8 Green's, Stokes' and divergence theorems

8.1 The three big theorems

1. *Green's theorem in the plane*

Let $P(x,y)$ and $Q(x,y)$, $(x,y) \in \mathbb{R}^2$, be continuously differentiable scalar fields in 2 dimensions. Then

$$\boxed{\oint_C \big(P(x,y)\,dx + Q(x,y)\,dy\big) = \int_A \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dx\,dy} \tag{8.1}$$

where C is the boundary of A, traversed in a positive (anti-clockwise) direction. A good way to remember which way is positive is to imagine you are walking around the boundary with the area of integration to your left; then you are walking in a positive direction!

Green's theorem in the plane can also be written in vector form, embedding the x,y plane into $\mathbb{R}^3$ (with $z = 0$) and setting $\underline{F}(x,y,z) = (P(x,y), Q(x,y), R)$ with R arbitrary:

$$\oint_C \underline{F} \cdot d\underline{x} = \int_A (\underline{\nabla} \times \underline{F}) \cdot \underline{e}_3\,dA\,.$$

It's a good exercise to check that you agree with this statement!

2. *Stokes' theorem*

This generalises the vector form of Green's theorem to arbitrary surfaces in $\mathbb{R}^3$.

Take a continuously differentiable vector field $\underline{F}(x,y,z)$ in $\mathbb{R}^3$, and a surface S also in $\mathbb{R}^3$ with area elements $d\underline{A} = \widehat{\underline{n}}\,dA$ and boundary curve $C \equiv \partial S$. Then

$$\boxed{\oint_C \underline{F} \cdot d\underline{x} = \int_S (\underline{\nabla} \times \underline{F}) \cdot d\underline{A}} \tag{8.2}$$

As with Green's theorem, we need to make a comment about orientations. The surface S has two choices of normal vector (say $\widehat{\underline{n}}$ and $-\widehat{\underline{n}}$), and the curve $C = \partial S$ also has two possible choices of orientation. In either integral, changing from one orientation to the other (changing from $\widehat{\underline{n}}$ to $-\widehat{\underline{n}}$, or vice versa, in the case of the surface integral) changes the overall sign of the answer. So, given a surface with a choice of normal, how do we know which orientation we should take for the boundary (or equivalently, given a choice of boundary orientation, which normal should we take for the surface) in order to get the equality of Stokes' theorem?

The answer is given by the **right hand rule:**

Curl the fingers of your right hand, and extend your thumb. If you imagine placing your hand on the surface, near the boundary, with your thumb pointing in the direction of the surface normal, then your fingers curl in the direction of the orientation of the boundary.

Equivalently, if you were to stand on the boundary with your head pointing in the direction of the normal, and walk around the boundary such that the surface is on your left, then you are walking in the direction the boundary should be oriented.

3. *The divergence theorem:*

As the name suggests, this theorem involves the divergence operator. If $\underline{F}$ is a continuously differentiable vector field defined over the volume V with bounding surface S, then

$$\boxed{\int_S \underline{F} \cdot d\underline{A} = \int_V \underline{\nabla} \cdot \underline{F}\, dV} \tag{8.3}$$

As in Stokes' theorem, $d\underline{A} = \widehat{\underline{n}}\, dA$, while $\widehat{\underline{n}}$ is the outward unit normal.

Proofs? See later!

These theorems can be considered as higher-dimensional analogues of the fundamental theorem of calculus: in each case the integral of a differentiated object over some region is equated to the integral of the undifferentiated object over the boundary of that region, as illustrated in figure 41.

8.2 Examples

Example 47. (Green) Check Green's theorem when $P(x, y) = y^2 - 7y$, $Q(x, y) = 2xy + 2x$ and the area of integration is bounded by the unit circle $x^2 + y^2 = 1$. Calculating the RHS of equation (8.1) first:

$$\begin{aligned}\int_A (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dx\, dy &= \int_A (2y + 2) - (2y - 7)\, dx\, dy \\ &= 9 \int_A dx\, dy = 9\pi\,,\end{aligned}$$

since $\int_A dx\, dy$ is the area of the unit circle (recall from section 1). Next we parametrise the circle as before, so that $dx = -\sin t\, dt$, $dy = \cos t\, dt$, and calculate the LHS of equation (8.1):

$$\begin{aligned}\oint_C (P\frac{dx}{dt} + Q\frac{dy}{dt})\, dt &= \int_0^{2\pi} \big((\sin^2 t - 7\sin t)(-\sin t) + (2\cos t \sin t + 2\cos t)(\cos t)\big)\, dt \\ &= \int_0^{2\pi} (-\sin^3 t + 7\sin^2 t + 2\cos^2 t \sin t + 2\cos^2 t)\, dt \;=\; 9\pi\end{aligned}$$

This agrees with the previous calculation, but was more arduous. Usually (not always) the area integral is the easier part, so if faced with a tough closed line integral, see if Green's Theorem is applicable.

Example 48. (Stokes) Evaluate $I = \oint_C \underline{F} \cdot d\underline{x}$ where

$$\underline{F} = x^2 e^{5z} \underline{e}_1 + x\, \cos y\, \underline{e}_2 + 3y\, \underline{e}_3$$

and C is the circle defined by $x = 0$, $y = 2 + 2\cos\theta$, $z = 2 + 2\sin\theta$, $0 \le \theta \le 2\pi$.

We'll do this in two ways. First, the direct route. We have

$$\underline{x}(\theta) = (2 + 2\cos\theta)\, \underline{e}_2 + (2 + 2\sin\theta)\, \underline{e}_3$$

so

$$\frac{d\underline{x}}{d\theta} = -2\sin\theta\, \underline{e}_2 + 2\cos\theta\, \underline{e}_3$$

and

$$\underline{F}(\underline{x}(\theta)) \cdot \frac{d\underline{x}}{d\theta} = 3(2 + 2\cos\theta)2\cos\theta\,.$$

Hence

$$I = \int_0^{2\pi} 12(1 + \cos\theta)\cos\theta\, d\theta = 12\pi\,.$$

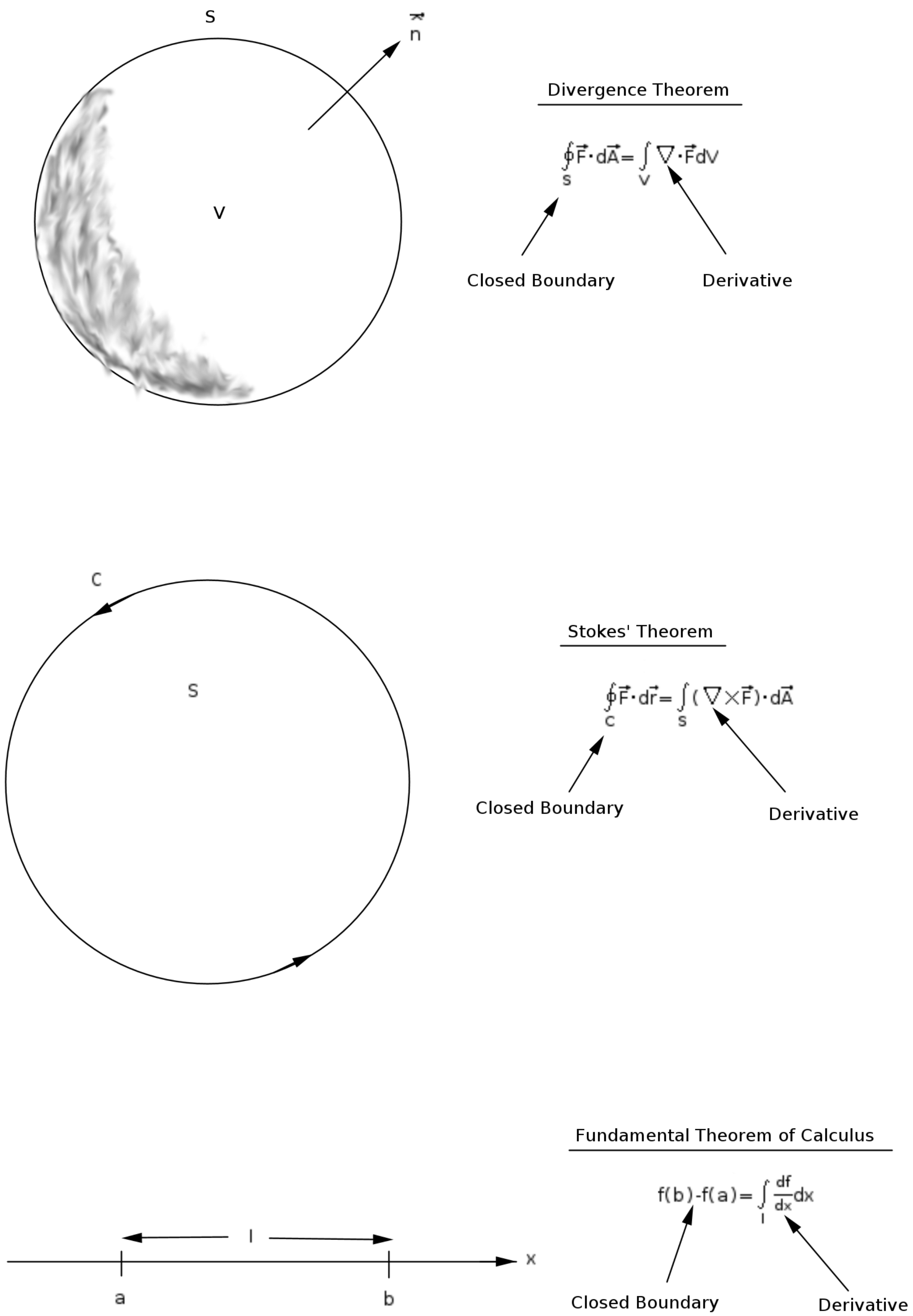


Figure 41: The integration regions for the divergence theorem, Stokes' theorem, and the fundamental theorem of calculus. Note the similarities between the formulae: the left hand side of each has one less integral than the right hand side, and this is 'compensated' by the presence of a derivative on each right hand side. Also notice that the domains of the right hand side integrations, V, S and I, are bounded by the domains of the left hand sides, S, C and $\{a, b\}$ respectively.

Alternatively, Stokes' theorem can be used. Calculating,

$$\underline{\nabla} \times \underline{F} = 3\,\underline{e}_1 + 5x^2 e^{5z}\underline{e}_2 + \cos y\,\underline{e}_3\,.$$

Take S to be the planar (flat) disk spanning C. Note that C is such that y initially decreases as θ increases from 0, and that z increases similarly. By the right hand rule, we should therefore take the normal on S in the positive x direction. So $\widehat{\underline{n}} = \underline{e}_1$ everywhere on S, $d\underline{A} = \underline{e}_1\,dA$, and $(\underline{\nabla} \times \underline{F})\cdot d\underline{A} = 3\,dA$. Hence

$$I = \iint_A 3\,dA = 3 \times (\text{area of } S) = 12\pi \quad \text{as before.}$$

Clearly, if we had taken the normal as $-\underline{e}_1$ instead, we would have got the opposite sign for the surface integral, and the two answers wouldn't have agreed.

Example 49. (Stokes again) Let S be the upper ($z \geq 0$) half of the sphere $x^2 + y^2 + z^2 = 1$; evaluate the surface integral

$$I = \int_S (x^3 e^y \underline{e}_1 - 3x^2 e^y \underline{e}_2) \cdot d\underline{A}\,.$$

To do this directly is tricky; instead we'll attempt to use Stokes' theorem. Observe that if $\underline{F} = x^3 e^y \underline{e}_3$, then $\underline{\nabla} \times \underline{F} = (x^3 e^y \underline{e}_1 - 3x^2 e^y \underline{e}_2)$, which is the thing we want to integrate over the hemisphere. Hence by Stokes' theorem

$$I = \oint_C \underline{F} \cdot d\underline{x}$$

where C is the boundary of S, which can be parametrised via $\underline{x}(\theta) = \cos\theta\,\underline{e}_1 + \sin\theta\,\underline{e}_2$, $0 \leq \theta \leq 2\pi$. Then $\frac{dx}{d\theta} = -\sin\theta\,\underline{e}_1 + \cos\theta\,\underline{e}_2$, which has zero scalar product with $\underline{F}$ for all θ. Hence $I = 0$.

Example 50. (Divergence) Evaluate $I = \int_S (3x\,\underline{e}_1 + 2y\,\underline{e}_2) \cdot d\underline{A}$ where S is the sphere $x^2 + y^2 + z^2 = 9$. Here $\underline{F} = (3x, 2y, 0)$ so $\underline{\nabla} \cdot \underline{F} = 3 + 2 = 5$ and so, by the divergence theorem,

$$I = \int_V 5\,dV = 5\frac{4}{3}\pi\,3^3 = 180\pi$$

where we used the fact that the volume of V, the interior of S, is $\frac{4}{3}\pi\,3^3$.

Example 51. (Divergence again) Let R be the region of $\mathbb{R}^3$ bounded by the surface $z = x^2 + y^2$ (a paraboloid) and the plane $z = 1$, and let S be its boundary. Evaluate $I = \int_S \underline{F} \cdot d\underline{A}$ where $\underline{F} = (y, x, z^2)$. Using the divergence theorem,

$$\begin{aligned} I &= \int_V \underline{\nabla} \cdot \underline{F}\,dV \\ &= \int_V 2z\,dV \\ &= \int_0^1 2z\,\pi\,r_z^2\,dz \quad \text{where } r_z = \sqrt{z}\text{, as in example 41} \\ &= \int_0^1 2\,\pi\,z^2\,dz = \frac{2\pi}{3}\,. \end{aligned}$$

8.3 Conservation laws and the continuity equation

This is an important application of the divergence theorem; it might also help you to get an intuition for the meaning of the surface integral $\int_S \underline{F} \cdot d\underline{A}$, which is usually interpreted as the flux of some quantity $\underline{F}$ through the surface of interest, S. Flux is the rate of flow per unit area, which is measured perpendicular to the face of the surface, so it is dA multiplied by the unit normal vector $\widehat{\underline{n}}$ with which the dot product with $\underline{F}$ is taken. Typical fluxes of interest are the magnetic flux ($\underline{F} = \underline{B}$), the electric flux ($\underline{F} = \underline{E}$), and transport fluxes such as mass ($\underline{F} = \rho\underline{v}$) (of fluid perhaps), heat or energy.

To give a concrete example, let us consider a swarm of bees and suppose that at time t and position $\underline{x} \in \mathbb{R}^3$ their density (number of bees per unit volume) is $\rho(\underline{x}, t)$, and that near that point they are flying

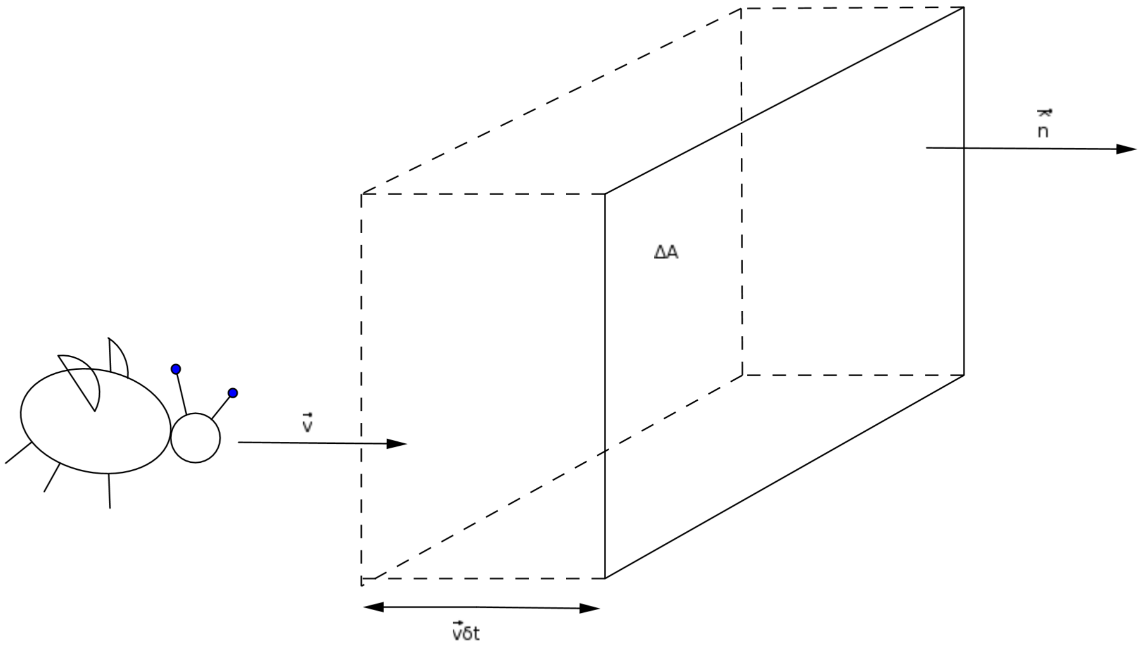


Figure 42: Cartoon of flying bees, travelling through the surface ΔA at a velocity $\underline{v}$.

with average velocity $\underline{v}(\underline{x}, t)$. Then the flux of bees is $\underline{j} = \rho \underline{v}$, meaning that the number of bees per unit time passing through a small area ΔA with unit normal $\widehat{\underline{n}}$ is $\underline{j} \cdot \widehat{\underline{n}}\, \Delta A$.

To see this, imagine you are one of the flying bees near to the area element ΔA; in a small time interval δt this element will appear to move with a distance $\underline{v}\delta t$ and so the element will have swept out a volume $\Delta A \widehat{\underline{n}} \cdot \underline{v}\delta t$, as shown in figure 42. The total number of bees in this volume is $\rho \Delta A \widehat{\underline{n}} \cdot \underline{v}\delta t$, and they will all cross the area element ΔA in this time δt; this implies that the rate at which the bees fly through ΔA is $\rho \Delta A \widehat{\underline{n}} \cdot \underline{v}$. Now imagine we have a larger surface S made up of many small area elements $\widehat{\underline{n}}\, dA$. The total flux of bees across the whole surface S can be found by subdividing S into its constiuent area elements and adding up the fluxes across each of these. Taking the limit as the sizes of these small area elements tend to zero, the total flux through S is

$$\lim_{\Delta A \to 0} \sum_{\text{all } \Delta A\text{'s}} \rho\, \Delta A\, \widehat{\underline{n}} \cdot \underline{v} = \int_S \underline{j} \cdot d\underline{A}\,.$$

We can now think about the implications of the "conservation of bees" to this situation. The total number of bees in some fixed volume V at time t is

$$B(t) = \int_V \rho\, dV = \int_V \rho(x, y, z, t)\, dx\, dy\, dz$$

and the rate of change of $B(t)$ is

$$\frac{dB}{dt} = \frac{d}{dt} \int_V \rho\, dV = \int_V \frac{\partial \rho}{\partial t}\, dV$$

(where the derivative can be taken inside the integral since the volume V is fixed). At the same time, the rate of flow of bees out of V is

$$R = \int_S \underline{j} \cdot d\underline{A}$$

where S is the boundary of V. Assuming no bees are being born (hatched?) or killed, it must be true that

$$\frac{dB}{dt} = -R$$

where the minus sign is because R is the rate of flow *out* of V, while it is flow *into* V which causes B to increase. Substituting in the formulae obtained above for B and R,

$$\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_S \underline{j} \cdot d\underline{A} = -\int_V \underline{\nabla} \cdot \underline{j}\, dV$$

using the divergence theorem for the second equality. It's important to remember that it is the *three-*dimensional divergence (involving x, y and z derivatives) that appears on the right-hand side, since it arose from applying the divergence theorem to a three-dimensional surface integral in x, y and z. Hence,

$$\int_V \left(\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{j}\right) dV = 0$$

must hold, and must do so for *all* volumes V. The only way for this to be true is for the thing being integrated to be zero everywhere, or in other words

$$\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot \underline{j}\,.$$

This is called the *continuity equation* and besides its relevance to beekeeping, it crops up in fluid mechanics, electromagnetism, and many other situations where some conservation law is in play.

Bees are perhaps not the best example, since they are not continuous and so the dot product in the integral will consider only the part of the bee crossing the surface perpendicularly. So what happens to the other part of the bee? (This is why we should really consider continuous quantities e.g. flow of water, magnetic fluxes etc.)

To quote Monty Python, "Eric the half a bee":

"Half a bee, philosophically, must *ipso facto* half not be. But half the bee has got to be, *vis-à-vis* its entity - d'you see? But can a bee be said to be or not to be an entire bee when half the bee is not a bee, due to some ancient injury?"

8.4 Path independence of line integrals

Having seen an application of the divergence theorem, we now discuss an important special case of Stokes' theorem.

In general, line integrals will depend on the path taken between their end points. This is not always the case, though, and vector fields for which the line integral is path independent are rather special. They are called *conservative* vector fields. We begin with a short example.

Bonus example 3. Calculate the line integral $\int_C \underline{F} \cdot d\underline{x}$, for $\underline{F} = (y\cos(x), \sin(x))$ between $(0,0)$ and $(1,1)$ over the two paths shown in figure 43 below.

Let us first calculate $\int_{C_1} \underline{F} \cdot d\underline{x}$. The parametrisation (recall the table of parametrisations given in section 7.3) of this path can be taken to be $\underline{x}(t) = (t,t)$ $(0 \le t \le 1)$, giving $d\underline{x} = (1,1)\, dt$ and $\underline{F}(\underline{x}(t)) = (t\cos(t), \sin(t))$. We can now perform the integration:

$$\int_{C_1} \underline{F} \cdot d\underline{x} = \int_0^1 (t\cos(t) + \sin(t))\, dt = \int_0^1 \frac{d(t\sin(t))}{dt}\, dt = \sin(1)\,.$$

Okay, now let's try along the second path. It can be split into two sub-paths C_{21} and C_{22}, parametrised as $\underline{x}(t) = (t,0)$ and $\underline{x}(t) = (1,t)$ respectively, with t running from 0 to 1 in each case. Using this setup we obtain for C_{21}: $d\underline{x} = (1,0)\, dt$, $\underline{F}(\underline{x}(t)) = (0, \sin(t))$, and so $\underline{F} \cdot d\underline{x} = 0$. For C_{22}: $d\underline{x} = (0,1)\, dt$, $\underline{F}(\underline{x}(t)) = (t\cos(1), \sin(1))$, and so $\underline{F} \cdot d\underline{x} = \sin(1)$. Putting these together to perform the line integral along C_2 gives

$$\int_{C_2} \underline{F} \cdot d\underline{x} = \int_{C_{21}} \underline{F} \cdot d\underline{x} + \int_{C_{22}} \underline{F} \cdot d\underline{x} = \int_{C_{22}} \underline{F} \cdot d\underline{x} = \int_0^1 \sin(1)\, dt = \sin(1)\,.$$

Both paths give us the same answer!

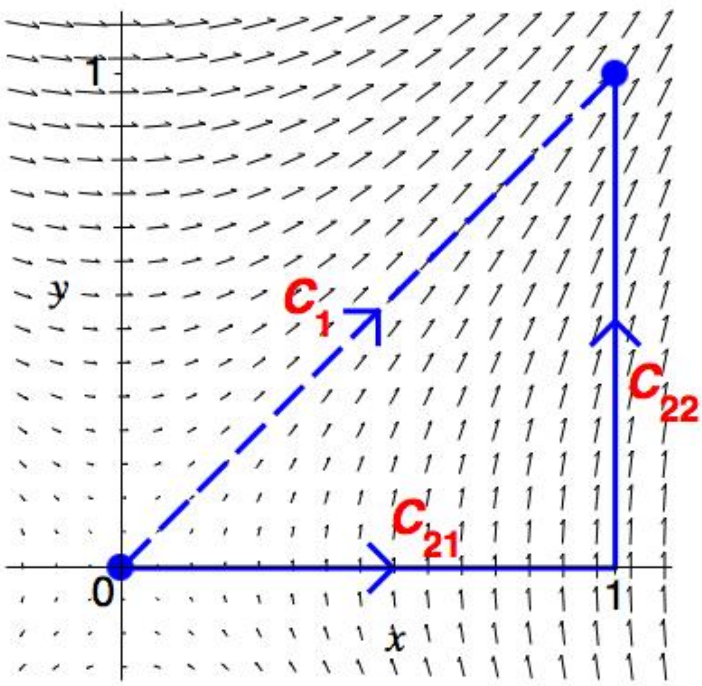


Figure 43: The field plot of $\underline{F}$ with two distinct paths shown, C_1 and $C_2 = C_{21} + C_{22}$.

Now, we only showed path independence for two very simple paths between two particular points, and to get the general result, in other words to prove that the field is conservative, we would need to check all possible paths joining all pairs of points. This would be both tediously (infinitely!) time consuming, and rather hard in the case of very wiggly paths. We will now show that there is a simple method to show path independence for all possible paths, which uses Stokes' theorem in a crucial way.

Suppose that $\underline{F}(\underline{x})$ is a continuously differentiable vector field defined on an open subset D of $\mathbb{R}^3$. Let C_1 and C_2 be any two paths from point $\underline{a}$ to point $\underline{b}$ in D, as shown in figure 44.

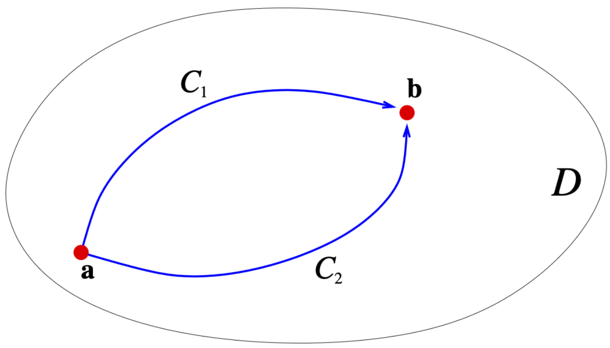


Figure 44: Two paths between points $\underline{a}$ and $\underline{b}$ in a domain D.

Let us investigate the conditions required for

$$\Delta I := \int_{C_1} \underline{F} \cdot d\underline{x} - \int_{C_2} \underline{F} \cdot d\underline{x}$$

to be zero, i.e. path independence. Suppose we have parametrised C_2 by a parameter t running from $t_{\underline{a}}$

to $t_{\underline{b}}$, and that $\overline{C}_2$ is the path along C_2 taken in the opposite direction. Then

$$\int_{\overline{C}_2} \underline{F} \cdot d\underline{x} = \int_{t_{\underline{b}}}^{t_{\underline{a}}} \underline{F}(\underline{x}(t)) \cdot \frac{d\underline{x}}{dt} dt = -\int_{t_{\underline{a}}}^{t_{\underline{b}}} \underline{F}(\underline{x}(t)) \cdot \frac{d\underline{x}}{dt}\, dt = -\int_{C_2} \underline{F} \cdot d\underline{x}\,.$$

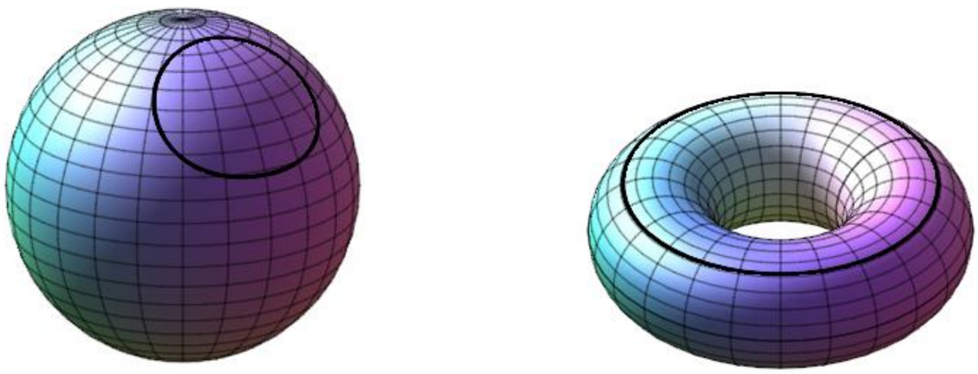
Combining these equations we have

$$\Delta I = \int_{C_1} \underline{F} \cdot d\underline{x} + \int_{\overline{C}_2} \underline{F} \cdot d\underline{x} = \oint_C \underline{F} \cdot d\underline{x}\,,$$

where C is the closed path consisting of C_1 followed by $\overline{C}_2$. Now if C is the boundary of a surface S in D then, by Stokes' theorem,

$$\Delta I = \int_S \underline{\nabla} \times \underline{F} \cdot d\underline{A}\,.$$

and so ΔI will be zero if $\underline{\nabla} \times \underline{F} = \underline{0}$ throughout D, giving us path independence. However for this to work we need not only that $\underline{\nabla} \times \underline{F} = \underline{0}$ in D, but also that every closed path in D is the boundary of some surface in D. Fortunately, there is a standard condition which ensures that this is the case. A region D is called *simply connected* if any closed curve in D can be continuously shrunk to a point in D, which in particular implies the condition we need, namely that every closed curve in D is the boundary of a surface in D. Two examples: the two-dimensional surface of a sphere is simply connected, while the two-dimensional surface of a torus is not simply connected – see figure 45.



(a) (b)

Figure 45: Examples of simply and non simply connected spaces: the surface of a sphere (a) and of a torus (b). The closed curve on (a) can be shrunk to a point and is the boundary of a part of the surface, while the closed curve on (b) can't be shrunk to a point, and is not the boundary of a part of the surface.

Three-dimensional spaces are harder to visualise, but for example $D = \mathbb{R}^3$ is simply-connected, while if we drill out a cylinder around the z axis to leave the set of points

$$D' = \left\{(x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 > 1\right\}$$

then D' is not simply connected. (Check that you can see why – can you identify a closed curve in D' which can't be contracted to a point?)

To summarise: if a vector field $\underline{F}$ satisfies $\underline{\nabla} \times \underline{F} = \underline{0}$ in some simply-connected region D, and if C_1 and C_2 are two paths in D joining points $\underline{a}$ and $\underline{b}$, then

$$\int_{C_1} \underline{F} \cdot d\underline{x} = \int_{C_2} \underline{F} \cdot d\underline{x}$$

and the line integral depends only on the end points – it is *path independent* and the vector field $\underline{F}$ is conservative.

The scalar potential

A further observation: last term we saw that one way for $\underline{\nabla} \times \underline{F}$ to be zero in some region D is for $\underline{F}$ to be the gradient of a scalar field there. A natural question is to ask whether it is possible to go the other way, and deduce from $\underline{\nabla} \times \underline{F} = \underline{0}$ that $\underline{F} = \underline{\nabla}\phi$ for some ϕ. If the region D is simply connected, then the answer is yes. Suppose that $\underline{\nabla} \times \underline{F} = \underline{0}$ in D, and define a scalar field ϕ by

$$\phi(\underline{x}) = \int_{\underline{x}_0}^{\underline{x}} \underline{F} \cdot d\underline{x}$$

where $\underline{x}_0$ is some (fixed) reference point in D. By path independence, this is well-defined. But what is its gradient?

We can prove this in a few different ways. One method (which is covered in the lectures), is just to calculate the partial derivatives directly. Consider the partial derivative of ϕ with respect to x,

$$\frac{\partial \phi}{\partial x} = \lim_{h \to 0} \frac{\phi(\underline{x} + h\underline{e}_1) - \phi(\underline{x})}{h} = \lim_{h \to 0} \frac{\int_{\underline{x}_0}^{\underline{x}+h\underline{e}_1} \underline{F} \cdot d\underline{x} - \int_{\underline{x}_0}^{\underline{x}} \underline{F} \cdot d\underline{x}}{h}.$$

Since we have that $\underline{\nabla} \times \underline{F} = 0$ over a simply-connected region D, then the line integral of $\underline{F}$ is path independent over D. For the first of the two integrals in the expression above, let us therefore choose to integrate along a path that goes first from $\underline{x}_0$ to $\underline{x}$, then from $\underline{x}$ to $\underline{x} + h\underline{e}_1$ in a straight line, as shown in figure 46 (where $\underline{h}$ is taken to be $h\underline{e}_1$).

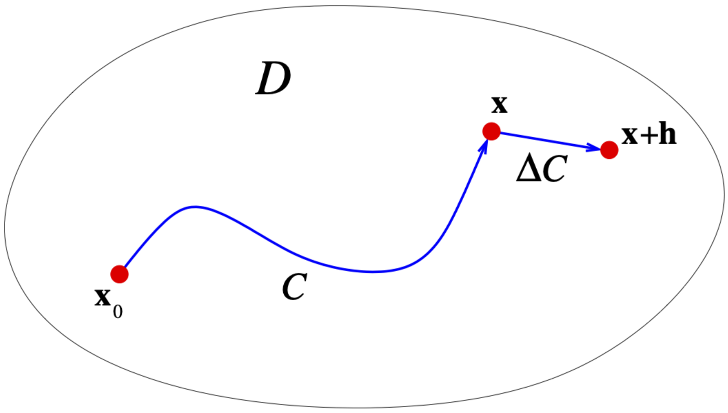


Figure 46: A path C from $\underline{x}_0$ to $\underline{x}$, and ΔC from $\underline{x}$ to $\underline{x} + \underline{h}$, where $\underline{h}$ is a small increment.

We can therefore compute the line integral of $\underline{F}$ from $\underline{x}_0$ to $\underline{x} + h\underline{e}_1$ as the sum of the line integrals along the two parts of this path, and hence

$$\frac{\partial \phi}{\partial x} = \lim_{h \to 0} \frac{\int_{\underline{x}_0}^{\underline{x}} \underline{F} \cdot d\underline{x} + \int_{\underline{x}}^{\underline{x}+h\underline{e}_1} \underline{F} \cdot d\underline{x} - \int_{\underline{x}_0}^{\underline{x}} \underline{F} \cdot d\underline{x}}{h} = \lim_{h \to 0} \frac{1}{h} \int_{\underline{x}}^{\underline{x}+h\underline{e}_1} \underline{F} \cdot d\underline{x}\,.$$

If we parameterise this path as $\underline{x}(t) = (x + th, y, z)$, $0 \le t \le 1$, then we have $\underline{F}(\underline{x}(t)) = \underline{F}(x + th, y, z)$ and $\frac{d\underline{x}}{dt} = (h, 0, 0)$, so

$$\int_{\underline{x}}^{\underline{x}+h\underline{e}_1} \underline{F} \cdot d\underline{x} = \int_0^1 \underline{F}(x + th, y, z) \cdot h\underline{e}_1 \, dt = h \int_0^1 F_1(x + th, y, z)\, dt,$$

where F_1 is the $\underline{e}_1$ component of $\underline{F}$. Substituting this back in to our expression for $\frac{\partial \phi}{\partial x}$ gives

$$\frac{\partial \phi}{\partial x} = \lim_{h \to 0} \int_0^1 F_1(x + th, y, z)\, dt.$$

A nice way to evaluate this expression is using the mean value theorem for integrals, which in general states that $\int_a^b G(x)\,dx = G(x^*)(b-a)$ for some $x^* \in [a,b]$. In this case, this says that there exists a $t^* \in [0,1]$ such that $\int_0^1 F_1(x+th,y,z) = F_1(z+t^*h,y,z)$, and then taking the limit gives $\frac{\partial\phi}{\partial x} = F_1(x,y,z)$ by continuity.

Alternatively, we can consider a Taylor expansion of $F_1(x+th,y,z)$, and then we see that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= \lim_{h\to 0}\int_0^1 \left(F_1(x,y,z) + th\frac{\partial}{\partial x}F_1(x,y,z) + \ldots\right) dt \\ &= \lim_{h\to 0}\left[tF_1(x,y,z) + \frac{1}{2}t^2h\frac{\partial}{\partial x}F_1(x,y,z) + \ldots\right]_0^1 \\ &= \lim_{h\to 0}\left(F_1(x,y,z) + \frac{1}{2}h\frac{\partial}{\partial x}F_1(x,y,z) + \ldots\right),\end{aligned}$$

where all the terms contained within $\ldots$ are of order at least h^2, and hence upon taking the limit we obtain

$$\frac{\partial\phi}{\partial x} = F_1(x,y,z).$$

The partial derivatives with respect to y and z can be calculated following exactly the same procedure, and so we find

$$\underline{\nabla}\phi(\underline{x}) = F_1(x,y,z)\underline{e}_1 + F_2(x,y,z)\underline{e}_2 + F_3(x,y,z)\underline{e}_3 = \underline{F}(\underline{x}),$$

as required.

Here's an alternative (though similar) approach, which doesn't require us to consider the different components separately, and has the bonus of showing that ϕ is fully differentiable in the sense seen last term. If we return to 46, then (similarly to before), for $\underline{x}+\underline{h} \in D$, we can write:

$$\phi(\underline{x}+\underline{h}) - \phi(\underline{x}) = \int_{\Delta C} \underline{F}\cdot d\underline{x}\,.$$

Choosing the small increment to the path to be a straight line, we can parametrise ΔC as $\underline{x}(t) = \underline{x}+\underline{h}\,t$, where $0 \le t \le 1$. Now

$$\int_{\Delta C} \underline{F}\cdot d\underline{x} = \int_0^1 \underline{F}(\underline{x}+\underline{h}\,t)\cdot\underline{h}\,dt = \underline{F}(\underline{x}+t_*\,\underline{h})\cdot\underline{h}\,, \qquad \text{for some } t^* \in [0,1]\,,$$

where again the last step comes from the integral form of the mean value theorem. So, we have

$$\phi(\underline{a}+\underline{h}) - \phi(\underline{a}) = \underline{F}(\underline{a})\cdot\underline{h} + \underbrace{\underline{F}(\underline{a}+t_*\,\underline{h})\cdot\underline{h} - \underline{F}(\underline{a})\cdot\underline{h}}_{R(\underline{h})}.$$

Denoting the unit vector in the direction $\underline{h}$ by $\widehat{\underline{n}}_h$,

$$\frac{R(\underline{h})}{|\underline{h}|} = \left(\underline{F}(\underline{a}+t_*\,\underline{h}) - \underline{F}(\underline{a})\right)\cdot\widehat{\underline{n}}_h$$

which tends to zero as $\underline{h}\to\underline{0}$ if $\underline{F}$ is continuous. Hence ϕ is differentiable at $\underline{a}$, and the piece linear in $\underline{h}$ allows us to identify

$$\underline{\nabla}\phi(\underline{a}) = \underline{F}(\underline{a})\,.$$

To conclude: if $\underline{\nabla}\times\underline{F} = \underline{0}$ in D and D is simply connected, then

$$\boxed{\exists\,\phi \;\text{ s.t. }\; \underline{F} = \underline{\nabla}\phi}$$

in D. The function ϕ (or sometimes its negative) is called the *scalar potential*.

One more question: starting with $\underline{F} = \underline{\nabla}\phi$, can we show path independence directly? The answer is again yes. This follows from the answer to exercise 68 on problem sheet 7, but for completeness the relevant calculation is repeated here. Let C be any curve running from $\underline{a}$ to $\underline{b}$, and $t \mapsto \underline{x}(t)$, $t = t_{\underline{a}} \dots t_{\underline{b}}$ be any parametrisation of it, so that $\underline{x}(t_{\underline{a}}) = \underline{a}$ and $\underline{x}(t_{\underline{b}}) = \underline{b}$. Then

$$\begin{aligned}
\int_C \underline{F} \cdot d\underline{x} &= \int_{t_{\underline{b}}}^{t_{\underline{a}}} \underline{\nabla}\phi(\underline{x}(t)) \cdot \frac{d\underline{x}}{dt}\, dt \\
&= \int_{t_{\underline{b}}}^{t_{\underline{a}}} \frac{\partial \phi}{\partial x_i} \frac{dx_i}{dt}\, dt && \text{(Einstein notation!)} \\
&= \int_{t_{\underline{b}}}^{t_{\underline{a}}} \frac{d\phi(\underline{x}(t))}{dt}\, dt && \text{(Chain rule)} \\
&= \phi(\underline{x}(t_{\underline{a}})) - \phi(\underline{x}(t_{\underline{b}})) && \text{(Fundamental Theorem of Calculus)} \\
&= \phi(\underline{a}) - \phi(\underline{b})\,.
\end{aligned}$$

Since we have not specified C beyond giving its end points $\underline{a}$ and $\underline{b}$, the integral is path independent.

Finally we can use our double ended arrows! In a simply connected region D,

$$\boxed{\underline{\nabla} \times \underline{F} = \underline{0} \;\Leftrightarrow\; \text{path independence of } I \;\Leftrightarrow\; \exists\,\phi \text{ s.t. } \underline{F} = \underline{\nabla}\phi.}$$

Example 52. Compute $I = \int_C \underline{F} \cdot d\underline{x}$ for $\underline{F} = (y\,\cos(xy), x\,\cos(xy) - z\,\sin(yz), -y\,\sin(yz))$, where C is specified by

$$t \mapsto \underline{x}(t) = \left(\frac{\sin(t)}{\sin(1)}, \frac{\log(1+t)}{\log(2)}, \frac{1-e^t}{1-e}\right)\,, \quad 0 \le t \le 1\,.$$

Answer: first note that $\underline{x}(0) = (0,0,0)$ and $\underline{x}(1) = (1,1,1)$. Then compute $\underline{\nabla} \times \underline{F} = \cdots = \underline{0}$, on all of $\mathbb{R}^3$ (which is simply connected). Hence $\underline{F} = \underline{\nabla}\phi$ for some ϕ. To find ϕ we need to solve

$$(1)\ \phi_x = F_1\,; \qquad (2)\ \phi_y = F_2\,; \qquad (3)\ \phi_z = F_3\,.$$

From (1), $\phi = \sin(xy) + f(y,z)$ for some function f of y and z; then from (2), $f(y,z) = \cos(yz) + g(z)$. Finally from (3), $g(z) = \text{constant} = A$, say. Hence

$$\phi(x,y,z) = \sin(xy) + \cos(yz) + A$$

and

$$I = \phi(1,1,1) - \phi(0,0,0) = (\sin(1) + \cos(1) + A) - (1 + A) = \sin(1) + \cos(1) - 1\,.$$

Some vocabulary: if $\underline{\nabla} \times \underline{F} = \mathbf{0}$, $\underline{F}$ is said to be *closed*; and if $\underline{F} = \underline{\nabla}\phi$, $\underline{F}$ is said to be *exact*. Even though it wasn't phrased in this way, we saw in subsection 3.3 that exact $\Rightarrow$ closed, and we have just shown that closed $\Rightarrow$ exact whenever the space D on which $\underline{F}$ is defined is simply connected. However, if D is not simply connected, this may fail, and the extent of this failure gives some information about the 'shape' (or topology) of D. This features in a scene in the film 'A beautiful mind'. . .

Example 53. A quick example to see what can go wrong on a non-simply connected space.

Let $\underline{F}(\underline{x}) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$ be a vector field defined on $D = \mathbb{R}^3 - \{\underline{0}\}$. D is a non-simply connected region, as can be seen by considering any circle around the origin. Then $\underline{F}$ is closed, since we have

$$\partial_x F_2 = \partial_y F_1 = \frac{y^2 - x^2}{(x^2+y^2)^2}\,,$$

and hence $\underline{\nabla} \times \underline{F} = \underline{0}$.

Now consider $\int_C \underline{F} \cdot d\underline{x}$, where C is the circle of radius 1 in the x, y-plane centred on the origin, which we can parameterise as $\underline{x}(t) = (\cos(t), \sin(t), 0)$ for $0 \leq t \leq 2\pi$. Then using our standard method for computing line integrals, we have $\frac{d\underline{x}(t)}{dt} = (-\sin(t), \cos(t), 0)$ and $\underline{F}(\underline{x}(t)) = (-\sin(t), \cos(t), 0)$, and hence

$$\int_C \underline{F} \cdot d\underline{x} = \int_0^{2\pi} \underline{F}(\underline{x}(t)) \cdot \frac{d\underline{x}(t)}{dt} dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

If we now consider the left and right semicircles C_L and C_R as shown in Figure 47, then we must have $\int_{C_L} \underline{F} \cdot d\underline{x} \neq \int_{C_R} \underline{F} \cdot d\underline{x}$, as otherwise $\int_C \underline{F} \cdot d\underline{x} = \int_{C_R} \underline{F} \cdot d\underline{x} - \int_{C_L} \underline{F} \cdot d\underline{x}$, in contradiction to the direct calculation we did above.

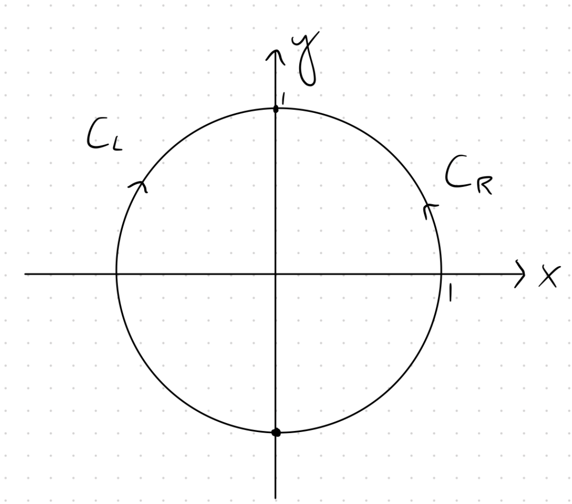


Figure 47: The curve C, and the left and right semicircles C_L and C_R.

Line integrals of $\underline{F}$ on D are therefore not necessarily path independent, and we cannot find a scalar field ϕ such that $\underline{F} = \underline{\nabla}\phi$ on D. $\underline{F}$ is therefore not exact.

The existence of a vector field which is closed, but not exact, is due to the topology of D.

Aside: Green ⇒ Cauchy

One final remark, which might be relevant if you are taking the complex variable course (otherwise, don't worry about it). In two dimensions Stokes becomes Green, stating that if $P(x, y)$ and $Q(x, y)$ are continuously differentiable on $A \subset \mathbb{R}^2$ then

$$\oint_{C=\partial A} P(x, y)\, dx + Q(x, y)\, dy = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA\,.$$

A special ('zero-curl') case is the fact that that if $P(x, y)$ and $Q(x, y)$ are continuously differentiable on $A \subset \mathbb{R}^2$ and satisfy $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ there, then

$$\oint_{C=\partial A} P(x, y)\, dx + Q(x, y)\, dy = 0\,.$$

This might remind you of Cauchy's integral theorem, which states that if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on some region $A \subset \mathbb{C}$, then

$$\oint_{C=\partial A} f(z)\, dz = 0\,.$$

This isn't a coincidence. In fact Cauchy is a consequence of Green, at least if f is assumed to have continuous partial derivatives[1]. To prove this, write

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)\,.$$

Given that f is holomorphic, the following (Cauchy-Riemann) equations hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{(CR1)}\,; \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{(CR2)}\,.$$

Since $dz = dx + i\,dy$, we have

$$\oint_C f(z)\,dz = \oint_C (u+iv)\,(dx + i\,dy) = \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)\,.$$

Using Green with $P = u$ and $Q = -v$ the first integral on the right-hand side can be rewritten as

$$\oint_C (u\,dx - v\,dy) = \int_A \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA$$

which vanishes since the integrand on the RHS is zero by (CR2). Likewise the second integral is

$$\oint_C (v\,dx + u\,dy) = \int_A \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA$$

and this is also zero, by (CR1). Hence $\oint_C f(z)\,dz = 0$, as stated by Cauchy.

8.5 Changing integration variables

Simply put, changing variables can make life easier. How and why a change of variables makes calculations easier can be seen by revisiting example 40, but this time, somewhat perversely, using Cartesian coordinates:

Bonus example 4. Integrate $f(x,y) = x^2 + y^2$ over the unit disc, using Cartesian coordinates.

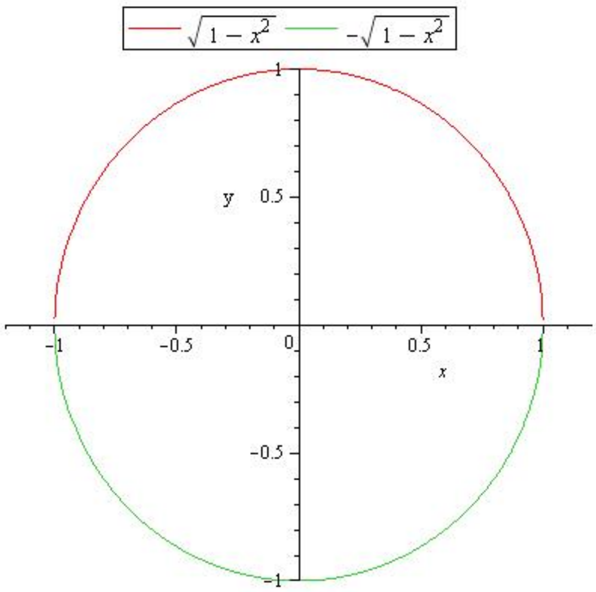


Figure 48: The region of integration is the area between two curves, which together make a circle.

[1] For a more powerful result special to complex analysis, look up *Goursat's theorem*.

The equation of a unit circle is given implicitly by $x^2 + y^2 = 1$, but in order to perform our iterated integral (with respect to y first) we must consider the upper and lower halves of the circle as individual curves given by $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$ respectively, with $-1 \leq x \leq 1$.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(x^2+y^2)\,dy\,dx = \int_{-1}^{1}\left[x^2y+\tfrac{1}{3}y^3\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dx = \int_{-1}^{1}\tfrac{2}{3}(1+2x^2)\sqrt{1-x^2}\,dx$$

Using the substitution $x = \sin\theta$, $\sqrt{1-x^2}dx = \cos^2\theta\,d\theta$:

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tfrac{2}{3}(\cos^2\theta + 2\sin^2\theta\cos^2\theta)\,d\theta$$

Use trig double angle identities to simplify and get:

$$= \tfrac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\tfrac{3}{4}+\tfrac{1}{2}\cos 2\theta - \tfrac{1}{4}\cos 4\theta)\,d\theta$$

$$= \left[\tfrac{1}{2}\theta + \tfrac{1}{6}\sin 2\theta - \tfrac{1}{24}\sin 4\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \tfrac{1}{2}\pi\,.$$

That was quite hard work, so let's recap on how changing variables can simplify life. We'll change to polar coordinates: $(x,y) \to (r,\theta)$ where $x = r\cos\theta$, $y = r\sin\theta$. Our limits must also change to $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, and also $dA = rdrd\theta$; we will see later why this is so. This form of dA is very important and the extra r, which may look out of place, is crucial. Therefore we can write:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(x^2+y^2)\,dy\,dx = \int_{0}^{2\pi}\int_{0}^{1} r^3 drd\theta$$

$$= \int_{0}^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^1 d\theta = \left[\tfrac{1}{4}\theta\right]_0^{2\pi} = \tfrac{1}{2}\pi\,.$$

The change of variables made our computation much simpler, but how do we change from one set of variables to another in general? A change of variables is a change of coordinate systems, and usually (though not always) the good choice is pretty obvious – e.g. if you are dealing with a circular area of integration, polar coordinates will probably be best. One further word of warning: if we start with two variables in the integration (e.g. x, y) then we need to keep two variables free after the coordinate change (e.g. r, θ). So NEVER, when changing variables to make an integration easier, substitute actual values for the variables (so NEVER do this: $r = 3$)!!! The values are for the integration limits.

This small section is to give some intuition for what will follow:

We briefly look at the change from Cartesian coordinates to polar coordinates from a geometric point of view before giving the general method.

If we start at the point $B = (r,\theta)$ on the diagram shown in figure 49, small increases in r and θ will give us the region traced out by the points $B = (r,\theta)$, $C = (r+dr,\theta)$, $D = (r+dr,\theta+d\theta)$ and $E = (r,\theta+d\theta)$. When $d\theta$ and dr are very small we can approximate this by a rectangle, and its area dA is approximately equal to the product of the lengths $|BC|$ and $|BE|$. It is clear that $|BC| = dr$, while $|BE|$ is approximately equal to the length of the corresponding circular arc, which can be obtained by considering it as a proportion of the full circumference, $C = 2\pi r$. The proportion is simply given by the ratio of the angle of the arc to that of the whole circle, so $|BE| = 2\pi r\frac{d\theta}{2\pi} = rd\theta$ and the infinitesimal area is given by

$$dA = |BC||BE| = rdrd\theta\,. \tag{8.4}$$

You can also see that this holds by calculating the area of the sector $OCDO$ and subtracting the area of $OBEO$, and then neglecting higher order terms as they become negligible when dr is small: $dA = \frac{d\theta}{2\pi}(\pi(r+dr)^2 - \pi r^2) \sim rdrd\theta$.

Now back to the general case:

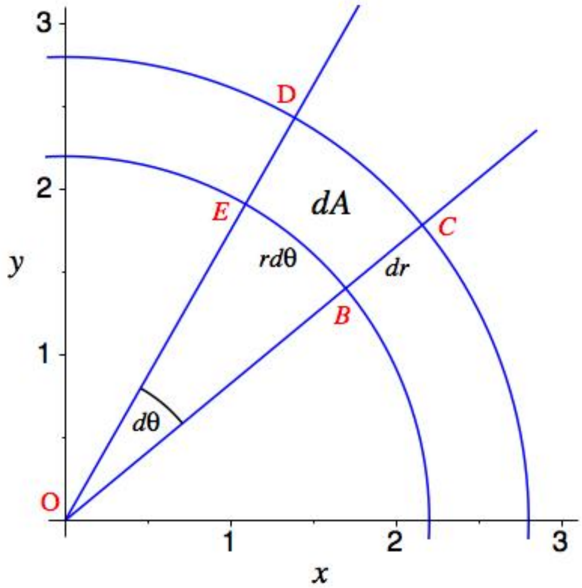


Figure 49: A small area dA in terms of polar coordinates, i.e. small changes in r and θ instead of in x and y.

In the above example the change of variables from (x, y) to (r, θ) was (apart from the special point at the origin, which is exceptional enough to be ignored) a diffeomorphism from one set of coordinates to another. Consider a general diffeomorphism $\underline{v} : U \to V = \underline{v}(U)$, with inverse $\underline{w}$, and suppose that V, with coordinates (x, y), is an area over which we wish to integrate some scalar field $f\colon V \to \mathbb{R}$. The big picture is shown in figure 50(a), while figure 50(b) zooms in on a single infinitesimal area element in U and its image in V.

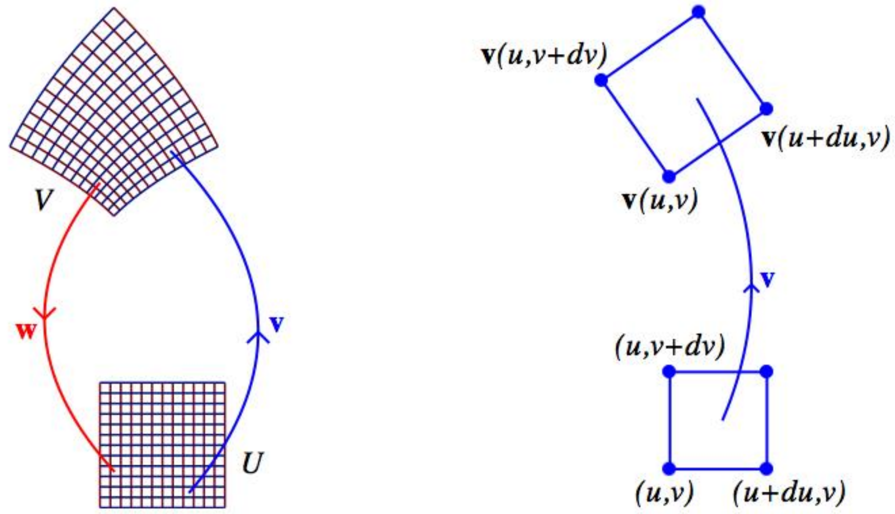
The integral of f over V can be found as the limit of a Riemann sum using the initial integration variables $(x, y) \in V$, as in section 7.1:

$$
\begin{aligned}
I &= \int_V f(x, y)\, dxdy \\
&= \lim_{N\to\infty} \sum_{k=0}^{N} f(\underline{x}_k^*)\, \Delta A_k \\
&= \lim_{\Delta x_i,\, \Delta y_j \to 0} \sum_{i,j} f(\underline{x}_{ij}^*)\, \Delta x_i \Delta y_j \, .
\end{aligned}
$$

where the choice of partition of V is taken to be parallel to the x and y axes in V, i.e. $\Delta A_k = \Delta x_i \Delta y_j$.

However, the value of the integral does not depend on the partition taken to compute it, so let us use another one, the image under $\underline{u}$ of a partition of U parallel to the u and v axes in U. From the lower panel of figure 50(b), a typical area element from such a partition can be specified by its lower left, lower right, and upper left corners, which have position vectors (u, v), $(u + du, v)$, and $(u, v + dv)$ respectively.

These three corners are mapped to the points $\underline{v}(u, v)$, $\underline{v}(u + du, v)$, and $\underline{v}(u, v + dv)$ in V by the diffeomorphism $\underline{v}$, which define the in general skewed area element dA_k in V. As this area element is small we can treat it as an approximate parallelogram and compute its area via the cross product of vectors along two neighbouring sides (temporarily embedding the x,y plane into $\mathbb{R}^3$ in order to define the cross



(a) (b)

Figure 50: On the left is a diffeomorphism, $\underline{v}$, and its inverse, $\underline{w}$, mapping to and from an area of integration V. On the right are infinitesimal areas in the coordinate systems $(u, v) \in U$ (lower) and $(x, y) \in V$ (upper). Note that the transformation illustrated is rather special in that small squares in U are mapped to small squares in V – it is a *conformal* mapping. In general the image of a small square in U would be a parallelogram in V.

product). Thus

$$\begin{aligned}
\Delta A_k &\approx \|(\underline{v}(u+du, v) - \underline{v}(u, v)) \times (\underline{v}(u, v+du) - \underline{v}(u, v))\| && \text{(area of parallelogram)}\\
&\approx \|du\frac{\partial \underline{v}}{\partial u} \times dv\frac{\partial \underline{v}}{\partial v}\| && \text{(from the definition of the partial derivatives)}\\
&= \left|\,\begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial v_1}{\partial u} & \frac{\partial v_2}{\partial u} & 0 \\ \frac{\partial v_1}{\partial v} & \frac{\partial v_2}{\partial v} & 0 \end{vmatrix}\,\right| du\, dv && \text{(definition of cross product)}\\
&= \left|\,\begin{vmatrix} \frac{\partial v_1}{\partial u} & \frac{\partial v_2}{\partial u} \\ \frac{\partial v_1}{\partial v} & \frac{\partial v_2}{\partial v} \end{vmatrix}\,\right| du\, dv && \text{(expanding out the determinant)}\\
&= \left|\,\begin{vmatrix} \frac{\partial v_1}{\partial u} & \frac{\partial v_1}{\partial v} \\ \frac{\partial v_2}{\partial u} & \frac{\partial v_2}{\partial v} \end{vmatrix}\,\right| du\, dv && \text{(rearranging using properties of } 2\times 2 \text{ determinants)}\\
&= |J(\underline{v})|\, du\, dv\,,
\end{aligned}$$

where $J(\underline{v})$ is the Jacobian matrix of the transformation $\underline{v}$. Now that we have calculated the area element under the diffeomorphism we can recalculate the integral using a Riemann sum with our new partition, with $\Delta A_k = |J(\underline{v})|\, du\, dv$:

$$\begin{aligned}
I &= \lim_{N\to\infty} \sum_{k=0}^{N} f(\underline{v}(u_k^*, v_k^*))\, \Delta A_k \\
&= \lim_{du_i dv_j \to 0} \sum_{i,j} f(\underline{v}(u_k^*, v_k^*))\, |J(\underline{v})|\, du_i dv_j \\
&= \int_U f(\underline{v}(u, v))\, |J(\underline{v})|\, du\, dv\,.
\end{aligned}$$

Note that $U = \underline{w}(V)$ is the area of integration in the new coordinate system, and that the limits of integration must be altered accordingly.

Let's use this result to check the form of dA used in example 40. In this case our 'V' coordinates were (x, y) and our 'U' ones were (r, θ) with $x = r\cos\theta$, $y = r\sin\theta$, and so

$$\begin{aligned} dA_{new} &= |J(\underline{v})|\, dr\, d\theta \\ &= |\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}|\, dr\, d\theta \\ &= |\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}|\, dr\, d\theta \\ &= r\, dr\, d\theta\,. \end{aligned}$$

This is the formula quoted in example 40 and agrees with equation (8.4). The Jacobian allows us to convert complicated integrals into easier ones by means of a variable change.

In summary:

$$\boxed{dA = dxdy = |J|\, du\, dv}$$

Changing variables for volume (triple) integrals is very similar to double integrals. Again we assume that there is a diffeomorphism $\underline{U}$ which maps the volume between the two coordinate systems (W_1, W_2, W_3) and (U_1, U_2, U_3), and an inverse $\underline{W}$ which maps in the other direction, from (U_1, U_2, U_3) to (W_1, W_2, W_3).

Then, much as before,

$$\int_V f(\underline{W})\, dW_1\, dW_2\, dW_3 = \int_{\underline{U}(V)} f(\underline{W}(\underline{U}))\, |J(\underline{W})|\, dU_1\, dU_2\, dU_3\,,$$

where $|J(\underline{W})|$ is the magnitude of Jacobian, and J, the Jacobian matrix or differential, is now the determinant of a 3-by-3 matrix of partial derivatives:

$$|J(\underline{W})| = |\begin{vmatrix} \frac{\partial W_1}{\partial U_1} & \frac{\partial W_1}{\partial U_2} & \frac{\partial W_1}{\partial U_3} \\ \frac{\partial W_2}{\partial U_1} & \frac{\partial W_2}{\partial U_2} & \frac{\partial W_2}{\partial U_3} \\ \frac{\partial W_3}{\partial U_1} & \frac{\partial W_3}{\partial U_2} & \frac{\partial W_3}{\partial U_3} \end{vmatrix}|\,.$$

For example, for the usual Cartesian $\longrightarrow$ Spherical Polars coordinate change, take $\underline{W}(r, \theta, \phi) = (x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ and $\underline{U} = (r, \theta, \phi)$. As can be calculated (exercise!), the magnitude of the Jacobian of this mapping is $|J| = r^2\sin\theta$.

Just to re-iterate, this is a change of variables in all three dimensions (and NOT a parametrisation of a surface), so we need all three coordinates. We only fix r to specific values in, for example, the limits of an integration.

Example 54. Let us use the coordinate change to re-evaluate the volume of a sphere of radius a, found before in bonus example 1:

$$\begin{aligned} \int_V dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= \int_0^{2\pi}\int_0^{\pi} \frac{a^3}{3}\sin\theta\, d\theta\, d\phi \\ &= \int_0^{2\pi} \frac{2a^3}{3}\, d\phi = \frac{4\pi a^3}{3}\,. \end{aligned}$$

This agrees with our previous calculation but the steps were fewer and easier!

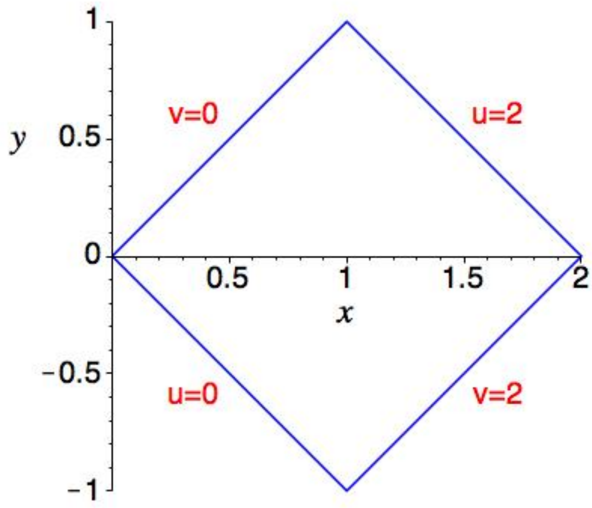


Figure 51: The area of integration, a square with sides of length $\sqrt{2}$, shown with old coordinates (x, y) and new ones (u, v).

One last example:

Example 55. Integrate $f(x, y) = x^2 + y^2$ over the square shown in figure 51. Doing this in Cartesian coordinates (x, y) is possible, but would involve splitting the integral into two and having limits which are functions, for example

$$I = \int_0^1 \int_{y=-x}^{y=x} (x^2 + y^2)\, dy\, dx + \int_1^2 \int_{y=x-2}^{y=-x+2} (x^2 + y^2)\, dy\, dx.$$

If we instead use coordinates (u, v), which are given by taking the axes starting at the origin, along the sides of the square labelled $u = 0$, $v = 0$, we get a transformation $x + y = u$, $x - y = v$. This gives us $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$, which we can use to find the absolute value of the Jacobian:

$$|J| = | \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} | = | - \frac{1}{2} | = \frac{1}{2}\,.$$

So now we have the easier integral

$$I = \int_0^2 \int_0^2 \frac{1}{2}(u^2 + v^2)\frac{1}{2}\, du\, dv = \int_0^2 \int_0^2 \frac{1}{2} u^2\, du\, dv = \frac{8}{3}\,.$$

Note this illustrates the need to take the absolute value of the Jacobian when changing variables – clearly the integral must be positive since it is the integral of an everywhere-positive function, while our answer would have been negative if we had used J instead of $|J|$ in the above.