Calculus I, Tutorial Problem Sheet, Week 3

Functions: Even, odd and inverse functions

Q1. Are the following functions even, odd or neither? Justify your answers.

(a) $f(x) = (x-1)(x-2)$

(b) $f(x) = \sum_{k=0}^{n} x^{2k+1}$

(c) $f(x) = \frac{x}{(x^2+1)\cos x}$

Solution. *(a) $f(x) = x^2 - 3x + 2$, so $f(-x) = x^2 + 3x + 2$. Since $f(x) \neq f(-x)$ and $f(x) \neq -f(-x)$ this function is neither even nor odd.*

(b) $f(-x) = \sum_{k=0}^{n}(-1)^{2k+1}x^{2k+1} = -\sum_{k=0}^{n} x^{2k+1} = -f(x)$ hence this function is odd.

(c) x is odd, but both x^2+1 and $\cos x$ are even, hence $f(x)$ is the product of one odd function and two even functions and is therefore an odd function.

Q2. If $f : \mathbb{R} \mapsto \mathbb{R}$ is an even function and $g : \mathbb{R} \mapsto \mathbb{R}$ is an odd function then determine whether the following functions are even, odd or neither? Justify your answers.

(a) $f_1(x) = \begin{cases} f(x) & \text{if } \ x > 0 \\ -f(x) & \text{if } \ x < 0 \end{cases}$

(d) $f_2(x) = (g \circ g)(x)$

Solution.

(a) On $\mathbb{R}\backslash\{0\}$

$f_1(-x) = \begin{cases} f(-x) & \textit{if } \ -x > 0 \\ -f(-x) & \textit{if } \ -x < 0 \end{cases} = \begin{cases} f(x) & \textit{if } \ x < 0 \\ -f(x) & \textit{if } \ x > 0 \end{cases} = -f_1(x)$, *since f is even. Hence this function is odd.*

(b) $f_2(-x) = (g \circ g)(-x) = g(g(-x)) = g(-g(x)) = -g(g(x)) = -(g \circ g)(x) = -f_2(x)$, hence this function is odd.

Q3. Which of the following functions are injective? Find the inverses of those which are and specify the domain of the inverse.

(a) $f(x) = (1-x)^2$ in $[1, 2]$

(b) $f(x) = (x-1)/(x+2)$ in $\mathbb{R}\backslash\{-2\}$

(c) $f(x) = x^2 + 2x - 1$ in $[-2, 2]$

Solution.

(a) One can check $f(x)$ is injective on this domain using the horizontal line test.

To find the inverse function, we write $y = f^{-1}(x)$ and use $f(y) = x$. We therefore have $f(y) = (1-y)^2 = x$. Now since we have $y \in$ Dom f, we need $y \in [1, 2]$, so $(1-y) \leq 0$

and we therefore need to take the negative square root to obtain $1 - y = -\sqrt{x}$. *We therefore find* $y = 1 + \sqrt{x} = f^{-1}(x)$.

Dom f^{-1} = *Ran* $f = [0, 1]$.

(b) One can check $f(x)$ *is injective on this domain using the horizontal line test.*

To find the inverse function, we write $y = f^{-1}(x)$ *and use* $f(y) = x$.

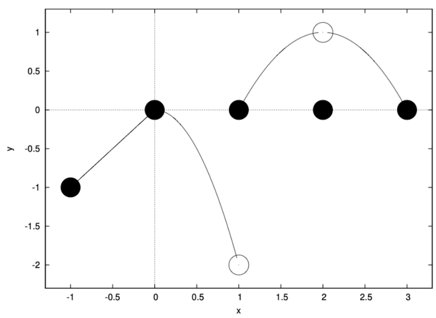
So $f(y) = (y-1)/(y+2) = x$, *and by rearranging we obtain* $y = (2x+1)/(1-x) = f^{-1}(x)$.

Dom f^{-1} = *Ran* $f = \mathbb{R}\backslash\{1\}$.

(c) The function $f(x)$ *is not injective. We can see this by applying horizontal line test or for example noting that* $f(-2) = -1 = f(0)$.

Limits

Q4. Consider the given graph of the function $f(x)$. Are the following statements true or false?



$f(x)$ for Q4

(a) $\lim_{x\to 2} f(x)$ does not exist, (b) $\lim_{x\to 2} f(x) = 1$, (c) $\lim_{x\to 1} f(x)$ does not exist,

(d) $\lim_{x\to a} f(x)$ exists $\forall\ a \in (-1, 1)$ (e) $\lim_{x\to a} f(x)$ exists $\forall\ a \in (1, 3)$.

Solution.

(a) false, (b) true, (c) true, (d) true, (e) true.

Q5. In each case either evaluate the limit, or state that no limit exists

(a) $\lim_{x\to\pi/2} x \sin x$, (b) $\lim_{x\to 1} \frac{x^4-1}{x^3-1}$, (c) $\lim_{x\to\pi} \frac{\cos x}{1-\pi}$, (d) $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2}$,

(e) $\lim_{x\to 0} \frac{x^2}{1-\cos 2x}$ (f) $\lim_{x\to 3} \frac{(x^2+x-12)}{(x-3)^2}$, (g) $\lim_{h\to 0} \frac{1+1/h}{1+1/h^2}$.

Solution.

(a) $\lim_{x\to\pi/2} x \sin x, = \pi/2$, *since the function is continuous at the point* $x = \pi/2$.

(b) $\lim_{x\to 1} \frac{x^4-1}{x^3-1} = \lim_{x\to 1} \frac{(x^2+1)(x+1)(x-1)}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{(x^2+1)(x+1)}{x^2+x+1} = 4/3$, *as this final expression is continuous at* $x = 1$.

(c) $\lim_{x\to\pi} \frac{\cos x}{1-\pi} = \frac{1}{\pi-1}$, *since the function is continuous at the point* $x = \pi$.

(d) $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2} = \lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}+2)} = \lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{x+3-4} = 4$, *as this final expression is continuous at* $x = 1$.

(e) $\lim_{x\to 0} \frac{x^2}{1-\cos 2x} = \lim_{x\to 0} \frac{x^2(1+\cos 2x)}{(1-\cos 2x)(1+\cos 2x)} = \lim_{x\to 0} \frac{(2x)^2(1+\cos 2x)}{4\sin^2 2x} = 1/2$, *as this final expression is continuous at* $x = 0$.

(f) $\lim_{x\to 3} \frac{(x^2+x-12)}{(x-3)^2} = \lim_{x\to 3} \frac{(x+4)(x-3)}{(x-3)^2} = \lim_{x\to 3} \frac{x+4}{x-3}$. *Therefore no limit exists, as in any small interval around* $x = 3$, *one can make the function arbitrarily large by considering values close to 3.*

(g) $\lim_{h\to 0} \frac{1+1/h}{1+1/h^2} = \lim_{h\to 0} \frac{h^2+h}{h^2+1} = 0$, *as this final expression is continuous at* $h = 0$.

Q6. If $f(x) > 0 \ \forall\ x \neq a$ and $\lim_{x\to a} f(x) = L$, can we conclude that $L > 0$? Justify your answer.

Solution.

No. An example is provided by $f(x) = x^2$ *with* $a = 0$ *so that* $L = 0$ *which is not positive.*

Q7. Does $\lim_{x\to 0} \frac{\sin(x+|x|)}{x}$ exist?

If the limit exists then find it.

Solution. *For* $x > 0$, $\frac{\sin(x+|x|)}{x} = \frac{\sin 2x}{x}$.

Hence $\lim_{x\to 0^+} \frac{\sin(x+|x|)}{x} = \lim_{x\to 0^+} \frac{\sin 2x}{x} = \lim_{x\to 0^+} \frac{2\sin 2x}{2x} = 2$.

For $x < 0$, $\frac{\sin(x+|x|)}{x} = 0$. *Hence* $\lim_{x\to 0^-} \frac{\sin(x+|x|)}{x} = 0$.

The left-sided and right-sided limits exist but are not equal, hence the limit does not exist.

Q8. Calculate the limit as $x \to \infty$ of the following

(a) $\frac{6x+7}{1-2x}$, (b) $\frac{x^2}{x^2+\sin^2 x}$.

Solution.

(a) $\lim_{x\to\infty} \frac{6x+7}{1-2x} = \lim_{x\to\infty} \frac{6+\frac{7}{x}}{\frac{1}{x}-2} = -3$.

(b) First note that $0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$.

As $\lim_{x\to\infty} \frac{1}{x^2} = 0$ *then by the pinching theorem* $\lim_{x\to\infty} \frac{\sin^2 x}{x^2} = 0$.

Thus $\lim_{x\to\infty} \frac{x^2}{x^2+\sin^2 x} = \lim_{x\to\infty} \frac{1}{1+\frac{\sin^2 x}{x^2}} = 1$.