57 *Let A be the region bounded by the positive x- and y-axes and the line $3x + 4y = 10$. Compute $\iint_A (x^2 + y^2)\, dx\, dy$, taking the integrals in both orders and checking that your answers agree.*

58 *In the following integrals sketch the integration regions and then evaluate the integrals. Next interchange the order of integrations and re-evaluate.*

(a) $\int_0^1 (\int_x^1 xy\, dy)\, dx$,

(b) $\int_0^{\pi/2} (\int_0^{\cos\theta} \cos\theta\, dr)\, d\theta$,

(c) $\int_0^1 (\int_1^{2-y} (x+y)^2\, dx)\, dy$.

59 *Exam question 2010 (Section A) Q4: Calculate the double integral*

$$\iint_A (|x| + |y|)\, dx\, dy.$$

where A is the region defined by $|x| + |y| \leq 1$.

60 *Exam question 2011 (Section A) Q4: Change the order of integration in the double integral*

$$\int_0^2 \int_x^{2x} f(x,y)\, dy\, dx\,.$$

61 *Exam question May 2017 (Section A): A solid cylinder C of radius 1 and height 1 is defined by*

$$C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1,\ 0 \leq z \leq 1\}\,.$$

Show that the paraboloid $P = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2\}$ cuts C into two pieces of equal volume.

62 *Compute the iterated integral*

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dy\, dx\,.$$

Now reverse the order of integrations and re-evaluate. Why doesn't your answer contradict Fubini's theorem? (Hints: for the first integral, with respect to y, it might help to aim at something that can be integrated by parts using $\frac{d}{dy}\left(\frac{1}{x^2+y^2}\right) = -\frac{2y}{(x^2+y^2)^2}$; and in answering to the last part of the question, the fact that $\int_0^1 \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| dy \geq \int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\, dy$ could be useful.)

63 *Let B be the region bounded by the five planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$, and $z = x + y$.*

(a) *Find the volume of B.*

(b) *Evaluate $\int_B x\, dV$.*

(c) *Evaluate $\int_B y\, dV$.*

64 *A function $f(x, y)$ is defined by*

$$f(x,y) = \begin{cases} 1 & \text{if } -1 < x - y < 0 \\ -1 & \text{if } 0 < x - y < 1 \\ 0 & \text{otherwise}. \end{cases}$$

Compute $\int_0^\infty \left(\int_{-\infty}^\infty f(x,y)\,dx \right) dy$ and also $\int_{-\infty}^\infty \left(\int_0^\infty f(x,y)\,dy \right) dx$ (for the second case it might help to draw a picture). Comment on your two answers – why does Fubini fail?

65 *A function $f(x, y)$ is defined by*

$$f(x,y) = \begin{cases} 2^{2(n+1)} & \text{if } 2^{-(n+1)} < x < 2^{-n}, \quad 2^{-(n+1)} < y < 2^{-n} \\ -2^{2n+3} & \text{if } 2^{-(n+1)} < x < 2^{-n}, \quad 2^{-(n+2)} < y < 2^{-(n+1)} \\ 0 & \text{otherwise}, \end{cases}$$

for $n \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$.

Compute $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$, and $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$. Does this contradict Fubini's theorem?

66 *Write the line integral*

$$\int_C xdx + ydy + (xz - y)dz$$

in the form $\int_C \mathbf{v} \cdot d\mathbf{x}$ for a suitable vector field $\mathbf{v}(\mathbf{x})$, and compute its value when C is the curve given by $\mathbf{x}(t) = t^2\,\mathbf{e_1} + 2t\,\mathbf{e_2} + 4t^3\,\mathbf{e_3}$ with $0 \leq t \leq 1$.

67 *Evaluate $\int_\sigma \mathbf{F} \cdot d\mathbf{x}$, where $\mathbf{F} = y\,\mathbf{e_1} + 2x\,\mathbf{e_2} + y\,\mathbf{e_3}$ and the path σ is given by $\mathbf{x}(t) = t\,\mathbf{e_1} + t^2\,\mathbf{e_2} + t^3\,\mathbf{e_3}$, $0 \leq t \leq 1$.*

68 *Let $\underline{A}(\underline{x})$ be the vector field $\underline{A}(x, y, z) = x\,\underline{e}_1 + y\,\underline{e}_2 + z\,\underline{e}_3$.*

(a) Compute the line integral $\int_C \underline{A} \cdot d\underline{x}$ where C is the straight line from the origin to the point $(1, 1, 1)$.

(b) Show (by finding f) that the vector field $\underline{A}$ from part (a) is equal to $\nabla f(\underline{x})$ for some scalar field f, and that your answer to part (a) is equal to $f(1, 1, 1) - f(0, 0, 0)$.

69 *Show that the result from question 68 applies in general: if the vector field $\underline{v}(\underline{x})$ in $\mathbb{R}^n$ is the gradient of a scalar field $f(\underline{x})$, so that $\underline{v} = \nabla f$, and if C is a curve in $\mathbb{R}^n$ running from $\underline{x} = \underline{a}$ to $\underline{x} = \underline{b}$, then $\int_C \underline{v} \cdot d\underline{x} = f(\underline{b}) - f(\underline{a})$. (Hint: use the chain rule.)*

70 *Use the result from question 69 to evaluate $\int_C 2xyzdx + x^2zdy + x^2ydz$, where C is any regular curve connecting (1,1,1) to (1,2,4).*

71 *Use method 2 from lectures to compute the surface integral, $\int_S \mathbf{F} \cdot d\mathbf{A}$, of the vector field $\mathbf{F} = (3x^2, -2yx, 8)$ over the surface given by the plane $z = 2x - y$ with $0 \leq x \leq 2$, $0 \leq y \leq 2$.*

72 *Let* $\mathbf{F}(x, y, z) = (z, x, y)$*, and* S *be the part of the surface of the sphere* $x^2 + y^2 + (z-1)^2 = r^2$ *above the plane* $z = 0$*. Assume that* $r > 1$ *so that the boundary* C *of* S*, where* S *intersects the plane* $z = 0$*, is non-empty.*

(a) *Compute the line integral* $\oint_C \mathbf{F} \cdot d\mathbf{x}$.

(b) *By parameterising the surface* S *with spherical coordinates centred on the point* $(0, 0, 1)$*, compute the surface integral of the curl of* $\mathbf{F}$*,* $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{A}$.

73 *Let* $\underline{A}(\underline{x})$ *be the vector field* $\underline{A}(x, y, z) = z\,\underline{e}_1 + x\,\underline{e}_2 + y\,\underline{e}_3$*,* C *be the circle in the* x, y*-plane of radius* r *centred on the origin, and* S *the disk in the* x, y*-plane whose boundary is* C.

(a) *Compute the line integral* $\oint_C \underline{A} \cdot d\underline{x}$.

(b) *Compute the surface integral of the curl of* $\underline{A}$*,* $\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{A}$.

74 *Let* $\underline{A}(\underline{x})$ *be the vector field* $\underline{A}(x, y, z) = z\,\underline{e}_1 + x\,\underline{e}_2 + y\,\underline{e}_3$*,* C *be the* a *by* a *square* $\underline{abcd}$ *in the* y, z*-plane with vertices* $\underline{a} = \underline{0}$*,* $\underline{b} = a\,\underline{e}_2$*,* $\underline{c} = a\,\underline{e}_2 + a\,\underline{e}_3$ *and* $\underline{d} = a\,\underline{e}_3$*, and* S *be the region of the* y, z*-plane bounded by* C.

(a) *Compute the line integral* $\oint_C \underline{A} \cdot d\underline{x}$.

(b) *Compute the surface integral of the curl of* $\underline{A}$*,* $\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{A}$.

75 *Based on exam question May 2015 (Section A) Q5 (which didn't contain part (b)):*

(a) *Calculate* $\int_V (\underline{\nabla} \cdot \underline{U})\, dV$ *where* V *is the solid cube with faces* $x = \pm 1$*,* $y = \pm 1$ *and* $z = \pm 1$ *and*

$$\underline{U}(x, y, z) = (\,x\,y^2, y\,x^2, z\,)\,.$$

(b) *Calculate* $\int_S \underline{U} \cdot d\underline{A}$*, where* S *is the surface of* V.