6 Differentiability of vector fields

6.1 Differentiable maps $\mathbb{R}^n \to \mathbb{R}^n$

We will now generalise the idea of differentiability of scalar fields from section 5. Recall that for a scalar field $f(\underline{x}) : U \to \mathbb{R}$, with U open in $\mathbb{R}^n$, f is differentiable at $\underline{a} \in U$ if

$$\begin{aligned} f(\underline{a}+\underline{h}) - f(\underline{a}) &= \underline{h}.\underline{\nabla} f(\underline{a}) + R(\underline{h}) && (a) \\ \text{with} \qquad \lim_{\underline{h}\to\underline{0}} \frac{R(\underline{h})}{|\underline{h}|} &= 0, && (b) \end{aligned}$$

where you should note that the first term on RHS of (a), $\underline{h}.\underline{\nabla} f(\underline{a})$, is *linear* in $\underline{h}$.

Definition 6.1. Consider a vector field $\underline{F}(\underline{x}) : U \to \mathbb{R}^n$, U open in $\mathbb{R}^n$. Then $\underline{F}$ is defined to be *differentiable* at $\underline{a} \in U$ if there is a linear function $\underline{L} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\begin{aligned} \underline{F}(\underline{a}+\underline{h}) - \underline{F}(\underline{a}) &= \underline{L}(\underline{h}) + \underline{R}(\underline{h}) && (A) \\ \text{with} \qquad \lim_{\underline{h}\to\underline{0}} \frac{\underline{R}(\underline{h})}{|\underline{h}|} &= \underline{0}\,. && (B) \end{aligned}$$

Now linear functions $\mathbb{R}^n \to \mathbb{R}^n$ are matrices. To see what matrix use the standard basis,

$$\begin{aligned} \underline{F}(\underline{x}) &= F_1(\underline{x})\underline{e}_1 + F_2(\underline{x})\underline{e}_2 + \cdots + F_n(\underline{x})\underline{e}_n \\ \underline{L}(\underline{h}) &= L_1(\underline{h})\underline{e}_1 + L_2(\underline{h})\underline{e}_2 + \cdots + L_n(\underline{h})\underline{e}_n \\ \underline{R}(\underline{h}) &= R_1(\underline{h})\underline{e}_1 + R_2(\underline{h})\underline{e}_2 + \cdots + R_n(\underline{h})\underline{e}_n, \end{aligned}$$

so the jth components of (A) and (B) are

$$\begin{aligned} F_j(\underline{a}+\underline{h}) - F_j(\underline{a}) &= L_j(\underline{h}) + R_j(\underline{h}) && (A)_j \\ \text{with} \qquad \lim_{\underline{h}\to\underline{0}} \frac{R_j(\underline{h})}{|\underline{h}|} &= 0\,. && (B)_j \end{aligned}$$

These are just conditions (a) and (b) for $F_j(\underline{x})$ to be differentiable as a map $U \to \mathbb{R}$, i.e. as a scalar field. So we can use results from section 5 to see that

$$L_j(\underline{h}) = \underline{h}.\underline{\nabla} F_j(\underline{a}),$$

that is

$$\begin{aligned} L_1 &= \underline{h}.\underline{\nabla} F_1(\underline{a}) = h_1\frac{\partial F_1}{\partial x_1} + h_2\frac{\partial F_1}{\partial x_2} + \cdots + h_n\frac{\partial F_1}{\partial x_n} \\ L_2 &= \underline{h}.\underline{\nabla} F_2(\underline{a}) = h_1\frac{\partial F_2}{\partial x_1} + h_2\frac{\partial F_2}{\partial x_2} + \cdots + h_n\frac{\partial F_2}{\partial x_n} \\ &\vdots \quad \vdots \\ L_n &= \underline{h}.\underline{\nabla} F_n(\underline{a}) = h_1\frac{\partial F_n}{\partial x_1} + h_2\frac{\partial F_n}{\partial x_2} + \cdots + h_n\frac{\partial F_n}{\partial x_n}, \end{aligned}$$

or

$$\begin{aligned} \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_n \end{pmatrix} &= \begin{pmatrix} h_1\frac{\partial F_1}{\partial x_1} + h_2\frac{\partial F_1}{\partial x_2} + \ldots + h_n\frac{\partial F_1}{\partial x_n} \\ h_1\frac{\partial F_2}{\partial x_1} + h_2\frac{\partial F_2}{\partial x_2} + \ldots + h_n\frac{\partial F_2}{\partial x_n} \\ \vdots \\ h_1\frac{\partial F_n}{\partial x_1} + h_2\frac{\partial F_n}{\partial x_2} + \ldots + h_n\frac{\partial F_n}{\partial x_n} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}. \end{aligned}$$

The $n \times n$ matrix on the RHS of the last equation is called the Jacobian matrix, or differential, of $\underline{F}(\underline{x})$ at $\underline{x} = \underline{a}$; it is written as $D\underline{F}(\underline{a})$, or $d\underline{F}(\underline{a})$ (or $D\underline{F}_{\underline{a}}$ or $d\underline{F}_{\underline{a}}$ or even J_{ij}).

Definition 6.2. The determinant of the differential,

$$\det(D\underline{v}) \equiv |D\underline{v}|$$

is called the Jacobian, $J(\underline{v})$.

Example 33. If

$$\underline{v}(\underline{x}) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

then

$$D\underline{v}(\underline{x}) = \begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

The Jacobian is then given by

$$\begin{aligned} J(\underline{v}) &= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} \\ &= 4(x^2 + y^2) = 4|\underline{x}|^2 \end{aligned}$$

Example 34. If $\underline{x} \in \mathbb{R}^n$ and $\underline{v}(\underline{x}) = \underline{x}$ then

$$D\underline{v}(\underline{x}) = \begin{pmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial x_2} & \cdots & \frac{\partial x_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial x_1} & \frac{\partial x_n}{\partial x_2} & \cdots & \frac{\partial x_n}{\partial x_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = \mathbb{I}_n,$$

the $n \times n$ identity matrix, and

$$J(\underline{v}) = |\mathbb{I}_n| = 1$$

6.2 Diffeomorphisms and the inverse function theorem

We can think of a vector field $\underline{v}(\underline{x})$ as a mapping $\mathbb{R}^n \to \mathbb{R}^n$, or equivalently as a coordinate transformation on $\mathbb{R}^n$. If we think of the components of $\underline{h}$ as the coordinates of a point $\underline{x} = \underline{a} + \underline{h}$ relative to an "origin" at $\underline{a}$, then the components of $\underline{v}(\underline{a}+\underline{h}) - \underline{v}(\underline{a})$ are the transformed coordinates relative to the transformed origin $\underline{v}(\underline{a})$. Then for differentiable $\underline{v}$ (and small $\underline{h}$)

$$\begin{aligned} \underline{v}(\underline{a}+\underline{h}) - \underline{v}(\underline{a}) &\simeq D\underline{v}(\underline{a}) \,.\, \underline{h} \\ \text{new coordinates} &\simeq \text{matrix} \times \text{old coordinates}, \end{aligned}$$

which is a *linear* transformation, invertible if the determinant of $D\underline{v}(\underline{a})$ (i.e. the Jacobian) is non-zero.

The inverse function theorem says that this invertibility can be extended beyond the linear behaviour:

Theorem 6.3. Let $\underline{v} : U \to \mathbb{R}^n$ (with U open in $\mathbb{R}^n$) be a differentiable vector field with continuous partial derivatives, and let $\underline{a} \in U$. Then if $J(\underline{v}(\underline{a})) \neq 0$, $\exists$ an open set $\tilde{U} \subseteq U$ containing $\underline{a}$ such that

(i) $\underline{v}(\tilde{U})$ is open

(ii) The mapping $\underline{v}$ from $\tilde{U}$ to $\underline{v}(\tilde{U})$ has a differentiable inverse - i.e. there exists a differentiable vector field $\underline{w} : \underline{v}(\tilde{U}) \to \mathbb{R}^n$ such that $\underline{w}(\underline{v}(\underline{x})) = \underline{x}$ and $\underline{v}(\underline{w}(\underline{y})) = \underline{y}$

Definition 6.4.

- A mapping $\underline{v} : \tilde{U} \to V \subset \mathbb{R}^n$ satisfying (i) and (ii) above is called a diffeomorphism of $\tilde{U}$ onto $\tilde{V} = \underline{v}(\tilde{U})$, and $\tilde{U}$ and $\tilde{V}$ are said to be diffeomorphic.
- More generally, a mapping $\underline{v} : U \to V$ is called a local diffeomorphism if for every point $\underline{a} \in U$ there is an open set $\tilde{U} \subset U$ containing $\underline{a}$ such that $\underline{v} : \tilde{U} \to \underline{v}(\tilde{U})$ is a diffeomorphism.

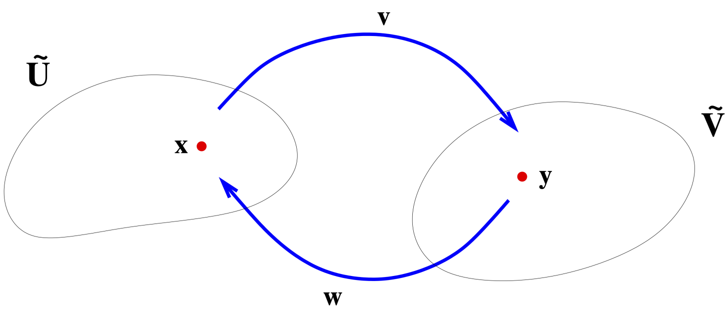

Figure 29: A diffeomorphism from $\tilde{U}$ to $\tilde{V}$

Remarks In general, suppose that

$$\underline{v} : U \to V \subset \mathbb{R}^n$$
$$\underline{w} : V \to W \subset \mathbb{R}^n$$

(with U and V both open in $\mathbb{R}^n$) are both continuously differentiable vector fields (not necessarily diffeomorphisms).

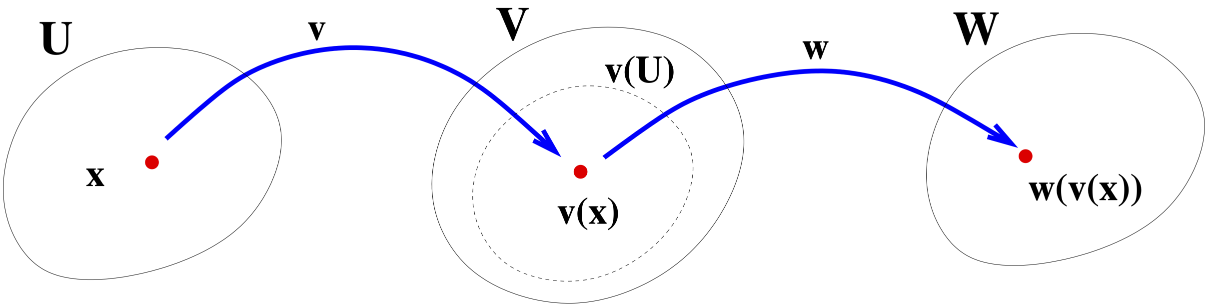

Figure 30: Composition of maps $\underline{v}$ and $\underline{w}$

Then $\underline{w}(\underline{v}(\underline{x}))$ is a mapping $U \to W \subset \mathbb{R}^n$ and its differential can be calculated using the chain rule:

$$\begin{aligned}
D\underline{w}(\underline{v}(\underline{x})) &= \begin{pmatrix} \frac{\partial w_1(\underline{v}(\underline{x}))}{\partial x_1} & \frac{\partial w_1(\underline{v}(\underline{x}))}{\partial x_2} & \cdots & \frac{\partial w_1(\underline{v}(\underline{x}))}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial w_n(\underline{v}(\underline{x}))}{\partial x_1} & \frac{\partial w_n(\underline{v}(\underline{x}))}{\partial x_2} & \cdots & \frac{\partial w_n(\underline{v}(\underline{x}))}{\partial x_n} \end{pmatrix} \\
&= \begin{pmatrix} \sum_j \frac{\partial w_1}{\partial v_j}\frac{\partial v_j}{\partial x_1} & \sum_j \frac{\partial w_1}{\partial v_j}\frac{\partial v_j}{\partial x_2} & \cdots & \sum_j \frac{\partial w_1}{\partial v_j}\frac{\partial v_j}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \sum_j \frac{\partial w_n}{\partial v_j}\frac{\partial v_j}{\partial x_1} & \sum_j \frac{\partial w_n}{\partial v_j}\frac{\partial v_j}{\partial x_2} & \cdots & \sum_j \frac{\partial w_n}{\partial v_j}\frac{\partial v_j}{\partial x_n} \end{pmatrix}
\end{aligned}$$

using chain rule on each matrix element,

$$\begin{aligned}
&= \begin{pmatrix} \frac{\partial w_1}{\partial v_1} & \frac{\partial w_1}{\partial v_2} & \cdots & \frac{\partial w_1}{\partial v_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial w_n}{\partial v_1} & \frac{\partial w_n}{\partial v_2} & \cdots & \frac{\partial w_n}{\partial v_n} \end{pmatrix} \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \cdots & \frac{\partial v_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial v_n}{\partial x_1} & \frac{\partial v_n}{\partial x_2} & \cdots & \frac{\partial v_n}{\partial x_n} \end{pmatrix} \\
&= D\underline{w}(\underline{v})\, D\underline{v}(\underline{x}) \qquad \text{i.e. matrix multiplication}
\end{aligned}$$

For the special case when $\underline{v}$ is a local diffeomorphism and $\underline{w}$ is its inverse map,

$$\begin{aligned}
\underline{w}(\underline{v}(\underline{x})) &= \underline{x} \\
\Rightarrow D\underline{w}\, D\underline{v} &= D\underline{w}(\underline{v}(\underline{x})) = D\underline{x}(\underline{x}) = \mathbb{I}_n,
\end{aligned}$$

using Example 34 above. Likewise

$$\begin{aligned} \underline{v}(\underline{w}(\underline{y})) &= \underline{y} \\ \Rightarrow D\underline{v}\, D\underline{w} &= D\underline{v}(\underline{w}(\underline{y})) = D\underline{y}(\underline{y}) = \mathbb{I}_n. \end{aligned}$$

So $D\underline{v}$ is an invertible matrix, with inverse $(D\underline{v})^{-1} = D\underline{w}$. Taking determinants,

$$J(\underline{w}) = \frac{1}{J(\underline{v})},$$

and in particular, $J(\underline{v}) \neq 0$, which was the main condition of the inverse function theorem.

Definition 6.5. Such a $\underline{v}$ is called orientation preserving if $J(\underline{v}) > 0$, and orientation reversing if $J(\underline{v}) < 0$.

Example 35. Continuing Example 33, we had

$$\underline{v}(\underline{x}) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} \qquad \Rightarrow \qquad J(\underline{v}) = 4(x^2 + y^2).$$

So for $(x, y) \neq (0, 0)$, $J(\underline{v}) > 0$.

Hence if $U = \mathbb{R}^2 - \{\underline{0}\}$, $\underline{v} : U \to U$ is an orientation preserving local diffeomorphism. However it is not a global diffeomorphism since $\underline{v}(-\underline{x}) = \underline{v}(\underline{x})$ (so no inverse can exist globally). But $\underline{v}$ does map $\{(x, y) : x > 0\}$ onto $\mathbb{R}^2 - \{(x, 0), x \leq 0\}$ diffeomorphically.

Example 36. Consider the transformation from polar coordinates (r, θ) back to cartesians (x, y). We have

$$\underline{v}(r, \theta) = \begin{pmatrix} x(r, \theta) \\ y(r, \theta) \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

Differential:

$$D\underline{v} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \tag{6.1}$$

Jacobian:

$$\begin{aligned} J(\underline{v}(r, \theta)) &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta = r \end{aligned}$$

For $r > 0$, $J(\underline{v}) > 0$, and the transformation is therefore orientation preserving.

The inverse mapping is

$$\underline{w}(x, y) = \begin{pmatrix} r(x, y) \\ \theta(x, y) \end{pmatrix} = \begin{pmatrix} \sqrt{x^2 + y^2} \\ \tan^{-1}\left(\frac{y}{x}\right) \end{pmatrix}$$

Exercise: check this!