Calculus I, Tutorial Problem Sheet, Week 7

The fundamental theorem of calculus

Q1. Let $F(x) = \int_{\pi}^{x} t \sin t \, dt$. Calculate $F(\pi)$, $F'(x)$ and $F'(\pi/2)$.

Solution.

$F(\pi) = 0$, $F'(x) = x \sin x$ *and* $F'(\pi/2) = \pi/2$.

Q2. Let

$$F(x) = -\int_{0}^{x^2} \frac{2}{3 + e^t} \, dt.$$

Find all critical points of $F(x)$ and determine whether they are local minima, maxima or points of inflection. Prove that $F(300) > F(310)$.

Solution.

$F'(x) = -4x/(3 + e^{x^2})$ *hence* $F'(x) = 0$ *iff* $x = 0$.

For $x < 0$, $F'(x) > 0$ *whereas for* $x > 0$, $F'(x) < 0$, *hence* $x = 0$ *is a local maximum.*

As $F'(x) < 0$ *for* $x > 0$ *then* $F(x)$ *is strictly monotonic decreasing in* $(0, \infty)$

hence $F(310) < F(300)$.

Q3. Calculate the derivatives of the following functions:

$$(a)\ F(x) = \int_{x^2}^{1} (t - \sin^2 t) \, dt,$$

$$(b)\ G(t) = \int_{t^2}^{t^4} \sqrt{u} \, du.$$

Solution. *In each case we use the Fundamental Theorem of Calculus with the chain rule,*

(a) $F'(x) = -\frac{d}{dx} \int_{1}^{x^2} (t - \sin^2 t) \, dt = -2x(x^2 - \sin^2(x^2))$.

(b) $G'(t) = t^2(4t^3) - |t|2t = 4t^5 - 2t|t|$.

Integration using a recurrence relation

Q4. For integer $n \geq 0$ define

$$I_n = \int_{0}^{\pi/4} \cos^{n+1} x \, dx.$$

Find a recurrence relation between I_n and I_{n-2} and hence evaluate I_2 and I_4.

Solution.

Integration by parts is the clear way to produce a recurrence relation. There is only one obvious way to split the integrand so we have a factor we can integrate and the new integral will only involve trig functions.

$I_n = \int_0^{\pi/4} \cos^{n+1} x\, dx = \int_0^{\pi/4} \cos^n x \cos x\, dx = [\cos^n x \sin x]_0^{\pi/4} + \int_0^{\pi/4} n \cos^{n-1} x \sin^2 x\, dx$

$= \frac{1}{\sqrt{2}^{n+1}} + n \int_0^{\pi/4} (\cos^{n-1} x\,(1 - \cos^2 x))\, dx = \frac{1}{\sqrt{2}^{n+1}} + n(I_{n-2} - I_n)$ *hence*

$I_n = (\frac{1}{\sqrt{2}^{n+1}} + nI_{n-2})/(n+1).$ *From above* $I_0 = 1/\sqrt{2}$ *therefore*

$I_2 = (\frac{1}{\sqrt{2}^3} + 2I_0)/3 = 5/(6\sqrt{2})$ *and* $I_4 = (\frac{1}{\sqrt{2}^5} + 4I_2)/5 = 43/(60\sqrt{2}).$

Double integrals

Q5. Calculate $\iint\limits_D x^3 y\, dxdy,$ where D is the triangle with vertices $(0,0), (1,0), (1,1)$.

Solution.

$$\iint\limits_D x^3 y\, dxdy = \int_0^1 \left(\int_0^x x^3 y\, dy \right) dx = \int_0^1 \left[\frac{x^3 y^2}{2} \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \frac{x^5}{2}\, dx = \left[\frac{x^6}{12} \right]_0^1 = \frac{1}{12}.$$

Q6. Calculate $\iint\limits_D \sqrt{xy}\, dxdy,$

where D is the finite region between the curves $y = x$ and $y = x^2$.

Solution.

$$\iint\limits_D \sqrt{xy}\, dxdy = \int_0^1 \left(\int_{x^2}^x \sqrt{xy}\, dy \right) dx = \int_0^1 \left[\frac{2}{3} x^{\frac{1}{2}} y^{\frac{3}{2}} \right]_{y=x^2}^{y=x} dx$$

$$= \frac{2}{3} \int_0^1 (x^2 - x^{\frac{7}{2}})\, dx = \frac{2}{3} \left[\frac{x^3}{3} - \frac{2}{9} x^{\frac{9}{2}} \right]_0^1 = \frac{2}{27}.$$

Q7. Calculate

$$\int_0^{\pi/2} \left(\int_x^{\pi/2} \frac{\sin y}{y}\, dy \right) dx.$$

Solution.

The given iterated integral can be written as a double integral over the region D between the curves $y = x$ and $y = \pi/2$ for $0 \le x \le \pi/2$, hence

$$\int_0^{\pi/2} \left(\int_x^{\pi/2} \frac{\sin y}{y}\, dy \right) dx = \iint\limits_D \frac{\sin y}{y}\, dxdy = \int_0^{\pi/2} \left(\int_0^y \frac{\sin y}{y}\, dx \right) dy$$

$$= \int_0^{\pi/2} \left[\frac{x \sin y}{y} \right]_{x=0}^{x=y} dy = \int_0^{\pi/2} \sin y\, dy = \Big[-\cos y \Big]_0^{\pi/2} = 1.$$

Q8. Use polar coordinates to calculate $\iint\limits_D e^{-(x^2+y^2)}\,dxdy$, where D is the unit disc centred at the origin.

Solution.

$$\iint\limits_D e^{-(x^2+y^2)})\,dxdy = \int_0^1 \left(\int_0^{2\pi} e^{-r^2}\,d\theta\right) rdr = 2\pi \int_0^1 re^{-r^2}\,dr = -\pi\left[e^{-r^2}\right]_0^1 = \pi\left(1-\frac{1}{e}\right).$$