

19 Compute the divergence,  $\nabla \cdot \mathbf{A}$ , of the following vector fields:

- (a)  $\mathbf{A}(x, y, z) = yz\mathbf{e}_1 + xz\mathbf{e}_2 + xy\mathbf{e}_3$ ,
- (b)  $\mathbf{A}(x, y, z) = (x^2 + y^2 + z^2)(3\mathbf{e}_1 + 4\mathbf{e}_2 + 5\mathbf{e}_3)$ ,
- (c)  $\mathbf{A}(x, y, z) = (x + y)\mathbf{e}_1 + (y + z)\mathbf{e}_2 + (z + x)\mathbf{e}_3$ .

20 Compute the curl,  $\nabla \times \mathbf{A}$ , of each of the vector fields,  $\mathbf{A}$ , in the previous question.

21 If  $f(r)$  is a differentiable function of  $r = |\mathbf{x}|$ , show that

- (a)  $\text{grad } f(r) = f'(r) \mathbf{x} / r$ ,
- (b)  $\text{curl } [f(r)\mathbf{x}] = 0$ .

22 Let  $\mathbf{x}$  be the position vector in three dimensions, with  $r = |\mathbf{x}|$ , and let  $\mathbf{a}$  be a constant vector. Show that

- (a)  $\text{div } \mathbf{x} = 3$ ,
- (b)  $\text{curl } \mathbf{x} = 0$ ,
- (c)  $\text{grad } r = \mathbf{x} / r$ ,
- (d)  $\text{div } (r^n \mathbf{x}) = (n + 3) r^n$ ,
- (e)  $\text{grad } (\mathbf{a} \cdot \mathbf{x}) = \mathbf{a}$ ,
- (f)  $\text{div } (\mathbf{a} \times \mathbf{x}) = 0$ ,
- (g)  $\text{curl } (\mathbf{a} \times \mathbf{x}) = 2 \mathbf{a}$ ,
- (h)  $\text{curl } (r^2 \mathbf{a}) = 2 (\mathbf{x} \times \mathbf{a})$ ,
- (i)  $\nabla^2(1/r) = 0$ , if  $r \neq 0$ ,
- (j)  $\nabla^2(\log r) = 1/r^2$ , if  $r \neq 0$ ,
- (k)  $\text{div } [(\mathbf{a} \cdot \mathbf{x})\mathbf{x}] = 4 \mathbf{a} \cdot \mathbf{x}$ ,
- (l)  $\text{div } [\mathbf{x} \times (\mathbf{x} \times \mathbf{a})] = 2 \mathbf{a} \cdot \mathbf{x}$ ,
- (m)  $\text{curl } (\mathbf{a} \times \mathbf{x} / r^3) = 3 (\mathbf{a} \cdot \mathbf{x})\mathbf{x} / r^5 - \mathbf{a} / r^3$ ,
- (n) Exam question June 2002 (Section A): calculate the curl of  $(\mathbf{a} \cdot \mathbf{x}) \mathbf{x}$ .

23 If  $\mathbf{x}$  is the position vector,  $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ ,  $\mathbf{a}$  is a constant vector,  $\mathbf{F} = (\mathbf{a} \cdot \mathbf{x}) \mathbf{x}$  and  $\mathbf{G} = r^2 \mathbf{a}$ , (with  $r = |\mathbf{x}|$ ), show that

- (a)  $\text{div } \mathbf{F} = 2 \text{div } \mathbf{G} = 4 \mathbf{a} \cdot \mathbf{x}$ ,
- (b)  $\text{curl } \mathbf{G} = -2 \text{curl } \mathbf{F} = 2 \mathbf{x} \times \mathbf{a}$ ,
- (c)  $\text{div curl } \mathbf{F} = \text{div curl } \mathbf{G} = 0$ ,
- (d)  $\text{curl curl } \mathbf{G} = -2 \text{curl curl } \mathbf{F} = -4 \mathbf{a}$ .

24 Exam question June 2001 (Section A):

- (a) Give a representation of the vector function  $\mathbf{A}(x, y) = y\mathbf{e}_1$  as a collection of arrows in the region of the  $(x, y)$ -plane bounded by  $(x_1, y_1) = (-2, 2)$ ,  $(x_2, y_2) = (2, 2)$ ,  $(x_3, y_3) = (2, -2)$ ,  $(x_4, y_4) = (-2, -2)$ .

- (b) Calculate the curl of the vector field  $\mathbf{A}(x, y) = (-y\mathbf{e}_1 + x\mathbf{e}_2)/(x^2 + y^2)$  defined everywhere in the  $(x, y)$ -plane except at the origin. (You can consider  $\mathbf{A}$  to be embedded in three dimensions, independent of  $z$  and with zero  $z$  component.)
- (c) Give the unit vector normal to the surface of equation  $ax + by = cz$ , where  $a, b, c$ , are three real constants.
- (d) (Slightly modified from exam) Let  $\mathbf{x}$  be the position vector in 3-dimensions and  $\mathbf{a}$  be a constant vector. Use the result  $\mathbf{x} \times (\mathbf{x} \times \mathbf{a}) = \mathbf{x}(\mathbf{x} \cdot \mathbf{a}) - \mathbf{a}(\mathbf{x} \cdot \mathbf{x})$  to show that  $\text{div} [\mathbf{x} \times (\mathbf{x} \times \mathbf{a})] = 2\mathbf{a} \cdot \mathbf{x}$ .