6 Differentiability of vector fields

6.1 Differentiable maps $\mathbb{R}^n \to \mathbb{R}^n$

We will now generalise the idea of differentiability of scalar fields from last term. Recall that for a scalar field $f(\underline{x}) : U \to \mathbb{R}$, with U open in $\mathbb{R}^n$, f is differentiable at $\underline{a} \in U$ if

$$f(\underline{a} + \underline{h}) - f(\underline{a}) \;=\; \underline{h}.\nabla f(\underline{a}) \;+\; R(\underline{h}) \qquad\qquad (a)$$

with $$\lim_{\underline{h} \to \underline{0}} \frac{R(\underline{h})}{|\underline{h}|} \;=\; 0, \qquad\qquad (b)$$

where you should note that the first term on RHS of (a), $\underline{h}.\nabla f(\underline{a})$, is *linear* in $\underline{h}$.

Definition 6.1. Consider a <u>vector</u> field $\underline{F}(\underline{x}) : U \to \mathbb{R}^n$, U open in $\mathbb{R}^n$. Then $\underline{F}$ is defined to be *differentiable* at $\underline{a} \in U$ if there is a linear function $\underline{L} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\underline{F}(\underline{a} + \underline{h}) - \underline{F}(\underline{a}) \;=\; \underline{L}(\underline{h}) \;+\; \underline{R}(\underline{h}) \qquad\qquad (A)$$

with $$\lim_{\underline{h} \to \underline{0}} \frac{R(\underline{h})}{|\underline{h}|} \;=\; \underline{0}. \qquad\qquad (B)$$

Now linear functions $\mathbb{R}^n \to \mathbb{R}^n$ are matrices. To see what matrix use the standard basis,

$$\underline{F}(\underline{x}) = F_1(\underline{x})\underline{e}_1 + F_2(\underline{x})\underline{e}_2 + \cdots + F_n(\underline{x})\underline{e}_n$$
$$\underline{L}(\underline{h}) = L_1(\underline{h})\underline{e}_1 + L_2(\underline{h})\underline{e}_2 + \cdots + L_n(\underline{h})\underline{e}_n$$
$$\underline{R}(\underline{h}) = R_1(\underline{h})\underline{e}_1 + R_2(\underline{h})\underline{e}_2 + \cdots + R_n(\underline{h})\underline{e}_n,$$

so the jth components of (A) and (B) are

$$F_j(\underline{a} + \underline{h}) - F_j(\underline{a}) \;=\; L_j(\underline{h}) \;+\; R_j(\underline{h}) \qquad\qquad (A)_j$$

with $$\lim_{\underline{h} \to \underline{0}} \frac{R_j(\underline{h})}{|\underline{h}|} \;=\; 0. \qquad\qquad (B)_j$$

These are just conditions (a) and (b) for $F_j(\underline{x})$ to be differentiable as a map $U \to \mathbb{R}$, i.e. as a scalar field. So we can use results from last term to see that

$$L_j(\underline{h}) \;=\; \underline{h}.\nabla F_j(\underline{a}),$$

that is

$$L_1 \;=\; \underline{h}.\nabla F_1(\underline{a}) \;=\; h_1\frac{\partial F_1}{\partial x_1} + h_2\frac{\partial F_1}{\partial x_2} + \cdots + h_n\frac{\partial F_1}{\partial x_n}$$

$$L_2 \;=\; \underline{h}.\nabla F_2(\underline{a}) \;=\; h_1\frac{\partial F_2}{\partial x_1} + h_2\frac{\partial F_2}{\partial x_2} + \cdots + h_n\frac{\partial F_2}{\partial x_n}$$

$$\vdots \quad \vdots$$

$$L_n \;=\; \underline{h}.\nabla F_n(\underline{a}) \;=\; h_1\frac{\partial F_n}{\partial x_1} + h_2\frac{\partial F_n}{\partial x_2} + \cdots + h_n\frac{\partial F_n}{\partial x_n},$$

or

$$\begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_n \end{pmatrix} = \begin{pmatrix} h_1\frac{\partial F_1}{\partial x_1} + h_2\frac{\partial F_1}{\partial x_2} + \ldots + h_n\frac{\partial F_1}{\partial x_n} \\ h_1\frac{\partial F_2}{\partial x_1} + h_2\frac{\partial F_2}{\partial x_2} + \ldots + h_n\frac{\partial F_2}{\partial x_n} \\ \vdots \\ h_1\frac{\partial F_n}{\partial x_1} + h_2\frac{\partial F_n}{\partial x_2} + \ldots + h_n\frac{\partial F_n}{\partial x_n} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}.$$

The $n \times n$ matrix on the RHS of the last equation is called the <u>Jacobian matrix</u>, or <u>differential</u>, of $\underline{F}(\underline{x})$ at $\underline{x} = \underline{a}$; it is written as $D\underline{F}(\underline{a})$, or $d\underline{F}(\underline{a})$ (or $D\underline{F}_{\underline{a}}$ or $d\underline{F}_{\underline{a}}$ or even J_{ij}).

Definition 6.2. The determinant of the differential,

$$\det(D\underline{v}) \equiv |D\underline{v}|$$

is called the <u>Jacobian</u>, $J(\underline{v})$.

Example 33. If

$$\underline{v}(\underline{x}) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

then

$$D\underline{v}(\underline{x}) = \begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

The Jacobian is then given by

$$\begin{aligned} J(\underline{v}) &= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} \\ &= 4(x^2 + y^2) = 4|\underline{x}|^2 \end{aligned}$$

Example 34. If $\underline{x} \in \mathbb{R}^n$ and $\underline{v}(\underline{x}) = \underline{x}$ then

$$D\underline{v}(\underline{x}) = \begin{pmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial x_2} & \cdots & \frac{\partial x_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial x_1} & \frac{\partial x_n}{\partial x_2} & \cdots & \frac{\partial x_n}{\partial x_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \mathbb{I}_n,$$

the $n \times n$ identity matrix, and

$$J(\underline{v}) = |\mathbb{I}_n| = 1$$

6.2 Diffeomorphisms and the inverse function theorem

We can think of a vector field $\underline{v}(\underline{x})$ as a mapping $\mathbb{R}^n \to \mathbb{R}^n$, or equivalently as a <u>coordinate transformation</u> on $\mathbb{R}^n$. If we think of the components of $\underline{h}$ as the coordinates of a point $\underline{x} = \underline{a} + \underline{h}$ relative to an "origin" at $\underline{a}$, then the components of $\underline{v}(\underline{a} + \underline{h}) - \underline{v}(\underline{a})$ are the transformed coordinates relative to the transformed origin $\underline{v}(\underline{a})$. Then for differentiable $\underline{v}$ (and small $\underline{h}$)

$$\begin{aligned} \underline{v}(\underline{a} + \underline{h}) - \underline{v}(\underline{a}) &\simeq D\underline{v}(\underline{a}) \cdot \underline{h} \\ \text{new coordinates} &\simeq \text{matrix} \times \text{old coordinates}, \end{aligned}$$

which is a *<u>linear</u>* transformation, invertible if the determinant of $D\underline{v}(\underline{a})$ (i.e. the Jacobian) is non-zero.

The <u>inverse function theorem</u> says that this invertibility can be extended beyond the linear behaviour:

Theorem 6.3. Let $\underline{v} : U \to \mathbb{R}^n$ (with U open in $\mathbb{R}^n$) be a differentiable vector field with continuous partial derivatives, and let $\underline{a} \in U$. Then if $J(\underline{v}(\underline{a})) \neq 0$, $\exists$ an open set $\tilde{U} \subseteq U$ containing $\underline{a}$ such that

(i) $\underline{v}(\tilde{U})$ is open

(ii) The mapping $\underline{v}$ from $\tilde{U}$ to $\underline{v}(\tilde{U})$ has a differentiable inverse - i.e. there exists a differentiable vector field $\underline{w} : \underline{v}(\tilde{U}) \to \mathbb{R}^n$ such that $\underline{w}(\underline{v}(\underline{x})) = \underline{x}$ and $\underline{v}(\underline{w}(\underline{y})) = \underline{y}$

Definition 6.4.

- A mapping $\underline{v} : \tilde{U} \to V \subset \mathbb{R}^n$ satisfying (i) and (ii) above is called a <u>diffeomorphism</u> of $\tilde{U}$ onto $\tilde{V} = \underline{v}(\tilde{U})$, and $\tilde{U}$ and $\tilde{V}$ are said to be <u>diffeomorphic</u>.

- More generally, a mapping $\underline{v} : U \to V$ is called a <u>local diffeomorphism</u> if for every point $\underline{a} \in U$ there is an open set $\tilde{U} \subset U$ containing $\underline{a}$ such that $\underline{v} : \tilde{U} \to \underline{v}(\tilde{U})$ is a diffeomorphism.

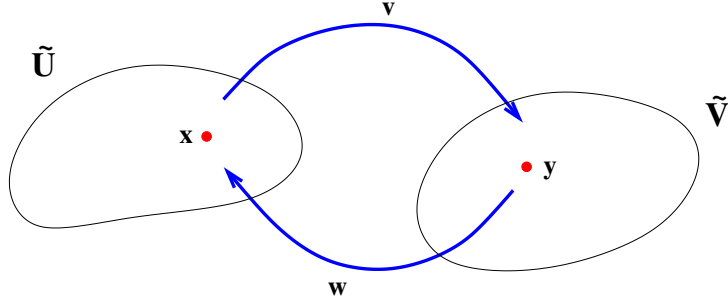Figure 29: A diffeomorphism from $\tilde{U}$ to $\tilde{V}$

Remarks In general, suppose that

$$\underline{v} : U \to V \subset \mathbb{R}^n$$
$$\underline{w} : V \to W \subset \mathbb{R}^n$$

(with U and V both open in $\mathbb{R}^n$) are both continuously differentiable vector fields (not necessarily diffeomorphisms).
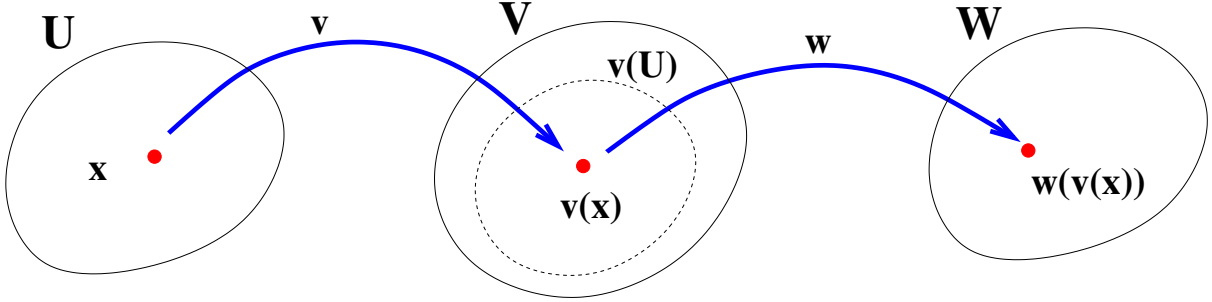

Figure 30: Composition of maps $\underline{v}$ and $\underline{w}$

Then $\underline{w}(\underline{v}(\underline{x}))$ is a mapping $U \to W \subset \mathbb{R}^n$ and its differential can be calculated using the chain rule:

$$
D\underline{w}(\underline{v}(\underline{x})) =
\begin{pmatrix}
\frac{\partial w_1(\underline{v}(\underline{x}))}{\partial x_1} & \frac{\partial w_1(\underline{v}(\underline{x}))}{\partial x_2} & \cdots & \frac{\partial w_1(\underline{v}(\underline{x}))}{\partial x_n} \\
\vdots & \vdots & & \vdots \\
\frac{\partial w_n(\underline{v}(\underline{x}))}{\partial x_1} & \frac{\partial w_n(\underline{v}(\underline{x}))}{\partial x_2} & \cdots & \frac{\partial w_n(\underline{v}(\underline{x}))}{\partial x_n}
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
\sum_j \frac{\partial w_1}{\partial v_j}\frac{\partial v_j}{\partial x_1} & \sum_j \frac{\partial w_1}{\partial v_j}\frac{\partial v_j}{\partial x_2} & \cdots & \sum_j \frac{\partial w_1}{\partial v_j}\frac{\partial v_j}{\partial x_n} \\
\vdots & \vdots & & \vdots \\
\sum_j \frac{\partial w_n}{\partial v_j}\frac{\partial v_j}{\partial x_1} & \sum_j \frac{\partial w_n}{\partial v_j}\frac{\partial v_j}{\partial x_2} & \cdots & \sum_j \frac{\partial w_n}{\partial v_j}\frac{\partial v_j}{\partial x_n}
\end{pmatrix}
$$

using chain rule on each matrix element,

$$
=
\begin{pmatrix}
\frac{\partial w_1}{\partial v_1} & \frac{\partial w_1}{\partial v_2} & \cdots & \frac{\partial w_1}{\partial v_n} \\
\vdots & \vdots & & \vdots \\
\frac{\partial w_n}{\partial v_1} & \frac{\partial w_n}{\partial v_2} & \cdots & \frac{\partial w_n}{\partial v_n}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \cdots & \frac{\partial v_1}{\partial x_n} \\
\vdots & \vdots & & \vdots \\
\frac{\partial v_n}{\partial x_1} & \frac{\partial v_n}{\partial x_2} & \cdots & \frac{\partial v_n}{\partial x_n}
\end{pmatrix}
$$

$$= D\underline{w}(\underline{v})\, D\underline{v}(\underline{x}) \qquad \text{i.e. matrix multiplication}$$

For the special case when $\underline{v}$ is a local diffeomorphism and $\underline{w}$ is its inverse map,

$$\underline{w}(\underline{v}(\underline{x})) = \underline{x}$$
$$\Rightarrow D\underline{w}\, D\underline{v} = D\underline{w}(\underline{v}(\underline{x})) = D\underline{x}(\underline{x}) = \mathbb{I}_n,$$

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using Example 34 above. Likewise

$$v(w(y)) = y$$
$$\Rightarrow Dv \, Dw = Dv(w(y)) = Dy(y) = \mathbb{I}_n.$$

So Dv is an invertible matrix, with inverse $(Dv)^{-1} = Dw$. Taking determinants,

$$J(w) = \frac{1}{J(v)},$$

and in particular, $J(v) \neq 0$, which was the main condition of the inverse function theorem.

Definition 6.5. Such a v is called <u>orientation preserving</u> if $J(v) > 0$, and <u>orientation reversing</u> if $J(v) < 0$.

Example 35. Continuing Example 33, we had

$$v(x) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} \qquad \Rightarrow \qquad J(v) = 4(x^2 + y^2).$$

So for $(x, y) \neq (0, 0)$, $J(v) > 0$.

Hence if $U = \mathbb{R}^2 - \{0\}$, $v : U \to U$ is an orientation preserving local diffeomorphism. However it is not a global diffeomorphism since $v(-x) = v(x)$ (so no inverse can exist globally). But v <u>does</u> map $\{(x, y) : x > 0\}$ onto $\mathbb{R}^2 - \{(x, 0), x \leq 0\}$ diffeomorphically.

Example 36. Consider the transformation from polar coordinates (r, θ) back to cartesians (x, y). We have

$$v(r, \theta) = \begin{pmatrix} x(r, \theta) \\ y(r, \theta) \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$$

<u>Differential:</u>

$$Dv = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \tag{6.1}$$

<u>Jacobian:</u>

$$J(v(r, \theta)) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$
$$= r \cos^2 \theta + r \sin^2 \theta = r$$

For $r > 0$, $J(v) > 0$, and the transformation is therefore orientation preserving.

The inverse mapping is

$$w(x, y) = \begin{pmatrix} r(x, y) \\ \theta(x, y) \end{pmatrix} = \begin{pmatrix} \sqrt{x^2 + y^2} \\ \tan^{-1}\left(\frac{y}{x}\right) \end{pmatrix}$$

<u>Exercise:</u> check this!