Calculus I, Background Material Problem Sheet

Example problems using assumed background material

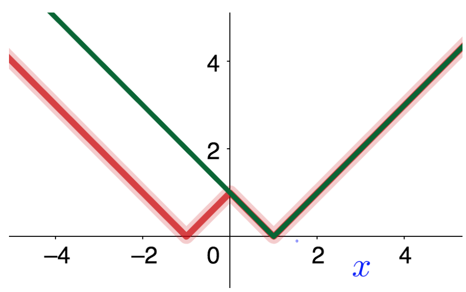
Q1. Let $f(x) = x^2 + ax + 1$, where $a \in \mathbb{R}$.
Derive the allowed values of a such that $\forall\ x \in \mathbb{R},\ f(x) = |f(x)|$.

Solution:

$f = |f|$ iff $f \geq 0$ so the required condition is equivalent to $\forall x \in \mathbb{R},\ f(x) \geq 0$.
By completing the square $f(x) = (x + \frac{1}{2}a)^2 + 1 - \frac{1}{4}a^2 \geq 0$
hence $a^2 \leq 4$ giving $a \in [-2, 2]$.

Q2. Demonstrate graphically that $||x| - 1| \leq |x - 1|,\ \forall\ x \in \mathbb{R}$. Prove this inequality (the graphical demonstration should provide a hint).

Solution:



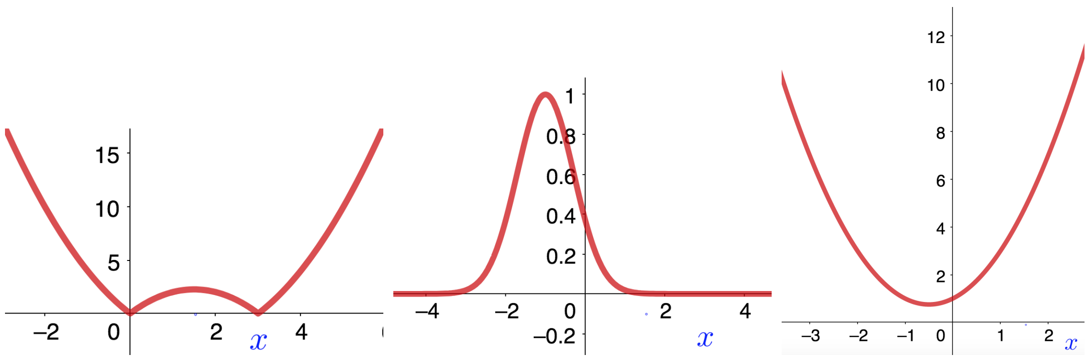
Solution for Q2

The red curve is the graph of $||x| - 1|$. This is obtained by shifting the graph of $|x|$ vertically down by 1 unit and then reflecting in the x-axis any portion of the curve below this axis.
The green curve is the graph of $|x - 1|$. This is obtained by shifting the graph of $|x|$ to the right by 1 unit.
The inequality corresponds to the fact that the red curve is never above the green curve.
Define $f(x) = |x - 1| - ||x| - 1|$. We need to show that $f(x) \geq 0,\ \forall\ x \in \mathbb{R}$.
The graph suggests we consider three regions.
If $x \geq 0$ then $f(x) = |x - 1| - |x - 1| = 0 \geq 0$.
If $-1 \leq x \leq 0$ then $f(x) = -(x - 1) - |-x - 1| = -x + 1 - (x + 1) = -2x \geq 0$.
If $x \leq -1$ then $f(x) = -(x - 1) - |-x - 1| = -x + 1 - (-x - 1) = 2 \geq 0$.
Together these three regions cover $\mathbb{R}$ hence we have shown that $f(x) \geq 0,\ \forall\ x \in \mathbb{R}$.

Q3. Graph the following functions

(a) $f(x) = |-x^2 + 3x|$, (b) $f(x) = e^{-(x+1)^2}$, (c) $f(x) = |-x^2 - x - 1|$

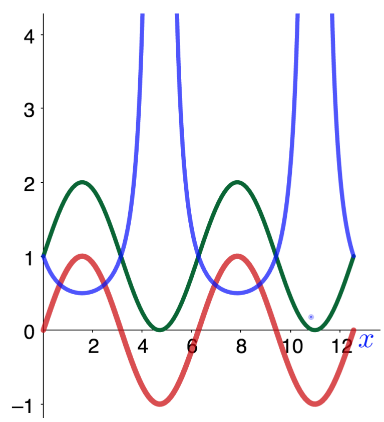
Solution:



Solution: Graph of $f(x)$ for Q3 (a),(b),(c)

Q4. For $x \in [0, 4\pi]$, on the same drawing, graph the following three functions $h(x) = \sin x$, $g(x) = 1 + \sin x$, $f(x) = 1/(1 + \sin x)$.

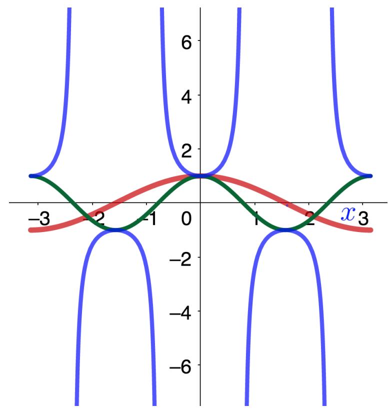
Solution:



Solution: Graphs of $h(x)$ (red), $g(x)$ (green), $f(x)$ (blue) for Q4

Q5. For $x \in [-\pi, \pi]$, on the same drawing, graph the following three functions
$h(x) = \cos x,\ g(x) = \cos(2x),\ f(x) = 1/\cos(2x).$

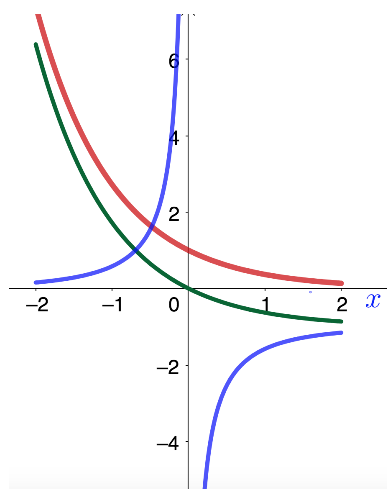
Solution:



Solution: Graphs of $h(x)$ (red), $g(x)$ (green), $f(x)$ (blue) for Q5

Q6. For $x \in [-2, 2]$, on the same drawing, graph the following three functions
$h(x) = e^{-x},\ g(x) = e^{-x} - 1,\ f(x) = 1/(e^{-x} - 1)$

Solution:



Solution: Graphs of $h(x)$ (red), $g(x)$ (green), $f(x)$ (blue) for Q6

Q7. State any vertical and horizontal asymptotes of the following functions

(a) $f(x) = -2x^2/(x^2-9)$

(b) $f(x) = 3x/(x^2+1)$

(c) $f(x) = x^3/(x^2+2)$

(d) $f(x) = \frac{3x^5+5x^2}{(x-1)(7x^4+9)}$

Solution:

(a) Vertical asymptotes $x = \pm 3$, horizontal asymptote $y = -2$.

(b) Horizontal asymptote $y = 0$.

(c) No asymptotes.

(d) Vertical asymptote $x = 1$, horizontal asymptote $y = 3/7$.

Q8. A rational function is a ratio of two polynomials and is called proper if the degree of the numerator is less than the degree of the denominator. Write each of the following rational functions as the sum of a polynomial and a proper rational function

$$\text{(a)} \quad \frac{x^6+x^4-x^3+2x^2+5}{x^2+1},$$

$$\text{(b)} \quad \frac{x^6+2x^4+3x^2+5}{x^4+2},$$

$$\text{(c)} \quad \frac{2x^5+4x^3-1}{2x^3-1},$$

$$\text{(d)} \quad \frac{x^7+6x^4+8x+1}{x^3+4}.$$

Solution:

(a) $\frac{x^6+x^4-x^3+2x^2+5}{x^2+1} = x^4 - x + 2 + \frac{x+3}{x^2+1}$

(b) $\frac{x^6+2x^4+3x^2+5}{x^4+2} = x^2 + 2 + \frac{x^2+1}{x^4+2}$

(c) $\frac{2x^5+4x^3-1}{2x^3-1} = x^2 + 2 + \frac{x^2+1}{2x^3-1}$

(d) $\frac{x^7+6x^4+8x+1}{x^3+4} = x^4 + 2x + \frac{1}{x^3+4}$

Q9. Evaluate each of the following expressions (without using a calculator). Here $\log$ denotes the natural logarithm, $\log_e$, sometimes denoted $\ln$.

(a) $\log e$, (b) $\log_3 \frac{1}{9}$, (c) $\log_{\frac{1}{16}} 4$, (d) $\log_8 8^{-3}$, (e) $\log_5 625$

(f) $\log_{10} 10^n$, $n \in \mathbb{Z}$, (g) $\frac{\log(ne)}{m \log n + \log e^m}$, $n, m > 0$

Solution:

(a) 1. (b) -2. (c) $-\frac{1}{2}$.

(d) -3. (e) 4. (f) n. (g) $\frac{\log(ne)}{m\log n+\log e^m} = \frac{\log n+1}{m\log n+m} = \frac{1}{m}$.

Q10. Find the Cartesian equation for the tangent to the graph of $f : \mathbb{R} \mapsto \mathbb{R} : x \mapsto 5x^2 - 4x$ at the point $(1, 1)$.

Solution:

$f(x) = 5x^2 - 4x$, hence $f'(x) = 10x - 4$, with $f'(1) = 6$. The Cartesian equation for the tangent at $(1, 1)$ is $y = f(1) + f'(1)(x - 1) = 1 + 6(x - 1) = 6x - 5$.

Q11. Find the slope of the straight line which passes through the point $(-2, 0)$ and is also tangential to the graph of $f(x) = \sqrt{x}$ at some point.

Solution:

Let $x = a > 0$ be the point at which the line is tangential to the graph of $f(x) = \sqrt{x}$. As $f'(x) = 1/(2\sqrt{x})$ then the tangent line is given by
$y = f(a) + f'(a)(x - a) = \sqrt{a} + \frac{1}{2\sqrt{a}}(x - a)$. For $(-2, 0)$ to be on this point requires that $0 = \sqrt{a} + \frac{1}{2\sqrt{a}}(-2 - a)$, with solution $a = 2$. The slope is then $f'(2) = 1/(2\sqrt{2})$.

Q12. Evaluate the following definite integrals

$$\text{(a)} \int_{-4}^{-2} (x + 3)^{14}\, dx, \qquad \text{(b)} \int_1^2 \frac{6 - t}{t^3}\, dt, \qquad \text{(c)} \int_0^{\frac{1}{2}} \frac{8}{1 + 4q^2}\, dq.$$

Solution:

(a) Put $u = x + 3$ then $\int_{-4}^{-2}(x + 3)^{14}\, dx = \int_{-1}^{1} u^{14}\, du = \left[\frac{u^{15}}{15}\right]_{-1}^{1} = \frac{2}{15}$.

(b) $\int_1^2 \frac{6-t}{t^3}\, dt = \int_1^2 (6t^{-3} - t^{-2})\, dt = \left[-3t^{-2} + t^{-1}\right]_1^2 = -\frac{3}{4} + \frac{1}{2} + 3 - 1 = \frac{7}{4}$.

(c) Put $2q = \tan u$ then $2dq = \sec^2 u\, du$ and $\int_0^{\frac{1}{2}} \frac{8}{1+4q^2}\, dq = \int_0^{\pi/4} 4\, du = \left[4u\right]_0^{\pi/4} = \pi$.

Q13. Calcuate the following indefinite integrals

$$\text{(a)} \int \frac{x}{\sqrt{2 + 3x^2}}\, dx, \qquad \text{(b)} \int \cot x\, dx, \qquad \text{(c)} \int \frac{1}{1 + \sqrt{x}}\, dx.$$

Solution:

(a) Put $u = 2 + 3x^2$ then $du = 6x\, dx$
$\int \frac{x}{\sqrt{2+3x^2}}\, dx = \int \frac{1}{6}u^{-1/2}\, du = \frac{1}{3}u^{1/2} + c = \frac{1}{3}\sqrt{2 + 3x^2} + c$.

(b) Put $u = \sin x$ then $du = \cos x\, dx$

$\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx = \int \frac{du}{u} = \log|u| + c = \log|\sin x| + c.$

(c) Put $u = \sqrt{x}$ then $du = \frac{1}{2\sqrt{x}}\, dx$
$\int \frac{1}{1+\sqrt{x}}\, dx = \int \frac{2u}{1+u}\, du = \int (2 - \frac{2}{1+u})\, du = 2u - 2\log|1+u| + c = 2\sqrt{x} - 2\log|1+\sqrt{x}| + c.$

Q14. Calculate the following integrals

(a) $\displaystyle\int x\sqrt{1+x}\, dx,$ (b) $\displaystyle\int x^2 \cos x\, dx,$ (c) $\displaystyle\int x^n \log x\, dx,$ where n is a positive integer,

(d) $\displaystyle\int e^x \sin(3x)\, dx,$ (e) $\displaystyle\int e^{-x} \sinh x\, dx.$

Solution:

(a) $\int x\sqrt{1+x}\, dx = \frac{2}{3}x(1+x)^{3/2} - \int \frac{2}{3}(1+x)^{3/2}\, dx = \frac{2}{3}x(1+x)^{3/2} - \frac{4}{15}(1+x)^{5/2} + c$

(b) $\int x^2 \cos x\, dx = x^2 \sin x - \int 2x \sin x\, dx = x^2 \sin x + 2x\cos x - \int 2\cos x\, dx$
$= x^2 \sin x + 2x\cos x - 2\sin x\ + c.$

(c) $\int x^n \log x\, dx = \log x\, \frac{x^{n+1}}{n+1} - \int \frac{1}{x}\frac{x^{n+1}}{n+1}\, dx = \log x\, \frac{x^{n+1}}{n+1} - \frac{x^{n+1}}{(n+1)^2}\ + c.$

(d) $\int e^x \sin(3x)\, dx = -\frac{1}{3}e^x\cos(3x) + \int \frac{1}{3}e^x \cos(3x)\, dx$
$= -\frac{1}{3}e^x\cos(3x) + \frac{1}{9}e^x \sin(3x) - \int \frac{1}{9}e^x \sin(3x)\, dx$ and hence
$\int e^x \sin(3x)\, dx = \frac{1}{10}e^x(-3\cos(3x) + \sin(3x)) + c.$

(e) $\int e^{-x}\sinh x\, dx = \int \frac{1}{2}e^{-x}(e^x - e^{-x}) = \int (\frac{1}{2} - \frac{1}{2}e^{-2x})\, dx = \frac{1}{2}x + \frac{1}{4}e^{-2x} + c.$

Q15. Write the following in partial fraction form

(a) $\displaystyle\frac{7x^2 - x - 2}{(x^2-1)(2x-1)},$ (b) $\displaystyle\frac{7-2x}{(x+1)(x-2)^2},$ (c) $\displaystyle\frac{8}{(x-1)^2(x+1)(x^2+1)}.$

Solution:

$$\text{(a)}\ \frac{7x^2 - x - 2}{(x^2-1)(2x-1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x-1}$$

$$A = \frac{7x^2 - x - 2}{(x+1)(2x-1)}\bigg|_{x=1} = 2, \quad B = \frac{7x^2 - x - 2}{(x-1)(2x-1)}\bigg|_{x=-1} = 1, \quad C = \frac{7x^2 - x - 2}{(x-1)(x+1)}\bigg|_{x=1/2} = 1.$$

$$\text{(b)}\ \frac{7-2x}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$A = \frac{7-2x}{(x-2)^2}\bigg|_{x=-1} = 1, \quad C = \frac{7-2x}{x+1}\bigg|_{x=2} = 1,$$

Evaluating at $x = 0$ gives $\frac{7}{4} = 1 - \frac{B}{2} + \frac{1}{4}$ hence $B = -1$.

$$\text{(c)}\ \frac{8}{(x-1)^2(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{Dx+E}{x^2+1}$$

$$B = \left.\frac{8}{(x+1)(x^2+1)}\right|_{x=1} = 2, \quad C = \left.\frac{8}{(x-1)^2(x^2+1)}\right|_{x=-1} = 1,$$

Evaluating at $x = 0$ gives $E = A + 5$.
Evaluating at $x = 2$ gives $3A + D = -7$. Evaluating at $x = -2$ gives $A + 3D = 3$.
Hence $D = 2, A = -3, E = 2$.

$$\frac{8}{(x-1)^2(x+1)(x^2+1)} = \frac{-3}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{x+1} + \frac{2x+2}{x^2+1}.$$

Q16. Calculate the following integrals

$$\text{(a)} \int \frac{1+x^4}{x^2-x}\,dx, \quad \text{(b)} \int \frac{x^2+1}{x(x^2-1)}\,dx, \quad \text{(c)} \int \frac{1}{x^3+x^5}\,dx.$$

Solution:

$$\text{(a)}\ \frac{1+x^4}{x^2-x} = x^2 + x + 1 + \frac{1+x}{x^2-x} = x^2 + x + 1 - \frac{1}{x} + \frac{2}{x-1}$$

$$\int \frac{1+x^4}{x^2-x}\,dx = \int x^2+x+1-\frac{1}{x}+\frac{2}{x-1}\,dx = \frac{1}{3}x^3+\frac{1}{2}x^2+x-\log|x|+2\log|x-1|+c.$$

$$\text{(b)}\ \frac{x^2+1}{x(x^2-1)} = -\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1}$$

$$\int \frac{x^2+1}{x(x^2-1)}\,dx = \int -\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1}\,dx = -\log|x| + \log|x-1| + \log|x+1| + c.$$

$$\text{(c)}\ \frac{1}{x^3+x^5} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1}.$$

$1 = C + Bx + (A+C)x^2 + (B+E)x^3 + (A+D)x^4$ hence $C = 1, B = 0, A = -1, E = 0, D = 1$.

$$\int \frac{1}{x^3+x^5}\,dx = \int -\frac{1}{x} + \frac{1}{x^3} + \frac{x}{x^2+1}\,dx = -\log|x| - \frac{1}{2x^2} + \frac{1}{2}\log(x^2+1) + c.$$

Q17. Show that $\cosh^2 x - \sinh^2 x = 1$.

Solution:

$\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$ hence

$$\cosh^2 x - \sinh^2 x = \frac{1}{4}\Big((e^x+e^{-x})^2 - (e^x-e^{-x})^2\Big) = \frac{4e^xe^{-x}}{4} = 1.$$

Q18. Differentiate $\cosh x$.

Solution:

$\cosh x = (e^x + e^{-x})/2$ hence $\frac{d}{dx}\cosh x = (e^x - e^{-x})/2 = \sinh x$.

Q19. Differentiate $\tanh x$.

Solution:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

hence

$$\frac{d}{dx}\tanh x = \frac{e^x + e^{-x}}{e^x + e^{-x}} - \frac{e^x - e^{-x}}{(e^x + e^{-x})^2}(e^x - e^{-x}) = \frac{1}{(e^x + e^{-x})^2}((e^x + e^{-x})^2 - (e^x - e^{-x})^2)$$

$$= \frac{4}{(e^x + e^{-x})^2} = \frac{1}{((e^x + e^{-x})/2)^2} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x.$$