76 *For a simple closed curve C in the (x, y)−plane, show by Green's theorem that the area enclosed is $A = \frac{1}{2} \oint_C (x dy - y dx)$. (Note, 'simple' means that C does not cross itself, which means that it encloses a well-defined area A.)*

77 *Evaluate $\oint \mathbf{F} \cdot \mathbf{dx}$ around the circle $x^2 + y^2 + 2x = 0$, where $\mathbf{F} = y\mathbf{e_1} - x\mathbf{e_2}$, both directly and by using Green's theorem in the plane.*

78 *Evaluate $\oint_C 2x dy - 3y dx$ around the square with vertices at $(x, y) = (0, 2)$, $(2, 0)$, $(-2, 0)$ and $(0, -2)$.*

79 *Let $\underline{v}$ be the radial vector field $\underline{v}(\underline{x}) = \underline{x}$.*

   (a) *Compute $l_1 = \int_{C_1} \underline{v} \cdot d\underline{x}$, where C_1 is the straight-line contour from the origin to the point $(2, 0, 0)$.*

   (b) *Compute $l_2 = \int_{C_2} \underline{v} \cdot d\underline{x}$, where C_2 is the semi-circular contour from the origin to the point $(2, 0, 0)$ defined by $0 \le x \le 2$, $y = +\sqrt{1 - (x-1)^2}$, $z = 0$. [It may help to start by sketching C_2, and then to parameterize it as $\underline{x}(t) = (1 - \cos t, \sin t, 0)$ with $0 \le t \le \pi$.]*

   (c) *You should have found that $l_1 = l_2$. Explain this result using Stokes' theorem.*

80 *Integrate $\operatorname{curl} \mathbf{F}$, where $\mathbf{F} = 3y\mathbf{e_1} - xz\mathbf{e_2} + yz^2\mathbf{e_3}$, over the portion S of the surface $2z = x^2 + y^2$ below the plane $z = 2$, both directly and by using Stokes' theorem. Take the area elements of S to point outwards, so that their z components are negative.*

81 *The paraboloid of equation $z = x^2 + y^2$ intersects the plane $z = y$ in a curve C. Calculate $\oint_C \mathbf{v} \cdot \mathbf{dx}$ for $\mathbf{v} = 2z\mathbf{e_1} + x\mathbf{e_2} + y\mathbf{e_3}$ using Stokes' theorem. Check your answer by evaluating the line integral directly.*

82 *For an open surface S with boundary C, show that $2 \int_S \mathbf{u} \cdot d\mathbf{A} = \oint_C (\mathbf{u} \times \mathbf{x}) \cdot \mathbf{dx}$, where $\mathbf{u}$ is a fixed vector.*

83 *Verify Stokes' theorem for the upper hemispherical surface S: $z = \sqrt{1 - x^2 - y^2}$, $z \ge 0$, with $\mathbf{F}$ equal to the radial vector field, i.e. $\mathbf{F}(x, y, z) = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$.*

84 *Let $\mathbf{F} = ye^z\,\mathbf{e_1} + xe^z\,\mathbf{e_2} + xye^z\,\mathbf{e_3}$. Show that the integral of $\mathbf{F}$ around a regular closed curve C that is the boundary of a surface S is zero.*

85 *By applying Stokes' theorem to the vector field $\mathbf{G} = |\mathbf{x}|^2\mathbf{a}$ (with $\mathbf{a}$ a constant vector), or otherwise, show that $\int_S \mathbf{x} \times d\mathbf{A} = -\frac{1}{2} \oint_C |\mathbf{x}|^2\, \mathbf{dx}$, where S is the area bounded by the closed curve C.*

86 *Exam question June 2002 (Section B): Evaluate the line integral $I = \int_C \mathbf{F} \cdot \mathbf{dx}$ where $\mathbf{F}(\mathbf{x}) = 2y\mathbf{e_1} + z\mathbf{e_2} + 3y\mathbf{e_3}$ and the path C is the intersection of the surface of equation $x^2 + y^2 + z^2 = 4z$ and the surface of equation $z = x + 2$, taken in a clockwise direction to an observer at the origin. A picture of the path is required, as well as full justifications of the theoretical results you might use.*

87 *Exam question 2012 (Section B) Q9(a)(ii): Use Stokes' theorem to calculate the line integral $\oint_C y\,dx + z\,dy + x\,dz$, where C is the intersection of the surfaces $x^2 + y^2 + z^2 = a^2$ and $x + y + z = 0$ and is orientated anticlockwise when viewed from above.*
Suggestion: Instead of one of the two standard methods for surface integrals, just use $d\mathbf{A} = \frac{\nabla f}{|\nabla f|} dA$ and think about how the surfaces intersect.

88 *Consider the vector field $\mathbf{F} = y\,\mathbf{e_1} + (z-x)\,\mathbf{e_2} + (x^3 + y)\,\mathbf{e_3}$. Evaluate $\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$ explicitly, where*

  (i) *S is the disk $x^2 + y^2 \le 4$, $z = 1$,*

  (ii) *S is the surface of a paraboloid $x^2 + y^2 = 5 - z$ above the plane $z = 1$.*

  Verify that each of these agrees with Stokes' theorem by considering the integral of $\mathbf{F}$ around each bounding contour.

89 *Evaluate the line integral*

$$\int_{P_1}^{P_2} yz\,dx + xz\,dy + (xy + z^2)\,dz$$

from P_1 with co-ordinates (1,0,0) to P_2 at (1,0,1) explicitly,

  (a) *along a straight line in the z–direction, and*

  (b) *along a helical path parametrised by $\mathbf{x}(t) = \mathbf{e_1} \cos t + \mathbf{e_2} \sin t + \mathbf{e_3}\, t/2\pi$, where t varies along the path.*

  Compare these two results – does this suggest that there might be a general formula for the integral from $P_1 = (1,0,0)$ to any point P with coordinates (x, y, z)? Check your answer when $P = P_2$.

90 *State a necessary and sufficient condition for a vector field $\mathbf{F}$ to be expressible in the form $\mathbf{F} = \nabla \phi$ in some simply-connected region. The scalar field ϕ is called a scalar potential, though sometimes the opposite sign is used.*

  Determine whether the following vector fields are expressible in this form

  (i) $(\mathbf{a} \cdot \mathbf{x})\,\mathbf{x}$, (ii) $(\mathbf{a} \cdot \mathbf{x})\,\mathbf{a}$, (iii) $(\mathbf{a} \cdot \mathbf{a})\,\mathbf{x}$, (iv) $\mathbf{a} \times \mathbf{x}$, (v) $\mathbf{a} \times (\mathbf{a} \times \mathbf{x})$,

  and find the vector fields, $\mathbf{F}$, for which the corresponding potentials are

  (a) $\frac{1}{2}(\mathbf{a} \cdot \mathbf{x})^2$, (b) $\frac{1}{2}a^2|\mathbf{x}|^2 - \frac{1}{2}(\mathbf{a} \cdot \mathbf{x})^2$.

  Here $\mathbf{a}$ is a constant non-zero vector, and $a^2 = |\mathbf{a}|^2$.

91 *Exam question June 2002 (Section B):*

  (a) *State the conditions for the line integral $\int_c \mathbf{F} \cdot d\mathbf{x}$ from $\mathbf{x}_0$ to $\mathbf{x}_1$ to be independent of the path connecting these two points.*

  (b) *Determine the value of the line integral $\int_c \mathbf{F} \cdot d\mathbf{x}$ where*

$$\mathbf{F} = (e^{-y} - ze^{-x})\,\mathbf{e_1} + (e^{-z} - xe^{-y})\,\mathbf{e_2} + (e^{-x} - ye^{-z})\,\mathbf{e_3},$$

and C is the path

$$x = \frac{1}{\ln 2} \ln(1 + p), \quad y = \sin\frac{\pi p}{2}, \quad z = \frac{1 - e^p}{1 - e},$$

with the parameter p in the range $0 \leq p \leq 1$, from $(0, 0, 0)$ to $(1, 1, 1)$.

92 *If $\mathbf{F} = x\,\mathbf{e_1} + y\,\mathbf{e_2}$, calculate $\int_S \mathbf{F} \cdot \mathbf{dA}$, where S is the part of the surface $z = 9 - x^2 - y^2$ that is above the x,y plane, by applying the divergence theorem to the volume bounded by the surface and the piece it cuts out of the x, y plane.*
Hint: what is $\mathbf{F} \cdot \mathbf{dA}$ on the x,y plane?

93 *Evaluate each of the integrals below as **either** a volume integral **or** a surface integral, whichever is easier:*

(a) *$\int_S \mathbf{x} \cdot \mathbf{dA}$ over the whole surface of a cylinder bounded by $x^2 + y^2 = R^2$, $z = 0$ and $z = L$. Note that $\mathbf{x}$ means $x\,\mathbf{e_1} + y\,\mathbf{e_2} + z\,\mathbf{e_3}$.*

(b) *$\int_S \mathbf{F} \cdot \mathbf{dA}$, where $\mathbf{F} = x\cos^2 y\,\mathbf{e_1} + xz\,\mathbf{e_2} + z\sin^2 y\,\mathbf{e_3}$, over the surface of a sphere with centre at the origin and radius π.*

(c) *$\int_V \underline{\nabla} \cdot \mathbf{F}\,dV$, where $\mathbf{F} = \sqrt{x^2 + y^2}(x\,\mathbf{e_1} + y\,\mathbf{e_2})$, over the three-dimensional volume $x^2 + y^2 \leq R^2$, $0 \leq z \leq L$.*

94 *Suppose the vector field $\mathbf{F}$ is everywhere tangent to the closed surface S, which encloses the volume V. Prove that*

$$\int_V \underline{\nabla} \cdot \mathbf{F}\,dV = 0.$$

95 *Exam question June 2003 (Section A): Evaluate $\int_S \mathbf{B}(\mathbf{x}) \cdot \mathbf{dA}$, where*

$$\mathbf{B}(\mathbf{x}) = (8x + \alpha\,y - z)\,\mathbf{e_1} + (x + 2y + \beta\,z)\,\mathbf{e_2} + (\gamma\,x + y - z)\,\mathbf{e_3}$$

and S is the surface of the sphere having centre at (α, β, γ) and radius γ, where $\alpha, \beta \in \mathbb{R}$, and γ is an arbitrary positive real number.

96 *Let A be the interior of the circle of unit radius centred on the origin. Evaluate $\iint_A \exp(x^2 + y^2)\,dx\,dy$ by making a change of variables to polar co-ordinates.*

97 *Let A be the region $0 \leq y \leq x$ and $0 \leq x \leq 1$. Evaluate $\int_A (x + y)\,dx\,dy$ by making the change of variables $x = u + v$, $y = u - v$. Check your answer by evaluating the integral directly.*

98 *The expressions below are called Green's first and second identities. Derive them from the divergence theorem with $\mathbf{F} = f\underline{\nabla}g$ or $g\underline{\nabla}f$ as appropriate, where f and g are differentiable functions:*

$$\int_V (f\underline{\nabla}^2 g + \underline{\nabla}f \cdot \underline{\nabla}g)\,dV = \int_S (f\underline{\nabla}g) \cdot \mathbf{dA},$$

$$\int_V (f\underline{\nabla}^2 g - g\underline{\nabla}^2 f)\,dV = \int_S (f\underline{\nabla}g - g\underline{\nabla}f) \cdot \mathbf{dA}.$$

where the volume V is that enclosed by the closed surface S.

99 *A gas holder has the form of a vertical cylinder of radius R and height H with hemispherical top also of radius R. The density, ρ, (i.e. the mass per unit volume) of the gas inside varies with height z above the base according to the relation $\rho = C \exp(-z)$, where C is a constant. Calculate the total mass of gas in the holder, taking care to define any coordinate system used and the range of the corresponding variables.*

Use this result to find the integral of the field

$$\mathbf{F} \;=\; B\,z\,e^{-y}\,\mathbf{e_1} \;+\; C\,y\,e^{-z}\,\mathbf{e_2}$$

over the curved surface of the gas holder.
Hint: it may help to note first what the integral over the base of the holder is.

100 *Exam question June 2001 (Section B): In electrostatic theory, Gauss' law states that the flux of an electrostatic (vector) field $\mathbf{E}(\mathbf{x})$ over some closed surface S is equal to the enclosed charge q divided by a constant ϵ_0 :*

$$\int_S \mathbf{E}(\mathbf{x}) \cdot \mathbf{dA} \;=\; \frac{q}{\epsilon_0}\,.$$

If the electrostatic field is given by $\mathbf{E}(x,y) = \alpha x \mathbf{e_1} + \beta y \mathbf{e_2}$, use Gauss' law to find the total charge in the compact region bounded by the surface S consisting of S_1, the curved portion of the half-cylinder $z = (R^2 - y^2)^{1/2}$ of length H; S_2 and S_3 the two semi-circular plane end pieces; and S_4, the rectangular portion of the (x, y)-plane. (In equations, the relevant bounded region may be described by $z^2 + y^2 \leq R^2, z \geq 0, -\frac{H}{2} \leq x \leq \frac{H}{2}$). Express your result in terms of α, β, R and H.

101 *Exam question June 2002 (Section A): In electrostatic theory, Gauss' law states that the flux of an electrostatic field $\mathbf{E}(\mathbf{x})$ over a closed surface S is given by the ratio of the enclosed charge q and a constant ϵ_0. Calculate the electric charge enclosed in the ellipsoid of equation $x^2 + \frac{1}{2}y^2 + z^2 = 1$ in the presence of*

(a) *an electrostatic field $\mathbf{E}(\mathbf{x}) = yz\mathbf{e_1} + xz\mathbf{e_2} + xy\mathbf{e_3}$,*

(b) *an electrostatic field $\mathbf{E}(\mathbf{x}) = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$.*