28 *Let* $\mathbf{x}$ *be the position vector in three dimensions, with* $r = |\mathbf{x}|$*, and let* $\mathbf{a}$ *be a constant vector. Using index notation, show that*

(a) $\operatorname{div}\mathbf{x} \;=\; 3$,

Solution:

$$\operatorname{div}\mathbf{x} = \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3\,.$$

(b) $\operatorname{curl}\mathbf{x} \;= 0$,

Solution:

$$(\operatorname{curl}\mathbf{x})_i = \varepsilon_{ijk}\frac{\partial x_k}{\partial x_j} = \varepsilon_{ijk}\delta_{jk} = 0\,,$$

using the fact that ε_{ijk} *is antisymmetric under swaps of* j *and* k *while* δ_{jk} *is symmetric.*

(c) $\operatorname{grad} r \;=\; \mathbf{x}/r$,

Solution:

$$\begin{aligned}(\operatorname{grad} r)_i = \frac{\partial}{\partial x_i}r &= \frac{\partial}{\partial x_i}(x_jx_j)^{1/2} \\ &= \frac{1}{2}(x_jx_j)^{-1/2}\,\frac{\partial}{\partial x_i}x_kx_k \\ &= \frac{1}{2}(x_jx_j)^{-1/2}\,(2\delta_{ik}x_k) \\ &= (x_jx_j)^{-1/2}\,x_i \\ &= x_i/r\,.\end{aligned}$$

(d) $\operatorname{div}(r^n\mathbf{x}) \;=\; (n+3)\,r^n$,

Solution:

$$\begin{aligned}\operatorname{div}(r^n\underline{x}) = \frac{\partial}{\partial x_i}(r^nx_i) &= (\frac{\partial}{\partial x_i}r^n)x_i + r^n(\frac{\partial}{\partial x_i}x_i) = nr^{n-1}(\frac{\partial}{\partial x_i}r)x_i + r^n\delta_{ii} \\ &= nr^{n-1}(x_i/r)\,x_i + 3r^n = nr^n + 3r^n = (n+3)r^n.\end{aligned}$$

(e) $\operatorname{grad}(\mathbf{a}\cdot\mathbf{x}) \;=\; \mathbf{a}$,

Solution:

$$(\operatorname{grad}(\underline{a}\cdot\underline{x}))_i = \frac{\partial}{\partial x_i}(a_jx_j) = a_j\frac{\partial}{\partial x_i}x_j = a_j\delta_{ij} = a_i\,.$$

(f) $\operatorname{div}(\mathbf{a}\times\mathbf{x}) = 0$,

Solution:

$$\operatorname{div}(\underline{a}\times\underline{x}) = \frac{\partial}{\partial x_i}(\varepsilon_{ijk}a_jx_k) = \varepsilon_{ijk}a_j\delta_{ik} = 0\,,$$

using the fact that ε_{ijk} *is antisymmetric under swaps of* i *and* k *while* δ_{ik} *is symmetric.*

(g) $\operatorname{curl}(\mathbf{a}\times\mathbf{x}) \;=\; 2\,\mathbf{a}$,

Solution:

$$
\begin{aligned}
(\operatorname{curl}(\underline{a}\times\underline{x}))_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\varepsilon_{klm}a_l x_m) = \varepsilon_{ijk}\varepsilon_{klm}a_l\frac{\partial}{\partial x_j}x_m \\
&= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})a_l\frac{\partial}{\partial x_j}x_m = (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})a_l\delta_{jm} \\
&= a_i\delta_{jj}-\delta_{ij}a_j = 3a_i - a_i = 2a_i\,.
\end{aligned}
$$

(h) $\operatorname{curl}(r^2\mathbf{a}) \;=\; 2\,(\mathbf{x}\,\times\mathbf{a})$,

Solution:

$$
\begin{aligned}
(\operatorname{curl}(r^2\mathbf{a}))_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}(r^2\mathbf{a}_k) = \varepsilon_{ijk}\Big(\frac{\partial}{\partial x_j}r^2\Big)\mathbf{a}_k \\
&= \varepsilon_{ijk}2r\frac{\partial r}{\partial x_j}\,\mathbf{a}_k = \varepsilon_{ijk}2r\frac{x_j}{r}\,\mathbf{a}_k = (2\,(\mathbf{x}\,\times\mathbf{a}))_i\,.
\end{aligned}
$$

(i) $\nabla^2(1/r) \;=\; 0$ *if* $r\neq 0$*: using* $\frac{\partial}{\partial x_i}r = x_i/r$ *from part (c),*

Solution:

$$
\begin{aligned}
\nabla^2(1/r) &= \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}(r^{-1}) = \frac{\partial}{\partial x_i}\big(-r^{-2}\frac{\partial}{\partial x_i}r\big) = \frac{\partial}{\partial x_i}\big(-r^{-3}x_i\big) \\
&= -\big(\frac{\partial}{\partial x_i}r^{-3}\big)x_i - r^{-3}\big(\frac{\partial}{\partial x_i}x_i\big) = 3r^{-4}\big(\frac{\partial}{\partial x_i}r\big)x_i - r^{-3}\delta_{ii} \\
&= 3r^{-5}x_ix_i - 3r^{-3} = (3-3)r^{-3} = 0\,.
\end{aligned}
$$

Notice that this is only valid for $r\neq 0$ *since the calculation involves division by* r*.*

(j) $\nabla^2(\log r) \;=\; 1/r^2$ *if* $r\neq 0$*:*

Solution: *as in part (i), and using* $\frac{\partial}{\partial x_i}r = x_i/r$ *again,*

$$
\begin{aligned}
\nabla^2(\log r) &= \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}(\log r) = \frac{\partial}{\partial x_i}\big(r^{-1}\frac{\partial}{\partial x_i}r\big) = \frac{\partial}{\partial x_i}\big(r^{-2}x_i\big) \\
&= \big(\frac{\partial}{\partial x_i}r^{-2}\big)x_i + r^{-2}\big(\frac{\partial}{\partial x_i}x_i\big) = -2r^{-3}\big(\frac{\partial}{\partial x_i}r\big)x_i + r^{-2}\delta_{ii} \\
&= -2r^{-4}x_ix_i + 3r^{-2} = (-2+3)r^{-2} = 1/r^2.
\end{aligned}
$$

Again, this is only valid for $r\neq 0$ *since the calculation involves division by* r*.*

(k) $\operatorname{div}[(\mathbf{a}\cdot\mathbf{x})\mathbf{x}] \;=\; 4\,\mathbf{a}\cdot\mathbf{x}$,

Solution:

$$
\begin{aligned}
\operatorname{div}[(\mathbf{a}\cdot\mathbf{x})\mathbf{x}] &= \frac{\partial}{\partial x_i}\big(a_jx_jx_i\big) = a_j\frac{\partial}{\partial x_i}\big(x_jx_i\big) = a_j\Big(\big(\frac{\partial}{\partial x_i}x_j\big)x_i + x_j\big(\frac{\partial}{\partial x_i}x_i\big)\Big) \\
&= a_j\big(\delta_{ij}x_i + x_j\delta_{ii}\big) = a_j\big(x_j+3x_j\big) = 4a_jx_j = 4\underline{a}\cdot\underline{x}\,.
\end{aligned}
$$

Using $\operatorname{div} f\mathbf{V} = (\nabla f)\cdot\mathbf{V} + f\operatorname{div}\mathbf{V}$*, with* $f=\mathbf{a}\cdot\mathbf{x}$ *so* $\nabla f = \mathbf{a}$ *from part (e) and* $\mathbf{V}=\mathbf{x}$ *so* $\operatorname{div}\mathbf{x} = 3$ *from part (a) hence* $\operatorname{div}[(\mathbf{a}\cdot\mathbf{x})\mathbf{x}] = \mathbf{a}\cdot\mathbf{x} + \mathbf{a}\cdot\mathbf{x}\,3 = 4\mathbf{a}\cdot\mathbf{x}$.

(l) $\operatorname{div}[\mathbf{x} \times (\mathbf{x} \times \mathbf{a})] = 2\,\mathbf{a}\cdot\mathbf{x}$,

Solution:

$$\begin{aligned}
\operatorname{div}[\mathbf{x} \times (\mathbf{x} \times \mathbf{a})] &= \frac{\partial}{\partial x_i}\left(\varepsilon_{ijk}x_j\varepsilon_{klm}x_l a_m\right) = \varepsilon_{ijk}\varepsilon_{klm}\frac{\partial}{\partial x_i}\left(x_j x_l\right)a_m \\
&= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\frac{\partial}{\partial x_i}\left(x_j x_l\right)a_m \\
&= \frac{\partial}{\partial x_i}\left(x_j x_i\right)a_j - \frac{\partial}{\partial x_i}\left(x_j x_j\right)a_i \\
&= \delta_{ij}x_i a_j + x_j\delta_{ii}a_j - \delta_{ij}x_j a_i - x_j\delta_{ij}a_i \\
&= x_j a_j + 3x_j a_j - x_j a_j - x_j a_j = 2x_j a_j = 2\underline{a}\cdot\underline{x}\,.
\end{aligned}$$

(m) $\operatorname{curl}\,(\mathbf{a}\times\mathbf{x}/r^3) = 3\,(\mathbf{a}\cdot\mathbf{x})\mathbf{x}/r^5 - \mathbf{a}/r^3$,

Solution:

$$\begin{aligned}
\left(\operatorname{curl}\,\left(\mathbf{a}\times\mathbf{x}/r^3\right)\right)_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\varepsilon_{klm}a_l x_m/r^3\right) = \varepsilon_{ijk}\varepsilon_{klm}a_l\frac{\partial}{\partial x_j}\left(x_m/r^3\right) \\
&= \varepsilon_{ijk}\varepsilon_{klm}a_l\left(\delta_{jm}r^{-3} + x_m\frac{\partial}{\partial x_j}r^{-3}\right) \\
&= \varepsilon_{ijk}\varepsilon_{klm}a_l\left(\delta_{jm}r^{-3} - 3x_m r^{-4}\frac{\partial}{\partial x_j}r\right) \\
&= \varepsilon_{ijk}\varepsilon_{klm}a_l\left(\delta_{jm}r^{-3} - 3x_m r^{-5}x_j\right) \\
&= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)a_l\left(\delta_{jm}r^{-3} - 3x_m r^{-5}x_j\right) \\
&= a_i\delta_{jj}r^{-3} - a_j\delta_{ij}r^{-3} - 3a_i x_j r^{-5}x_j + 3a_j x_i r^{-5}x_j \\
&= 3a_i r^{-3} - a_i r^{-3} - 3a_i r^{-3} + 3x_i a_j x_j r^{-5} \\
&= -a_i r^{-3} + 3x_i a_j x_j r^{-5} = \left(-\underline{a}/r^3 + 3\underline{x}(\underline{a}\cdot\underline{x})/r^5\right)_i\,.
\end{aligned}$$

Remark: in this question, and also in some of the earlier ones, there is some freedom in the order in which you, for example, use the product rule on derivatives, use the very useful formula, and simplify δs involving summed indices. In such cases any order is fine – the final answer will be the same.

(n) *Exam question June 2002 (Section A): calculate the curl of* $(\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}$.

Solution:

$$\begin{aligned}
\left(\operatorname{curl}\left((\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}\right)\right)_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(a_l x_l x_k\right) \\
&= \varepsilon_{ijk}a_l\left(\delta_{jl}x_k + x_l\delta_{jk}\right) \\
&= \varepsilon_{ijk}a_j x_k + \varepsilon_{ijj}a_l x_l = \varepsilon_{ijk}a_j x_k = \left(\underline{a}\times\underline{x}\right)_i\,,
\end{aligned}$$

using the fact that $\varepsilon_{ijj} = 0$, by the antisymmetry of the Levi-Civita symbol. Note: in the first line $\underline{a}\cdot\underline{x}$ was written using l as the repeated index rather than j or k, since j and k were already being used to write the curl.

General remark: Even though some of these index notation calculations look a little long, they are (almost) all self-contained, while in the version of this question on the Topic 3 sheet, there was much more quoting of results from other parts.

29 *The vector $\underline{a}$ has components $(a_r) = (1, 1, 1)$ and the vector $\underline{b}$ has components $(b_r) = (2, 3, 4)$. In the following expressions state which indices are free and which are dummy, and give the numerical values of the expressions for each value that the free variable takes (e.g. for $a_r - b_r$ the free variable is r and it takes the values $1,\ 2,\ 3$ so $a_1 - b_1 = -1$, $a_2 - b_2 = -2$, $a_3 - b_3 = -3$)*

(a) $a_r + b_r$,

Solution: *r is a free index so we need to specify 3 pieces of information. $a_r + b_r$ is the r-th component of the vector $\underline{a} + \underline{b}$ so we can give the information in the form $(a_r + b_r) = (3, 4, 5)$ or as $a_1 + b_1 = 3$, $a_2 + b_2 = 4$, $a_3 + b_3 = 5$.*

(b) $a_r\, b_r$,

Solution: *r is a dummy index and $a_r\, b_r = a_1\, b_1 + a_2\, b_2 + a_3\, b_3 = 9$, which is the scalar product of $\underline{a}$ and $\underline{b}$.*

(c) $a_r\, b_s\, a_r$,

Solution: *r is a dummy index, and s is free so to specify $a_r\, b_s\, a_r$ we need to specify 3 pieces of information, these are the components of the vector $\underline{b}\, \underline{a} \cdot \underline{a}$. Now $a_r\, a_r = a_1^2 + a_2^2 + a_3^2 = 3$ so $(a_r\, b_s\, a_r) = (3b_s) = (6, 9, 12)$*

(d) $a_r\, b_s\, a_r\, b_s - a_r\, b_r\, a_s\, b_s$.

Solution: *r and s are both dummy indices. $a_r\, b_s\, a_r\, b_s - a_r\, b_r\, a_s\, b_s = (a_r\, a_r)(b_s\, b_s) - (a_r\, b_r)(a_s\, b_s)$ now $a_r\, a_r = 3$ and $b_s\, b_s = 4+9+16 = 29$ so $a_r\, b_s\, a_r\, b_s - a_r\, b_r\, a_s\, b_s = 3 \times 29 - 9^2 = 6$.*

30 *If δ_{rs} is the three-dimensional Kronecker delta, evaluate*

(a) $\delta_{rs}\, \delta_{sr}\, \delta_{pq}\, \delta_{pq}$,

Solution: $\delta_{rs}\, \delta_{sr}\, \delta_{pq}\, \delta_{pq} = \delta_{rr}\, \delta_{pq}\, \delta_{pq} = 3\, \delta_{pp} = 3 \times 3 = 9$.

(b) $\delta_{rs}\, \delta_{sk}\, \delta_{kl}\, \delta_{lr}$,

Solution: $\delta_{rs}\, \delta_{sk}\, \delta_{kl}\, \delta_{lr} = \delta_{rk}\, \delta_{kl}\, \delta_{lr} = \delta_{rl}\, \delta_{lr} = \delta_{rr} = 3$.

(c) $\delta_{rs}\, \delta_{qr}\, \delta_{pq}\, \delta_{sp}$.

Solution: $\delta_{rs}\, \delta_{qr}\, \delta_{pq}\, \delta_{sp} = \delta_{qs}\, \delta_{pq}\, \delta_{sp} = \delta_{ps}\, \delta_{sp} = \delta_{ss} = 3$.

31 *If δ_{rs} is the three-dimensional Kronecker delta, simplify*

(a) $(\delta_{rp}\, \delta_{sq} - \delta_{rq}\, \delta_{sp})\, a_p\, b_q$,

Solution: $(\delta_{rp}\, \delta_{sq} - \delta_{rq}\, \delta_{sp})\, a_p\, b_q = \delta_{rp}\, \delta_{sq}\, a_p\, b_q - \delta_{rq}\, \delta_{sp}\, a_p\, b_q = a_r\, b_s - a_s\, b_r$,

(b) $(\delta_{rp}\, \delta_{sq} - \delta_{rq}\, \delta_{sp})\, \delta_{pq}$.

Solution: $(\delta_{rp}\, \delta_{sq} - \delta_{rq}\, \delta_{sp})\, \delta_{pq} = \delta_{rp}\, \delta_{sq}\, \delta_{pq} - \delta_{rq}\, \delta_{sp}\, \delta_{pq} = \delta_{rs} - \delta_{rs} = 0$.

32 *If δ_{rs} is the Kronecker delta in n dimensions, calculate*

(a) δ_{rr},

Solution: $\delta_{rr} = 1 + 1 + .. + 1$ *(n terms), so* $\delta_{rr} = n$,

(b) $\delta_{rs}\,\delta_{rs}$,

Solution: $\delta_{rs}\,\delta_{rs} = \delta_{rr} = n$,

(c) $\delta_{rs}\,\delta_{st}\,\delta_{tr}$.

Solution: $\delta_{rs}\,\delta_{st}\,\delta_{tr} = \delta_{rs}\,\delta_{sr} = \delta_{rr} = n$.

33 *Starting from* $\varepsilon_{ijk}\,\varepsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$ *simplify as much as possible:*

(a) $\varepsilon_{ijk}\varepsilon_{ijp}$,

Solution: *First use the cyclic symmetry of* ε_{ijk} *to rewrite the expression with the repeated indices next to each other:* $\varepsilon_{ijk}\varepsilon_{ijp} = \varepsilon_{jki}\varepsilon_{ijp}$. *Then, relabelling in the starting formula,*

$$\varepsilon_{jki}\varepsilon_{ijp} = \delta_{jj}\,\delta_{kp} - \delta_{jp}\,\delta_{kj} = 3\,\delta_{kp} - \delta_{kp} = 2\,\delta_{kp}$$

(b) $\varepsilon_{ijk}\varepsilon_{ijk}$.

Solution: *Substituting* $p = k$ *into part (a),*

$$\varepsilon_{ijk}\varepsilon_{ijk} = 2\,\delta_{kk} = 6$$

34 *Calculate* ε_{ijj} .

Solution: ε_{ijk} vanishes whenever two indices take the same value, so ε_{ijj} also vanishes, being a sum of terms in each of which the last two indices take the same value.

35 *Show, using index notation, that*

(a) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = 0$,

Solution: *The* i*-th component of* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ *is*

$$\begin{aligned}\varepsilon_{ijk}\,a_j\,(\varepsilon_{krs}\,b_r\,c_s) &= (\varepsilon_{ijk}\,\varepsilon_{rsk})\,a_j\,b_r\,c_s = (\delta_{ir}\,\delta_{js} - \delta_{is}\,\delta_{jr})\,a_j\,b_r\,c_s \\ &= \delta_{ir}\,\delta_{js}\,a_j\,b_r\,c_s - \delta_{is}\,\delta_{jr}\,a_j\,b_r\,c_s = a_s\,b_i\,c_s - a_r\,b_r\,c_i\end{aligned}$$

so the i*-th component of* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b})$ *is*

$$\begin{aligned}&(a_s\,b_i\,c_s - a_r\,b_r\,c_i) + (b_s\,c_i\,a_s - b_r\,c_r\,a_i) + (c_s\,a_i\,b_s - c_r\,a_r\,b_i) \\ &\quad = a_i\,(c_s\,b_s - b_r\,c_r) + b_i\,(a_s\,c_s - c_r\,a_r) + c_i\,(b_s\,a_s - a_r\,b_r) = 0\end{aligned}$$

since the names of dummy indices can be changed.

(b) $$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= [\mathbf{d}\cdot(\mathbf{a}\times\mathbf{b})]\,\mathbf{c} - [\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})]\,\mathbf{d} \\ &= [\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})]\,\mathbf{b} - [\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})]\,\mathbf{a},\end{aligned}$$

Solution: *The* i*-th component of* $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ *is*

$$\begin{aligned}\varepsilon_{ijk}\,(\varepsilon_{jrs}\,a_r\,b_s)\,(\varepsilon_{kpq}\,c_p\,d_q) &= \varepsilon_{ijk}\,\varepsilon_{jrs}\,\varepsilon_{kpq}\,a_r\,b_s\,c_p\,d_q \\ &= (\varepsilon_{jki}\,\varepsilon_{jrs})\,\varepsilon_{kpq}\,a_r\,b_s\,c_p\,d_q \\ &= \delta_{kr}\,\delta_{is}\,\varepsilon_{kpq}\,a_r\,b_s\,c_p\,d_q - \delta_{ks}\,\delta_{ir}\,\varepsilon_{kpq}\,a_r\,b_s\,c_p\,d_q \\ &= \varepsilon_{rpq}\,a_r\,b_i\,c_p\,d_q - \varepsilon_{spq}\,a_i\,b_s\,c_p\,d_q\end{aligned}$$

which is the same as the i-th component of $[\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})]\,\mathbf{b} \;-\; [\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})]\,\mathbf{a}$*, because this is*

$$[\,a_r\,(\varepsilon_{rpq}\,c_p\,d_q)]\;b_i \;-\; [\,b_s\,(\varepsilon_{spq}\,c_p\,d_q)]\;a_i$$

We can also expand the i-th component of $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ *as*

$$\varepsilon_{ijk}\;(\varepsilon_{jrs}\,a_r\,b_s)\;(\varepsilon_{kpq}\,c_p\,d_q) = \varepsilon_{ijk}\,\varepsilon_{jrs}\,\varepsilon_{kpq}\,a_r\,b_s\,c_p\,d_q$$

$$\begin{aligned}(\varepsilon_{ijk}\,\varepsilon_{pqk})\,\varepsilon_{jrs}\,a_r\,b_s\,c_p\,d_q &= \delta_{ip}\,\delta_{jq}\,\varepsilon_{jrs}\,a_r\,b_s\,c_p\,d_q - \delta_{iq}\,\delta_{jp}\,\varepsilon_{jrs}\,a_r\,b_s\,c_p\,d_q \\ &= \varepsilon_{qrs}\,a_r\,b_s\,c_i\,d_q\varepsilon_{prs}\,a_r\,b_s\,c_p\,d_i\end{aligned}$$

which is the same as the i-th component of $[\mathbf{d}\cdot(\mathbf{a}\times\mathbf{b})]\,\mathbf{c} \;-\; [\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})]\,\mathbf{d}$*, because this is*

$$[\,d_q\,(\varepsilon_{qrs}\,a_r\,b_s)]\;c_i \;-\; [\,c_p\,(\varepsilon_{prs}\,a_r\,b_s\,)]\;d_i\,.$$

(c) $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) \;=\; (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) \;-\; (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$,

Solution:

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= (\mathbf{a}\times\mathbf{b})_i(\mathbf{c}\times\mathbf{d})_i \\ &= \epsilon_{ijk}a_jb_k\epsilon_{ilm}c_ld_m \\ &= \epsilon_{ijk}\epsilon_{ilm}a_jb_kc_ld_m \\ &= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})a_jb_kc_ld_m \\ &= a_jc_jb_kd_k - a_jd_jb_kc_k \\ &= (\mathbf{a}\cdot\mathbf{c})(\mathbf{d}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}).\end{aligned}$$

(d) $\mathbf{a}\times[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})] \;=\; a^2\,(\mathbf{b}\times\mathbf{a})$,

Solution: *Note that the left-hand side of this expression is a vector, so we will write down the ith component of this vector, and check that this gives the ith component of the right-hand side of the expression. If this holds for all i, the expression is then true as a statement about vectors.*

$$\begin{aligned}(\mathbf{a}\times[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})])_i &= \epsilon_{ijk}a_j\epsilon_{klm}a_l\epsilon_{mno}a_nm_o \\ &= \epsilon_{ijk}(\delta_{kn}\delta_{lo} - \delta_{ko}\delta_{ln})a_ja_la_nb_o \\ &= \epsilon_{ijk}a_ja_ka_lb_l - \epsilon_{ijk}a_jb_ka_la_l \\ &= (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\times\mathbf{a})_i - (\mathbf{a}\cdot\mathbf{a})(\mathbf{a}\times\mathbf{b})_i.\end{aligned}$$

Now note that $\mathbf{a}\times\mathbf{a} = 0$ *for all vectors* $\mathbf{a}$*, and recall that* $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$*, so we have*

$$(\mathbf{a}\times[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})]) = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\times\mathbf{a}),$$

and letting $a = |\mathbf{a}|$, we have the required result.

(e) $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) + (\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d}) + (\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d}) = 0$.

Solution: *Using part (a), we have*

$$\begin{aligned}&(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) + (\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d}) + (\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d}) \\ &\quad= (\mathbf{a}\cdot\mathbf{c})(\mathbf{d}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) + (\mathbf{b}\cdot\mathbf{a})(\mathbf{c}\cdot\mathbf{d}) \\ &\qquad - (\mathbf{b}\cdot\mathbf{d})(\mathbf{c}\cdot\mathbf{a}) + (\mathbf{c}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{d}) - (\mathbf{c}\cdot\mathbf{d})(\mathbf{a}\cdot\mathbf{b}) \\ &\quad= 0.\end{aligned}$$

36 *Exam question June 2002 (Section A): Evaluate* $\varepsilon_{ijk}\varepsilon_{ijl}x_k x_l$.

Solution:

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{ijl}x_k x_l &= (\delta_{jj}\delta_{kl} - \delta_{jl}\delta_{kj})x_k x_l \\ &= (3\delta_{kl} - \delta_{kl})x_k x_l = 2\delta_{kl}x_k x_l = 2x_l x_l = 2|\mathbf{x}|^2.\end{aligned}$$

37 *Exam question June 2001 (Section A): Evaluate* $\varepsilon_{ijk}\partial_i\partial_j(x_l x_l)^{1/2}$ *away from the origin.*

Solution: The answer is zero, because the mixed second partial derivatives are symmetric in i and j while the ε_{ijk} is antisymmetic. For full marks you should say that the mixed partial derivatives are symmetric in i and j because they are continuous functions at all points away from the origin (which are the only points which need to be considered).

38 *Exam question June 2003 (Section A): Calculate* $\partial_i\,(\varepsilon_{ijk}\,\varepsilon_{jkl}\,x_l\,)$. *(Hint: use the connection between* $\partial_i x_j = \frac{\partial x_j}{\partial x_i}$ *and the Kronecker delta.)*

Solution: Use $\partial_i x_j = \delta_{ij}$ in $\partial_i\,(\varepsilon_{ijk}\,\varepsilon_{jkl}\,x_l\,) = \varepsilon_{ijk}\,\varepsilon_{jkl}\,\partial_i\,x_l = \varepsilon_{ijk}\,\varepsilon_{jkl}\,\delta_{il} = \varepsilon_{ijk}\,\varepsilon_{jki} = \varepsilon_{ijk}\,\varepsilon_{ijk} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji} = 3\times 3 - \delta_{ii} = 9 - 3 = 6$.

39 *The functions* f, g *are scalars, while* $\mathbf{A}$ *and* $\mathbf{B}$ *are vector functions with components* A_i *and* B_i *respectively. Verify the following identities using index notation:*

(a) $\mathrm{grad}\,(fg) = f\,\mathrm{grad}\,g + g\,\mathrm{grad}\,f$,

Solution: *The* i*-th component of the LHS is* $\partial_i(fg) = f\partial_i g + g\partial_i f$ *by the product-rule. This is the* i*-th component of the RHS, which proves the identity.*

(b) $\mathrm{grad}\,(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times\mathrm{curl}\,\mathbf{B} + \mathbf{B}\times\mathrm{curl}\,\mathbf{A} + (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B}$,

Solution: *This time it's best to start on the RHS (general message: it's always a good idea to start with the most complicated side and then simplify). We have*

$$\begin{aligned}(\mathbf{A}\times\mathrm{curl}\,\mathbf{B} + \mathbf{B}\times\mathrm{curl}\,\mathbf{A})_i &= \varepsilon_{ijk}A_j\varepsilon_{klm}\partial_l B_m + \varepsilon_{ijk}B_j\varepsilon_{klm}\partial_l A_m \\ &= \varepsilon_{ijk}\varepsilon_{klm}\,(A_j\partial_l B_m + B_j\partial_l A_m) \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,(A_j\partial_l B_m + B_j\partial_l A_m) \\ &= A_j\partial_i B_j + B_j\partial_i A_j - A_j\partial_j B_i - B_j\partial_j A_i\end{aligned}$$

The last two terms in the final line are exactly cancelled by the i*-th component of* $(\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B}$*, while the first two are reasonably-easily seen to be the* i*-th component of* $\mathrm{grad}\,(\mathbf{A}\cdot\mathbf{B})$*, proving the identity.*

(c) $\mathrm{div}\,(f\mathbf{A}) = f\,\mathrm{div}\,\mathbf{A} + (\mathrm{grad} f)\cdot\mathbf{A}$,

Solution: *The LHS is a scalar equal to* $\partial_i(fA_i) = f\partial_i A_i + (\partial_i f)A_i$ *by the product rule for derivatives. This equals the RHS, which proves the identity.*

(d) $\mathrm{curl}\,(f\mathbf{A}) = f\,\mathrm{curl}\,\mathbf{A} + (\mathrm{grad}\,f)\times\mathbf{A}$,

Solution: *The LHS is a vector whose* i*-th component is*

$$\begin{aligned}\varepsilon_{ijk}\partial_j(fA_k) &= \varepsilon_{ijk}\,(f\partial_j A_k + (\partial_j f)A_k) \\ &= f\varepsilon_{ijk}\partial_j A_k + \varepsilon_{ijk}(\partial_j f)A_k\end{aligned}$$

This is the i*-th component of the RHS, which proves the identity.*

(e) $\operatorname{div}(\mathbf{A}\times\mathbf{B}) \;=\; \mathbf{B}\cdot\operatorname{curl}\mathbf{A} \;-\; \mathbf{A}\cdot\operatorname{curl}\mathbf{B}$,

Solution: *The LHS is a scalar equal to $\partial_i(\varepsilon_{ijk}A_jB_k) = \varepsilon_{ijk}\partial_i(A_jB_k)$ since ε_{ijk} is just a number, independent of $\mathbf{x}$. Using the product-rule this becomes*

$$\varepsilon_{ijk}(\partial_i A_j)B_k + \varepsilon_{ijk}A_j(\partial_i B_k) = B_k\left(\varepsilon_{kij}(\partial_i A_j)\right) - A_j(\varepsilon_{jik}\partial_i B_k)$$

This equals the RHS, which proves the identity.

(f) $\operatorname{curl}(\mathbf{A}\times\mathbf{B}) \;=\; (\operatorname{div}\mathbf{B})\mathbf{A} \;-\; (\operatorname{div}\mathbf{A})\mathbf{B} \;+\; (\mathbf{B}\cdot\nabla)\mathbf{A} \;-\; (\mathbf{A}\cdot\nabla)\mathbf{B}$,

Solution: *The LHS is a vector whose i-th component is $\varepsilon_{ijk}\partial_j(\varepsilon_{krs}A_rB_s) = \varepsilon_{ijk}\varepsilon_{rsk}\partial_j(A_rB_s)$ since $\varepsilon_{krs} = \varepsilon_{rsk}$ is just a number, independent of $\mathbf{x}$. Using the product-rule, and the very useful formula, this becomes*

$$(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr})\left((\partial_j A_r)B_s + A_r(\partial_j B_s)\right)$$

$$= \delta_{ir}\delta_{js}(\partial_j A_r)B_s + \delta_{ir}\delta_{js}A_r(\partial_j B_s) - \delta_{is}\delta_{jr}(\partial_j A_r)B_s - \delta_{is}\delta_{jr}A_r(\partial_j B_s)$$

$$= (\partial_s A_i)B_s + A_i(\partial_s B_s) - (\partial_r A_r)B_i - A_r(\partial_r B_i) = A_i(\partial_s B_s) - (\partial_r A_r)B_i + (B_s\partial_s)A_i - (A_r\partial_r)B_i$$

This is the i-th component of the RHS, which proves the identity.

(g) $\operatorname{div}\operatorname{curl}\mathbf{A} \;=\; 0$,

Solution: *The LHS is a scalar equal to $\partial_i(\varepsilon_{ijk}\partial_j A_k) = \varepsilon_{ijk}\partial_i\partial_j A_k$. On the assumption that $\partial_i\partial_j A_k$ is continuous then $\partial_i\partial_j A_k = \partial_j\partial_i A_k$ so $\varepsilon_{ijk}\partial_i\partial_j A_k = \varepsilon_{ijk}\partial_j\partial_i A_k = \varepsilon_{jik}\partial_i\partial_j A_k$ by re-naming dummy variables, but this is $-\varepsilon_{ijk}\partial_i\partial_j A_k$ by skew symmetry of ε_{ijk}, i.e. $\varepsilon_{ijk}\partial_i\partial_j A_k = -\varepsilon_{ijk}\partial_i\partial_j A_k$ which implies $\varepsilon_{ijk}\partial_i\partial_j A_k = 0$.*

(h) $\operatorname{curl}\,\operatorname{curl}\mathbf{A} \;=\; \operatorname{grad}\operatorname{div}\mathbf{A} - \nabla^2\mathbf{A}$.

Solution: *The LHS is a vector whose i-th component is*

$$\varepsilon_{ijk}\partial_j(\varepsilon_{krs}\partial_r A_s) = \varepsilon_{ijk}\varepsilon_{rsk}\partial_j(\partial_r A_s)$$

since $\varepsilon_{krs} = \varepsilon_{rsk}$ is just a number, independent of $\mathbf{x}$. Using the very useful formula, this becomes

$$(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr})\partial_j\partial_r A_s = \delta_{ir}\delta_{js}\partial_j\partial_r A_s - \delta_{is}\delta_{jr}\partial_j\partial_r A_s = \partial_s\partial_i A_s - \partial_r\partial_r A_i$$

On the assumption that $\partial_j\partial_r A_s$ is continuous we can write this as $\partial_i\partial_s A_s - \partial_r\partial_r A_i$. This is the i-th component of the RHS, which proves the identity.

40 *What is the divergence of the vector function $\mathbf{A}(\mathbf{x}) = r\,\mathbf{x} + \nabla r$ where $\mathbf{x}$ is the position vector in 3 dimensions and $r = |\mathbf{x}|$? What is the corresponding result in n dimensions?*

Solution: Using components, $\nabla\cdot\mathbf{A} = \partial_i\left(rx_i + \partial_i(r)\right) = \partial_i(r)x_i + r\partial_i x_i + \partial_i(\partial_i(r))\,.$ Using $\partial_i(r) = \partial_i(x_jx_j)^{1/2} = \frac{1}{2}(x_jx_j)^{-1/2}2x_i = x_i/r$ (as in 28(c)) this is

$$\begin{aligned} x_ix_i/r + r\delta_{ii} + \partial_i(x_i/r) &= r + 3r + (\partial_i x_i)/r - x_ir^{-2}\partial_i(r) \\ &= 4r + \delta_{ii}/r - x_ir^{-2}x_i/r \\ &= 4r + 3/r - 1/r = 4r + 2/r\,. \end{aligned}$$

In n dimensions $\delta_{ii} = n$ which changes the result to $(n+1)r + (n-1)/r$.