Calculus I, Chapter 1 Problems

Functions, domain and range

Q1. By applying the vertical line test, or otherwise, determine whether each of the following equations gives a function $y(x)$

(a) $x^2 + (y-1)^2 = 1$

(b) $y = x^2 - 2x + 1$

(c) $x + |y| = 1$

(d) $|x| + y = 1$

(e) $y^2 = 4x^2$

Solution.

Applying the vertical line test yields gives the following answers to whether the equation determines a function:

(a) no, (b) yes, (c) no, (d) yes, (e) no.

Q2. State the domain and range of each of the following functions

(a) $f(x) = |x-1| - 7$

(b) $f(x) = 5 - \sqrt{2x}$

(c) $f(x) = 2\sqrt{x^2 - 3}$

(d) $f(x) = 2x^2/(x^2 + 4)$

(e) $f(x) = -x/(x^2 - 16)$

(f) $f(x) = e^{1/(x^2-4)}$

Solution.

(a) $\textit{Dom}\, f = \mathbb{R},\ \textit{Ran}\, f = [-7, \infty)$

(b) $\textit{Dom}\, f = [0, \infty),\ \textit{Ran}\, f = (-\infty, 5]$

(c) $\textit{Dom}\, f = (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty),\ \textit{Ran}\, f = [0, \infty)$

(d) $\textit{Dom}\, f = \mathbb{R},\ \textit{Ran}\, f = [0, 2)$

(e) $\textit{Dom}\, f = \mathbb{R}\backslash\{\pm 4\},\ \textit{Ran}\, f = \mathbb{R}$

(f) $\textit{Dom}\, f = \mathbb{R}\backslash\{\pm 2\},\ \textit{Ran}\, f = (0, e^{-1/4}] \cup (1, \infty)$

Even and odd functions

Q3. Are the following functions even, odd or neither? Justify your answers.

(a) $f(x) = (x-1)(x-2)$

(b) $f(x) = \sum_{k=0}^{n} x^{2k+1}$

(c) $f(x) = \sin(x^2)$

(d) $f(x) = \frac{x}{(x^2+1)\cos x}$

Solution.

(a) $f(x) = x^2 - 3x + 2$, *so* $f(-x) = x^2 + 3x + 2$. *Since* $f(x) \neq f(-x)$ *and* $f(x) \neq -f(-x)$ *this function is neither even nor odd.*

(b) $f(-x) = \sum_{k=0}^{n}(-1)^{2k+1}x^{2k+1} = -\sum_{k=0}^{n} x^{2k+1} = -f(x)$ *hence this function is odd.*

(c) $f(-x) = \sin((-x)^2) = \sin(x^2) = f(x)$ *hence this function is even.*

(d) x *is odd, but both* x^2+1 *and* $\cos x$ *are even, hence* $f(x)$ *is the product of one odd function and two even functions and is therefore an odd function.*

Q4. Are the following functions even, odd or neither? Justify your answers.

(a) $f(x) = e^x$

(b) $g(x) = \tan x$

(c) $h(x) = xe^{-\frac{1}{2}|\log x^2|}$

(d) $k(x) = \log|x|$

(e) $p(x) = (x^3+x)/(x^3-x)$

(f) $q(x) = \sin^2(4x)$

(g) $r(x) = x^2 - 4\sin x$

Solution.

(a) $f(-x) = e^{-x}$. *Since* $f(x) \neq f(-x)$ *and* $f(x) \neq -f(-x)$ *this function is neither even nor odd.*

(b) $g(-x) = \sin(-x)/\cos(-x) = -\sin x/\cos x = -\tan x = -g(x)$ *hence this function is odd.*

(c) $h(-x) = -xe^{-\frac{1}{2}|\log(-x)^2|} = -xe^{-\frac{1}{2}|\log x^2|} = -h(x)$ *hence this function is odd.*

(d) $k(-x) = \log|-x| = \log|x| = k(x)$ *hence this function is even.*

(e) $x^3 + x$ *is an odd function and so is* $x^3 - x$ *hence* $p(x)$ *is the ratio of two odd functions and is therefore even.*

(f) $q(-x) = \sin^2(-4x) = (\sin(-4x))^2 = (-\sin(4x))^2 = \sin^2(4x)$ *hence this function is even.*

(g) $r(-x) = (-x)^2 - 4\sin(-x) = x^2 + 4\sin x$. *Since* $r(x) \neq r(-x)$ *and* $r(x) \neq -r(-x)$ *this function is neither even nor odd.*

Q5. If $f : \mathbb{R} \mapsto \mathbb{R}$ is an even function and $g : \mathbb{R} \mapsto \mathbb{R}$ is an odd function then determine whether the following functions are even, odd or neither? Justify your answers.

(a) $f_1(x) = \begin{cases} f(x) & \text{if } x > 0 \\ -f(x) & \text{if } x < 0 \end{cases}$

(b) $f_2(x) = f(x) + |f(x)|$

(c) $f_3(x) = (g \circ f)(x)$

(d) $f_4(x) = (f \circ g)(x)$

(e) $f_5(x) = (g \circ g)(x)$

Solution.

(a) On $\mathbb{R}\backslash\{0\}$

$f_1(-x) = \begin{cases} f(-x) & \textit{if } -x > 0 \\ -f(-x) & \textit{if } -x < 0 \end{cases} = \begin{cases} f(x) & \textit{if } x < 0 \\ -f(x) & \textit{if } x > 0 \end{cases} = -f_1(x)$ *hence this function is odd.*

(b) $f_2(-x) = f(-x) + |f(-x)| = f(x) + |f(x)| = f_2(x)$ *hence this function is even.*

(c) $f_3(-x) = (g \circ f)(-x) = g(f(-x)) = g(f(x)) = (g \circ f)(x) = f_3(x)$ *hence this function is even.*

(d) $f_4(-x) = (f \circ g)(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = (f \circ g)(x) = f_4(x)$ *hence this function is even.*

(e) $f_5(-x) = (g \circ g)(-x) = g(g(-x)) = g(-g(x)) = -g(g(x)) = -(g \circ g)(x) = -f_5(x)$ *hence this function is odd.*

Q6. Write each of the following functions as the sum of an even function $f_{even}(x)$ and an odd function $f_{odd}(x)$.

(a) $f(x) = x^3 - 5x + 4$

(b) $f(x) = e^{x^3}$

(c) $f(x) = \log|x + 2|$

(d) $f(x) = x^3/(x^2 - 1)$

Solution.

(a) By inspection $f_{even}(x) = 4$ *and* $f_{odd}(x) = x^3 - 5x$.

(b) $f_{even}(x) = \frac{1}{2}(f(x) + f(-x)) = \frac{1}{2}(e^{x^3} + e^{-x^3})$ *and* $f_{odd}(x) = \frac{1}{2}(f(x) - f(-x)) = \frac{1}{2}(e^{x^3} - e^{-x^3})$

(c) $f_{even}(x) = \frac{1}{2}(f(x) + f(-x)) = \frac{1}{2}(\log|2 + x| + \log|2 - x|)$ *and* $f_{odd}(x) = \frac{1}{2}(f(x) - f(-x)) = \frac{1}{2}(\log|2 + x| - \log|2 - x|)$

(d) Since $f(-x) = -x^3/(x^2 - 1) = -f(x)$ *is an odd function* $f_{even}(x) = 0$ *and* $f_{odd}(x) = f(x) = x^3/(x^2 - 1)$

Composition, inverse and injective functions

Q7. Write a formula for the functions $f \circ g$ and $g \circ f$ and find the domain and range of each, where $f(x) = x^2$ and $g(x) = 1 - \sqrt{x}$.

Solution. $(f \circ g)(x) = f(g(x)) = f(1 - \sqrt{x}) = (1 - \sqrt{x})^2 = 1 - 2\sqrt{x} + x$

Dom $(f \circ g) = [0, \infty)$, *Ran* $(f \circ g) = [0, \infty)$.

$(g \circ f)(x) = g(f(x)) = g(x^2) = 1 - \sqrt{x^2} = 1 - |x|$

Dom $(g \circ f) = \mathbb{R}$, *Ran* $(g \circ f) = (-\infty, 1]$.

Q8. Given $f(x) = x - 1$ and $g(x) = 1/(1 + x)$, find

(a) $(f \circ g)(\frac{1}{2})$

(b) $(f \circ f)(2)$

(c) $(g \circ f)(x)$

(d) $(g \circ g)(2)$

Solution.

(a) $(f \circ g)(\frac{1}{2}) = f(2/3) = -1/3$

(b) $(f \circ f)(2) = f(1) = 0$

(c) $(g \circ f)(x) = g(x - 1) = 1/x$

(d) $(g \circ g)(2) = g(1/3) = 3/4$

Q9. Given $u(x) = 2x - 3$, $v(x) = x^4$ and $f(x) = 1/x$, find $(f \circ (v \circ u))(x)$ and $(v \circ (f \circ u))(x)$

Solution.

$(f \circ (v \circ u))(x) = f(v(2x - 3)) = f((2x - 3)^4) = 1/(2x - 3)^4$

$(v \circ (f \circ u))(x) = v(f(2x - 3)) = v(1/(2x - 3)) = 1/(2x - 3)^4$

Q10. For each $f(x)$ given below, find the inverse function $f^{-1}(x)$ and identify its domain and range.

(a) $f(x) = x^5$

(b) $f(x) = x^4, \ x \geq 0$

(c) $f(x) = \frac{1}{2}x - \frac{7}{2}$

(d) $f(x) = 1/x^3, \ x \neq 0$

Solution.

Write $y = f^{-1}(x)$ *and use* $f(y) = x$.

(a) $f(y) = y^5 = x$ *hence* $y = x^{\frac{1}{5}} = f^{-1}(x)$.

Dom f^{-1} = *Ran* $f = \mathbb{R}$ *and* *Ran* f^{-1} = *Dom* $f = \mathbb{R}$.

(b) $f(y) = y^4 = x$ *hence* $y = x^{\frac{1}{4}} = f^{-1}(x)$.

Dom f^{-1} = *Ran* $f = [0, \infty)$ *and* *Ran* f^{-1} = *Dom* $f = [0, \infty)$.

(c) $f(y) = \frac{1}{2}y - \frac{7}{2} = x$ *hence* $y = 2x + 7 = f^{-1}(x)$.

Dom $f^{-1} =$ *Ran* $f = \mathbb{R}$ *and Ran* $f^{-1} =$ *Dom* $f = \mathbb{R}$.

(d) $f(y) = 1/y^3 = x$ *hence* $y = x^{-\frac{1}{3}} = f^{-1}(x)$.

Dom $f^{-1} =$ *Ran* $f = \mathbb{R}\backslash\{0\}$ *and Ran* $f^{-1} =$ *Dom* $f = \mathbb{R}\backslash\{0\}$.

Q11. Which of the following functions are injective? Find the inverses of those which are and specify the domain of the inverse.

(a) $f(x) = (1-x)^2$ in $[1, 2]$

(b) $f(x) = (x-1)/(x+2)$ in $\mathbb{R}\backslash\{-2\}$

(c) $f(x) = x^2 + 2x - 1$ in $[-1, 1]$

(d) $f(x) = x^2 + 2x - 1$ in $[-2, 2]$

Solution.

(a). It is injective. Apply horizontal line test.

Write $y = f^{-1}(x)$ *and use* $f(y) = x$. *So* $f(y) = (y-1)^2 = x$ *hence* $y = 1 + \sqrt{x} = f^{-1}(x)$.

Dom $f^{-1} =$ *Ran* $f = [0, 1]$.

(b). It is injective. Apply horizontal line test.

Write $y = f^{-1}(x)$ *and use* $f(y) = x$.

So $f(y) = (y-1)/(y+2) = x$ *hence* $y = (2x+1)/(1-x) = f^{-1}(x)$.

Dom $f^{-1} =$ *Ran* $f = \mathbb{R}\backslash\{1\}$.

(c) It is injective. Apply horizontal line test or $f(x) = x^2+2x-1 = (x+1)^2-2$ *so* $[-1, 1]$ *is only on one side of the turning point* $x = -1$ *of the quadratic. Write* $y = f^{-1}(x)$ *and use* $f(y) = x$. *So* $f(y) = (y+1)^2 - 2 = x$ *hence* $y = \sqrt{x+2} - 1 = f^{-1}(x)$.

Dom $f^{-1} =$ *Ran* $f = [-2, 2]$.

(d) It is not injective. Apply horizontal line test or eg. $f(-2) = -1 = f(0)$.

Q12. Complete the table.

$g(x)$	$f(x)$	$(f \circ g)(x)$
$1/x$		x
$\frac{1}{x-1}$	$\lvert x\rvert$	
	$\frac{x-1}{x}$	$\frac{x}{x+1}$
	$\sqrt{x}$	$\lvert x\rvert$
$\sqrt{x}$		$\lvert x\rvert$

Solution.

$g(x)$	$f(x)$	$(f \circ g)(x)$
$1/x$	$\mathbf{\frac{1}{x}}$	x
$\frac{1}{x-1}$	$\lvert x\rvert$	$\mathbf{\frac{1}{\lvert x-1\rvert}}$
$\mathbf{x+1}$	$\frac{x-1}{x}$	$\frac{x}{x+1}$
$\mathbf{x^2}$	$\sqrt{x}$	$\lvert x\rvert$
$\sqrt{x}$	$\mathbf{x^2}$	$\lvert x\rvert$

Note that we take the domains of the functions to be the maximal allowed subsets of $\mathbb{R}$ and for $f \circ g$ this is always a subset of Dom g. Also, note that knowing g and $f \circ g$ gives information about $f(x)$ only for $x \in$ Ran g. E.g. in the last line we can only determine $f(x) = x^2$ for $x \in [0, \infty)$ and also note that since we must take Ran $g = [0, \infty)$, $(f \circ g)(x)$ is only defined for $x \in [0, \infty)$ in which case $|x| = x$.

Q13. For $x \neq 0, 1$, define the following six functions

$$f_1(x) = x,\ f_2(x) = \frac{1}{x},\ f_3(x) = 1-x,\ f_4(x) = \frac{1}{1-x},\ f_5(x) = \frac{x-1}{x},\ f_6(x) = \frac{x}{x-1}.$$

These have the property that the composition of any two of these functions is again one of these functions.

Complete the following table

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1						
f_2			f_4			
f_3						
f_4						
f_5						
f_6						

Solution.

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_4	f_3	f_6	f_5
f_3	f_3	f_5	f_1	f_6	f_2	f_4
f_4	f_4	f_6	f_2	f_5	f_1	f_3
f_5	f_5	f_3	f_6	f_1	f_4	f_2
f_6	f_6	f_4	f_5	f_2	f_3	f_1