1 *For the function $f(x, y) = \cos(x + y)\exp(x - y)$ calculate $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x^2$, $\partial^2 f/\partial y^2$, $\partial^2 f/\partial x\partial y$, $\partial^2 f/\partial y\partial x$. Use your results to show that $\partial^2 f/\partial x^2 = -\partial^2 f/\partial y^2$ and $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$.*

Solution: $\partial f/\partial x = -\sin(x + y)\exp(x - y) + \cos(x + y)\exp(x - y)$, $\partial f/\partial y = -\sin(x + y)\exp(x - y) - \cos(x + y)\exp(x - y)$, $\partial^2 f/\partial x^2 = -2\sin(x + y)\exp(x - y)$, $\partial^2 f/\partial y^2 = 2\sin(x + y)\exp(x - y)$, $\partial^2 f/\partial x\partial y = -2\cos(x + y)\exp(x - y)$, $\partial^2 f/\partial y\partial x = -2\cos(x + y)\exp(x - y)$. By inspection $\partial^2 f/\partial x^2 = -\partial^2 f/\partial y^2$ and $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$.

2 *Let $F(t)$ be the value of the function $f(x, y, z) = \cos(xy)\, z$ restricted to the helix $x = \cos(t)$, $y = \sin(t)$, $z = t$ which is parametrised by t and $-\infty < t < \infty$. Calculate dF/dt as a function of t (i) directly by substituting the equations of the helix into $f(x, y, z)$ to calculate $F(t)$ as a function of t and then differentiating, and (ii) using the chain rule. Note how similar these approaches are.*

Solution: (i) Substitution gives $F = \cos(\cos(t)\sin(t))\, t$. Differentiating: $dF/dt = \sin(\cos(t)\sin(t))\, \sin^2(t)\, t - \sin(\cos(t)\sin(t))\, \cos^2(t)\, t + \cos(\cos(t)\sin(t))$. (ii) Using the chain rule:

$$\begin{aligned} dF/dt &= (\partial f/\partial x)(dx/dt) + (\partial f/\partial y)(dy/dt) + (\partial f/\partial z)(dz/dt) \\ &= \sin(xy)\, yz\, \sin(t) - \sin(xy)\, xz\, \cos(t) + \cos(xy) \\ &= \sin(\cos(t)\sin(t))\, \sin^2(t)\, t - \sin(\cos(t)\sin(t))\, \cos^2(t)\, t + \cos(\cos(t)\sin(t)). \end{aligned}$$

3 *If $\mathbf{a} = \sin 2t\, \mathbf{e_1} + e^t\, \mathbf{e_2} - (t^3 - 5t)\, \mathbf{e_3}$, find*
(a) $d\mathbf{a}/dt$, (b) $||d\mathbf{a}/dt||$, (c) $d^2\mathbf{a}/dt^2$, (d) $||d^2\mathbf{a}/dt^2||$, all at $t = 0$.

Solution: (a) $d\mathbf{a}/dt = 2\cos(2t)\, \mathbf{e_1} + e^t\, \mathbf{e_2} - (3t^2 - 5)\, \mathbf{e_3}$. At $t = 0$ this is $2\, \mathbf{e_1} + \mathbf{e_2} + 5\, \mathbf{e_3}$. (b) which has length $||d\mathbf{a}/dt|| = \sqrt{2^2 + 1^2 + 5^2} = \sqrt{30}$, (c) $d^2\mathbf{a}/dt^2 = -4\sin(2t)\, \mathbf{e_1} + e^t\, \mathbf{e_2} - 6t\, \mathbf{e_3}$, at $t = 0$ this is $\mathbf{e_2}$, (d) so $||d^2\mathbf{a}/dt^2|| = 1$ at $t = 0$.

4 *Find a unit vector tangent to the space curve $x = t^3, y = t, z = t^2$ at $t = 2$.*

Solution: If $\mathbf{x}(t) = t^3\mathbf{e_1} + t\mathbf{e_2} + t^2\mathbf{e_3}$ then $d\mathbf{x}(t)/dt = 3t^2\mathbf{e_1} + \mathbf{e_2} + 2t\mathbf{e_3}$ is parallel to the tangent. At $t = 2$ this is $12\mathbf{e_1} + \mathbf{e_2} + 4\mathbf{e_3}$ which has length $\sqrt{161}$ so a unit vector in this direction is $(12\mathbf{e_1} + \mathbf{e_2} + 4\mathbf{e_3})/\sqrt{161}$.

5 *Use the chain rule to calculate df/dt when $f(\mathbf{x}) = \exp(-||\mathbf{x}||^2)$ is restricted to the curves:*

(a) $\mathbf{x} = \mathbf{e_1}\log t + \mathbf{e_2}t\log t + \mathbf{e_3}t$,

(b) $(x, y, z) = (\cosh t, \sinh t, 0)$.

Solution: The chain rule to use here is

$$\frac{df}{dt} = \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} + \frac{dz}{dt}\frac{\partial f}{\partial z}\,.$$

Since $f(\mathbf{x}) = \exp(-||\mathbf{x}||^2) = \exp(-(x^2 + y^2 + z^2))$ we have

$$\frac{\partial f}{\partial x} = -2x\, f\,, \quad \frac{\partial f}{\partial y} = -2y\, f\,, \quad \frac{\partial f}{\partial z} = -2z\, f\,.$$

(a) $\mathbf{x} = \mathbf{e_1} \log t + \mathbf{e_2} t \log t + \mathbf{e_3} t$, so $d\mathbf{x}/dt = \mathbf{e_1}/t + \mathbf{e_2}(\log t + 1) + \mathbf{e_3}$ and

$$\frac{df}{dt} = (1/t)(-2x)\,f + (\log t + 1)(-2y)\,f + (-2z)\,f$$

$$= -2((\log t)/t + (\log t + 1)t\log t + t)\ \exp -((\log t)^2 + (t\log t)^2 + t^2)\,.$$

(b) $(x, y, z) = (\cosh t, \sinh t, 0)$, so $(dx/dt, dy/dt, dz/dt) = (\sinh t, \cosh t, 0)$ giving

$$\begin{aligned}\frac{df}{dt} &= (\sinh t)(-2x)\,f + (\cosh t)(-2y)\,f + 0.(-2z)\,f \\ &= -4\sinh t\cosh t\ \exp(-\cosh^2 t - \sinh^2 t) = -4\sinh t\cosh t\ \exp(-\cosh 2t)\,.\end{aligned}$$

6 *Show that the curve C, given as points $\underline{x}(s) = \left(\sin(s/\sqrt{2}), \cos(s/\sqrt{2}), s/\sqrt{2}\right)$, is the arc-length parameterisation of a helix, that is that* $\left|\frac{d\underline{x}}{ds}\right| = 1 \quad \forall s$.

Solution: To show that the curve is parameterised by arc-length, we need to show that $|\frac{d\underline{x}}{ds}| = 1 \quad \forall s$. We have

$$\frac{d\underline{x}}{ds} = \left(\frac{1}{\sqrt{2}}\cos(s/\sqrt{2}), -\frac{1}{\sqrt{2}}\sin(s/\sqrt{2}), 1/\sqrt{2}\right),$$

and therefore

$$\begin{aligned}\left|\frac{d\underline{x}}{ds}\right| &= \frac{1}{2}\cos^2(s/\sqrt{2}) + \frac{1}{2}\sin^2(s/\sqrt{2}) + \sqrt{\frac{1}{2}} \\ &= \sqrt{\frac{1}{2} + \frac{1}{2}} \\ &= 1.\end{aligned}$$

7 *Describe the curve $\gamma : \underline{x}(t) = (2t+1, t-3, 6-2t)$. Find the arc-length parameterisation of γ, that is, re-parameterise the curve in terms of a parameter s, such that* $\left|\frac{d\underline{x}(s)}{ds}\right| = 1\,\forall s$.

Solution: We can write $\underline{x}(t) = (1, -3, 6) + t(2, 1, -2)$, which we recognise as a parameterisation of a straight line through $\underline{a} = (1, -3, 6)$ parallel to $\underline{b} = (2, 1, -2)$.

Before trying to re-parameterise γ, we should check whether the curve is *already* parameterised in terms of arc-length. We have

$$\frac{d\underline{x}}{dt} = (2, 1, -2),$$

and so

$$\left|\frac{d\underline{x}}{dt}\right| = \sqrt{4 + 1 + 4} = 3.$$

Since this is not equal to 1 $\forall t$, t does not give the arc length parameterisation of γ.

Note that we cannot just multiply $\underline{x}(t)$ by $1/3$, as although the resulting curve would have tangent vectors of unit length, this would be a straight line through $\underline{a}/3$ rather than through $\underline{a}$, and so this is a parameterisation of a different curve. However, we can see that if we let $t = s/3$, then we will find that the resulting tangent vector $\frac{d\underline{x}}{ds}$ is unit length:

$$\begin{aligned}\underline{x}(s) &= \left(2\frac{s}{3}+1, \frac{s}{3}-3, 6-2\frac{s}{3}\right),\\ \frac{d\underline{x}}{ds} &= \left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right),\\ \left|\frac{d\underline{x}}{ds}\right| &= \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{4}{9}} = 1.\end{aligned}$$

We can see that since the only changes we have made are to the parameter in the curve, this is still a parameterisation of a curve through $\underline{a}$ parallel to $\underline{b}$. This is therefore the arc-length parameterisation of γ that was required. If you choose to study Differential Geometry III, you will learn the general method for parameterising a curve in terms of its arc-length.

8 ***Harder:*** *Let* $\mathbf{t}$ *denote the unit tangent vector to a space curve* $\mathbf{a} = \mathbf{a}(s)$ *in* $\mathbb{R}^3$*, where* $\mathbf{a}(s)$ *is assumed differentiable, and where* s *measures the arclength from some fixed point on the curve. Define the unit vector* $\mathbf{n} = \frac{1}{\kappa}\frac{d\mathbf{t}}{ds}$*, where* κ *is a scalar. Also define* $\mathbf{b} = \mathbf{t}\times\mathbf{n}$ *as the* unit binormal vector *to the space curve.*

By considering the derivative of the product $\mathbf{t}.\mathbf{t}$*, show that the the 3 vectors* $\mathbf{t}, \mathbf{n}, \mathbf{b}$ *form an orthonormal basis of* $\mathbb{R}^3$*.*

Hence, prove that

$$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{n}, \quad \text{and} \quad \frac{d\mathbf{n}}{ds} = \tau\mathbf{b} - \kappa\mathbf{t},$$

where τ *is some real constant.*

These formulae are of fundamental importance in differential geometry. They involve the curvature κ *and the torsion* τ*. The reciprocals of these are the radius of curvature (*$\rho = \frac{1}{\kappa}$*) and the radius of torsion (*$\sigma = \frac{1}{\tau}$*).*

Solution: $\mathbf{t}$ is a unit tangent, and hence we have $\mathbf{t}.\mathbf{t} = 1$. Differentiating this expression with respect to s using the product rule gives $2\mathbf{t}.\frac{d\mathbf{t}}{ds} = 0$, and hence $\frac{d\mathbf{t}}{ds}$ is orthogonal to $\mathbf{t}$. Since $\mathbf{n} = \frac{1}{\kappa}\frac{d\mathbf{t}}{ds}$, we have $\mathbf{t}$ and $\mathbf{n}$ orthogonal, and since we define $\mathbf{b} = \mathbf{t}\times\mathbf{n}$, $\mathbf{b}$ is orthogonal to the plane containing $\mathbf{t}$ and $\mathbf{n}$. We have therefore shown that these three vectors form an orthogonal basis of $\mathbb{R}^3$, and since both $\mathbf{t}$ and $\mathbf{n}$ are orthogonal vectors, then so is their vector product, and hence these vectors form an orthonormal basis of $\mathbb{R}^3$.

Since these vectors are all mutually orthogonal, alongside the relation $\mathbf{b} = \mathbf{t}\times\mathbf{n}$, we also have $\mathbf{t} = \mathbf{n}\times\mathbf{b}$ and $\mathbf{n} = \mathbf{b}\times\mathbf{t}$, as can easily be seen using the right-hand rule.

Now the derivative $\frac{d\mathbf{b}}{ds}$ is given by

$$\begin{aligned}\frac{d\mathbf{b}}{ds} = \frac{d}{ds}(\mathbf{t}\times\mathbf{n}) &= \left(\frac{d\mathbf{t}}{ds}\right)\times\mathbf{n} + \mathbf{t}\times\left(\frac{d\mathbf{n}}{ds}\right)\\ &= (\kappa\mathbf{n})\times\mathbf{n} + \mathbf{t}\times\left(\frac{d\mathbf{n}}{ds}\right)\\ &= \mathbf{t}\times\left(\frac{d\mathbf{n}}{ds}\right),\end{aligned}$$

since the cross product of $\mathbf{n}$ with itself is 0.

Since $\mathbf{n}$ is also a unit vector, we have $\mathbf{n}.\mathbf{n} = 1$, and as before differentiating this with respect to s gives $2\mathbf{n}.\frac{d\mathbf{n}}{ds} = 0$. $\frac{d\mathbf{n}}{ds}$ is therefore orthogonal to $\mathbf{n}$, and since $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ form a basis of $\mathbb{R}^3$, we therefore have that

$$\frac{d\mathbf{n}}{ds} = \sigma\mathbf{t} + \tau\mathbf{b}.$$

Substituting this into our equation for the derivative of $\mathbf{b}$ gives

$$\begin{aligned}\frac{d\mathbf{b}}{ds} &= \mathbf{t} \times (\sigma\mathbf{t} + \tau\mathbf{b}) \\ &= \tau\mathbf{t} \times \mathbf{b} \\ &= -\tau\mathbf{n},\end{aligned}$$

which is the first of the equations we were asked to show.

Finally, since $\mathbf{n} = \mathbf{b} \times \mathbf{t}$, we have

$$\begin{aligned}\frac{d\mathbf{n}}{ds} &= \frac{d\mathbf{b}}{ds} \times \mathbf{t} + \mathbf{b} \times \frac{d\mathbf{t}}{ds}, \\ &= (-\tau\mathbf{n}) \times \mathbf{t} + \mathbf{b} \times (\kappa\mathbf{n}) \quad = \tau\mathbf{b} - \kappa\mathbf{t},\end{aligned}$$

which is the second equation we were asked to show.

These are the Serret-Frenet formulas; see e.g. G.E.Hay, Vector and Tensor Analysis.

Bonus 1 *If $f(x, y) = F(r, \theta)$ with $x = r\cos\theta$ and $y = r\sin\theta$, use the chain rule to compute $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2$ in terms of partial r- and θ- derivatives of F, and hence find the general rotationally-symmetric solution to $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = 0$ in two dimensions which is non-singular away from the origin.*
Suggestion: Begin by writing $\partial r/\partial x$, $\partial r/\partial y$, $\partial\theta/\partial x$ and $\partial\theta/\partial y$ as functions of r and θ.

Solution: Following the suggestion, and borrowing some of the calculations from section 0.2 of lectures, we have

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta\,; \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta\,,$$
$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2} = -\frac{\sin\theta}{r}\,; \qquad \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}\,,$$

and so by the chain rule

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\,, \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\,.\end{aligned}$$

Hence

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\,,$$

or, a little more explicitly

$$\frac{\partial^2 f}{\partial x^2} = \left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\right)\left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\right)F\,.$$

Now for a key point – each derivative in this formula acts on *everything* to its right. So when expanding the brackets on the RHS you'll get terms like

$$-\frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\cos\theta\,\frac{\partial}{\partial r}F = -\frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\left(\cos\theta\,\frac{\partial F}{\partial r}\right) = -\frac{\sin\theta}{r}\,\left(-\sin\theta\,\frac{\partial F}{\partial r} + \cos\theta\,\frac{\partial^2 F}{\partial\theta\partial r}\right)$$

where the first equality simply emphasises the fact that the $\partial/\partial\theta$ acts on everything to its right, while the second comes from using the product rule for derivatives. With practice you should be able to do these calculations in 'operator' notation, not bothering to write f and F explicitly but remembering that they are there implicitly, so the full calculation of $\partial^2/\partial x^2$ would look like

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\right)\left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\right) \\
&= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} \\
&\quad + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} \\
&= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}
\end{aligned}$$

(Using the equality of the mixed second partial derivatives to gather the terms together.) A similar calculation (exercise!) gives

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta\frac{\partial^2}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Adding the two, many terms cancel and/or simplify to leave the final result

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\,.$$

For the final part: for a rotationally-symmetric solution, F will be independent of θ and so the equation (which is called Laplace's equation) simplifies to

$$0 = \frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right)$$

(Since F only depends on one variable, there's no need to use the partial derivative symbol.) This equation is easily solved: $F(r) = C\log r + D$ with C and D two constants, and this is the general rotationally-symmetric solution to Laplace's equation in two dimensions.