1 *For the function* $f(x,y) = \cos(x+y)\,\exp(x-y)$ *calculate* $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x^2$, $\partial^2 f/\partial y^2$, $\partial^2 f/\partial x\partial y$, $\partial^2 f/\partial y\partial x$. *Use your results to show that* $\partial^2 f/\partial x^2 = -\partial^2 f/\partial y^2$ *and* $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$.

2 *Let* $F(t)$ *be the value of the function* $f(x,y,z) = \cos(xy)\,z$ *restricted to the helix* $x = \cos(t)$, $y = \sin(t)$, $z = t$ *which is parametrised by* t *and* $-\infty < t < \infty$. *Calculate* dF/dt *as a function of* t *(i) directly by substituting the equations of the helix into* $f(x,y,z)$ *to calculate* $F(t)$ *as a function of* t *and then differentiating, and (ii) using the chain rule. Note how similar these approaches are.*

3 *If* $\mathbf{a} = \sin 2t\,\mathbf{e_1} + e^t\,\mathbf{e_2} - (t^3 - 5t)\,\mathbf{e_3}$, *find*
(a) $d\mathbf{a}/dt$, *(b)* $||d\mathbf{a}/dt||$, *(c)* $d^2\mathbf{a}/dt^2$, *(d)* $||d^2\mathbf{a}/dt^2||$, *all at* $t = 0$.

4 *Find a unit vector tangent to the space curve* $x = t^3, y = t, z = t^2$ *at* $t = 2$.

5 *Use the chain rule to calculate* df/dt *when* $f(\mathbf{x}) = \exp(-||\mathbf{x}||^2)$ *is restricted to the curves:*

(a) $\mathbf{x} = \mathbf{e_1}\log t + \mathbf{e_2} t\log t + \mathbf{e_3} t$,

(b) $(x,y,z) = (\cosh t, \sinh t, 0)$.

6 *Show that the curve* C, *given as points* $\underline{x}(s) = \left(\sin(s/\sqrt{2}), \cos(s/\sqrt{2}), s/\sqrt{2}\right)$, *is the arc-length parameterisation of a helix, that is that* $\left|\frac{d\underline{x}}{ds}\right| = 1 \quad \forall s$.

7 *Describe the curve* $\gamma : \underline{x}(t) = (2t+1, t-3, 6-2t)$. *Find the arc-length parameterisation of* γ, *that is, re-parameterise the curve in terms of a parameter* s, *such that* $\left|\frac{d\underline{x}(s)}{ds}\right| = 1\ \forall s$.

8 ***Harder:*** *Let* $\mathbf{t}$ *denote the unit tangent vector to a space curve* $\mathbf{a} = \mathbf{a}(s)$ *in* $\mathbb{R}^3$, *where* $\mathbf{a}(s)$ *is assumed differentiable, and where* s *measures the arclength from some fixed point on the curve. Define the unit vector* $\mathbf{n} = \frac{1}{\kappa}\frac{d\mathbf{t}}{ds}$, *where* κ *is a scalar. Also define* $\mathbf{b} = \mathbf{t}\times\mathbf{n}$ *as the* unit binormal vector *to the space curve.*

By considering the derivative of the product $\mathbf{t}.\mathbf{t}$, *show that the the 3 vectors* $\mathbf{t}, \mathbf{n}, \mathbf{b}$ *form an orthonormal basis of* $\mathbb{R}^3$.

Hence, prove that

$$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{n}, \quad \text{and} \quad \frac{d\mathbf{n}}{ds} = \tau\mathbf{b} - \kappa\mathbf{t},$$

where τ *is some real constant.*

These formulae are of fundamental importance in differential geometry. They involve the curvature κ *and the torsion* τ. *The reciprocals of these are the radius of curvature* ($\rho = \frac{1}{\kappa}$) *and the radius of torsion* ($\sigma = \frac{1}{\tau}$).

Bonus 1 *If* $f(x,y) = F(r,\theta)$ *with* $x = r\cos\theta$ *and* $y = r\sin\theta$, *use the chain rule to compute* $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2$ *in terms of partial* r- *and* θ- *derivatives of* F, *and hence find the general rotationally-symmetric solution to* $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = 0$ *in two dimensions which is non-singular away from the origin.*
Suggestion: Begin by writing $\partial r/\partial x$, $\partial r/\partial y$, $\partial\theta/\partial x$ *and* $\partial\theta/\partial y$ *as functions of* r *and* θ.