58 *Compute the differential, or Jacobian matrix, and the Jacobian of the function* $\underline{V} : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $\underline{V}(x, y) = (x \cos y, x \sin y)$ *. State where* $\underline{V}$ *defines an orientation preserving local diffeomorphism, and where it defines an orientation reversing local diffeomorphism.*

Solution: $\underline{V}(x, y) = (x \cos y, x \sin y)$ has Jacobian matrix

$$D\underline{V}_{(x,y)} = \begin{pmatrix} \cos y\,, & -x \sin y \\ \sin y\,, & x \cos y \end{pmatrix}$$

and so the Jacobian is

$$J(\underline{V}) = \begin{vmatrix} \cos y\,, & -x \sin y \\ \sin y\,, & x \cos y \end{vmatrix} = x\,.$$

Since $J(\underline{V}) > 0$ for $x > 0$, $\underline{V}$ is an orientation preserving local diffeomorphism on $R = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. $\underline{V}$ is an orientation reversing local diffeomorphism on $L = \{(x, y) \in \mathbb{R}^2 : x < 0\}$.

59 *Repeat question 58 for* $\underline{V}(x, y) = (e^x \cos y, e^x \sin y)$ *.*

Solution: $\underline{V}(x, y) = (e^x \cos y, e^x \sin y)$ has Jacobian matrix

$$D\underline{V}_{(x,y)} = \begin{pmatrix} e^x \cos y\,, & -e^x \sin y \\ e^x \sin y\,, & e^x \cos y \end{pmatrix}$$

and so the Jacobian is

$$J(\underline{V}) = \begin{vmatrix} e^x \cos y\,, & -e^x \sin y \\ e^x \sin y\,, & e^x \cos y \end{vmatrix} = e^{2x}\,.$$

Since $J(\underline{V}) > 0$ for all $x \in \mathbb{R}^2$, $\underline{V}$ defines an orientation preserving local diffeomorphism on all of $\mathbb{R}^2$ (to $\underline{V}(\mathbb{R}^2) = \mathbb{R}^2 - \{\underline{0}\}$).

60 *Calculate the differential, or Jacobian matrix, and the Jacobian of the following transformations:*

(a) $\underline{U}(u, v) = (x(u, v), y(u, v))$ *where* $x(u, v) = \frac{1}{2}(u + v)$ *and* $y(u, v) = \frac{1}{2}(u - v)$ *;*

(b) $\underline{V}(r, \theta) = (x(r, \theta), y(r, \theta))$ *where* $x(r, \theta) = r \cos \theta$ *and* $y(r, \theta) = r \sin \theta$ *;*

(c) $\underline{W}(r, \theta, \phi) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$ *.*

Solution:

(a)

$$D\underline{U} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\ ; \quad J(\underline{U}) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}\,.$$

(b)

$$D\underline{V} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r \sin\theta \\ \sin\theta & r \cos\theta \end{pmatrix}\ ; \quad J(\underline{V}) = \begin{vmatrix} \cos\theta & -r \sin\theta \\ \sin\theta & r \cos\theta \end{vmatrix} = r\,.$$

(c)

$$D\underline{W} = \begin{pmatrix} \frac{\partial W_1}{\partial r} & \frac{\partial W_1}{\partial \theta} & \frac{\partial W_1}{\partial \phi} \\ \frac{\partial W_2}{\partial r} & \frac{\partial W_2}{\partial \theta} & \frac{\partial W_2}{\partial \phi} \\ \frac{\partial W_3}{\partial r} & \frac{\partial W_3}{\partial \theta} & \frac{\partial W_3}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix};$$

$$J(\underline{W}) = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta\,.$$

61 *Adapted from exam question 2009 (Section B) Q7:*

(a) Let $\underline{V} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. Give the definition of $\underline{V}$ being differentiable at a point $\underline{a}$.

(b) Let $\underline{V}(x)$ and $\underline{W}(x)$ be two differentiable vector fields in $\mathbb{R}^2$. Give formulae for the two differentials $D\underline{V}_{\underline{x}}$ and $D\underline{W}_{\underline{x}}$.

(c) Use the chain rule to show that the differential of the composite map $\underline{U}(\underline{x}) := \underline{V}(\underline{W})$ satisfies

$$D\underline{U}_{\underline{x}} = D\underline{V}_{\underline{W}} D\underline{W}_{\underline{x}}\,.$$

Solution:

(a) $\underline{V}$ is differentiable at a point if it can be well enough linearly approximated near that point. In particular, $\underline{V}$ is differentiable at $\underline{a}$ if there exists a linear function $\underline{L}(\underline{h}) : \mathbb{R}^n \to \mathbb{R}^n$ such that

i) $\underline{V}(\underline{a}+\underline{h}) - \underline{V}(\underline{a}) = \underline{L}(\underline{h}) + \underline{R}(\underline{h})$

ii) $\lim_{\underline{h}\to\underline{0}} \frac{\underline{R}(\underline{h})}{|\underline{h}|} = 0$.

(b)

$$D\underline{V}_{\underline{x}} = \begin{pmatrix} \frac{\partial V_1}{\partial x} & \frac{\partial V_1}{\partial y} \\ \frac{\partial V_2}{\partial x} & \frac{\partial V_2}{\partial y} \end{pmatrix}$$

and

$$D\underline{W}_{\underline{x}} = \begin{pmatrix} \frac{\partial W_1}{\partial x} & \frac{\partial W_1}{\partial y} \\ \frac{\partial W_2}{\partial x} & \frac{\partial W_2}{\partial y} \end{pmatrix}.$$

(c)

$$\begin{aligned} D(\underline{U})_{\underline{x}} &= \begin{pmatrix} \frac{\partial V_1(\underline{W})}{\partial x} & \frac{\partial V_1(\underline{W})}{\partial y} \\ \frac{\partial V_2(\underline{W})}{\partial x} & \frac{\partial V_2(\underline{W})}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial V_1}{\partial W_1}\frac{\partial W_1}{\partial x} + \frac{\partial V_1}{\partial W_2}\frac{\partial W_2}{\partial x} & \frac{\partial V_1}{\partial W_1}\frac{\partial W_1}{\partial y} + \frac{\partial V_1}{\partial W_2}\frac{\partial W_2}{\partial y} \\ \frac{\partial V_2}{\partial W_1}\frac{\partial W_1}{\partial x} + \frac{\partial V_2}{\partial W_2}\frac{\partial W_2}{\partial x} & \frac{\partial V_2}{\partial W_1}\frac{\partial W_1}{\partial y} + \frac{\partial V_2}{\partial W_2}\frac{\partial W_2}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial V_1}{\partial W_1} & \frac{\partial V_1}{\partial W_2} \\ \frac{\partial V_2}{\partial W_1} & \frac{\partial V_2}{\partial W_2} \end{pmatrix} \begin{pmatrix} \frac{\partial W_1}{\partial x} & \frac{\partial W_1}{\partial y} \\ \frac{\partial W_2}{\partial x} & \frac{\partial W_2}{\partial y} \end{pmatrix}, \end{aligned}$$

as required.

62 *Adapted from exam question 2018 (Section B) Q8:*

(a) *Given a vector field $\underline{u}(\underline{x}) = \underline{w}(\underline{v}(\underline{x}))$, use the chain rule to show that $D\underline{u}(\underline{x}) = D\underline{w}(\underline{v})D\underline{v}(\underline{x})$, and hence $J(\underline{u}) = J(\underline{w})J(\underline{v})$.*

(b) *Let*

$$\begin{aligned}\underline{v}(\underline{x}) &= (v_1, v_2) = (\cos y, \sin x)\\ \underline{w}(\underline{x}) &= (w_1, w_2) = (x^2 + y^3, x^2 y),\end{aligned}$$

and define $\underline{u}(\underline{x}) = \underline{w}(\underline{v}(\underline{x}))$. Use the result from part (a) to calculate $J(\underline{u})$. Verify your answer by direct substitution.

Solution:

(a) This follows from problem 61 part (c), but explicitly we have

$$\begin{aligned}D\underline{u}(\underline{x}) &= \begin{pmatrix} \frac{\partial w_1(\underline{v})}{\partial x} & \frac{\partial w_1(\underline{v})}{\partial y} \\ \frac{\partial w_2(\underline{v})}{\partial x} & \frac{\partial w_2(\underline{v})}{\partial y} \end{pmatrix}\\
&= \begin{pmatrix} \frac{\partial w_1}{\partial v_1}\frac{\partial v_1}{\partial x} + \frac{\partial w_1}{\partial v_2}\frac{\partial v_2}{\partial x} & \frac{\partial w_1}{\partial v_1}\frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial v_2}\frac{\partial v_2}{\partial y} \\ \frac{\partial w_2}{\partial v_1}\frac{\partial v_1}{\partial x} + \frac{\partial w_2}{\partial v_2}\frac{\partial v_2}{\partial x} & \frac{\partial w_2}{\partial v_1}\frac{\partial v_1}{\partial y} + \frac{\partial w_2}{\partial v_2}\frac{\partial v_2}{\partial y} \end{pmatrix}\\
&= \begin{pmatrix} \frac{\partial w_1}{\partial v_1} & \frac{\partial w_1}{\partial v_2} \\ \frac{\partial w_2}{\partial v_1} & \frac{\partial w_2}{\partial v_2} \end{pmatrix} \begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} \end{pmatrix}\\
&= D\underline{w}(\underline{v})D\underline{v}(\underline{x}).\end{aligned}$$

Now by taking determinants of both sides we have

$$\begin{aligned}|D\underline{u}(\underline{x})| &= |D\underline{w}(\underline{v})D\underline{v}(\underline{x})|\\ J(\underline{u}) &= |D\underline{w}(\underline{v})|\,|D\underline{v}(\underline{x})|\\ &= J(\underline{w})J(\underline{v}),\end{aligned}$$

as required.

(b) We have

$$D\underline{v}(\underline{x}) = \begin{pmatrix} 0 & -\sin y \\ \cos x & 0 \end{pmatrix},$$

so

$$J(\underline{v}) = \cos x \sin y,$$

and

$$D\underline{w}(\underline{v}) = \begin{pmatrix} 2v_1 & 3v_2^2 \\ 2v_1 v_2 & v_1^2 \end{pmatrix},$$

and so

$$J(\underline{w}) = 2v_1(v_1^2 - 3v_2^3).$$

We therefore have

$$\begin{aligned}
J(\underline{u}) &= J(\underline{w})J(\underline{v}) \\
&= 2v_1(v_1^2 - 3v_2^3)(\cos x \sin y) \\
&= 2\cos y(\cos^2 y - 3\sin^3 x)(\cos x \sin y) \\
&= \sin 2y \cos x(\cos^2 y - 3\sin^3 x).
\end{aligned}$$

Checking by direct substitution gives

$$\underline{u}(\underline{x}) = \underline{w}(\underline{v}(\underline{x})) = (\cos^2 y + \sin^3 x, \cos^2 y \sin x),$$

and hence

$$D\underline{u}(\underline{x}) = \begin{pmatrix} 3\cos x \sin^2 x & -2\cos y \sin y \\ \cos x \cos^2 y & -2\sin y \cos y \sin x \end{pmatrix},$$

and therefore

$$\begin{aligned}
J(\underline{u}) &= -6\sin^3 x \cos x \sin y \cos y + 2\cos x \cos^3 y \sin y \\
&= 2\cos x \sin y \cos y(\cos^2 y - 3\sin^3 x) \\
&= \sin 2y \cos x(\cos^2 y - 3\sin^3 x),
\end{aligned}$$

which agrees with the previous calculation.

63 *(a) Let $\underline{v} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. Give the definition of $\underline{V}$ being differentiable on an open set $U \subseteq \mathbb{R}^n$.*

(b) For $\underline{x} = x\underline{e}_1 + y\underline{e}_2$, let $\underline{v} : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\underline{v}(\underline{x}) = (x^2 + y^2, x + y).$$

Using the definition of differentiability, show that $\underline{v}$ is differentiable on $\mathbb{R}^2$.

(c) Draw a diagram to show where $\underline{v}$ defines an orientation preserving local diffeomorphism (on $U \subseteq \mathbb{R}^2$), and where $\underline{v}$ defines an orientation reversing local diffeomorphism (on $V \subseteq \mathbb{R}^2$).

Solution:

(a) A vector field $\underline{v} : U \to \mathbb{R}^n$, with $U \subseteq \mathbb{R}^n$ open is differentiable at a point $\underline{a} \in U$ if $\exists$ a linear function $\underline{L} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\underline{v}(\underline{a} + \underline{h}) - \underline{v}(\underline{a}) = \underline{L}(\underline{h}) + \underline{R}(\underline{h})$$

$$\text{with} \qquad \lim_{\underline{h}\to\underline{0}} \frac{\underline{R}(\underline{h})}{|\underline{h}|} = \underline{0}.$$

A vector field is differentiable on an open set $U \subseteq \mathbb{R}^n$ if it is differentiable at every point $\underline{a} \in U$.

(b) We know from the lecture notes, that when $\underline{L}(\underline{h})$ exists, it is given by $D\underline{v}_{\underline{a}} \cdot \underline{h}$. In this case we have

$$D\underline{v}_{\underline{a}} = \begin{pmatrix} 2a_1 & 2y_1 \\ 1 & 1 \end{pmatrix},$$

and so, for any $\underline{a} \in \mathbb{R}^2$,

$$\begin{aligned} \underline{R}(\underline{h}) &= \underline{v}(\underline{a}+\underline{h}) - \underline{v}(\underline{a}) - D\underline{v}_{\underline{a}} \cdot h \\ &= \left((a_1+h_1)^2 + (a_2+h_2)^2, a_1+h_1+a_2+h_2\right)^t \\ &\quad - \left(a_1^2+a_2^2, a_1+a_2\right)^t - (2a_1h_1+2a_2h_2, h_1+h_2)^t \\ &= (h_1^2+h_2^2, 0)^t. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{\underline{h}\to\underline{0}} \frac{\underline{R}(\underline{h})}{|\underline{h}|} &= \lim_{\underline{h}\to\underline{0}} \left(\frac{h_1^2+h_2^2}{|\underline{h}|}, \frac{0}{|\underline{h}|}\right) \\ &= \lim_{\underline{h}\to\underline{0}} \left(\frac{|\underline{h}|^2}{|\underline{h}|}, 0\right) \\ &= \lim_{\underline{h}\to\underline{0}} (|\underline{h}|, 0) = \underline{0}, \end{aligned}$$

and hence $\underline{v}$ is differentiable at any $\underline{a} \in \mathbb{R}^2$, and therefore $\underline{v}$ is differentiable on $\mathbb{R}^2$ as required.

(c) Since, by the previous part of this question, $\underline{v}$ is differentiable on all of $\mathbb{R}^2$, we can apply the inverse function theorem. We have that $J(\underline{v}) = |D\underline{v}_{\underline{x}}| = 2(x-y)$, and so $\underline{v}$ defines a local diffeomorphism around all points where $x \neq y$. This local diffeomorphism is orientation preserving when $x > y$, and orientation reversing when $x < y$. A diagram indicating this is given below.

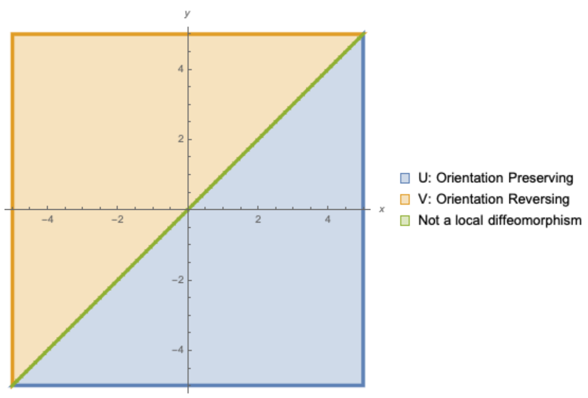

Figure 1: The vector field $\underline{v}$ defines an orientation preserving local diffeomorphism around all points in the region where $x > y$, it defines an orientation reversing local diffeomorphism around all points in the region where $y > x$, and it does not define a local diffeomorphism in open regions around points on the line $x = y$.