21 *Compute the divergence,* $\nabla \cdot \mathbf{A}$, *of the following vector fields:*

(a) $\mathbf{A}(x,y,z) \;=\; yz\mathbf{e_1} + xz\mathbf{e_2} + xy\mathbf{e_3}$,

(b) $\mathbf{A}(x,y,z) \;=\; (x^2+y^2+z^2)\,(3\mathbf{e_1} + 4\mathbf{e_2} + 5\mathbf{e_3})$,

(c) $\mathbf{A}(x,y,z) \;=\; (x+y)\mathbf{e_1} + (y+z)\mathbf{e_2} + (z+x)\mathbf{e_3}$.

22 *Compute the curl,* $\nabla \times \mathbf{A}$, *of each of the vector fields,* $\mathbf{A}$, *in the previous question.*

23 *If* $f(r)$ *is a differentiable function of* $r = |\mathbf{x}\,|$, *for* $\mathbf{x} \in \mathbb{R}^n,\ r \neq 0$, *show that*

(a) $\mathrm{grad}\, f(r) \;=\; f'(r)\,\mathbf{x}\,/r$,

(b) $\mathrm{curl}\,[f(r)\mathbf{x}\,] \;=\; 0$, *where now we let* $n = 3$.

24 *Let* $\mathbf{x}$ *be the position vector in three dimensions, with* $r = |\mathbf{x}\,|$, *and let* $\mathbf{a}$ *be a constant vector. Show that*

(a) $\mathrm{div}\,\mathbf{x} \;=\; 3$,

(b) $\mathrm{curl}\,\mathbf{x} \;=0$,

(c) $\mathrm{grad}\, r \;=\; \mathbf{x}\,/r$,

(d) $\mathrm{div}\,(r^n\mathbf{x}\,) \;=\; (n+3)\,r^n$,

(e) $\mathrm{grad}\,(\mathbf{a}\cdot\mathbf{x}\,) \;=\; \mathbf{a}$,

(f) $\mathrm{div}\,(\mathbf{a}\times\mathbf{x}\,) = 0$,

(g) $\mathrm{curl}\,(\mathbf{a}\times\mathbf{x}\,) \;=\; 2\,\mathbf{a}$,

(h) $\mathrm{curl}\,(r^2\mathbf{a}) \;=\; 2\,(\mathbf{x}\,\times\mathbf{a})$,

(i) $\nabla^2(1/r) \;=\; 0 \;\;,\quad \text{if } r \neq 0$,

(j) $\nabla^2(\log r) \;=\; 1/r^2\,,\quad \text{if } r \neq 0$,

(k) $\mathrm{div}\,[(\mathbf{a}\cdot\mathbf{x}\,)\mathbf{x}\,] \;=\; 4\,\mathbf{a}\cdot\mathbf{x}\,$,

(l) $\mathrm{div}\,[\mathbf{x}\,\times(\mathbf{x}\,\times\mathbf{a})] \;=\; 2\,\mathbf{a}\cdot\mathbf{x}\,$,

(m) $\mathrm{curl}\;(\mathbf{a}\times\mathbf{x}\,/r^3) \;=\; 3\,(\mathbf{a}\cdot\mathbf{x}\,)\mathbf{x}\,/r^5 \;-\; \mathbf{a}/r^3$,

(n) *Exam question June 2002 (Section A): calculate the curl of* $(\mathbf{a}\cdot\mathbf{x}\,)\,\mathbf{x}\,$.

25 *If* $\mathbf{x}$ *is the position vector,* $\mathbf{x} \;=\; x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$, $\mathbf{a}$ *is a constant vector,* $\mathbf{F} = (\mathbf{a}\cdot\mathbf{x}\,)\,\mathbf{x}$ *and* $\mathbf{G} = r^2\mathbf{a}$, *(with* $r = |\mathbf{x}\,|$*), show that*

(a) $\mathrm{div}\,\mathbf{F} = 2\;\mathrm{div}\,\mathbf{G} = 4\,\mathbf{a}\cdot\mathbf{x}\,$,

(b) $\mathrm{curl}\,\mathbf{G} = -2\;\mathrm{curl}\,\mathbf{F} = 2\,\mathbf{x}\;\times\mathbf{a}$,

(c) $\mathrm{div}\,\mathrm{curl}\,\mathbf{F} = \mathrm{div}\,\mathrm{curl}\,\mathbf{G} = 0$,

(d) $\mathrm{curl}\,\mathrm{curl}\,\mathbf{G} = -2\;\mathrm{curl}\,\mathrm{curl}\,\mathbf{F} = -4\,\mathbf{a}$.

26 *Exam question June 2001 (Section A):*

(a) *Give a representation of the vector function* $\mathbf{A}(x,y) \;=\; y\mathbf{e_1}$ *as a collection of arrows in the region of the* (x,y)*-plane bounded by* $(x_1,y_1) \;=\; (-2,2), (x_2,y_2) \;=\; (2,2), (x_3,y_3) = (2,-2), (x_4,y_4) = (-2,-2)$.

(b) *Calculate the curl of the vector field* $\mathbf{A}(x, y) = (-y\mathbf{e_1} + x\mathbf{e_2})/(x^2 + y^2)$ *defined everywhere in the* (x, y)*-plane except at the origin. (You can consider* $\mathbf{A}$ *to be embedded in three dimensions, independent of* z *and with zero* z *component.)*

(c) *Give the unit vector normal to the surface of equation* $ax + by = cz$*, where* a, b, c*, are three real constants.*

(d) *(Slightly modified from exam) Let* $\mathbf{x}$ *be the position vector in 3-dimensions and* $\mathbf{a}$ *be a constant vector. Use the result* $\mathbf{x} \times (\mathbf{x} \times \mathbf{a}) = \mathbf{x}\,(\mathbf{x} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{x} \cdot \mathbf{x})$ *to show that* $\mathrm{div}\,[\,\mathbf{x} \times (\mathbf{x} \times \mathbf{a})] = 2\mathbf{a} \cdot \mathbf{x}$.

27 *Let* $\underline{v} : \mathbb{R}^3 \to \mathbb{R}^3$ *be a vector field. Prove the vector identity*

$$\underline{v} \times (\underline{\nabla} \times \underline{v}) = \underline{\nabla}(|\underline{v}|^2/2) - (\underline{v} \cdot \underline{\nabla})\underline{v}.$$