1 Functions

1.1 Functions, domain and range

Defn: A **function** f is a correspondence between two sets D (called the *domain*) and C (called the *codomain*), that assigns to each element of D one and only one element of C. The notation to indicate the domain and codomain is $f : D \mapsto C$.

For $x \in D$ we write $f(x)$ to denote the assigned element in C, and call this the value of f at x, or the image of x under f, where x is called the argument of the function. Extending the above notation we write

$$f : D \mapsto C : x \mapsto f(x)$$

ie. "f is a function from D to C that associates x in D to $f(x)$ in C".

The set of all images is called the **range** of f and our notation for the domain and range is

$$\text{Dom } f \qquad \text{and} \qquad \text{Ran } f = \{f(x) : x \in \text{Dom } f\}.$$

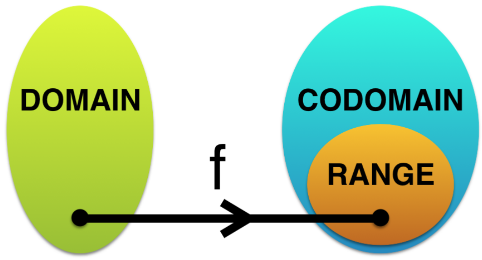

Figure 1: Illustration of the domain, codomain and range of a function.

In (almost all of) this term of the module we shall only deal with real-valued functions of a real variable, meaning that our functions assign real numbers to real numbers ie. both the domain and codomain are subsets of $\mathbb{R}$.

There are various ways of representing a function, but perhaps the most familiar is through an explicit expression.

Eg. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2, \ \forall x \in \mathbb{R}$.
In this case Dom $f = \mathbb{R}$ and Ran $f = [0, \infty)$.
We say that f *maps* the real line onto $[0, \infty)$.

Eg. $f(x) = \sqrt{2x + 4}, \ x \in [0, 6]$. Here Dom $f = [0, 6]$ and Ran $f = [2, 4]$.

If the domain of a function f is not explicitly given, then it is taken to be the maximal set of real numbers x for which $f(x)$ is a real number.

Eg. $f(x) = \sqrt{2x+4}$. Here Dom $f = [-2, \infty)$ and Ran $f = [0, \infty)$.

Eg. $f(x) = 1/(1-x)$. Here Dom $f = \mathbb{R}\backslash\{1\} = (-\infty, 1) \cup (1, \infty)$ and Ran $f = \mathbb{R}\backslash\{0\}$.

In addition to explicit expressions, a function f may be represented by other means, for example, as a set of ordered pairs (x, y), where $x \in$ Dom f and $y \in$ Ran f.

Note: $y(x)$ is another common notation for a function. The element x in the domain is called the **independent variable** and the element y in the range is called the **dependent variable**. This notation is often used if the function is defined by an equation in two variables, including differential equations (see later).

The function $f(x) = x^2$ is defined by the equation $y = x^2$, as we have simply written $y = f(x)$. However, not all equations will define a function.

Eg. The equation $y^2 = x$ does *not* define a non-trivial function $y(x)$, because for $x < 0$ there are no real solutions for y, and for $x > 0$ there are two solutions for y, whereas a function must assign only one element of the range for each element of the domain.

1.2 The graph of a function

Defn: The **graph** of a function f is the set of all points (x, y) in the xy-plane with $x \in \text{Dom}\, f$ and $y = f(x)$, ie.

$$\text{graph f} = \{(x, y) : x \in \text{Dom}\, f \text{ and } \; y = f(x)\}.$$

The graph of a function over the interval $[a, b]$ is the portion of the graph where the argument is restricted to this interval.

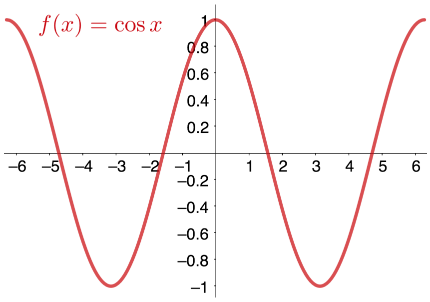

Figure 2: The graph of $f(x) = \cos x$ over $[-2\pi, 2\pi]$.

Note: If you are asked to graph a function, but no interval is given, then try to choose an appropriate interval that includes all the interesting behaviour eg. turning points.

To indicate on a graph exactly which points are included we use a closed circle to denote an included point, so if it is at the end of a curve segment this denotes that this end is a closed interval. We use an open circle to denote an excluded point associated with an open interval.

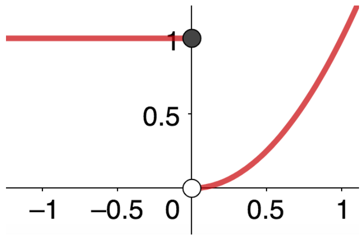

Figure 3: A graph illustating the use of open (white) and closed (black) circles.

The graph of a function is a curve in the plane, but not every curve is the graph of a function. The following simple test determines whether a curve is the graph of a function.

The vertical line test.
If any vertical line intersects the curve more than once then the curve is not the graph of a function, otherwise it is.

The proof is obvious, given the defining property of a function that only one element in the range is associated with an element in the domain.

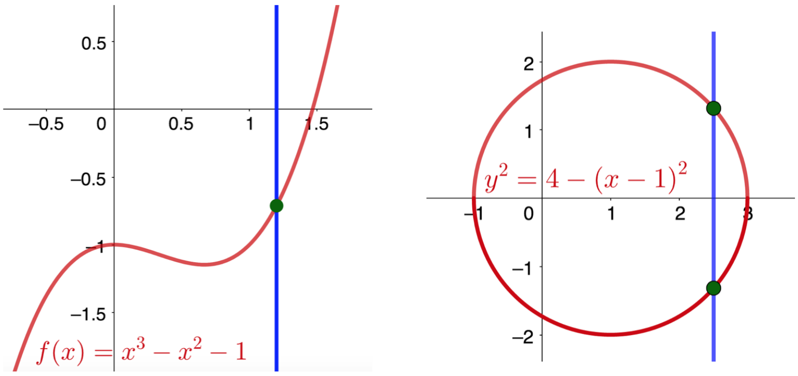

Figure 4: The vertical line test applied to a cubic curve and a circle.

In the figure the first curve is the graph of a function: in fact the cubic function $f(x) = x^3 - x^2 - 1$. The second curve is not the graph of a function as the vertical line shown intersects the curve twice. In fact the curve is given by the equation $y^2 = 4 - (x-1)^2$ ie. the circle of radius 2 with centre $(x, y) = (1, 0)$. Note that even when a curve is not the graph of a function, some sections of the curve may be graphs of functions. For the above circle example the section given by $y \geq 0$ is the graph of the function $f(x) = \sqrt{4 - (x-1)^2}$ with Dom $f = [-1, 3]$.

1.3 Even and odd functions

Defn. A function f is **even** if $f(x) = f(-x)\ \forall \pm x \in$ Dom f.
Defn. A function f is **odd** if $f(x) = -f(-x)\ \forall \pm x \in$ Dom f.

The graph of an even function is symmetric under a reflection about the y-axis.
The graph of an odd function is symmetric under a rotation by $180°$ about the origin (equivalent to a reflection in the y-axis followed by a reflection in the x-axis).

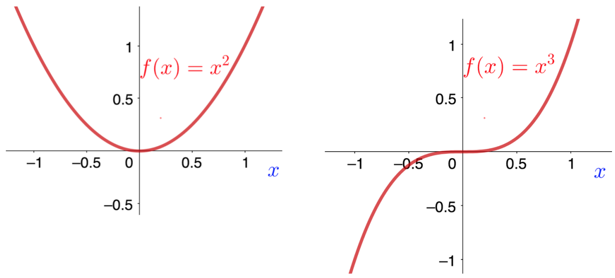

Figure 5: Graphs of the even function x^2 and the odd function x^3.

All functions $f : \mathbb{R} \mapsto \mathbb{R}$ can be written as the sum of an even and an odd function

$$f(x) = f_{even}(x) + f_{odd}(x),$$

$$\text{where} \quad f_{even}(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x) \quad \text{and} \quad f_{odd}(x) = \frac{1}{2}f(x) - \frac{1}{2}f(-x).$$

This decomposition is often useful, as is the ability to spot an even or odd function, as it can simplify some calculations, as we shall see in later sections.

Eg. $f(x) = (1 + x)\sin x$ with $f_{even}(x) = x \sin x$ and $f_{odd}(x) = \sin x$.

Eg. $f(x) = e^x$ with $f_{even}(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$ and $f_{odd}(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh x$.

1.4 Piecewise functions

Some functions are defined **piecewise** ie. different expressions are given for different intervals in the domain.

Eg. The absolute value (or modulus) function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

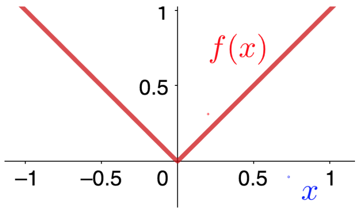

Figure 6: The graph of $|x|$ over $[-1, 1]$.

Eg.

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ x - 1 & \text{if } 1 \leq x \leq 2. \end{cases}$$

In this example Dom $f = [0, 2]$ but the function has a discontinuity at $x = 1$ (more about this later).

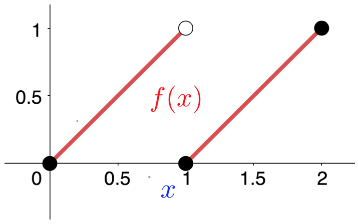

Figure 7: The graph of a piecewise function.

A **step function** is a piecewise function which is constant on each piece. An example is the Heaviside step function $H(x)$ defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Note that with this definition Dom $H = \mathbb{R} \backslash \{0\}$. It is sometimes convenient to extend the domain to $\mathbb{R}$ by defining the value of $H(0)$ (some obvious candidates are $0, \frac{1}{2}, 1$).

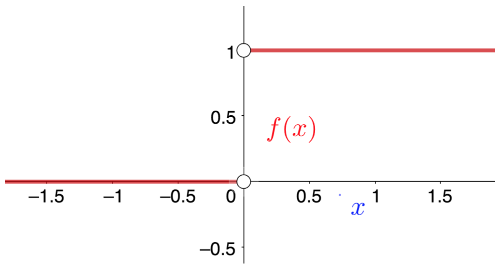

Figure 8: The Heaviside step function.

1.5 Operations with functions

Given two functions f and g we can define the following:

- the **sum** is $(f+g)(x) = f(x) + g(x)$, with domain $\mathrm{Dom}\, f \cap \mathrm{Dom}\, g$.

- the **difference** is $(f-g)(x) = f(x) - g(x)$, with domain $\mathrm{Dom}\, f \cap \mathrm{Dom}\, g$.

- the **product** is $(fg)(x) = f(x)g(x)$, with domain $\mathrm{Dom}\, f \cap \mathrm{Dom}\, g$.

- the **ratio** is $(f/g)(x) = f(x)/g(x)$, with domain $(\mathrm{Dom}\, f \cap \mathrm{Dom}\, g)\backslash\{x : g(x) = 0\}$.

- the **composition** is $(f \circ g)(x) = f(g(x))$, with domain $\{x \in \mathrm{Dom}\, g : g(x) \in \mathrm{Dom}\, f\}$.

Note that $f \circ g$ and $g \circ f$ are usually different functions.
Eg. $f(x) = \sin x,\ g(x) = x^2$ then $(f \circ g)(x) = \sin(x^2)$ but $(g \circ f)(x) = \sin^2 x$.

Note that the sum and difference, along with **scalar multiplication** $(cf)(x) = c \times f(x)$ for any constant c are special case of **linear combinations** of functions. The most general linear combination of f and g is $(af + bg)(x) = a \times f(x) + b \times g(x)$ for some constants a and b, with domain $\mathrm{Dom}\, f \cap \mathrm{Dom}\, g$..

1.6 Inverse functions

Defn. A function $f : D \mapsto C$ is **surjective** (or onto) if Ran $f = C$,
ie. if $\forall\ y \in C\ \exists\ x \in D$ s.t. $f(x) = y$.

Eg. $f : \mathbb{R} \mapsto \mathbb{R}$ given by $f(x) = 2x + 1$ is surjective.

Eg. $f : \mathbb{R} \mapsto \mathbb{R}$ given by $f(x) = x^2$ is not surjective, since any negative number in the codomain is not the image of an element in the domain.

Defn. A function $f : D \mapsto C$ is **injective** (or one-to-one) if $\forall\ x_1, x_2 \in D$ with $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

Eg. $f(x) = 2x + 1$ is injective since $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Eg. $f(x) = x^2$ is not injective since $f(x) = f(-x)$ and $x \neq -x$ if $x \neq 0$.

The following simple test can be applied to see if a function is injective from its graph.

The horizontal line test.
If no horizontal line intersects the graph of f more than once then f is injective, otherwise it is not.

Eg. The function $f(x) = x^3 - 3x$ is surjective as Ran $f = \mathbb{R}$ but it is not injective as the horizontal line $y = 1$ intersects the graph of f at 3 points (see the figure).

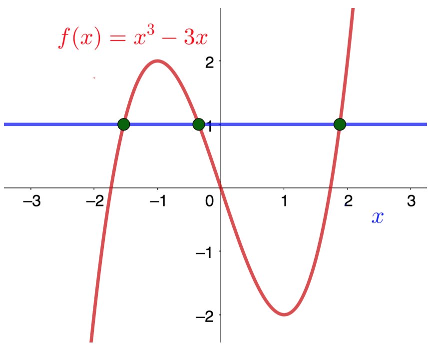

Figure 9: Graph of $f(x) = x^3 - 3x$ and the horizontal line $y = 1$.

Defn. A function $f : D \mapsto C$ is **bijective** if it is both surjective and injective.

Eg. From above we have seen that the function $f(x) = 2x + 1$ is bijective.

Theorem of inverse functions.
A bijective function f admits a unique inverse, denoted f^{-1}, such that

$$(f^{-1} \circ f)(x) = x = (f \circ f^{-1})(x).$$

It is clear from the definition that Dom f^{-1} = Ran f and Ran f^{-1} = Dom f.

As the inverse undoes the effect of the function an equivalent definition is

$$f(x) = y \quad \text{iff} \quad f^{-1}(y) = x.$$

This second definition is the same as the fact that the graph of the inverse function f^{-1} is given by reflecting the graph of f in the line $y = x$. However, note that in general the curve given by such a reflection is not the graph of a function – in particular it will not be single-valued if f is not injective.

Eg. We have seen that $f(x) = 2x + 1$ is bijective, so lets find its inverse. To simplify notation first write $y = f^{-1}(x)$. Using the property $x = f(f^{-1}(x))$, we have $x = f(y) = 2y+1$ and hence $y = \frac{1}{2}(x - 1) = f^{-1}(x)$.

Note that given an injective function which is not surjective, we can make a bijective function by taking the codomain equal to the range (this automatically makes a function surjective), and then an inverse exists. This is essentially a harmless modification of the function since the function does not really care what its codomain is, the only requirement is that the range is a subset of the codomain.

Eg. We have seen that $f(x) = x^2$ is not an injective function if Dom $f = \mathbb{R}$, but it is injective if we take Dom $f = [0, \infty)$. With this choice we can take the codomain equal to Ran $f = [0, \infty)$ and f is now a bijective function. The inverse is $f^{-1}(x) = \sqrt{x}$ with Dom f^{-1} = Ran $f = [0, \infty)$.

Note that here we made the function injective by restricting its domain. While we can always do this, in general doing so loses information about the function since the original non-injective function would have given values for other inputs values. However, in special cases such as even functions or periodic functions, we can restrict the domain and have a simple description of how to generate the original function (say with domain $\mathbb{R}$) from the function with the restricted domain. We used this above to find an inverse for $f(x) = x^2$ and this is also how we define an inverse for $f(x) = \sin x$ etc.

Warning: Don't confuse the notation for the inverse with the same notation for the reciprocal.

1.7 Summary: Functions

You should have a good and precise mathematical understanding of functions and various definitions, particularly focussing on real-valued functions of a single real variable. Here are some key points:

- A function f is a mapping from its *domain*, Dom f, to a *codomain*. Its *range*, Ran f, is the image of Dom f, i.e. the set of all values that $f(x)$ can actually take for $x \in$ Dom f.
- Functions are single-valued, i.e. $f(x)$ must have a unique value for all $x \in$ Dom f. This can be checked graphically by the *vertical line test*.
- Functions can be *even* – symmetric under reflection in the y-axis.
- Functions can be *odd* – symmetric under rotation by 180° around the origin, or equivalently by two reflections, in the x- and y-axes (in either order).
- Typical functions are neither even nor odd, but can be uniquely written as a sum of an even and an odd function, i.e. $f(x) = f_{even}(x) + f_{odd}(x)$. You should know how to define f_{even} and f_{odd} in terms of f.
- We can define *piecewise* functions using different expressions for different parts of the domain. At boundaries of intervals we use filled or empty circles to indicate that a point is or is not included.
- You should be familiar with the operations on functions: *linear combinations*, *product*, *ratio* and *composition*, and know what the domain of the resulting functions is in terms of the domains of the original functions.
- You should know the conditions for the *inverse* of a function to exist. You should know the definitions of *surjective*, *injective* and *bijective*.