******The following questions are concerned with Chapter 3 of the notes - Codes as Images.******

28 Let code $C_5 \subseteq \mathbb{F}_5^4$ be the span of the set $\{(0,1,2,3),(1,1,1,1),(3,1,4,2)\}$. Find a generator-matrix for C_5. What is the dimension of C_5?

29 Let code $C_7 \subseteq \mathbb{F}_7^4$ be the span of the set $\{(0,1,2,3),(1,1,1,1),(3,1,4,2)\}$. Find a generator-matrix for C_7. What is the dimension of C_7? (- so, identical to the previous question, except that we are over a different field.)

30 For each of the codes above, C_5 and C_7, write down an alternative generator-matrix.

31 a) Draw 49 points in a square grid, to represent $\mathbb{F}_7^2$. (You could label just the "axes", $S((0,0),1)$). Find the points corresponding to the code C with generator-matrix $(2\ 1)$. Does it look like a "line" in a "plane"? Can you think of a better way to draw (or model?) these vector spaces?
b) Perhaps on a new grid, draw the code C' with generator-matrix $(1\ 3)$. Can you see two different ways to draw the "line"? Is one better than the other?

32 The code $C \subseteq \mathbb{F}_7^5$ has generator matrix $G_1 = \begin{pmatrix} 1 & 2 & 3 & 3 & 3 \\ 0 & 2 & 1 & 5 & 5 \\ 4 & 5 & 0 & 6 & 3 \end{pmatrix}$.
Use this to encode the message $(3,2,1) \in \mathbb{F}_7^3$ to a codeword $\mathbf{c}$. Also, channel-decode codeword $\mathbf{c}' = (4,5,0,0,2)$ to find the corresponding message. (You will need to solve a set of five equations - possibly by row-reducing a suitable augmented matrix.)

33 For the code C of Q32, find an alternative generator-matrix, G_2, in RREF. Use this to encode the message $(3,2,1)$. Also, use G_2 to channel-decode the codeword $(2,1,1,4,0)$.

34 There is a code C' which is equivalent to code C of Q32 but has a generator matrix G_3 in standard form. Use this matrix to encode $(3,2,1)$ to a codeword of C', and channel-decode the codeword $(2,1,4,1,0)$.

35 Equivalent codes have the same rank, redundancy and rate. Find these values for C' and C above.

36 Let C be an (n,M,d) over an alphabet of order q, not necessarily linear. If C_2 is equivalent to C_1, show that C_2 is also an (n,M,d) over an alphabet of order q.

37 The codes C_1 and C_2 in $\mathbb{F}_5^6$ have generator-matrices G_1 and G_2 respectively, where
$G_1 = \begin{pmatrix} 0 & 3 & 1 & 0 & 3 & 1 \\ 1 & 4 & 0 & 2 & 3 & 4 \\ 2 & 0 & 3 & 4 & 3 & 0 \end{pmatrix}$ and $G_2 = \begin{pmatrix} 3 & 1 & 4 & 1 & 0 & 0 \\ 4 & 4 & 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 1 & 3 & 2 \end{pmatrix}$.
Show that C_1 and C_2 are monomially equivalent.

38 Given a code $C \subseteq \mathbb{F}_q^n$, prove that $\mathsf{PAut}(C)$ is a group.

39 Let $C \subseteq \mathbb{F}_3^4$ be the code with generator matrix

$$G = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Let $g = (134) \in S_4$. Show that $g \in \mathsf{PAut}(C)$.

40 Consider two maps $\pi : \mathbb{F}_q^n \longrightarrow \mathbb{F}_q^n$. Map $\pi_{s(i,j)}$ swaps the i^{th} and j^{th} entry of each vector, and map $\pi_{m(i,\mu)}$ multiplies the i^{th} entry by $\mu \in \mathbb{F}_q$. Show that for each of these maps, and for any $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$ and $\lambda \in \mathbb{F}_q$, we have $\pi(\mathbf{x}+\mathbf{y}) = \pi(\mathbf{x}) + \pi(\mathbf{y})$, and $\pi(\lambda\mathbf{x}) = \lambda\pi(\mathbf{x})$. For this reason we say that these maps "preserve linear structure".

41 Suppose an $[n, k, d]$ code C has a generator-matrix G in RREF. By considering the weights of the rows of G, find a new proof that $d \leq n - k + 1$ (the Singleton bound for linear codes).

42 We know that any generator-matrix for a code C can be row-reduced to a generator-matrix G in RREF, and that this RREF generator-matrix is unique. Thus, if C does have a generator-matrix in standard form $(I \mid A)$, it will be this matrix G. Again by considering weights of rows, show that if C is maximum distance separable then it has a generator-matrix in standard form. (Hint: Prove the contrapositive.)

43 Show (by example or argument) that the converse of Q42,
"If C has a generator-matrix in standard form then it is MDS."
is false.