3 $\underline{\nabla}$ acting on vector fields

So far we've introduced scalar fields and vector fields, and seen how the differential operator $\underline{\nabla}$ can act on a scalar field $f(\underline{x}) : \mathbb{R}^n \to \mathbb{R}$ to give a vector field known as the gradient of the scalar field, $\underline{\nabla} f$. We also saw that the gradient tells us the direction in which the scalar field increases fastest. As the title of this section suggests, we're now going to see how we can define an action of $\underline{\nabla}$ on vector fields $\underline{v}(\underline{x}) : \mathbb{R}^n \to \mathbb{R}^n$, and what these quantities can tell us about the original vector field.

3.1 Divergence (div)

Since $\underline{\nabla}$ is a vector operator, and vector fields assign a vector $\underline{v}(\underline{x})$ for each point $\underline{x} \in \mathbb{R}^n$, we can take the dot product between $\underline{\nabla}$ and $\underline{v}(\underline{x})$. This quantity $\underline{\nabla}.\underline{v}$ is the known as the *divergence* of $\underline{v}$, and is also written as div $\underline{v}$.

In the standard cartesian basis for $\mathbb{R}^n$

$$\begin{aligned}
& \underline{v}(\underline{x}) = \underline{e}_1 v_1(\underline{x}) + \underline{e}_2 v_2(\underline{x}) + \cdots + \underline{e}_n v_n(\underline{x}) \\
\Rightarrow \quad \underline{\nabla}.\underline{v} \;\equiv\; \text{div}\,\underline{v} & \equiv (\underline{e}_1 \frac{\partial}{\partial x_1} + \cdots + \underline{e}_n \frac{\partial}{\partial x_n}) \,.\, (\underline{e}_1 v_1(\underline{x}) + \cdots + \underline{e}_n v_n(\underline{x})) \\
& = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \cdots + \frac{\partial v_n}{\partial x_n} \\
& = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} \qquad \text{(index notation)} \\
& = \frac{\partial v_i}{\partial x_i}, \qquad \text{(Einstein Summation Convention)}
\end{aligned}$$

since the $\underline{e}_a$ are orthonormal and constant.

Beware! In other coordinate systems the basis vectors $\underline{e}_a$ might vary with $\underline{x}$ and hence the formula needs more care. We may return to the divergence in other coordinate systems later in the term, depending on time.

Note: $\underline{\nabla}.\underline{v}(\underline{x})$ is a scalar field, as can be seen in the following example.

Example 9.

$$\begin{aligned}
\underline{v}(\underline{x}) &= (v_1(\underline{x}), v_2(\underline{x}), v_3(\underline{x})) \;=\; (x^2, y^2, z^2) \\
\underline{\nabla}.\underline{v} &= \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z} \\
&= 2(x + y + z),
\end{aligned}$$

a number for each point $\underline{x} = (x, y, z)$ not a vector.

Although we treat $\underline{\nabla}$ like a vector, note that it is a vector differential operator, and therefore is not actually a vector in $\mathbb{R}^n$ like $\underline{x}$ is. Therefore although the inner product on $\mathbb{R}^n$ is symmetric, note that

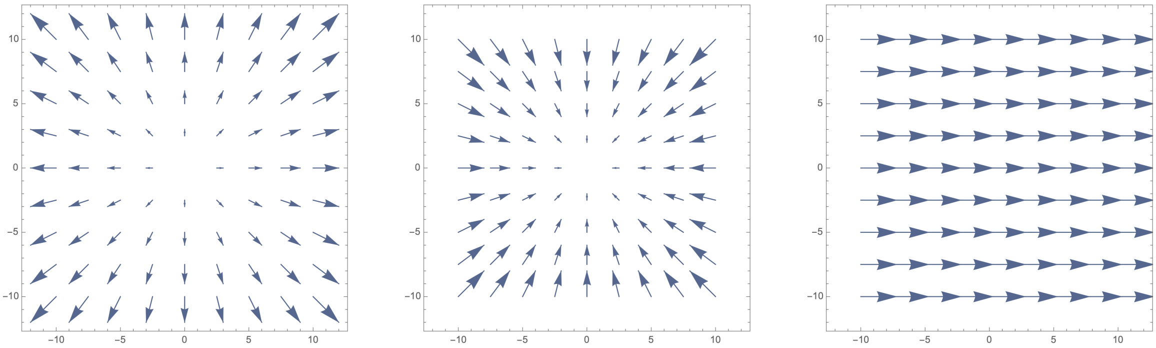
$$\underline{\nabla}.\underline{v} \;\neq\; \underline{v}.\underline{\nabla},$$

as the left-hand side of this is a scalar field, whereas the right-hand side is the (scalar) differential operator

$$\left(x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z} \right)$$

acting on scalar fields.

To get an intuition of what the divergence tells us about our vector field, we should think of our vector field as if it were a fluid, where the direction of a vector at a point tells us the direction of fluid flow at that point, and the magnitude of the vector tells us how fast the fluid is flowing at the point. The divergence of the vector field at a point then tells us whether the point is acting like a *source* (corresponding to positive divergence) or a *sink* (negative divergence) for the fluid. That is, whether more of the 'fluid' is entering than the point than is leaving. This is shown for a vector field $\underline{v}$ in Figure 11, where the divergence at the origin is positive, negative and zero respectively.



$\underline{\nabla}.\underline{v} > 0$ $\qquad$ $\underline{\nabla}.\underline{v} < 0$ $\qquad$ $\underline{\nabla}.\underline{v} = 0$

Figure 11: The divergence of a vector field at a point tells us whether the point acts like a source or a sink for the vector field, if we think of the vector field as describing the flow of a fluid.

Properties of div:

Let a, b be constants, f, g be scalar fields and $\underline{v}, \underline{w}$ vector fields, all in $\mathbb{R}^n$. Then

$$\begin{aligned} (i)\underline{\nabla}.(a\underline{v} + b\underline{w}) &= a\underline{\nabla}.\underline{v} + b\underline{\nabla}.\underline{w} \\ (ii)\underline{\nabla}.(f\underline{v}) &= (\underline{\nabla}f).\underline{v} + f\underline{\nabla}.\underline{v} \end{aligned}$$

<u>Proof:</u> (i) This follows from linearity of the partial derivative:

$$\begin{aligned} \underline{\nabla}.(a\underline{v} + b\underline{w}) &= \frac{\partial(av_1 + bw_1)}{\partial x_1} + \cdots + \frac{\partial(av_n + bw_n)}{\partial x_n} \\ &= a\frac{\partial v_1}{\partial x_1} + b\frac{\partial w_1}{\partial x_1} + \cdots + a\frac{\partial v_n}{\partial x_n} + b\frac{\partial w_n}{\partial x_n} \\ &= a\left(\frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n}\right) + b\left(\frac{\partial w_1}{\partial x_1} + \cdots + \frac{\partial w_n}{\partial x_n}\right) \\ &= a\underline{\nabla}.\underline{v} + b\underline{\nabla}.\underline{w}. \end{aligned}$$

(ii) First note that $f\underline{v}$ is a vector field with components $(fv_1, fv_2, fv_3, \ldots, fv_n)$, so

$$\begin{aligned} \underline{\nabla}.(f\underline{v}) &= \frac{\partial fv_1}{\partial x_1} + \cdots + \frac{\partial fv_n}{\partial x_n} \\ &= \frac{\partial f}{\partial x_1}v_1 + f\frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial f}{\partial x_n}v_n + f\frac{\partial v_n}{\partial x_n} \\ &= \frac{\partial f}{\partial x_1}v_1 + \cdots + \frac{\partial f}{\partial x_n}v_n \ + \ f\frac{\partial v_1}{\partial x_1} + \cdots + f\frac{\partial v_n}{\partial x_n} \\ &= (\underline{\nabla}f).\underline{v} \ + \ f(\underline{\nabla}.\underline{v}). \end{aligned}$$

Once we finish section 4, we'll be able to derive this more effectively by using index notation.

Example 10. Suppose

$$\begin{aligned}
f(\underline{x}) &= \underline{a}.\underline{x} \\
\underline{v}(\underline{x}) &= \underline{a} \qquad \text{a constant} \\
\Rightarrow \qquad f\underline{v} &= (\underline{a}.\underline{x})\underline{a} \\
&= (a_1 x + a_2 y + a_3 z)(a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3) \\
\Rightarrow \quad \underline{\nabla}.((\underline{a}.\underline{x})\underline{a}) &= \frac{\partial}{\partial x}(a_1(a_1 x + a_2 y + a_3 z)) \\
&+ \frac{\partial}{\partial y}(a_2(a_1 x + a_2 y + a_3 z)) \\
&+ \frac{\partial}{\partial z}(a_3(a_1 x + a_2 y + a_3 z)) \\
&= a_1^2 + a_2^2 + a_3^2 \;=\; \|\underline{a}\|^2
\end{aligned}$$

by direct calculation. Or, using property (ii):

$$\underline{\nabla}.((\underline{a}.\underline{x})\underline{a}) \;=\; (\underline{\nabla}\underline{a}.\underline{x}).\underline{a} + (\underline{a}.\underline{x})\underline{\nabla}.\underline{a}.$$

But $\underline{\nabla}.\underline{a} = 0$, while

$$\begin{aligned}
\underline{\nabla}(\underline{a}.\underline{x}) &= (\underline{e}_1\frac{\partial}{\partial x} + \underline{e}_2\frac{\partial}{\partial y} + \underline{e}_3\frac{\partial}{\partial z})(a_1 x + a_2 y + a_3 z) \\
&= \underline{e}_1 a_1 + \underline{e}_2 a_2 + \underline{e}_3 a_3 \;=\; \underline{a} \\
\underline{\nabla}.((\underline{a}.\underline{x})\underline{a}) &= \underline{a}.\underline{a} \;=\; \|\underline{a}\|^2
\end{aligned}$$

agreeing with the direct calculation.

3.2 Curl

In 3 dimensions, there is a second type of product one can take between vectors, and this is the vector cross product. Recall that we define the vector product of two vectors $\underline{A}$ and $\underline{B}$ as

$$\begin{aligned}
\underline{A} \times \underline{B} &= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \\
&= \underline{e}_1(A_2B_3 - A_3B_2) + \underline{e}_2(A_3B_1 - A_1B_3) + \underline{e}_3(A_1B_2 - A_2B_1).
\end{aligned}$$

Then, for a vector field $\underline{v}(\underline{x})$ in 3 dimensions, define the <u>curl</u> of $\underline{v}$ as

$$\underline{\nabla} \times \underline{v}(\underline{x}) \;\equiv\; \text{curl}\,\underline{v} \;\equiv\; \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix},$$

where we have to expand this *always making sure that the derivatives* $\frac{\partial}{\partial x_i}$ *are on the left of the* v_i. So therefore we have

$$\underline{\nabla} \times \underline{v}(\underline{x}) = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \quad = \underline{e}_1(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}) + \underline{e}_2(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}) + \underline{e}_3(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}).$$

Note that the curl of a vector field is therefore a *new* <u>vector</u> field.

Example 11. if $\underline{v} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field, and can be expressed in terms of its components as

$\underline{v} = (x^2 z, xyz, x)$, then

$$\begin{aligned}
\text{curl } \underline{v} = \underline{\nabla} \times \underline{v} \; &= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z & xyz & x \end{vmatrix} \\
&= \underline{e}_1 (\frac{\partial x}{\partial y} - \frac{\partial xyz}{\partial z}) + \underline{e}_2 (\frac{\partial x^2 z}{\partial z} - \frac{\partial x}{\partial x}) + \underline{e}_3 (\frac{\partial xyz}{\partial x} - \frac{\partial x^2 z}{\partial y}) \\
&= -xy\underline{e}_1 + (x^2 - 1)\underline{e}_2 + yz\underline{e}_3
\end{aligned}$$

Note that since $\underline{\nabla} \times \underline{v}$ is a vector field, we can calculate its divergence. In the case of $\underline{v}$ being the vector field from Example 11,

$$\begin{aligned}
\underline{\nabla}.(\underline{\nabla} \times \underline{v}) &= \frac{\partial}{\partial x}(-xy) + \frac{\partial}{\partial y}(x^2 - 1) + \frac{\partial}{\partial z}(yz) \\
&= -y + 0 + y \;=\; 0.
\end{aligned}$$

In turns out that this is always true so long as $\underline{v}$ has components with continuous second partial derivatives. We'll come back to this in subsection 3.3.

The curl of a vector field $\underline{v}$, tells us how much $\underline{v}$ is 'curling around' at a point. If we imagine our vector field $\underline{v}$ as a fluid, like when we thought about the meaning of the divergence (see subsection 3.1), the magnitude of the curl then tells us about the rotational speed of the fluid, and the direction of the curl then tells us which axis the fluid is rotating around. This axis is determined using the so-called 'right-hand rule': If you curl the fingers of your right hand, such that your fingers represent the rotation of the fluid, then your thumb points in the direction of the curl vector.

Example 12. Consider the vector field $\underline{v}$ with components $\underline{v} = (-y, x, 0)$. To imagine this vector field, realise that it is independent of z, and so you can imagine the vector field in the $z = 0$ plane, and the vector field in any other plane of z is then just a translation of the vector field at $z = 0$. This $z = 0$ plane is shown in Figure 12.

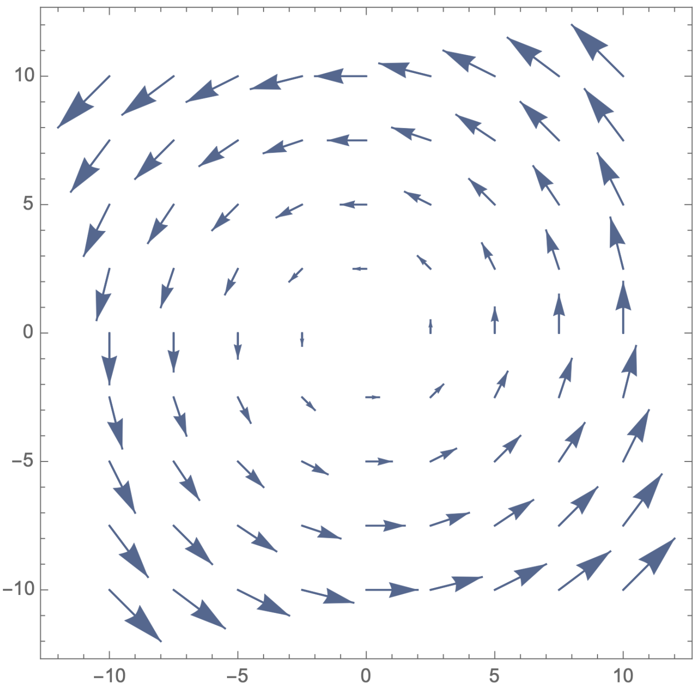


Figure 12: The plane $z = 0$ of the vector field $\underline{v} = (-y, x, 0)$, where the horizontal axis of the image is the x-axis, and the vertical axis is the y-axis. You should imagine the z-axis as coming straight out of the page.

The curl of this vector field can then easily be checked to be

$$\underline{\nabla} \times \underline{v} = (0,0,2),$$

so is a vector field of constant magnitude pointing in the positive z direction, as one would expect using the right-hand rule.

Properties of curl:

Let a, b be constants, $\underline{v}, \underline{w}$ be vector fields, and f be a scalar field, all in $\mathbb{R}^3$. Then:

$$\begin{aligned} (i)\underline{\nabla} \times (a\underline{u} + b\underline{v}) &= a\underline{\nabla} \times \underline{u} + b\underline{\nabla} \times \underline{v} \\ (ii)\underline{\nabla} \times (f\underline{v}) &= (\underline{\nabla} f) \times \underline{v} + f\underline{\nabla} \times \underline{v} \end{aligned}$$

Proof:

(i) follows from linearity of derivatives, as before.

Lets check (ii):

$$\begin{aligned} \underline{\nabla} \times (f\underline{v}) =& \underline{e}_1 \left[\frac{\partial}{\partial y}(fv_3) - \frac{\partial}{\partial z}(fv_2)\right] + \underline{e}_2 \left[\frac{\partial}{\partial z}(fv_1) - \frac{\partial}{\partial x}(fv_3)\right] \\ &+ \underline{e}_3 \left[\frac{\partial}{\partial x}(fv_2) - \frac{\partial}{\partial y}(fv_1)\right] \\ &= \underline{e}_1 \left[\left(\frac{\partial f}{\partial y}\right) v_3 + f\frac{\partial v_3}{\partial y} - \left(\frac{\partial f}{\partial z}\right) v_2 - f\frac{\partial v_2}{\partial z}\right] + [\dots] + [\dots] \\ &= \underline{e}_1 \left[\left(\frac{\partial f}{\partial y}\right) v_3 - \left(\frac{\partial f}{\partial z}\right) v_2\right] \; + \; \underline{e}_1 f \left[\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right] + \dots + \dots \\ &= \underline{\nabla} f \times \underline{v} \; + \; f\underline{\nabla} \times \underline{v} \end{aligned}$$

This is tedious. Once we develop our index notation in section 4 we will be able to do this in a slicker way.

3.3 Applying $\underline{\nabla}$ twice

If $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar field then $\underline{\nabla} f$ is a vector field and we can take its divergence. This is

$$\begin{aligned} \underline{\nabla}.(\underline{\nabla} f) &= \text{div grad } f \equiv \nabla^2 f \equiv \Delta f \\ &= \left(\underline{e}_1 \frac{\partial}{\partial x_1} + \dots + \underline{e}_n \frac{\partial}{\partial x_n}\right) \cdot \left(\underline{e}_1 \frac{\partial}{\partial x_1} + \dots + \underline{e}_n \frac{\partial}{\partial x_n}\right) f \\ &= \frac{\partial^2 f}{\partial x_1^2} + \dots + \frac{\partial^2 f}{\partial x_n^2} \end{aligned}$$

in cartesian coordinates - note that it will look different in other coordinates if the basis vectors are not constant (as in the case of polar coordinates for example).

$\Delta f \equiv \nabla^2 f$ is called the Laplacian of f. Since the Laplacian is the divergence of a vector field, it is a scalar field.

Note that you actually met the Laplacian in Calculus I, when you considered *Linear Partial Differential Equations*. In particular, the Laplacian is a differential operator that appears in *Laplace's Equation*, the *Heat Equation* and the *Wave Equation*. More on this next term.

Example 13. For $n = 2$, let $f = log(x^2 + y^2) = log(\underline{x}.\underline{x})$. Then

$$\begin{aligned}
\Rightarrow \quad \frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2}, \quad \frac{\partial^2 f}{\partial x^2} &= \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2} \\
\Rightarrow \quad \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2}, \quad \frac{\partial^2 f}{\partial y^2} &= \frac{2(x^2-y^2)}{(x^2+y^2)^2} \qquad \text{(just swap } x \text{ and } y\text{)} \\
\Rightarrow \quad \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= \frac{2(y^2-x^2)}{(x^2+y^2)^2} + \frac{2(x^2-y^2)}{(x^2+y^2)^2} \\
= 0 \qquad \text{if } \underline{x} \neq \underline{0}
\end{aligned}$$

Example 14. Again, with $n = 2$ let $f = x^3 - 3xy^2$. Then

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}(3x^2 - 3y^2) = 6x \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}(-6xy) = -6x \\
\Rightarrow \quad \nabla^2 f &= 0
\end{aligned}$$

Notice: If we let $z = x + iy$, then $f = \text{Re}[(x+iy)^3] = \text{Re}[z^3]$. If you study Complex Analysis II, you will see that this is a differentiable complex function. As such, the real and imaginary parts of the function satisfy the Cauchy-Riemann equations. One can show (exercise) that in this case the Laplacian of both the real and imaginary parts of the function have Laplacian equal to 0.

For $n = 3$ (i.e. in $\mathbb{R}^3$) there are a couple of other natural combinations. Firstly, since the gradient of a scalar field is a vector field, we can take its curl:

$$\begin{aligned}
\underline{\nabla} \times \underline{\nabla} f &= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
&= \underline{e}_1(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}) + \underline{e}_2(\frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z}) + \underline{e}_3(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}) \\
&= \underline{0},
\end{aligned}$$

assuming the 2nd partial derivatives of f are continuous (so we can equate $\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$ etc.). Also in $\mathbb{R}^3$, we can find the divergence of a curl. We did this once follow example 11, but in general

$$\begin{aligned}
\underline{\nabla}.(\underline{\nabla} \times \underline{v}) = \underline{\nabla}.&\begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= \frac{\partial}{\partial x}(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}) \\
&= 0,
\end{aligned}$$

again, assuming the 2nd partial derivatives of f are continuous. Later we will redo these cases in a more elegant way using index notation.