8.5 Summary: Taylor Series

You should know how to calculate Taylor polynomials $P_n(x)$ and what the integral form and Lagrange form of the remainder $R_n(x)$ are, as well as how to estimate the maximum error in using a Taylor polynomial as an approximation of a function. Here are some key points:

- The *Taylor polynomial* of degree n about $x = a$ for a function f is the degree n polynomial in $(x-a)$, $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$. At the point $x = a$, P_n and its first n derivatives will agree with f and its first n derivatives. In this sense the Taylor polynomial tries to approximate f close to $x = a$. Note that if we don't explicitly state "about $x = a$" then it is assumed we are working about $x = 0$.

- *Taylor's theorem* states that $f(x) = P_n(x) + R_n(x)$ where $R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt$ is the *remainder*.

- The *Lagrange form of the remainder* says that $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some (unknown) c between a and x. (Note the similarity to the next term in the Taylor polynomial.) This can often be used to place an upper bound on *the error* $|R_n(x)|$ in using $P_n(x)$ as an approximation of $f(x)$.

- The *Taylor series* about $x = a$ is $\lim_{n\to\infty} P_n(x)$. If $\lim_{n\to\infty} R_n(x) = 0$ (i.e. the error vanishes in this limit) for all x in some interval containing a we say that the Taylor series converges to $f(x)$ on that interval. In such cases $P_n(x)$ approximates $f(x)$ as well as we want on that interval by taking n large enough.

- You should be familiar with the form of the Taylor series (about $x = 0$) for e^x, $\sinh x$, $\cosh x$, $\sin x$, $\cos x$ and $\log(1+x)$. The Taylor series for $\log(1+x)$ converges to $\log(1+x)$ for $x \in (-1, 1]$. The other ones converge to their functions $\forall x \in \mathbb{R}$.

- We can find Taylor polynomials of degree n for sums of products of functions by taking sums of products of Taylor polynomials of degree n (and ignoring any terms with degree higher than n which arise from products). We can also use substitution, e.g. to find the Taylor polynomial of degree n for $e^{\sin x}$ replace x in the Taylor polynomial of degree n for e^x with the Taylor polynomial of degree n for $\sin x$ (and again ignore all powers higher than n). It is sometimes possible to divide by Taylor series using $1/(1-x) = 1 + x + x^2 + x^3 + \cdots$ for $x \in (-1, 1)$ and appropriate substitution.

- We can evaluate limits as $x \to a$ by replacing functions with their Taylor polynomials of sufficiently high order about $x = a$ provided the Taylor series converge to the functions on some interval containing a. This is because in such cases the higher order terms (higher powers of $(x-a)$) will vanish in the limit $x \to a$.