Calculus I, Chapter 3 Problems

Differentiable functions

Q1. Use the limit definition of the derivative to calculate the derivative of the following functions

(a) $f(x) = \sin x$, (b) $f(x) = x\sqrt{x}$, (c) $f(x) = \cos^2 x$.

Solution.

(a) $f'(x) = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{\sin x\cos h+\cos x\sin h-\sin x}{h} = \lim_{h\to 0} \frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$
$= -\sin x \lim_{h\to 0} \frac{1-\cos h}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} = 0\sin x + 1\cos x = \cos x.$

(b) $f'(x) = \lim_{h\to 0} \frac{(x+h)\sqrt{x+h}-x\sqrt{x}}{h} = \lim_{h\to 0} \frac{\left((x+h)\sqrt{x+h}-x\sqrt{x}\right)\left((x+h)\sqrt{x+h}+x\sqrt{x}\right)}{h\left((x+h)\sqrt{x+h}+x\sqrt{x}\right)}$

$= \lim_{h\to 0} \frac{(x+h)^3-x^3}{h\left((x+h)\sqrt{x+h}+x\sqrt{x}\right)} = \lim_{h\to 0} \frac{3x^2+3xh+h^2}{(x+h)\sqrt{x+h}+x\sqrt{x}} = \frac{3x^2}{2x\sqrt{x}} = \frac{3}{2}\sqrt{x}.$

(c) $f'(x) = \lim_{h\to 0} \frac{\cos^2(x+h)-\cos^2 x}{h} = \lim_{h\to 0} \frac{(\cos x\cos h-\sin x\sin h)^2-\cos^2 x}{h}$
$= \lim_{h\to 0} \frac{\cos^2 x(\cos^2 h-1)-2\sin x\cos x\sin h\cos h+\sin^2 x\sin^2 h}{h}$
$= \lim_{h\to 0} \frac{(\sin^2 x-\cos^2 x)\sin^2 h-2\sin x\cos x\sin h\cos h}{h} = \lim_{h\to 0} \frac{(\sin^2 x-\cos^2 x)(1-\cos(2h))-2\sin x\cos x\sin(2h)}{2h}$
$= 0(\sin^2 x - \cos^2 x) + 1(-2\sin x\cos x) = -2\sin x\cos x.$

Q2. Show that if $g(x)$ is continuous at $x = 0$ then $g(x)\tan x$ is differentiable at $x = 0$.

Solution. *Let* $f(x) = g(x)\tan x$ *then we need to show that* $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ *exists.*

$\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{g(h)\tan h-g(0)\tan 0}{h} = \lim_{h\to 0} \frac{g(h)\tan h}{h}$

$= \left(\lim_{h\to 0} g(h)\right)\left(\lim_{h\to 0} \frac{1}{\cos h}\right)\left(\lim_{h\to 0} \frac{\sin h}{h}\right) = g(0)(1)(1) = g(0)$

where we have made use of the continuity of $g(x)$ *and* $1/\cos x$ *at* $x = 0$.

Hence $f(x) = g(x)\tan x$ *is differentiable at* $x = 0$ *with* $f'(0) = g(0)$.

Q3. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a differentiable function that satisfies $2f(x) + e^{x^2 f(x)} - \sin f(x) = 1$ and has a continuous derivative. Find $f(0)$ and $f'(0)$.

Solution.

Evaluating the given equation at $x = 0$ *yields* $2f(0) + 1 - \sin f(0) = 1$, *that is*

$2f(0) = \sin f(0)$. *The only solution of this equation is* $f(0) = 0$.

Differentiating the given equation with respect to x *gives*

$2f'(x) + (2xf(x) + x^2 f'(x))e^{x^2 f(x)} - f'(x)\cos x = 0$, *and after setting* $x = 0$ *this becomes*

$2f'(0) - f'(0) = 0$, *that is,* $f'(0) = 0$.

Q4. Explicitly write out the Leibniz rule for $\frac{d^4}{dx^4}(f(x)g(x))$ and use this to calculate the fourth derivative of $x^4\cos x$.

Solution. $\frac{d^4}{dx^4}(f(x)g(x)) = f^{(4)}(x)g(x)+4f'''(x)g'(x)+6f''(x)g''(x)+4f'(x)g'''(x)+f(x)g^{(4)}(x)$.

$f(x)=x^4,\ f'(x)=4x^3,\ f''(x)=12x^2,\ f'''(x)=24x,\ f^{(4)}(x)=24.$

$g(x)=\cos x,\ g'(x)=-\sin x,\ g''(x)=-\cos x,\ g'''(x)=\sin x,\ g^{(4)}(x)=\cos x.$

$\frac{d^4}{dx^4}(x^4\cos x)=(24-72x^2+x^4)\cos x+(-96x+16x^3)\sin x.$

Q5. Given $f(x)=4x+3$ and $g(x)=1/(4+x^2)^2$, find $(f\circ g)(x)$ and $(g\circ f)(x)$. Calculate $(f\circ g)'(0)$ and $(g\circ f)'(0)$.

Solution.

$(f\circ g)(x)=f(1/(4+x^2)^2)=\frac{4}{(4+x^2)^2}+3, \qquad (g\circ f)(x)=g(4x+3)=\frac{1}{(4+(4x+3)^2)^2}.$

$f'(x)=4$ *and* $g'(x)=\frac{-4x}{(4+x^2)^3}$, *hence* $(f\circ g)'(x)=f'(g(x))g'(x)=\frac{-16x}{(4+x^2)^3}$, *giving* $(f\circ g)'(0)=0$.

$(g\circ f)'(x)=g'(f(x))f'(x)=\frac{-16(4x+3)}{(4+(4x+3)^2)^3}$, *giving* $(g\circ f)'(0)=-48/13^3=-48/2197$.

Q6. Use L'Hopital's rule to calculate the limit as $x\to 0$ of the following

(a) $\frac{1-\cos 2x}{x}$, (b) $\frac{1-\cos x}{x^2}$, (c) $\frac{\tan 2x}{x}$, (d) $\frac{x^2}{1-\cos 2x}$, (e) $\frac{x^2}{1-\cos 4x}$.

Solution.

(a) $f(x)=1-\cos(2x),\ g(x)=x$, *are differentiable and satisfy* $f(0)=g(0)=0$.

$f'(x)=2\sin(2x),\ f'(0)=0,\ g'(x)=1\neq 0.$

$\lim_{x\to 0}f(x)/g(x)=\lim_{x\to 0}f'(x)/g'(x)=f'(0)/g'(0)=0/1=0.$

(b) $f(x)=1-\cos x,\ g(x)=x^2$, *are twice differentiable and satisfy* $f(0)=g(0)=0$.

$f'(x)=\sin x,\ f'(0)=0,\ g'(x)=2x,\ g'(0)=0.$

$f''(x)=\cos x,\ f''(0)=1,\ g''(x)=2\neq 0.$

$\lim_{x\to 0}f(x)/g(x)=\lim_{x\to 0}f'(x)/g'(x)=\lim_{x\to 0}f''(x)/g''(x)=f''(0)/g''(0)=1/2.$

(c) $f(x)=\tan(2x),\ g(x)=x$, *are differentiable and satisfy* $f(0)=g(0)=0$.

$f'(x)=2\sec^2(2x),\ f'(0)=2,\ g'(x)=1\neq 0.$

$\lim_{x\to 0}f(x)/g(x)=\lim_{x\to 0}f'(x)/g'(x)=f'(0)/g'(0)=2/1=2.$

(d) $f(x)=x^2,\ g(x)=1-\cos(2x)$, *are twice differentiable and satisfy* $f(0)=g(0)=0$.

$f'(x)=2x,\ f'(0)=0,\ g'(x)=2\sin(2x),\ g'(0)=0.$

$f''(x)=2,\ g''(x)=4\cos(2x)\neq 0$ *for* x *sufficiently close to* $x=0$. *Also,* $g''(0)=4$.

$\lim_{x\to 0}f(x)/g(x)=\lim_{x\to 0}f'(x)/g'(x)=\lim_{x\to 0}f''(x)/g''(x)=f''(0)/g''(0)=\frac{2}{4}=\frac{1}{2}.$

(e) $f(x)=x^2,\ g(x)=1-\cos(4x)$, *are twice differentiable and satisfy* $f(0)=g(0)=0$.

$f'(x)=2x,\ f'(0)=0,\ g'(x)=4\sin(4x),\ g'(0)=0.$

$f''(x)=2,\ g''(x)=16\cos(4x)\neq 0$ *for* x *sufficiently close to* $x=0$. *Also,* $g''(0)=16$.

$\lim_{x\to 0}f(x)/g(x)=\lim_{x\to 0}f'(x)/g'(x)=\lim_{x\to 0}f''(x)/g''(x)=f''(0)/g''(0)=\frac{1}{8}.$

Q7. Find an expression for $\frac{dy}{dx}$ in terms of x and y in the following cases

(a) $xy^2 - 4x^{3/2} - y = 0$, (b) $x + \sin y = xy$,

(c) $(3xy+7)^2 = 6y$, (d) $x + \tan(xy) = 0$, (e) $\cosh x + \sinh(xy) = 0$.

Solution.

(a) $y^2 + 2xyy' - 6\sqrt{x} - y' = 0$ *hence* $y' = (6\sqrt{x} - y^2)/(2xy-1)$.

(b) $1 + y'\cos y = y + xy'$ *hence* $y' = (y-1)/(\cos y - x)$.

(c) $2(3xy+7)(3y+3xy') = 6y'$ *hence* $y' = y(3xy+7)/(1-7x-3x^2y)$.

(d) $1 + \sec^2(xy)(y+xy') = 0$ *hence* $y' = -(\cos^2(xy)+y)/x$.

(e) $\sinh x + \cosh(xy)(y+xy') = 0$ *hence* $y' = -\frac{1}{x}(y + \sinh x/\cosh(xy))$.

Q8. In each of the following cases, assume that y is a differentiable function of x and satisfies the given equation. Calculate $\frac{dy}{dx}$ at the given point.

(a) $xy + y^2 - 3x - 3 = 0$, $(-1,1)$.

(b) $xe^y + \sin(xy) + y = \log 2$, $(0, \log 2)$.

Solution.

(a) $y + xy' + 2yy' - 3 = 0$ *at* $(x,y) = (-1,1)$ *this becomes* $1 - y' + 2y' - 3 = 0$ *so* $y' = 2$.

(b) $e^y + xy'e^y + (y+xy')\cos(xy) + y' = 0$ *at* $(x,y) = (0,\log 2)$ *this becomes* $2 + \log 2 + y' = 0$ *so* $y' = -2 - \log 2$.

Extreme values

Q9. Find the global extreme values of $f(x) = \frac{1}{3}x^3 - 3x + |x^2-4|$ in $[-2,4]$.

Solution. *Note that* $f(x) = \frac{x^3}{3} - 3x + |x^2-4|$ *is differentiable everywhere except at* $x = \pm 2$.

For $x \in (-2,2)$, $f(x) = \frac{x^3}{3} - 3x + 4 - x^2$ *with*

$f'(x) = x^2 - 3 - 2x = (x-3)(x+1) = 0$ *iff* $x = -1$. $f(-1) = \frac{17}{3}$.

For $x \in (2,4)$, *then* $f(x) = \frac{x^3}{3} - 3x + x^2 - 4$ *with*

$f'(x) = x^2 - 3 + 2x = (x+3)(x-1) \neq 0$.

Now $f(2) = -\frac{10}{3}$, $f(-2) = \frac{10}{3}$, $f(4) = \frac{64}{3}$.

The global maximum is $\frac{64}{3}$ *and the global minimum is* $-\frac{10}{3}$.

Q10. Either find the global maximum or justify that it does not exist for of each of the following

(a) $f(x) = x^4 - 2x^2$ in $[\frac{1}{3}, \frac{4}{3}]$, (b) $f(x) = 1 - |1-x^2|$ in $[0, \sqrt{2}]$,

(c) $f(x) = x/(x^2+1)$ in $x \geq 0$, (d) $f(x) = x\cos(\frac{1}{x})/(x+1)$ in $x \geq 1$.

Solution.

(a) $f'(x) = 4x(x^2-1) = 0$ *in* $(\frac{1}{3}, \frac{4}{3})$ *iff* $x = 1$.

$f(1) = -1,\ f(\frac{1}{3}) = -\frac{17}{81},\ f(\frac{4}{3}) = -\frac{32}{81}$. *Global maximum is* $-\frac{17}{81}$.

(b) $f(x) = 1 - |1-x^2|$ *is differentiable in* $(0, \sqrt{2})$ *except at* $x = 1$.

For $x \in (0,1),\ f(x) = 1-(1-x^2) = x^2$, *so* $f'(x) = 2x \neq 0$.

For $x \in (1,\sqrt{2}),\ f(x) = 1+(1-x^2) = 2-x^2$, *so* $f'(x) = -2x \neq 0$.

$f(1) = 1,\ f(0) = 0,\ f(\sqrt{2}) = 0,$ *hence the global maximum is* 1.

(c) For $x > 0,\ f'(x) = (1-x^2)/(1+x^2)^2 = 0$ *iff* $x = 1$.

$f(1) = \frac{1}{2},\ f(0) = 0,\ \lim_{x\to\infty} f(x) = 0,$ *hence the global maximum is* $\frac{1}{2}$.

(d) For $x \geq 1,\ f'(x) = \frac{\cos(\frac{1}{x})}{(1+x)^2} + \frac{\sin(\frac{1}{x})}{x(1+x)} > 0$, *thus* $f(x)$ *is increasing for* $x \geq 1$.

In fact $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\{x\cos(\frac{1}{x})/(x+1)\} = \lim_{u\to 0}\frac{\cos u}{1+u} = \cos 0 = 1$.

There is no global maximum in $x \geq 1$.

Q11. A group of Chilean miners are trapped underground at a depth of 300 metres. A rescue team starts at the bottom of an abandoned mine shaft that is 600 metres West of the trapped miners and has a depth of 100 metres. The rescue team must dig a tunnel to the trapped miners that has an initial horizontal segment followed by a segment directly towards the trapped miners. At a depth of 100 metres the rock is soft and it takes only 5 minutes to dig one horizontal metre. However, at any depth below this, the rock is hard and it takes 13 minutes to dig a distance of one metre.

Calculate the minimal number of hours that it takes to tunnel to the trapped miners.

Solution.

Use length units of metres and time units of minutes. Let the horizontal tunnel have a length $600-x$, *where* $x \in [0,600]$. *Then the distance from the end of the horizontal tunnel to the trapped miners is* $\sqrt{x^2+(200)^2}$. *The time taken is* $T(x) = 5(600-x) + 13\sqrt{x^2+(200)^2}$.

$\frac{dT}{dx} = -5 + \frac{13x}{\sqrt{x^2+(200)^2}}$, *therefore* $\frac{dT}{dx} = 0$ *iff* $25(x^2+40000) = 169x^2$, *ie.* $10^6 = 144x^2$ *giving* $x = 1000/12 = 250/3$. *At this value* $T(250/3) = \frac{5}{3}(1800-250) + \frac{130}{3}(5)\sqrt{5^2+(12)^2} = \frac{5}{3}(1550+(130)13) = \frac{50}{3}(155+169) = \frac{50}{3}(324) = 5400$.

Check the endpoints: $T(0) = 5600$ *and* $T(600) = 2600\sqrt{10} > 5400$.

Thus the minimal time is $T = 5400$ *minutes ie.* $5400/60 = 90$ *hours.*

Partial derivatives

Q12. Given the function $f(x,y) = \log(1+xy)$ calculate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$.

Solution.

$$\frac{\partial f}{\partial x} = \frac{y}{1+xy}, \qquad \frac{\partial f}{\partial y} = \frac{x}{1+xy}, \qquad \frac{\partial^2 f}{\partial x^2} = \frac{-y^2}{(1+xy)^2}, \qquad \frac{\partial^2 f}{\partial y^2} = \frac{-x^2}{(1+xy)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{1}{(1+xy)^2}$$

Q13. Calculate $f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}$ for the function $f(x,y) = xe^{xy}$.

Solution.

$f_x = e^{xy}(1+xy), \qquad f_y = e^{xy}x^2, \qquad f_{xx} = e^{xy}(2y+xy^2), \qquad f_{yy} = e^{xy}x^3,$
$f_{xy} = e^{xy}(2x+x^2y).$

Q14. Show that, for any constants A and B, the function $f(x,y) = A\cos x \sinh y + B\sin x \cosh y$ satisfies the equation $f_{xx} + f_{yy} = 0$.

Solution.

$f_x = -A\sin x \sinh y + B\cos x \cosh y, \qquad f_{xx} = -A\cos x \sinh y - B\sin x \cosh y$

$f_y = A\cos x \cosh y + B\sin x \sinh y, \qquad f_{yy} = A\cos x \sinh y + B\sin x \cosh y$

Therefore $f_{xx} + f_{yy} = 0$.