# 7 Volume, line and surface integrals

## 7.1 Double integrals and Fubini's theorem

We'll start with the familiar case of one-dimensional integrals. Recall the single integral computes the area under a curve, as illustrated in figure 31.

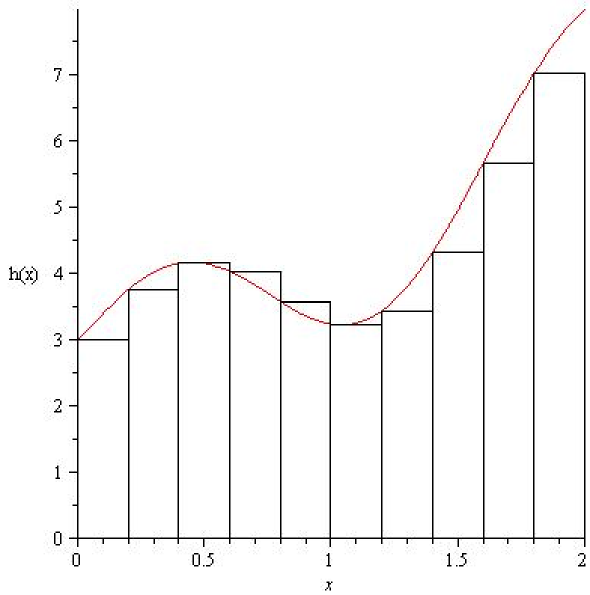



Figure 31: Graph showing a section of the (red) curve given by $y = h(x)$, with the discretised area under the curve given by the sum of the black rectangles of width $\Delta x_i$.

As the widths of the rectangles tend to zero, so the sum of their areas tends to the integral of the curve over the desired range, defining the integral via a Riemann sum:

$$\int_a^b h(x)\,dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} h(x_i^*)\Delta x_i\,,$$

where the interval $[a, b]$ is partitioned as $a = x_0 < x_1 < ... < x_i < ... < x_n = b$, $\Delta x_i = x_{i+1} - x_i$, and the choice of $x_i^* \in [x_i, x_{i+1}]$ is arbitrary as long as the limit exists (in the figure they were all taken at the left-hand ends).

In general the $\Delta x_i$'s can be different sizes, but if they are, the limit $n \to \infty$ must include the extra requirement that $\sup(\Delta x_i) \to 0$. From here on we will assume equally-spaced partitions, and mostly assume that the functions we deal with are such that the relevant limits all exist.

### Double integrals

As for the single integral, the double integral, written as $\int_R f(x, y)\,dA$ (or sometimes $\iint_R f(x, y)\,dA$), can be used to calculate the volume under the surface defined by the equation $z = f(x, y)$ where $f(x, y)$ is continuous over $R \subset \mathbb{R}^2$, the region over which we wish to perform the integration. This is illustrated in figure 32.

This double integral can be defined in a similar fashion to the single integral using a Riemann sum. We start by splitting the region of integration $R$ into $N$ smaller areas $\Delta A_k$ (see figure 33, where the areas $\Delta A_k$ are chosen to be rectangles).

As for single integrals, where we add the areas of the small rectangles, we can add up the smaller volumes (prisms, with volumes (area of base $= \Delta A_k) \times$ (height)) to give an approximation to the double integral.

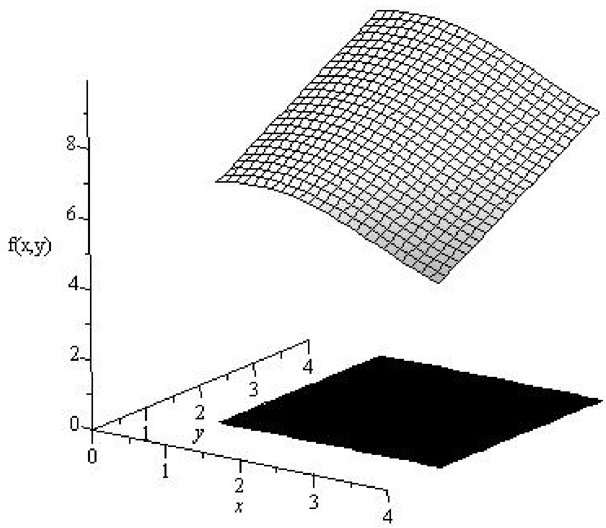



Figure 32: The section of the surface $z = f(x, y)$ lying above the region $R = \{ (x, y) \in \mathbb{R}^2 \mid x \in [1, 4],\, y \in [1, 4] \}$.

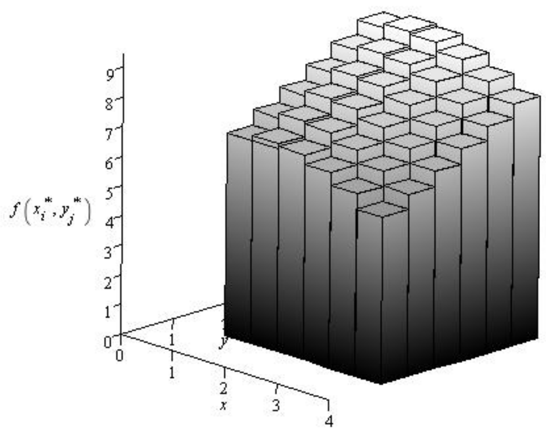



Figure 33: A sketch of the discretised surface $z = f(x, y)$. The double integral can be approximated by the sum of the volumes of all the small cuboids; if we take the limit of infinitely many cuboids, the sum tends to the integral.

As we increase the number of prisms the approximation becomes more and more accurate, and the integral can be defined as the limit of the sum:

$$\int_R f(x,y)\,dA = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k^*, y_k^*)\,\Delta A_k\,.$$

Here $(x_k^*, y_k^*)$ is in the base of the $k^{\text{th}}$ prism. If we choose the small areas to be rectangles on a regular grid, then $\Delta A_k = \Delta x_i \Delta y_j$ with $\Delta x_i = x_{i+1} - x_i$, $\Delta y_j = y_{j+1} - y_j$ and $x$ and $y$ are partitioned in a similar way to that used before: $x_0 < x_1 < ... < x_i < ... < x_n$, $y_0 < y_1 < ... < y_j < ... < y_m$. We then obtain

$$\int_R f(x,y)\,dA = \lim_{n,m\to\infty} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f(x_i^*, y_j^*)\Delta x_i \Delta y_j\,,$$

where $x_i^*$ and $y_j^*$ are the coordinates of the points at which the function is evaluated and are in the ranges $x_i^* \in [x_i, x_{i+1}]$, $y_j^* \in [y_j, y_{j+1}]$. These points are often taken to be the mid-points. If we take the limit $m \to \infty$ first, and only then take $n \to \infty$, we get:

$$\begin{aligned}
\lim_{n,m\to\infty} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f(x_i^*, y_j^*)\Delta y_j \Delta x_i &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left( \int_y f(x_i^*, y) dy \right) \Delta x_i \\
&= \int_x \left( \int_y f(x,y)\,dy \right) dx\,.
\end{aligned}$$

Let's see an example of calculating a double integral:

**Example 37.** Integrate the function $f(x,y) = 6xy^2$ over $R = [2,4] \times [1,2]$.

$$\begin{aligned}
\int_2^4 \int_1^2 6xy^2\,dy\,dx &= \int_2^4 \left[2xy^3\right]_1^2 dx \\
&= \int_2^4 (16x - 2x)\,dx \\
&= \left[7x^2\right]_2^4 = 84\,.
\end{aligned}$$

If we'd taken $n \to \infty$ first instead, we would have ended up with the opposite order of integrations, but the final result is unchanged:

$$\begin{aligned}
\int_1^2 \int_2^4 6xy^2 dx\,dy &= \int_1^2 \left[3x^2y^2\right]_2^4 dy \\
&= \int_1^2 36y^2 dy \\
&= \left[12y^3\right]_1^2 = 84\,.
\end{aligned}$$

As we can see, we obtain the same final result as before.

Things become a little more complicated if the region of integration, let's call it $A$, is not a rectangle. Suppose that it is defined to be the set of points in the $x, y$ plane lying between two curves $y_0(x)$ and $y_1(x)$ with $a \le x \le b$:

$$A = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b\,,\ y_0(x) \le y \le y_1(x)\}\,.$$

(To visualise this, it might help to draw a picture!) Then taking $m \to \infty$ first:

$$\int_A f(x,y)\,dx\,dy = \int_a^b \int_{y_0(x)}^{y_1(x)} f(x,y)\,dy\,dx\,.$$

Instead we could take $n \to \infty$ first, taking care to rearrange the limits (see example 38 below for this in action). Again, the final answer will be the same.

**Fubini's theorem** sums this all up.

**Theorem 7.1.** If the function $f(x, y)$ is continuous over a bounded and closed (i.e. compact) region of integration $A$, then the double integral over that region can be written as an **iterated integral**, with the integrals in either order:

$$\boxed{\int_A f(x,y)\, dA = \int_y \int_x f(x,y)\, dx\, dy = \int_x \int_y f(x,y)\, dy\, dx\,.}$$

In order to calculate an integral in this form we take the inner integration first while treating the outer variable as a constant, and then do the outer integration. In this way the problem of two- (or more!) dimensional integration has been reduced to doing a bunch of one-dimensional integrals, one after the other.

**Important note:** If the region and/or the function is unbounded (the latter option arising, for example, if $A$ is open and $f(x, y)$, while continuous on $A$, tends to infinity somewhere on its boundary), then Fubini's theorem still holds **provided that the double integral is absolutely convergent**, meaning that the integral of $|f(x, y)|$ over $A$ must be finite. If this doesn't hold then the result might not be true, with the iterated integrals in the two orders giving different answers – see questions 61, 63 and 64 from the problem sheets for some examples.

**Example 38.** Consider integrating $f(x, y) = 4xy - y^3$ over the region drawn in figure 34 below, where the region we wish to integrate over is the area between two curves.

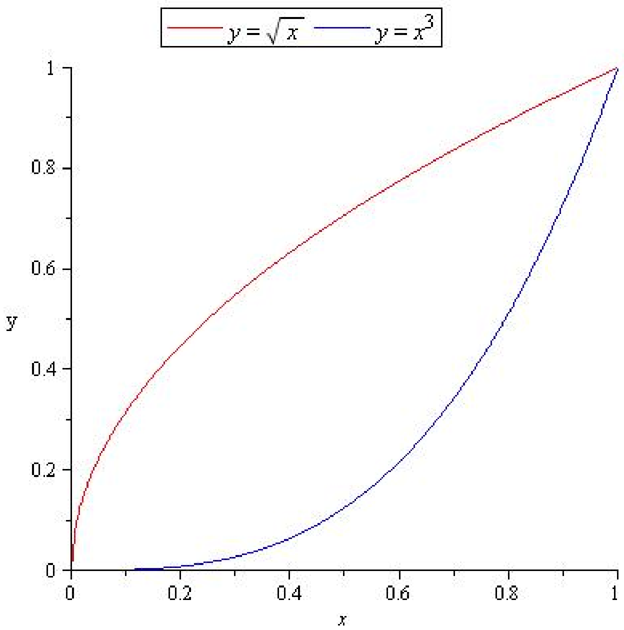



Figure 34: Graph of the region of integration, lying between the two curves.

From the graph we can see that $0 \le x \le 1$ and $x^3 \le y \le \sqrt{x}$, so in the notations of the previous page we should take $a = 0$, $b = 1$, $y_0(x) = x^3$, and $y_1(x) = \sqrt{x}$. The integral of $f(x, y) = 4xy - y^3$ over this region can be calculated as:

$$\int_0^1 \int_{x^3}^{\sqrt{x}} (4xy - y^3)\,dy\,dx = \int_0^1 \left[2xy^2 - \frac{y^4}{4})\right]_{x^3}^{\sqrt{x}} dx$$
$$= \int_0^1 (\frac{7}{4}x^2 - 2x^7 + \frac{x^{12}}{4})\,dx$$
$$= \left[\frac{7x^3}{12} - \frac{x^8}{4} + \frac{x^{13}}{52}\right]_0^1 = \frac{55}{156}\,.$$

If we want to change the order of integration, we must take care over the limits, as they are functions. To integrate with respect to $x$ first so we must calculate the limits of the inner integral in the form $x = g(y)$; on inspection of the graph, figure 34, we can see that the lower limit will be given by the line $y = \sqrt{x}$ which must be re-written as $x = y^2$, similarly the upper limit will be $x = y^{1/3}$. The outer integral, with respect to $y$, has lower and upper limits given by $y = 0$ and $y = 1$ respectively. Now we are ready to integrate!

$$\int_0^1 \int_{y^2}^{y^{1/3}} (4xy - y^3)\,dx\,dy = \int_0^1 \left[2x^2y - y^3x\right]_{y^2}^{y^{1/3}} dy$$
$$= \int_0^1 (2y^{5/3} - y^{10/3} - y^5)dy$$
$$= \left[\frac{3y^{8/3}}{4} - \frac{3y^{13/3}}{13} - \frac{y^6}{6}\right]_0^1 = \frac{55}{156}\,.$$

In the example just treated, changing the order of integration didn't make much difference to the calculation, but sometimes it can be crucial:

**Example 39.** Evaluate the integral of the function $f(x, y) = e^{-x^2}$ over the triangle $A$ shown in figure 35.

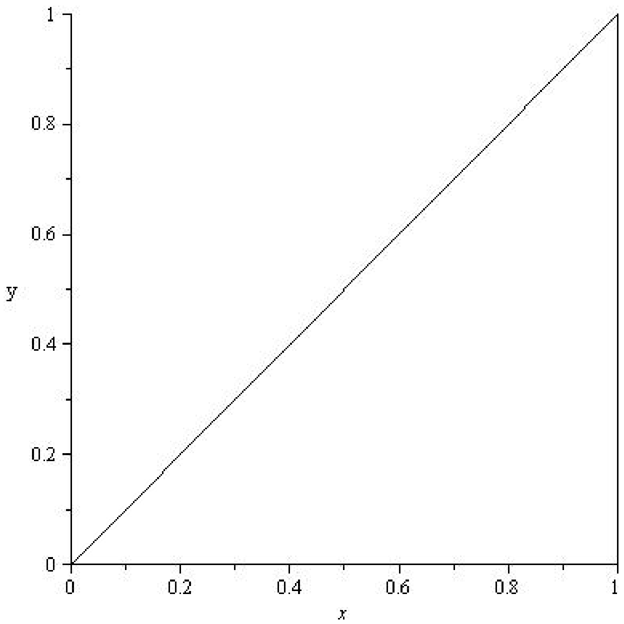



Figure 35: Graph of the region of integration $A$, a triangle with base and height = 1.

Attempting to take the $x$ integral first, we have

$$I = \int_A e^{-x^2} dA = \int_0^1 \int_y^1 e^{-x^2} dx\,dy\,,$$

and then we seem to be stuck: the integral w.r.t. $x$ has no elementary solution, and can only be given in terms of the error function. Thankfully all is not lost: using Fubini we can change the order of integration and obtain the answer:

$$
\begin{aligned}
I &= \int_A e^{-x^2} dA \\
&= \int_0^1 \int_0^x e^{-x^2} dy\, dx \\
&= \int_0^1 x e^{-x^2} dx \\
&= \left[-\tfrac{1}{2}\, e^{-x^2}\right]_0^1 = \tfrac{1}{2}\,(1-e^{-1})\,.
\end{aligned}
$$

*Note:* to calculate an area in the plane, for example between two curves, simply set $f(x,y)=1$:

$$
\boxed{\int_R 1\, dA = \int_R dA = \text{Area of } R}
$$

an integral which can be evaluated in whichever order is easiest.

Sometimes it is better to use a non-rectangular grid of elementary areas $\Delta A_k$, but the idea is the same.

**Example 40.** Integrate $f(x,y)=x^2+y^2$ over the unit circle centred on the origin.

Using polar coordinates, the elementary areas are $\Delta A = \Delta r_i \times (r_j \Delta\theta_j)$, and so

$$
\begin{aligned}
\int\!\!\int_A (x^2+y^2)\, dA &= \int_0^{2\pi} \int_0^1 r^2\, r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 r^3\, dr\, d\theta \\
&= \int_0^{2\pi} \left[\tfrac{1}{4} r^4\right]_0^1 d\theta \\
&= \int_0^{2\pi} \tfrac{1}{4}\, d\theta \;=\; \pi/2\,.
\end{aligned}
$$

There will be more on the systematics of this later in the term.

## 7.2 Volume integrals

**Volume integrals** are the obvious extension to all of this, and Fubini's theorem applies to them too (and also to four, five and so on dimensional integrals, though we'll stop at three). To begin, let's divide up our volume and define the integral as the limit of a Riemann sum.

Let $f(\underline{x})$ be a continuous scalar field defined on a volume $V \in \mathbb{R}^3$ enclosed by the surface $S$. We define the volume integral of $f$ over $V$ ($I = \int_V f(\underline{x})\, dV$) as the limit of a Riemann sum associated with a partition of $V$ into many, $N$, small volumes $\Delta V_i$, as in figure 36, as the number of these small volumes tends to infinity and their sizes all tend to zero:

$$
I = \int_V f(\underline{x})\, dV = \lim_{N\to\infty} \sum_{i=1}^{N} f(\underline{x}_i) \Delta V_i\,.
$$

Some notes:

- The limit should be independent of the partition taken, as we saw with double integrals.

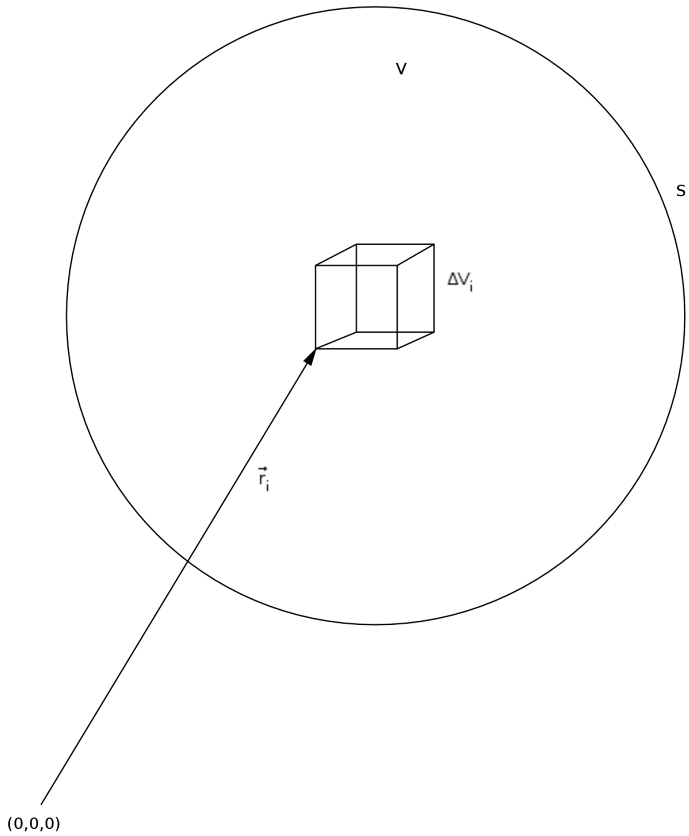



Figure 36: Diagram showing a volume to be integrated over, $V$, with its bounding surface, $S$. $\Delta V_i$ is a volume element in the partition, situated at $\underline{x}_i$.

- Geometrically the volume (triple) integral is a 4-dimensional 'hypervolume' lying 'below' the graph of $f(x, y, z)$ in $\mathbb{R}^4$. This is a natural extension of single integrals giving the area under the curve and double integrals giving the (3-D) volume under the surface, but it is hard to visualise!
- Physically, if $f(\underline{x})$ is the density of some quantity (e.g. number of flying ants per unit volume), then $I = \int_V f(\underline{x})\, dV$ is the amount of 'stuff' (total number of flying ants) inside all of $V$.
- As in the double integral case, we can calculate the volume inside a surface by setting $f(x, y, z) = 1$:

$$\boxed{\int_V 1\, dV = \int_V dV = \text{Volume inside } S.}$$

Now consider a simple shape such as the sphere shown in figure 37, which can be split into an upper and lower surface $z = g_U(x, y)$ and $z = g_L(x, y)$ respectively. (Note that the sphere does not need to be centred on the origin for this to be possible.)

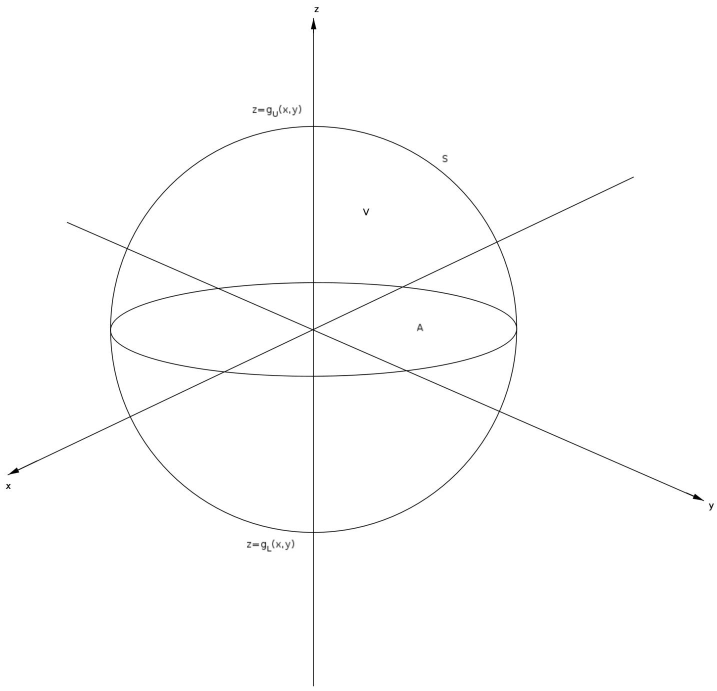



Figure 37: Diagram showing the volume to be integrated over, $V$ with its bounding surface, $S$. The circle on the $x, y$ plane is the projection of the volume onto the plane, and it is labelled $A$. Simple shapes like the sphere can be split into an upper surface $z = g_U(x, y)$ and a lower surface $z = g_L(x, y)$.

If we consider in this case a simple partition which is parallel to the coordinate planes then

$$\Delta V_i = \Delta x_r \Delta y_s \Delta z_t\,,$$

where $\Delta x_r = x_{r+1} - x_r$, $\Delta y_s = y_{s+1} - y_s$, $\Delta z_t = z_{t+1} - z_t$.

Taking the limits of the Riemann sum,

$$
\begin{aligned}
I &= \lim_{N\to\infty} \sum_{i=1}^{N} f(\underline{x}_i)\Delta V_i \\
&= \lim_{N\to\infty} \sum_{r,s,t} f(x_r^*, y_s^*, z_t^*)\Delta x_r \Delta y_s \Delta z_t \\
&= \lim_{\Delta x_r, \Delta y_s \to 0} \sum_{r,s} \left( \lim_{\Delta z_t \to 0} \sum_t f(x_r^*, y_s^*, z_t^*)\Delta z_t \right) \Delta x_r \Delta y_s \quad \text{take } z \text{ limit first} \\
&= \lim_{\Delta x_r, \Delta y_s \to 0} \sum_{r,s} \left( \int\limits_{z=g_L(x_r^*, y_s^*)}^{z=g_U(x_r^*, y_s^*)} f(x_r^*, y_s^*, z) dz \right) \Delta x_r \Delta y_s \quad \text{and now } x,\, y \text{ limits} \\
&= \int\limits_A \left( \int\limits_{z=g_L(x,y)}^{z=g_U(x,y)} f(x,y,z) dz \right) dx\, dy\,,
\end{aligned}
$$

where $x_r^* \in [x_r, x_{r+1}]$, $y_s^* \in [y_s, y_{s+1}]$, $z_t^* \in [z_t, z_{t+1}]$. We may also disentangle the area integral to write $I$ as a three-times iterated integral:

$$
I \;=\; \int_x \int_y \int_z f(\underline{x})\, dz\, dy\, dx\,.
$$

By Fubini's theorem, we can change the order of integration as long as $f(\underline{x})$ is continuous over the closed volume, as in the following example.

**Example 41.** Find the mass $M$ of air inside a hemispherical volume of radius $r$ centred on the origin, when the air density varies with height as $\rho = cz + \rho_0$ (and $c$ and $\rho_0$ are constants).

Doing the integrals in the order $x$ first, then $y$, then $z$, we have

$$
M = \int_0^r \int_{-r_z}^{r_z} \int_{-\sqrt{r_z^2 - y^2}}^{\sqrt{r_z^2 - y^2}} \rho(z)\, dx\, dy\, dz
$$

where $r_z = \sqrt{r^2 - z^2}$. Noting that $\rho(z)$ can be taken outside the $x$ and $y$ integrals,

$$
M = \int_0^r \rho(z) \left( \int_{-r_z}^{r_z} \int_{-\sqrt{r_z^2 - y^2}}^{\sqrt{r_z^2 - y^2}} dx\, dy \right) dz\,.
$$

Now the quantity in round brackets is just the area $A(z)$ of the horizontal 'slice' of the hemisphere at height $z$, so it's equal to $\pi r_z^2$, and

$$
\begin{aligned}
M &= \pi \int_0^r (cz + \rho_0) r_z^2\, dz \\
&= \pi \int_0^r (cz + \rho_0)(r^2 - z^2)\, dz \\
&= \pi \int_0^r cr^2 z + \rho_0 r^2 - cz^3 - \rho_0 z^2\, dz \\
&= \pi \left[ \tfrac{1}{2} cr^2 z^2 + \rho_0 r^2 z - \tfrac{1}{4} cz^4 - \tfrac{1}{3}\rho_0 z^3 \right]_0^r \\
&= \pi (\tfrac{1}{4} cr^4 + \tfrac{2}{3}\rho_0 r^3)\,.
\end{aligned}
$$

**Bonus example 1.** Use the volume integral to calculate the volume of a sphere of radius $a$ (which we know to be $V = \frac{4}{3}\pi a^3$).

We can calculate this volume as $\int_V f\,dV$, where $V$ is the set of points $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le a^2\}$ and $f=1$. We have:

$$\begin{aligned}
\int_V dV &= \int_A \left( \int_{z=g_L(x,y)}^{z=g_U(x,y)} dz \right) dx\,dy \\
&= \int_A \left( \int_{z=-\sqrt{a^2-x^2-y^2}}^{z=\sqrt{a^2-x^2-y^2}} dz \right) dx\,dy \\
&= \int_A 2\sqrt{a^2-x^2-y^2}\,dx\,dy \\
&= \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} 2\sqrt{a^2-x^2-y^2}\,dx\,dy
\end{aligned}$$

Use a substitution to complete the x-integration: $x = \sqrt{a^2-y^2}\cos\theta$, $dx = -\sqrt{a^2-y^2}\sin\theta\,d\theta$

$$\begin{aligned}
&= \int_{-a}^{a} \left( \int_0^{\pi} 2(a^2-y^2)\sin^2\theta\,d\theta \right) dy \\
&= \int_{-a}^{a} (a^2-y^2)\pi\,dy \;=\; \pi(2a^3 - \frac{2a^3}{3}) \;=\; \frac{4}{3}\pi a^3, \text{ as expected.}
\end{aligned}$$

Later, we'll rederive this result slightly less painfully, via a change of variables.

## 7.3 Line integrals

So far the integrals have been over 'flat' regions in one, two or three dimensions, but for applications it is important to generalise to curved lines and surfaces.

A *regular arc* $C \subset \mathbb{R}^n$ is a parametrised curve $\underline{x}(t)$ for which the Cartesian components $x_a(t)$, $a = 1 \dots n$ are continuous with continuous first derivatives, where $t$ lies in some (maybe infinite) interval $[\alpha, \beta]$. A *regular curve* consists of a finite number of regular arcs joined end to end.

If $\underline{v}(\underline{x})$ is a vector field in $\mathbb{R}^n$, then its restriction to a regular arc, $\underline{v}(\underline{x}(t))$, is a vector function of $t$ and its scalar product with the tangent $d\underline{x}(t)/dt$ to the arc is a scalar function of $t$. We can therefore integrate it along the arc to get a real number, called the *line integral* of $\underline{v}$ along the arc $C$: $t \mapsto \underline{x}(t),\ t = \alpha \dots \beta$, denoted $\int_C \underline{v} \cdot d\underline{x}$:

$$\boxed{\int_C \underline{v} \cdot d\underline{x} = \int_{\alpha}^{\beta} \underline{v}(\underline{x}(t)) \cdot \frac{d\underline{x}(t)}{dt}\,dt\,.} \tag{7.1}$$

An important fact is that, as suggested by the notation, the line integral does not depend on the choice of parametrisation of $C$. This follows from the chain rule, but it is instructive to check it for yourself in some examples.

If $C$ is a regular curve made up of one or more regular arcs, then the line integral of $\underline{v}$ along $C$ is the sum of the line integrals over these arcs; it is also denoted as $\int_C \underline{v} \cdot d\underline{x}$.

Finally, if the integral is performed over a *closed* regular curve then it is often written as $\oint_C \underline{v} \cdot d\underline{x}$.

Some variants:

| Interpretation | Form |
|---|---|
| Length of curve, $C$ | $\int_C ds = \int_a^b \lvert\lvert \frac{d\underline{x}(t)}{dt} \rvert\rvert\,dt$ |
| Mass, if $f$ is a density function | $\int_C f ds = \int_a^b f(\underline{x}(t))\,\lvert\lvert \frac{d\underline{x}(t)}{dt} \rvert\rvert\,dt$ |
| Work done, if $\underline{F}$ is a force | $\int_C \underline{F} \cdot d\underline{x}$ |

Here is a quick list of useful parametrisations (with a positive orientation i.e. anti-clockwise) for some common curves in two dimensions:

| Curve | Parametric Equations |
|---|---|
| Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ | $x(t) = a\cos(t)$, $y(t) = b\sin(t)$ |
| $y = f(x)$ | $x(t) = t$, $y(t) = f(t)$ |
| $x = g(y)$ | $x(t) = g(t)$, $y(t) = t$ |
| Straight line segment from $(x_0, y_0)$ to $(x_1, y_1)$ | $x(t) = (1-t)x_0 + tx_1$, $y(t) = (1-t)y_0 + ty_1$ |

**Example 42.** Let $\underline{u}(\underline{x})$ be the vector field $(x^2z, xyz, x)$ and $C$ be the circle parallel to the $x$, $y$ plane with centre $\underline{a} = (a_1, a_2, a_3)$ and radius $r$, given by

$$\underline{x}(t) = \underline{a} + r\cos t\,\underline{e}_1 + r\,\sin t\,\underline{e}_2\,, \quad 0 \le t \le 2\pi\,.$$

Evaluate $\oint_C \underline{u} \cdot d\underline{x}$.

Calculating, we have

$$\frac{d\underline{x}}{dt} = -r\,\sin t\,\underline{e}_1 + r\,\cos t\,\underline{e}_2 = (-r\,\sin t, r\,\cos t, 0)$$

and

$$\underline{x}(t) = (a_1 + r\,\cos t, a_2 + r\,\sin t, a_3)$$

so

$$\underline{u}(\underline{x}(t)) = ((a_1 + r\,\cos t)^2 a_3, (a_1 + r\,\cos t)(a_2 + r\,\sin t)a_3, a_1 + r\,\cos t)$$

and

$$\underline{u}(\underline{x}(t)) \cdot \frac{d\underline{x}}{dt} = -r\,\sin t(a_1 + r\,\cos t)^2 a_3 + r\,\cos t(a_1 + r\,\cos t)(a_2 + r\,\sin t)a_3\,.$$

Hence

$$\begin{aligned}\oint_C \underline{u} \cdot d\underline{x} &= \int_0^{2\pi} \left(-r\,\sin t(a_1 + r\,\cos t)^2 a_3 + r\,\cos t(a_1 + r\,\cos t)(a_2 + r\,\sin t)a_3\right) dt \\ &= \ldots \\ &= \pi r^2 a_2 a_3\,.\end{aligned}$$

Note: if we had instead parametrized $C$ as

$$\underline{x}(t) = \underline{a} + r\cos 2t\,\underline{e}_1 + r\,\sin 2t\,\underline{e}_2\,, \quad 0 \le t \le \pi$$

the steps of the calculation would have changed slightly, but the final answer is unchanged – it's worthwhile to check this for yourself.

**Bonus example 2.** Consider a circle of radius 1, centred at the origin. Let $C$ be the arc of this circle which lies in the first quadrant, taken in an anticlockwise direction. Find the value of the line integral of $\underline{F}(x, y) = (-y, -xy)$ along $C$.

First we must parametrise the path, $C$: $\underline{x}(t) = (\cos(t), \sin(t))$, $0 \le t \le \pi/2$ (the restriction on the range of $t$ is because the arc lies only in the first quadrant). Therefore $\frac{d\underline{x}(t)}{dt} = (-\sin(t), \cos(t))$. Next note that $\underline{F}(\underline{x}(t)) = \underline{F}(x(t), y(t)) = (-\sin(t), -\cos(t)\sin(t))$, so

$$\begin{aligned}\int_C \underline{F} \cdot d\underline{x} &= \int_0^{\frac{\pi}{2}} (-\sin(t), -\cos(t)\sin(t)) \cdot (-\sin(t), \cos(t))\,dt \\ &= \int_0^{\frac{\pi}{2}} (\sin^2(t) - \cos^2(t)\sin(t))\,dt \\ &= \int_0^{\frac{\pi}{2}} \left(\tfrac{1}{2}(1 - \cos(2t)) - \cos^2(t)\sin(t)\right) dt \\ &= \left[\tfrac{1}{2}(t - \tfrac{1}{2}\sin(2t)) + \tfrac{1}{3}\cos^3(t)\right]_0^{\frac{\pi}{2}} = \tfrac{1}{4}\pi - \tfrac{1}{3}\,.\end{aligned}$$

Once you have understood Green's theorem [next chapter!], try to use it to double-check this result.

To summarize: the steps for performing a line integral of a vector field $\underline{v}(\underline{x})$ along a regular arc $C$ are as follows:

- parametrise $C$ somehow, as $t \mapsto \underline{x}(t)$, with $t$ in some range;
- compute $\frac{d\underline{x}(t)}{dt}$ and $\underline{v}(\underline{x}(t))$;
- compute their scalar product and re-write the integrand in terms of $t$;
- perform the integration between the limits identified in the first part.

For a regular curve made up of a number of regular arcs, just do the above steps for each arc, and then add up the results.

## 7.4 Surface integrals I: defining a surface

As with a line integral where the integration of a vector field along a curve yields a real number, a three-dimensional vector field can be integrated over a two-dimensional surface $S$ sitting in $\mathbb{R}^3$, to give a double integral analogue of the line integral. Surface integrals are of particular importance in electromagnetism and fluid mechanics.

The first task is to specify the surface. There are (at least) two standard ways to do this:

**Method 1:** Give the surface in *parametric form* as $\underline{x}(u, v)$ where the real parameters $u$ and $v$ lie in some region $U \subset \mathbb{R}^2$ called the *parameter domain*.

**Example 43.** Points on a sphere of radius $a$ centred on the origin can be parametrised in spherical (polar) coordinates, with $u$ and $v$ usually written as $\theta$ and $\phi$, as

$$\underline{x}(\theta, \phi) = (x(\theta, \phi), y(\theta, \phi), z(\theta, \phi))$$

where

$$x(\theta, \phi) = a \sin\theta \, \cos\phi \,, \quad y(\theta, \phi) = a \sin\theta \, \sin\phi \,, \quad z(\theta, \phi) = a \cos\theta \,,$$

and $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ .

*Small aside:* The parametrisation for the surface of a sphere comes from the spherical polar coordinate system, a three-dimensional extension of two-dimensional polar coordinates. Figure 38 shows how we get these coordinates: first calculate $r' = r\sin(\theta)$. Then $x = r'\cos(\phi) = r\sin(\theta)\cos(\phi)$, $y = r'\sin(\phi) = r\sin(\theta)\sin(\phi)$, while $z = r\cos(\theta)$. As ever, $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin, while $\theta$ and $\phi$ are known as the polar and azimuthal angles respectively, with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. Beware that some texts have these the other way around, so that $\theta$ is the azimuthal and $\phi$ the polar angle. Note though that here we fix $r$ to be equal to $a$, the radius of the sphere, as we want to stay on its surface.

The following table gives some more examples:

| Surface | Parametric Equations |
|---|---|
| General | $\underline{x}(u, v) = x(u, v)\underline{e}_1 + y(u, v)\underline{e}_2 + z(u, v)\underline{e}_3$ |
| Sphere/part sphere, radius $a$ | $x = a\sin(u)\cos(v)$, $y = a\sin(v)\sin(u)$, $z = a\cos(u)$ |
| Cylinder, radius $a$, centred on $z$-axis | $x = a\cos(u)$, $y = a\sin(u)$, $z = v$ |
| $z = f(x, y)$ | $x = u$, $y = v$, $z = f(u, v)$ |
| $y = g(x, z)$ | $x = u$, $y = g(u, v)$, $z = v$ |
| $x = h(y, z)$ | $x = h(u, v)$, $z = v$, $y = u$ |

Returning to the general case, $\frac{\partial \underline{x}}{\partial u}$ and $\frac{\partial \underline{x}}{\partial v}$ are two tangent vectors to $S$ at $\underline{x}(u, v)$, and so their cross product

$$\frac{\partial \underline{x}(u, v)}{\partial u} \times \frac{\partial \underline{x}(u, v)}{\partial v}$$

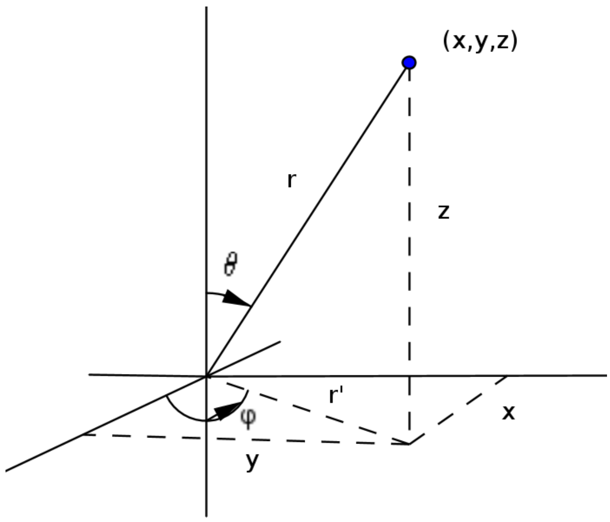



Figure 38: Spherical polar coordinates $(r, \theta, \phi)$.

is a normal vector to $S$ there, and

$$\widehat{\underline{n}} = \Big(\frac{\partial \underline{x}(u,v)}{\partial u} \times \frac{\partial \underline{x}(u,v)}{\partial v}\Big) \Big/ \Big|\frac{\partial \underline{x}(u,v)}{\partial u} \times \frac{\partial \underline{x}(u,v)}{\partial v}\Big|$$

is a unit normal to $S$ at $\underline{x}(u,v)$. If $u$ and $v$ are swapped over, then we also get a unit normal to $S$ at $\underline{x}(u,v)$, but one pointing in the opposite direction (which for a surface sitting in $\mathbb{R}^3$ is the only other option).

It's a good exercise to check for example 43 that this recipe gives the answer you'd expect, that is a unit vector pointing in the radial direction. For this case the relevant partial derivatives are

$$\frac{\partial \underline{x}}{\partial \theta} = (a\cos\theta\cos\phi, a\cos\theta\sin\phi, -a\sin\theta)$$
$$\frac{\partial \underline{x}}{\partial \phi} = (-a\sin\theta\sin\phi, a\cos\phi\sin\theta, 0)$$

and so

$$\underline{x}_\theta \times \underline{x}_\phi = (a^2\sin^2\theta\cos\phi, a^2\sin^2\theta\sin\phi, a^2\sin\theta\cos\theta) = a\sin\theta\,\underline{x}\,.$$

Finally constructing the unit normal vector gives

$$\widehat{\underline{n}} = \frac{a\sin\theta\,\underline{x}}{a\sin\theta\,|\underline{x}|} = \frac{(x,y,z)}{a}\,,$$

which is indeed the expected answer, if you think about it.

**Method 2:** Express the surface as (part of) a **level surface** (recall from term 1) of a scalar field $f$, i.e. give the surface implicitly as $f(x,y,z) = \text{const}$. Then the gradient of $f$, $\underline{\nabla} f$, is a normal vector to $S$, and

$$\widehat{\underline{n}} = \frac{\underline{\nabla} f}{|\underline{\nabla} f|}\,, \tag{7.2}$$

is a unit normal.

**Example 44.** For the sphere discussed above, we can take $f(x,y,z) := x^2 + y^2 + z^2 - a^2 = 0$ (or

$f(x,y,z) := x^2+y^2+z^2 = a^2$ would work just as well). Then

$$\begin{aligned}\underline{\widehat{n}} &= \frac{\nabla f}{|\nabla f|} \\ &= \frac{(2x,2y,2z)}{\sqrt{4x^2+4y^2+4z^2}} \\ &= \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \frac{(x,y,z)}{a}\,.\end{aligned}$$

This agrees with the result from method 1.

Note: if we'd instead defined the surface of the sphere by $h(x,y,z) = -f(x,y,z) = a^2 - x^2 - y^2 - z^2 = 0$, then the normal vector obtained by using $f$ (which we'll call $\underline{\widehat{n}}_f$) would have been replaced by

$$\underline{\widehat{n}}_h = \frac{(-x,-y,-z)}{\sqrt{x^2+y^2+z^2}} = -\underline{\widehat{n}}_f$$

This reflects the same ambiguity in the sign of the normal vector that was seen when swapping $u$ and $v$ in method 1.

## 7.5 Surface integrals II: evaluating the integral

We will once again use a Riemann sum to define the surface integral of a continuous vector field $\underline{F}(\underline{x})$ over a surface $S$ lying in $\mathbb{R}^3$. The parametrised position vector of a point on the surface is $\underline{x} = \underline{x}(u,v)$ with $(u,v) \in U$, the parameter domain. We will also assume that the partial derivatives of $\underline{x}$ exist and are continuous, and that the unit normal vector $\underline{\widehat{n}}(u,v)$ is continuous – this means that the surface is what we call orientable. (This last point may seem obvious but it is not always true e.g. the Möbius strip is a non-orientable surface.) The general setup is illustrated in figure 39. Then the surface integral is *defined* as

$$\int_S \underline{F}\cdot d\underline{A} = \lim_{\Delta A_k \to 0} \sum_k \underline{F}(\underline{x}_k^*)\cdot \underline{\widehat{n}}_k\, \Delta A_k\,. \tag{7.3}$$

Note the dot product between the normal to the surface and the vector field: this implies that we are looking at the vector field contributions which are perpendicular to the surface.

To turn this rather-formal definition into something which can be used in practice, we use either method 1 or method 2 to specify the surface, and then convert everything into a 'flat' two-dimensional area integral of the sort seen in section 7.1 above.

**Method 1:** We construct the area elements $\Delta A_k$ by approximating them as parallelograms given by the partitioning of the surface along lines of constant $u$ and $v$. Remember $u$ and $v$ are in the parameter domain $U$, split as shown in figure 39(b) and indexed $i$ and $j$ respectively, and that the modulus of the cross product of two vectors in equal to the area of the parallelogram that they span. Thus

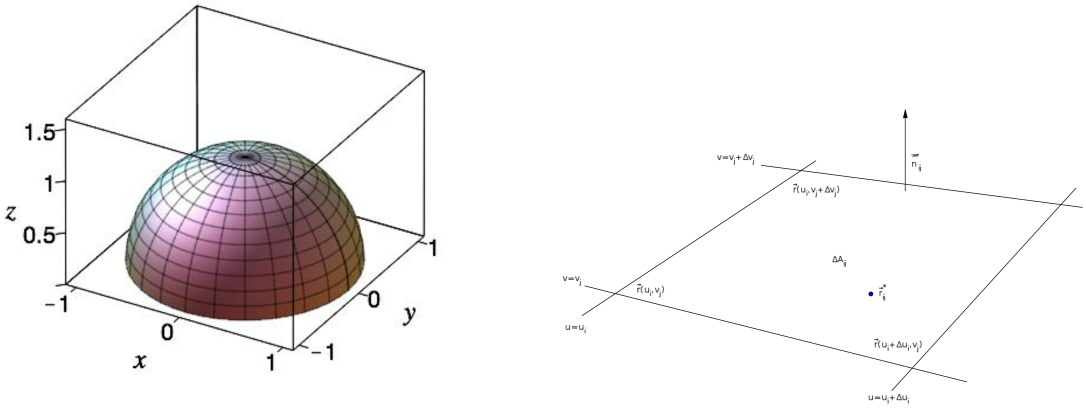
$$\begin{aligned}\underline{\widehat{n}}_k \Delta A_k &\approx (\underline{x}(u_i+\Delta u_i, v_j) - \underline{x}(u_i,v_j)) \times (\underline{x}(u_i, v_j+\Delta v_j) - \underline{x}(u_i,v_j)) \\ &\approx (\Delta u_i \frac{\partial \underline{x}}{\partial u}) \times (\Delta v_j \frac{\partial \underline{x}}{\partial v}).\end{aligned} \tag{7.4}$$

Substituting equation (7.4) into equation (7.3) gives us

$$\int_S \underline{F}\cdot d\underline{A} = \lim_{\Delta u_i, \Delta v_j \to 0} \sum_{i,j} \underline{F}(\underline{x}_{ij}^*)\cdot \left(\frac{\partial \underline{x}(u_i,v_j)}{\partial u} \times \frac{\partial \underline{x}(u_i,v_j)}{\partial v}\right) \Delta u_i \Delta v_j$$

and, taking the limit, we get the key formula, the 'surface' version of the line integral definition (7.1):

$$\boxed{\int_S \underline{F}\cdot d\underline{A} = \int_U \underline{F}(\underline{x}(u,v))\cdot \left(\frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v}\right) du\, dv\,.} \tag{7.5}$$




(a) (b)

Figure 39: Plot (a) shows a hemispherical surface, over which we may wish to integrate a vector field; plot (b) shows an area element, $\Delta A_{ij}$, of the surface with unit normal vector $\underline{\widehat{n}}_{ij}$
.

This is a double integral over the parameter domain $U$, and can be written neatly as

$$\int_S \underline{F} \cdot d\underline{A} = \int_U \underline{F} \cdot \underline{N}\, du\, dv\,,$$

where $\underline{N} = \underline{x}_u \times \underline{x}_v$ is a normal vector to $S$ (but not of unit length).

**Example 45.** Find the integral of $\underline{F} = \underline{e}_3$ over the surface, $S$, given by the hemisphere of radius 1, centred at the origin, with $z > 0$, as shown in figure 39(a). We have

$$\int_S \underline{F} \cdot d\underline{A} = \int_U \underline{F}(\underline{x}) \cdot \left( \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right) du\, dv\,,$$

where $r = 1$, and taking $U = \{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq \frac{\pi}{2}, 0 \leq v \leq 2\pi\}$ captures the part of the surface of the unit sphere with $z \geq 0$. We also calculated $\underline{x}_u \times \underline{x}_v$ in the discussion following example 43, and substituting all our values and completing the calculation,

$$\begin{aligned}
\int_S \underline{F} \cdot d\underline{A} &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \underline{F}(\underline{x}) \cdot \left( \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right) du\, dv \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \underline{e}_3 \cdot (\sin^2(u)\cos(v)\underline{e}_1 + \sin^2(u)\sin(v)\underline{e}_2 + \sin(u)\cos(u)\underline{e}_3)\, du\, dv \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin(u)\cos(u)\, du\, dv \\
&= \int_0^{2\pi} \left[\tfrac{1}{2}\sin^2(u)\right]_0^{\frac{\pi}{2}} dv = \int_0^{2\pi} \tfrac{1}{2}\, dv = \pi\,.
\end{aligned}$$

**Method 2:** Suppose that $S$ is given as the level set (or part of the level set) of a function $f(x, y, z)$, and furthermore that $\partial f/\partial z \neq 0$ on $S$. Then, by the implicit function theorem for surfaces, the points of $S$ can be written as $(x, y, g(x, y))$ for some function $g(x, y)$, where the $(x, y)$ ranges over some region $A$ of the $x$,$y$ plane, which is the projection of $S$ onto that plane. (This is not possible for every surface; for example the $x$,$z$ plane cannot be parametrised this way, and neither can the surface of a whole sphere.) We can then apply method 1 taking the parameters to be the $x$, $y$ coordinates, so that $\mathbf{x}(x, y) = (x, y, g(x, y)) = x\,\underline{e}_1 + y\,\underline{e}_2 + g(x, y)\,\underline{e}_3$, and (after a short calculation)

$$\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y} = -\mathbf{e}_1 \frac{\partial g}{\partial x} - \mathbf{e}_2 \frac{\partial g}{\partial y} + \mathbf{e}_3\,.$$

The partial derivatives of $g$ can be calculated as in the implicit function theorem for surfaces, noting that the function of two variables, $F(x,y)$, defined by $F(x,y) = f(x,\, y,\, g(x,y))$, is constant, so that

$$0 = \frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}$$

$$0 = \frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}$$

Using these equations,

$$\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y} = \mathbf{e}_1 \frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial z} + \mathbf{e}_2 \frac{\partial f}{\partial y}\Big/\frac{\partial f}{\partial z} + \mathbf{e}_3 = (\underline{\nabla} f)/(\mathbf{e}_3 \cdot \underline{\nabla} f)\,,$$

enabling the surface integral of $\mathbf{F}$ over $S$ to be written as an area integral over the region $A$ in the $x,y$-plane:

$$\int_S \underline{F} \cdot d\underline{A} = \int_A \frac{\underline{F} \cdot \underline{\nabla} f}{\underline{e}_3 \cdot \underline{\nabla} f}\, dx\, dy\,.$$

*Note:* whenever we compute a surface integral, there is a choice as to the direction to take the normal vectors to the surface – 'in' or 'out' for a closed surface, or 'upwards' or 'downwards' for a surface lying above the $x$,$y$ plane, for example. In the derivation of the formula just given, $z$ component of $\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y}$ was equal to 1, so it corresponds to the 'upwards' choice of normals. If the downwards option was the one you were after, the formula should be negated.

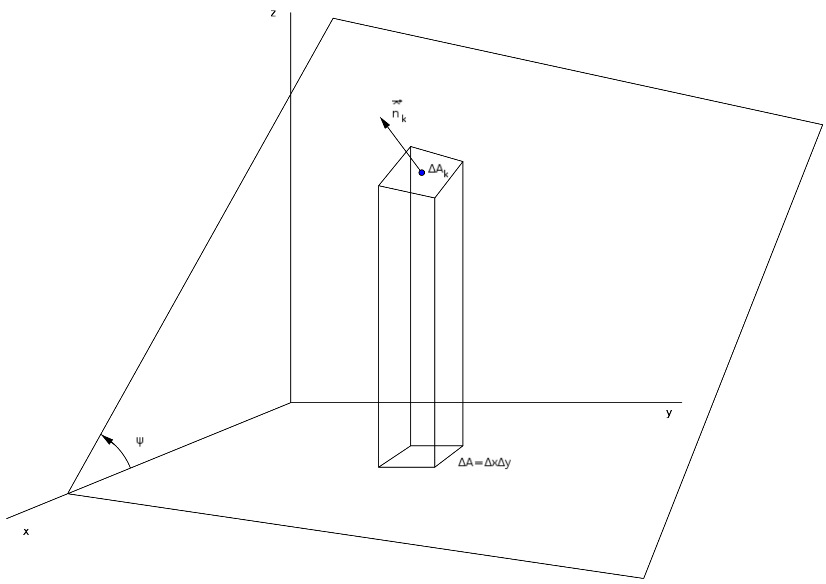



Figure 40: Plot of the tangent plane to a surface at a point $\underline{x}_k$, making an angle $\psi$ with the $x$,$y$ plane.

We can give an alternative derivation of the method 2 formula starting from the definition of the surface integral in terms of a Riemann sum, equation (7.3). We shall use equation (7.2) for the unit normal, $\underline{\widehat{n}}_k = \frac{\nabla f}{|\nabla f|}$ , where $f(x,y,z) = \text{const}$ defines the surface. Instead of approximating the small area element by a parallelogram we can use a geometrical argument. Figure 40 shows the tangent plane to $S$ at a point of interest, $\underline{x}_k$, on $S$, and an area $\Delta A_k$ on that plane together with the 'shadow' $\Delta A = \Delta x \Delta y$ that it casts on the $x$,$y$ plane. From the figure we can see that $\frac{\Delta A}{\Delta A_k} = \cos(\psi)$ , while $\cos(\psi) = \underline{e}_3 \cdot \underline{\widehat{n}}_k$ from definition of dot product. Hence we can write

$$\Delta A_k = \frac{\Delta x \Delta y}{\cos(\psi)} = \frac{\Delta x \Delta y}{\underline{e}_3 \cdot \underline{\widehat{n}}_k} \quad \text{and} \quad \underline{\widehat{n}}_k\, \Delta A_k = \frac{\underline{\widehat{n}}_k\, \Delta x \Delta y}{\underline{e}_3 \cdot \underline{\widehat{n}}_k}\,.$$

Returning to the Riemann sum, and substituting $\widehat{\underline{n}}_k = \frac{\underline{\nabla} f}{|\underline{\nabla} f|}$ in the formula just given for $\widehat{\underline{n}}_k\, \Delta_k$, we have

$$\begin{aligned}
\int_S \underline{F} \cdot d\underline{A} &= \lim_{\Delta A_k \to 0} \sum_k \underline{F}(\underline{x}_k^*) \cdot \widehat{\underline{n}}_k \Delta A_k \\
&= \lim_{\Delta x_i, \Delta y_j \to 0} \sum_{i,j} \underline{F}(\underline{x}_{i,j}^*) \cdot \frac{\underline{\nabla} f}{|\underline{\nabla} f|} \frac{\Delta x_i \Delta y_j}{\underline{e}_3 \cdot \frac{\underline{\nabla} f}{|\underline{\nabla} f|}} \\
&= \lim_{\Delta x_i, \Delta y_j \to 0} \sum_{i,j} \frac{\underline{F}(\underline{x}_{i,j}^*) \cdot \underline{\nabla} f}{\underline{e}_3 \cdot \underline{\nabla} f} \Delta x_i \Delta y_j \\
&= \int_A \frac{\underline{F} \cdot \underline{\nabla} f}{\underline{e}_3 \cdot \underline{\nabla} f}\, dx\, dy\,,
\end{aligned}$$

where $A$ is the area of the surface projected onto the $x$,$y$ plane. Let's return to the previous example and calculate the surface integral using method 2.

**Example 46.** Recompute the integral of $\underline{F} = \underline{e}_3$ over the surface, $S$, given by the hemisphere of radius 1 and centred at the origin with $z > 0$, as shown in figure 39(a), using method 2. Here the surface can be represented by $f(x, y, z) = x^2 + y^2 + z^2 = 1$, $A$ is the unit circle centred at the origin, and $\underline{\nabla} f = (2x, 2y, 2z)$. Hence

$$\begin{aligned}
\int_S \underline{F} \cdot d\underline{A} &= \int_A \frac{\underline{F} \cdot \underline{\nabla} f}{\underline{e}_3 \cdot \underline{\nabla} f}\, dx\, dy \\
&= \int_A \frac{\underline{e}_3 \cdot \underline{\nabla} f}{\underline{e}_3 \cdot \underline{\nabla} f}\, dx\, dy \\
&= \int_A dx\, dy = \pi\,,
\end{aligned}$$

since the area of the unit disk is $\pi$. This agrees with our previous calculation, as of course it had to.

To recap: there are two ways to evaluate a surface integral, and which one is the best to use depends on the form of the surface you are integrating over:

| Equation of surface | Form of surface integral |
|---|---|
| Parametric: $\underline{x}(u, v)$ as in example 45 | $\int_S \underline{F} \cdot d\underline{A} = \int_U \underline{F} \cdot (\underline{x}_u \times \underline{x}_v)\, du\, dv$ |
| Implicit/level surface: $f(x, y, z) = \text{const}$ as in example 46 | $\int_S \underline{F} \cdot d\underline{A} = \int_A \frac{\underline{F} \cdot \underline{\nabla} f}{\underline{e}_3 \cdot \underline{\nabla} f}\, dx\, dy$ |

These two forms of the surface integral are equivalent, but one may be easier than the other for any given example so you must practise! The problem sheets contain plenty of examples, many of them in the context of the three theorems that will be discussed next.