4 Integration

4.1 Indefinite and definite integrals

Defn: A function $F(x)$ is called an **indefinite integral** or **antiderivative** of a function $f(x)$ in the interval (a,b) if $F(x)$ is differentiable with
$F'(x) = f(x)$ throughout (a,b). We then write $F(x) = \int f(x)\,dx$.

Eg. $\int \cos x\,dx = \sin x$ in $\mathbb{R}$.

Eg. $\int \frac{1}{x^2}\,dx = -\frac{1}{x}$ in $\mathbb{R}\backslash\{0\}$.

Eg. $\int \text{sgn}x\,dx = |x|$ in $\mathbb{R}\backslash\{0\}$.

Note1: If $F(x)$ is an indefinite integral of $f(x)$ in (a,b) then so is $F(x)+c$ for any constant c. In applications of integration it is important to include this arbitrary constant. We therefore write eg. $\int \cos x\,dx = \sin x + c$.

Note2: If $F_1(x)$ and $F_2(x)$ are both indefinite integrals of $f(x)$ in (a,b) then $F_1(x) - F_2(x) = c$ for some constant c.

Defn: We say that $f(x)$ is **integrable in** (a,b) if it has an indefinite integral $F(x)$ in (a,b) that is continuous in $[a,b]$.

Eg. $\cos x$ is integrable in any finite interval (a,b) because $\sin x$ is continuous in $\mathbb{R}$.

Eg. $\frac{1}{x^2}$ is not integrable in $(0,1)$ because its indefinite integral $-\frac{1}{x}$ is not continuous at $x=0$.

Eg. sgn x is integrable in $(0,1)$ because its indefinite integral $|x|$ is continuous in $[0,1]$.

Defn: A **subdivision** S of $[a,b]$ is a partition into a finite number of subintervals

$$[a,x_1],[x_1,x_2],...,[x_{n-1},b]$$

where $a = x_0 < x_1 < x_2 < ... < x_n = b$. The **norm** $|S|$ of the subdivision is the maximum of the subinterval lengths $|a-x_1|,|x_1-x_2|,..,|x_{n-1}-b|$. (Thus a small value of $|S|$ means that the interval $[a,b]$ has been chopped up into small pieces.) The numbers $z_1,z_2,...,z_n$ form a set of **sample points** from S if $z_j \in [x_{j-1},x_j]$ for $j=1,..,n$.

Defn: Suppose that $f(x)$ is a function defined for $x \in [a, b]$. The **Riemann sum** is

$$\mathcal{R} = \sum_{j=1}^{n} (x_j - x_{j-1}) f(z_j).$$

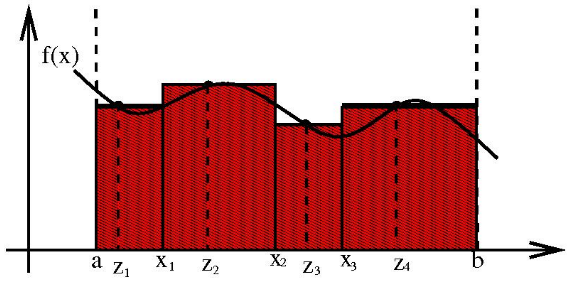

Figure 27: An illustration of a Riemann sum.

The Riemann sum is equal to the sum of the (signed) areas of rectangles of height $f(z_j)$ and width $x_j - x_{j-1}$. Here signed means that the areas of rectangles below the x-axis are counted negatively. If $f(x)$ is continuous in $[a, b]$ and $|S|$ is small then we expect $\mathcal{R}$ to be a good approximation to the (signed) area under the graph of $f(x)$ above the interval $[a, b]$. This turns out to be correct and the error in the approximation can be reduced to zero by taking the limit in which $|S|$ tends to zero. This leads to the definition of the **definite integral** as

$$\int_a^b f(x)\, dx = \lim_{|S| \to 0} \mathcal{R}$$

and it can be shown that this limit exists if $f(x)$ is continuous in $[a, b]$.

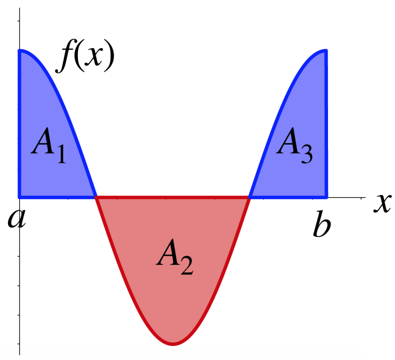

Figure 28: The definite integral is the signed area under the curve: $\int_a^b f(x)\, dx = A_1 - A_2 + A_3$.

By definition we set $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ and therefore $\int_a^a f(x)\,dx = 0$.
There are a number of fairly obvious properties of the definite integral that are not too hard to prove. In the following let $f(x)$ and $g(x)$ both be integrable in (a,b).

(i) Linearity. If λ, μ are any constants then $\lambda f(x) + \mu g(x)$ is integrable in (a,b) with

$$\int_a^b \Big(\lambda f(x) + \mu g(x)\Big)\,dx = \lambda \int_a^b f(x)\,dx + \mu \int_a^b g(x)\,dx.$$

(ii) If $c \in [a,b]$ then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

(iii) If $f(x) \geq g(x)\ \forall\ x \in (a,b)$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

(iv) If $m \leq f(x) \leq M\ \forall\ x \in [a,b]$ then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

4.2 The fundamental theorem of calculus

So far, we don't have any connection between indefinite and definite integrals. This is provided by the following theorem:

The fundamental theorem of calculus
If $f(x)$ is continuous on $[a, b]$ then the function

$$F(x) = \int_a^x f(t)\, dt$$

defined for $x \in [a, b]$ is continuous on $[a, b]$ and differentiable on (a, b) and is an indefinite integral of $f(x)$ on (a, b) ie.

$$F'(x) = \frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$

throughout (a, b). Furthermore if $\widetilde{F}(x)$ is any indefinite integral of $f(x)$ on $[a, b]$ then

$$\int_a^b f(t)\, dt = \widetilde{F}(b) - \widetilde{F}(a) = [\widetilde{F}(x)]_a^b.$$

We shall sketch the important points of the proof.
For $a \leq x < x + h < b$ we have that

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt = \int_x^{x+h} f(t)\, dt$$

where we have used property (ii).
Let $m(h)$ and $M(h)$ denote the minimum and maximum values of $f(x)$ on the interval $[x, x+h]$. Then by property (iv) we have that

$$m(h)h \leq \int_x^{x+h} f(t)\, dt \leq M(h)h.$$

Thus

$$m(h) \leq \frac{F(x+h) - F(x)}{h} \leq M(h).$$

Since $f(x)$ is continuous on $[x, x+h]$ we have that $\lim_{h\to 0^+} m(h) = \lim_{h\to 0^+} M(h) = f(x)$ and so by the pinching theorem

$$\lim_{h\to 0^+} \frac{F(x+h) - F(x)}{h} = f(x).$$

A similar argument applies to the limit from below and together they give

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

which proves that $F(x)$ is an indefinite integral of $f(x)$. The proof of the first part of the theorem is completed by showing that $F(x)$ is continuous from the right at $x = a$ and from the left at $x = b$. Both these follow by a simple consideration of the limit of the relevant quotient. Finally, to prove the last part of the theorem the key observation is that any indefinite integral $\widetilde{F}(x)$ is related to $F(x)$ by the addition of a constant.

The fundamental theorem of calculus provides a simple rule for differentiating a definite integral with respect to its limits.

$$Eg. \qquad \frac{d}{dx}\int_0^x \frac{1}{1+\sin^2 t}\,dt = \frac{1}{1+\sin^2 x}.$$

We can combine this result with the chain rule if the limit is a more complicated expression

$$Eg. \qquad \frac{d}{dx}\int_0^{x^2} \frac{1}{1+e^t}\,dt = \frac{1}{1+e^{x^2}}\left(\frac{d}{dx}x^2\right) = \frac{2x}{1+e^{x^2}}.$$

4.3 Limits with logarithms, powers and exponentials

There are some important results concerning limits as $x \to \infty$ for the logarithm and exponential functions. To derive these results we begin with the following,

Lemma 1: $\forall\, x \geq 0, \quad e^x \geq 1 + x$
Proof: Consider $f(x) = e^x - (1+x)$ then $f(0) = 0$ and $f'(x) = e^x - 1 \geq 0$.
Hence $f(x)$ is monotonic increasing in $[0, \infty)$ so $f(x) \geq 0$ for all $x \geq 0$.

Lemma 2: $\forall\, x \geq 0,$ and for any positive integer $n, \quad e^x \geq \sum_{j=0}^{n} x^j / j!$.
Note that the case $n = 1$ corresponds to Lemma 1, and the proof for general n is similar to the proof of Lemma 1.

Result 1: powers beat logs
For any constant $a > 0$

$$\lim_{x \to \infty} \frac{\log x}{x^a} = 0.$$

This result is encapsulated by the phrase *powers beat logs.*

Proof: Put $x = e^y$ then

$$\lim_{x \to \infty} \frac{\log x}{x^a} = \lim_{y \to \infty} \frac{y}{e^{ay}}.$$

For $y > 0$ then by Lemma 2 with $n = 2$

$$0 \leq \frac{y}{e^{ay}} \leq \frac{y}{1 + ay + \frac{1}{2}a^2y^2} \leq \frac{y}{\frac{1}{2}a^2y^2} = \frac{2}{a^2 y}$$

As $\lim_{y \to \infty} \frac{2}{a^2 y} = 0$ then by the pinching theorem

$$\lim_{y \to \infty} \frac{y}{e^{ay}} = 0 = \lim_{x \to \infty} \frac{\log x}{x^a}.$$

Result 2: exponentials beat powers
For any constant $a > 0$

$$\lim_{x\to\infty} \frac{x^a}{e^x} = 0.$$

This result is encapsulated by the phrase *exponentials beat powers.*

Proof: Let n be the smallest integer such that $n > a$. By Lemma 2, for $x > 0$ we have that

$$0 \leq \frac{x^a}{e^x} \leq \frac{x^a}{1 + x + \ldots + x^n/n!} = \frac{x^{a-n}}{x^{-n} + x^{1-n} + \ldots 1/n!}$$

As $a - n < 0$ then $\lim_{x\to\infty} \frac{x^{a-n}}{x^{-n}+x^{1-n}+\ldots 1/n!} = 0$, hence by the pinching theorem $\lim_{x\to\infty} \frac{x^a}{e^x} = 0.$

Result 3: the exponential as a limit
For any constant a

$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a.$$

Proof (for the case $a > 0$):
Recall the definition of the derivative of a function $f(x)$ at $x = b$.

$$f'(b) = \lim_{h\to 0} \frac{f(b+h) - f(b)}{h}.$$

Apply this to $f(x) = \log x$ so that $f'(x) = \frac{1}{x}$ and take $b = 1$.

$$1 = \lim_{h\to 0} \frac{\log(1+h) - \log(1)}{h} = \lim_{h\to 0} \frac{\log(1+h)}{h}.$$

With the change of variable $h = \frac{a}{x}$ this becomes

$$1 = \lim_{x\to\infty} \frac{\log(1 + \frac{a}{x})}{\frac{a}{x}} \quad \text{ie.} \quad a = \lim_{x\to\infty} (x \log(1 + \frac{a}{x})).$$

Since e^x is continuous at a we have that

$$e^a = \lim_{x\to\infty} \exp(x \log(1 + \frac{a}{x})) = \lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x.$$

4.4 Integration using a recurrence relation

Eg. Calculate $\int_0^1 x^3 e^x \, dx$.
Define $I_n = \int_0^1 x^n e^x \, dx$ for all integer $n \geq 0$.

$$I_{n+1} = \int_0^1 x^{n+1} e^x \, dx = \Big[x^{n+1} e^x \Big]_0^1 - \int_0^1 (n+1) x^n e^x \, dx = e - (n+1) I_n.$$

The recurrence relation $I_{n+1} = e - (n+1)I_n$ can be used to calculate I_n for any positive integer n from the starting value $I_0 = \int_0^1 e^x \, dx = [e^x]_0^1 = e - 1$.
$I_1 = e - I_0 = e - (e-1) = 1, \quad I_2 = e - 2I_1 = e - 2(1) = e - 2,$
$I_3 = e - 3I_2 = e - 3(e-2) = 6 - 2e$ is the required integral.

Eg. Calculate $\int \tan^4 x \, dx$.
Define $F_n(x) = \int \tan^n x \, dx$ for all integer $n \geq 0$.

$$F_{n+2}(x) + F_n(x) = \int \tan^n x \, (1 + \tan^2 x) \, dx = \int \tan^n x \, \mathsf{sec}^2 x \, dx$$

Put $u = \tan x$ then $du = \mathsf{sec}^2 x \, dx$

$$F_{n+2}(x) + F_n(x) = \int u^n \, du = \frac{u^{n+1}}{n+1} = \frac{\tan^{n+1} x}{n+1}.$$

The recurrence relation $F_{n+2}(x) = \frac{1}{n+1} \tan^{n+1}(x) - F_n(x)$ can be used to calculate the required integral for n even from the starting value $F_0(x) = \int 1 \, dx = x$ and for n odd from the starting value $F_1(x) = -\log |\cos x|$.
In particular, $F_2(x) = \tan x - x$ and $F_4(x) = \frac{1}{3} \tan^3 x - \tan x + x$ thus
$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + c.$

4.5 Definite integrals using even and odd functions

The definite integral of an odd function over a symmetric interval is zero.
If $f_{odd}(x)$ is an integrable odd function on $[-a,a]$ then

$$\int_{-a}^{a} f_{odd}(x)\,dx = 0.$$

Proof:

$$\int_{-a}^{a} f_{odd}(x)\,dx = \int_{-a}^{0} f_{odd}(u)\,du + \int_{0}^{a} f_{odd}(x)\,dx$$

$$= -\int_{a}^{0} f_{odd}(-x)\,dx + \int_{0}^{a} f_{odd}(x)\,dx = \int_{0}^{a} (f_{odd}(-x) + f_{odd}(x))\,dx = 0.$$

A similarly argument shows that if $f_{even}(x)$ is an integrable even function on $[-a,a]$ then

$$\int_{-a}^{a} f_{even}(x)\,dx = 2\int_{0}^{a} f_{even}(x)\,dx.$$

Recall that any function $f(x)$ can be decomposed as a sum of even and odd functions $f(x) = f_{even}(x) + f_{odd}(x)$. This can be a useful technique to calculate definite integrals over symmetric intervals as

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} f_{even}(x) + f_{odd}(x)\;dx = 2\int_{0}^{a} f_{even}(x)\,dx.$$

Eg. Calculate $\int_{-1}^{1} \frac{e^x x^4}{\cosh x}\,dx$.
For $f(x) = \frac{e^x x^4}{\cosh x}$ we have $f_{even}(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x) = \frac{1}{2}\frac{x^4(e^x+e^{-x})}{\cosh x} = x^4$. Thus

$$\int_{-1}^{1} \frac{e^x x^4}{\cosh x}\,dx = 2\int_{0}^{1} x^4 = 2\left[\frac{1}{5}x^5\right]_0^1 = \frac{2}{5}.$$

Eg. Calculate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(1+2x)^3\cos x}{1+12x^2}\,dx$.
For $f(x) = \frac{(1+2x)^3\cos x}{1+12x^2}$ we have
$f_{even}(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x) = \frac{(\cos x)\{(1+2x)^3+(1-2x)^3\}}{2(1+12x^2)} = \frac{(\cos x)(2+24x^2)}{2(1+12x^2)} = \cos x$. Thus

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(1+2x)^3\cos x}{1+12x^2}\,dx = 2\int_{0}^{\frac{\pi}{4}} \cos x\,dx = 2\Big[\sin x\Big]_0^{\frac{\pi}{4}} = 2\frac{1}{\sqrt{2}} = \sqrt{2}.$$