

Michaelmas AMV Notes

The lecture notes for this term were originally prepared by Drs. Emma Coutts and Ian Vernon and Professor Patrick Dorey, and have been edited slightly since. If you find any typos, please let me know.

Sam Fearn

Admin Details

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- Use the Topic forums (available from the Blackboard Ultra Discussions Page) to ask questions about the course. This way other students will also be able to see any answers. You can post anonymously if you prefer. If you think you can answer another student's question you should try to do so - trying to explain something to others is often the best way to learn.
- There will also be Question and Answer sessions held during the Tuesday and Thursday timetabled lecture slots. These will be available synchronously, both face-to-face in TLC042, as well as online via Zoom for those unable to attend in person.
- The lectures will all be available as videos through Blackboard Ultra. Lecture notes will be provided to go along with the video lectures. You are expected to spend time studying the lecture notes yourself as well as watching the video lectures.
- All the material for this term will be available via the Ultra course. I will try to make everything as easy to find as possible, but please also familiarise yourself with what is available.
- Tutorials: Tutorials will be held in odd numbered weeks, starting in week 3. The suggested problems for these tutorials will be available through the 'Michaelmas Tutorials' block in Ultra. You should always make sure that you have both the lecture notes and any relevant problems sheets available in your tutorials.
- Problems Classes: In even numbered teaching weeks, the Thursday 5pm timetabled slot will be a problems class rather than a lecture. These sessions will be available synchronously, both face-to-face in TLC042, as well as online via Zoom for those unable to attend in person. In these classes I will focus on working through examples rather than introducing any new material.
- Assignments: Assignments will be set weekly, and will alternate between work to be done on paper, and handed in via Gradescope, and e-assessments to be completed on the Maths Quiz Server. Further details, as well as the necessary links, are available in the Assignments block on Ultra.
- Additional Reading: The notes available in each topic, along with the lecture videos and problems classes will contain all the material that you will be examined on in this course. If you feel that you would like some additional reading to support the lecture notes then I recommend the book *Mathematical Methods for Physics and Engineering* by Riley, Hobson and Bence. This is available for you to read online via the 'Reading List', which is available as a link in Ultra.

0 Introduction

Most of this term of this course is actually about Vector Calculus. Vector calculus is a generalisation of the calculus you studied in Calculus I, which focussed mostly on functions of one or two variables, to studying functions defined from and to higher dimensional real spaces. Although the notation we use today was first introduced in the study of Electromagnetism (Maxwell, 1831-1879) in the 1860's and 1870's, the language of vector calculus is now fundamental in many parts of pure and applied mathematics, as is shown in Figure 1.

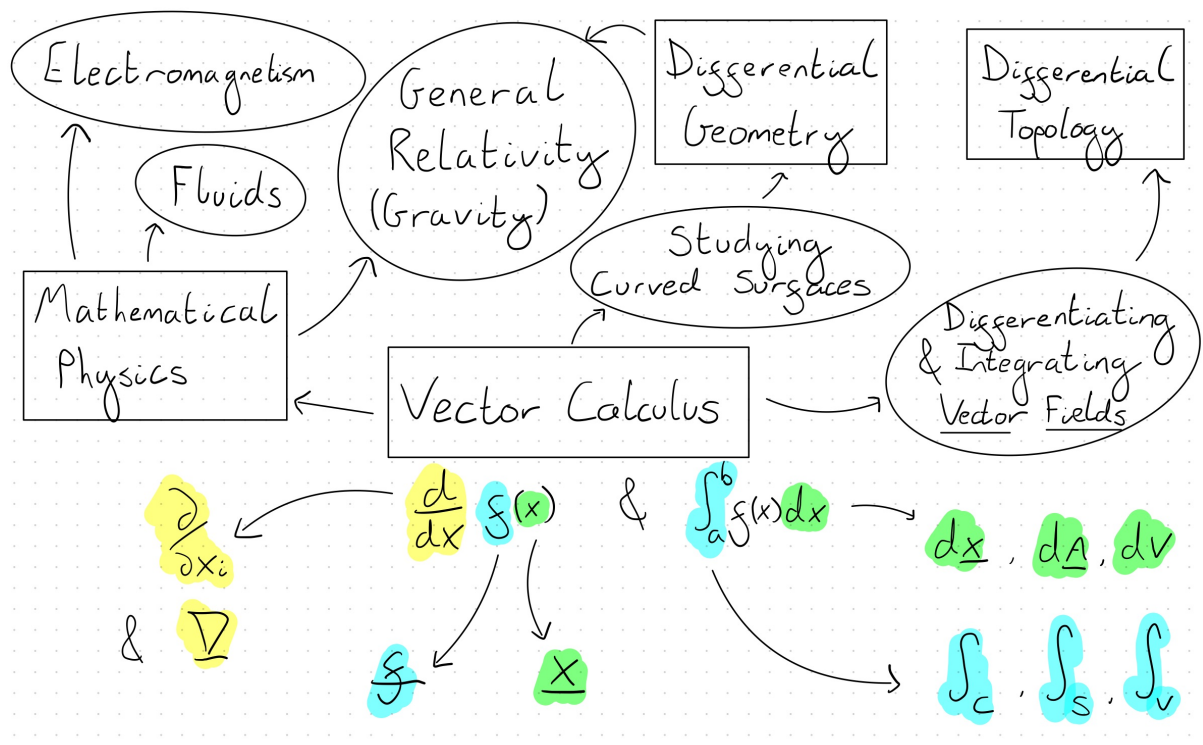


Figure 1: Not only is vector calculus fundamental for studying most topics in mathematical physics, it's also important for the study of the geometry and topology of differentiable spaces (more specifically, differentiable manifolds, which you'll meet in a later course), and the study of differential equations.

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