1. For the function $f(x,y) = \cos(x+y)\exp(x-y)$ calculate $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x^2$, $\partial^2 f/\partial y^2$, $\partial^2 f/\partial x\partial y$, $\partial^2 f/\partial y\partial x$. Use your results to show that $\partial^2 f/\partial x^2 = -\partial^2 f/\partial y^2$ and $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$.

2. Let $F(t)$ be the value of the function $f(x,y,z) = \cos(xy)\,z$ restricted to the helix $x = \cos(t), y = \sin(t)\,, z = t$ which is parametrised by t and $-\infty < t < \infty$. Calculate dF/dt as a function of t (i) directly by substituting the equations of the helix into $f(x,y,z)$ to calculate $F(t)$ as a function of t and then differentiating, and (ii) using the chain rule. Note how similar these approaches are.

3. If $\mathbf{a} = \sin 2t\,\mathbf{e_1} + e^t\,\mathbf{e_2} - (t^3 - 5t)\,\mathbf{e_3}$, find
(a) $d\mathbf{a}/dt$, (b) $||d\mathbf{a}/dt||$, (c) $d^2\mathbf{a}/dt^2$, (d) $||d^2\mathbf{a}/dt^2||$, all at $t = 0$.

4. Find a unit vector tangent to the space curve $x = t^3, y = t, z = t^2$ at $t = 2$.

5. Use the chain rule to calculate df/dt when $f(\mathbf{x}) = \exp(-||\mathbf{x}||^2)$ is restricted to the curves:

   (a) $\mathbf{x} = \mathbf{e_1}\log t + \mathbf{e_2} t\log t + \mathbf{e_3} t,$

   (b) $(x,y,z) = (\cosh t, \sinh t, 0).$

6. **Harder:** Let $\mathbf{t}$ denote the unit tangent vector to a space curve $\mathbf{a} = \mathbf{a}(s)$ in $\mathbb{R}^3$, where $\mathbf{a}(s)$ is assumed differentiable, and where s measures the arclength from some fixed point on the curve. Define the unit vector $\mathbf{n} = \frac{1}{\kappa}\frac{d\mathbf{t}}{ds}$, where κ is a scalar. Also define $\mathbf{b} = \mathbf{t}\times\mathbf{n}$ as the *unit binormal vector* to the space curve.

   By considering the derivative of the product $\mathbf{t}.\mathbf{t}$, show that the the 3 vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ form an orthonormal basis of $\mathbb{R}^3$.

   Hence, prove that
$$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{n}, \quad \text{and} \quad \frac{d\mathbf{n}}{ds} = \tau\mathbf{b} - \kappa\mathbf{t},$$
   where τ is some real constant.

   These formulae are of fundamental importance in differential geometry. They involve the curvature κ and the torsion τ. The reciprocals of these are the radius of curvature ($\rho = \frac{1}{\kappa}$) and the radius of torsion ($\sigma = \frac{1}{\tau}$).

Bonus 1. If $f(x,y) = F(r,\theta)$ with $x = r\cos\theta$ and $y = r\sin\theta$, use the chain rule to compute $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2$ in terms of partial r- and θ- derivatives of F, and hence find the general rotationally-symmetric solution to $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = 0$ in two dimensions which is non-singular away from the origin.
Suggestion: Begin by writing $\partial r/\partial x$, $\partial r/\partial y$, $\partial\theta/\partial x$ and $\partial\theta/\partial y$ as functions of r and θ.

7. Compute the gradient, $\underline{\nabla} f$, for the following functions:

   (a) $f(x,y,z) = \sqrt{x^2 + y^2 + z^2},$

   (b) $f(x,y,z) = xy + yz + xz,$

   (c) $f(x,y,z) = 1/(x^2 + y^2 + z^2).$

8. Show that $\underline{h}(s) = \left(s/\sqrt{2}, \cos(s/\sqrt{2}), \sin(s/\sqrt{2})\right)$ is the arc-length parameterisation of a helix, that is that $|\frac{dh}{ds}| = 1 \quad \forall s$.

   Calculate the directional derivative of the scalar field $f(\underline{x}) = (\log(x^2 + y^2 + z^2))$ along $\underline{h}(s)$ at $s = \sqrt{2}\pi$.

9. Draw a sketch of the contour plot of the scalar field on $\mathbb{R}^2$ $f(\underline{x}) = xy$, as well as the gradient of f. What do you notice?

10. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be scalar fields on $\mathbb{R}^3$, $h : \mathbb{R} \to \mathbb{R}$ be a function on $\mathbb{R}$ and a be a constant in $\mathbb{R}$. Show (using the definition of $\underline{\nabla}$) that

$$\underline{\nabla}(af(\underline{x})g(\underline{x}) + h(f(\underline{x}))) = a(\underline{\nabla} f)g + af\underline{\nabla} g + \underline{\nabla} f \frac{dh}{df}.$$

11. Exam question June 2001 (Section B): You are given the following family of scalar functions labelled by a real parameter $\lambda : \Phi_\lambda(x, y, z) = (y - \lambda)\cos x + zxy$.
    (a) What are their derivatives in the direction $\mathbf{V} = \mathbf{e_1} + 2(\mathbf{e_2} + \mathbf{e_3})$?
    (b) Which member of the family has its gradient at the point $(\frac{\pi}{2}, 1, 1)$ equal to $\frac{\pi}{2}(\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3})$?
    (c) Calling this particular member of the family Φ_{λ_0}, in which direction is Φ_{λ_0} decreasing most rapidly when starting at the point $(\frac{\pi}{2}, 1, 1)$?

12. Exam question June 2002 (Section A): Give the unit vector normal to the surface of equation $x^2/a^2 + y^2/b^2 + z^2/c^2 = 4$ where a, b, c are three real constants.
    What is the unit vector normal to a sphere of radius 2 at the point $(x, y, z) = (\sqrt{2}, 0, \sqrt{2})$?

13. Find the vector equations of tangent and normal lines in $\mathbb{R}^2$ to the following curves at the given points
    (a) $x^2 + 2y^2 = 3$ at $(1, 1)$,
    (b) $xy = 1$ at $(2, 1/2)$,
    (c) $x^2 - y^3 = 3$ at $(2, 1)$.

14. Exam question June 2003 (Section A): Find the directional derivative of the function $\phi(x, y, z) = xy^2z^3$ at the point $P = (1, 1, 1)$ in the direction from P towards $Q = (3, 1, -1)$. Starting from P, in which direction is the directional derivative maximum and what is the value of this maximum?

15. Exam question June 2002 (Section A): What is the derivative of the scalar function $\phi(x, y, z) = x\cos z - y$ in the direction $\mathbf{V} = \mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}$? What is the gradient at the point $(x, y, z) = (0, 1, \pi/2)$? In which direction is ϕ increasing the most when moving away from this point?

16. A marble is released from the point $(1, 1, c - a - b)$ on the elliptic paraboloid defined by $z = c - ax^2 - by^2$, where a, b, c are positive real numbers and the z-coordinate is vertical. In which direction in the (x, y) plane does the marble begin to roll?

17. In which direction does the function $f(x, y) = x^2 - y^2$ increase fastest at the points (a) $(1, 0)$, (b) $(-1, 0)$, (c) $(2, 1)$? Illustrate with a sketch.

18. Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$.
    (a) In which direction is the directional derivative of f at $(1, 1)$ equal to zero?
    (b) What about at an arbitrary point (x_0, y_0) in the first quadrant?
    (c) Describe the level curves of f and discuss them in the light of the result in (b).

19. Compute the divergence, $\nabla \cdot \mathbf{A}$, of the following vector fields:
    (a) $\mathbf{A}(x, y, z) = yz\mathbf{e_1} + xz\mathbf{e_2} + xy\mathbf{e_3}$,
    (b) $\mathbf{A}(x, y, z) = (x^2 + y^2 + z^2)\,(3\mathbf{e_1} + 4\mathbf{e_2} + 5\mathbf{e_3})$,
    (c) $\mathbf{A}(x, y, z) = (x + y)\mathbf{e_1} + (y + z)\mathbf{e_2} + (z + x)\mathbf{e_3}$.

20. Compute the curl, $\nabla \times \mathbf{A}$, of each of the vector fields, $\mathbf{A}$, in the previous question.

21. If $f(r)$ is a differentiable function of $r = |\mathbf{x}\,|$, show that
    (a) $\mathrm{grad}\, f(r) = f'(r)\,\mathbf{x}\,/r$,
    (b) $\mathrm{curl}\,[f(r)\mathbf{x}\,] = 0$.

22. Let $\mathbf{x}$ be the position vector in three dimensions, with $r = |\mathbf{x}\,|$, and let $\mathbf{a}$ be a constant vector. Show that
    (a) $\mathrm{div}\,\mathbf{x} = 3$,
    (b) $\mathrm{curl}\,\mathbf{x} = 0$,
    (c) $\mathrm{grad}\, r = \mathbf{x}\,/r$,
    (d) $\mathrm{div}\,(r^n\mathbf{x}\,) = (n + 3)\,r^n$,
    (e) $\mathrm{grad}\,(\mathbf{a} \cdot \mathbf{x}\,) = \mathbf{a}$,
    (f) $\mathrm{div}\,(\mathbf{a} \times \mathbf{x}\,) = 0$,
    (g) $\mathrm{curl}\,(\mathbf{a} \times \mathbf{x}\,) = 2\,\mathbf{a}$,
    (h) $\mathrm{curl}\,(r^2\mathbf{a}) = 2\,(\mathbf{x} \times \mathbf{a})$,
    (i) $\nabla^2(1/r) = 0$, if $r \neq 0$,
    (j) $\nabla^2(\log r) = 1/r^2$, if $r \neq 0$,
    (k) $\mathrm{div}\,[(\mathbf{a} \cdot \mathbf{x}\,)\mathbf{x}\,] = 4\,\mathbf{a} \cdot \mathbf{x}$,
    (l) $\mathrm{div}\,[\mathbf{x} \times (\mathbf{x} \times \mathbf{a})] = 2\,\mathbf{a} \cdot \mathbf{x}$,
    (m) $\mathrm{curl}\,(\mathbf{a} \times \mathbf{x}\,/r^3) = 3\,(\mathbf{a} \cdot \mathbf{x}\,)\mathbf{x}\,/r^5 - \mathbf{a}/r^3$,
    (n) Exam question June 2002 (Section A): calculate the curl of $(\mathbf{a} \cdot \mathbf{x}\,)\,\mathbf{x}$.

23. If $\mathbf{x}$ is the position vector, $\mathbf{x} = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$, $\mathbf{a}$ is a constant vector, $\mathbf{F} = (\mathbf{a} \cdot \mathbf{x})\,\mathbf{x}$ and $\mathbf{G} = r^2\mathbf{a}$, (with $r = |\mathbf{x}\,|$), show that

    (a) $\operatorname{div}\mathbf{F} = 2\,\operatorname{div}\mathbf{G} = 4\,\mathbf{a}\cdot\mathbf{x}\,,$

    (b) $\operatorname{curl}\mathbf{G} = -2\,\operatorname{curl}\mathbf{F} = 2\,\mathbf{x}\,\times\mathbf{a},$

    (c) $\operatorname{div}\operatorname{curl}\mathbf{F} = \operatorname{div}\operatorname{curl}\mathbf{G} = 0,$

    (d) $\operatorname{curl}\operatorname{curl}\mathbf{G} = -2\,\operatorname{curl}\operatorname{curl}\mathbf{F} = -4\,\mathbf{a}.$

24. Exam question June 2001 (Section A):

    (a) Give a representation of the vector function $\mathbf{A}(x, y) = y\mathbf{e_1}$ as a collection of arrows in the region of the (x, y)-plane bounded by $(x_1, y_1) = (-2, 2), (x_2, y_2) = (2, 2), (x_3, y_3) = (2, -2), (x_4, y_4) = (-2, -2)$.

    (b) Calculate the curl of the vector field $\mathbf{A}(x, y) = (-y\mathbf{e_1} + x\mathbf{e_2})/(x^2 + y^2)$ defined everywhere in the (x, y)-plane except at the origin. (You can consider $\mathbf{A}$ to be embedded in three dimensions, independent of z and with zero z component.)

    (c) Give the unit vector normal to the surface of equation $ax + by = cz$, where a, b, c, are three real constants.

    (d) (Slightly modified from exam) Let $\mathbf{x}$ be the position vector in 3-dimensions and $\mathbf{a}$ be a constant vector. Use the result $\mathbf{x}\times(\mathbf{x}\times\mathbf{a}) = \mathbf{x}\,(\mathbf{x}\cdot\mathbf{a}) - \mathbf{a}(\mathbf{x}\cdot\mathbf{x}\,)$ to show that $\operatorname{div}\,[\,\mathbf{x}\times(\mathbf{x}\times\mathbf{a})] = 2\mathbf{a}\cdot\mathbf{x}\,.$

25. Let $\mathbf{x}$ be the position vector in three dimensions, with $r = |\mathbf{x}\,|$, and let $\mathbf{a}$ be a constant vector. Using index notation, show that

    (a) $\operatorname{div}\mathbf{x} \;=\; 3,$

    (b) $\operatorname{curl}\mathbf{x} \;= 0,$

    (c) $\operatorname{grad} r \;=\; \mathbf{x}\,/r,$

    (d) $\operatorname{div}(r^n\mathbf{x}\,) \;=\; (n+3)\,r^n,$

    (e) $\operatorname{grad}(\mathbf{a}\cdot\mathbf{x}\,) \;=\; \mathbf{a},$

    (f) $\operatorname{div}(\mathbf{a}\times\mathbf{x}\,) \;= 0,$

    (g) $\operatorname{curl}(\mathbf{a}\times\mathbf{x}\,) \;=\; 2\,\mathbf{a},$

    (h) $\operatorname{curl}(r^2\mathbf{a}) \;=\; 2\,(\mathbf{x}\,\times\mathbf{a}),$

    (i) $\nabla^2(1/r) \;=\; 0$ if $r \neq 0$: using $\frac{\partial}{\partial x_i} r = x_i/r$ from part (c),

    (j) $\nabla^2(\log r) \;=\; 1/r^2$ if $r \neq 0$:

    (k) $\operatorname{div}[(\mathbf{a}\cdot\mathbf{x}\,)\mathbf{x}\,] \;=\; 4\,\mathbf{a}\cdot\mathbf{x}\,,$

    (l) $\operatorname{div}[\mathbf{x}\,\times(\mathbf{x}\,\times\mathbf{a})] \;=\; 2\,\mathbf{a}\cdot\mathbf{x}\,,$

    (m) $\operatorname{curl}\,(\mathbf{a}\times\mathbf{x}\,/r^3) \;=\; 3\,(\mathbf{a}\cdot\mathbf{x}\,)\mathbf{x}\,/r^5 \;-\; \mathbf{a}/r^3,$

    (n) Exam question June 2002 (Section A): calculate the curl of $(\mathbf{a}\cdot\mathbf{x}\,)\,\mathbf{x}\,.$

26. The vector $\underline{a}$ has components $(a_r) = (1, 1, 1)$ and the vector $\underline{b}$ has components $(b_r) = (2, 3, 4)$. In the following expressions state which indices are free and which are dummy, and give the numerical values of the expressions for each value that the free variable takes (e.g. for $a_r - b_r$ the free variable is r and it takes the values 1, 2, 3 so $a_1 - b_1 = -1$, $a_2 - b_2 = -2$, $a_3 - b_3 = -3$)
   (a) $a_r + b_r$,
   (b) $a_r\, b_r$,
   (c) $a_r\, b_s\, a_r$,
   (d) $a_r\, b_s\, a_r\, b_s - a_r\, b_r\, a_s\, b_s$.

27. If δ_{rs} is the three-dimensional Kronecker delta, evaluate
   (a) $\delta_{rs}\, \delta_{sr}\, \delta_{pq}\, \delta_{pq}$,
   (b) $\delta_{rs}\, \delta_{sk}\, \delta_{kl}\, \delta_{lr}$,
   (c) $\delta_{rs}\, \delta_{qr}\, \delta_{pq}\, \delta_{sp}$.

28. If δ_{rs} is the three-dimensional Kronecker delta, simplify
   (a) $(\delta_{rp}\, \delta_{sq} - \delta_{rq}\, \delta_{sp})\, a_p\, b_q$,
   (b) $(\delta_{rp}\, \delta_{sq} - \delta_{rq}\, \delta_{sp})\, \delta_{pq}$.

29. If δ_{rs} is the Kronecker delta in n dimensions, calculate
   (a) δ_{rr},
   (b) $\delta_{rs}\, \delta_{rs}$,
   (c) $\delta_{rs}\, \delta_{st}\, \delta_{tr}$.

30. Starting from $\varepsilon_{ijk}\, \varepsilon_{klm} = \delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl}$ simplify as much as possible:
   (a) $\varepsilon_{ijk}\varepsilon_{ijp}$,
   (b) $\varepsilon_{ijk}\varepsilon_{ijk}$.

31. Calculate ε_{ijj} .

32. Show, using index notation, that
   (a) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$,
   (b) $$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= [\mathbf{d} \cdot (\mathbf{a} \times \mathbf{b})]\, \mathbf{c} - [\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})]\, \mathbf{d} \\ &= [\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})]\, \mathbf{b} - [\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})]\, \mathbf{a},\end{aligned}$$
   (c) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$,
   (d) $\mathbf{a} \times [\mathbf{a} \times (\mathbf{a} \times \mathbf{b})] = a^2\, (\mathbf{b} \times \mathbf{a})$,
   (e) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{d}) + (\mathbf{c} \times \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{d}) = 0$.

33. Exam question June 2002 (Section A): Evaluate $\varepsilon_{ijk}\varepsilon_{ijl}x_k x_l$.

34. Exam question June 2001 (Section A): Evaluate $\varepsilon_{ijk}\partial_i\partial_j(x_l x_l)^{1/2}$ away from the origin.

35. Exam question June 2003 (Section A): Calculate $\partial_i\,(\varepsilon_{ijk}\,\varepsilon_{jkl}\,x_l\,)$. (Hint: use the connection between $\partial_i x_j = \frac{\partial x_j}{\partial x_i}$ and the Kronecker delta.)

36. The functions f, g are scalars, while $\mathbf{A}$ and $\mathbf{B}$ are vector functions with components A_i and B_i respectively. Verify the following identities using index notation:

    (a) $\mathrm{grad}\,(fg) \;=\; f\,\mathrm{grad}\,g \;+\; g\,\mathrm{grad}\,f,$

    (b) $\mathrm{grad}\,(\mathbf{A}\cdot\mathbf{B}) \;=\; \mathbf{A}\times\mathrm{curl}\,\mathbf{B} + \mathbf{B}\times\mathrm{curl}\,\mathbf{A} \;+\; (\mathbf{B}\cdot\nabla)\mathbf{A} \;+\; (\mathbf{A}\cdot\nabla)\mathbf{B},$

    (c) $\mathrm{div}\,(f\mathbf{A}) \;=\; f\,\mathrm{div}\,\mathbf{A} \;+\; (\mathrm{grad}f)\cdot\mathbf{A},$

    (d) $\mathrm{curl}\,(f\mathbf{A}) \;=\; f\,\mathrm{curl}\,\mathbf{A} \;+\; (\mathrm{grad}\,f)\times\mathbf{A},$

    (e) $\mathrm{div}\,(\mathbf{A}\times\mathbf{B}) \;=\; \mathbf{B}\cdot\mathrm{curl}\,\mathbf{A} \;-\; \mathbf{A}\cdot\mathrm{curl}\,\mathbf{B},$

    (f) $\mathrm{curl}\,(\mathbf{A}\times\mathbf{B}) \;=\; (\mathrm{div}\,\mathbf{B})\mathbf{A} \;-\; (\mathrm{div}\,\mathbf{A})\mathbf{B} \;+\; (\mathbf{B}\cdot\nabla)\mathbf{A} \;-\; (\mathbf{A}\cdot\nabla)\mathbf{B},$

    (g) $\mathrm{div}\,\mathrm{curl}\,\mathbf{A} \;=\; 0,$

    (h) $\mathrm{curl}\;\mathrm{curl}\,\mathbf{A} \;=\; \mathrm{grad}\,\mathrm{div}\,\mathbf{A} - \nabla^2\mathbf{A}.$

37. What is the divergence of the vector function $\mathbf{A}(\mathbf{x}\,) = r\,\mathbf{x} + \nabla\,r$ where $\mathbf{x}$ is the position vector in 3 dimensions and $r = |\mathbf{x}\,|$? What is the corresponding result in n dimensions?

38. For which values of (x, y) are the following continuous:

    (a) $x/(x^2 + y^2 + 1),$

    (b) $x/(x^2 + y^2),$

    (c) $(x + y)/(x - y),$

    (d) $x^3/(y - x^2)$?

39. Which of the following sets are open:

    (a) $\{(x, y, z) : x > 0\},$

    (b) $\{(x, y, z) : y \geq 0\},$

    (c) $\{(x, y, z) : 1 > (x^2 + y^2)/z\},$

    (d) $\{(x, y, z) : 1 \geq (x^2 + y^2)/z\}$?

40. Prove that an open ball, as defined in lectures, is an open set.

41. Prove that the intersection of two open sets, as defined in lectures, is another open set. (Note that the empty set is an open set: since it contains no points, the statement that every point in it sits inside an open ball which is also in the set is *vacuously* true.) What about the intersection of a finite number of open sets? And what about the intersection of an *infinite* number of open sets?

42. Exam question June 2014 (Section A):

(a) Give the definition of the open ball $B_\delta(\mathbf{a})$ with centre $\mathbf{a} \in \mathbb{R}^n$ and radius $\delta > 0$, and define what it means for a subset S of $\mathbb{R}^n$ to be open.

(b) Which of the following subsets of $\mathbb{R}^2$ are open? In each case, justify your answer in terms of the definition you gave in part (a).

(i) $S_1 = \{(x, y) : x > 2\}$,

(ii) $S_2 = \{(x, y) : x > 2, y = 2\}$,

(iii) $S_3 = \{(x, y) : x > 2, y > 2\}$.

43. Exam question (last part) June 2014 (Section B): Determine the points of $\mathbb{R}^2$ at which the function $f(x, y) = |xy + x + y + 1|$ is
(a) continuously differentiable; (b) differentiable. (Hint: first factorise f.)

44. Determine the points of $\mathbb{R}^2$ at which the function $f(x, y) = |x^2 - y^2|$ is
(a) continuously differentiable; (b) differentiable.

45. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(\mathbf{0}) = 0$ whilst for $\mathbf{x} \neq \mathbf{0}$:

$$f(\mathbf{x}) = \frac{x^3}{x^2 + y^2}.$$

Calculate the partial derivatives of f with respect to x and y at $\mathbf{x} = \mathbf{0}$ *using their definitions as limits*. Defining $R(\mathbf{h})$ at the origin by $R(\mathbf{h}) = f(\mathbf{h}) - f(\mathbf{0}) - \mathbf{h} \cdot \nabla f$ as usual, show that $R(\mathbf{h})/|\mathbf{h}|$ does not tend to zero as $\mathbf{h}$ tends to $\mathbf{0}$, so that f is not differentiable at the origin.

On the line through the origin, $\mathbf{x} = \mathbf{b}t$, (with $\mathbf{b}$ a constant vector), f becomes a function of the single variable t, $f(\mathbf{b}t)$. Write $\mathbf{b} = \mathbf{e_1}b_1 + \mathbf{e_2}b_2$ and use this to write $f(\mathbf{b}t)$ explicitly as a function of t. Show that this function is differentiable at the origin, i.e. df/dt exists at $t = 0$ despite $f(\mathbf{x})$ not being differentiable at $\mathbf{0}$.

46. If $y = 1 + xy^5$ show that y may be written in the form $y = f(x)$ in a neighbourhood of $(0, 1)$ and find the gradient of the graph of f at the point $(0, 1)$.

47. Show that the equation $xy^3 - y^2 - 3x^2 + 1 = 0$ can be written in the form $y = f(x)$ in a neighbourhood of the point $(0, 1)$, and in the form $y = g(x)$ in a neighbourhood of the point $(0, -1)$. Is it true that $f(x)$ and $g(x)$ are equivalent as functions of x? What are the critical values of the curve $H(x, y) = xy^3 - y^2 - 3x^2 + 1$, and what are the regular values of this curve?

48. Determine whether or not the equation $x^2 + y + \sin(xy) = 0$ can be written in the form $y = f(x)$ or in the form $x = g(y)$ in some small open disc about the origin for some suitable continuously differentiable functions f, g.

49. Exam question May 2015 (Section B, lightly edited):
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the scalar function $f(x, y) = e^{xy} - x + y$.
   (a) Find the vector equations of the tangent and normal lines to the curve $f(x, y) = 0$ at the points $(1, 0)$ and $(0, -1)$.
   (b) Use the implicit function theorem for functions of two variables to determine whether or not the curve $f(x, y) = 2$ can be written in the form $y = g(x)$ for some differentiable function $g(x)$ in the neighbourhoods of the points (i) $(0, 1)$; (ii) $(-1, 0)$. Determine also whether the curve can be written as $x = h(y)$ for some differentiable function $h(y)$, in the neighbourhoods of the same two points.
   (c) Does the function $f(x, y)$ have any critical points? Justify your answer. (You can quote without proof that $|xe^{-x^2}| < 1$ for all $x \in \mathbb{R}$.)

50. Part of Exam Question May 2017 (Section B):
   (c) Consider the function
$$f(x, y) = (3x + y)e^{3xy}.$$
   Determine whether or not the curve $f(x, y) = c$ can be written in the form $y = g(x)$, and if not, state clearly the points (x_0, y_0) and corresponding values of c where problems occur. You may assume that f is differentiable on $\mathbb{R}^2$.
   (d) Using $f(x, y)$ as given in the previous part, determine whether or not the curve $(f(x, y) = c)$ can be written in the form $x = y(h)$, and if not, state clearly the points (x_0, y_0) where problems occur.
   (e) Using $f(x, y)$ as in the previous parts of this question, are there any points where the curve $f(x, y) = c$ can neither be written as $y = g(x)$, nor as $x = h(y)$?

51. Compute the differential, or Jacobian matrix, and the Jacobian of the function $\underline{V} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\underline{V}(x, y) = (x \cos y, x \sin y)$. State where $\underline{V}$ defines an orientation preserving local diffeomorphism, and where it defines an orientation reversing local diffeomorphism.

52. Repeat question 51 for $\underline{V}(x, y) = (e^x \cos y, e^x \sin y)$.

53. Calculate the differential, or Jacobian matrix, and the Jacobian of the following transformations:
   (a) $\underline{U}(u, v) = (x(u, v), y(u, v))$ where $x(u, v) = \frac{1}{2}(u + v)$ and $y(u, v) = \frac{1}{2}(u - v)$;
   (b) $\underline{V}(r, \theta) = (x(r, \theta), y(r, \theta))$ where $x(r, \theta) = r \cos\theta$ and $y(r, \theta) = r \sin\theta$;
   (c) $\underline{W}(r, \theta, \phi) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$.

54. Adapted from exam question 2009 (Section B) Q7:

   (a) Let $\underline{V} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. Give the definition of $\underline{V}$ being differentiable at a point $\underline{a}$.

   (b) Let $\underline{V}(x)$ and $\underline{W}(x)$ be two differentiable vector fields in $\mathbb{R}^2$. Give formulae for the two differentials $D\underline{V}_{\underline{x}}$ and $D\underline{W}_{\underline{x}}$.

   (c) Use the chain rule to show that the differential of the composite map $\underline{U}(\underline{x}) := \underline{V}(\underline{W})$ satisfies
$$D\underline{U}_{\underline{x}} = D\underline{V}_{\underline{W}} D\underline{W}_{\underline{x}} \,.$$

55. Adapted from exam question 2018 (Section B) Q8:

   (a) Given a vector field $\underline{u}(\underline{x}) = \underline{w}(\underline{v}(\underline{x}))$, use the chain rule to show that $D\underline{u}(\underline{x}) = D\underline{w}(\underline{v})D\underline{v}(\underline{x})$, and hence $J(\underline{u}) = J(\underline{w})J(\underline{v})$.

   (b) Let
$$\underline{v}(\underline{x}) = (v_1, v_2) = (\cos y, \sin x)$$
$$\underline{w}(\underline{x}) = (w_1, w_2) = (x^2 + y^3, x^2 y),$$
   and define $\underline{u}(\underline{x}) = \underline{w}(\underline{v}(\underline{x}))$. Use the result from part (a) to calculate $J(\underline{u})$. Verify your answer by direct substitution.

56. (a) Let $\underline{v} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. Give the definition of $\underline{V}$ being differentiable on an open set $U \subseteq \mathbb{R}^n$.

   (b) For $\underline{x} = x\underline{e}_1 + y\underline{e}_2$, let $\underline{v} : \mathbb{R}^2 \to \mathbb{R}^2$ be given by
$$\underline{v}(\underline{x}) = (x^2 + y^2, x + y).$$
   Using the definition of differentiability, show that $\underline{v}$ is differentiable on $\mathbb{R}^2$.

   (c) Draw a diagram to show where $\underline{v}$ defines an orientation preserving local diffeomorphism (on $U \subseteq \mathbb{R}^2$), and where $\underline{v}$ defines an orientation reversing local diffeomorphism (on $V \subseteq \mathbb{R}^2$).

57. Let A be the region bounded by the positive x- and y-axes and the line $3x + 4y = 10$. Compute $\iint_A (x^2 + y^2)\, dx\, dy$, taking the integrals in both orders and checking that your answers agree.

58. In the following integrals sketch the integration regions and then evaluate the integrals. Next interchange the order of integrations and re-evaluate.

   (a) $\int_0^1 \left(\int_x^1 xy\, dy\right) dx,$

   (b) $\int_0^{\pi/2} \left(\int_0^{\cos\theta} \cos\theta\, dr\right) d\theta,$

   (c) $\int_0^1 \left(\int_1^{2-y} (x + y)^2\, dx\right) dy.$

59. Exam question 2010 (Section A) Q4: Calculate the double integral

$$\iint_A (|x| + |y|)\, dx\, dy.$$

where A is the region defined by $|x| + |y| \leq 1$.

60. Exam question 2011 (Section A) Q4: Change the order of integration in the double integral

$$\int_0^2 \int_x^{2x} f(x, y)\, dy\, dx\,.$$

61. Exam question May 2017 (Section A): A solid cylinder C of radius 1 and height 1 is defined by

$$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1,\ 0 \leq z \leq 1\}\,.$$

Show that the paraboloid $P = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}$ cuts C into two pieces of equal volume.

62. Compute the iterated integral

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy\, dx\,.$$

Now reverse the order of integrations and re-evaluate. Why doesn't your answer contradict Fubini's theorem? (Hints: for the first integral, with respect to y, it might help to aim at something that can be integrated by parts using $\frac{d}{dy}\left(\frac{1}{x^2+y^2}\right) = -\frac{2y}{(x^2+y^2)^2}$; and in answering to the last part of the question, the fact that $\int_0^1 \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| dy \geq \int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\, dy$ could be useful.)

63. Let B be the region bounded by the five planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$, and $z = x + y$.

    (a) Find the volume of B.

    (b) Evaluate $\int_B x\, dV$.

    (c) Evaluate $\int_B y\, dV$.

64. A function $f(x, y)$ is defined by

$$f(x, y) = \begin{cases} 1 & \text{if } -1 < x - y < 0 \\ -1 & \text{if } 0 < x - y < 1 \\ 0 & \text{otherwise}\,. \end{cases}$$

Compute $\int_0^\infty \left(\int_{-\infty}^\infty f(x, y)\, dx\right) dy$ and also $\int_{-\infty}^\infty \left(\int_0^\infty f(x, y)\, dy\right) dx$ (for the second case it might help to draw a picture). Comment on your two answers – why does Fubini fail?

65. A function $f(x,y)$ is defined by

$$f(x,y) = \begin{cases} 2^{2(n+1)} & \text{if } 2^{-(n+1)} < x < 2^{-n}, \quad 2^{-(n+1)} < y < 2^{-n} \\ -2^{2n+3} & \text{if } 2^{-(n+1)} < x < 2^{-n}, \quad 2^{-(n+2)} < y < 2^{-(n+1)} \\ 0 & \text{otherwise}, \end{cases}$$

for $n \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$.

Compute $\int_0^1 \int_0^1 f(x,y)\, dx\, dy$, and $\int_0^1 \int_0^1 f(x,y)\, dy\, dx$. Does this contradict Fubini's theorem?

66. Write the line integral

$$\int_C x dx + y dy + (xz - y) dz$$

in the form $\int_C \mathbf{v} \cdot d\mathbf{x}$ for a suitable vector field $\mathbf{v}(\mathbf{x})$, and compute its value when C is the curve given by $\mathbf{x}(t) = t^2\,\mathbf{e_1} + 2t\,\mathbf{e_2} + 4t^3\,\mathbf{e_3}$ with $0 \leq t \leq 1$.

67. Evaluate $\int_\sigma \mathbf{F} \cdot d\mathbf{x}$, where $\mathbf{F} = y\,\mathbf{e_1} + 2x\,\mathbf{e_2} + y\,\mathbf{e_3}$ and the path σ is given by $\mathbf{x}(t) = t\,\mathbf{e_1} + t^2\,\mathbf{e_2} + t^3\,\mathbf{e_3}$, $0 \leq t \leq 1$.

68. Let $\underline{A}(\underline{x})$ be the vector field $\underline{A}(x,y,z) = x\,\underline{e}_1 + y\,\underline{e}_2 + z\,\underline{e}_3$.

    (a) Compute the line integral $\int_C \underline{A} \cdot d\underline{x}$ where C is the straight line from the origin to the point $(1,1,1)$.

    (b) Show (by finding f) that the vector field $\underline{A}$ from part (a) is equal to $\nabla f(\underline{x})$ for some scalar field f, and that your answer to part (a) is equal to $f(1,1,1) - f(0,0,0)$.

69. Show that the result from question 68 applies in general: if the vector field $\underline{v}(\underline{x})$ in $\mathbb{R}^n$ is the gradient of a scalar field $f(\underline{x})$, so that $\underline{v} = \nabla f$, and if C is a curve in $\mathbb{R}^n$ running from $\underline{x} = \underline{a}$ to $\underline{x} = \underline{b}$, then $\int_C \underline{v} \cdot d\underline{x} = f(\underline{b}) - f(\underline{a})$. (Hint: use the chain rule.)

70. Use the result from question 69 to evaluate $\int_C 2xyz dx + x^2 z dy + x^2 y dz$, where C is any regular curve connecting (1,1,1) to (1,2,4).

71. Use method 2 from lectures to compute the surface integral, $\int_S \mathbf{F} \cdot d\mathbf{A}$, of the vector field $\mathbf{F} = (3x^2, -2yx, 8)$ over the surface given by the plane $z = 2x - y$ with $0 \leq x \leq 2$, $0 \leq y \leq 2$.

72. Let $\mathbf{F}(x,y,z) = (z,x,y)$, and S be the part of the surface of the sphere $x^2 + y^2 + (z-1)^2 = r^2$ above the plane $z = 0$. Assume that $r > 1$ so that the boundary C of S, where S intersects the plane $z = 0$, is non-empty.

    (a) Compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{x}$.

    (b) By parameterising the surface S with spherical coordinates centred on the point $(0,0,1)$, compute the surface integral of the curl of $\mathbf{F}$, $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{A}$.

73. Let $\underline{A}(\underline{x})$ be the vector field $\underline{A}(x, y, z) = z\,\underline{e}_1 + x\,\underline{e}_2 + y\,\underline{e}_3$, C be the circle in the x, y-plane of radius r centred on the origin, and S the disk in the x, y-plane whose boundary is C.

    (a) Compute the line integral $\oint_C \underline{A} \cdot d\underline{x}$.

    (b) Compute the surface integral of the curl of $\underline{A}$, $\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{A}$.

74. Let $\underline{A}(\underline{x})$ be the vector field $\underline{A}(x, y, z) = z\,\underline{e}_1 + x\,\underline{e}_2 + y\,\underline{e}_3$, C be the a by a square $\underline{abcd}$ in the y, z-plane with vertices $\underline{a} = \underline{0}$, $\underline{b} = a\,\underline{e}_2$, $\underline{c} = a\,\underline{e}_2 + a\,\underline{e}_3$ and $\underline{d} = a\,\underline{e}_3$, and S be the region of the y, z-plane bounded by C.

    (a) Compute the line integral $\oint_C \underline{A} \cdot d\underline{x}$.

    (b) Compute the surface integral of the curl of $\underline{A}$, $\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{A}$.

75. Based on exam question May 2015 (Section A) Q5 (which didn't contain part (b)):

    (a) Calculate $\int_V (\underline{\nabla} \cdot \underline{U})\, dV$ where V is the solid cube with faces $x = \pm 1$, $y = \pm 1$ and $z = \pm 1$ and
    $$\underline{U}(x, y, z) = (\, x\, y^2, y\, x^2, z\,)\,.$$

    (b) Calculate $\int_S \underline{U} \cdot d\underline{A}$, where S is the surface of V.

76. For a simple closed curve C in the $(x, y)-$plane, show by Green's theorem that the area enclosed is $A = \frac{1}{2} \oint_C (xdy - ydx)$. (Note, 'simple' means that C does not cross itself, which means that it encloses a well-defined area A.)

77. Evaluate $\oint \mathbf{F} \cdot \mathbf{dx}$ around the circle $x^2 + y^2 + 2x = 0$, where $\mathbf{F} = y\mathbf{e_1} - x\mathbf{e_2}$, both directly and by using Green's theorem in the plane.

78. Evaluate $\oint_C 2xdy - 3ydx$ around the square with vertices at $(x, y) = (0, 2)$, $(2, 0)$, $(-2, 0)$ and $(0, -2)$.

79. Let $\underline{v}$ be the radial vector field $\underline{v}(\underline{x}) = \underline{x}$.

    (a) Compute $l_1 = \int_{C_1} \underline{v} \cdot d\underline{x}$, where C_1 is the straight-line contour from the origin to the point $(2, 0, 0)$.

    (b) Compute $l_2 = \int_{C_2} \underline{v} \cdot d\underline{x}$, where C_2 is the semi-circular contour from the origin to the point $(2, 0, 0)$ defined by $0 \leq x \leq 2$, $y = +\sqrt{1 - (x-1)^2}$, $z = 0$. [It may help to start by sketching C_2, and then to parameterize it as $\underline{x}(t) = (1 - \cos t, \sin t, 0)$ with $0 \leq t \leq \pi$.]

    (c) You should have found that $l_1 = l_2$. Explain this result using Stokes' theorem.

80. Integrate $\operatorname{curl}\mathbf{F}$, where $\mathbf{F} = 3y\mathbf{e_1} - xz\mathbf{e_2} + yz^2\mathbf{e_3}$, over the portion S of the surface $2z = x^2 + y^2$ below the plane $z = 2$, both directly and by using Stokes' theorem. Take the area elements of S to point outwards, so that their z components are negative.

81. The paraboloid of equation $z = x^2+y^2$ intersects the plane $z = y$ in a curve C. Calculate $\oint_C \mathbf{v} \cdot \mathbf{dx}$ for $\mathbf{v} = 2z\mathbf{e_1} + x\mathbf{e_2} + y\mathbf{e_3}$ using Stokes' theorem. Check your answer by evaluating the line integral directly.

82. For an open surface S with boundary C, show that $2\int_S \mathbf{u}\cdot d\mathbf{A} = \oint_C (\mathbf{u}\times\mathbf{x})\cdot d\mathbf{x}$, where $\mathbf{u}$ is a fixed vector.

83. Verify Stokes' theorem for the upper hemispherical surface S: $z = \sqrt{1 - x^2 - y^2}$, $z \geq 0$, with $\mathbf{F}$ equal to the radial vector field, i.e. $\mathbf{F}(x, y, z) = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$.

84. Let $\mathbf{F} = ye^z\,\mathbf{e_1} + xe^z\,\mathbf{e_2} + xye^z\,\mathbf{e_3}$. Show that the integral of $\mathbf{F}$ around a regular closed curve C that is the boundary of a surface S is zero.

85. By applying Stokes' theorem to the vector field $\mathbf{G} = |\mathbf{x}|^2\mathbf{a}$ (with $\mathbf{a}$ a constant vector), or otherwise, show that $\int_S \mathbf{x} \times d\mathbf{A} = -\frac{1}{2}\oint_C |\mathbf{x}|^2\, d\mathbf{x}$, where S is the area bounded by the closed curve C.

86. Exam question June 2002 (Section B): Evaluate the line integral $I = \int_C \mathbf{F} \cdot d\mathbf{x}$ where $\mathbf{F}(\mathbf{x}) = 2y\mathbf{e_1} + z\mathbf{e_2} + 3y\mathbf{e_3}$ and the path C is the intersection of the surface of equation $x^2 + y^2 + z^2 = 4z$ and the surface of equation $z = x + 2$, taken in a clockwise direction to an observer at the origin. A picture of the path is required, as well as full justifications of the theoretical results you might use.

87. Exam question 2012 (Section B) Q9(a)(ii): Use Stokes' theorem to calculate the line integral $\oint_C ydx+zdy+xdz$, where C is the intersection of the surfaces $x^2+y^2+z^2 = a^2$ and $x + y + z = 0$ and is orientated anticlockwise when viewed from above.
Suggestion: Instead of one of the two standard methods for surface integrals, just use $d\mathbf{A} = \frac{\nabla f}{|\nabla f|}dA$ and think about how the surfaces intersect.

88. Consider the vector field $\mathbf{F} = y\,\mathbf{e_1} + (z-x)\,\mathbf{e_2} + (x^3+y)\,\mathbf{e_3}$. Evaluate $\int_S (\underline{\nabla}\times\mathbf{F})\cdot\mathbf{dA}$ explicitly, where

    (i) S is the disk $x^2 + y^2 \leq 4$, $z = 1$,

    (ii) S is the surface of a paraboloid $x^2 + y^2 = 5 - z$ above the plane $z = 1$.

    Verify that each of these agrees with Stokes' theorem by considering the integral of $\mathbf{F}$ around each bounding contour.

89. Evaluate the line integral

$$\int_{P_1}^{P_2} yz\,dx + xz\,dy + (xy + z^2)\,dz$$

from P_1 with co-ordinates (1,0,0) to P_2 at (1,0,1) explicitly,

(a) along a straight line in the z–direction, and

(b) along a helical path parametrised by $\mathbf{x}(t) = \mathbf{e_1}\,\cos t + \mathbf{e_2}\,\sin t + \mathbf{e_3}\,t/2\pi$, where t varies along the path.

Compare these two results – does this suggest that there might be a general formula for the integral from $P_1 = (1,0,0)$ to any point P with coordinates (x,y,z)? Check your answer when $P = P_2$.

90. State a necessary and sufficient condition for a vector field $\mathbf{F}$ to be expressible in the form $\mathbf{F} = \underline{\nabla}\,\phi$ in some simply-connected region. The scalar field ϕ is called a *scalar potential*, though sometimes the opposite sign is used.

Determine whether the following vector fields are expressible in this form

(i) $(\mathbf{a}\cdot\mathbf{x})\,\mathbf{x}$, (ii) $(\mathbf{a}\cdot\mathbf{x})\,\mathbf{a}$, (iii) $(\mathbf{a}\cdot\mathbf{a})\,\mathbf{x}$, (iv) $\mathbf{a}\times\mathbf{x}$, (v) $\mathbf{a}\times(\mathbf{a}\times\mathbf{x})$,

and find the vector fields, $\mathbf{F}$, for which the corresponding potentials are

(a) $\frac{1}{2}(\mathbf{a}\cdot\mathbf{x})^2$, (b) $\frac{1}{2}a^2|\mathbf{x}|^2 - \frac{1}{2}(\mathbf{a}\cdot\mathbf{x})^2$.

Here $\mathbf{a}$ is a constant non-zero vector, and $a^2 = |\mathbf{a}|^2$.

91. Exam question June 2002 (Section B):

(a) State the conditions for the line integral $\int_c \mathbf{F}\cdot d\mathbf{x}$ from $\mathbf{x}_0$ to $\mathbf{x}_1$ to be independent of the path connecting these two points.

(b) Determine the value of the line integral $\int_c \mathbf{F}\cdot d\mathbf{x}$ where

$$\mathbf{F} = (e^{-y} - ze^{-x})\,\mathbf{e_1} + (e^{-z} - xe^{-y})\,\mathbf{e_2} + (e^{-x} - ye^{-z})\,\mathbf{e_3},$$

and C is the path

$$x = \frac{1}{\ln 2}\ln(1+p), \quad y = \sin\frac{\pi p}{2}, \quad z = \frac{1-e^p}{1-e},$$

with the parameter p in the range $0 \le p \le 1$, from $(0,0,0)$ to $(1,1,1)$.

92. If $\mathbf{F} = x\,\mathbf{e_1} + y\,\mathbf{e_2}$, calculate $\int_S \mathbf{F}\cdot\mathbf{dA}$, where S is the part of the surface $z = 9 - x^2 - y^2$ that is above the x,y plane, by applying the divergence theorem to the volume bounded by the surface and the piece it cuts out of the x, y plane.
Hint: what is $\mathbf{F}\cdot\mathbf{dA}$ *on* the x,y plane?

93. Evaluate each of the integrals below as **either** a volume integral **or** a surface integral, whichever is easier:

    (a) $\int_S \mathbf{x} \cdot \mathbf{dA}$ over the whole surface of a cylinder bounded by $x^2 + y^2 = R^2$, $z = 0$ and $z = L$. Note that $\mathbf{x}$ means $x\,\mathbf{e_1} + y\,\mathbf{e_2} + z\,\mathbf{e_3}$.

    (b) $\int_S \mathbf{F} \cdot \mathbf{dA}$, where $\mathbf{F} = x\cos^2 y\,\mathbf{e_1} + xz\,\mathbf{e_2} + z\sin^2 y\,\mathbf{e_3}$, over the surface of a sphere with centre at the origin and radius π.

    (c) $\int_V \underline{\nabla} \cdot \mathbf{F}\, dV$, where $\mathbf{F} = \sqrt{x^2+y^2}(x\,\mathbf{e_1} + y\,\mathbf{e_2})$, over the three-dimensional volume $x^2 + y^2 \leq R^2$, $0 \leq z \leq L$.

94. Suppose the vector field $\mathbf{F}$ is everywhere tangent to the closed surface S, which encloses the volume V. Prove that
$$\int_V \underline{\nabla} \cdot \mathbf{F}\, dV = 0\,.$$

95. Exam question June 2003 (Section A): Evaluate $\int_S \mathbf{B}(\mathbf{x}) \cdot \mathbf{dA}$, where
$$\mathbf{B}(\mathbf{x}) = (8x + \alpha\, y - z)\,\mathbf{e_1} + (x + 2y + \beta\, z)\,\mathbf{e_2} + (\gamma\, x + y - z)\,\mathbf{e_3}$$
and S is the surface of the sphere having centre at (α, β, γ) and radius γ, where $\alpha, \beta \in \mathbb{R}$, and γ is an arbitrary positive real number.

96. Let A be the interior of the circle of unit radius centred on the origin. Evaluate $\iint_A \exp(x^2 + y^2)\, dx\, dy$ by making a change of variables to polar co-ordinates.

97. Let A be the region $0 \leq y \leq x$ and $0 \leq x \leq 1$. Evaluate $\int_A (x + y)\, dx\, dy$ by making the change of variables $x = u + v$, $y = u - v$. Check your answer by evaluating the integral directly.

98. The expressions below are called Green's first and second identities. Derive them from the divergence theorem with $\mathbf{F} = f\underline{\nabla} g$ or $g\underline{\nabla} f$ as appropriate, where f and g are differentiable functions:
$$\int_V (f\underline{\nabla}^2 g + \underline{\nabla} f \cdot \underline{\nabla} g)\, dV = \int_S (f\underline{\nabla} g) \cdot \mathbf{dA}\ ,$$
$$\int_V (f\underline{\nabla}^2 g - g\underline{\nabla}^2 f)\, dV = \int_S (f\underline{\nabla} g - g\underline{\nabla} f) \cdot \mathbf{dA}\ .$$
where the volume V is that enclosed by the closed surface S.

99. A gas holder has the form of a vertical cylinder of radius R and height H with hemispherical top also of radius R. The density, ρ, (i.e. the mass per unit volume) of the gas inside varies with height z above the base according to the relation $\rho = C\exp(-z)$, where C is a constant. Calculate the total mass of gas in the holder, taking care to define any coordinate system used and the range of the corresponding variables.

Use this result to find the integral of the field

$$\mathbf{F} = B\,z\,e^{-y}\,\mathbf{e_1} + C\,y\,e^{-z}\,\mathbf{e_2}$$

over the curved surface of the gas holder.

Hint: it may help to note first what the integral over the base of the holder is.

100. Exam question June 2001 (Section B): In electrostatic theory, Gauss' law states that the flux of an electrostatic (vector) field $\mathbf{E}(\mathbf{x})$ over some closed surface S is equal to the enclosed charge q divided by a constant ϵ_0 :

$$\int_S \mathbf{E}(\mathbf{x})\cdot \mathbf{dA} = \frac{q}{\epsilon_0}\,.$$

If the electrostatic field is given by $\mathbf{E}(x,y) = \alpha x\mathbf{e_1} + \beta y\mathbf{e_2}$, use Gauss' law to find the total charge in the compact region bounded by the surface S consisting of S_1, the curved portion of the half-cylinder $z = (R^2 - y^2)^{1/2}$ of length H; S_2 and S_3 the two semi-circular plane end pieces; and S_4, the rectangular portion of the (x,y)-plane. (In equations, the relevant bounded region may be described by $z^2 + y^2 \le R^2, z \ge 0, -\frac{H}{2} \le x \le \frac{H}{2}$). Express your result in terms of α, β, R and H.

101. Exam question June 2002 (Section A): In electrostatic theory, Gauss' law states that the flux of an electrostatic field $\mathbf{E}(\mathbf{x})$ over a closed surface S is given by the ratio of the enclosed charge q and a constant ϵ_0. Calculate the electric charge enclosed in the ellipsoid of equation $x^2 + \frac{1}{2}y^2 + z^2 = 1$ in the presence of

(a) an electrostatic field $\mathbf{E}(\mathbf{x}) = yz\mathbf{e_1} + xz\mathbf{e_2} + xy\mathbf{e_3}$,

(b) an electrostatic field $\mathbf{E}(\mathbf{x}) = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$.