41 *For which values of (x, y) are the following continuous:*

(a) $x/(x^2 + y^2 + 1)$,

(b) $x/(x^2 + y^2)$,

(c) $(x + y)/(x - y)$,

(d) $x^3/(y - x^2)$?

42 *Let the scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as*

$$f(\underline{x}) = \begin{cases} 1 + \dfrac{x^2}{y} & y \neq 0, \\ 1 & y = 0. \end{cases}$$

(a) *Show that, along any straight line through the origin, $\lim_{\underline{x}\to\underline{0}} f(\underline{x}) = f(\underline{0})$.*

(b) *Is $f(\underline{x})$ continuous at $\underline{0}$? Explain your answer, with reference to the first part of this question.*

43 *Let the scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as*

$$f(\underline{x}) = \begin{cases} \dfrac{x^2y^2}{x^4 + y^4} & \underline{x} \neq \underline{0}, \\ 0 & \underline{x} = \underline{0}. \end{cases}$$

(a) *Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist at $\underline{0}$, and find their values at this point.*

(b) *Show that $\lim_{\underline{x}\to\underline{0}} f(\underline{x})$ does not exist, and hence $f(\underline{x})$ is not continuous at the origin. Comment on this in relation to the previous part of this question.*

44 *Which of the following sets are open?.*

(a) $\{(x, y, z) : x > 0\}$,

(b) $\{(x, y, z) : y \geq 0\}$,

(c) $\{(x, y, z) : 1 > (x^2 + y^2)/z\}$,

(d) $\{(x, y, z) : 1 \geq (x^2 + y^2)/z\}$?

45 *Prove that an open ball, as defined in lectures, is an open set.*

46 *Prove that the intersection of two open sets, as defined in lectures, is another open set. (Note that the empty set is an open set: since it contains no points, the statement that every point in it sits inside an open ball which is also in the set is vacuously true.) What about the intersection of a finite number of open sets? And what about the intersection of an infinite number of open sets?*

47 *Exam question June 2014 (Section A):*

(a) *Give the definition of the open ball $B_\delta(\mathbf{a})$ with centre $\mathbf{a} \in \mathbb{R}^n$ and radius $\delta > 0$, and define what it means for a subset S of $\mathbb{R}^n$ to be open.*

(b) *Which of the following subsets of $\mathbb{R}^2$ are open? In each case, justify your answer in terms of the definition you gave in part (a).*

(i) $S_1 = \{(x, y) : x > 2\}$,
(ii) $S_2 = \{(x, y) : x > 2, y = 2\}$,
(iii) $S_3 = \{(x, y) : x > 2, y > 2\}$.

48 *Exam question (last part) June 2014 (Section B): Determine the points of* $\mathbb{R}^2$ *at which the function* $f(x, y) = |xy + x + y + 1|$ *is*
(a) continuously differentiable; (b) differentiable.

Hint: *first factorise* f.

49 *Determine the points of* $\mathbb{R}^2$ *at which the function* $f(x, y) = |x^2 - y^2|$ *is*
(a) continuously differentiable; (b) differentiable.

50 *Define* $f : \mathbb{R}^2 \to \mathbb{R}$ *by* $f(\mathbf{0}) = 0$ *whilst for* $\mathbf{x} \neq \mathbf{0}$*:*

$$f(\mathbf{x}) = \frac{x^3}{x^2 + y^2}\,.$$

Calculate the partial derivatives of f *with respect to* x *and* y *at* $\mathbf{x} = \mathbf{0}$ *using their definitions as limits. Defining* $R(\mathbf{h})$ *at the origin by* $R(\mathbf{h}) = f(\mathbf{h}) - f(\mathbf{0}) - \mathbf{h} \cdot \nabla f$ *as usual, show that* $R(\mathbf{h})/|\mathbf{h}|$ *does not tend to zero as* $\mathbf{h}$ *tends to* $\mathbf{0}$*, so that* f *is not differentiable at the origin.*
On the line through the origin, $\mathbf{x} = \mathbf{b}t$*, (with* $\mathbf{b}$ *a constant vector),* f *becomes a function of the single variable* t*,* $f(\mathbf{b}t)$*. Write* $\mathbf{b} = \mathbf{e_1}b_1 + \mathbf{e_2}b_2$ *and use this to write* $f(\mathbf{b}t)$ *explicitly as a function of* t*. Show that this function is differentiable at the origin, i.e.* df/dt *exists at* $t = 0$ *despite* $f(\mathbf{x})$ *not being differentiable at* $\mathbf{0}$.

51 *Exam Question June 2022 (Part B)*

(a) *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a scalar field on* $\mathbb{R}^n$*. Define what it means for the limit of* $f(\underline{x})$ *as* $\underline{x}$ *tends to* $\underline{a}$ *to be* L.

(b) *Define what it means for* f *to be continuous at* $\underline{a}$.

(c) *Let* f *be a scalar field on* $\mathbb{R}^2$ *given by*

$$f(x, y) = \begin{cases} xy(x^2 - y^2)/(x^2 + y^2) & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0). \end{cases}$$

Show that f *is continuous on* $\mathbb{R}^2$*, stating any results that you use.*

(d) *Is* f *differentiable at the origin?*

(e) *Show that, at the origin,*

$$\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}.$$

52 *If* $y = 1 + xy^5$ *show that* y *may be written in the form* $y = f(x)$ *in a neighbourhood of* $(0, 1)$ *and find the gradient of the graph of* f *at the point* $(0, 1)$.

53 *Show that the equation $xy^3 - y^2 - 3x^2 + 1 = 0$ can be written in the form $y = f(x)$ in a neighbourhood of the point $(0, 1)$, and in the form $y = g(x)$ in a neighbourhood of the point $(0, -1)$. Is it true that $f(x)$ and $g(x)$ are equivalent as functions of x? What are the critical values of the curve $H(x, y) = xy^3 - y^2 - 3x^2 + 1$, and what are the regular values of this curve?*

54 *Determine whether or not the equation $x^2 + y + \sin(xy) = 0$ can be written in the form $y = f(x)$ or in the form $x = g(y)$ in some small open disc about the origin for some suitable continuously differentiable functions f, g.*

55 *Exam question May 2015 (Section B, lightly edited):*
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the scalar function $f(x, y) = e^{xy} - x + y$.

(a) *Find the vector equations of the tangent and normal lines to the curve $f(x, y) = 0$ at the points $(1, 0)$ and $(0, -1)$.*

(b) *Use the implicit function theorem for functions of two variables to determine whether or not the curve $f(x, y) = 2$ can be written in the form $y = g(x)$ for some differentiable function $g(x)$ in the neighbourhoods of the points (i) $(0, 1)$; (ii) $(-1, 0)$. Determine also whether the curve can be written as $x = h(y)$ for some differentiable function $h(y)$, in the neighbourhoods of the same two points.*

(c) *Does the function $f(x, y)$ have any critical points? Justify your answer. (You can quote without proof that $|xe^{-x^2}| < 1$ for all $x \in \mathbb{R}$.)*

56 *Part of Exam Question May 2017 (Section B):*

(c) *Consider the function*
$$f(x, y) = (3x + y)e^{3xy}.$$
Determine whether or not the curve $f(x, y) = c$ can be written in the form $y = g(x)$, and if not, state clearly the points (x_0, y_0) and corresponding values of c where problems occur. You may assume that f is differentiable on $\mathbb{R}^2$.

(d) *Using $f(x, y)$ as given in the previous part, determine whether or not the curve $(f(x, y) = c)$ can be written in the form $x = y(h)$, and if not, state clearly the points (x_0, y_0) where problems occur.*

(e) *Using $f(x, y)$ as in the previous parts of this question, are there any points where the curve $f(x, y) = c$ can neither be written as $y = g(x)$, nor as $x = h(y)$?*

57 *For each of the following two surfaces, show that the surface can be parameterised as $\underline{x}(x, y) = x\underline{e}_1 + y\underline{e}_2 + g(x, y)\underline{e}_3$, and show that the normal vector $\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y}$ can be written as*
$$\frac{\partial \underline{x}}{\partial x} \times \frac{\partial \underline{x}}{\partial y} = \frac{\underline{\nabla} f}{\underline{e}_3 \cdot \underline{\nabla} f},$$
for a scalar field $f\underline{x}$ which you should specify.

- *The upper hemisphere of radius two, centred on the origin (where $z > 0$).*
- *The surface defined by $\mathrm{e}^(x + y + z) = 1 - (xy)^2$.*