Calculus I, Chapter 3 Problems

Differentiable functions

Q1. Use the limit definition of the derivative to calculate the derivative of the following functions

(a) $f(x) = \sin x$, (b) $f(x) = x\sqrt{x}$, (c) $f(x) = \cos^2 x$.

Q2. Show that if $g(x)$ is continuous at $x = 0$ then $g(x) \tan x$ is differentiable at $x = 0$.

Q3. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a differentiable function that satisfies $2f(x) + e^{x^2 f(x)} - \sin f(x) = 1$ and has a continuous derivative. Find $f(0)$ and $f'(0)$.

Q4. Explicitly write out the Leibniz rule for $\frac{d^4}{dx^4}(f(x)g(x))$ and use this to calculate the fourth derivative of $x^4 \cos x$.

Q5. Given $f(x) = 4x + 3$ and $g(x) = 1/(4 + x^2)^2$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Calculate $(f \circ g)'(0)$ and $(g \circ f)'(0)$.

Q6. Use L'Hopital's rule to calculate the limit as $x \to 0$ of the following

(a) $\frac{1-\cos 2x}{x}$, (b) $\frac{1-\cos x}{x^2}$, (c) $\frac{\tan 2x}{x}$, (d) $\frac{x^2}{1-\cos 2x}$, (e) $\frac{x^2}{1-\cos 4x}$.

Q7. Find an expression for $\frac{dy}{dx}$ in terms of x and y in the following cases

(a) $xy^2 - 4x^{3/2} - y = 0$, (b) $x + \sin y = xy$,

(c) $(3xy + 7)^2 = 6y$, (d) $x + \tan(xy) = 0$, (e) $\cosh x + \sinh(xy) = 0$.

Q8. In each of the following cases, assume that y is a differentiable function of x and satisfies the given equation. Calculate $\frac{dy}{dx}$ at the given point.

(a) $xy + y^2 - 3x - 3 = 0$, $(-1, 1)$.

(b) $xe^y + \sin(xy) + y = \log 2$, $(0, \log 2)$.

Extreme values

Q9. Find the global extreme values of $f(x) = \frac{1}{3}x^3 - 3x + |x^2 - 4|$ in $[-2, 4]$.

Q10. Either find the global maximum or justify that it does not exist for of each of the following

(a) $f(x) = x^4 - 2x^2$ in $[\frac{1}{3}, \frac{4}{3}]$, (b) $f(x) = 1 - |1 - x^2|$ in $[0, \sqrt{2}]$,

(c) $f(x) = x/(x^2 + 1)$ in $x \geq 0$, (d) $f(x) = x\cos(\frac{1}{x})/(x + 1)$ in $x \geq 1$.

Q11. A group of Chilean miners are trapped underground at a depth of 300 metres. A rescue team starts at the bottom of an abandoned mine shaft that is 600 metres West of the trapped miners and has a depth of 100 metres. The rescue team must dig a tunnel to the trapped miners that has an initial horizontal segment followed by a segment directly towards the trapped miners. At a depth of 100 metres the rock is soft and it takes only 5 minutes to dig one horizontal metre. However, at any depth below this, the rock is hard and it takes 13 minutes to dig a distance of one metre.

Calculate the minimal number of hours that it takes to tunnel to the trapped miners.

Partial derivatives

Q12. Given the function $f(x, y) = \log(1 + xy)$ calculate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$.

Q13. Calculate $f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}$ for the function $f(x, y) = xe^{xy}$.

Q14. Show that, for any constants A and B, the function $f(x, y) = A \cos x \sinh y + B \sin x \cosh y$ satisfies the equation $f_{xx} + f_{yy} = 0$.