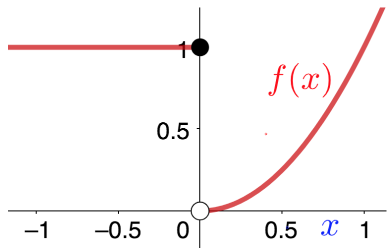


Figure 12: A graph illustrating the use of open and closed circles. No limit exists at $x = 0$.

In this case no limit exists at $x = 0$. This is clear because if we consider $x < 0$ then to get arbitrarily close to the function for x arbitrarily close to 0 would require the limit to be 1. However, if we consider $x > 0$ then to get arbitrarily close to the function for x arbitrarily close to 0 would require the limit to be 0. These two requirements are incompatible and so no limit exists at $x = 0$.

Eg. For the function $f(x) = \sin(1/x)$ no limit exists at $x = 0$ because as x approaches zero this function oscillates between 1 and -1 over smaller and smaller intervals. In particular, in any given interval $0 < x < \delta$ it is always possible to find values x_1 and x_2 s.t. $f(x_1) = 1$ and $f(x_2) = -1$, thus $f(x)$ cannot remain close to any given constant L.

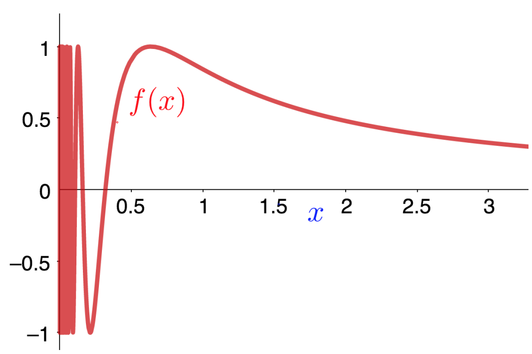


Figure 13: A graph of the function $\sin(1/x)$.

2.2 Facts about limits and continuity

There are some simple **facts about limits** that follow from the definition:

• The limit is unique.

• If $f(x) = g(x)$ (except possibly at $x = a$) on some open interval containing a then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

• If $f(x) \geq K$ on either an interval (a, b) or an interval (c, a)
and if $\lim_{x\to a} f(x) = L$ then $L \geq K$.
(A similar result holds by replacing all $\geq$ with $\leq$).

• If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then
(i) $\lim_{x\to a}(f(x) + g(x)) = L + M$
(ii) $\lim_{x\to a}(f(x)g(x)) = LM$
(iii) if $M \neq 0$ then $\lim_{x\to a}(f(x)/g(x)) = L/M$.
Note: In Durham the results (i),(ii),(iii) are sometimes known as the
Calculus Of Limits Theorem (COLT). However, this seems to be a rather grand name for these results and you will not find this name in any books.

There are also some simple **facts about continuous functions** that follow from the definition and the above results:

• If $f(x)$ and $g(x)$ are continuous functions then so are $f(x) + g(x)$, $f(x)g(x)$ and $f(x)/g(x)$.

• All polynomial, rational, trigonometric and hyperbolic functions are continuous.

• If $f(x)$ is continuous then so is $|f(x)|$.

• If $\lim_{x\to a} g(x) = L$ exists and $f(x)$ is continuous at $x = L$ then $\lim_{x\to a}(f \circ g)(x) = f(L)$.

Eg. All the following functions are continuous:
$2x^3 + x + 7, \quad 3x/(x - 1), \quad |(1 + x^2)/\sin x|, \quad \tan x.$

Note: Although $\tan x$ is continuous, it is not continuous on the interval $[0, \pi]$, because this interval includes the point $x = \pi/2$ which is not in the domain of $\tan x$.

Eg. $\lim_{x\to\pi/2} x^2 \sin x = (\pi/2)^2 \sin(\pi/2) = \pi^2/4$.

Eg. $\lim_{x\to 0} \frac{1}{x}$ does not exist. The value of $f(x)$ can be made greater than any finite constant by taking x sufficiently close to zero, so no limit exists.

Eg. Calculate $\lim_{x\to 3} \frac{2x^2-18}{x-3}$.

$$\frac{2x^2-18}{x-3} = \frac{2(x+3)(x-3)}{x-3} = 2(x+3) \text{ if } x \neq 3.$$

The value of the function at $x = 3$ is irrelevant in defining the limit as $x \to 3$ so

$$\lim_{x\to 3} \frac{2x^2-18}{x-3} = \lim_{x\to 3} 2(x+3) = 12.$$

Eg. Calculate $\lim_{x\to 25} \frac{\sqrt{x}-5}{x-25}$.

$$\frac{\sqrt{x}-5}{x-25} = \frac{(\sqrt{x}-5)(\sqrt{x}+5)}{(x-25)(\sqrt{x}+5)} = \frac{x-25}{(x-25)(\sqrt{x}+5)} = \frac{1}{\sqrt{x}+5} \text{ if } x \neq 25.$$

$$\lim_{x\to 25} \frac{\sqrt{x}-5}{x-25} = \lim_{x\to 25} \frac{1}{\sqrt{x}+5} = \frac{1}{10}.$$

2.3 The pinching theorem

The pinching (squeezing) theorem: If $g(x) \leq f(x) \leq h(x)$ for all $x \neq a$ in some open interval containing a and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} f(x) = L$.

Eg. Calculate $\lim_{x\to 0} x^2 \sin(\frac{1}{x})$.

As $-1 \leq \sin(\frac{1}{x}) \leq 1$ then $-x^2 \leq x^2 \sin(\frac{1}{x}) \leq x^2$. Also $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$.
Hence by the pinching theorem $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$.

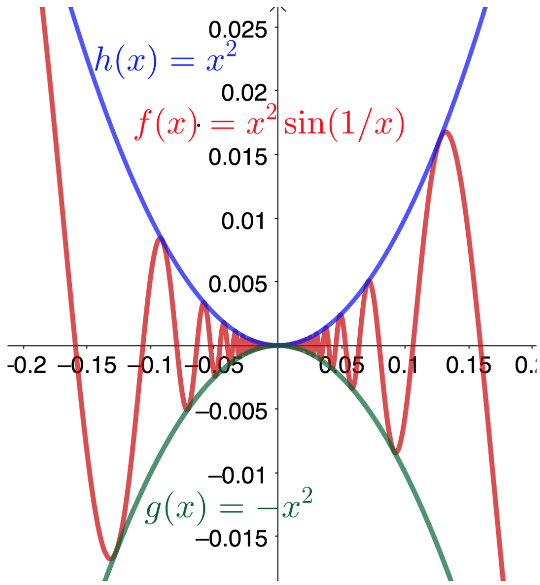


Figure 14: Graphs of $f(x) = x^2 \sin(1/x)$ and the bounding functions $g(x) = -x^2$ and $h(x) = x^2$.

2.4 Two trigonometric limits

Two important trigonometric limits that you may assume to be true and use are:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0}\frac{1-\cos x}{x} = 0.$$

The graphs in Figure 15 provide some evidence to support these results.

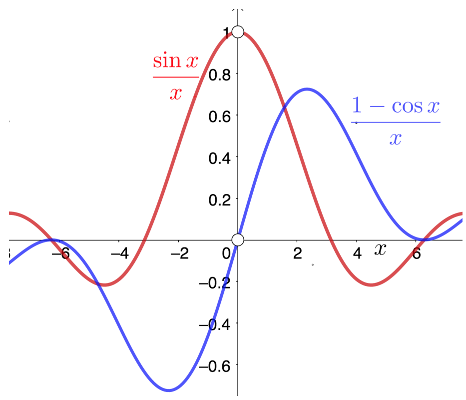


Figure 15: Graphs of $\frac{\sin x}{x}$ and $\frac{1-\cos x}{x}$.

The first of these limits can be proved by applying the pinching theorem but to do this we will need to prove the inequalities $\sin x < x < \tan x$ for $0 < x < \pi/2$.

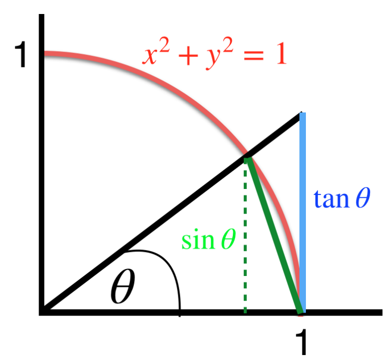


Figure 16: Some geometry to prove some inequalities

Consider the geometry illustrated in the above figure, where $0 < \theta < \pi/2$.
Let T_1 be the area of the triangle with two black sides and a solid green side.
Let S be the area of the sector with two black sides and a red arc of the unit circle.
Let T_2 be the area of the triangle with two black sides and a blue side.
Clearly we have the inequalities $T_1 < S < T_2$.
Now $T_1 = \frac{1}{2}\sin\theta$, $\quad S = \pi\frac{\theta}{2\pi} = \frac{1}{2}\theta$, $\quad T_2 = \frac{1}{2}\tan\theta$ hence $\sin\theta < \theta < \tan\theta$ as required.

Next consider upper and lower bounding functions for $\frac{x}{\sin x}$ where $x \in (0, \pi/2)$. Using the above inequalities for x we have

$$1 = \frac{\sin x}{\sin x} < \frac{x}{\sin x} < \frac{\tan x}{\sin x} = \frac{1}{\cos x}.$$

As all the combinations of functions in this inequality produce even functions then the result extends to $x \in (-\pi/2, 0) \cup (0, \pi/2)$. The limits of the bounding functions are $\lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0}(1/\cos x) = 1$ hence by the pinching theorem $\lim_{x\to 0}(x/\sin x) = 1$ and therefore $\lim_{x\to 0}((\sin x)/x) = 1$.

Now that we have proved the first trigonometric limit we can use it to prove the second,

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \frac{x}{(1+\cos x)} = 0.$$

Here are some more examples that use the first trigonometric limit.

Eg. Calculate $\lim_{x\to 0} \frac{\sin(2x)}{x}$.

$$\lim_{x\to 0} \frac{\sin(2x)}{x} = \lim_{x\to 0} \frac{2\sin(2x)}{2x} = \lim_{u\to 0} \frac{2\sin(u)}{u} = 2 \cdot 1 = 2.$$

The above used the change of variable $u = 2x$ and then the first of the two important trigonometric limits given above.

Eg. Calculate $\lim_{x\to 0} \frac{\tan x}{x}$.

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x\cos x} = \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{\cos x}\right) = 1 \cdot 1 = 1.$$

Eg. Calculate $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.
For $-\pi < x < \pi$

$$\frac{1-\cos x}{x^2} = \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x}$$

Hence by using the first important trigonometric limit we have that

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \left\{ \left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x} \right\} = (1)^2 \cdot \frac{1}{1+1} = \frac{1}{2}.$$

2.5 Classification of discontinuities

If a function $f(x)$ is not continuous at a point $x = a$ then we say it has a discontinuity at $x = a$. There are different types of discontinuity and these are best classified by considering how the function behaves on each side of the point $x = a$. This motivates the definition of the following one-sided limits.

Defn: $f(x)$ has a **right-sided limit** $L^+ = \lim_{x \to a^+} f(x)$ as x tends to a from above if

$$\forall \ \ \varepsilon > 0 \ \ \exists \ \ \delta > 0 \ \ \text{s.t} \ \ |f(x) - L^+| < \varepsilon \ \ \text{when} \ \ 0 < x - a < \delta.$$

Note that the difference between this definition and the definition of the limit is the removal of the modulus sign in $|x - a|$. This means that we only need to worry about points to the right of a when requiring the function to be close to L^+.

Similarly, we have the definition
Defn: $f(x)$ has a **left-sided limit** $L^- = \lim_{x \to a^-} f(x)$ as x tends to a from below if

$$\forall \ \ \varepsilon > 0 \ \ \exists \ \ \delta > 0 \ \ \text{s.t} \ \ |f(x) - L^-| < \varepsilon \ \ \text{when} \ \ 0 < a - x < \delta.$$

Here we only need to worry about points to the left of a when requiring the function to be close to L^-.

It should be obvious that $L = \lim_{x \to a} f(x)$ exists iff L^+ and L^- both exist and are equal. In which case $L = L^+ = L^-$.

There are 3 types of discontinuity:

(i) **Removable discontinuity**.
In this case L exists but $f(a) \neq L$.
The discontinuity can be removed to make the continuous function

$$g(x) = \begin{cases} f(x) & \text{if} \ \ x \neq a \\ L & \text{if} \ \ x = a. \end{cases}$$

Eg. The following function (Figure 17i) has a removable discontinuity at $x = 0$,

$$f(x) = \begin{cases} x^2 & \text{if} \ \ x \neq 0 \\ 1 & \text{if} \ \ x = 0. \end{cases}$$

Removing this discontinuity yields the continuous function $g(x) = x^2$.

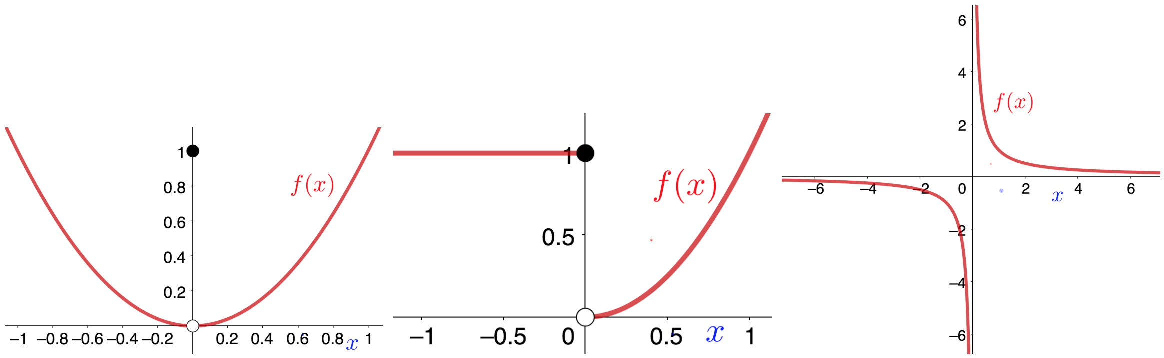


Figure 17: 3 types of discontinuities at $x = 0$: (i) removable; (ii) jump; (iii) infinite.

(ii) **Jump discontinuity.**
In this case both L^+ and L^- exist but $L^+ \neq L^-$.

Eg. The following function (Figure 17ii) has a jump discontinuity at $x = 0$,

$$f(x) = \begin{cases} 1 & \text{if } \ x \leq 0 \\ x^2 & \text{if } \ x > 0. \end{cases}$$

In this example $L^+ = 0 \neq 1 = L^-$.

Eg. The signum function

$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

has a jump discontinuity at $x = 0$. In this example $L^+ = 1 \neq -1 = L^-$.

(iii) **Infinite discontinuity.**
In this case at least one of L^+ or L^- does not exist.

Eg. The function $f(x) = 1/x$ (Figure 17iii) has an infinite discontinuity at $x = 0$.
In this example neither L^+ nor L^- exist.

2.6 Limits as $x \to \infty$.

So far we have only been concerned with the limit of a function $f(x)$ as x approaches a finite point a. However, it is also possible to define a limit as $x \to \infty$.

Rough idea #4:
A function $f(x)$ has a limit L as $x \to \infty$ if $f(x)$ can be kept arbitrarily close to L by making x sufficiently large.

Defn: $f(x)$ has a **limit** L **as** x **tends to** ∞ if

$$\forall \ \ \varepsilon > 0 \ \ \exists \ \ S > 0 \ \ \text{s.t} \ \ |f(x) - L| < \varepsilon \ \ \text{when} \ \ x > S.$$

We then write

$$\lim_{x\to\infty} f(x) = L \quad \text{or equivalently} \quad f(x) \to L \ \ \text{as} \ \ x \to \infty.$$

If there is no such L then we say that *no limit exists*.

The corresponding definition associated with $\lim_{x\to-\infty} f(x) = L$ should be obvious
Defn: $f(x)$ has a **limit** L **as** x **tends to** $-\infty$ if

$$\forall \ \ \varepsilon > 0 \ \ \exists \ \ S < 0 \ \ \text{s.t} \ \ |f(x) - L| < \varepsilon \ \ \text{when} \ \ x < S.$$

Eg. $\lim_{x\to\infty} \frac{1}{x} = 0$. This should be clear because $\frac{1}{x}$ can be made as close to zero as required by making x sufficiently large. An easy way to calculate limits as $x \to \infty$ is to first make the substitution $u = 1/x$ and then use the fact that if $\lim_{x\to\infty} f(x)$ exists then $\lim_{x\to\infty} f(x) = \lim_{u\to 0^+} f(1/u)$. This has transformed the limit into one that we are already familiar with.
Eg. $\lim_{x\to\infty} \frac{1}{x} = \lim_{u\to 0^+} u = 0$.

Eg. Calculate $\lim_{x\to\infty} \frac{x\cos(1/x)+2}{x}$. By using the substitution $u = 1/x$ we have that

$$\lim_{x\to\infty} \frac{x\cos(1/x)+2}{x} = \lim_{u\to 0^+} \frac{\frac{1}{u}\cos u + 2}{\frac{1}{u}} = \lim_{u\to 0^+} (\cos u + 2u) = \cos(0) + 0 = 1.$$

Our earlier discussion of a horizontal asymptote can now be made more precise, in that the graph of a function $f(x)$ will have a horizontal asymptote $y = L$ if $\lim_{x\to\infty} f(x) = L$.

Eg. $$\lim_{x\to\infty} \frac{2x+3}{x+5} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{1+\frac{5}{x}} = \frac{2+0}{1+0} = 2.$$

Here we have used the result that $\lim_{x\to\infty} \frac{1}{x} = 0$, together with the earlier facts about limits. We see that calculating this limit shows that the graph of this rational function has the horizontal asymptote $y = 2$, reproducing our earlier observations about the horizontal asymptotes of rational functions.

2.7 The intermediate value theorem

The intermediate value theorem states that if $f(x)$ is continuous on $[a, b]$ and u is any number between $f(a)$ and $f(b)$ (ie. either $f(a) < u < f(b)$ or $f(b) < u < f(a)$) then $\exists$ (at least one) $c \in (a, b)$ s.t. $f(c) = u$.

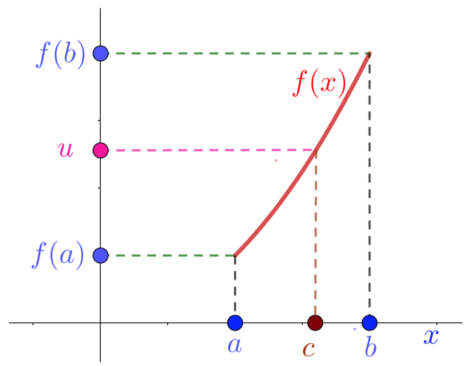


Figure 18: An illustration of the intermediate value theorem.

Eg. $f(x) = \sin x$ is continuous on $[0, \pi/2]$ and $f(0) = 0 < \frac{1}{2} < 1 = f(\pi/2)$ so by the intermediate value theorem there is (at least) one point x in $(0, \pi/2)$ s.t. $\sin x = \frac{1}{2}$.

It is important that $f(x)$ is continuous throughout the interval, otherwise the theorem does not apply.

Eg. $f(x) = \frac{\text{sgn}(x)}{1+x^2}$ has $f(-1) = -\frac{1}{2} < \frac{1}{5} < \frac{1}{2} = f(1)$ but there is no x in $(-1, 1)$ s.t. $f(x) = \frac{1}{5}$. The intermediate value theorem does not apply because $f(x)$ is not continuous at $x = 0$. There is a jump discontinuity at $x = 0$ with $\lim_{x\to 0^+} f(x) = 1 \neq -1 = \lim_{x\to 0^-} f(x)$.

An application of the intermediate value theorem is locating the zeros of a function. If the function $f(x)$ is continuous on $[a, b]$ and we know that either $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$ then by the intermediate value theorem the equation $f(x) = 0$ has at least one root between a and b.

Eg. The function $f(x) = x^2 - 2$ is continuous on $[1, 2]$ with $f(1) = -1 < 0$ and $f(2) = 2 > 0$, so there is at least one root of the equation $x^2 - 2 = 0$ in $(1, 2)$ (of course the root is $x = \sqrt{2} = 1.4142...$).

A repeated iterated application of this approach gives the *bisection method*, which can be used to locate the roots of a wide variety of equations to any desired accuracy.

2.8 Summary: Limits and continuity

You should have a good understanding of the concept of a limit as well as knowing the precise definition of a limit, and how continuity is defined in terms of limits. You should be familiar with methods to calculate limits. Here are some key points:

- You should know the ε-δ definition of a limit $\lim_{x\to a} f(x)$ and understand the concept of taking x closer and closer to a but not reaching a (so the limit does not depend in any way on $f(a)$).
- A function $f(x)$ is *continuous* at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.
- Any algebraic combination of continuous functions gives a continuous function assuming the combination does to involve division by zero since limits of algebraic combinations of functions can be evaluated as the algebraic combinations of the limits (assuming they exist and don't involve dividing by zero).
- Indeterminate limits of the form $0/0$ or ∞/∞ may be well-defined, and if so can be calculated by cancelling common factors in the numerator and denominator.
- You should know how to use the *Pinching (Squeezing) theorem*.
- $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.
- You should know the definitions of *left-sided* and *right-sided limits* and how these are used to classify *discontinuities* at $x = a$.
- You should understand the concept of limits as $x \to \pm\infty$ and that they are equivalent to one-sided limits $u \to 0^{\pm}$ by changing variable to $u = 1/x$.
- You should know what the *Intermediate Value Theorem (IVT)* is and how it can be used to prove the existence of, and numerically approximate, zeros of continuous functions.

3 Differentiation

3.1 Derivative as a limit

Geometrically the derivative $f'(a)$ of a function $f(x)$ at $x = a$ is equal to the slope of the tangent to the graph of $f(x)$ at $x = a$.

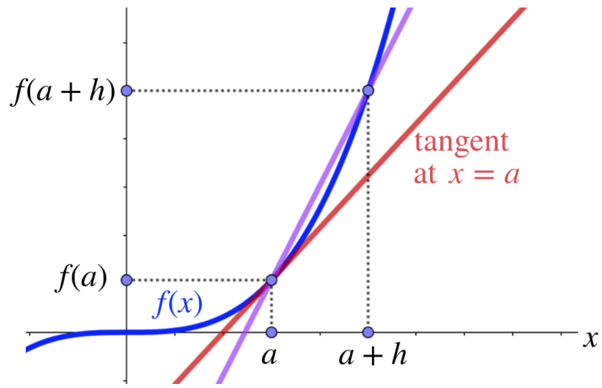


Figure 19: The graph of a function $f(x)$, the tangent at $x = a$ and the secant through the points on the graph at $x = a$ and $x = a + h$.

A secant is a line that intersects a curve at two points (the part of the secant that is between the two intersection points is called a chord).
Consider a secant that intersects the graph of $f(x)$ at the two (x, y) points $(a, f(a))$ and $(a + h, f(a + h))$. The slope of this secant is given by the difference quotient $(f(a + h) - f(a))/h$. As the two points approach each other ie. as $|h|$ decreases, the secant approaches the tangent at $x = a$. The tangent is obtained in the limit as $h \to 0$ and the derivative at a is the slope of the secant in this limit ie.

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

providing this limit exists.
If this limit exists then we say that $f(x)$ **is differentiable at** $x = a$.
Eg. Use the limit definition of the derivative to calculate $f'(a)$ for $f(x) = x^2$.

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{(a + h)^2 - a^2}{h} = \lim_{h \to 0} \frac{2ah + h^2}{h} = \lim_{h \to 0}(2a + h) = 2a.$$

This is, of course, the expected result from the knowledge that $f'(x) = 2x$.

Eg. Use the limit definition of the derivative to calculate $f'(\pi)$ for $f(x) = \sin x$.

$$f'(\pi) = \lim_{h \to 0} \frac{f(\pi + h) - f(\pi)}{h} = \lim_{h \to 0} \frac{\sin(\pi + h) - \sin \pi}{h} = \lim_{h \to 0} \frac{\sin \pi \cos h + \cos \pi \sin h}{h}$$

$$= \lim_{h\to 0} \frac{-\sin h}{h} = -1$$

where the final equality uses the earlier important trigonometric limit result.
This is, of course, the expected answer from the knowledge that $f'(x) = \cos x$ giving $f'(\pi) = \cos\pi = -1$.

If $f'(a)$ exists for all a in Dom f we say that $f(x)$ **is differentiable** and then $f'(a)$ defines a function $f'(x)$ called the derivative.

Eg. Use the limit definition of the derivative to calculate the derivative of the function $f(x) = x\cos x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)\cos(x+h) - x\cos x}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)(\cos x\cos h - \sin x \sin h) - x\cos x}{h}$$

$$= \lim_{h\to 0} \left(x\cos x \frac{\cos h - 1}{h} - x\sin x \frac{\sin h}{h} + \cos x \cos h - \sin x \sin h \right) = -x\sin x + \cos x$$

where we have used both of the earlier important trigonometric limit results.

The fact that the derivative is equal to the slope of the tangent of the graph allows us to determine a Cartesian equation for the tangent by calculating the derivative. Explicitly, the tangent line at the point $(a, f(a))$ is given by $y = f(a) + f'(a)(x-a)$.

Eg. For the function $f(x) = 4x^3 - x$, find a Cartesian equation for the tangent to the graph at the point $(1, 3)$.
$f'(x) = 12x^2 - 1$ so $f'(1) = 11$. The tangent line is given by
$y = f(1) + f'(1)(x-1) = 3 + 11(x-1) = 11x - 8$

A necessary condition for a function to be differentiable at a point is that it is continuous at that point, but this is not a sufficient condition.

Geometrically, there are two ways that a function can fail to be differentiable at a point where it is continuous:

(i). The tangent line is vertical at that point.

Eg. The function $f(x) = x^{1/3}$ is continuous in $\mathbb{R}$ but it is not differentiable at $x = 0$. The difference quotient at $x = 0$ is

$$\frac{f(0+h) - f(0)}{h} = \frac{h^{1/3}}{h} = \frac{1}{h^{2/3}}$$

and no limit exists as $h \to 0$ because the above grows without bound.

(ii). There is no tangent line at that point.

Eg. The function $f(x) = |x|$ is continuous in $\mathbb{R}$ but it is not differentiable at $x = 0$. The difference quotient at $x = 0$ is

$$\frac{f(0+h) - f(0)}{h} = \frac{|h|}{h} = \begin{cases} -1 & \text{if } h < 0 \\ 1 & \text{if } h > 0 \end{cases}$$

The left and right limits therefore both exist

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = 1$$

and

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = -1$$

but these are not equal, so

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist.

3.2 The Leibniz and chain rules

If $f(x)$ is differentiable then its derivative $f'(x)$ may also be differentiable, in which case we denote its derivative by $f''(x)$ ie. the **second derivative** of $f(x)$. Similarly the third derivative is $f'''(x)$ and in general the n^{th} derivative is $f^{(n)}(x)$ so that $f^{(3)}(x) = f'''$. This notation is due to Lagrange (1736-1813), but other notations are also common.

$f'(x) = \frac{df}{dx},\ f''(x) = \frac{d^2 f}{dx^2},\ f^{(n)}(x) = \frac{d^n f}{dx^n}$ is due to Leibniz (1646-1716).

$f'(x) = Df(x),\ f''(x) = D^2 f(x),\ f^{(n)}(x) = D^n f(x)$ is due to Euler (1707-1783).

Finally, if the independent variable represents time, say $f(t)$, then $f'(t) = \dot{f}$ and $f''(t) = \ddot{f}$ is due to Newton (1642-1727). As we have seen, the derivative $\frac{df}{dx}$ is the slope of the tangent to the graph of $f(x)$. This means that the derivative $\frac{df}{dx}$ measures the rate of change of the dependent variable f with respect to changes in the independent variable x. In the case that the independent variable is time t, the derivative measures the rate of change with time, so if the dependent variable, say $X(t)$ represents the position of an object confined to a line then $\dot{X}$ is the velocity of the object, with $|\dot{X}|$ the speed, and $\ddot{X}$ is the acceleration of the object.

The product rule for differentiation is just the first case of the more general **Leibniz rule**:
If $f(x)$ and $g(x)$ are both differentiable n times then so is the product $f(x)g(x)$ with

$$D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} (D^k f)(D^{n-k} g)$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient familiar from Pascal's triangle

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Figure 20: The binomial coefficients arranged into Pascal's triangle.

Eg. Use the Leibniz rule to calculate $D^3(x^2 \sin x)$ using
$D^3(fg) = f(D^3 g) + 3(Df)(D^2 g) + 3(D^2 f)(Dg) + (D^3 f)g.$
$f = x^2, \quad Df = 2x, \quad D^2 f = 2, \quad D^3 f = 0.$
$g = \sin x, \quad Dg = \cos x, \quad D^2 g = -\sin x, \quad D^3 g = -\cos x.$
$D^3(x^2 \sin x) = -x^2 \cos x - (3)2x \sin x + (3)2 \cos x + 0 \sin x = (6 - x^2)\cos x - 6x \sin x.$

To handle the differentiation of composite functions $(f \circ g)(x) = f(g(x))$ we turn to the following theorem:

The **chain rule theorem** states that if $g(x)$ is differentiable at x and $f(x)$ is differentiable at $g(x)$ then the composition $(f \circ g)(x)$ is differentiable at x with

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Recall that prime denotes differentiation with respect to the argument so in Leibniz notation the above formula may be written more crudely as

$$\frac{d}{dx}f(g(x)) = \frac{df}{dg}\,\frac{dg}{dx}$$

where we need to be aware that on the right hand side the argument of the first factor is $g(x)$ and the argument of the second factor is x.
This form is useful as a mnemonic (think of 'cancelling' the dg factor) but it is no more than this.

Eg. Calculate $\dfrac{d}{dx}\left(\left(x + \dfrac{1}{x}\right)^{-3}\right)$.

In terms of the above notation $g(x) = x + \frac{1}{x}$ and $f(x) = x^{-3}$
giving $g'(x) = 1 - \frac{1}{x^2}$ and $f'(x) = -3x^{-4}$ thus

$$\frac{d}{dx}\left(\left(x + \frac{1}{x}\right)^{-3}\right) = f'(g(x))g'(x) = -3\left(x + \frac{1}{x}\right)^{-4}\left(1 - \frac{1}{x^2}\right).$$

Eg. $\frac{d}{dt}\cos(t^2) = -2t\sin(t^2)$.

3.3 L'Hôpital's rule

*** Only use this method in assignments/exam questions when told to do so ***

Recall that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M \neq 0$ then $\lim_{x\to a} f(x)/g(x) = L/M$. Clearly this result is not applicable to the situation where $L = M = 0$, which is called an **indeterminant form**. We have already seen how to deal with situations like this directly, but an alternative (and often easier) approach is sometimes available by making use of the following.

L'Hôpital's rule

Let $f(x)$ and $g(x)$ be differentiable on $I = (a-h,a) \cup (a,a+h)$ for some $h > 0$, with $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$.

If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists and $g'(x) \neq 0 \ \forall \ x \in I$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Proof: We shall only consider the proof for the slightly easier situation in which $f(x)$ and $g(x)$ are both differentiable at $x = a$ and $g'(a) \neq 0$.

In this case

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

We can apply L'Hôpital's rule to calculate the two important trigonometric limits from earlier.

Eg. Calculate $\lim_{x\to 0} \frac{\sin x}{x}$.

$f(x) = \sin x$ and $g(x) = x$ are both differentiable. Also $\lim_{x\to 0} \sin x = \sin 0 = 0$ and $\lim_{x\to 0} x = 0$. Furthermore, $f'(x) = \cos x$ and $g'(x) = 1 \neq 0$. Thus L'Hôpital's rule applies to give

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1.$$

Eg. Calculate $\lim_{x\to 0} \frac{1-\cos x}{x}$.
$f(x) = 1 - \cos x$ and $g(x) = x$ are both differentiable. Also $\lim_{x\to 0}(1 - \cos x) = 1 - \cos 0 = 0$ and $\lim_{x\to 0} x = 0$. Furthermore, $f'(x) = \sin x$ and $g'(x) = 1 \neq 0$. Thus by L'Hôpital's rule

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{\sin x}{1} = \sin 0 = 0.$$

If applying L'Hôpital's rule yields another indeterminant form then L'Hôpital's rule can be reapplied to this form and so on.

Eg. Calculate $\lim_{x\to 0} \frac{2\sin x - \sin(2x)}{x - \sin x}$.

$$\begin{aligned}\lim_{x\to 0} \frac{2\sin x - \sin(2x)}{x-\sin x} &= \lim_{x\to 0} \frac{2\cos x - 2\cos(2x)}{1-\cos x} = \lim_{x\to 0} \frac{-2\sin x + 4\sin(2x)}{\sin x} \\ &= \lim_{x\to 0} \frac{-2\cos x + 8\cos(2x)}{\cos x} = \frac{-2+8}{1} = 6.\end{aligned}$$

3.4 Boundedness and monotonicity

The following definitions apply to a function $f(x)$ defined in some interval I.

Defn: If $\exists$ a constant k_1 s.t. $f(x) \le k_1 \;\; \forall \;\; x$ in I we say that $f(x)$ is **bounded above in** I and we call k_1 an **upper bound** of $f(x)$ in I.

Furthermore, if $\exists$ a point x_1 in I s.t. $f(x_1) = k_1$ we say that the upper bound k_1 is **attained** and we call k_1 the **global maximum value** of $f(x)$ in I.

Similarly, we have the following:

Defn: If $\exists$ a constant k_2 s.t. $f(x) \ge k_2 \;\; \forall \;\; x$ in I we say that $f(x)$ is **bounded below in** I and we call k_2 a **lower bound** of $f(x)$ in I.

Furthermore, if $\exists$ a point x_2 in I s.t. $f(x_2) = k_2$ we say that the lower bound k_2 is **attained** and we call k_2 the **global minimum value** of $f(x)$ in I.

Defn: $f(x)$ is **bounded in** I if it is both bounded above and bounded below in I ie. if $\exists$ a constant k s.t. $|f(x)| \le k \;\; \forall \;\; x$ in I.

If no interval I is specified then it is taken to be the domain of the function.

Eg. $\cos x$ is bounded (in $\mathbb{R}$) because $|\cos x| \le 1 \; \forall x \in \mathbb{R}$. Both these upper and lower bounds are attained as $\cos 0 = 1$ and $\cos \pi = -1$. The global maximum value in $\mathbb{R}$ is 1 and the global minimum value in $\mathbb{R}$ is -1.

Eg. Consider $f(x) = \frac{\text{sgn}(x)}{1+x^2}$ for $-1 \le x \le 1$. Then $f(x)$ is bounded in $[-1, 1]$ because $|f(x)| \le 1$ for all x in $[-1, 1]$ but neither of the bounds is attained, so it has no global maximum value and no global minimum value in $[-1, 1]$.

Eg. On $[0, \pi/2)$ the function $\tan x$ is bounded below but not bounded above.

A condition that guarantees the existence of both a global maximum and minimum value (called **extreme values**) is provided by the following:

The extreme value theorem states that if f is a continuous function on a *closed* interval $[a, b]$ then it is bounded on that interval and has upper and lower bounds that are attained ie. $\exists$ points x_1 and x_2 in $[a, b]$ such that $f(x_2) \le f(x) \le f(x_1) \; \forall \; x \in [a, b]$.

Defn: $f(x)$ is **monotonic increasing** in $[a, b]$ if $f(x_1) \le f(x_2)$ for all pairs x_1, x_2 with $a \le x_1 < x_2 \le b$.

Defn: $f(x)$ is **strictly monotonic increasing** in $[a, b]$ if $f(x_1) < f(x_2)$ for all pairs x_1, x_2 with $a \le x_1 < x_2 \le b$.

Obvious definitions apply with increasing replaced by decreasing.

Eg. sgn(x) is monotonic increasing in $[-1, 1]$ but not strictly.

Eg. x^2 is strictly monotonic increasing in $[0, b]$ for any $b > 0$.

3.5 Critical points

Defn: We say that $f(x)$ has a **local maximum** at $x = a$ if $\exists\ h > 0$ s.t. $f(a) \geq f(x)\ \ \forall\ x \in (a-h, a+h)$.

Remark: If $f(x)$ has a local maximum at $x = a$ and is differentiable at this point then $f'(a) = 0$.

Proof: As $f(x)$ is differentiable at a then

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = f'(a).$$

$f(x)$ has a local maximum at a implies

$$\frac{f(x) - f(a)}{x - a} \geq 0 \quad \text{for } x \in (a-h, a)$$

$$\text{hence} \quad f'(a) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \geq 0.$$

Similarly, $f(x)$ has a local maximum at a also implies

$$\frac{f(x) - f(a)}{x - a} \leq 0 \quad \text{for } x \in (a, a+h)$$

$$\text{hence} \quad f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0.$$

Putting these two together gives $f'(a) \leq 0 \leq f'(a)$ and thus $f'(a) = 0$.

Defn: We say that $f(x)$ has a **local minimum** at $x = a$ if $\exists\ h > 0$ s.t. $f(x) \geq f(a)\ \ \forall\ x \in (a-h, a+h)$.

Remark: If $f(x)$ has a local minimum at $x = a$ and is differentiable at this point then $f'(a) = 0$.

Defn: $f(x)$ has a **stationary point** at $x = a$ if it is differentiable at $x = a$ with $f'(a) = 0$.

Defn: An interior point $x = a$ of the domain of $f(x)$ is called a **critical point** if either $f'(a) = 0$ or $f'(a)$ does not exist.

Every local maximum and local minimum of a differentiable function is a stationary point but there may be other stationary points too.

Eg. $f(x) = x^3$ has $f'(0) = 0$ so $x = 0$ is a stationary point but it is neither a local maximum nor a local minimum because $f(x) > f(0)$ for $x \in (0, h)$ but $f(x) < f(0)$ for $x \in (-h, 0)$.

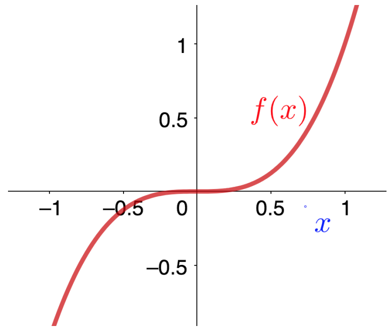


Figure 21: The graph of $f(x) = x^3$ indicating the point of inflection at $x = 0$.

Defn: If $f(x)$ is twice differentiable in an open interval around $x = a$ with $f''(a) = 0$ and if $f''(x)$ changes sign at $x = a$ then we say that $x = a$ is a **point of inflection.**

Eg. $f(x) = x^3$ has $f''(x) = 6x$ so $f''(0) = 0$. As $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$ then $f''(x)$ changes sign at $x = 0$ so it is a point of inflection.

The first derivative test
Suppose $f(x)$ is continuous at a critical point $x = a$.

(i). If $\exists\, h > 0$ s.t. $f'(x) < 0\ \forall\, x \in (a-h, a)$ and $f'(x) > 0\ \forall\, x \in (a, a+h)$ then $x = a$ is a local minimum.

(ii). If $\exists\, h > 0$ s.t. $f'(x) > 0\ \forall\, x \in (a-h, a)$ and $f'(x) < 0\ \forall\, x \in (a, a+h)$ then $x = a$ is a local maximum.

(iii). If $\exists\, h > 0$ s.t. $f'(x)$ has a constant sign $\forall\, x \neq a$ in $(a-h, a+h)$ then $x = a$ is a not a local extreme value (minimum/maximum).

Eg. $f(x) = |x|$ is continuous but $f'(x) \neq 0$ so there are no stationary points. The derivative does not exist at $x = 0$ so this is a critical point. $f'(x) < 0$ for $x \in (-1, 0)$ and $f'(x) > 0$ for $x \in (0, 1)$ so by the first derivative test $x = 0$ is a local minimum. In fact $f(0) = 0$ is a global minimum as $|x| \geq 0$.

Eg. $f(x) = x^4 - 2x^3$ has $f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3)$. The only critical points are $x = 0, 3/2$. Now $f'(x) < 0$ for $x < 3/2$ and $f'(x) > 0$ for $x > 3/2$. Thus $f'(x)$ has constant sign on both (sufficiently close) sides of $x = 0$ so this is not a local extreme value. However, $f'(x)$ changes sign from negative to positive as x passes through $x = 3/2$ so this is a local minimum.

The second derivative test
Suppose $f(x)$ is twice differentiable at $x=a$ with $f'(a)=0$.
(i). If $f''(a)>0$ then $x=a$ is a local minimum.
(ii). If $f''(a)<0$ then $x=a$ is a local maximum.

Note that the theorem doesn't say anything about the case $f''(a)=0$.

Eg. $f(x)=2x^3-9x^2+12x$ has $f'(x)=6x^2-18x+12=6(x^2-3x+2)=6(x-1)(x-2)$ giving stationary points at $x=1$ and $x=2$. Furthermore, $f''(x)=6(2x-3)$ giving $f''(1)=-6<0$ and $f''(2)=6>0$ so by the second derivative test $x=1$ is a local maximum and $x=2$ is a local minimum.

Note that the first derivative test is more general than the second derivative test as it does not require the function to be differentiable at the critical point.

Determining all critical points and asymptotes is the key to sketching the graph of a function.
Eg. Sketch the graph of the function $f(x)=\frac{4x-5}{x^2-1}$.

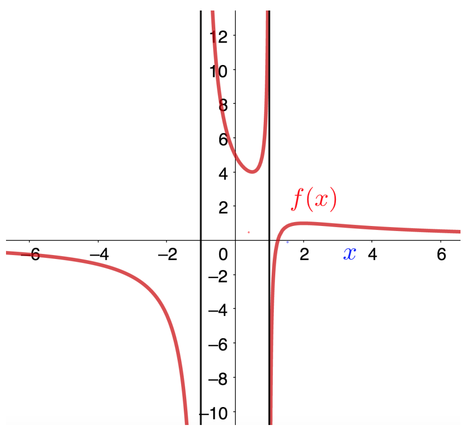


Figure 22: Graph of $f(x)=\frac{4x-5}{x^2-1}$.

$y=0$ is a horizontal asymptote along both the positive and negative x-axis because

$$\lim_{x\to\pm\infty}\frac{4x-5}{x^2-1}=\lim_{x\to\pm\infty}\frac{1}{x}\left(\frac{4-5/x}{1-\frac{1}{x^2}}\right)=0\left(\frac{4-0}{1-0^2}\right)=0$$

There are vertical asymptotes at $x=\pm 1$. Also

$$f'(x)=\frac{4(x^2-1)-2x(4x-5)}{(x^2-1)^2}=\frac{-4x^2+10x-4}{(x^2-1)^2}=\frac{-(4x-2)(x-2)}{(x^2-1)^2}$$

so there are stationary points at $x=\frac{1}{2},2$ with $f(\frac{1}{2})=4$ and $f(2)=1$.

x		-1		$\frac{1}{2}$		1		2	
f'	$-$		$-$		$+$		$+$		$-$

As x passes through $\frac{1}{2}$ then $f'(x)$ changes sign from negative to positive so this is a local minimum. As x passes through 2 then $f'(x)$ changes sign from positive to negative so this is a local maximum.

If a function is defined on an interval then extreme values can occur at the endpoints of the interval. Here an **endpoint** $x = c$ is a point at which the function is defined but where the function is undefined either to the left or right of this point. For example, if the domain of a function is $[a, b]$ then a and b are both endpoints.

Defn: If c is an endpoint of $f(x)$ then f has an **endpoint maximum** at $x = c$ if $f(x) \leq f(c)$ for x sufficiently close to c.

The obvious similar definition applies for an **endpoint minimum**.

If a function is differentiable sufficiently close to an endpoint then examining the sign of the derivative can determine if there is an endpoint maximum or minimum.

If $f(x)$ is continuous on an interval $[a, b]$ then the global extreme values in this interval are attained at either critical points or endpoints, so we can determine them by examining all the possibilities.

Eg. Find the global extreme values of $f(x) = 1 + 4x^2 - \frac{1}{2}x^4$ for $x \in [-1, 3]$.

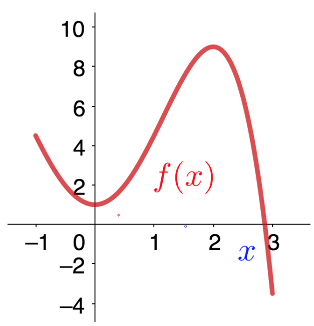


Figure 23: Graph of $f(x) = 1 + 4x^2 - \frac{1}{2}x^4$

$f'(x) = 8x - 2x^3 = 2x(4 - x^2)$ which is zero at $x = 0, 2$; recall we only consider $x \in [-1, 3]$.
Critical point: $f(0) = 1$ and $f(2) = 9$
End points: $f(-1) = \frac{9}{2}$ and $f(3) = -\frac{7}{2}$.
The smallest of these four values is $-\frac{7}{2}$, which is the global minimum, whereas the largest is 9 which is the global maximum.

Eg. Find the global extreme values of $x^2 - 2|x| + 2$ for $x \in [-\frac{1}{2}, 2]$.
First of all we can write

$$f(x) = \begin{cases} x^2 + 2x + 2 & \text{if} \quad -\frac{1}{2} \le x < 0 \\ x^2 - 2x + 2 & \text{if} \quad\quad 0 \le x \le 2 \end{cases}$$

$f'(x) = 2x + 2$ for $x \in [-\frac{1}{2}, 0)$ so no critical points here.
$f'(x) = 2x - 2$ for $x \in (0, 2]$ so a critical point at $x = 1$ with $f(1) = 1$.
$\lim_{x\to 0^+} f'(x) = -2 \neq 2 = \lim_{x\to 0^-} f'(x)$ hence $f'(x)$ does not exist at $x = 0$, making this a critical point with $f(0) = 2$.
Endpoints: $f(-\frac{1}{2}) = \frac{5}{4}$ and $f(2) = 2$.
Thus the global minimum is 1 and the global maximum is 2.

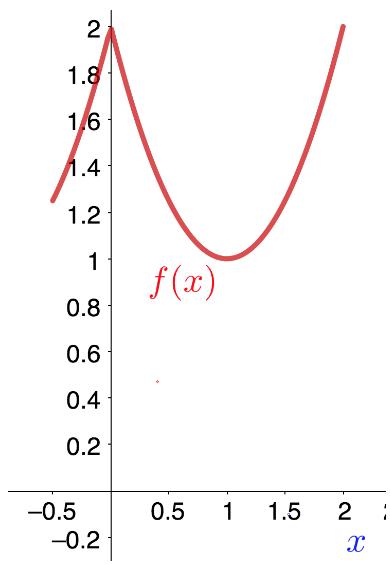


Figure 24: Graph of a piecewise function with two quadratic pieces.

3.6 Rolle's theorem

Rolle's theorem states that if f is differentiable on the open interval (a, b) and continuous on the closed interval $[a, b]$, with $f(a) = f(b)$, then there is at least one $c \in (a, b)$ for which $f'(c) = 0$.

Proof: By the extreme value theorem $\exists\ x_1$ and x_2 in $[a, b]$ s.t.
$f(x_1) \leq f(x) \leq f(x_2)\ \forall\ x \in [a, b]$.
If $x_1 \in (a, b)$ then x_1 is a local minimum and $f'(x_1) = 0$ so we are done.
If $x_2 \in (a, b)$ then x_2 is a local maximum and $f'(x_2) = 0$ so we are done.
The only case that is left is if both x_1 and x_2 are endpoints, a, b. But since $f(a) = f(b)$ then in this case $f(x_1) = f(x_2) = f(a)$ so the above bound becomes $f(a) \leq f(x) \leq f(a)\ \forall\ x \in [a, b]$. Thus $f(x) = f(a)$ is constant in the interval and $f'(x) = 0\ \forall\ x \in [a, b]$.

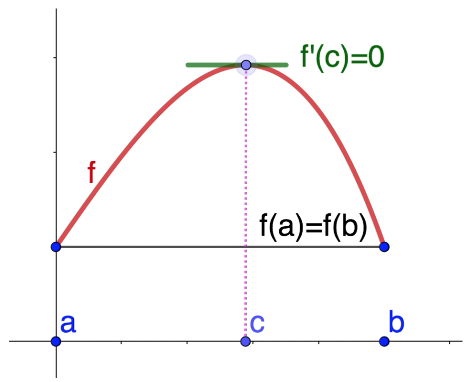


Figure 25: Illustration of Rolle's theorem.

An obvious corollary of Rolle's theorem is that if $f(x)$ is differentiable at every point of an open interval I then each pair of zeros of the function in I is separated by at least one zero of $f'(x)$. This is simply the special case of Rolle's theorem with a and b taken to be the the zeros of $f(x)$ so that $f(a) = f(b) = 0$.

An application of the above result is to provide a limit on the number of distinct real zeros of a function.

Eg. Show that $f(x) = \frac{1}{7}x^7 - \frac{1}{5}x^6 + x^3 - 3x^2 - x$ has no more than 3 distinct real roots.
Suppose the required result is false and there are (at least) 4 distinct real roots of $f(x)$. Then by Rolle's theorem there are (at least) 3 distinct real roots of $f'(x)$. Then by applying Rolle's theorem to $f'(x)$ there are (at least) 2 distinct real roots of $f''(x)$. However, $f''(x) = 6x^5 - 6x^4 + 6x - 6 = 6(x-1)(x^4+1)$ has only one real root (at $x = 1$). This contradiction proves the required result.

3.7 The mean value theorem

The mean value theorem states that if f is differentiable on the open interval (a, b) and continuous on the closed interval $[a, b]$, then there is at least one $c \in (a, b)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

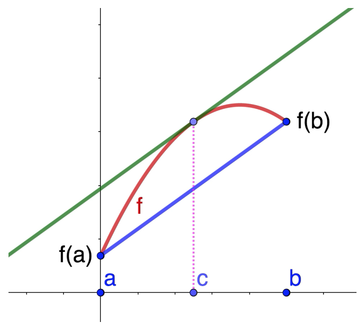


Figure 26: Illustration of the mean value theorem.

Geometrically this theorem states that there is at least one point in the interval at which the tangent to the curve is parallel to the line joining the two points $(b, f(b))$ and $(a, f(a))$ at the ends of the interval. The geometry makes the result clear but here is the proof.
Proof: Define the function

$$g(x) = (b - a)(f(x) - f(a)) - (x - a)(f(b) - f(a)).$$

Then $g(a) = g(b) = 0$ and as $g(x)$ satisfies the requirements of Rolle's theorem then $\exists\ c \in (a, b)$ for which $g'(c) = 0$. As $g'(x) = (b - a)f'(x) - (f(b) - f(a))$ then setting $x = c$ yields the required result $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Note that Rolle's theorem is just a special case of the mean value theorem with $f(a) = f(b)$.

The mean value theorem can be used to prove some obvious results relating monotonicity to the sign of the derivative. Here is an example:

Suppose $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) with $f'(x) \geq 0$ throughout this interval. Then $f(x)$ is monotonic increasing in (a, b).
Proof: If $a \leq x_1 < x_2 \leq b$ then by the mean value theorem $\exists\ c \in (a, b)$ s.t. $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. However, since $f'(x) \geq 0$ in (a, b) then $f'(c) \geq 0$ which imples that $f(x_2) \geq f(x_1)$ ie. f is monotonic increasing.

Similar obvious results and proofs follow for the cases of monotonic decreasing and strictly monotonic increasing/decreasing.

3.8 The inverse function rule

The derivative of the inverse of a function can be obtained from the following:

The inverse function rule states that if $f(x)$ is continuous in $[a,b]$ and differentiable in (a,b) with $f'(x) > 0$ throughout this interval then its inverse function $g(y)$ (recall $g(f(x)) = x$) is differentiable for all $f(a) < y < f(b)$ with $g'(y) = 1/f'(g(y))$.

Note: A similar theorem exists for the case $f'(x) < 0$.

Eg. $f(x) = \sin x$ has $f'(x) = \cos x > 0$ for $x \in (-\pi/2, \pi/2)$.
Its inverse function $g(y) = \sin^{-1} y$ is therefore differentiable in $(-1,1)$ with

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{\cos g(y)} = \frac{1}{\sqrt{1-\sin^2 g(y)}}$$

but $y = f(g(y)) = \sin(g(y))$ hence $g'(y) = \frac{1}{\sqrt{1-y^2}}$. Thus we have shown that

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}.$$

Eg. $f(x) = \tan x$ has $f'(x) = \sec^2 x > 0$ for $x \in (-\pi/2, \pi/2)$.
With this domain then Ran $f = \mathbb{R}$ hence the domain of the inverse function $g(y) = \tan^{-1}(y)$ is $\mathbb{R}$.
By the inverse function rule

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{\sec^2(g(y))} = \frac{1}{1+\tan^2(g(y))} = \frac{1}{1+y^2}$$

where we have used $y = f(g(y)) = \tan(g(y))$.
Thus we have shown that

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}.$$

3.9 Partial derivatives

So far we have only considered functions of a single variable eg. $f(x)$. Later we shall study functions of two variables eg. $f(x, y)$. For this situation we shall need the concept of a **partial derivative**. The partial derivative of f with respect to x is written as $\frac{\partial f}{\partial x}$ and is obtained by differentiating f with respect to x, keeping y fixed ie. y may be treated as a constant in performing the partial differentiation with respect to x.

$$\text{Eg.} \quad f(x, y) = x^2y^3 - \sin x \cos y \, + y \quad \text{has} \quad \frac{\partial f}{\partial x} = 2xy^3 - \cos x \cos y.$$

Similarly, the partial derivative of f with respect to y is written as $\frac{\partial f}{\partial y}$ and is obtained by differentiating f with respect to y, keeping x fixed.

$$\text{Eg.} \quad f(x, y) = x^2y^3 - \sin x \cos y \, + y \quad \text{has} \quad \frac{\partial f}{\partial y} = 3x^2y^2 + \sin x \sin y \, + 1.$$

Partial derivatives are defined in terms of the following limits (which we assume exist)

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}, \quad \frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

An alternative notation for partial derivatives is to write f_x for $\frac{\partial f}{\partial x}$ etc.

By applying partial differentiation to these partial derivatives we may obtain the second order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

For the earlier example $f(x, y) = x^2y^3 - \sin x \cos y \, + y$ we obtain

$$\frac{\partial^2 f}{\partial x^2} = 2y^3 + \sin x \cos y, \quad \frac{\partial^2 f}{\partial y^2} = 6x^2y + \sin x \cos y,$$

$$\frac{\partial^2 f}{\partial y \partial x} = 6xy^2 + \cos x \sin y, \quad \frac{\partial^2 f}{\partial x \partial y} = 6xy^2 + \cos x \sin y.$$

Note the equality, in this example, of the two mixed partial derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

It can be shown that this result is true in general, if f and all first and second order partial derivatives are continuous.

An alternative notation for partial derivatives is

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}$$

etc

3.10 Summary: Differentiation

You should have a good understanding of the definition of the derivative, the general Leibniz rule, L'Hôpital's rule, local and global extreme values and how to find them, the Mean Value Theorem (MVT) and the equivalent Rolle's theorem, the inverse function rule and partial derivatives. Here are some key points:

- The *derivative* of $f(x)$ at $x = a$ is (if it exists) $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.
- The *Leibniz rule* is $(fg)' = f'g + fg'$ and extends in an obvious way to products of more than two functions, e.g. $(fgh)' = f'gh + fg'h + fgh'$.
- The *general Leibniz rule* gives an efficient way to calculate higher order derivatives of a product of two functions, specifically $D^n(fg) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (D^k f)(D^{n-k} g)$.
- The *chain rule* tells us how to differentiate a function of a function, $(f \circ g)'(x) = f'(g(x))g'(x)$.
- It may be possible to evaluate an indeterminate limit $0/0$ or ∞/∞ by *L'Hôpital's rule* $\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(x)/g'(x)$.
- The *extreme value theorem* states that continuous functions on closed intervals always attain upper and lower bounds (*global maxima and minima*).
- A *monotonically increasing/decreasing* function $f(x)$ on an interval is a function which never decreases/increases as x increases on that interval. *Strictly monotonic* excludes the possibility that the function may be constant on some subinterval.
- A function $f(x)$ has a *local maximum/minimum* at $x = a$ if $f(a)$ is a global maximum/minimum on some open interval containing a.
- If f is differentiable and has a local max/min at $x = a$ then it has a *stationary point* at $x = a$, i.e. $f'(a) = 0$. You can test whether a stationary point is a local max or local min or neither with the *first derivative test*, or often with the *second derivative test*.
- All *extreme values* (maxima and minima) of a function on an interval will be at *critical points* (stationary points or points where the function is not differentiable) or at the *endpoints* of the interval. If you are looking for the global extreme values, just evaluate the function at all those points – no need to identify whether each point is a local max or min.
- If f is continuous on $[a, b]$ and differentiable on (a, b) then the *Mean Value Theorem (MVT)* says that $\exists c \in (a, b)$ s.t. $f'(c)$ equals the gradient of the straight line from $(a, f(a))$ to $(b, f(b))$. Rolle's theorem is the MVT in the case where $f(a) = f(b)$ so $f'(c) = 0$.
- If f is continuous on $[a, b]$ and differentiable on (a, b) with $f'(x) \neq 0$ on (a, b) then the *inverse function rule* tells that the derivative of f^{-1} exists on (a, b) and tells us how to relate it to the derivative of f. Letting $g = f^{-1}$ we have $g'(x) = 1/f'(g(x))$.

- If we have a function of more than one variable we can define *partial derivatives* with respect to one of the variables as the ordinary derivative when we treat all the other variables as constants. E.g. for $f(x,y)$ we have the partial derivative of f wrt. x at (x,y): $f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$ etc.

4 Integration

4.1 Indefinite and definite integrals

Defn: A function $F(x)$ is called an **indefinite integral** or **antiderivative** of a function $f(x)$ in the interval (a,b) if $F(x)$ is differentiable with
$F'(x) = f(x)$ throughout (a,b). We then write $F(x) = \int f(x)\,dx$.

Eg. $\int \cos x\,dx = \sin x \quad$ in $\mathbb{R}$.

Eg. $\int \frac{1}{x^2}\,dx = -\frac{1}{x} \quad$ in $\mathbb{R}\backslash\{0\}$.

Eg. $\int$ sgn$x\,dx = |x| \quad$ in $\mathbb{R}\backslash\{0\}$.

Note1: If $F(x)$ is an indefinite integral of $f(x)$ in (a,b) then so is $F(x)+c$ for any constant c. In applications of integration it is important to include this arbitrary constant. We therefore write eg. $\int \cos x\,dx = \sin x + c$.

Note2: If $F_1(x)$ and $F_2(x)$ are both indefinite integrals of $f(x)$ in (a,b) then $F_1(x) - F_2(x) = c$ for some constant c.

Defn: We say that $f(x)$ is **integrable in** (a,b) if it has an indefinite integral $F(x)$ in (a,b) that is continuous in $[a,b]$.

Eg. $\cos x$ is integrable in any finite interval (a,b) because $\sin x$ is continuous in $\mathbb{R}$.

Eg. $\frac{1}{x^2}$ is not integrable in $(0,1)$ because its indefinite integral $-\frac{1}{x}$ is not continuous at $x=0$.

Eg. sgn x is integrable in $(0,1)$ because its indefinite integral $|x|$ is continuous in $[0,1]$.

Defn: A **subdivision** S of $[a,b]$ is a partition into a finite number of subintervals

$$[a,x_1],[x_1,x_2],...,[x_{n-1},b]$$

where $a = x_0 < x_1 < x_2 < ... < x_n = b$. The **norm** $|S|$ of the subdivision is the maximum of the subinterval lengths $|a-x_1|, |x_1-x_2|, .., |x_{n-1}-b|$. (Thus a small value of $|S|$ means that the interval $[a,b]$ has been chopped up into small pieces.) The numbers $z_1, z_2, ..., z_n$ form a set of **sample points** from S if $z_j \in [x_{j-1}, x_j]$ for $j = 1,..,n$.

Defn: Suppose that $f(x)$ is a function defined for $x \in [a,b]$. The **Riemann sum** is

$$\mathcal{R} = \sum_{j=1}^{n} (x_j - x_{j-1}) f(z_j).$$

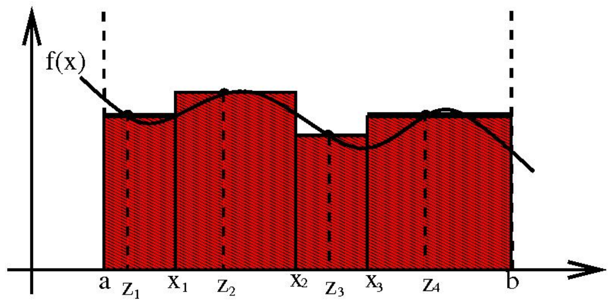


Figure 27: An illustration of a Riemann sum.

The Riemann sum is equal to the sum of the (signed) areas of rectangles of height $f(z_j)$ and width $x_j - x_{j-1}$. Here signed means that the areas of rectangles below the x-axis are counted negatively. If $f(x)$ is continuous in $[a,b]$ and $|S|$ is small then we expect $\mathcal{R}$ to be a good approximation to the (signed) area under the graph of $f(x)$ above the interval $[a,b]$. This turns out to be correct and the error in the approximation can be reduced to zero by taking the limit in which $|S|$ tends to zero. This leads to the definition of the **definite integral** as

$$\int_a^b f(x)\,dx = \lim_{|S| \to 0} \mathcal{R}$$

and it can be shown that this limit exists if $f(x)$ is continuous in $[a,b]$.

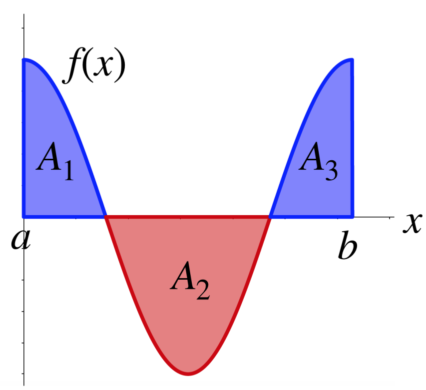


Figure 28: The definite integral is the signed area under the curve: $\int_a^b f(x)\,dx = A_1 - A_2 + A_3$.

By definition we set $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ and therefore $\int_a^a f(x)\,dx = 0$.
There are a number of fairly obvious properties of the definite integral that are not too hard to prove. In the following let $f(x)$ and $g(x)$ both be integrable in (a,b).

(i) Linearity. If λ, μ are any constants then $\lambda f(x) + \mu g(x)$ is integrable in (a,b) with

$$\int_a^b \Big(\lambda f(x) + \mu g(x)\Big)\,dx = \lambda \int_a^b f(x)\,dx + \mu \int_a^b g(x)\,dx.$$

(ii) If $c \in [a,b]$ then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

(iii) If $f(x) \geq g(x)\ \forall\ x \in (a,b)$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

(iv) If $m \leq f(x) \leq M\ \forall\ x \in [a,b]$ then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

4.2 The fundamental theorem of calculus

So far, we don't have any connection between indefinite and definite integrals. This is provided by the following theorem:

The fundamental theorem of calculus
If $f(x)$ is continuous on $[a,b]$ then the function

$$F(x) = \int_a^x f(t)\, dt$$

defined for $x \in [a,b]$ is continuous on $[a,b]$ and differentiable on (a,b) and is an indefinite integral of $f(x)$ on (a,b) ie.

$$F'(x) = \frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$

throughout (a,b). Furthermore if $\widetilde{F}(x)$ is any indefinite integral of $f(x)$ on $[a,b]$ then

$$\int_a^b f(t)\, dt = \widetilde{F}(b) - \widetilde{F}(a) = [\widetilde{F}(x)]_a^b.$$

We shall sketch the important points of the proof.
For $a \le x < x+h < b$ we have that

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt = \int_x^{x+h} f(t)\, dt$$

where we have used property (ii).
Let $m(h)$ and $M(h)$ denote the minimum and maximum values of $f(x)$ on the interval $[x, x+h]$.
Then by property (iv) we have that

$$m(h)h \le \int_x^{x+h} f(t)\, dt \le M(h)h.$$

Thus

$$m(h) \le \frac{F(x+h) - F(x)}{h} \le M(h).$$

Since $f(x)$ is continuous on $[x, x+h]$ we have that $\lim_{h\to 0^+} m(h) = \lim_{h\to 0^+} M(h) = f(x)$ and so by the pinching theorem

$$\lim_{h\to 0^+} \frac{F(x+h) - F(x)}{h} = f(x).$$

A similar argument applies to the limit from below and together they give

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

which proves that $F(x)$ is an indefinite integral of $f(x)$. The proof of the first part of the theorem is completed by showing that $F(x)$ is continuous from the right at $x = a$ and from the left at $x = b$. Both these follow by a simple consideration of the limit of the relevant quotient. Finally, to prove the last part of the theorem the key observation is that any indefinite integral $\widetilde{F}(x)$ is related to $F(x)$ by the addition of a constant.

The fundamental theorem of calculus provides a simple rule for differentiating a definite integral with respect to its limits.

$$Eg. \qquad \frac{d}{dx}\int_0^x \frac{1}{1+\sin^2 t}\,dt = \frac{1}{1+\sin^2 x}.$$

We can combine this result with the chain rule if the limit is a more complicated expression

$$Eg. \qquad \frac{d}{dx}\int_0^{x^2} \frac{1}{1+e^t}\,dt = \frac{1}{1+e^{x^2}}\left(\frac{d}{dx}x^2\right) = \frac{2x}{1+e^{x^2}}.$$

4.3 Limits with logarithms, powers and exponentials

There are some important results concerning limits as $x \to \infty$ for the logarithm and exponential functions. To derive these results we begin with the following,

Lemma 1: $\forall\ x \geq 0, \quad e^x \geq 1 + x$
Proof: Consider $f(x) = e^x - (1 + x)$ then $f(0) = 0$ and $f'(x) = e^x - 1 \geq 0$.
Hence $f(x)$ is monotonic increasing in $[0, \infty)$ so $f(x) \geq 0$ for all $x \geq 0$.

Lemma 2: $\forall\ x \geq 0,$ and for any positive integer $n, \quad e^x \geq \sum_{j=0}^{n} x^j/j!$.
Note that the case $n = 1$ corresponds to Lemma 1, and the proof for general n is similar to the proof of Lemma 1.

Result 1: powers beat logs
For any constant $a > 0$

$$\lim_{x \to \infty} \frac{\log x}{x^a} = 0.$$

This result is encapsulated by the phrase *powers beat logs.*

Proof: Put $x = e^y$ then

$$\lim_{x \to \infty} \frac{\log x}{x^a} = \lim_{y \to \infty} \frac{y}{e^{ay}}.$$

For $y > 0$ then by Lemma 2 with $n = 2$

$$0 \leq \frac{y}{e^{ay}} \leq \frac{y}{1 + ay + \frac{1}{2}a^2y^2} \leq \frac{y}{\frac{1}{2}a^2y^2} = \frac{2}{a^2y}$$

As $\lim_{y \to \infty} \frac{2}{a^2y} = 0$ then by the pinching theorem

$$\lim_{y \to \infty} \frac{y}{e^{ay}} = 0 = \lim_{x \to \infty} \frac{\log x}{x^a}.$$

Result 2: exponentials beat powers
For any constant $a > 0$

$$\lim_{x \to \infty} \frac{x^a}{e^x} = 0.$$

This result is encapsulated by the phrase *exponentials beat powers.*

Proof: Let n be the smallest integer such that $n > a$. By Lemma 2, for $x > 0$ we have that

$$0 \leq \frac{x^a}{e^x} \leq \frac{x^a}{1 + x + \ldots + x^n/n!} = \frac{x^{a-n}}{x^{-n} + x^{1-n} + \ldots 1/n!}$$

As $a - n < 0$ then $\lim_{x \to \infty} \frac{x^{a-n}}{x^{-n} + x^{1-n} + \ldots 1/n!} = 0$, hence by the pinching theorem $\lim_{x \to \infty} \frac{x^a}{e^x} = 0$.

Result 3: the exponential as a limit
For any constant a

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a.$$

Proof (for the case $a > 0$):
Recall the definition of the derivative of a function $f(x)$ at $x = b$.

$$f'(b) = \lim_{h \to 0} \frac{f(b+h) - f(b)}{h}.$$

Apply this to $f(x) = \log x$ so that $f'(x) = \frac{1}{x}$ and take $b = 1$.

$$1 = \lim_{h \to 0} \frac{\log(1+h) - \log(1)}{h} = \lim_{h \to 0} \frac{\log(1+h)}{h}.$$

With the change of variable $h = \frac{a}{x}$ this becomes

$$1 = \lim_{x \to \infty} \frac{\log(1 + \frac{a}{x})}{\frac{a}{x}} \quad \text{ie.} \quad a = \lim_{x \to \infty} (x \log(1 + \frac{a}{x})).$$

Since e^x is continuous at a we have that

$$e^a = \lim_{x \to \infty} \exp(x \log(1 + \frac{a}{x})) = \lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x.$$

4.4 Integration using a recurrence relation

Eg. Calculate $\int_0^1 x^3 e^x \, dx$.
Define $I_n = \int_0^1 x^n e^x \, dx$ for all integer $n \geq 0$.

$$I_{n+1} = \int_0^1 x^{n+1} e^x \, dx = \Big[x^{n+1} e^x \Big]_0^1 - \int_0^1 (n+1) x^n e^x \, dx = e - (n+1) I_n.$$

The recurrence relation $I_{n+1} = e - (n+1) I_n$ can be used to calculate I_n for any positive integer n from the starting value $I_0 = \int_0^1 e^x \, dx = [e^x]_0^1 = e - 1$.
$I_1 = e - I_0 = e - (e-1) = 1, \quad I_2 = e - 2I_1 = e - 2(1) = e - 2,$
$I_3 = e - 3I_2 = e - 3(e-2) = 6 - 2e$ is the required integral.

Eg. Calculate $\int \tan^4 x \, dx$.
Define $F_n(x) = \int \tan^n x \, dx$ for all integer $n \geq 0$.

$$F_{n+2}(x) + F_n(x) = \int \tan^n x \, (1 + \tan^2 x) \, dx = \int \tan^n x \sec^2 x \, dx$$

Put $u = \tan x$ then $du = \sec^2 x \, dx$

$$F_{n+2}(x) + F_n(x) = \int u^n \, du = \frac{u^{n+1}}{n+1} = \frac{\tan^{n+1} x}{n+1}.$$

The recurrence relation $F_{n+2}(x) = \frac{1}{n+1} \tan^{n+1}(x) - F_n(x)$ can be used to calculate the required integral for n even from the starting value $F_0(x) = \int 1 \, dx = x$ and for n odd from the starting value $F_1(x) = -\log|\cos x|$.
In particular, $F_2(x) = \tan x - x$ and $F_4(x) = \frac{1}{3} \tan^3 x - \tan x + x$ thus
$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + c.$

4.5 Definite integrals using even and odd functions

The definite integral of an odd function over a symmetric interval is zero.
If $f_{odd}(x)$ is an integrable odd function on $[-a,a]$ then

$$\int_{-a}^{a} f_{odd}(x)\,dx = 0.$$

Proof:

$$\int_{-a}^{a} f_{odd}(x)\,dx = \int_{-a}^{0} f_{odd}(u)\,du + \int_{0}^{a} f_{odd}(x)\,dx$$

$$= -\int_{a}^{0} f_{odd}(-x)\,dx + \int_{0}^{a} f_{odd}(x)\,dx = \int_{0}^{a} (f_{odd}(-x) + f_{odd}(x))\,dx = 0.$$

A similarly argument shows that if $f_{even}(x)$ is an integrable even function on $[-a,a]$ then

$$\int_{-a}^{a} f_{even}(x)\,dx = 2\int_{0}^{a} f_{even}(x)\,dx.$$

Recall that any function $f(x)$ can be decomposed as a sum of even and odd functions $f(x) = f_{even}(x) + f_{odd}(x)$. This can be a useful technique to calculate definite integrals over symmetric intervals as

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} f_{even}(x) + f_{odd}(x)\;dx = 2\int_{0}^{a} f_{even}(x)\,dx.$$

Eg. Calculate $\int_{-1}^{1} \frac{e^x x^4}{\cosh x}\,dx$.
For $f(x) = \frac{e^x x^4}{\cosh x}$ we have $f_{even}(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x) = \frac{1}{2}\frac{x^4(e^x+e^{-x})}{\cosh x} = x^4$. Thus

$$\int_{-1}^{1} \frac{e^x x^4}{\cosh x}\,dx = 2\int_{0}^{1} x^4 = 2\left[\frac{1}{5}x^5\right]_0^1 = \frac{2}{5}.$$

Eg. Calculate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(1+2x)^3\cos x}{1+12x^2}\,dx$.
For $f(x) = \frac{(1+2x)^3\cos x}{1+12x^2}$ we have
$f_{even}(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x) = \frac{(\cos x)\{(1+2x)^3+(1-2x)^3\}}{2(1+12x^2)} = \frac{(\cos x)(2+24x^2)}{2(1+12x^2)} = \cos x$. Thus

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(1+2x)^3\cos x}{1+12x^2}\,dx = 2\int_{0}^{\frac{\pi}{4}} \cos x\,dx = 2\Big[\sin x\Big]_0^{\frac{\pi}{4}} = 2\frac{1}{\sqrt{2}} = \sqrt{2}.$$

4.6 Summary: Integration

You should have a good understanding of the definitions of indefinite and definite integrals, and how they are related via the Fundamental Theorem of Calculus. You should also know how to use methods to calculate integrals such as integration by parts, recurrence relations and simplifications by considering even and odd (parts of) functions integrated over an interval symmetric about $x = 0$. Here are some key points:

- A function F is an *indefinite integral* or *antiderivative* of a function f if the derivative of F is f, i.e. if $F'(x) = f(x)$. Note that for a given f, F is not unique, but is fixed up to an *integration constant*.

- The *definite integral* of f over an interval $[a, b]$ is the area under the graph of f over this interval and can be defined as a limit of *Riemann sums*. When this limit exists we say f is *integrable* on $[a, b]$ (and if f is continuous on $[a, b]$ then it is integrable on $[a, b]$) and we denote the integral by $\int_a^b f(x)dx$.

- The *Fundamental theorem of calculus* says that if we define $F(x) = \int_a^x f(t)dt$ then F is an antiderivative of f, i.e. $F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$.

- You should know the definition of *even* and *odd functions* and that any function f can be uniquely split into its even and odd parts $f = f_{even} + f_{odd}$. The integral of any odd function over $[-a, a]$ is zero and the integral of any even function over $[-a, a]$ is twice the integral of that function over $[0, a]$.

- Sometimes we have a family of definite or indefinite integrals parametrised by n which we can denote I_n. If we can find a *recurrence relation* relating the integrals for different values of n, we can use this to calculate I_n. E.g. if we can determine I_{n+1} in terms of I_n and we can calculate I_1 then the recurrence relation determines I_n for all $n \in \mathbb{N}$.

- This section also contained some results for limits which are summarised as "*exponentials beat powers beat logs*" as $x \to \infty$. We also saw that $\lim_{y\to\infty}\left(1 + \frac{x}{y}\right)^y = e^x$.

5 Double integrals

5.1 Rectangular regions

Recall that for a function $f(x)$ the definite integral $\int_a^b f(x)\,dx$ is the signed area under the curve $y = f(x)$ between $x = a$ and $x = b$. This interpretation of the definite integral can be generalized for a function of two variables to a volume under a surface.
Given a function $f(x,y)$ and a region D in the (x,y) plane, the double integral

$$\iint_D f(x,y)\,dxdy$$

is the signed volume of the region between the surface $z = f(x,y)$ and the region D in the $z = 0$ plane (see Figure 29).

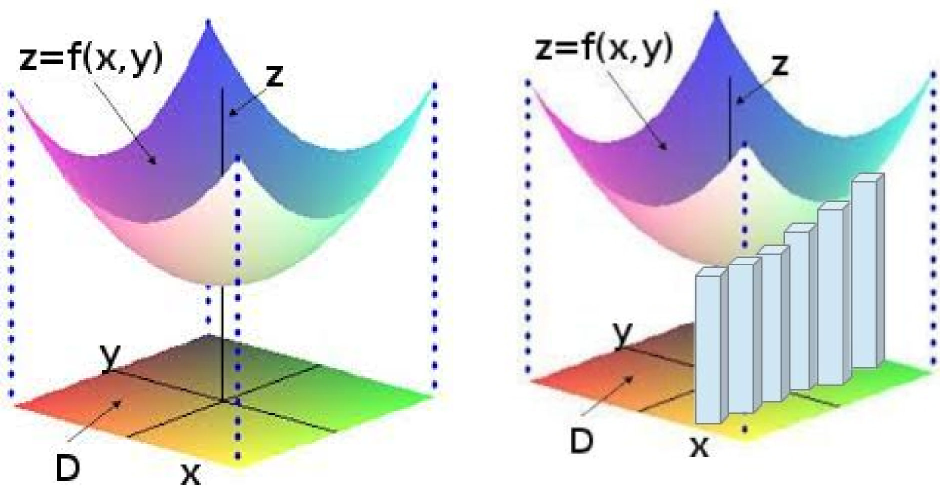


Figure 29: The double integral is the signed volume of the region between the surface $z = f(x,y)$ and the region D in the $z = 0$ plane. It is obtained as the limit of a Riemann sum of cuboid volumes.

In a similar manner to our earlier construction of the definite integral, in terms of the limit of a Riemann sum of areas of rectangles, we can define the double integral as the limit of a Riemann sum of volumes of cuboids.
For simplicity, consider the case of a rectangular region $D = [a_0, a_1] \times [b_0, b_1]$. We construct a subdivision S of D by defining the points of a two-dimensional lattice as $a_0 = x_0 < x_1 < \ldots < x_n = a_1$ and $b_0 = y_0 < y_1 < \ldots < y_m = b_1$. The edge lengths of the lattice are equal to $dx_i = x_i - x_{i-1}$ for $i = 1, \ldots, n$ and $dy_j = y_j - y_{j-1}$ for $j = 1, \ldots, m$. The norm of the subdivision $|S|$ is given by the maximum of all the dx_i and dy_j.
We introduce sample points $(p_i, q_j) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ inside each rectangle of the lattice. The area of the rectangle is $dx_i dy_j$ and we associate to this rectangle a cuboid of height $f(p_i, q_j)$.

The volume of all the cuboids is the Riemann sum

$$\mathcal{R} = \sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i, q_j) dx_i dy_j$$

and the double integral is the limit

$$\iint_D f(x,y)\, dxdy = \lim_{|S|\to 0} \mathcal{R}.$$

In the special case that $f(x,y)$ is a constant, say $f(x,y) = c$, then
$\iint_D f(x,y)\, dxdy = c \times \mathrm{area}(D)$.

For a rectangular region $D = [a_0, a_1] \times [b_0, b_1]$ a practical way to compute the double integral is to use the following iterated integral result (valid providing f is continuous on D)

$$\iint_D f(x,y)\, dxdy = \int_{a_0}^{a_1} \left(\int_{b_0}^{b_1} f(x,y)\, dy \right) dx = \int_{b_0}^{b_1} \left(\int_{a_0}^{a_1} f(x,y)\, dx \right) dy.$$

Eg. Given the region $D = [-2, 1] \times [0, 1]$, calculate $\iint_D (x^2 + y^2)\, dxdy$.

$$\iint_D (x^2 + y^2)\, dxdy = \int_{-2}^{1} \left(\int_0^1 (x^2 + y^2)\, dy \right) dx = \int_{-2}^{1} \left(\left[yx^2 + \frac{1}{3} y^3 \right]_{y=0}^{y=1} \right) dx$$

$$= \int_{-2}^{1} (x^2 + \frac{1}{3})\, dx = \left[\frac{x^3}{3} + \frac{x}{3} \right]_{-2}^{1} = \frac{1}{3}(1 + 1 + 8 + 2) = 4.$$

As an exercise, check that the same result is obtained if the integration is performed in the other order.

5.2 Beyond rectangular regions

To compute double integrals beyond rectangular regions we need to consider the following definition.

Defn: A region D of the plane is called $\mathbf{y-simple}$ if every line that is parallel to the y-axis and intersects D, does so in a single line segment (or a single point if this is on the boundary of D).

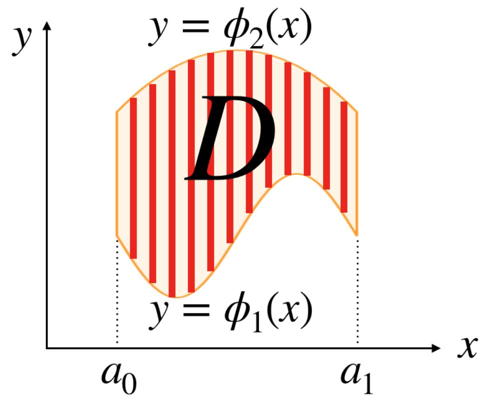


Figure 30: A y-simple region.

See Figure 30 for an example of a y-simple region. The boundaries of this region are given by the curves $y = \phi_1(x)$ and $y = \phi_2(x)$ for $a_0 \leq x \leq a_1$.

The double integral can be calculated as an iterated integral by integrating over y first

$$\iint\limits_D f(x,y)\,dxdy = \int_{a_0}^{a_1} \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) dx.$$

Eg. Sketch the region D that lies between the curves $y = x$ and $y = x^2$ for $0 \le x \le 1$ and calculate $\iint\limits_D 6xy\,dxdy$.

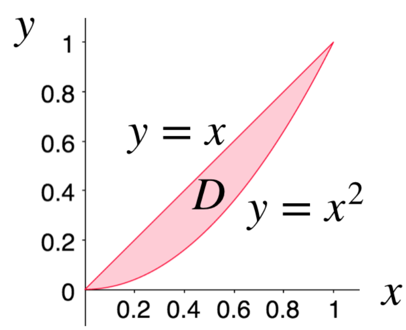


Figure 31: A sketch of the region D in the example.

As the region D is y-simple then

$$\iint\limits_D 6xy\,dxdy = \int_0^1 \left(\int_{x^2}^{x} 6xy\,dy \right) dx = \int_0^1 \Big[3xy^2 \Big]_{y=x^2}^{y=x} dx = 3\int_0^1 (x^3 - x^5)\,dx$$

$$= 3\left[\frac{x^4}{4} - \frac{x^6}{6} \right]_0^1 = 3\left(\frac{1}{4} - \frac{1}{6} \right) = \frac{1}{4}.$$

There is a similar definition of an x-simple region.

Defn: A region D of the plane is called $\mathbf{x-simple}$ if every line that is parallel to the x-axis and intersects D, does so in a single line segment (or a single point if this is on the boundary of D).

Figure 32 shows an example of an x-simple region that is not y-simple.

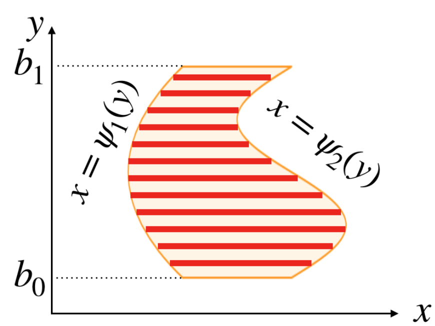


Figure 32: A region that is x-simple but is not y-simple.

The boundaries of this region are the curves $x = \psi_1(y)$ and $x = \psi_2(y)$ for $b_0 \le y \le b_1$. The double integral can be calculated as an iterated integral by integrating over x first

$$\iint\limits_D f(x,y)\,dxdy = \int_{b_0}^{b_1} \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right) dy.$$

A region can be both x-simple and y-simple, in which case the double integral can be calculated as an iterated integral by integrating over x first or y first. The region in the previous example is x-simple, so we can recalculate this double integral using this property.

Same Eg. D is the region between the curves $y = x$ and $y = x^2$ for $0 \le x \le 1$. Use the fact that this region is x-simple to calculate $\iint\limits_D 6xy\,dxdy$.

The boundaries of D are given by the curves $x = y$ and $x = \sqrt{y}$ for $0 \le y \le 1$. Note from Figure 31 that moving along a line (that passes through D) parallel to the x-axis with x increasing, intersects the curve $x = y$ before it intersects the curve $x = \sqrt{y}$. Thus y is the lower limit in the x integration and $\sqrt{y}$ is the upper limit.

$$\iint\limits_D 6xy\,dxdy = \int_0^1 \left(\int_y^{\sqrt{y}} 6xy\,dx \right) dy = \int_0^1 \Big[3x^2y \Big]_{x=y}^{x=\sqrt{y}} dy = 3\int_0^1 (y^2 - y^3)\,dy$$

$$= 3\left[\frac{y^3}{3} - \frac{y^4}{4} \right]_0^1 = 3\left(\frac{1}{3} - \frac{1}{4} \right) = \frac{1}{4}.$$

Sometimes a region may be both x-simple and y-simple but we can only perform the integration explicitly for one of the ordering choices ie. either integrating over x first or y first.

Eg. The boundaries of D are given by the curves $y = x$ and $y = \sqrt{x}$ for $0 \le x \le 1$.
Calculate $\iint\limits_D \frac{e^y}{y}\,dxdy$.
If we use the fact that D is y-simple then we have

$$\iint\limits_D \frac{e^y}{y}\,dxdy = \int_0^1 \left(\int_x^{\sqrt{x}} \frac{e^y}{y}\,dy \right) dx,$$

but we can't do the integration. However, if we use the fact that D is x-simple and the bounding curves are $x = y$ and $x = y^2$ then

$$\iint\limits_D \frac{e^y}{y}\,dxdy = \int_0^1 \left(\int_{y^2}^{y} \frac{e^y}{y}\,dx \right) dy = \int_0^1 \left[\frac{xe^y}{y} \right]_{x=y^2}^{x=y} dy = \int_0^1 (e^y - ye^y)\,dy$$

$$= [e^y]_0^1 - [ye^y]_0^1 + \int_0^1 e^y\,dy = e - 2.$$

5.3 Integration using polar coordinates

Sometimes it may be useful to describe the region of integration D in terms of polar coordinates r and θ, where $x = r\cos\theta$ and $y = r\sin\theta$. To convert a double integral into polar coordinates we need to know what to do with the area element $dA = dxdy$. Recall that this area element came from the definition of the integral in terms of a division of D into small rectangles. The area of a small rectangle with side lengths dx and dy is $dA = dxdy$ which is the area element in the infinitesimal limit.

We therefore need to know the area dA of a small region obtained by taking the point with polar cooridnates (r, θ) and extending r by dr and θ by $d\theta$, as shown in Figure 33. We see that the area is approximately a rectangle with area $dA \approx rd\theta dr$ and this approximation improves as the area decreases. In the infinitesimal limit that defines the integral the result is $dA = dxdy = rd\theta dr$.

We now know how to convert a double integral to polar coordinates, we replace $dxdy$ with $rd\theta dr$. Note the crucial factor of r here. We can then calculate the double integral as an iterated integral if the region D is θ-simple or r-simple (with the obvious definitions of these terms).

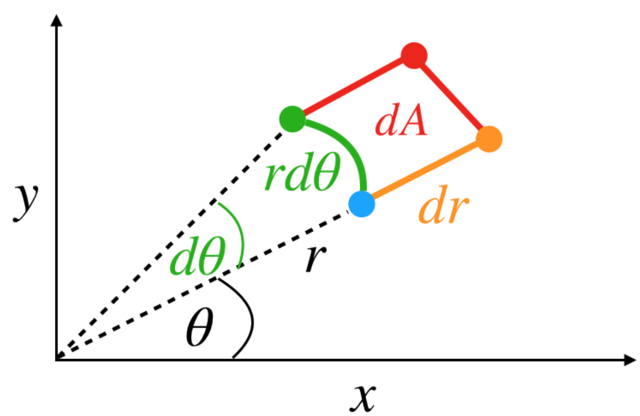


Figure 33: The area element in polar coordinates.

Eg. Let D be the region between the curves $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ satisfying $x \geq 0$ and $y \geq 0$. Calculate $\iint\limits_D xy\, dxdy$.

In polar coordinates the region D is given by $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$.
The area element is $dxdy = rd\theta dr$ and the integrand is $xy = r^2 \cos\theta \sin\theta$.

$$\iint\limits_D xy\, dxdy = \iint\limits_D r^2 \sin\theta \cos\theta\, r dr d\theta = \int_1^2 \left(\int_0^{\pi/2} r^3 \sin\theta \cos\theta\, d\theta \right) dr$$

$$= \int_1^2 \left(\int_0^{\pi/2} \frac{1}{2} \sin(2\theta)\, d\theta \right) r^3\, dr = \int_1^2 \left[-\frac{1}{4} \cos(2\theta) \right]_{\theta=0}^{\theta=\pi/2} r^3\, dr = \int_1^2 \frac{1}{2} r^3\, dr$$

$$= \left[\frac{1}{8} r^4 \right]_1^2 = \frac{15}{8}.$$

As an exercise, you can check that you obtain the same result if you perform the calculation without changing to polar coordinates.

5.4 Change of variables and the Jacobian

Using polar coordinates r, θ rather than Cartesian coordiantes x, y is a particular example of a change of variables. In general we might want to change variables from x, y to new variables u, v given by relations of the form $x = g(u, v)$ and $y = h(u, v)$ for some functions g and h. Not only do we need to know what the integration region D looks like in the new variables, but we also need the area element $dxdy$ in the new variables. The following definition plays the central role in this issue.

Defn. The **Jacobian** of the transformation from the variables x, y to u, v is

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

ie. it is the determinant of the 2×2 matrix of partial derivatives.

To obtain the area element $dxdy$ in terms of the new variables we first compute the Jacobian J and then use the result that

$$dxdy = |J|dudv$$

where $|J|$ is the absolute value of the Jacobian.

As an example, we can rederive the area element in polar coordinates. We have $x = r\cos\theta$ and $y = r\sin\theta$, so in this case r, θ are the new variables, playing the role of u, v in the general setting.

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

giving $dxdy = |J|drd\theta = rdrd\theta$ as before.

Eg. Let D be the square in the (x, y) plane with vertices $(0, 0), (1, 1), (2, 0), (1, -1)$. Calculate $\iint\limits_D (x + y)\, dxdy$ by making the change of variables $x = u + v$ and $y = u - v$.

First we need to determine the region D in terms of the variables $[u, v]$. Using $u = (x + y)/2$ and $v = (x - y)/2$ we find that the vertices map to the $[u, v]$ coordinates $[0, 0], [1, 0], [1, 1], [0, 1]$. The edges of the square lie along the lines $y = x,\ y = -x,\ y = 2 - x,\ y = x - 2$ which map to the lines $v = 0,\ u = 0,\ u = 1,\ v = 1$. Therefore in the $[u, v]$ plane D is again a square, with the vertices given above.
The Jacobian of the transformation is

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$$

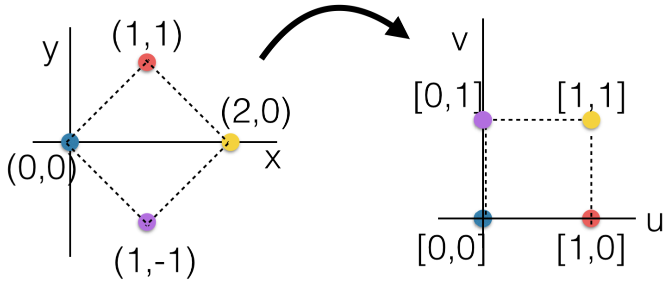


Figure 34: A square in the (x, y) plane is transformed into a square in the $[u, v]$ plane.

Thus $dxdy = |J|dudv = 2dudv$.
Finally, the integrand is $x + y = 2u$. Putting all this together we get

$$\iint_D (x+y)\,dxdy = \iint_D 4u\,dudv = \int_0^1 \left(\int_0^1 4u\,dv \right) du$$

$$= \int_0^1 \Big[4uv \Big]_{v=0}^{v=1} du = \int_0^1 4u\,du = \Big[2u^2 \Big]_0^1 = 2.$$

5.5 The Gaussian integral

An important integral that appears in a wide range of mathematical contexts is the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$$

where a is a positive constant.

We can derive this result using a trick that involves calculating a double integral, as follows. We define I to be the required integral

$$I = \int_{-\infty}^{\infty} e^{-ax^2}\,dx$$

and observe that

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-ax^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-ay^2}\,dy\right) = \iint\limits_{\mathbb{R}^2} e^{-a(x^2+y^2)}\,dxdy.$$

We now use polar coordinates to evaluate this double integral

$$I^2 = \iint\limits_{\mathbb{R}^2} e^{-ar^2} r\,drd\theta = \int_0^{\infty}\left(\int_0^{2\pi} re^{-ar^2}\,d\theta\right)dr = 2\pi\int_0^{\infty} re^{-ar^2}\,dr$$

$$= -2\pi\left[\frac{e^{-ar^2}}{2a}\right]_0^{\infty} = \frac{\pi}{a}.$$

Hence $I = \sqrt{\pi/a}$, which is the required result.

5.6 Summary: Double integration

You should have a good understanding of how to calculate double integrals and methods such as swapping the order of integration or changing variables. Here are some key points:

- The *double integral* of a function of two variables $f(x, y)$ over a region D is the definite integral $\int\int_D f(x, y)dxdy$ which can be defined as a limit of Riemann sums. This limit always exists if f is continuous on a closed region D. This has the interpretation as the (signed) volume between the surface $f(x, y)$ and the region D in the (x, y)-plane. If $f(x, y) = 1$ the integral gives the area of D.

- You should know how to calculate double integrals for x-*simple* and y-*simple* regions which are regions which can be divided into strips parallel to the x-axis or y-axis respectively with each strip being a single interval. You should know how to change the order of integration for regions which are both x-simple and y-simple – note that this process is trivial for rectangular regions but otherwise requires some care.

- Sometimes for a region which is both x-simple and y-simple we can only calculate the integral for one order of integration, sometimes both ways are possible but one way is easier.

- A change of variable from (x, y) to (u, v) can be performed but care must be taken so that the integration measure is correct, i.e. that the value of the integral doesn't depend on the choice of variables. This can be derived by ensuring that the double integral of 1 always gives the area of any region D. The result is $dxdy = |J|dudv$ where J is the *Jacobian* determinant $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$.

- A particular change of variables is from 2d Cartesian coordinates (x, y) to 2d polar coordinates (r, θ) where $dxdy = rdrd\theta$.

- Changes of variable may be motivated by the shape of the region D or by the form of the integrand but both factors will affect how easy or hard it is to calculate the integral.

- The 1d *Gaussian integral* $I = \int_{-\infty}^{\infty} e^{-ax^2}dx$ for $a > 0$ is an interesting example where we can square it to find a 2d Gaussian integral which is a double integral we can evaluate by changing to polar coordinates. This then gives the result for the original Gaussian integral $I = \sqrt{\pi/a}$ which we had no way to calculate as a single variable integral.

6 First order differential equations

6.1 First order separable ODEs

A **differential equation** is an equation that relates an unknown function (the dependent variable) to one or more of its derivatives. The **order** of a differential equation is the order of the highest derivative that appears in the equation. A function that satisfies the differential equation is called a **solution** and the process of finding a solution is called **solving** the differential equation. If the dependent variable depends on only a single unknown variable (the independent variable) then the differential equation is called an **ordinary differential equation** (ODE). We shall deal only with ODEs.
Differential equations for functions with several independent variables and involving partial derivatives are called **partial differential equations** (PDEs). These are generally much harder to solve and wont be discussed in this course.

We shall usually discuss ODEs using y to denote the dependent variable and x for the independent variable, so solving the ODE involves finding $y(x)$.

The generic first order ODE may be written in the form

$$\frac{dy}{dx} = f(x, y) \qquad\qquad (\star)$$

Generically this will have a one-parameter family of solutions ie. the **general solution** will contain an arbitrary constant. Assigning a particular value to this constant gives a **particular solution**. A particular solution will be singled out by the assignment of an **initial value** ie. requiring $y(x_0) = y_0$, for given x_0 and y_0. In this case the ODE is called an **initial value problem** (IVP). To solve an IVP, first solve the ODE to find the general solution and then determine the value of the constant so that the IVP is also satisfied.

Depending upon the form of the function $f(x, y)$ there are various methods that can be used to solve certain kinds of ODEs. We shall discuss these in turn.

The ODE $(\star)$ is **separable** if $f(x, y) = X(x)Y(y)$.
It can be solved by direct integration as

$$\int \frac{1}{Y(y)}\, dy = \int X(x)\, dx.$$

Eg. Solve the IVP

$$\frac{dy}{dx} = xe^{y-x} \quad \text{with} \quad y(0) = 0.$$

$$y' = xe^{-x}e^{y}, \qquad \int e^{-y}\, dy = \int xe^{-x}\, dx, \qquad -e^{-y} = -xe^{-x} + \int e^{-x}\, dx = -xe^{-x} - e^{-x} + c$$

$$y = -\log(e^{-x}(1 + x) - c). \quad y(0) = 0 = \log(1 - c), \quad \text{so} \quad c = 0.$$

$$y = -\log(e^{-x}(1+x)) = x - \log(1+x).$$

Eg. Solve

$$y' = \frac{2x(1+y^2)}{(1+x^2)^2}.$$

$$\int \frac{1}{1+y^2}\,dy = \int \frac{2x}{(1+x^2)^2}\,dx, \quad \tan^{-1} y = -\frac{1}{1+x^2} - c, \quad y = -\tan\left(\frac{1}{1+x^2} + c\right).$$

Eg. Solve the IVP

$$y' = \frac{x^2 y - y}{1+y}, \quad y(3) = 1.$$

$$\int \frac{1+y}{y}\,dy = \int (x^2-1)\,dx, \quad y + \log|y| = \frac{x^3}{3} - x + c. \quad \text{Put } x = 3 \text{ and } y = 1$$

$$1 = 6 + c, \quad c = -5, \quad \text{hence} \quad y + \log|y| = \frac{x^3}{3} - x - 5.$$

In this case the final solution can only be written in implicit form ie. it cannot be written in an explicit form $y = \ldots$ where the right hand side is a function of x only.

6.2 First order homogeneous ODEs

The ODE $(\star)$ is **homogeneous** if $f(tx,ty)=f(x,y)\ \ \forall\ t\in\mathbb{R}$.

In this case the substitution $y=xv$ produces a separable ODE for $v(x)$.

Eg. Solve

$$y'=\frac{y^2-x^2}{xy}.$$

$$f(tx,ty)=\frac{t^2y^2-t^2x^2}{txty}=\frac{y^2-x^2}{xy}=f(x,y)\quad \text{hence homogeneous.}$$

Put $y=xv$ then $y'=v+xv'$ and the ODE becomes

$$v+xv'=\frac{x^2v^2-x^2}{x^2v}=v-\frac{1}{v},\quad v'=-\frac{1}{xv},\quad \int v\,dv=\int -\frac{1}{x}\,dx,\quad \frac{v^2}{2}=-\log|x|+\log c,$$

$$v=\pm\sqrt{2\log\left|\frac{c}{x}\right|},\quad y=xv=\pm x\sqrt{2\log\left|\frac{c}{x}\right|}.$$

6.3 First order linear ODEs

The ODE ($\star$) is **linear** if $f(x,y) = -p(x)y + q(x)$, in which case we can write the standard form of a linear first order ODE as

$$y' + py = q$$

where p and q can be any functions of x.
In this case the ODE can be solved in terms of the **integrating factor**

$$I(x) = e^{\int p(x)\, dx}.$$

The solution is given by

$$y = \frac{1}{I(x)} \int I(x)q(x)\, dx.$$

Proof: We need to show that y satisfies the ODE $y' + py = q$.

The first step is to observe that $I' = e^{\int p\, dx} p = Ip$.

With $y = \frac{1}{I} \int Iq\, dx$ we have

$$y' = -\frac{I'}{I^2} \int Iq\, dx + \frac{1}{I} Iq = -\frac{I'}{I} y + q = -\frac{Ip}{I} y + q = -py + q,$$

as required.

Eg. Solve the IVP

$$y' - \frac{2}{x} y = 3x^3, \quad y(-1) = 2.$$

In the above notation $p = -\frac{2}{x}$ and $q = 3x^3$.

$$I = \exp\left(\int p\, dx \right) = \exp\left(\int -\frac{2}{x}\, dx \right) = \exp\Big(-2 \log x \Big) = \frac{1}{x^2}.$$

$$y = \frac{1}{I} \int Iq\, dx = x^2 \int \frac{1}{x^2} 3x^3\, dx = x^2 \int 3x\, dx = x^2 \left(\frac{3}{2} x^2 + c \right) = \frac{3}{2} x^4 + cx^2.$$

Using $y(-1) = 2$ we have $2 = \frac{3}{2} + c$ hence $c = \frac{1}{2}$ giving $y = \frac{3}{2} x^4 + \frac{1}{2} x^2$.

6.4 First order exact ODEs

An alternative form in which to write a first order ODE is

$$M(x,y)\,dx + N(x,y)\,dy = 0. \qquad (\star\star)$$

Rearranging gives

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

so this agrees with the form $(\star)$ with $f(x,y) = \frac{M(x,y)}{N(x,y)}$.
Note that a given $f(x,y)$ does not correspond to a unique choice of $M(x,y)$ and $N(x,y)$ as only their ratio determines the ODE, so there is a freedom to multiply both M and N by the same arbitrary function.

For any function $g(x,y)$ the **total differential** dg is defined to be

$$dg = \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy.$$

You may think of dg as the small change in the function g that arises in moving from the point (x,y) to $(x+dx, y+dy)$ where dx and dy are both small. In particular, if dg is identically zero then g is a constant.

The ODE $(\star\star)$ is called **exact** if there exists a function $g(x,y)$ such that the left hand side of $(\star\star)$ is equal to the total derivative dg ie.

$$M = \frac{\partial g}{\partial x} \quad \text{and} \quad N = \frac{\partial g}{\partial y}.$$

In this case the ODE says that $dg = 0$ hence $g = constant$ and this yields the solution of the ODE.

The equality of the mixed partial derivatives $\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y}$ requires that an exact equation satisfies $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
This results in the following **test for exactness**

$$\text{The ODE} \quad M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{is exact iff} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Eg. Show that $(3e^{3x}y + e^x)\,dx + (e^{3x}+1)\,dy = 0$ is an exact equation and hence solve it.
In this example $M = 3e^{3x}y + e^x$ and $N = e^{3x}+1$.
$\frac{\partial M}{\partial y} = 3e^{3x} = \frac{\partial N}{\partial x}$ hence this equation is exact.

$$\text{From } M = \frac{\partial g}{\partial x} \text{ we have } \frac{\partial g}{\partial x} = 3e^{3x}y + e^x \quad \text{giving} \quad g = e^{3x}y + e^x + \phi(y)$$

where the usual constant of integration is replaced by an arbitrary function of integration $\phi(y)$ because of the partial differentiation. To determine this function we use

$$\frac{\partial g}{\partial y} = N, \quad \frac{\partial g}{\partial y} = e^{3x} + 1 = e^{3x} + \phi', \quad \text{so } \phi' = 1, \text{ giving } \phi = y.$$

$$\text{Finally we have that} \quad g = e^{3x}y + e^x + y = \text{constant} = c.$$

In this case we can write the solution in explicit form $y = (c - e^x)/(e^{3x}+1)$.
As an exercise check that this does indeed solve the starting ODE
$y' = -(3e^{3x}y + e^x)/(e^{3x}+1)$. Note that this example is also a linear ODE so could also have been solved using an integrating factor.

Consider an ODE $\quad M\,dx + N\,dy = 0 \quad$ that is not exact ie. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.
By multiplying the ODE by a function $I(x,y)$ we can get an equivalent representation of the ODE as $\quad m\,dx + n\,dy = 0 \quad$ where $m = MI$ and $n = NI$. If this equation is exact ie. $\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$ then I is called an **integrating factor** for the original ODE.

Eg. Show that x is an integrating factor for $\quad (3xy - y^2)\,dx + x(x-y)\,dy = 0$.
$M = 3xy - y^2$ so $\frac{\partial M}{\partial y} = 3x - 2y$, but $N = x^2 - xy$ so $\frac{\partial N}{\partial x} = 2x - y$.
Therefore $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and the ODE is not exact.
However, with the integrating factor $I = x$ we get $m = MI = 3x^2y - y^2x$ and $n = NI = x^3 - x^2y$ so now $\frac{\partial m}{\partial y} = 3x^2 - 2yx = \frac{\partial n}{\partial x}$ which is exact.
We can now solve this exact equation using the above method.
$\frac{\partial g}{\partial x} = m = 3x^2y - y^2x$ giving $g = x^3y - \frac{1}{2}y^2x^2 + \phi(y)$.
$\frac{\partial g}{\partial y} = n = x^3 - x^2y = x^3 - x^2y + \phi'$ so $\phi' = 0$ and we can take $\phi = 0$.
This gives the final solution $g = \text{constant} = c$ ie. $x^3y - \frac{1}{2}y^2x^2 = c$.

6.5 Bernoulli equations

A **Bernoulli equation** is a nonlinear ODE of the form

$$y' + p(x)y = q(x)y^n$$

where $n \neq 0, 1,$ otherwise the equation is simply linear.

These ODEs can be solved by first performing the substitution $v = y^{1-n}$, which converts the equation to a linear ODE for $v(x)$.

Eg. Solve

$$y' - \frac{2y}{x} = -x^2y^2.$$

This is a Bernoulli equation with $n = 2$ hence put $v = \frac{1}{y},$ which gives $v' = -\frac{y'}{y^2}$.
Dividing the ODE by y^2 yields

$$\frac{y'}{y^2} - \frac{2}{xy} = -x^2$$

which in terms of v is

$$-v' - \frac{2v}{x} = -x^2$$

ie the linear equation

$$v' + \frac{2}{x}v = x^2.$$

We solve this with an integrating factor $I = \exp\left(\int \frac{2}{x}\, dx\right) = \exp(2\log x) = x^2$ as

$$v = \frac{1}{x^2}\int x^2x^2\, dx = \frac{1}{x^2}\left(\frac{1}{5}x^5 + \frac{c}{5}\right).$$

So finally

$$y = \frac{1}{v} = \frac{5x^2}{x^5 + c}.$$

6.6 Summary: First order ODEs

You should have a good understanding of how to solve the various types of first order ODEs we covered. We assume a first order ODE can be written in the form $\frac{dy}{dx} = f(x, y)$. Here are some key points:

- The *general solution* to a first order ODE will have 1 free real parameter which we can call the integration constant. If we are given an *initial condition* determining the value of y for some specific value of x then we can fix this integration constant.
- Often when solving an ODE we will get an *implicit solution*, i.e. and algebraic expression involving x and y. If it is reasonably straightforward to do so, solve this to write an *explicit solution*, i.e. an explicit expression for $y(x)$ as a function of x.
- If $f(x, y) = X(x)Y(y)$ the ODE is *separable*. Solve by separating the x and y dependence and integrating: $\int \frac{dy}{Y} = \int X dx$.
- If $f(tx, ty) = f(x, y)$ for all $x, y, t \in \mathbb{R}$ then the ODE is *first order homogeneous*. Solve by substituting $v = y/x$ for y and the OPE will become separable in variables x and v. Remember to substitute back after solving to give the solution for the original variable y.
- If $f(x, y) = -p(x)y + q(x)$, i.e. if $f(x, y)$ is linear in y, we have a *linear first order ODE* which can be written as $y + py' = q$. We can solve by multiplying both sides by an *integrating factor* $I = \exp(\int p(x)dx)$ so that the LHS becomes the total derivative $(Iy)'$ and we then solve by integrating the RHS Iq.
- A *Bernoulli equation* is a first order ODE of the form $y + p(x)y' = q(x)y^n$ with any $n \in \mathbb{R}\ \{0, 1\}$ where we exclude those values of n for which this is just a linear ODE. We solve by dividing bot sides by y^n and substituting $v = y^{1-n}$ which results in a first order linear ODE for $v(x)$.
- A first order *exact ODE* is one of the form $M(x, y)dx + N(x, y)dy = 0$ with $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which means we can write $M = \frac{\partial g}{\partial x}$ and $N = \frac{\partial g}{\partial y}$ for such function $g(x, y)$. The ODE is then simply $dg = 0$ which means that $g(x, y) = c$ for some $c \in \mathbb{R}$. To find g either integrate M wrt. x while treating y as a constant or integrate N wrt. y while treating x as a constant. The integration will give $g(x, y)$ up to an integration 'constant' which is an arbitrary function of y and x respectively (since these were treated as constants in the integration). Fix this integration 'constant' up to a real constant by substituting the expression for g into the other equation, $N = \frac{\partial g}{\partial y}$ or $M = \frac{\partial g}{\partial x}$ respectively.
- We can write a first order ODE in the form $M(x, y)dx + N(x, y)dy = 0$ in infinitely many ways since we can multiply through by any function $I(x, y)$. In general this will not be an exact ODE but it can be for specific *integrating factors* $I(x, y)$. Unfortunately there is no general method to finding these integrating factors (unlike the integrating factor for linear ODEs). There are some methods which work in some cases, but we did not cover that this term, so you just need to know the concept.

7 Second order differential equations

7.1 Linear constant coefficient homogeneous ODEs

The general form of a second order linear constant coefficient ODE is

$$\alpha_2 y'' + \alpha_1 y' + \alpha_0 y = \phi(x) \qquad (\dagger)$$

where $\alpha_2 \neq 0, \alpha_1, \alpha_0$ are constants (the constant coefficients) and $\phi(x)$ is an arbitrary function of x. The ODE is still second order and linear if these constants are replaced by functions of x, but then the ODE is not so easy to solve.

We first restrict to the case $\phi(x) = 0$ ie.

$$\alpha_2 y'' + \alpha_1 y' + \alpha_0 y = 0 \qquad (\dagger\dagger)$$

in which case the second order ODE is called **homogeneous**.

The first point to note is that if y_1 and y_2 are any two solutions of the homogeneous ODE $(\dagger\dagger)$ then so is any arbitrary linear combination
$y = Ay_1 + By_2$, where A and B are constants.
Proof: $y = Ay_1 + By_2$ so $y' = Ay_1' + By_2'$ and $y'' = Ay_1'' + By_2''$. Hence
$\alpha_2 y'' + \alpha_1 y' + \alpha_0 y = \alpha_2(Ay_1'' + By_2'') + \alpha_1(Ay_1' + By_2') + \alpha_0(Ay_1 + By_2) =$
$= A(\alpha_2 y_1'' + \alpha_1 y_1' + \alpha_0 y_1) + B(\alpha_2 y_2'' + \alpha_1 y_2' + \alpha_0 y_2) = 0 + 0 = 0.$
Note that the linearity of the ODE is crucial for this result.

The general solution of the ODE $(\dagger\dagger)$ contains two arbitrary constants (because the ODE is second order). If y_1 and y_2 are any two independent particular solutions then the general solution is given by $y = Ay_1 + By_2$, with A and B the two arbitrary constants. The task is therefore to find two independent particular solutions.

To find a particular solution, look for one in the form $y = e^{\lambda x}$.
Putting this into $(\dagger\dagger)$ yields the **characteristic** (or **auxiliary**) **equation**

$$\alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0 = 0. \qquad (Char)$$

There are 3 cases to consider, depending upon the type of roots of $(Char)$.

(i). Distinct real roots.
If $(Char)$ has real roots $\lambda_1 \neq \lambda_2$ then $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are the two required particular solutions and the general solution is

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}.$$

Eg. Solve $y'' - y' - 2y = 0$. Then $(Char)$ is $\lambda^2 - \lambda - 2 = 0 = (\lambda - 2)(\lambda + 1)$ with roots $\lambda = 2, -1$. The general solution is $y = Ae^{2x} + Be^{-x}$.

(ii). Repeated real root.
$(Char)$ has a repeated real root if $\alpha_1^2 = 4\alpha_0\alpha_2$ ie. when $\alpha_0 = \frac{\alpha_1^2}{4\alpha_2}$.
$(Char)$ then reduces to

$$\alpha_2\lambda^2 + \alpha_1\lambda + \frac{\alpha_1^2}{4\alpha_2} = 0 = \alpha_2\left(\lambda + \frac{\alpha_1}{2\alpha_2}\right)^2 \quad \text{with} \quad \lambda_1 = -\frac{\alpha_1}{2\alpha_2} \text{ the real double root.}$$

Thus $(Char)$ has produced only one solution $y_1 = e^{\lambda_1 x}$.
However, as we now show, $y = xe^{\lambda_1 x}$ is also a solution in this case.
The original ODE $(\dagger\dagger)$ now takes the form

$$\alpha_2 y'' + \alpha_1 y' + \frac{\alpha_1^2}{4\alpha_2} y = 0 = \alpha_2(y'' - 2\lambda_1 y' + \lambda_1^2 y).$$

With $y = xe^{\lambda_1 x}$ we have $y' = e^{\lambda_1 x} + \lambda_1 xe^{\lambda_1 x}$ and $y'' = 2\lambda_1 e^{\lambda_1 x} + \lambda_1^2 xe^{\lambda_1 x}$. Hence

$$y'' - 2\lambda_1 y' + \lambda_1^2 y = 2\lambda_1 e^{\lambda_1 x} + \lambda_1^2 xe^{\lambda_1 x} - 2\lambda_1(e^{\lambda_1 x} + \lambda_1 xe^{\lambda_1 x}) + \lambda_1^2 xe^{\lambda_1 x} = 0.$$

$y_1 = e^{\lambda_1 x}$ and $y_2 = xe^{\lambda_1 x}$ are the two required particular solutions and the general solution is

$$y = Ae^{\lambda_1 x} + Bxe^{\lambda_1 x}.$$

Eg. Solve $2y'' - 12y' + 18y = 0$. Then $(Char)$ is $2\lambda^2 - 12\lambda + 18 = 0 = 2(\lambda - 3)^2$ with double root $\lambda = 3$ The general solution is $y = Ae^{3x} + Bxe^{3x}$.

(iii). Complex roots.
If the roots of $(Char)$ are complex then they come in a complex conjugate pair ie. $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. We therefore have the two solutions $y_1 = e^{\lambda_1 x} = e^{\alpha x}e^{i\beta x}$ and $y_2 = e^{\lambda_2 x} = e^{\alpha x}e^{-i\beta x}$, but these are not the solutions we are looking for, since they are complex and we want real solutions. However, we have seen that we can take any linear combination of these solutions (even with complex coefficients) and it will still be a solution. In particular the combinations

$$Y_1 = \frac{1}{2}(y_1 + y_2) = \frac{1}{2}e^{\alpha x}(e^{i\beta x} + e^{-i\beta x}) = e^{\alpha x}\cos(\beta x)$$

$$Y_2 = \frac{1}{2i}(y_1 - y_2) = \frac{1}{2i}e^{\alpha x}(e^{i\beta x} - e^{-i\beta x}) = e^{\alpha x}\sin(\beta x)$$

are both real and provide the two particular solutions we want.
The general solution is an arbitrary real linear combination of Y_1 and Y_2 ie.

$$y = Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x) = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x))$$

with A and B real. Recall that α and β are both real and are the real and imaginary parts of the complex root of $(Char)$.

Eg. Solve $y'' - 6y' + 10y = 0$. Then $(Char)$ is $\lambda^2 - 6\lambda + 10 = 0$ with roots

$$\lambda = \frac{6 \pm \sqrt{36 - 40}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i = \alpha \pm i\beta.$$

Hence $\alpha = 3$ and $\beta = 1$ and the general solution is $y = e^{3x}(A\cos x + B\sin x)$.

7.2 The method of undetermined coefficients

We now return to the general inhomogeneous ODE

$$\alpha_2 y'' + \alpha_1 y' + \alpha_0 y = \phi(x) \qquad (\dagger)$$

where $\phi(x)$ is a function that is not identically zero.

The general solution to $(\dagger)$ is obtained as a sum of two parts

$$y = y_{CF} + y_{PI}$$

where y_{CF} is called the **complementary function** and is the general solution of the homogeneous version of the ODE ie. $(\dagger\dagger)$ which is obtained by removing the $\phi(x)$ term. This part of the solution contains the two arbitrary constants and we have already seen how to find this.
y_{PI} is called the **particular integral** and is any particular solution of $(\dagger)$. We haven't yet seen any methods to find this. However, before we look at a method, let us first show that the combination $y = y_{CF} + y_{PI}$ is indeed a solution of $(\dagger)$. We have that $y' = y'_{CF} + y'_{PI}$ and $y'' = y''_{CF} + y''_{PI}$ so put these into the left hand side of $(\dagger)$ to get

$$\begin{aligned}\alpha_2 y'' + \alpha_1 y' + \alpha_0 y &= \alpha_2(y''_{CF} + y''_{PI}) + \alpha_1(y'_{CF} + y'_{PI}) + \alpha_0(y_{CF} + y_{PI}) \\ &= \underbrace{\alpha_2 y''_{CF} + \alpha_1 y'_{CF} + \alpha_0 y_{CF}}_{=0 \text{ by } (\dagger\dagger)} + \underbrace{\alpha_2 y''_{PI} + \alpha_1 y'_{PI} + \alpha_0 y_{PI}}_{=\phi(x) \text{ by } (\dagger)} = \phi(x).\end{aligned}$$

To find a particular integral we shall use the **method of undetermined coefficients,** which can be applied if $\phi(x)$ takes certain forms such as polynomials, exponentials or trigonometric functions, including sums and products of these. The idea is to try an appropriate form of the solution that contains some unknown constant coefficients which are then determined by explicit calculation and the requirement that the try yields a solution. The table below lists the kind of terms that can be dealt with if they appear in $\phi(x)$ (they can come with any constant coefficients in front) together with the form to try for y_{PI}, containing undetermined constant coefficients a_i.

Term in $\phi(x)$	Form to try for y_{PI}
$e^{\gamma x}$	$a_1 e^{\gamma x}$
x^n	$a_0 + a_1 x + \dots + a_n x^n$
$\cos(\gamma x)$	$a_1 \cos(\gamma x) + a_2 \sin(\gamma x)$
$\sin(\gamma x)$	$a_1 \cos(\gamma x) + a_2 \sin(\gamma x)$

The motivation for the above is that the form of the try when differentiated zero, once or twice, should have a similar functional form to $\phi(x)$, as there is then a chance that the left hand side of $(\dagger)$ could equal the right hand side. If $\phi(x)$ contains sums/products of the kind of terms above then try sums/products of the listed forms for the try.

Special case rule: If the suggested form of the try for y_{PI} is contained in y_{CF} then we know that this form wont work as the left hand side of $(\dagger)$ will then be identically zero and so cant equal $\phi(x)$. In this case first multiply the suggested form of the try by x to get the correct try. In some special cases this rule may need to be applied twice as x times the suggested try may also be contained in y_{CF}.

Eg. Solve $y'' - y' - 2y = 7 - 2x^2$.
From earlier we already know that $y_{CF} = Ae^{2x} + Be^{-x}$.
To find y_{PI} we try the form $y = a_0 + a_1x + a_2x^2$
to simplify the notation for the calculation we have dropped the $_{PI}$ subscript here.
Now $y' = a_1 + 2a_2x$ and $y'' = 2a_2$, so putting these into the ODE gives $2a_2 - (a_1 + 2a_2x) - 2(a_0 + a_1x + a_2x^2) = 7 - 2x^2$
$= 2a_2 - a_1 - 2a_0 + x(-2a_2 - 2a_1) - 2a_2x^2$.
Comparing coefficients of x^2, x^1, x^0 gives $-2a_2 = -2,\ (a_2 = 1),\ -2a_2 - 2a_1 = 0,$
$(a_1 = -1),\ 2a_2 - a_1 - 2a_0 = 7,\ (a_0 = -2)$.
We have found $y_{PI} = -2 - x + x^2$ and the general solution is $y = y_{CF} + y_{PI}$ ie.
$y = Ae^{2x} + Be^{-x} - 2 - x + x^2$.

Eg. Solve $y'' + 4y' + 4y = 5e^{3x}$.
First find y_{CF} by solving $y'' + 4y' + 4y = 0$.
$(Char)$ is $\lambda^2 + 4\lambda + 4 = 0 = (\lambda + 2)^2$ with $\lambda = -2$ repeated,
hence $y_{CF} = e^{-2x}(A + Bx)$.
For y_{PI} try $y = a_1e^{3x}$ so $y' = 3a_1e^{3x}$ and $y'' = 9a_1e^{3x}$. Put these into the ODE
$9a_1e^{3x} + 12a_1e^{3x} + 4a_1e^{3x} = 5e^{3x} = 25a_1e^{3x}$ so $a_1 = \frac{1}{5}$ giving $y_{PI} = \frac{1}{5}e^{3x}$.
The general solution is $y = y_{CF} + y_{PI}$ ie $y = e^{-2x}(A + Bx) + \frac{1}{5}e^{3x}$.

Eg. Solve $y'' - 2y' + 2y = 10\cos(2x)$.
First find y_{CF} by solving $y'' - 2y' + 2y = 0$.
$(Char)$ is $\lambda^2 - 2\lambda + 2 = 0$ with $\lambda = 1 \pm i$. Hence $y_{CF} = e^x(A\cos x + B\sin x)$.
For y_{PI} try $y = a_1\cos(2x) + a_2\sin(2x)$ so $y' = -2a_1\sin(2x) + 2a_2\cos(2x)$ and $y'' = -4a_1\cos(2x) - 4a_2\sin(2x)$. Put these into the ODE $y'' - 2y' + 2y = -4a_1\cos(2x) - 4a_2\sin(2x) - 2(-2a_1\sin(2x) + 2a_2\cos(2x)) + 2(a_1\cos(2x) + a_2\sin(2x)) = 10\cos(2x) = (-2a_1 - 4a_2)\cos(2x) + (-2a_2 + 4a_1)\sin(2x)$.
Comparing coefficients of $\sin(2x)$ and $\cos(2x)$ gives $-2a_2 + 4a_1 = 0,\ (a_2 = 2a_1),\ -2a_1 - 4a_2 = 10,\ -10a_1 = 10,\ (a_1 = -1,\ a_2 = -2)$.
Thus $y_{PI} = -\cos(2x) - 2\sin(2x)$.
The general solution is $y = e^x(A\cos x + B\sin x) - \cos(2x) - 2\sin(2x)$.

Eg. Solve $y'' - y' - 2y = 6e^{-x}$.
From earlier we already know that $y_{CF} = Ae^{2x} + Be^{-x}$.
To find y_{PI} we first think that we should try the form $y = a_1 e^{-x}$ but note that this is contained in y_{CF} so instead we try $y = a_1 x e^{-x}$. Then
$y' = a_1 e^{-x} - a_1 x e^{-x}$ and $y'' = -2a_1 e^{-x} + a_1 x e^{-x}$. Put into ODE
$-2a_1 e^{-x} + a_1 x e^{-x} - (a_1 e^{-x} - a_1 x e^{-x}) - 2a_1 x e^{-x} = 6e^{-x} = -3a_1 e^{-x}$ so $a_1 = -2$ and $y_{PI} = -2xe^{-x}$.
The general solution is $y = Ae^{2x} + Be^{-x} - 2xe^{-x}$.

Eg. Solve $y'' - y' - 2y = e^{x}(8\sin(3x) - 14\cos(3x))$.
From earlier we already know that $y_{CF} = Ae^{2x} + Be^{-x}$.
For y_{PI} try $y = e^{x}(a_1\cos(3x) + a_2\sin(3x))$.
It is an exercise to show that $a_1 = 1,\ a_2 = -1$.
The general solution is then $y = Ae^{2x} + Be^{-x} + e^{x}(\cos(3x) - \sin(3x))$.

Eg. Solve $y'' - y' - 2y = -5 + 9x - 2x^3 + 4\cos x + 2\sin x$.
From earlier we already know that $y_{CF} = Ae^{2x} + Be^{-x}$.
For y_{PI} try $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4\cos x + a_5\sin x$.
It is an exercise to show that
$a_0 = 1,\ a_1 = 0,\ a_2 = -\frac{3}{2},\ a_3 = 1,\ a_4 = -1,\ a_5 = -1$.
The general solution is then $y = Ae^{2x} + Be^{-x} + 1 - \frac{3}{2}x^2 + x^3 - \cos x - \sin x$.

7.3 Initial and boundary value problems

The values of the two arbitrary constants in the general solution of a second order ODE are fixed by specifying two extra requirements on the solution and/or its derivative.

An **initial value problem** (IVP) is when we require $y(x_0) = y_0$ and $y'(x_0) = \delta$ for given constants x_0, y_0, δ. In this case the two constraints are given at the same value of the independent variable.

A **boundary value problem** (BVP) is when we require $y(x_0) = y_0$ and $y(x_1) = y_1$ for given constants $x_0 \neq x_1, y_0, y_1$. In this case the two constraints are given at two different values of the independent variable.

The way to solve an IVP or BVP is to first find the general solution of the ODE and then determine the values of the two constants in this solution so that the extra conditions are satisfied.

Eg. Solve the IVP $\quad y'' - y' - 2y = 7 - 2x^2, \quad y(0) = 5,\ y'(0) = 1.$
From earlier we already know that the general solution of the ODE is
$y(x) = Ae^{2x} + Be^{-x} - 2 - x + x^2.$
Therefore $y(0) = A + B - 2 = 5$, so $\ A + B = 7.$
Now $y'(x) = 2Ae^{2x} - Be^{-x} - 1 + 2x$, giving $y'(0) = 2A - B - 1 = 1$, so $2A - B = 2$.
The solution of these two equations for A and B is $A = 3,\ B = 4.$
Therefore the solution of the IVP is $y = 3e^{2x} + 4e^{-x} - 2 - x + x^2.$

Eg. Solve the BVP $\quad 4y'' + y = 0, \quad y(0) = 1,\ y(\pi) = 2.$
$(Char)$ is $4\lambda^2 + 1 = 0$ with roots $\lambda = \pm\frac{i}{2}$.
The general solution is $y = A\cos(\frac{x}{2}) + B\sin(\frac{x}{2})$.
$y(0) = A = 1$ and $y(\pi) = B = 2$.
Hence the solution of the BVP is $y = \cos(\frac{x}{2}) + 2\sin(\frac{x}{2})$.

7.4 The method of variation of parameters

So far, the only method we have seen to construct a particular integral is the method of undetermined coefficients. As we have seen, this method only applies if the inhomogeneous term ϕ takes particular forms. In this section we consider a more general method to find a particular integral, known as the method of variation of parameters.

Defn. Given two differentiable functions, $y_1(x), y_2(x)$, we define the **Wronskian** to be

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

It is obvious that if y_1 and y_2 are linearly dependent then $W(y_1, y_2)$ is identically zero, that is, zero for all x. Therefore, if $W(y_1, y_2)$ is not identically zero then this implies that y_1 and y_2 are linearly independent.

The task at hand is to find a particular integral for the inhomogeneous ODE (†),

$$\alpha_2 y'' + \alpha_1 y' + \alpha_0 y = \phi.$$

The starting point is to consider the two linearly independent solutions y_1, y_2 of the homogeneous version of the ODE (††). In other words $y_{CF} = Ay_1 + By_2$.
The method of variation of parameters is to look for a particular integral by replacing the constants in y_{CF} by functions. Namely, a solution to (†) is sought in the form

$$y = u_1 y_1 + u_2 y_2$$

for functions $u_1(x), u_2(x)$. Given this form then

$$y' = u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2'.$$

The form we have chosen has two arbitrary functions and the ODE will only give one relation, hence we need to impose a second condition. This condition is chosen to be

$$u_1' y_1 + u_2' y_2 = 0$$

so that the derivative now simplifies to

$$y' = u_1 y_1' + u_2 y_2'.$$

Differentiating once more gives

$$y'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''.$$

Putting these expressions into (†) yields

$$\alpha_2(u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'') + \alpha_1(u_1 y_1' + u_2 y_2') + \alpha_0(u_1 y_1 + u_2 y_2) = \phi.$$

Using the fact that both y_1 and y_2 solve (††) simplifies the above equation to

$$\alpha_2(u_1'y_1' + u_2'y_2') = \phi.$$

We now have two equations for the two unknown functions u_1', u_2' and this gives

$$u_1' = -\frac{y_2\phi/\alpha_2}{W(y_1,y_2)}, \qquad u_2' = \frac{y_1\phi/\alpha_2}{W(y_1,y_2)},$$

which are solved by direct integration

$$u_1 = -\int \frac{y_2\phi/\alpha_2}{W(y_1,y_2)}\,dx, \qquad u_2 = \int \frac{y_1\phi/\alpha_2}{W(y_1,y_2)}\,dx.$$

As a simple first example, we will apply the method of variation of parameters to find a particular integral that could also be found using the method of undetermined coefficients.

Eg. Solve $y'' - y = e^{2x}$.
We have that $y_{CF} = Ae^x + Be^{-x}$ hence $y_1 = e^x$ and $y_2 = e^{-x}$.

$$W(y_1,y_2) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

Using the above formulae

$$u_1 = -\int \frac{e^{-x}e^{2x}}{-2}\,dx = \int \frac{1}{2}e^x\,dx = \frac{1}{2}e^x.$$

$$u_2 = \int \frac{e^x e^{2x}}{-2}\,dx = \int \frac{-1}{2}e^{3x}\,dx = -\frac{1}{6}e^{3x}.$$

$$y_{PI} = u_1y_1 + u_2y_2 = \frac{1}{2}e^xe^x - \frac{1}{6}e^{3x}e^{-x} = \frac{1}{3}e^{2x}.$$

$$y = y_{CF} + y_{PI} = Ae^x + Be^{-x} + \frac{1}{3}e^{2x}.$$

Note that if you don't want to remember the formulae you can simply remember that the key step in the method is to remove the u_1', u_2' terms from y'.
Here is the same problem solved via this slightly longer way of proceeding.

$y = u_1e^x + u_2e^{-x}$ hence $y' = [u_1'e^x + u_2'e^{-x}] + u_1e^x - u_2e^{-x}$

we set $[u_1'e^x + u_2'e^{-x}] = 0$ and differentiate y' again to get $y'' = u_1'e^x - u_2'e^{-x} + u_1e^x + u_2e^{-x}$.

Putting all the expressions into the ODE gives $u_1'e^x - u_2'e^{-x} = e^{2x}$.
Adding and subtracting the two equations $u_1'e^x + u_2'e^{-x} = 0$ and $u_1'e^x - u_2'e^{-x} = e^{2x}$ yields $u_1' = \frac{1}{2}e^x$ and $u_2' = -\frac{1}{2}e^{3x}$, which are the same expressions as found above using the formulae.

Here is an example that needs the method of variation of parameters as the method of undetermined coefficients is inapplicable.

Eg. Solve

$$y'' - 2y' + y = \frac{e^x}{x^2+1}.$$

CF: (CHAR) $\lambda^2 - 2\lambda + 1 = 0 = (\lambda - 1)^2,\ y_{CF} = (A + Bx)e^x.$

$y_1 = e^x,\ y_2 = xe^x$, with $W(y_1, y_2) = y_1 y_2' - y_2 y_1' = e^{2x}$

$$u_1 = -\int \frac{e^x}{x^2+1}\frac{xe^x}{e^{2x}}\,dx = -\int \frac{x}{x^2+1}\,dx = -\frac{1}{2}\log(x^2+1)$$

$$u_2 = \int \frac{e^x}{x^2+1}\frac{e^x}{e^{2x}}\,dx = \int \frac{1}{x^2+1}\,dx = \tan^{-1} x$$

$y_{PI} = -\frac{1}{2}e^x \log(x^2+1) + xe^x \tan^{-1} x$

$y = y_{CF} + y_{PI} = (A + Bx)e^x - \frac{1}{2}e^x \log(x^2+1) + xe^x \tan^{-1} x$

7.5 Systems of first order linear ODEs

A system of n coupled first order linear ODEs for n dependent variables can be written as a single n^{th} order linear ODE for a single dependent variable by eliminating the other dependent variables. An associated IVP is obtained if all the values of the dependent variables are specified at a single value of the independent variable.

Eg. Find the solution $y(x), z(x)$ of the pair of first order linear ODEs
$y' = -y + z + x^2, \quad z' = -8y + 5z + 8x^2 - 7x + 1,$
satisfying the initial condition $y(0) = 2,\ z(0) = 0.$

We solve by first finding a second order ODE for $y(x)$.
From the first equation $z = y' + y - x^2$ hence $z' = y'' + y' - 2x.$
Substituting these expressions into the second equation gives
$y'' + y' - 2x = -8y + 5(y' + y - x^2) + 8x^2 - 7x + 1$
ie. the second order ODE $\quad y'' - 4y' + 3y = 3x^2 - 5x + 1$
We can now solve this ODE using our earlier methods.
$(Char)$ is $\lambda^2 - 4\lambda + 3 = 0 = (\lambda - 3)(\lambda - 1)$ with roots $\lambda = 1, 3$. Hence $y_{CF} = Ae^x + Be^{3x}$
For y_{PI} try $y = a_0 + a_1 x + a_2 x^2$ to give
$y'' - 4y' + 3y = 2a_2 - 4(a_1 + 2a_2 x) + 3(a_0 + a_1 x + a_2 x^2) = 3a_2 x^2 + (3a_1 - 8a_2)x + 2a_2 - 4a_1 + 3a_0 = 3x^2 - 5x + 1$
Comparing coefficients gives the solution $a_2 = 1,\ a_1 = 1,\ a_0 = 1.$
The general solution for y is therefore $y = Ae^x + Be^{3x} + x^2 + x + 1$
To obtain z we use the earlier relation $z = y' + y - x^2 = 2Ae^x + 4Be^{3x} + 3x + 2.$
We now have the general solution for both y and z.
From the general solution $y(0) = A + B + 1 = 2$ and $z(0) = 2A + 4B + 2 = 0$ with solution $A = 3,\ B = -2.$
Hence the solution of the IVP is $\quad y = 3e^x - 2e^{3x} + x^2 + x + 1, \quad z = 6e^x - 8e^{3x} + 3x + 2.$

7.6 Summary: Second order ODEs

You should have a good understanding of how to solve linear constant-coefficient second order ODEs which can be written as $\alpha_2 y'' + \alpha_1 y' + \alpha_0 y = \phi(x)$. Here are some key points:

- Second order ODEs will have two parameters (integration constants) in the general solution. These can be fixed with *initial conditions* (fixing the value of y and y' at the same point) or *boundary conditions* (fixing the value of y at two different points). Initial conditions will uniquely fix the integration constants. Boundary conditions will typically uniquely fix the integration constants but it is possible that there could be no solution or a one-parameter family of solutions – e.g. consider $y'' + y = 0$ with $y(0) = 0$ (which gives $y = A\sin x$) and the other boundary condition being $y(\pi/2) = 1$ (1 solution) or $y(\pi) = 1$ (no solutions) or $y(\pi) = 0$ (one-parameter family of solutions).

- If ϕ is zero we have a *homogeneous second order ODE*. We solve this by first solving the *Characteristic Equation* (CE) $\alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 = 0$.

- If the CE has two real solutions $\lambda_1 \neq \lambda_2$ then the ODE has general solution $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$.

- If the CE has only one solution λ_1 (repeated roots) then the ODE has general solution $y = Ae^{\lambda_1 x} + Bxe^{\lambda_1 x}$.

- If the CE has two complex solutions $\alpha + i\beta$ and $\alpha - i\beta$ then the ODE has general solution $y = Ce^{(\alpha+i\beta)x} + De^{(\alpha-i\beta)x} =^{\alpha x} (A\cos(\beta x) + B\sin(\beta x))$.

- In all case the general solution of the homogeneous ODE is of the form $y(x) = Ay_1(x) + By_2(x)$ where $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the ODE.

- The general solution to a linear ODE is $y = y_{CF} + y_{PI}$ where y_{PI} is a *particular integral* (PI) which is any solution of the ODE and y_{CF} is the *complementary function* (CF) which is the general solution of the homogeneous ODE formed by setting ϕ to zero.

- If $\phi(x)$ is formed from sums of products of exponentials $e^{\gamma x}$, sines and cosines $\sin(\gamma x)$ and $\cos(\gamma x)$, and polynomials we can find a PI using the *method of undetermined coefficients*. To do this we use an ansatz for y_{PI} which is of the same form as $\phi(x)$ with arbitrary coefficients for each term. Then substitute this ansatz into the ODE to solve for the coefficients. Note that if a term solves the homogeneous ODE we need to include an extra factor of x with that term, and another factor of x if that still solves the homogeneous ODE.

- If we cannot use the method of undetermined coefficients, we can try the *method of variation of parameters*. If $y_{CF} = Ay_1 + By_2$ we take $y = u_1y_1 + u_2y_2$ where u_1 and u_2 are functions of x and we impose the constraint $u_1'y_1 + u_2'y_2 = 0$ so that y' does not depend on u_1' or u_2'. Substituting this into the ODE, all terms with u_1 and u_2 (not differentiated) cancel as they must since constant u_1 and u_2 would give a solution of the homogeneous

ODE – the result is a second linear equation for u_1' and u_2', $\alpha_2(u_1'y_1' + u_2'y_2') = \phi$. Solve these two linear equation to find u_1' and u_2' and then integrate to find u_1 and u_2 (noting that the integration constants are exactly the parameters in y_{CF}).

- Two coupled first order linear ODEs for two variables give, by eliminating one of the variables, a linear second order ODE for one variable. This gives one method to solve coupled first order ODEs. (We can also go the other way by defining z as a linear combination of y and y' and using this to write a second order linear ODE for y as a first order linear ODE involving z', z and y.)

8 Taylor series

8.1 Taylor's theorem

Taylor's theorem states that if $f(x)$ has $n+1$ continuous derivatives in an open interval I that contains the point $x=a$, then $\forall\ x \in I$

$$f(x) = \left(\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \right) + R_n(x)$$

$$\text{where} \quad R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt \quad \text{is called the } \textbf{remainder}.$$

Proof:
Fix $x \in I$ then $\int_a^x f'(t)\,dt = f(x) - f(a)$, so

$$f(x) = f(a) + \int_a^x f'(t)\,dt.$$

If we think of the integrand in the above as $f'(t)\cdot 1$ then we can perform an integration by parts where we choose the constant of integration to be $-x$ so that $\int 1\,dt = (t-x)$. This gives

$$\begin{aligned} f(x) &= f(a) + \Big[f'(t)(t-x)\Big]_a^x - \int_a^x f''(t)(t-x)\,dt \\ &= f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)\,dt. \end{aligned}$$

Note that this proves the theorem for $n=1$. To prove the theorem for $n=2$ we perform another integration by parts on this equation,

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)\,dt \\ &= f(a) + f'(a)(x-a) - \left[f''(t)\frac{(x-t)^2}{2}\right]_a^x + \int_a^x f'''(t)\frac{(x-t)^2}{2}\,dt \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \int_a^x f'''(t)\frac{(x-t)^2}{2}. \end{aligned}$$

This proves the theorem for $n=2$. Continuing in this way the theorem follows after performing an integration by parts a total of n times. More formally, this can be written in the form of a proof by induction.

8.2 Taylor polynomials

The combination $P_n(x) = f(x) - R_n(x)$ is a polynomial in x of degree n called **the n^{th} order Taylor polynomial of $f(x)$ about $x = a$**

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If the term 'about $x = a$' is not included when referring to the n^{th} order Taylor polynomial of $f(x)$, then by default this is taken to mean the choice $a = 0$, as this is the most common case.

The Taylor polynomial $P_n(x)$ is an approximation to the function $f(x)$. Generically, it is a good approximation if x is close to a and the approximation improves with increasing order n. The remainder provides an exact expression for the error in the approximation.

Eg. Calculate the n^{th} order Taylor polynomial of e^x.
$f(x) = e^x,\ f'(x) = e^x,\ ...\ , f^{(k)}(x) = e^x$ so $f^{(k)}(0) = 1$ giving

$$P_n(x) = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Figure 35 shows the graph of e^x and its Taylor polynomials of order 0 to 3.

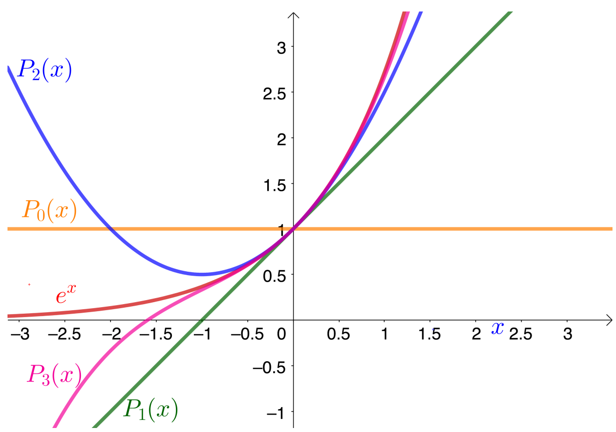


Figure 35: The function e^x (red) and its associated Taylor polynomials $P_0(x), P_1(x), P_2(x), P_3(x)$ of order 0 (orange), 1 (green), 2 (blue) and 3 (pink).

If $f(x)$ is infinitely differentiable on an open interval I that contains the point $x = a$ and in addition for each $x \in I$ we have that $\lim_{n\to\infty} R_n(x) = 0$, then we say that $f(x)$ can be expanded as a **Taylor series** about $x = a$ and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Infinite series and discussions/tests concerning their convergence are covered in the Analysis I course.

Note that the n^{th} order Taylor polynomial is obtained by taking just the first $(n+1)$ terms of the Taylor series (where we count terms even if they happen to be zero).

Eg. From our earlier calculation we have that the Taylor series of e^x is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

where $\cdots$ denotes an infinite number of terms with increasing powers of x.

Note that if we set $x = 1$ in the above Taylor series then we obtain a formula for Euler's number

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Eg. Calculate the Taylor series of $\sin x$.
$f(x) = \sin x,\ f(0) = 0, \quad f'(x) = \cos x,\ f'(0) = 1,$
$f''(x) = -\sin x,\ f''(0) = 0, \quad f'''(x) = -\cos x,\ f'''(0) = -1,$
$f^{(4)}(x) = \sin x$ so now the pattern repeats and we see that for all non-negative integers k we have that $f^{2k}(0) = 0$ and $f^{2k+1}(0) = (-1)^k$.
The Taylor series of $\sin x$ is therefore given by

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Observe that only odd powers of x appear in the Taylor series of $\sin x$ as this is an odd function.

A similar calculation (exercise) yields the Taylor series of $\cos x$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

in which only even powers of x appear, as $\cos x$ is an even function.

To calculate the Taylor series of $\sinh x$ observe that for all non-negative integer k we have that $f(x) = \sinh x$ satisfies

$$f^{(k)}(x) = \begin{cases} \sinh x & \text{if } k \text{ is even} \\ \cosh x & \text{if } k \text{ is odd} \end{cases}$$

thus $f^{(2k)}(0) = 0$ and $f^{(2k+1)}(0) = 1$ giving

$$\sinh x = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

A similar calculation (exercise) yields the Taylor series of $\cosh x$

$$\cosh x = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

These two results can also be obtained directly from the Taylor series of e^x by noting that $\cosh x$ and $\sinh x$ are the even and odd parts of e^x.

As $\log x$ is not defined at $x = 0$ it does not make sense to consider the Taylor series of $\log x$ about $x = 0$. Instead we consider the Taylor series of $\log x$ about $x = 1$.
$f(x) = \log x,\ f(1) = 0, \quad f'(x) = \frac{1}{x},\ f'(1) = 1,$
$f''(x) = -\frac{1}{x^2},\ f''(1) = -1, \quad f'''(x) = \frac{2}{x^3},\ f'''(1) = 2,$
$f^{(4)}(x) = -\frac{3\cdot 2}{x^4},\ f^{(4)}(1) = -3\cdot 2$ and in general for k a positive integer $f^{(k)}(x) = (-1)^{k-1}\frac{(k-1)!}{x^k},\ f^{(k)}(1) = (-1)^{k-1}(k-1)!$
Thus the Taylor series of $\log x$ about $x = 1$ is

$$\log x = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{k!}(x-1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k$$

This result is more often expressed in terms of the variable $X = x - 1$, when it becomes $\log(1+X) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} X^k$.
If we now rename the variable X as x we get

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

This result is not valid at $x = -1$, and this is to be expected because the logarithm is not defined when its argument is zero. A careful examination of the requirement $\lim_{n\to\infty} R_n(x) = 0$ reveals that x must satisfy $-1 < x \leq 1$ for the above Taylor series to be valid (ie. for the series to converge).

8.3 Lagrange form for the remainder

There is a more convenient expression for the remainder term in Taylor's theorem. The **Lagrange form for the remainder** is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}, \qquad \text{for some } \; c \in (a,x).$$

To prove this expression for the remainder we will first need to prove the following lemma:

Lemma
Let $h(t)$ be differentiable $n+1$ times on $[a,x]$ with $h^{(k)}(a) = 0$ for $0 \leq k \leq n$ and $h(x) = 0$.
Then $\exists\, c \in (a,x)$ s.t. $h^{(n+1)}(c) = 0$.

Proof:
$h(a) = 0 = h(x)$ so by Rolle's theorem $\exists\, c_1 \in (a,x)$ s.t. $h'(c_1) = 0$.
$h'(a) = 0 = h'(c_1)$ so by Rolle's theorem $\exists\, c_2 \in (a,c_1)$ s.t. $h''(c_1) = 0$.
Repeating this argument a total of n times we arrive at
$h^{(n)}(a) = 0 = h^{(n)}(c_n)$ so by Rolle's theorem $\exists\, c_{n+1} \in (a,c_n)$ s.t. $h^{(n+1)}(c_{n+1}) = 0$.
This proves the lemma with $c = c_{n+1} \in (a,x)$.

Proof of the Lagrange form of the remainder:
Consider the function

$$h(t) = (f(t) - P_n(t))(x-a)^{n+1} - (f(x) - P_n(x))(t-a)^{n+1}.$$

By construction $h(x) = 0$.
Also $\frac{d^k}{dt^k}(t-a)^{n+1}$ is zero when evaluated at $t = a$ for $0 \leq k \leq n$. Furthermore, by definition of the Taylor polynomial $P_n(t)$ we have that $f^{(k)}(a) = P_n^{(k)}(a)$ for $0 \leq k \leq n$.
Hence $h^{(n)}(a) = 0$ for $0 \leq k \leq n$.
$h(t)$ therefore satisfies the conditions of the lemma and we have that
$\exists\, c \in (a,x)$ s.t. $h^{(n+1)}(c) = 0$.
As $P_n(t)$ is a polynomial of degree n then $P_n^{(n+1)}(t) = 0$.
Also, $\frac{d^{n+1}}{dt^{n+1}}(t-a)^{n+1} = (n+1)!$ hence

$$0 = h^{(n+1)}(c) = (x-a)^{n+1} f^{(n+1)}(c) - (n+1)!(f(x) - P_n(x))$$

Rearranging this expression gives the required result

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

One use of the Lagrange form of the remainder is to provide an upper bound on the error of a Taylor polynomial approximation to a function.

Suppose that $|f^{(n+1)}(t)| \le M, \quad \forall\ t$ in the closed interval between a and x.
Then $|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!}$ provides a bound on the error.

Eg. Show that the error in approximating e^x by its 6^{th} order Taylor polynomial is always less than 0.0006 throughout the interval $[0,1]$.

In this example $f(x) = e^x$ and $n = 6$ so we first require an upper bound on $|f^{(7)}(t)| = |e^t|$ for $t \in [0,1]$. As e^t is monotonic increasing and positive then $|e^t| \le e^1 = e < 3$. Thus, in the above notation, we may take $M = 3$.
As $a = 0$ we now have that $|R_6(x)| < \frac{3|x|^7}{7!} \le \frac{3}{7!}$ for $x \in [0,1]$.
Evaluating $\frac{3}{7!} = \frac{1}{1680} < 0.0006$ and the required result has been shown.

8.4 Calculating limits using Taylor series

**** Only use this method in assignments/exam questions when told to do so ****

Defn: Let n be a positive integer.
We say that $f(x) = o(x^n)$ (as $x \to 0$) if $\lim_{x\to 0} \frac{f(x)}{x^n} = 0$.

In particular, if α is any non-zero constant then $\alpha x^m = o(x^n)$ iff $m > n$.

Egs. $3x^4 = o(x^3),\ 3x^4 = o(x^2),\ 3x^8 - x^6 = o(x^5),\ o(x^8) + o(x^5) = o(x^5)$.

If we know that a Taylor series is valid in a suitable open interval, then it may be useful in calculating certain limits.
It can be shown that the Taylor series we have already seen for $\sin x$ and $\cos x$ are valid for all $x \in \mathbb{R}$. Although we have not proved this result you may assume that it is true and use it in calculating limits. As examples, we shall now calculate the two important trigonometric limits that we saw earlier.

Eg. Use Taylor series to calculate $\lim_{x\to 0} \frac{\sin x}{x}$.
From the Taylor series of $\sin x$ we have that $\sin x = x - \frac{x^3}{3!} + o(x^4)$. Hence

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + o(x^4)}{x} = \lim_{x\to 0}(1 - \frac{x^2}{3!} + o(x^3)) = \lim_{x\to 0}(1 + o(x)) = 1.$$

Eg. Use Taylor series to calculate $\lim_{x\to 0} \frac{1-\cos x}{x}$.
From the Taylor series of $\cos x$ we have that $\cos x = 1 - \frac{x^2}{2} + o(x^3)$. Hence

$$\lim_{x\to 0} \frac{1 - \cos x}{x} = \lim_{x\to 0} \frac{1 - 1 + \frac{x^2}{2} + o(x^3)}{x} = \lim_{x\to 0} \frac{\frac{x^2}{2} + o(x^3)}{x} = \lim_{x\to 0} \left(\frac{x}{2} + o(x^2)\right) = 0.$$

8.5 Summary: Taylor Series

You should know how to calculate Taylor polynomials $P_n(x)$ and what the integral form and Lagrange form of the remainder $R_n(x)$ are, as well as how to estimate the maximum error in using a Taylor polynomial as an approximation of a function. Here are some key points:

- The *Taylor polynomial* of degree n about $x = a$ for a function f is the degree n polynomial in $(x-a)$, $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$. At the point $x = a$, P_n and its first n derivatives will agree with f and its first n derivatives. In this sense the Taylor polynomial tries to approximate f close to $x = a$. Note that if we don't explicitly state "about $x = a$" then it is assumed we are working about $x = 0$.

- *Taylor's theorem* states that $f(x) = P_n(x)+R_n(x)$ where $R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt$ is the *remainder*.

- The *Lagrange form of the remainder* says that $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some (unknown) c between a and x. (Note the similarity to the next term in the Taylor polynomial.) This can often be used to place an upper bound on *the error* $|R_n(x)|$ in using $P_n(x)$ as an approximation of $f(x)$.

- The *Taylor series* about $x = a$ is $\lim_{n\to\infty} P_n(x)$. If $\lim_{n\to\infty} R_n(x) = 0$ (i.e. the error vanishes in this limit) for all x in some interval containing a we say that the Taylor series converges to $f(x)$ on that interval. In such cases $P_n(x)$ approximates $f(x)$ as well as we want on that interval by taking n large enough.

- You should be familiar with the form of the Taylor series (about $x = 0$) for e^x, $\sinh x$, $\cosh x$, $\sin x$, $\cos x$ and $\log(1+x)$. The Taylor series for $\log(1+x)$ converges to $\log(1+x)$ for $x \in (-1, 1]$. The other ones converge to their functions $\forall x \in \mathbb{R}$.

- We can find Taylor polynomials of degree n for sums of products of functions by taking sums of products of Taylor polynomials of degree n (and ignoring any terms with degree higher than n which arise from products). We can also use substitution, e.g. to find the Taylor polynomial of degree n for $e^{\sin x}$ replace x in the Taylor polynomial of degree n for e^x with the Taylor polynomial of degree n for $\sin x$ (and again ignore all powers higher than n). It is sometimes possible to divide by Taylor series using $1/(1-x) = 1+x+x^2+x^3+\cdots$ for $x \in (-1, 1)$ and appropriate substitution.

- We can evaluate limits as $x \to a$ by replacing functions with their Taylor polynomials of sufficiently high order about $x = a$ provided the Taylor series converge to the functions on some interval containing a. This is because in such cases the higher order terms (higher powers of $(x-a)$) will vanish in the limit $x \to a$.

9 Fourier series

9.1 Fourier coefficients

In our study of Taylor series and Taylor polynomials we have seen how to write and approximate functions in terms of polynomials. If the function we are interested in is periodic, then it is more appropriate to use trigonometric functions rather than polynomials. This is the topic of Fourier series, which are important in a wide range of areas and applications. In particular, they play a vital role in the solution of certain partial differential equations.

Consider a function $f(x)$ of period $2L$ which is given on the interval $(-L, L)$. The functions $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$, with n any positive integer, are also periodic with period $2L$. It therefore seems reasonable to try and write $f(x)$ in terms of these trigonometric functions. Note that a constant is trivially a periodic function (for any period) so we can also include a constant term. We therefore aim to write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \qquad (FS)$$

where a_0, a_n, b_n are constants labelled by the positive integer n, and are called the **Fourier coefficients** of $f(x)$. (It is just a convenient notation to call the constant term $a_0/2$.). If these Fourier coefficients are such that this series converges then it is called the **Fourier series** of $f(x)$.

Eg. The function $f(x) = 2\cos^2\frac{\pi x}{L} + 3\sin\frac{\pi x}{L}$ has period $2L$.
Its Fourier series contains only a finite number of terms as
$2\cos^2\frac{\pi x}{L} + 3\sin\frac{\pi x}{L} = 1 + \cos\frac{2\pi x}{L} + 3\sin\frac{\pi x}{L}$.
Therefore $a_0 = 2,\ a_2 = 1,\ b_1 = 3$, and all the other Fourier coefficients are zero.

In general an infinite number of Fourier coefficients may be non-zero and we need a method to determine these from the given function $f(x)$.
In order to derive formulae for the Fourier coefficients we first need to prove some identities for integrals of trigonometric functions.

In the following let m and n be positive integers.

$$(i) \qquad \int_{-L}^{L} \cos\frac{n\pi x}{L}\,dx = 0.$$

$$(ii) \qquad \int_{-L}^{L} \sin\frac{n\pi x}{L}\,dx = 0.$$

$$(iii) \qquad \int_{-L}^{L} \sin\frac{m\pi x}{L}\,\cos\frac{n\pi x}{L}\,dx = 0.$$

$$(iv) \qquad \frac{1}{L}\int_{-L}^{L} \cos\frac{m\pi x}{L}\,\cos\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

$$(v) \qquad \frac{1}{L}\int_{-L}^{L} \sin\frac{m\pi x}{L}\,\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

The expression that occurs on the right hand side of the last two formulae arises so often that it is useful to introduce a shorthand notation for this. The Kronecker delta δ_{mn} is defined to be

$$\delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

So, for example, we may write $\frac{1}{L}\int_{-L}^{L} \sin\frac{m\pi x}{L}\,\sin\frac{n\pi x}{L}\,dx = \delta_{mn}$.

It is trivial to prove (i) : $\int_{-L}^{L} \cos\frac{n\pi x}{L}\,dx = \left[\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_{-L}^{L} = \frac{2L}{n\pi}\sin(n\pi) = 0.$

(ii) and (iii) are true by inspection, as they are both integrals of odd functions over a symmetric interval.

We can prove (iv) and (v) by using $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$.

First assume $m \neq n$ then

$$\frac{1}{L}\int_{-L}^{L} \cos\frac{m\pi x}{L}\,\cos\frac{n\pi x}{L}\,dx = \frac{1}{2L}\int_{-L}^{L}\left(\cos\frac{(m-n)\pi x}{L} + \cos\frac{(m+n)\pi x}{L}\right)dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{m-n}\sin\frac{(m-n)\pi x}{L} + \frac{1}{m+n}\sin\frac{(m+n)\pi x}{L}\right]_{-L}^{L} = 0.$$

Now if $m = n$

$$\frac{1}{L}\int_{-L}^{L} \cos\frac{m\pi x}{L}\,\cos\frac{n\pi x}{L}\,dx = \frac{1}{2L}\int_{-L}^{L}\left(1 + \cos\frac{2n\pi x}{L}\right)dx = \frac{1}{2L}\left[x + \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right]_{-L}^{L} = 1.$$

This proves (iv) and a similar calculation (exercise) proves (v).

We say that the set of functions $\{1, \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}\}$ form an **orthogonal set** on $[-L, L]$ because if ϕ, ψ are any two *different* functions from this set then $\frac{1}{L}\int_{-L}^{L} \phi\psi\, dx = 0$.

We now use the identities $(i), .., (v)$ to derive expressions for the Fourier coefficients in (FS). First of all, by integrating (FS) we have

$$\frac{1}{L}\int_{-L}^{L} f(x)\, dx = \frac{1}{L}\int_{-L}^{L} \left\{\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)\right\} dx = a_0$$

using (i) and (ii). Thus we have derived an expression for a_0

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx.$$

Let m be a positive integer and multiply (FS) by $\cos\frac{m\pi x}{L}$ and integrate to give

$$\frac{1}{L}\int_{-L}^{L} \cos\frac{m\pi x}{L} f(x)\, dx = \frac{1}{L}\int_{-L}^{L} \left\{\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)\right\} \cos\frac{m\pi x}{L}\, dx$$

$$= \sum_{n=1}^{\infty} a_n \delta_{mn} = a_m, \qquad \text{where we have used the identities } (i), (iii), (iv).$$

Thus we have derived an expression for a_n

$$a_n = \frac{1}{L}\int_{-L}^{L} \cos\frac{n\pi x}{L} f(x)\, dx.$$

Note that if we extend this relation to $n = 0$ then it reproduces the correct expression given above for a_0.

Similarly, multiply (FS) by $\sin\frac{m\pi x}{L}$ and integrate to give

$$\frac{1}{L}\int_{-L}^{L} \sin\frac{m\pi x}{L} f(x)\, dx = \frac{1}{L}\int_{-L}^{L} \left\{\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)\right\} \sin\frac{m\pi x}{L}\, dx$$

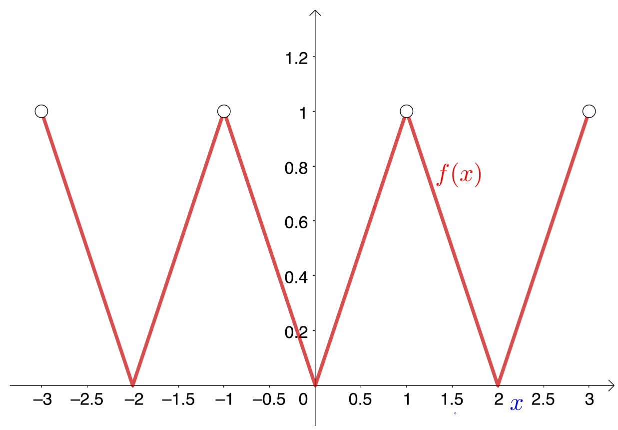
$$= \sum_{n=1}^{\infty} b_n \delta_{mn} = b_m, \qquad \text{where we have used the identities } (ii), (iii), (v).$$

Thus we have derived an expression for b_n

$$b_n = \frac{1}{L}\int_{-L}^{L} \sin\frac{n\pi x}{L} f(x)\, dx.$$

9.2 Examples of Fourier series

Eg. The function $f(x)$ has period 2, that is $f(x+2) = f(x)$, and is given by $f(x) = |x|$ for $-1 < x < 1$. The graph of this function is shown in Figure 9.2. To calculate the Fourier series



of this function we apply the earlier formulae with $L = 1$.

$a_0 = \int_{-1}^{1} |x|\, dx = 2\int_0^1 x\, dx = \Big[x^2\Big]_0^1 = 1.$

For $n > 0$

$$a_n = \int_{-1}^{1} |x| \cos(n\pi x)\, dx = 2\int_0^1 x\cos(n\pi x)\, dx$$

$$= 2\Big\{\Big[\frac{x}{n\pi}\sin(n\pi x)\Big]_0^1 - \int_0^1 \frac{1}{n\pi}\sin(n\pi x)\, dx\Big\} = \frac{2}{n^2\pi^2}\Big[\cos(n\pi x)\Big]_0^1$$

$$= \frac{2}{n^2\pi^2}(\cos(n\pi) - 1) = \frac{2}{n^2\pi^2}((-1)^n - 1).$$

Furthermore, for $n > 0$

$$b_n = \int_{-1}^{1} |x|\sin(n\pi x)\, dx = 0$$

because this is the integral of an odd function over a symmetric interval.

This demonstrates a general point that if $f(x)$ is an even function on the interval $(-L, L)$ then all $b_n = 0$ and the Fourier series contains only cosine terms (plus a constant term). This is called a cosine series.

Putting all this together we have the Fourier (cosine) series

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}((-1)^n - 1)\cos(n\pi x)$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\left(\cos(\pi x) + \frac{1}{9}\cos(3\pi x) + \frac{1}{25}\cos(5\pi x) + \cdots\right)$$

Note that in this example $a_{2n} = 0$ and $a_{2n-1} = -\frac{4}{\pi^2(2n-1)^2}$, so this Fourier (cosine) series could also be written as

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos((2n-1)\pi x)}{(2n-1)^2}.$$

To see how the Fourier series approaches the function $f(x)$ define the **partial sum**

$$S_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right).$$

In this example, $S_1(x) = \frac{1}{2} - \frac{4}{\pi^2}\cos(\pi x)$, $\quad S_3(x) = \frac{1}{2} - \frac{4}{\pi^2}(\cos(\pi x) + \frac{1}{9}\cos(3\pi x))$,
$S_5(x) = \frac{1}{2} - \frac{4}{\pi^2}(\cos(\pi x) + \frac{1}{9}\cos(3\pi x) + \frac{1}{25}\cos(5\pi x))$, etc.

In Figure 36 we plot the graph of $f(x)$ together with the partial sums $S_1(x), S_5(x), S_{11}(x)$. This helps to demonstrate that, in this case, $\lim_{m\to\infty} S_m(x) = f(x)$.

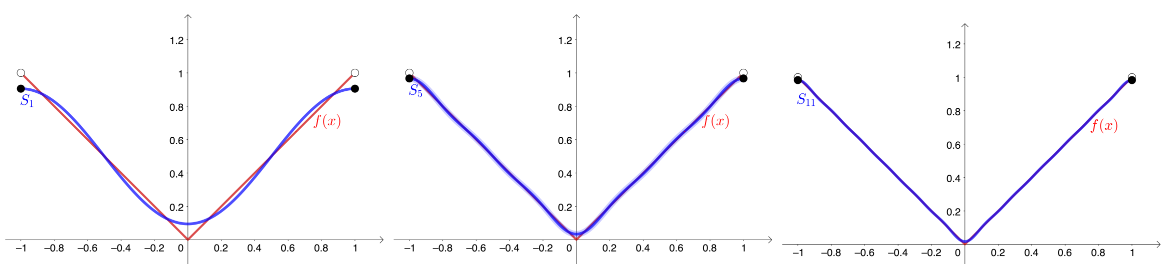


Figure 36: The graph of $f(x) = |x|$ for $-1 < x < 1$, together with the partial sums $S_1(x), S_5(x), S_{11}(x)$.

Eg. The function $f(x)$ has period 2π and is given by $f(x) = x$ for $-\pi < x < \pi$. Calculate the Fourier series of $f(x)$.

The graph of this function is shown in Figure 37. Note that the function is not continuous at the points $x = (2p+1)\pi, \quad p \in \mathbb{Z}$, where there are jump discontinuities.

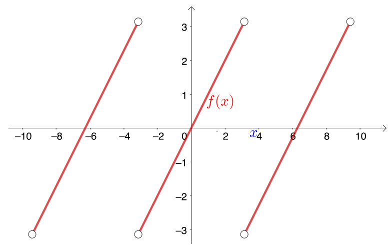


Figure 37: The graph of $f(x) = x$ for $-\pi < x < \pi$, with period 2π.

We apply the earlier formulae with $L = \pi$. For $n \geq 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = 0$$

because this is the integral of an odd function over a symmetric interval.

This demonstrates a general point that if $f(x)$ is an odd function on the interval $(-L, L)$ then all $a_n = 0$ and the Fourier series contains only sine functions. This is called a sine series.

For $n > 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{1}{\pi}\bigg\{\bigg[-\frac{x}{n}\cos(nx)\bigg]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{1}{n}\cos(nx)\,dx\bigg\}$$

$$= \frac{1}{\pi}\bigg\{-\frac{2\pi}{n}\cos(n\pi) + \bigg[\frac{1}{n^2}\sin(nx)\bigg]_{-\pi}^{\pi}\bigg\} = -\frac{2}{n}\cos(n\pi) = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}.$$

Putting all this together we have the Fourier (sine) series

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin(nx) = 2\bigg(\sin x - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x) + \cdots\bigg)$$

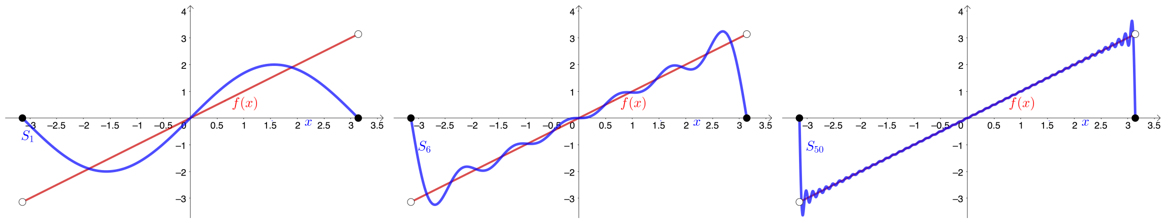


Figure 38: The graph of $f(x) = x$ for $-\pi < x < \pi$, together with the graphs of the partial sums $S_1(x), S_6(x), S_{50}(x)$.

In Figure 38 we plot the graph of $f(x) = x$ for $-\pi < x < \pi$, together with the graphs of the partial sums $S_1(x), S_6(x), S_{50}(x)$.
These graphs demonstrate that as more terms of the Fourier series are included it becomes an increasingly accurate approximation to $f(x)$ inside the interval $x \in (-\pi, \pi)$. However, notice what happens at the points $x = \pm\pi$, where $f(x)$ is not continuous. At these points the Fourier series converges to 0, which is the midpoint of the jump. Note the oscillations around the point of discontinuity, where the Fourier series under/overshoots. This is called the Gibbs phenomenon and the amount of under/overshoot tends to a constant (in fact about 9%) rather than dying away as more terms are included, but the under/overshoot region becomes more localized. In the limit of an infinite number of terms the undershoot and overshoot occur at exactly the same point and cancel each other out.

In general, we would like to know what happens to the partial sum

$$S_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

for all values of x in the limit as $m \to \infty$, and how this is related to $f(x)$. The following theorem provides an answer.

Dirichlet's theorem
Let $f(x)$ be a periodic function, with period $2L$, such that on the interval $(-L, L)$ it has a finite number of extreme values, a finite number of jump discontinuities and $|f(x)|$ is integrable on $(-L, L)$. Then its Fourier series converges for all values of x. Furthermore, it converges to $f(x)$ at all points where $f(x)$ is continuous and if $x = a$ is a jump discontinuity then it converges to $\frac{1}{2} \lim_{x \to a^-} f(x) + \frac{1}{2} \lim_{x \to a^+} f(x)$.

9.3 Parseval's theorem

We shall now derive a relation between the average of the square of a function and its Fourier coefficients.

Parseval's theorem
If $f(x)$ is a function of period $2L$ with Fourier coefficients a_n, b_n then

$$\frac{1}{2L}\int_{-L}^{L}(f(x))^2\,dx = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Proof:

$$\frac{1}{L}\int_{-L}^{L}(f(x))^2\,dx =$$

$$\frac{1}{L}\int_{-L}^{L}\left\{\left[\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)\right]\left[\frac{a_0}{2}+\sum_{m=1}^{\infty}\left(a_m\cos\frac{m\pi x}{L}+b_m\sin\frac{m\pi x}{L}\right)\right]\right\}dx$$

$$= \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_n a_m \delta_{mn} + b_n b_m \delta_{mn}) = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Parseval's theorem is useful in several contexts. One application is in finding the sum of certain infinite series.

Eg. Calculate $\sum_{n=1}^{\infty}\frac{1}{n^2}$ by applying Parseval's theorem to the Fourier series of $f(x) = x$ for $-\pi < x < \pi$.
From earlier we calculated that the Fourier coefficients are given by $a_n = 0$ and $b_n = \frac{2}{n}(-1)^{n+1}$.
Hence by Parseval's theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{4}{n^2}. \quad \text{Thus} \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Certain infinite sums can also be obtained by evaluating a Fourier series at a particular value of x.

Eg. Calculate $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$ by evaluating the Fourier series of $f(x) = |x|$ for $-1 < x < 1$, at the value $x = 0$.
From earlier we calculated the Fourier (cosine) series to be

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos((2n-1)\pi x)}{(2n-1)^2}.$$

As $f(x) = |x|$ satisfies the conditions of Dirichlet's theorem and is continuous at $x = 0$ then, by Dirichlet's theorem, evaluating the Fourier series at $x = 0$ gives $f(0) = 0$. Hence

$$0 = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}. \qquad \text{Thus} \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

9.4 Half range Fourier series

A half range Fourier series is a Fourier series defined on an interval $(0, L)$ rather than the interval $(-L, L)$.

Given a function $f(x)$, defined on the interval $(0, L)$, we obtain its **half range sine series** by calculating the Fourier sine series of its **odd extension**

$$f_o(x) = \begin{cases} f(x) & \text{if } \ 0 < x < L \\ -f(-x) & \text{if } \ -L < x < 0 \end{cases}$$

The Fourier coefficients of $f_o(x)$ are $a_n = 0$ and

$$b_n = \frac{1}{L}\int_{-L}^{L} f_o(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

This gives the half range sine series for $f(x)$ on $(0, L)$ as $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$.

Given a function $f(x)$, defined on the interval $(0, L)$, we obtain its **half range cosine series** by calculating the Fourier cosine series of its **even extension**

$$f_e(x) = \begin{cases} f(x) & \text{if } \ 0 < x < L \\ f(-x) & \text{if } \ -L < x < 0 \end{cases}$$

The Fourier coefficients of $f_e(x)$ are $b_n = 0$ and

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.$$

This gives the half range cosine series for $f(x)$ on $(0, L)$ as $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$.

For a given (physical) problem on an interval $(0, L)$ it is usually the boundary conditions that determine whether an odd extension and a half sine Fourier series or an even extension and a half cosine Fourier series is more appropriate.

9.5 Fourier series in complex form

Fourier series take a simpler form if written in terms of complex variables. Starting with the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

we use the relations $\cos\frac{n\pi x}{L} = \frac{1}{2}(e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}})$ and $\sin\frac{n\pi x}{L} = -\frac{i}{2}(e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}})$ to rewrite this as

$$f(x) = \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \left((a_n - ib_n)e^{\frac{in\pi x}{L}} + (a_n + ib_n)e^{-\frac{in\pi x}{L}} \right) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

where we have defined

$$c_0 = \frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$$

for $n > 0$

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2L}\int_{-L}^{L} f(x)(\cos\frac{n\pi x}{L} - i\sin\frac{n\pi x}{L})\, dx = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}\, dx$$

and for $n < 0$

$$c_n = \frac{1}{2}(a_{-n} + ib_{-n}) = \frac{1}{2L}\int_{-L}^{L} f(x)(\cos\frac{-n\pi x}{L} + i\sin\frac{-n\pi x}{L})\, dx = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}\, dx.$$

Note that $c_{-n} = \overline{c_n}$.
All 3 cases ($n = 0, n > 0, n < 0$) can be written as a single compact formula

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}\, dx, \qquad \text{with the Fourier series } \; f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

Eg. The function $f(x)$ has period 2π and is given by

$$f(x) = \begin{cases} -1 & \text{if } \; -\pi < x < 0 \\ \;\;1 & \text{if } \;\;\; 0 \le x < \pi \end{cases}$$

Obtain the complex form of the Fourier series for $f(x)$.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{0} -e^{-inx}\, dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx}\, dx$$

For $n = 0$

$$c_0 = -\frac{1}{2\pi}\int_{-\pi}^{0} dx + \frac{1}{2\pi}\int_{0}^{\pi} dx = -\frac{1}{2} + \frac{1}{2} = 0$$

For $n \neq 0$

$$c_n = \left[\frac{e^{-inx}}{2\pi in}\right]_{-\pi}^{0} - \left[\frac{e^{-inx}}{2\pi in}\right]_{0}^{\pi} = \frac{1}{2\pi in}(1 - e^{in\pi} - e^{-in\pi} + 1) = \frac{i}{\pi n}((-1)^n - 1)$$

Hence $c_{2m} = 0$ and $c_{2m+1} = \frac{-2i}{\pi(2m+1)}$ giving the complex form of the Fourier series

$$f(x) = \sum_{m=-\infty}^{\infty} \frac{-2i}{\pi(2m+1)} e^{i(2m+1)x}$$

This complex form can be converted back to the usual real form as follows.

Using $f(x) = \overline{f(x)}$ we have that

$$2f(x) = \sum_{m=-\infty}^{\infty} \frac{-2i}{\pi(2m+1)} e^{i(2m+1)x} + \sum_{m=-\infty}^{\infty} \frac{2i}{\pi(2m+1)} e^{-i(2m+1)x}$$

hence

$$f(x) = \sum_{m=-\infty}^{\infty} \frac{-i}{\pi(2m+1)}(e^{i(2m+1)x} - e^{-i(2m+1)x}) = \sum_{m=-\infty}^{\infty} \frac{-i}{\pi(2m+1)} 2i \sin((2m+1)x)$$

$$= \sum_{m=-\infty}^{\infty} \frac{2\sin((2m+1)x)}{\pi(2m+1)} = \sum_{m=0}^{\infty} \frac{4\sin((2m+1)x)}{\pi(2m+1)}$$

9.6 Summary: Fourier Series

Fourier series attempt to represent periodic functions as linear combinations of sines and cosines. You should know how to calculate Fourier coefficients so you can construct Fourier series (or Fourier partial sums). You should understand the interpretation of Fourier series as (infinite-dimensional) vectors where the orthogonal basis functions are the sines and cosines, and the components are the Fourier coefficients. Here are some key points:

- If $f(x)$ is a function of period $2L$, its (full-range) *Fourier series* is the infinite series $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L))$ where the Fourier coefficients are given by $a_n = \frac{1}{L}\int_{-L}^{L} \cos(n\pi x/L) f(x) dx$ (including $n = 0$) and $b_n = \frac{1}{L}\int_{-L}^{L} \sin(n\pi x/L) f(x) dx$. This arises for the vector interpretation with scalar product of $\phi(x)$ and $\psi(x)$ being $\frac{1}{L}\int_{-L}^{L} \phi\psi dx$, noting that the sines and cosines are orthonormal using this scalar product.

- If we include only the first N sines and cosines (i.e. have $\sum_{n=1}^{N}$ rather than $\sum_{n=1}^{\infty}$) we have a *Fourier partial sum*. The Fourier series is then given by the limit as $N \to \infty$.

- An odd function has a *sine Fourier series* (all $a_n = 0$) while an even function has a *cosine Fourier series* (all $b_n = 0$).

- *Dirichlet's theorem* says that if $f(x)$ is a periodic function with a finite number of extreme values and jump discontinuities in each period, and with $|f(x)|$ integrable over a period then the Fourier series converges (to some real number) for all x. It converges to $f(x)$ at all points where f is continuous, and at jump discontinuities it converges to the midpoint of the jump.

- The *Gibbs phenomenon* is the fact that at a jump discontinuity the Fourier series overshoots at the upper end and undershoots at the lower end by approximately 9% of the jump. This is visible in the Fourier partial sums with the peaks becoming both narrower and closer to the location of the jump as $N \to \infty$, but not reducing in height.

- Given any function $f(x)$ on $(-L, L]$ we can define its *periodic extension* to be the function of period $2L$ which equals $f(x)$ on $(-L, L]$. We can also define the *odd periodic extension* of $f(x)$ on $(0, L)$ to be the periodic extension of $f_o(x) = \left\{ \begin{array}{ll} f(x) & \text{if} x \in (0, L) \\ 0 & \text{if} x \in \{0, L\} \\ -f(-x) & \text{if} x \in (-L, 0) \end{array} \right.$. Similarly the *even periodic extension* is the periodic extension of $f_e(x) = \left\{ \begin{array}{ll} f(x) & \text{if} x \in [0, L] \\ f(-x) & \text{if} x \in (-L, 0) \end{array} \right.$. We call the Fourier series for these odd/even periodic extensions *half-range sine/cosine series*.

- *Parseval's theorem* says that $\frac{1}{L}\int_{-L}^{L} (f(x))^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$. It is the Fourier series equivalent of the fact that the magnitude squared of a vector is the sum of the squares of its components. Parseval's theorem can be used to evaluate infinite series if we can realise the series in terms of squares of Fourier coefficients for some function and we can calculate the integral of the function squared.

- We can also evaluate infinite series if we have a Fourier series which we know converges to a specific value at some point – e.g. at a point where the function is continuous and Dirichlet's theorem guarantees the Fourier series converges to the value of the function at that point.

- Using the orthonormal basis functions $e^{in\pi x/L}$ with scalar product of ϕ and ψ defined by $\frac{1}{2L}\int_{-L}^{L}\overline{\phi}\psi dx$, we can write the *Fourier series in complex form* $\sum_{-\infty}^{\infty} c_n e^{in\pi x/L}$. The Fourier coefficients are given by $c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\pi x/L} f(x)dx$. If f is a real-valued function then $c_{-n} = \overline{c}_n$. Note that the complex form of the Fourier series is just a different way to express the usual real Fourier series using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.