Calculus I, Tutorial Problem Sheet, Week 5

Differentiable functions

Q1. Use the limit definition of the derivative to calculate the derivative of $f(x) = \sqrt{x}$.

Solution. $f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$
$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$.

Q2. Show that if $g(x)$ is continuous at $x = 0$ then $g(x)\tan x$ is differentiable at $x = 0$.

Solution.
Let $f(x) = g(x)\tan x$ *then we need to show that* $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ *exists.*
$\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{g(h)\tan h - g(0)\tan 0}{h} = \lim_{h\to 0} \frac{g(h)\tan h}{h}$
$= \left(\lim_{h\to 0} g(h)\right)\left(\lim_{h\to 0} \frac{1}{\cos h}\right)\left(\lim_{h\to 0} \frac{\sin h}{h}\right) = g(0)(1)(1) = g(0)$
where we have made use of the continuity of $g(x)$ *and* $1/\cos x$ *at* $x = 0$, *along with one of our standard trigonometric results.*
Hence $f(x) = g(x)\tan x$ *is differentiable at* $x = 0$ *with* $f'(0) = g(0)$.

Q3. Use L'Hopital's rule to calculate the limit as $x \to 0$ of the following
(a) $\frac{1-\cos 2x}{x}$, (b) $\frac{x^2}{1-\cos 2x}$.

Solution.
(a) $f(x) = 1 - \cos(2x),\ g(x) = x,$ *are differentiable and satisfy* $f(0) = g(0) = 0$.
$f'(x) = 2\sin(2x),\ f'(0) = 0,\ g'(x) = 1 \neq 0$.
Therefore, by l'Hopital's rule, $\lim_{x\to 0} f(x)/g(x) = \lim_{x\to 0} f'(x)/g'(x) = f'(0)/g'(0) = 0/1 = 0$.

(b) $f(x) = x^2,\ g(x) = 1-\cos(2x),$ *are twice differentiable and satisfy* $f(0) = g(0) = 0$.
$f'(x) = 2x,\ f'(0) = 0,\ g'(x) = 2\sin(2x),\ g'(0) = 0$.
$f''(x) = 2,\ g''(x) = 4\cos(2x) \neq 0$ *for* x *sufficiently close to* $x = 0$. *Also,* $g''(0) = 4$.
Therefore, by l'Hopital's rule,
$\lim_{x\to 0} f(x)/g(x) = \lim_{x\to 0} f'(x)/g'(x) = \lim_{x\to 0} f''(x)/g''(x) = f''(0)/g''(0) = \frac{1}{2}$.

Q4. Assuming that y is a differentiable function of x and satisfies $xy + y^2 - 3x - 3 = 0$, Calculate $\frac{dy}{dx}$ at the point $(-1, 1)$.

Solution. *Differentiating the given expression, remembering that we have* y *a function of* x, $y + xy' + 2yy' - 3 = 0$. *At* $(x, y) = (-1, 1)$ *this becomes* $1 - y' + 2y' - 3 = 0$ *so* $y' = 2$.

Extreme values

Q5. Find the global extreme values of $f(x) = \frac{1}{3}x^3 - 3x + |x^2 - 4|$ in $[-2, 4]$.

Solution.

Note that $f(x) = \frac{x^3}{3} - 3x + |x^2 - 4|$ *is differentiable everywhere except at* $x = \pm 2$.

For $x \in (-2, 2)$, $f(x) = \frac{x^3}{3} - 3x + 4 - x^2$ *with*

$f'(x) = x^2 - 3 - 2x = (x-3)(x+1) = 0$ *iff* $x = -1$. $f(-1) = \frac{17}{3}$.

For $x \in (2, 4)$, *then* $f(x) = \frac{x^3}{3} - 3x + x^2 - 4$ *with*

$f'(x) = x^2 - 3 + 2x = (x+3)(x-1) \neq 0$.

Now $f(2) = -\frac{10}{3}$, $f(-2) = \frac{10}{3}$, $f(4) = \frac{64}{3}$.

The global maximum is $\frac{64}{3}$ *and the global minimum is* $-\frac{10}{3}$.

Q6. Either find the global maximum or justify that it does not exist for each of the following
(a) $f(x) = 1 - |1 - x^2|$ in $[0, \sqrt{2}]$, (b) $f(x) = x/(x^2+1)$ in $x \geq 0$,
(c) $f(x) = x\cos(\frac{1}{x})/(x+1)$ in $x \geq 1$.

Solution.

(a) $f(x) = 1 - |1 - x^2|$ *is differentiable in* $(0, \sqrt{2})$ *except at* $x = 1$.

For $x \in (0, 1)$, $f(x) = 1 - (1 - x^2) = x^2$, *so* $f'(x) = 2x \neq 0$.

For $x \in (1, \sqrt{2})$, $f(x) = 1 + (1 - x^2) = 2 - x^2$, *so* $f'(x) = -2x \neq 0$.

$f(1) = 1$, $f(0) = 0$, $f(\sqrt{2}) = 0$, *hence the global maximum is* 1.

(b) For $x > 0$, $f'(x) = (1 - x^2)/(1 + x^2)^2 = 0$ *iff* $x = 1$.

$f(1) = \frac{1}{2}$, $f(0) = 0$, $\lim_{x\to\infty} f(x) = 0$, *hence the global maximum is* $\frac{1}{2}$.

(c) For $x \geq 1$, $f'(x) = \frac{\cos(\frac{1}{x})}{(1+x)^2} + \frac{\sin(\frac{1}{x})}{x(1+x)} > 0$, *thus* $f(x)$ *is increasing for* $x \geq 1$.

In fact $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\{x\cos(\frac{1}{x})/(x+1)\} = \lim_{u\to 0} \frac{\cos u}{1+u} = \cos 0 = 1$.

There is no global maximum in $x \geq 1$.

Partial derivatives

Q7. Given the function $f(x, y) = \log(1 + xy)$ calculate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$.

Solution.

$$\frac{\partial f}{\partial x} = \frac{y}{1+xy}, \qquad \frac{\partial f}{\partial y} = \frac{x}{1+xy}, \qquad \frac{\partial^2 f}{\partial x^2} = \frac{-y^2}{(1+xy)^2}, \qquad \frac{\partial^2 f}{\partial y^2} = \frac{-x^2}{(1+xy)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{1}{(1+xy)^2}$$

Q8. Show that, for any constants A and B, the function $f(x,y) = A\cos x \sinh y + B\sin x \cosh y$ satisfies the equation $f_{xx} + f_{yy} = 0$.

Solution.

$f_x = -A\sin x \sinh y + B\cos x \cosh y, \qquad f_{xx} = -A\cos x \sinh y - B\sin x \cosh y$

$f_y = A\cos x \cosh y + B\sin x \sinh y, \qquad f_{yy} = A\cos x \sinh y + B\sin x \cosh y$

Therefore $f_{xx} + f_{yy} = 0$.