7.6 Summary: Second order ODEs

You should have a good understanding of how to solve linear constant-coefficient second order ODEs which can be written as $\alpha_2 y'' + \alpha_1 y' + \alpha_0 y = \phi(x)$. Here are some key points:

- Second order ODEs will have two parameters (integration constants) in the general solution. These can be fixed with *initial conditions* (fixing the value of y and y' at the same point) or *boundary conditions* (fixing the value of y at two different points). Initial conditions will uniquely fix the integration constants. Boundary conditions will typically uniquely fix the integration constants but it is possible that there could be no solution or a one-parameter family of solutions – e.g. consider $y'' + y = 0$ with $y(0) = 0$ (which gives $y = A\sin x$) and the other boundary condition being $y(\pi/2) = 1$ (1 solution) or $y(\pi) = 1$ (no solutions) or $y(\pi) = 0$ (one-parameter family of solutions).

- If ϕ is zero we have a *homogeneous second order ODE*. We solve this by first solving the *Characteristic Equation* (CE) $\alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 = 0$.

- If the CE has two real solutions $\lambda_1 \neq \lambda_2$ then the ODE has general solution $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$.

- If the CE has only one solution λ_1 (repeated roots) then the ODE has general solution $y = Ae^{\lambda_1 x} + Bxe^{\lambda_1 x}$.

- If the CE has two complex solutions $\alpha + i\beta$ and $\alpha - i\beta$ then the ODE has general solution $y = Ce^{(\alpha+i\beta)x} + De^{(\alpha-i\beta)x} =^{\alpha x} (A\cos(\beta x) + B\sin(\beta x))$.

- In all case the general solution of the homogeneous ODE is of the form $y(x) = Ay_1(x) + By_2(x)$ where $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the ODE.

- The general solution to a linear ODE is $y = y_{CF} + y_{PI}$ where y_{PI} is a *particular integral* (PI) which is any solution of the ODE and y_{CF} is the *complementary function* (CF) which is the general solution of the homogeneous ODE formed by setting ϕ to zero.

- If $\phi(x)$ is formed from sums of products of exponentials $e^{\gamma x}$, sines and cosines $\sin(\gamma x)$ and $\cos(\gamma x)$, and polynomials we can find a PI using the *method of undetermined coefficients*. To do this we use an ansatz for y_{PI} which is of the same form as $\phi(x)$ with arbitrary coefficients for each term. Then substitute this ansatz into the ODE to solve for the coefficients. Note that if a term solves the homogeneous ODE we need to include an extra factor of x with that term, and another factor of x if that still solves the homogeneous ODE.

- If we cannot use the method of undetermined coefficients, we can try the *method of variation of parameters*. If $y_{CF} = Ay_1 + By_2$ we take $y = u_1 y_1 + u_2 y_2$ where u_1 and u_2 are functions of x and we impose the constraint $u_1' y_1 + u_2' y_2 = 0$ so that y' does not depend on u_1' or u_2'. Substituting this into the ODE, all terms with u_1 and u_2 (not differentiated) cancel as they must since constant u_1 and u_2 would give a solution of the homogeneous

ODE – the result is a second linear equation for u_1' and u_2', $\alpha_2(u_1'y_1' + u_2'y_2') = \phi$. Solve these two linear equation to find u_1' and u_2' and then integrate to find u_1 and u_2 (noting that the integration constants are exactly the parameters in y_{CF}).

- Two coupled first order linear ODEs for two variables give, by eliminating one of the variables, a linear second order ODE for one variable. This gives one method to solve coupled first order ODEs. (We can also go the other way by defining z as a linear combination of y and y' and using this to write a second order linear ODE for y as a first order linear ODE involving z', z and y.)