57 *Let A be the region bounded by the positive x- and y-axes and the line $3x + 4y = 10$. Compute $\iint_A (x^2+y^2)\,dx\,dy$, taking the integrals in both orders and checking that your answers agree.*

Solution: This can be integrated either $dx\,dy$ or $dy\,dx$, finding the limits of integration via the intersection of the line with the x and y axes respectively:

$$\begin{aligned}I &= \int_0^{5/2}\left(\int_0^{(10-4y)/3}(x^2+y^2)\,dx\right)dy\\ &= \int_0^{5/2}\left[\tfrac{1}{3}x^3+y^2x\right]_0^{(10-4y)/3}\,dy\\ &= \int_0^{5/2}\left(\tfrac{1000}{81}-\tfrac{400}{27}y+\tfrac{250}{27}y^2-\tfrac{172}{81}y^3\right)\,dy \;=\; \frac{15625}{1296}\,.\end{aligned}$$

or

$$\begin{aligned}I &= \int_0^{10/3}\left(\int_0^{(10-3x)/4}(x^2+y^2)\,dy\right)dx\\ &= \int_0^{10/3}\left[yx^2+\tfrac{1}{3}y^3\right]_0^{(10-3x)/4}\,dx\\ &= \int_0^{10/3}\left(\tfrac{125}{24}-\tfrac{75}{16}x+\tfrac{125}{32}x^2-\tfrac{57}{64}x^3\right)\,dx \;=\; \frac{15625}{1296}\,.\end{aligned}$$

58 *In the following integrals sketch the integration regions and then evaluate the integrals. Next interchange the order of integrations and re-evaluate.*

(a) $\int_0^1 (\int_x^1 xy\,dy)\,dx$,

(b) $\int_0^{\pi/2} (\int_0^{\cos\theta} \cos\theta\,dr)\,d\theta$,

(c) $\int_0^1 (\int_1^{2-y} (x+y)^2\,dx)\,dy$.

Solution: (In all cases, sketches should also be included!)

(a)

$$I = \int_0^1 (\int_x^1 xy\,dy)\,dx = \int_0^1 [\tfrac{1}{2}xy^2]_x^1\,dx = \int_0^1 (\tfrac{1}{2}x - \tfrac{1}{2}x^3)\,dx = [\tfrac{1}{4}x^2 - \tfrac{1}{8}x^4]_0^1 = \tfrac{1}{8}\,,$$

or, interchanging the order of integrations,

$$I = \int_0^1 (\int_0^y xy\,dx)\,dy = \int_0^1 [\tfrac{1}{2}x^2y]_0^y\,dx = \int_0^1 (\tfrac{1}{2}y^3)\,dx = [\tfrac{1}{8}y^4]_0^1 = \tfrac{1}{8}\,.$$

(b)

$$I = \int_0^{\pi/2} (\int_0^{\cos\theta} \cos\theta\,dr)\,d\theta = \int_0^{\pi/2} [r\cos\theta]_0^{\cos\theta}\,d\theta = \int_0^{\pi/2} \cos^2\theta\,d\theta = \pi/4\,.$$

or, interchanging the order of integrations,

$$I = \int_0^1 \left(\int_0^{\cos^{-1} r} \cos\theta \, d\theta\right) dr = \int_0^1 \left[\sin\theta\right]_0^{\cos^{-1} r} \, dr = \int_0^1 \sqrt{1-r^2}\, dr\,.$$

Here $\sin(\cos^{-1} r)$ is the sine of the angle whose cosine is r, and so is given using the formula $\sin = \sqrt{1-\cos^2}$. Now substitute $r = \sin\varphi$ to get

$$I = \int_0^{\pi/2} \cos^2\varphi \, d\varphi = \pi/4\,.$$

(c)

$$\begin{aligned} I &= \int_0^1 \left(\int_1^{2-y} (x+y)^2\, dx\right) dy = \int_0^1 \left[\tfrac{1}{3}(x+y)^3\right]_1^{2-y} dy \\ &= \int_0^1 \left(\tfrac{8}{3} - \tfrac{1}{3}(1+y)^3\right) dy = \left[\tfrac{8}{3}y - \tfrac{1}{12}(1+y)^4\right]_0^1 dy = \tfrac{8}{3} - \tfrac{16}{12} + \tfrac{1}{12} = \tfrac{17}{12}\,. \end{aligned}$$

or, interchanging the order of integrations,

$$\begin{aligned} I &= \int_1^2 \left(\int_0^{2-x} (x+y)^2\, dy\right) dx = \int_1^2 \left[\tfrac{1}{3}(x+y)^3\right]_0^{2-x} dx \\ &= \int_1^2 \left(\tfrac{8}{3} - \tfrac{1}{3}x^3\right) dx = \left[\tfrac{8}{3}x - \tfrac{1}{12}x^4\right]_1^2 dx = \tfrac{16}{3} - \tfrac{16}{12} - \tfrac{8}{3} + \tfrac{1}{12} = \tfrac{17}{12}\,. \end{aligned}$$

59 *Exam question 2010 (Section A) Q4: Calculate the double integral*

$$\iint_A (|x| + |y|)\, dx\, dy.$$

where A is the region defined by $|x| + |y| \le 1$.

Solution: The integration region is shown in Figure 1.

Due to the symmetry of the region and of the function $f(x, y) = |x| + |y|$, the integral is simply $4\times$ the integral over one quadrant. Therefore

$$\begin{aligned} \iint_A (|x| + |y|)\, dx\, dy &= 4 \iint_{A \text{ in first quadrant}} (|x| + |y|)\, dx\, dy \\ &= 4 \int_0^1 \int_0^{1-y} (x+y)\, dx\, dy \\ &= 4 \int_0^1 \left[\tfrac{1}{2}x^2 + xy\right]_0^{1-y} dy \\ &= 4 \int_0^1 \left(\tfrac{1}{2}(1-y)^2 + (1-y)y\right) dy \\ &= 4 \int_0^1 \left(\tfrac{1}{2} - \tfrac{1}{2}y^2\right) dy \\ &= 2\Big[y - \tfrac{1}{3}y^3\Big]_0^1 = \frac{4}{3}\,. \end{aligned}$$

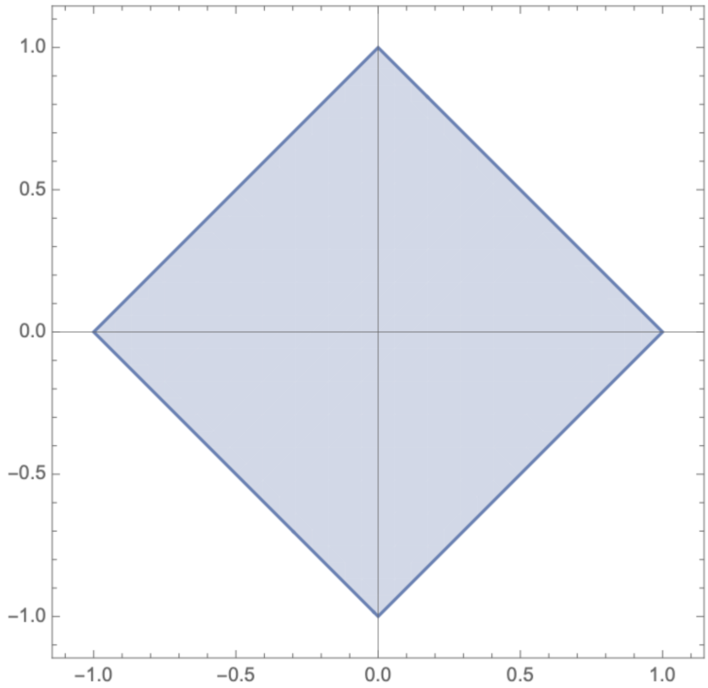

Figure 1: The region $|x| + |y| \leq 1$.

60 *Exam question 2011 (Section A) Q4: Change the order of integration in the double integral*

$$\int_0^2 \int_x^{2x} f(x,y)\, dy\, dx\,.$$

Solution: The area of integration is shown below in Figure 2. When we swap the order of integration, and perform x first, we require 2 integrals: one to accommodate part of triangle below the line $y = 2$, and one for the part above.

This is

$$\int_0^2 \int_{y/2}^{y} f(x,y)\, dx\, dy + \int_2^4 \int_{y/2}^{2} f(x,y)\, dx\, dy$$

61 *Exam question May 2017 (Section A): A solid cylinder C of radius 1 and height 1 is defined by*

$$C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1,\ 0 \leq z \leq 1\}\,.$$

Show that the paraboloid $P = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2\}$ cuts C into two pieces of equal volume.

Solution: First, calculate the volume above P but inside C. Opting to do the z integral last, this is

$$V_{\text{above}} = \int_0^1 \left(\int_{x^2+y^2 \leq z} dxdy \right) dz$$

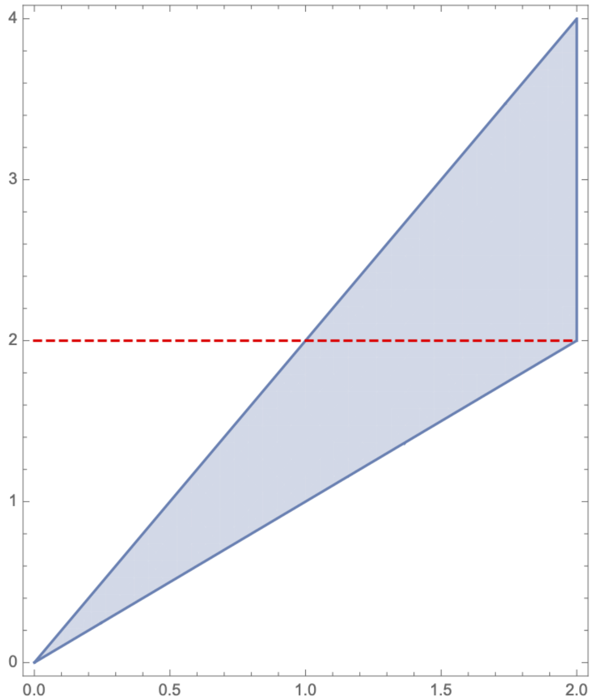

Figure 2: The required region. of integration

Recognising the integral in brackets as the area of a circle radius $\sqrt{z}$, we have

$$V_{\text{above}} = \int_0^1 (\pi z)\, dz = \left[\frac{\pi}{2} z^2\right]_0^1 = \frac{\pi}{2}\,.$$

(Other orders of integration would also be OK)

62 *Compute the iterated integral*

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dy\, dx\,.$$

Now reverse the order of integrations and re-evaluate. Why doesn't your answer contradict Fubini's theorem? (Hints: for the first integral, with respect to y, it might help to aim at something that can be integrated by parts using $\frac{d}{dy}\left(\frac{1}{x^2+y^2}\right) = -\frac{2y}{(x^2+y^2)^2}$; and in answering to the last part of the question, the fact that $\int_0^1 \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| dy \geq \int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\, dy$ could be useful.)

Solution: For the inner, y, integral one can either have a flash of inspriration, or else use the hint as follows:

$$\begin{aligned}
\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy &= \int_0^1 \left(\frac{x^2+y^2}{(x^2+y^2)^2} - \frac{2y^2}{(x^2+y^2)^2}\right) dy \\
&= \int_0^1 \frac{1}{x^2+y^2}\,dy - \int_0^1 \frac{2y^2}{(x^2+y^2)^2}\,dy \\
&= \int_0^1 \frac{1}{x^2+y^2}\,dy + \int_0^1 y\frac{d}{dy}\left(\frac{1}{x^2+y^2}\right) dy\,.
\end{aligned}$$

The last term can be integrated by parts:

$$\int_0^1 y\frac{d}{dy}\left(\frac{1}{x^2+y^2}\right) dy = \left[y\,\frac{1}{x^2+y^2}\right]_0^1 - \int_0^1 \frac{dy}{dy}\,\frac{1}{x^2+y^2}\,dy = \frac{1}{x^2+1} - \int_0^1 \frac{1}{x^2+y^2}\,dy\,.$$

Adding everything up and cancelling the left-over integrals,

$$\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy = \frac{1}{1+x^2}$$

and so the iterated integral is

$$\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx = \int_0^1 \frac{1}{1+x^2}\,dx = [\arctan(x)]_0^1 = \frac{\pi}{4}\,.$$

For the oppositely-ordered iterated integral the calculation is very similar, but take care with the signs! We have

$$\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx = -\frac{1}{1+y^2}$$

and so

$$\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy = -\int_0^1 \frac{1}{1+y^2}\,dy = -\left[\arctan(x)\right]_0^1 = -\frac{\pi}{4}\,.$$

As hinted in the question, this is not what Fubini would predict. But the conditions for Fubini require the iterated integral of the modulus of the integrand to be finite, and this isn't true: taking the hint from the question,

$$\begin{aligned}
\int_0^1 \int_0^1 \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| dy\,dx &\geq \int_0^1 \int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx \\
&= \int_0^1 \left[\frac{y}{x^2+y^2}\right]_0^x dx = \int_0^1 \frac{1}{2x}\,dx = \infty\,.
\end{aligned}$$

(Strictly speaking one should regulate the potentially improper x integrals on the right-hand sides of this calculation, and thereby keep everything well-defined at all stages, by running x from $\delta > 0$ to 1 and then letting $\delta \to 0$ at the end, but the conclusion would be the same.) The 'volume' under the second surface is infinite, which is why the conditions of Fubini's theorem are not met in this case.

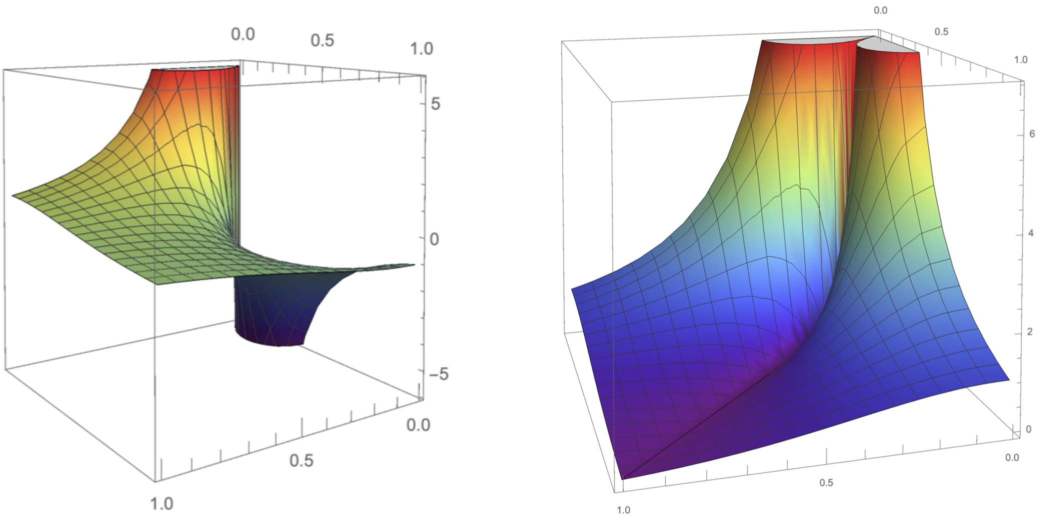

Figure 3: Plots of $\frac{x^2-y^2}{(x^2+y^2)^2}$ and of $\left|\frac{x^2-y^2}{(x^2+y^2)^2}\right|$.

63 *Let B be the region bounded by the five planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$, and $z = x + y$.*

(a) Find the volume of B.

(b) Evaluate $\int_B x\,dV$.

(c) Evaluate $\int_B y\,dV$.

Solution:

(a) Volume of B:

$$\begin{aligned}\int_B dV = \int_0^1 \int_0^{1-x} \int_0^{x+y} dzdydx &= \int_0^1 \int_0^{1-x} (x+y)dydx \\ &= \int_0^1 \left[xy + \tfrac{1}{2}y^2\right]_0^{1-x} dx \\ &= \int_0^1 \left(x(1-x) + \tfrac{1}{2}(1-x)^2\right) dx \\ &= \int_0^1 \left(x - x^2 + \tfrac{1}{2} - x + \tfrac{1}{2}x^2\right) dx \\ &= \left[\tfrac{1}{2}x - \tfrac{1}{6}x^3\right]_0^1 = \frac{1}{3}\,.\end{aligned}$$

(b)

$$\begin{aligned}\int_B x\,dV = \int_0^1 \int_0^{1-x} \int_0^{x+y} x\,dzdydx &= \int_0^1 x \int_0^{1-x} (x+y)dydx \\ &= \int_0^1 \left(x^2 - x^3 + \tfrac{1}{2}x - x^2 + \tfrac{1}{2}x^3\right) dx \\ &= \left[\tfrac{1}{4}x^2 - \tfrac{1}{8}x^4\right]_0^1 = \frac{1}{8}\,.\end{aligned}$$

(c) Evaluate $\int_B y\,dV$.

$$\begin{aligned}\int_B y\,dV = \int_0^1\int_0^{1-x}\int_0^{x+y} y\,dzdydx &= \int_0^1\int_0^{1-x} y(x+y)dydx\\ &= \int_0^1 \left[\tfrac{1}{2}xy^2+\tfrac{1}{3}y^3\right]_0^{1-x}dx\\ &= \int_0^1 \left(\tfrac{1}{2}x(1-x)^2+\tfrac{1}{3}(1-x)^3\right)dx\\ &= \int_0^1 \tfrac{1}{6}\left(2-3x+x^3\right)dx\\ &= \tfrac{1}{6}\left[2x-\tfrac{3}{2}x^2+\tfrac{1}{4}x^4\right]_0^1 = \frac{1}{8}\,.\end{aligned}$$

64 *A function $f(x,y)$ is defined by*

$$f(x,y) = \begin{cases} 1 & \textit{if } -1 < x-y < 0\\ -1 & \textit{if } 0 < x-y < 1\\ 0 & \textit{otherwise}\,.\end{cases}$$

Compute $\int_0^\infty \left(\int_{-\infty}^\infty f(x,y)\,dx\right)dy$ and also $\int_{-\infty}^\infty \left(\int_0^\infty f(x,y)\,dy\right)dx$ (for the second case it might help to draw a picture). Comment on your two answers – why does Fubini fail?

Solution: The first way round is easy! For any $y\in\mathbb{R}$ we have

$$\int_{-\infty}^\infty f(x,y)\,dx = \int_{y-1}^y 1\,dx + \int_y^{y+1} -1\,dx = 1-1 = 0$$

and so

$$\int_0^\infty \left(\int_{-\infty}^\infty f(x,y)\,dx\right)dy = \int_0^\infty 0\,dy = 0\,.$$

For the opposite order of integration, what enters into the y integral depends on the value of x, as follows:

$$\int_0^\infty f(x,y)\,dy = \begin{cases} 0 & \text{if } x<-1\\ \int_0^{x+1} 1\,dy = x{+}1 & \text{if } -1<x<0\\ \int_0^x -1\,dy + \int_x^{x+1} 1\,dy = -x{+}1 & \text{if } 0<x<1\\ \int_{x-1}^x -1\,dy + \int_x^{x+1} 1\,dy = 0 & \text{if } 1<x\,.\end{cases}$$

Hence, and only including the non-zero bits,

$$\begin{aligned}\int_{-\infty}^\infty \left(\int_0^\infty f(x,y)\,dy\right)dx &= \int_{-1}^0 (x{+}1)\,dx + \int_0^1 (-x{+}1)\,dx\\ &= \left[\tfrac{1}{2}x^2{+}x\right]_{-1}^0 + \left[-\tfrac{1}{2}x^2{+}x\right]_0^1 = (-\tfrac{1}{2}{+}1)+(-\tfrac{1}{2}{+}1) = 1\,.\end{aligned}$$

The two answers differ! This doesn't violate Fubini's theorem because if we were to replace $f(x,y)$ by $|f(x,y)|$ in the original double integral, the result would be infinite.

65 *A function $f(x,y)$ is defined by*

$$f(x,y) = \begin{cases} 2^{2(n+1)} & \text{if } 2^{-(n+1)} < x < 2^{-n}, \quad 2^{-(n+1)} < y < 2^{-n} \\ -2^{2n+3} & \text{if } 2^{-(n+1)} < x < 2^{-n}, \quad 2^{-(n+2)} < y < 2^{-(n+1)} \\ 0 & \text{otherwise}, \end{cases}$$

for $n \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$.

Compute $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$, and $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$. Does this contradict Fubini's theorem?

Solution: Let's start by sketching the function $f(x,y)$ for $0 \leq x \leq 1$, $0 \leq y \leq 1$.

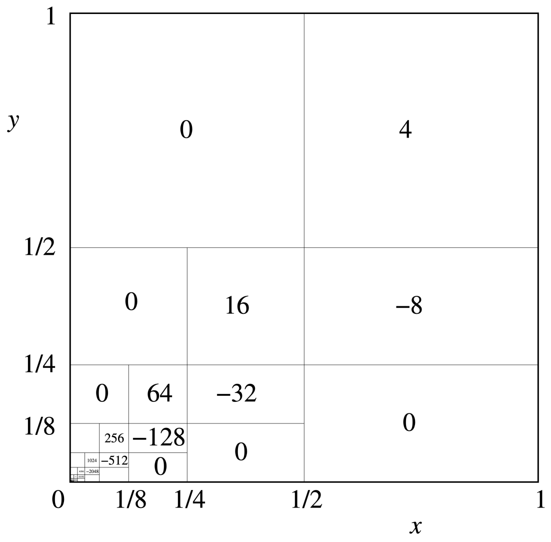

Figure 4: The function $f(x,y)$ for $0 \leq x \leq 1$, $0 \leq y \leq 1$.

Let's first compute $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$. From the picture we can see that the integrand is dependent on both the x and y values that we're integrating over. Specifically, we have

$$\begin{aligned} \int_0^1 \int_0^1 f(x,y)\,dy\,dx &= \int_{1/2}^1 \int_0^1 f(x,y)dy\,dx + \int_{1/4}^{1/2} \int_0^1 f(x,y)dy\,dx + \ldots \\ &= \sum_{n=0}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} \int_0^1 f(x,y)dy\,dx \\ &= \sum_{n=0}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} \left(\int_{2^{-(n+1)}}^{2^{-n}} 2^{2(n+1)}dy + \int_{2^{-(n+2)}}^{2^{-(n+1)}} (-2^{2n+3})dy \right) dx \\ &= \sum_{n=0}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} \left(2^{-(n+1)}2^{2(n+1)} - 2^{-(n+2)}2^{2n+3}\right) \\ &= \sum_{n=0}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} 0 \\ &= 0, \end{aligned}$$

where to go from the third line to the fourth we note that the integrands are simply constants.

Essentially, looking at the picture in vertical ‘strips’, we see that there are two rectangles where the function is non-zero. The upper rectangle is twice as long in the vertical direction but the integrand is half that of the integrand from the lower rectangle, and with opposite sign. Hence the two contributions cancel along each vertical slice of the integral.

The calculation for $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$ is similar.

$$\begin{aligned}
\int_0^1 \int_0^1 f(x,y)\,dy\,dx &= \int_{1/2}^1 \int_0^1 f(x,y)\,dx\,dy + \int_{1/4}^{1/2} \int_0^1 f(x,y)\,dx\,dy + \dots \\
&= \int_{1/2}^1 \int_0^1 f(x,y)\,dx\,dy + \sum_{n=1}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} \int_0^1 f(x,y)\,dx\,dy \\
&= \int_{1/2}^1 \int_{1/2}^1 4\,dx\,dy + \sum_{n=1}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} \left(\int_{2^{-(n+1)}}^{2^{-n}} 2^{2(n+1)}\,dx \right. \\
&\quad \left. - \int_{2^{-n}}^{2^{-n+1}} 2^{2n+1}\,dx \right) dy \\
&= 1 + \sum_{n=1}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} \left(2^{-(n+1)} 2^{2(n+1)} - 2^{-n} 2^{2n+1} \right) dy \\
&= 1 + \sum_{n=1}^{\infty} \int_{2^{-(n+1)}}^{2^{-n}} 0\,dy \\
&= 1.
\end{aligned}$$

The two answers differ. This doesn’t contradict Fubini’s theorem, as the double integral is not absolutely convergent.

66 *Write the line integral*

$$\int_C xdx + ydy + (xz - y)dz$$

in the form $\int_C \mathbf{v} \cdot d\mathbf{x}$ *for a suitable vector field* $\mathbf{v}(\mathbf{x})$*, and compute its value when* C *is the curve given by* $\mathbf{x}(t) = t^2\,\mathbf{e_1} + 2t\,\mathbf{e_2} + 4t^3\,\mathbf{e_3}$ *with* $0 \le t \le 1$.

Solution: The integral to be evaluated is

$$I = \int_C xdx + ydy + (xz - y)dz = \int_C \mathbf{v} \cdot d\mathbf{x} = \int_0^1 \mathbf{v}(\mathbf{x}(t)) \cdot \frac{d\mathbf{x}}{dt}\,dt\,.$$

where $\mathbf{v}(\mathbf{x}) = x\,\mathbf{e_1} + y\,\mathbf{e_2} + (xz - y)\,\mathbf{e_3}$. Following the recipe from lectures, we compute $\frac{d\mathbf{x}}{dt} = 2t\,\mathbf{e_1} + 2\,\mathbf{e_2} + 12t^2\,\mathbf{e_3}$ while $\mathbf{v}(\mathbf{x}(t)) = t^2\mathbf{e_1} + 2t\mathbf{e_2} + (4t^5 - 2t)\mathbf{e_3}$, so

$$I = \int_0^1 \left(2t^3 + 4t + 48t^7 - 24t^3\right) dt = \left[\tfrac{1}{2}t^4 + 2t^2 + 6t^8 - 6t^5\right]_0^1 = \tfrac{5}{2}\,.$$

Alteratively and a little more informally we can simply note that

$$x = t^2 \Rightarrow dx = 2t\,dt\,, \quad y = 2t \Rightarrow dy = 2\,dt\,, \quad z = 4t^3 \Rightarrow dz = 12t^2\,dt\,,$$

so $xdx = 2t^3\,dt$, $ydy = 4t$, and $(xz - y)dz = (4t^5 - 2t) \times 12t^2\,dt$, after which the calculation runs as before.

67 *Evaluate $\int_\sigma \mathbf{F} \cdot d\mathbf{x}$, where $\mathbf{F} = y\,\mathbf{e_1} + 2x\,\mathbf{e_2} + y\,\mathbf{e_3}$ and the path σ is given by $\mathbf{x}\,(t) = t\,\mathbf{e_1} + t^2\,\mathbf{e_2} + t^3\,\mathbf{e_3}$, $0 \le t \le 1$.*

Solution: We have $\frac{d\mathbf{x}}{dt} = \mathbf{e_1} + 2t\,\mathbf{e_2} + 3t^2\,\mathbf{e_3}$, and $\mathbf{F}(\mathbf{x}\,(t)) = t^2\mathbf{e_1} + 2t\mathbf{e_2} + t^2\mathbf{e_3}$. Hence

$$\int_\sigma \mathbf{F} \cdot \mathbf{dx} = \int_0^1 (t^2\mathbf{e_1} + 2t\mathbf{e_2} + t^2\mathbf{e_3}) \cdot (\mathbf{e_1} + 2t\mathbf{e_2} + 3t^2\mathbf{e_3})\,dt$$
$$= \int_0^1 (t^2 + 4t^2 + 3t^4)\,dt = \left[\tfrac{5}{3}t^3 + \tfrac{3}{5}t^5\right]_0^1 = \tfrac{34}{15}\,.$$

68 *Let $\underline{A}(\underline{x})$ be the vector field $\underline{A}(x,y,z) = x\,\underline{e}_1 + y\,\underline{e}_2 + z\,\underline{e}_3\,.$*

(a) Compute the line integral $\int_C \underline{A} \cdot d\underline{x}$ where C is the straight line from the origin to the point $(1,1,1)$.

(b) Show (by finding f) that the vector field $\underline{A}$ from part (a) is equal to $\nabla f(\underline{x})$ for some scalar field f, and that your answer to part (a) is equal to $f(1,1,1) - f(0,0,0)$.

Solution:

(a) Parametrise the curve as $\underline{x}(t) = (t,t,t)$ with t running from 0 to 1 (other choices would also be fine). Then $\frac{d\underline{x}(t)}{dt} = (1,1,1)$ and $\underline{A}(\underline{x}(t)) = (t,t,t)$ so $\frac{d\underline{x}(t)}{dt} \cdot \underline{A}(\underline{x}(t)) = 3t$. Hence

$$\int_C \underline{A} \cdot d\underline{x} = \int_0^1 \frac{d\underline{x}}{dt} \cdot \underline{A}(\underline{x}(t))\,dt = \int_0^1 3t\,dt = 3/2.$$

(b) If $\underline{A} = \nabla f$, $f_x = x$ so $f(x,y,z) = \frac{1}{2}x^2 + g(y,z)$; then $f_y = g_y = y$ so $g(y,z) = \frac{1}{2}y^2 + h(z)$; and finally $f_z = h_z = z$ so $h(z) = \frac{1}{2}z^2 + c$ where c is a constant. Hence $f(\underline{x}) = \frac{1}{2}\underline{x}^2 + c$, and it is easy to check that this works. Then $f(1,1,1) - f(0,0,0) = \frac{1}{2}(1^2 + 1^2 + 1^2) + c - c = 3/2$, which is indeed the same as the answer to part (a). (Aside: note that the procedure used here to find f does not always work: try it for $\underline{A}(x,y,z) = (z,x,y)$.)

69 *Show that the result from question 68 applies in general: if the vector field $\underline{v}(\underline{x})$ in $\mathbb{R}^n$ is the gradient of a scalar field $f(\underline{x})$, so that $\underline{v} = \nabla f$, and if C is a curve in $\mathbb{R}^n$ running from $\underline{x} = \underline{a}$ to $\underline{x} = \underline{b}$, then $\int_C \underline{v} \cdot d\underline{x} = f(\underline{b}) - f(\underline{a})$. (Hint: use the chain rule.)*

Solution: Let the curve C be parametrised by $\underline{x}(t)$ with t running from t_1 to t_2 so that $\underline{x}(t_1) = \underline{a}$ and $\underline{x}(t_2) = \underline{b}$. Then by the definition of the line integral,

$$\int_C \underline{v} \cdot d\underline{x} = \int_{t_1}^{t_2} \underline{v}(\underline{x}(t)) \cdot \frac{d\underline{x}(t)}{dt}\,dt = \int_{t_1}^{t_2} \nabla f(\underline{x}(t)) \cdot \frac{d\underline{x}(t)}{dt}\,dt\,.$$

But by the chain rule (see **2.2** in the Michaelmas Summary), if $F(t) = f(\underline{x}(t))$, then $\frac{dF}{dt} = \nabla f \cdot \frac{d\underline{x}}{dt}$. Hence

$$\int_C \underline{v} \cdot d\underline{x} = \int_{t_1}^{t_2} \frac{dF}{dt}\, dt = \big[F(t)\big]_{t_1}^{t_2} = F(t_2) - F(t_1) = f(\underline{b}) - f(\underline{a})$$

since $F(t_1) = f(\underline{x}(t_1)) = f(\underline{a})$ and $F(t_2) = f(\underline{x}(t_2)) = f(\underline{b})$.

70 *Use the result from question 69 to evaluate* $\int_C 2xyzdx + x^2zdy + x^2ydz$, *where* C *is any regular curve connecting (1,1,1) to (1,2,4).*

Solution: We look for ϕ such that $\phi_x = 2xyz$, $\Rightarrow \phi = x^2yz + f(y,z)$, then $\phi_y = x^2z$ $\Rightarrow f_y = 0 \Rightarrow f(y,z) = g(z)$, then $\phi_z = x^2y \Rightarrow g =$ const. So with $\phi(x,y,z) = x^2yz$ we have $(2xyz, x^2z, x^2y) = \nabla\phi$, and (using the result of question 79)

$$\int_C 2xyzdx + x^2zdy + x^2ydz = \int_C \nabla(x^2yz) \cdot d\mathbf{x} = \big[x^2yz\big]_{(1,1,1)}^{(1,2,4)} = 8 - 1 = 7\,.$$

Note, as we will see when discussing Stokes' theorem, a necessary condition for this tactic to work is the fact that $\nabla \times (2xyz, x^2z, x^2y) = (x^2 - x^2)\,\mathbf{e_1} + (2xy - 2xy)\,\mathbf{e_2} + (2xz - 2xz)\,\mathbf{e_3} = \underline{0}$.

71 *Use method 2 from lectures to compute the surface integral,* $\int_S \mathbf{F} \cdot d\mathbf{A}$, *of the vector field* $\mathbf{F} = (3x^2, -2yx, 8)$ *over the surface given by the plane* $z = 2x - y$ *with* $0 \le x \le 2$, $0 \le y \le 2$.

Solution: The surface as the (zero) level set of the function $f = z - 2x + y$, which we can use to find the vector normal to the surface:

$$\mathbf{N} = \nabla f = (-2, 1, 1)\,.$$

As in method 2 from lectures, since we have given the surface as a level set we can now write

$$\begin{aligned}
\int_S \mathbf{F} \cdot d\mathbf{A} &= \int_0^2 \int_0^2 \frac{\mathbf{F} \cdot \nabla f}{\mathbf{e_3} \cdot \nabla f} dx\, dy \\
&= \int_0^2 \int_0^2 (-6x^2 - 2yx + 8)\, dx\, dy \\
&= \int_0^2 \big[-2x^3 - yx^2 + 8x\big]_0^2\, dy \\
&= \int_0^2 (-4y)\, dy = -8\,.
\end{aligned}$$

It's instructive to check that you get the same result using 'method 1' instead. The question was a little remiss in not specifying that upward-pointing (ie positive z component) normals should be taken when evaluating the surface integral. With downward-pointing normals, the answer would have been $+8$, and this answer is also valid.

72 *Let* $\mathbf{F}(x, y, z) = (z, x, y)$*, and* S *be the part of the surface of the sphere* $x^2 + y^2 + (z-1)^2 = r^2$ *above the plane* $z = 0$*. Assume that* $r > 1$ *so that the boundary* C *of* S*, where* S *intersects the plane* $z = 0$*, is non-empty.*

(a) *Compute the line integral* $\oint_C \mathbf{F} \cdot d\mathbf{x}$.

(b) *By parameterising the surface* S *with spherical coordinates centred on the point* $(0, 0, 1)$*, compute the surface integral of the curl of* $\mathbf{F}$*,* $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{A}$.

Solution:

(a) The curve to be integrated over is the intersection of the sphere and the plane $z = 0$ in the picture below, which shows the situation for $r = 2$:

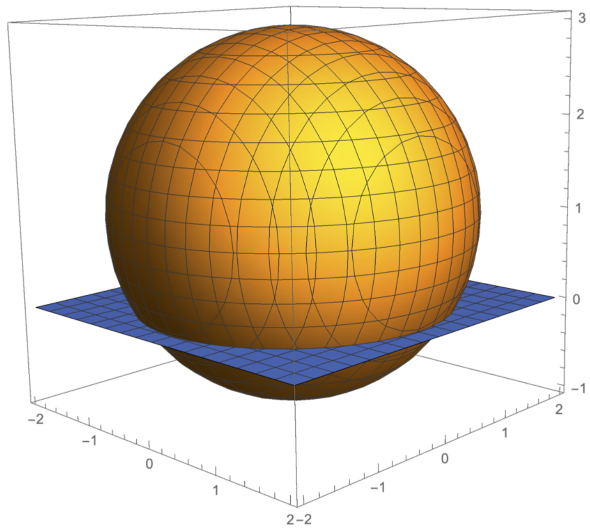

Figure 5: The intersection of the sphere $x^2 + y^2 + (z-1)^2 = r^2$ and the plane $z = 0$, shown for $r = 2$.

On C we have $z = 0$ and also $x^2 + y^2 + (z-1)^2 = r^2$, so $x^2 + y^2 = r^2 - 1$, and C is a circle in the x,y plane of radius $\rho = \sqrt{r^2 - 1}$. Parametrise it as $\mathbf{x}(t) = (\rho \cos t, \rho \sin t, 0)$ with $0 \le t \le 2\pi$, so $\mathbf{F}(\mathbf{x}(t)) = (0, \rho \cos t, \rho \sin t)$, $\frac{d\mathbf{x}(t)}{dt} = (-\rho \sin t, \rho \cos t, 0)$, and

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{x} &= \int_0^{2\pi} (-\rho \sin t, \rho \cos t, 0) \cdot (\rho \cos t, \rho \sin t, 0)\, dt \\ &= \int_0^{2\pi} \rho^2 \cos^2 t\, dt = \pi \rho^2 = \pi(r^2 - 1)\,. \end{aligned}$$

(b) For the surface integral, follow the suggestion and set

$$\mathbf{x}(\theta, \phi) = (x(\theta, \phi), y(\theta, \phi), z(\theta, \phi))$$

with (compare with example 43 in your notes)

$$
\begin{aligned}
x(\theta,\phi) &= r\sin\theta\,\cos\phi\,,\\
y(\theta,\phi) &= r\sin\theta\,\sin\phi\,,\\
z(\theta,\phi) &= 1 + r\cos\theta\,.
\end{aligned}
$$

To figure out the parameter domain U, note that $z(\theta,\phi)$ must be positive and hence $0 \le \theta \le \cos^{-1}(-1/r)$, while $0 \le \phi \le 2\pi$. Furthermore $\nabla \times \mathbf{F} = (1,1,1)$, and

$$
\begin{aligned}
\frac{\partial \mathbf{x}}{\partial \theta} \times \frac{\partial \mathbf{x}}{\partial \phi} &= (r\cos\theta\cos\phi, r\cos\theta\sin\phi, -r\sin\theta) \times (-r\sin\theta\sin\phi, r\sin\theta\cos\phi, 0)\\
&= r^2(\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \sin\theta\cos\theta)\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_S \nabla\times\mathbf{F}\cdot d\mathbf{A} &= \int_U \mathbf{F}(\mathbf{x}\,(u,v))\cdot\left(\frac{\partial \mathbf{x}}{\partial \theta} \times \frac{\partial \mathbf{x}}{\partial \phi}\right) d\theta\,d\phi\\
&= r^2\int_0^{\cos^{-1}(-1/r)}\int_0^{2\pi}\Big((1,1,1)\cdot(\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \sin\theta\cos\theta)\Big)d\phi\,d\theta\\
&= r^2\int_0^{\cos^{-1}(-1/r)}\int_0^{2\pi}\Big(\sin^2\theta\cos\phi + \sin^2\theta\sin\phi + \sin\theta\cos\theta\Big)d\phi\,d\theta\\
&= r^2\int_0^{\cos^{-1}(-1/r)}\Big(2\pi\sin\theta\cos\theta\Big)\,d\theta \quad \text{(the first two } \phi \text{ integrals vanishing)}\\
&= \pi r^2\int_0^{\cos^{-1}(-1/r)}\sin 2\theta\,d\theta\\
&= \pi r^2\Big[-\tfrac{1}{2}\cos 2\theta\Big]_0^{\cos^{-1}(-1/r)}\\
&= -\tfrac{1}{2}\pi r^2\Big[2\cos^2\theta - 1\Big]_0^{\cos^{-1}(-1/r)}\\
&= -\tfrac{1}{2}\pi r^2\Big(\frac{2}{r^2} - 2\Big) = \pi(r^2-1)\,.
\end{aligned}
$$

Remarks: Note that the answers to parts (a) and (b) are equal. We will see in Topic 8 that this result is predicted by Stokes' theorem. In this case it wouldn't have been appropriate to use method 2 for the surface integral (defining the surface implicitly and parametrising it by x and y) since for some parts of the surface there are two values of z for given values of x and y.

73 *Let $\underline{A}(\underline{x})$ be the vector field $\underline{A}(x,y,z) = z\,\underline{e}_1 + x\,\underline{e}_2 + y\,\underline{e}_3$, C be the circle in the x,y-plane of radius r centred on the origin, and S the disk in the x,y-plane whose boundary is C.*

(a) Compute the line integral $\oint_C \underline{A}\cdot d\underline{x}$.

(b) Compute the surface integral of the curl of $\underline{A}$, $\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{A}$.

Solution:

(a) Parameterising C as $\underline{x}(t) = r\cos t\,\underline{e}_1 + r\sin t\,\underline{e}_2$, for $0 \le t \le 2\pi$, we have $\underline{A}(\underline{x}(t)) = r\cos t\,\underline{e}_2 + r\sin t\,\underline{e}_3$ and $\frac{d\underline{x}(t)}{dt} = -r\sin t\,\underline{e}_1 + r\cos t\,\underline{e}_2$. Taking the dot product, $\underline{A}(\underline{x}(t))\cdot\frac{d\underline{x}(t)}{dt} = r^2\cos^2 t$ and so

$$\oint_C \underline{A}\cdot d\underline{x} = \int_0^{2\pi} r^2\cos^2 t\,dt = \frac{r^2}{2}\int_0^{2\pi}(\cos(2t)+1) = \frac{r^2}{2}\left[\frac{1}{2}\sin(2t)+t\right]_0^{2\pi} = \pi r^2.$$

(b) Parameterising the disk S as $\underline{x}(u,v) = (u\cos(v), u\sin(v), 0)$, for $0 \le u \le r$, $0 \le v \le 2\pi$, we have $\underline{x}_u = (\cos(v), \sin(v), 0)$ and $\underline{x}_v = (-u\sin(v), u\cos(v), 0)$, and hence $\underline{x}_u \times \underline{x}_v = u\underline{e}_3$.
$\underline{\nabla} \times \underline{A} = (1,1,1)$, and so

$$\begin{aligned}\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{A} &= \int_0^{2\pi}\int_0^r (1,1,1)\cdot(0,0,u)\,du\,dv\\ &= \int_0^{2\pi}\int_0^r u\,du\,dv\\ &= \pi r^2\end{aligned}$$

Remarks: Note that the answers to parts (a) and (b) are equal. We will see in Topic 8 that this result is predicted by Stokes' theorem.

74 *Let $\underline{A}(\underline{x})$ be the vector field $\underline{A}(x,y,z) = z\,\underline{e}_1 + x\,\underline{e}_2 + y\,\underline{e}_3$, C be the a by a square $\underline{abcd}$ in the y,z-plane with vertices $\underline{a} = \underline{0}$, $\underline{b} = a\,\underline{e}_2$, $\underline{c} = a\,\underline{e}_2 + a\,\underline{e}_3$ and $\underline{d} = a\,\underline{e}_3$, and S be the region of the y,z-plane bounded by C.*

(a) Compute the line integral $\oint_C \underline{A}\cdot d\underline{x}$.

(b) Compute the surface integral of the curl of $\underline{A}$, $\int_S(\nabla\times\mathbf{A})\cdot d\mathbf{A}$.

Solution:

(a) Splitting C into four straight lines $C_a = \underline{ab}$, $C_b = \underline{bc}$, $C_c = \underline{cd}$ and $C_d = \underline{da}$, we have

$$\oint_{C_2}\underline{A}\cdot d\underline{x} = \int_{C_a}\underline{A}\cdot d\underline{x} + \int_{C_b}\underline{A}\cdot d\underline{x} + \int_{C_c}\underline{A}\cdot d\underline{x} + \int_{C_d}\underline{A}\cdot d\underline{x}$$

and the task is to evaluate the line integrals along these four straight lines. Taking them in turn:

- C_a runs from $\underline{0}$ to $a\,\underline{e}_2$, and can be parametrized as $\underline{x}(t) = t\,\underline{e}_2$ with t running from 0 to a. Then $\underline{A}(\underline{x}(t)) = t\,\underline{e}_3$ and $\frac{d\underline{x}(t)}{dt} = \underline{e}_2$ so $\underline{A}(\underline{x}(t))\cdot\frac{d\underline{x}(t)}{dt} = 0$ and hence $\int_{C_a}\underline{A}\cdot d\underline{x} = 0$.
- C_b runs from $a\,\underline{e}_2$ to $a\,\underline{e}_2 + a\,\underline{e}_3$, and can be parametrized as $\underline{x}(t) = a\,\underline{e}_2 + t\,\underline{e}_3$ with t running from 0 to a. Then $\underline{A}(\underline{x}(t)) = t\,\underline{e}_1 + a\,\underline{e}_3$ and $\frac{d\underline{x}(t)}{dt} = \underline{e}_3$ so $\underline{A}(\underline{x}(t))\cdot\frac{d\underline{x}(t)}{dt} = a$ and hence $\int_{C_b}\underline{A}\cdot d\underline{x} = \int_0^a a\,dt = [at]_0^a = a^2$.
- C_c runs from $a\,\underline{e}_2 + a\,\underline{e}_3$ to $a\,\underline{e}_3$, and can be parametrized as $\underline{x}(t) = t\,\underline{e}_2 + a\,\underline{e}_3$ with t running from a to 0. Then $\underline{A}(\underline{x}(t)) = a\,\underline{e}_1 + t\,\underline{e}_3$ and $\frac{d\underline{x}(t)}{dt} = \underline{e}_2$ so $\underline{A}(\underline{x}(t))\cdot\frac{d\underline{x}(t)}{dt} = 0$ and hence $\int_{C_c}\underline{A}\cdot d\underline{x} = 0$.

- C_d runs from $a\,\underline{e}_3$ to $\underline{0}$, and can be parametrized as $\underline{x}(t) = t\,\underline{e}_3$ with t running from a to 0. Then $\underline{A}(\underline{x}(t)) = t\,\underline{e}_1$ and $\frac{d\underline{x}(t)}{dt} = \underline{e}_3$ so $\underline{A}(\underline{x}(t)) \cdot \frac{d\underline{x}(t)}{dt} = 0$ and hence $\int_{C_d} \underline{A} \cdot d\underline{x} = 0$.

Adding the four bits together gives the final result: $\oint_{C_2} \underline{A} \cdot d\underline{x} = a^2$.

(b) Using method 2, the surface S can be given as part of the level set $f(x, y, z) = x = 0$ for $0 \le y \le a\, 0 \le z \le a$. The surface S therefore has (right-pointing) normal given by $\underline{\nabla} f = \underline{e}_1$. Since we therefore have $\partial_x f \neq 0$ over the surface S, we can compute this integral by 'projecting' onto the y, z-plane using method 2.

We also have $\underline{\nabla} \times \underline{A} = (1, 1, 1)$, and hence

$$\begin{aligned}\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{A} &= \int_0^a \int_0^a \frac{(\nabla \times \underline{A}) \cdot \underline{\nabla} f}{\underline{e}_1 \cdot \underline{\nabla} f}\, dy\, dz \\ &= \int_0^a \int_0^a 1\, dy\, dz \\ &= a^2\end{aligned}$$

Remarks: Note that the answers to parts (a) and (b) are equal. We will see in Topic 8 that this result is predicted by Stokes' theorem.

75 *Based on exam question May 2015 (Section A) Q5 (which didn't contain part (b)):*

(a) Calculate $\int_V (\underline{\nabla} \cdot \underline{U})\, dV$ where V is the solid cube with faces $x = \pm 1$, $y = \pm 1$ and $z = \pm 1$ and

$$\underline{U}(x, y, z) = (\, x\, y^2, y\, x^2, z\,)\,.$$

(b) Calculate $\int_S \underline{U} \cdot d\underline{A}$, where S is the surface of V.

Solution:

(a) We have $\nabla \cdot \mathbf{U} = y^2 + x^2 + 1$ so (changing the orders of integrations as convenient)

$$\begin{aligned}\int_V (\underline{\nabla} \cdot \underline{U})\, dV &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (y^2 + x^2 + 1)\, dx dy dz \\ &= 4 \int_{-1}^1 y^2 dy + 4 \int_{-1}^1 x^2 dx + 8 \\ &= 4 \left[\tfrac{1}{3} y^3\right]_{-1}^1 + 4 \left[\tfrac{1}{3} x^3\right]_{-1}^1 + 8 \\ &= \tfrac{16}{3} + 8 = \tfrac{40}{3}\,.\end{aligned}$$

(b) The surface integral is the sum of six terms, one for each face of the cube:

• On the faces $x = \pm 1$, $\widehat{\mathbf{n}} = \pm \mathbf{e_1}$ and $\mathbf{U} = (\pm y^2, y, z)$, so $\mathbf{U} \cdot \widehat{\mathbf{n}} = y^2$ and each contributes $I_1 = \int_{-1}^1 \int_{-1}^1 y^2 dy dz = \frac{4}{3}$.

• On the faces $y = \pm 1$, $\widehat{\mathbf{n}} = \pm \mathbf{e_2}$ and $\mathbf{U} = (x, \pm x^2, z)$, so $\mathbf{U} \cdot \widehat{\mathbf{n}} = x^2$ and each contributes $I_2 = \int_{-1}^1 \int_{-1}^1 x^2 dx dz = \frac{4}{3}$.

• On the faces $z = \pm 1$, $\widehat{\mathbf{n}} = \pm \mathbf{e_3}$ and $\mathbf{U} = (xy^2, yx^2, \pm 1)$, so $\mathbf{U} \cdot \widehat{\mathbf{n}} = 1$ and each contributes $I_3 = \int_{-1}^1 \int_{-1}^1 dx dy = 4$.

The total is $2I_1 + 2I_2 + 2I_3 = \frac{8}{3} + \frac{8}{3} + 8 = \frac{40}{3}$ as before.
Aside: the calculation is quite close to the one in part (b) – this is related to how the general proof of the divergence theorem works.