Michaelmas AMV Notes

The lecture notes for this term were originally prepared by Drs. Emma Coutts and Ian Vernon and Professor Patrick Dorey, and have been edited slightly since. If you find any typos, please let me know.

Sam Fearn

Admin Details

- Dr Sam Fearn, s.m.fearn@durham.ac.uk
- Use the Topic forums (available from the Blackboard Ultra Discussions Page) to ask questions about the course. This way other students will also be able to see any answers. You can post anonymously if you prefer. If you think you can answer another student's question you should try to do so - trying to explain something to others is often the best way to learn.
- Lectures: Lectures will take place in TLC 042. These are timetabled for Monday at 1pm, Tuesday at 10am, and Thursday at 5pm (odd teaching weeks only). Lecture notes and the recorded videos from the previous year will be provided to go along with the lectures. You are expected to spend time studying the lecture notes yourself as well as attending the lectures.
- All the material for this term will be available via the Ultra course. I will try to make everything as easy to find as possible, but please also familiarise yourself with what is available.
- Tutorials: Tutorials will be held in odd numbered weeks, starting in week 3. The suggested problems for these tutorials will be available through the 'Michaelmas Tutorials' block in Ultra. You should always make sure that you have both the lecture notes and any relevant problems sheets available in your tutorials.
- Problems Classes: These will take place in TLC 042 on Thursdays at 5pm in even numbered teaching weeks. In these classes I will focus on working through examples rather than introducing any new material.
- Assignments: Assignments will be set weekly, and will alternate between work to be done on paper, and handed in via Gradescope, and e-assessments to be completed on the Maths Quiz Server. Further details, as well as the necessary links, are available in the Assignments block on Ultra.
- Additional Reading: The notes available in each topic, along with the lecture videos and problems classes will contain all the material that you will be examined on in this course. If you feel that you would like some additional reading to support the lecture notes then I recommend the book *Mathematical Methods for Physics and Engineering* by Riley, Hobson and Bence. This is available for you to read online via the 'Reading List', which is available as a link in Ultra.

0 Introduction

Most of this term of this course is actually about Vector Calculus. Vector calculus is a generalisation of the calculus you studied in Calculus I, which focussed mostly on functions of one or two variables, to studying functions defined from and to higher dimensional real spaces. Although the notation we use today was first introduced in the study of Electromagnetism (Maxwell, 1831-1879) in the 1860's and 1870's, the language of vector calculus is now fundamental in many parts of pure and applied mathematics, as is shown in Figure 2.

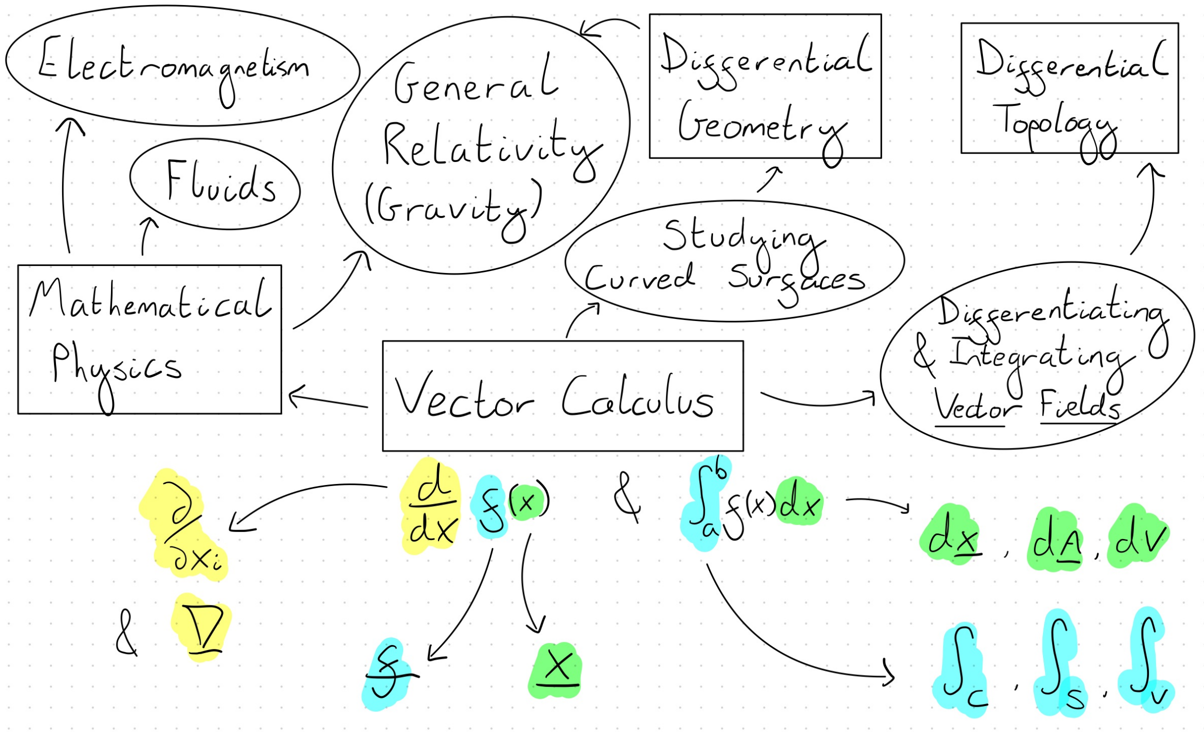

Figure 1: Not only is vector calculus fundamental for studying most topics in mathematical physics, it's also important for the study of the geometry and topology of differentiable spaces (more specifically, differentiable manifolds, which you'll meet in a later course), and the study of differential equations.

1 Introduction **5**

1 Introduction **5**

2 Maps between real vector spaces **6**
2.1 General notation . . . 6
2.2 Scalar fields, vector fields and curves . . . 7
2.3 Partial derivatives and the chain rule . . . 9

3 The gradient of a scalar field **12**
3.1 Differential operators and $\underline{\nabla}$ . . . 12
3.2 Directional derivatives . . . 13
3.3 Some properties of the gradient . . . 15

4 $\underline{\nabla}$ acting on vector fields **17**
4.1 Divergence (div) . . . 17
4.2 Curl . . . 19
4.3 Applying $\underline{\nabla}$ twice . . . 21

5 Index notation **23**
5.1 Einstein Summation Convention . . . 23
5.2 The Kronecker delta, δ_{ij} . . . 23
5.3 The Levi-Cevita symbol, ϵ_{ijk} . . . 24
5.4 The very useful ϵ_{ijk} formula . . . 25
5.5 Applications . . . 26

6 Differentiability of scalar fields **29**
6.1 Continuity . . . 30
6.2 Open sets . . . 31
6.3 Differentiable maps $\mathbb{R}^n \to \mathbb{R}$ . . . 34
6.4 Continuous Differentiability . . . 36
6.5 The chain rule revisited . . . 38
6.6 The implicit function theorem . . . 39

7 Differentiability of vector fields **46**
7.1 Differentiable maps $\mathbb{R}^n \to \mathbb{R}^n$ . . . 46
7.2 Diffeomorphisms and the inverse function theorem . . . 47

8 Volume, line and surface integrals **50**
8.1 Double integrals and Fubini's theorem . . . 50
8.2 Volume integrals . . . 55
8.3 Line integrals . . . 59
8.4 Surface integrals I: defining a surface . . . 61
8.5 Surface integrals II: evaluating the integral . . . 63

9 Green's, Stokes' and divergence theorems **67**
9.1 The three big theorems . . . 67
9.2 Examples . . . 68
9.3 Conservation laws and the continuity equation . . . 70
9.4 Path independence of line integrals . . . 72
9.5 Changing integration variables . . . 79

10 Proofs of the three big theorems (Non-Examinable) **85**
10.1 Green's theorem in the plane . . . 85
10.2 Stokes' theorem . . . 86
10.3 The divergence theorem . . . 89
10.4 Further examples . . . 91