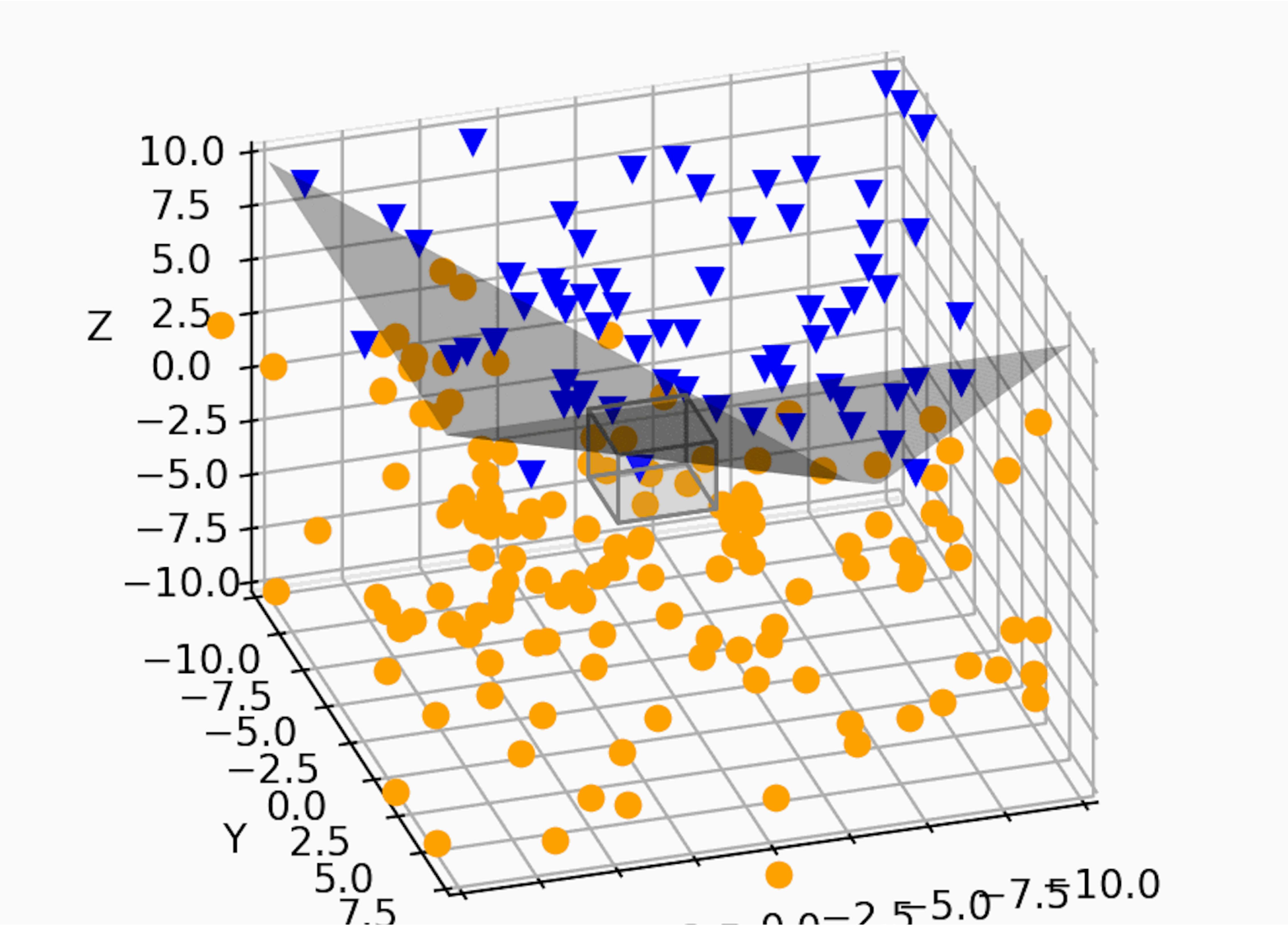
Tropical Support Vector Machines

MAP513 — Tuesday, December 19th



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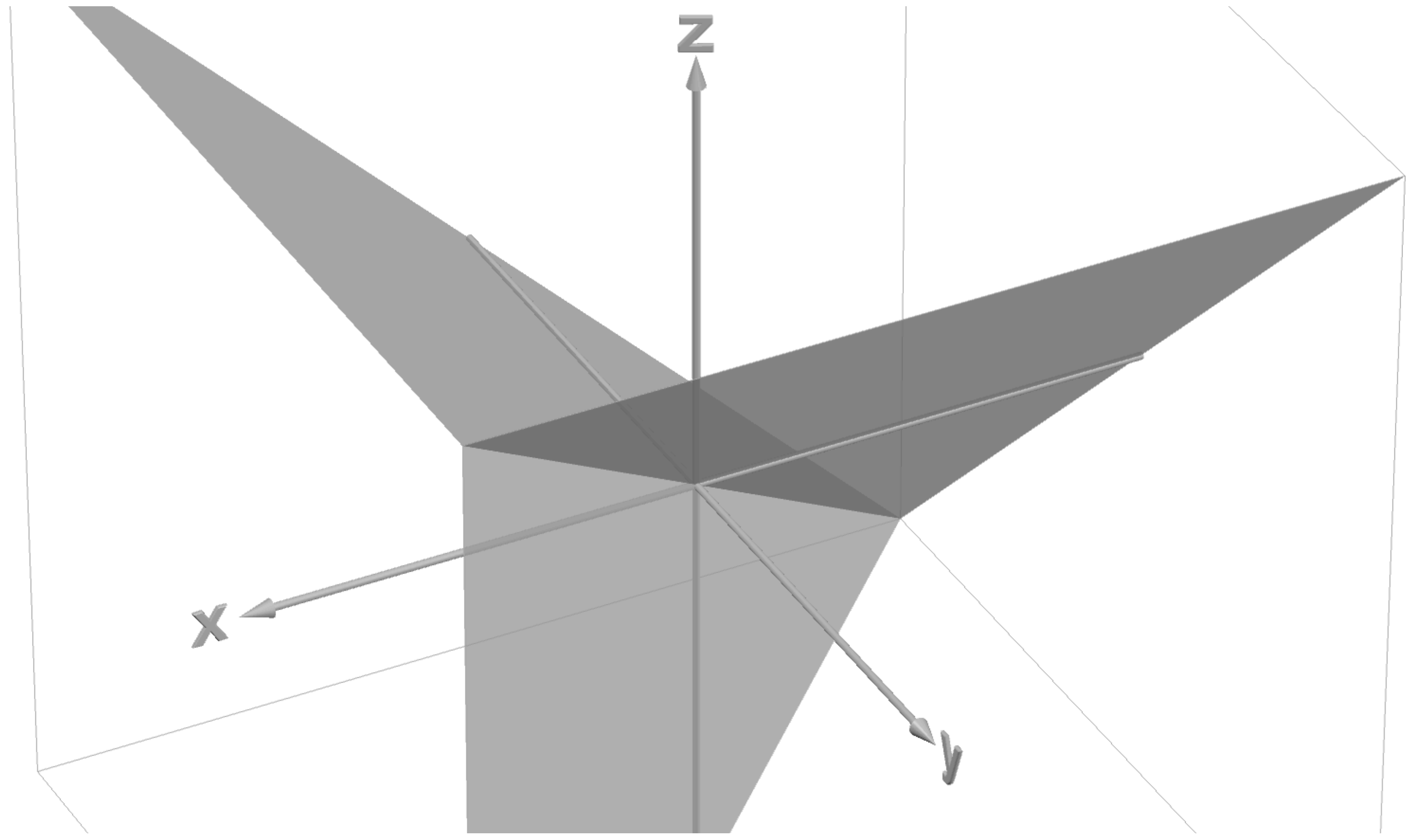
Supervised by Xavier Allamigeon, Stéphane Gaubert

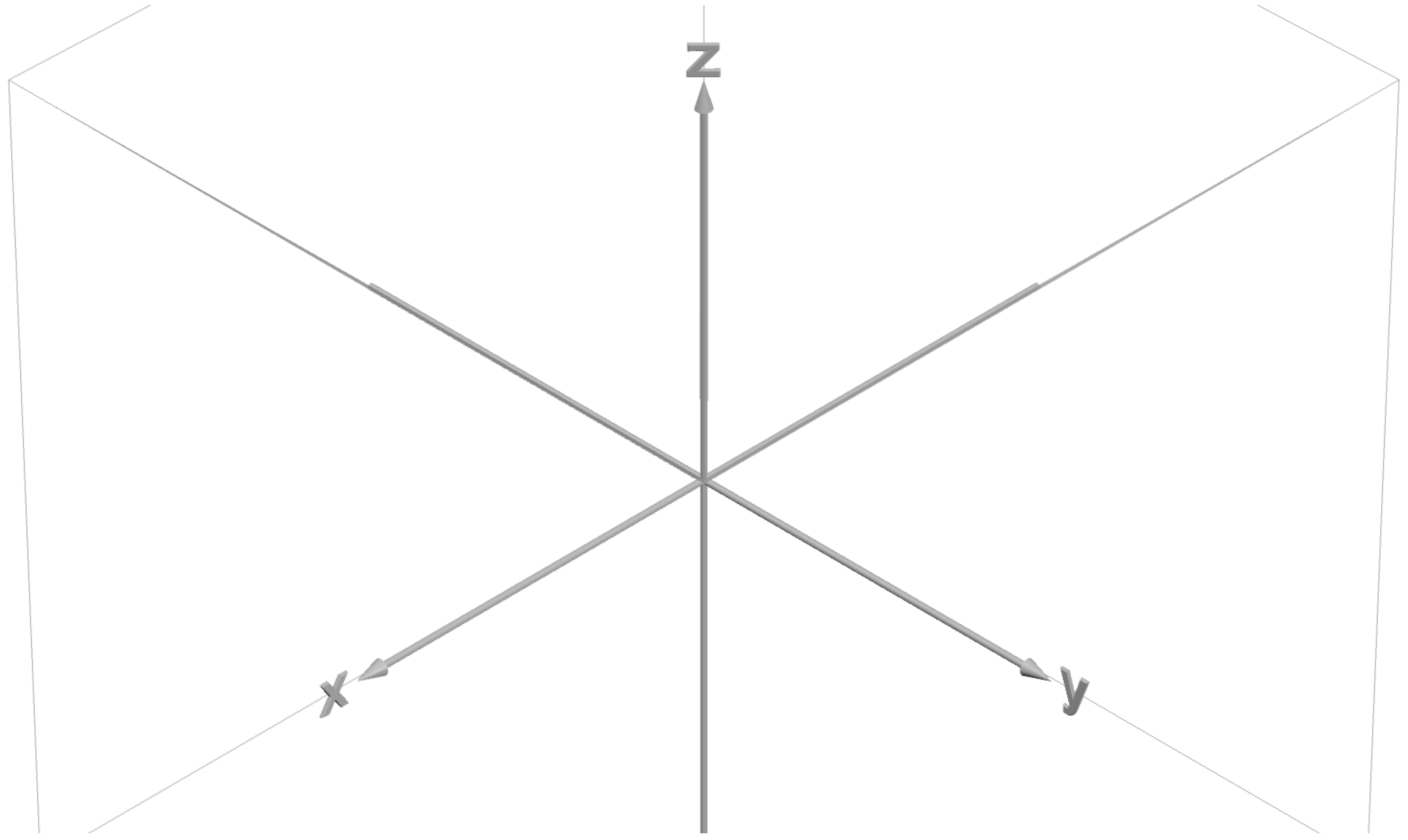
Tropical hyperplanes

- Max-plus semi-field: $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with addition $a \oplus b = \max(a, b)$ and multiplication $a \odot b = a + b$.

- Tropical hyperplane of apex $a \in \mathbb{R}^d_{\max}$: splits space depending on where $x - a$ reaches its maximum coordinate.



In 3D space

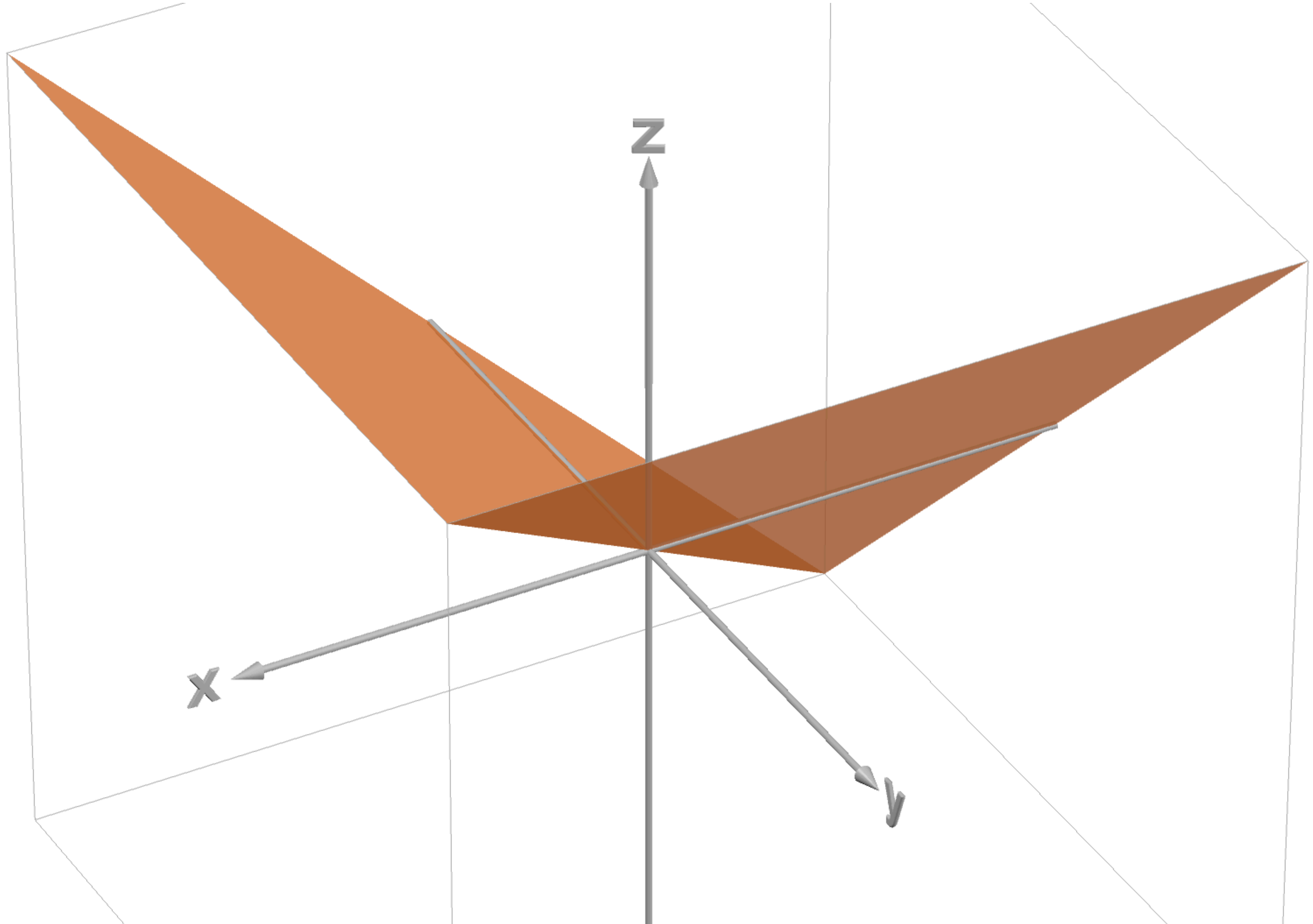


In the projective plane

- $H_a := \left\{ x \in \mathbb{R}^d_{\max}, \quad (x - a) \text{ reaches its max coordinate at least twice} \right\}$.

Tropical hyperplanes

- *Tropical parametrised hyperplane* of config. $\sigma = \{I^{\pm}\}$:
  $H_a^\sigma := \left\{x \in \mathbb{R}_{\max}^d, \quad (x-a) \text{ reaches its max coordinate in } I^+ \text{ and } I^-\right\}$



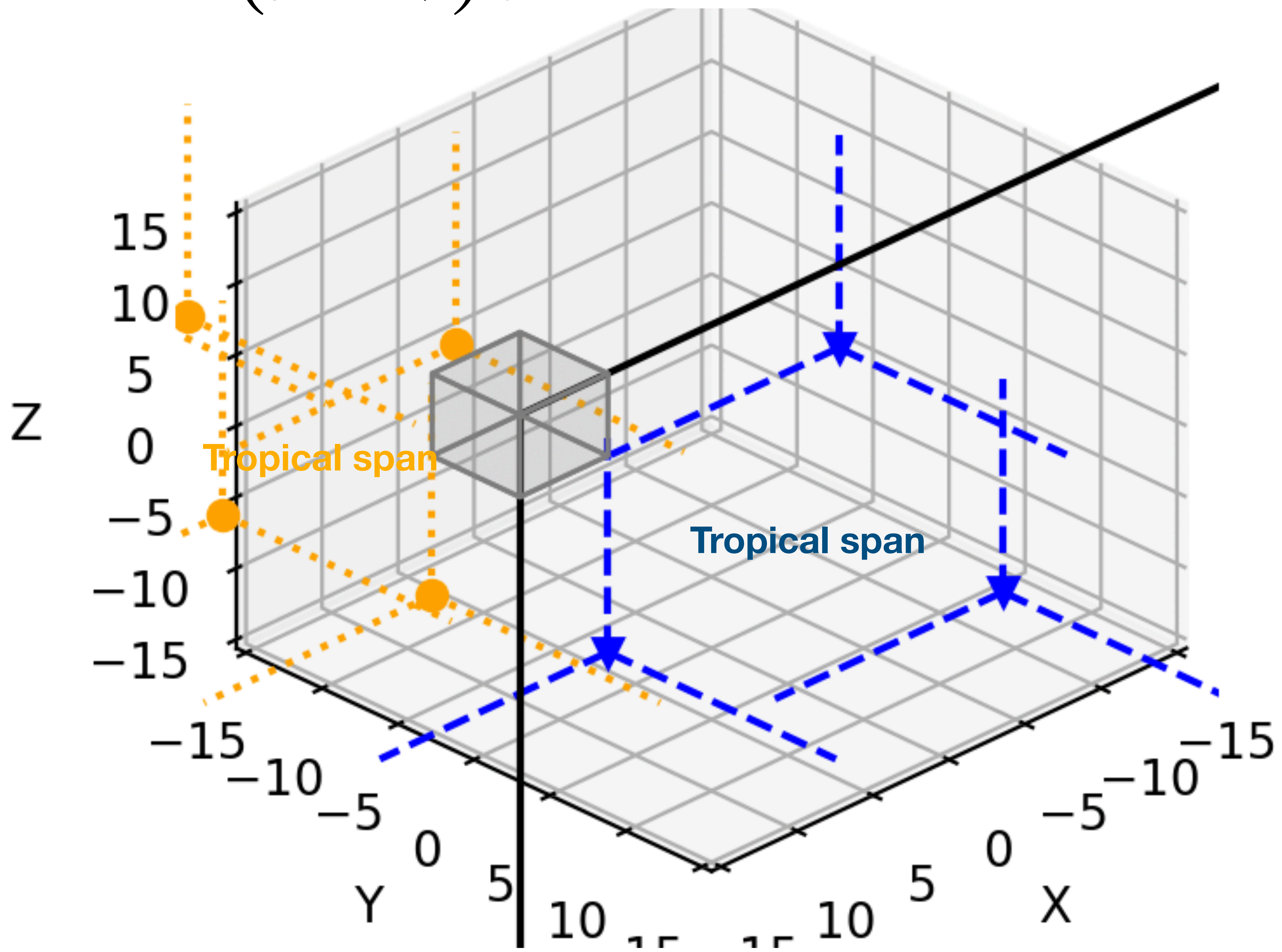
Tropical classification problem

- n classes of d-dimensional points: $X_1, \ldots, X_n \subset \mathbb{R}^d$.
- Tropical distance: $d(u, v) := \max(u - v) - \min(u - v)$.
- Hard separation with margin ν:

$$\forall x^k \in X^k, \quad \arg\max_{i \in [d]} x_i^k \in I^k,$$

$$d(H_a^\sigma, x^k) = \max(x^k - a) - \max_{\bigsqcup_{\ell \neq k} I^\ell} (x^k - a) \geq \nu \,.$$

- Amounts to separate tropical spans.



Tropical Binary Classification Example

Describing tropical spans

- **Shapley operator**: non-decreasing map over $\mathbb{R}^d_{\max}$ verifying $\forall \lambda \in \mathbb{R}_{\max}, T(\lambda + x) = \lambda + T(x)$.
- For T a Shapley map, we define $\mathcal{S}(T) = \{x \in \mathbb{R}^d, \quad x \leq T(x)\}$.
- Important example: *tropical projections:*

$$\forall i \in [p], \quad P_X(x)_i = \max_{j \in [p]} \left\{ X_{ij} + \min_{k \in [d]} (-X_{kj} + x_k) \right\} .$$

$$\boxed{\mathcal{S}(P_X) = \mathrm{Span}(X) \, .}$$

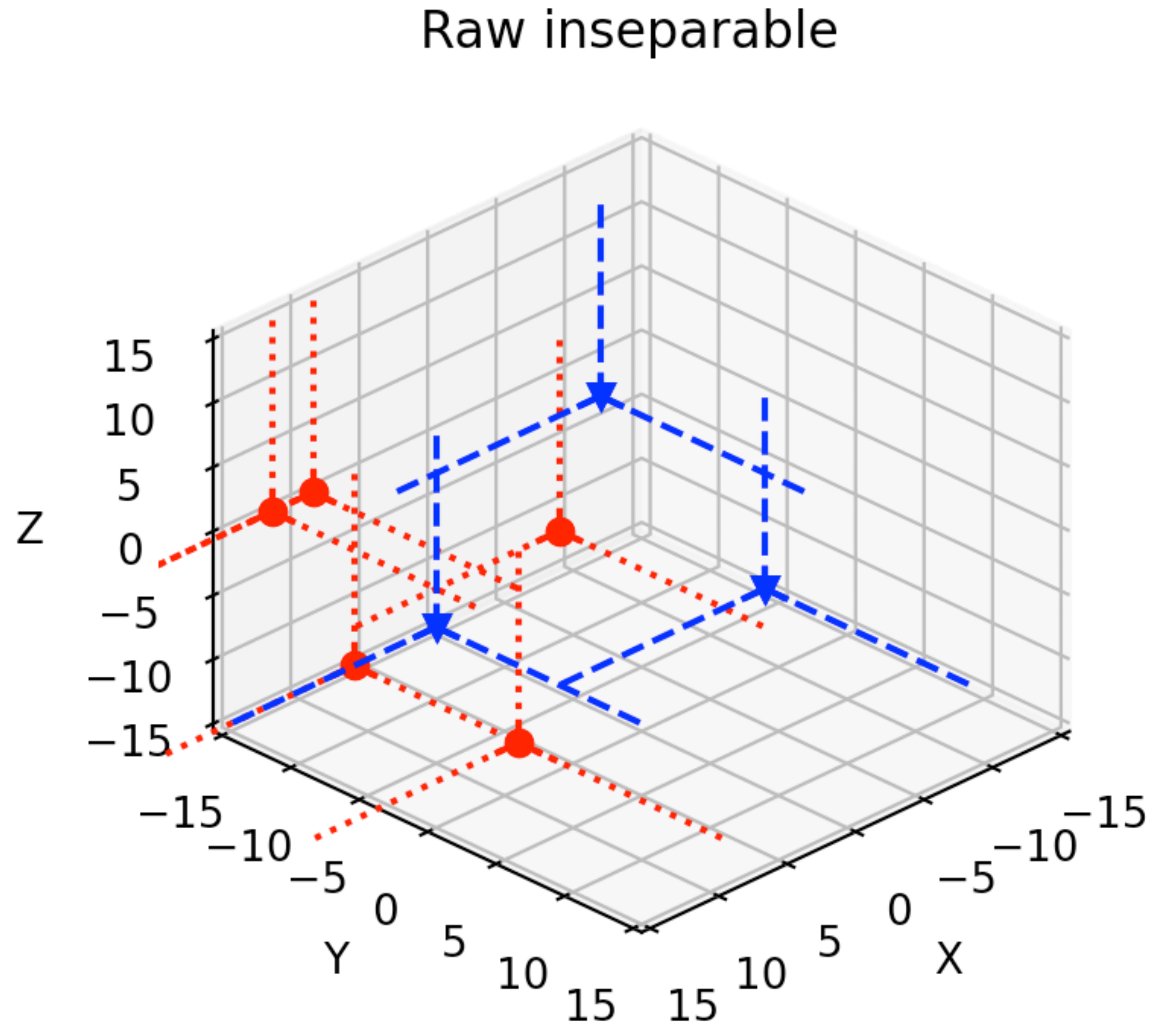
- Hence *tropical projections* describe the sets we want to separate.

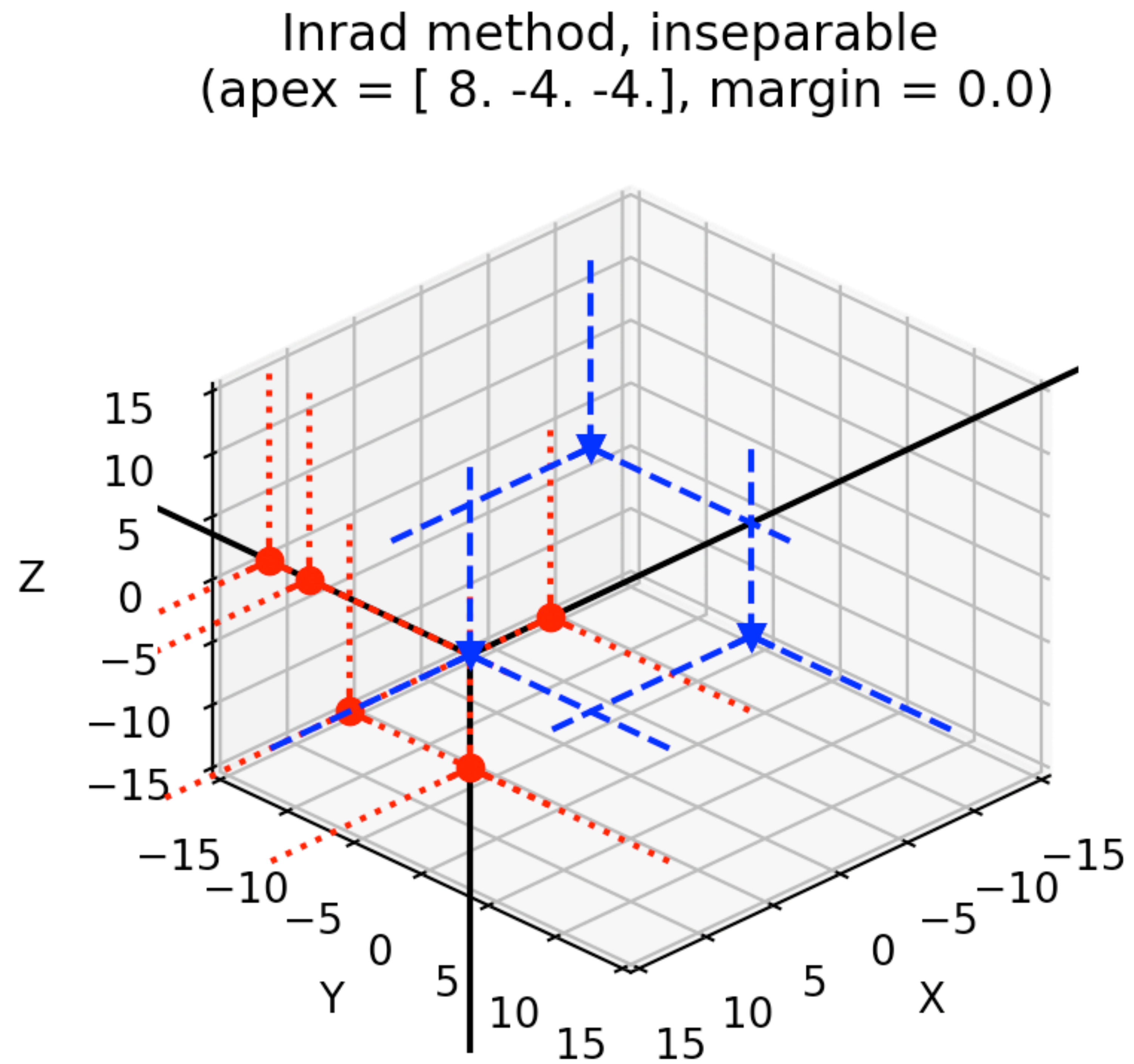
Dealing with non-separability

- **Objective:** Evaluating overlap $V^+ \cap V^-$ in terms of distance to separability.
- $\mathcal{S}(T^+ \wedge T^-) = \mathcal{S}(T^+) \cap \mathcal{S}(T^-)$: we have a Shapley for the intersection.
- We can make it *diagonal-free*.
- *[Allamigeon, Gaubert et al.]* $\rho(T) = \text{inrad}\, \mathcal{S}(T)$ when T DF.
- Eigenpairs computable in pseudo-polynomial time with Krasnoselskii-Mann.

Dealing with non-separability

- **Proposition.** Projecting points at distance less than $\rho(T)$ over H_a nullifies the interior of $V^+ \cap V^-$, with a eigenvector associated with $\rho(T)$.





Optimal margin in the separable case

- Let (a, λ) the unique eigenpair: in the separable case, $\lambda < 0$.
- Let the sectors $I^{\pm} := \{i \in [d], \quad T^{\pm}(a)_i > \lambda + a_i\}$.
- **Proposition.** H_a^{σ}, given the sectors defined above, separates $V^{\pm}$ with a margin of $-\lambda$. It is optimal in the case of finite point clouds.

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- Let (a, λ) the unique eigenpair: in the separable case, $\lambda < 0$.
- Let the sectors $I^{\pm} := \{i \in [d], \quad T^{\pm}(a)_i > \lambda + a_i\}$.
- **Proposition.** H_a^σ, given the sectors defined above, separates $V^{\pm}$ with a margin of $-\lambda$. It is optimal in the case of finite point clouds.
- *Proof.* As $T^{\pm}$ are non-expansive, we have for $x^{\pm} \in V^{\pm}$:

$$x_i^{\pm} \le \max(x^{\pm} - a) + T^{\pm}(a)_i \,.$$

- For instance, let $i \in I^-$. Then $T^+(a)_i = \lambda + a_i$, so for $x^+ \in V^+$:

$$x_i^+ - a_i \le \max(x^+ - a) + \lambda \,.$$

- And:

$$d(H_a^\sigma, x^+) = \max(x^+ - a) - \max(x^+ - a)_{I^-} \geq -\lambda\,.$$

- For the optimality, when $T^\pm$ is in the form:

$$T^\pm(x) := \sup_{v \in V^\pm} \big(v_i + \min(-v + x)\big)\,.$$

- For $\varepsilon > 0$, we can find $v \in V^+$ such that:

$$T^+(a)_i - \varepsilon \leq v_i - \max(v - a) \leq T^+(a)_i\,.$$

- Hence:

$$\lambda - \varepsilon \leq v_i - a_i - \max(v - a) \leq \lambda\,.$$

Topics being explored

1. Hard-Margin Multi-Classification

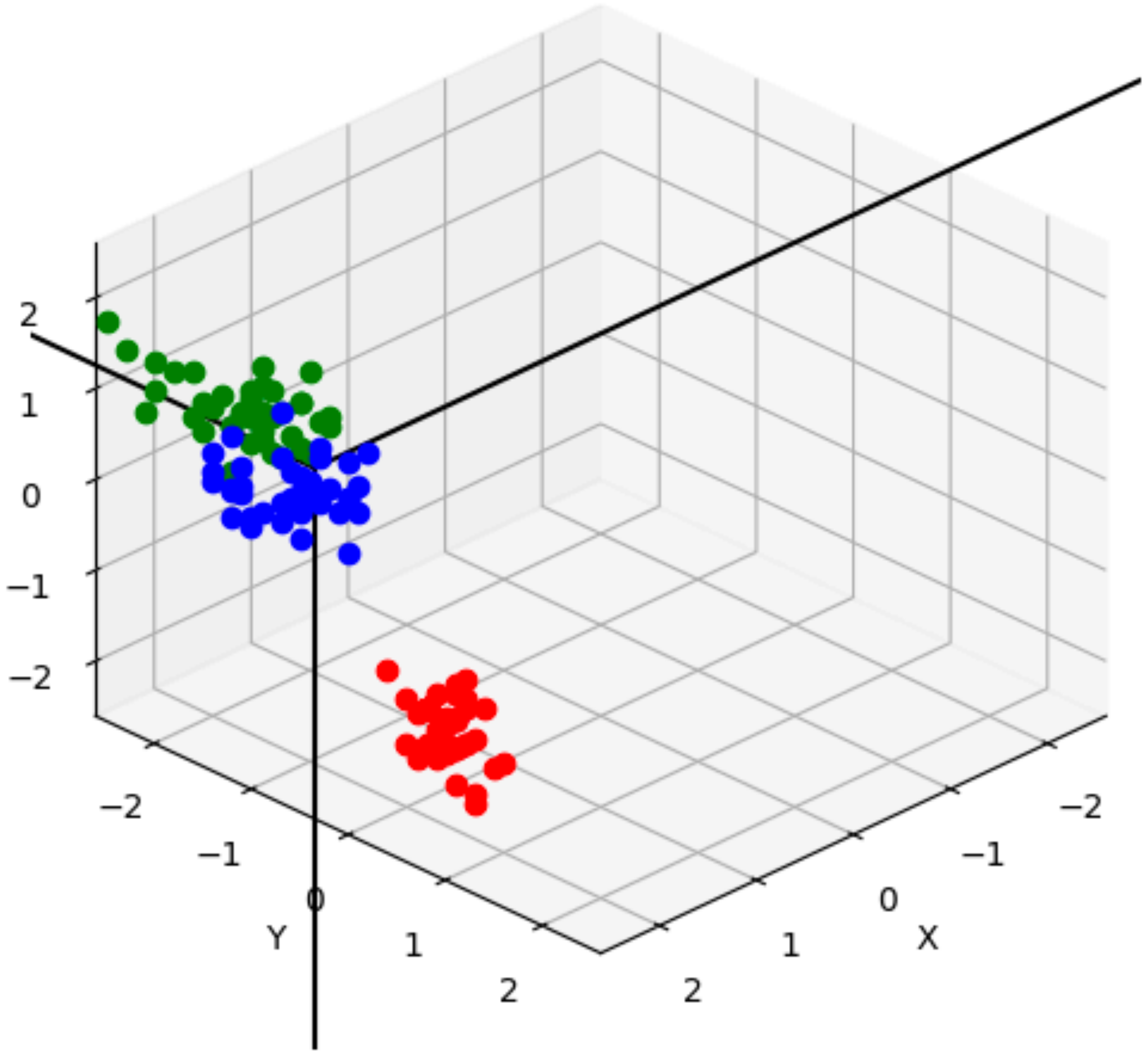
All-Vs-All

Same proofs!

$$T := \bigvee_{1 \leq k < l \leq n} T^k \wedge T^l$$

One-Vs-All?

$$T^{(1)} := T^1 \wedge \left(\bigvee_{j \neq 1} T^j \right).$$



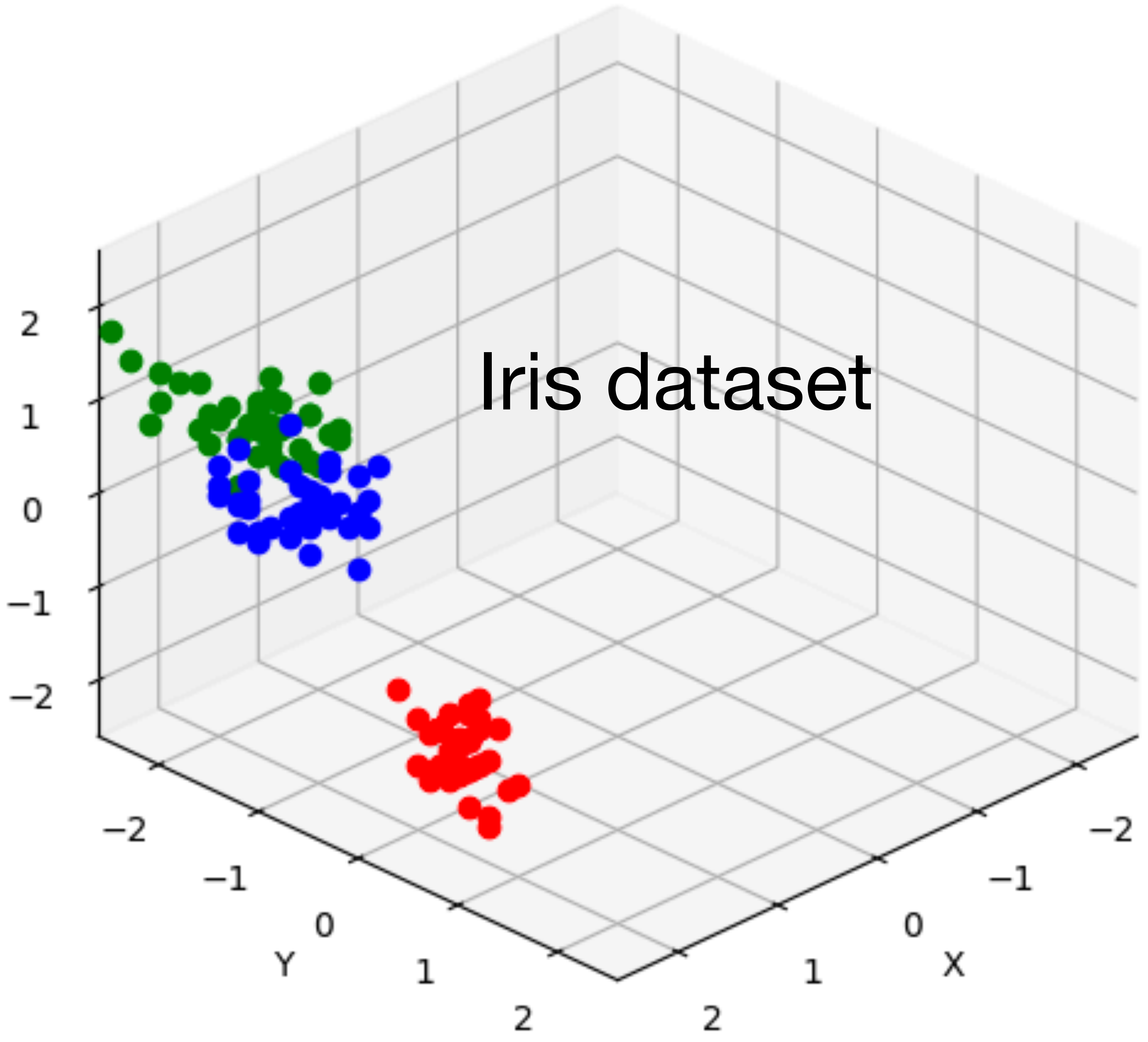
Topics being explored

2. Adding Features: Tropically Polynomial Decision Boundaries

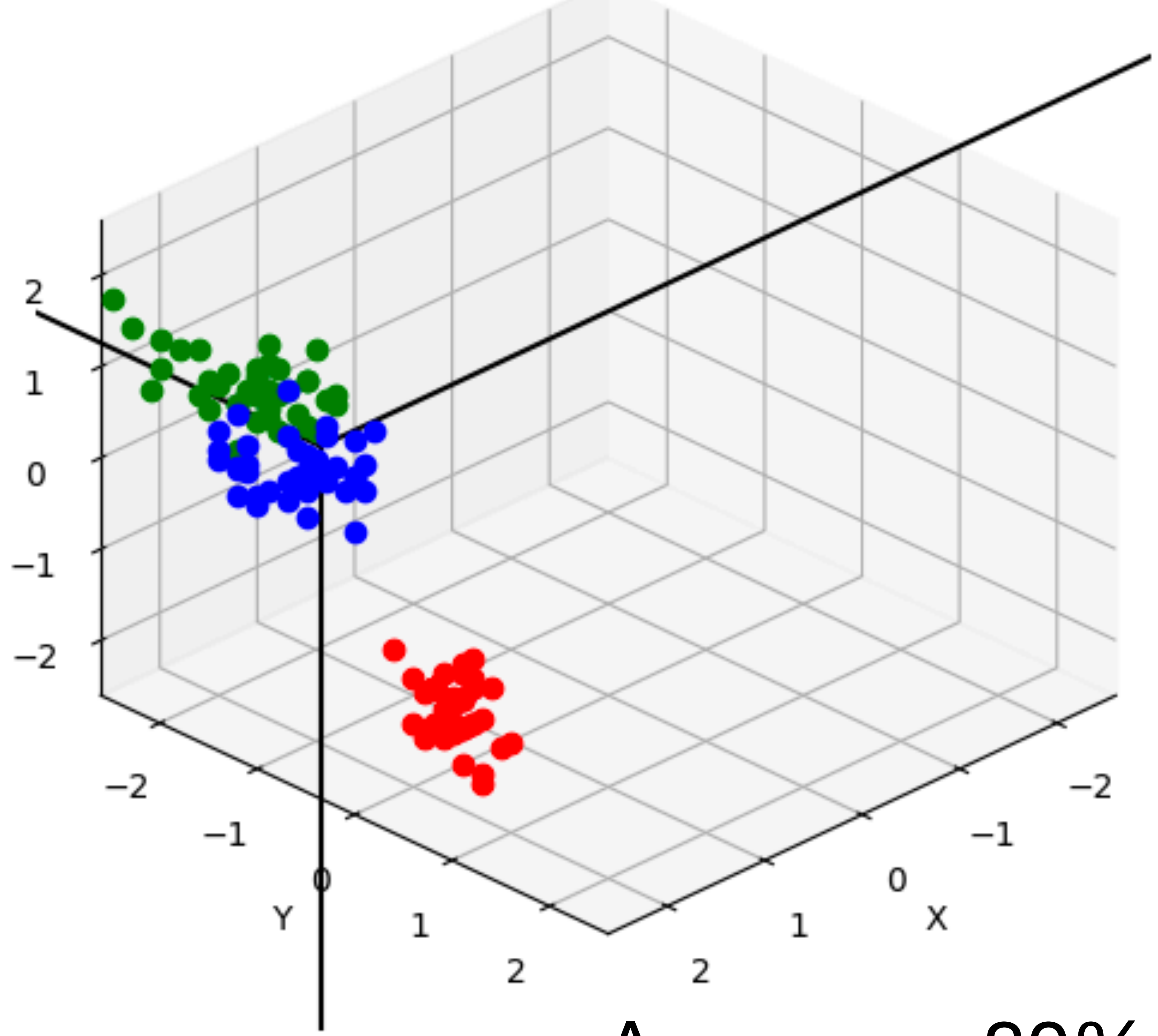
- Initial space: $\mathbb{R}^d$.
- New space: ver$(x) := \left(\langle x, \alpha \rangle\right)_{\alpha \in \mathscr{A}} \in \mathbb{R}^{\mathscr{A}}$.
- We consider integer combinations of features: $\mathscr{A}_s := (s\Delta_d) \cap \mathbb{Z}^d$.
- Ex: $\mathscr{A}_1$ corresponds to initial space, $\mathscr{A}_2$ also contains all sums of 2 features…
- *[Zhang, 18]* Feedforward neural networks + ReLU ≡ Tropical Rational Functions

Topics being explored

2. Adding Features: Tropically Polynomial Decision Boundaries



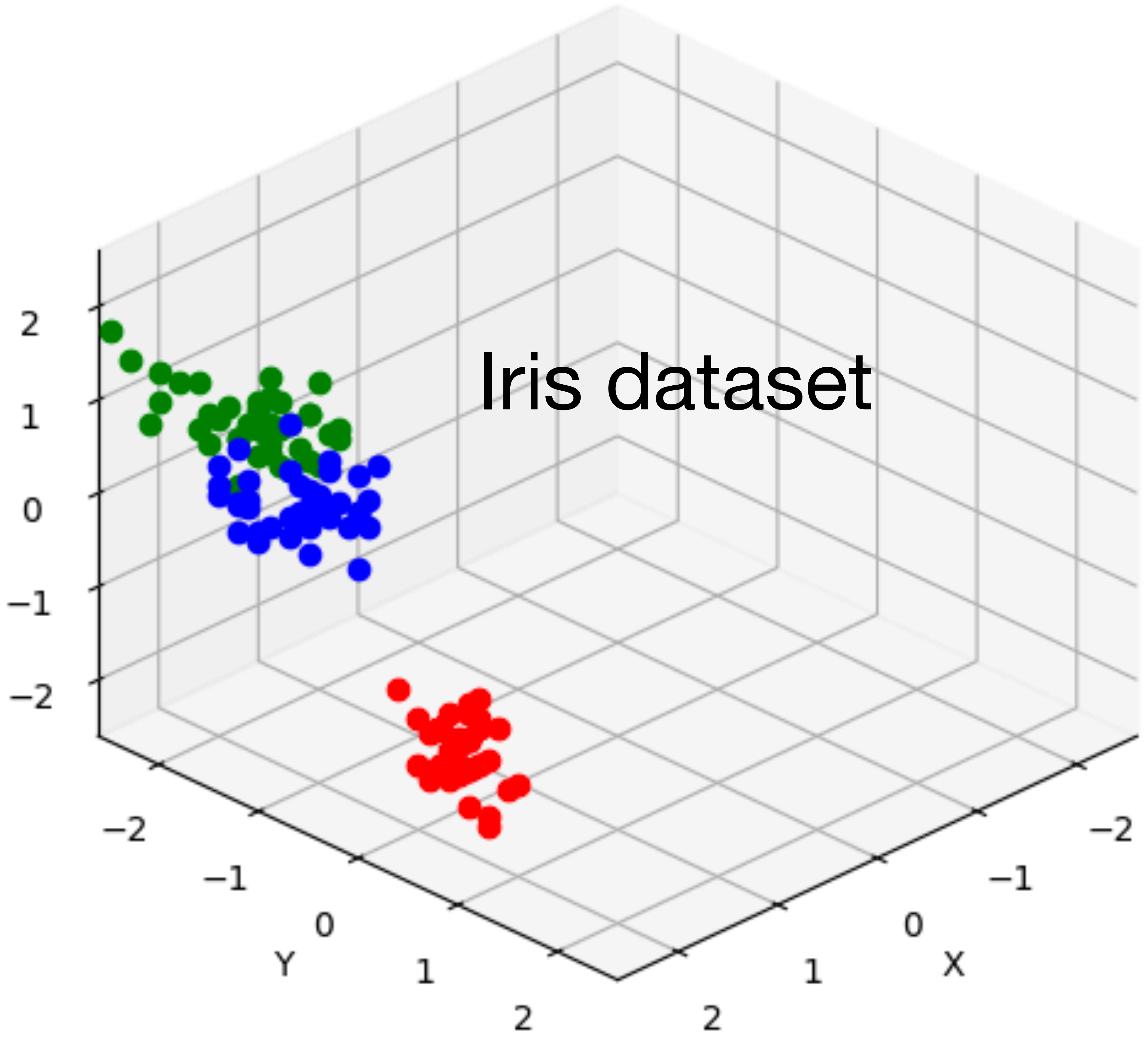
Tropical hyperplane ($\mathscr{A}_1$)



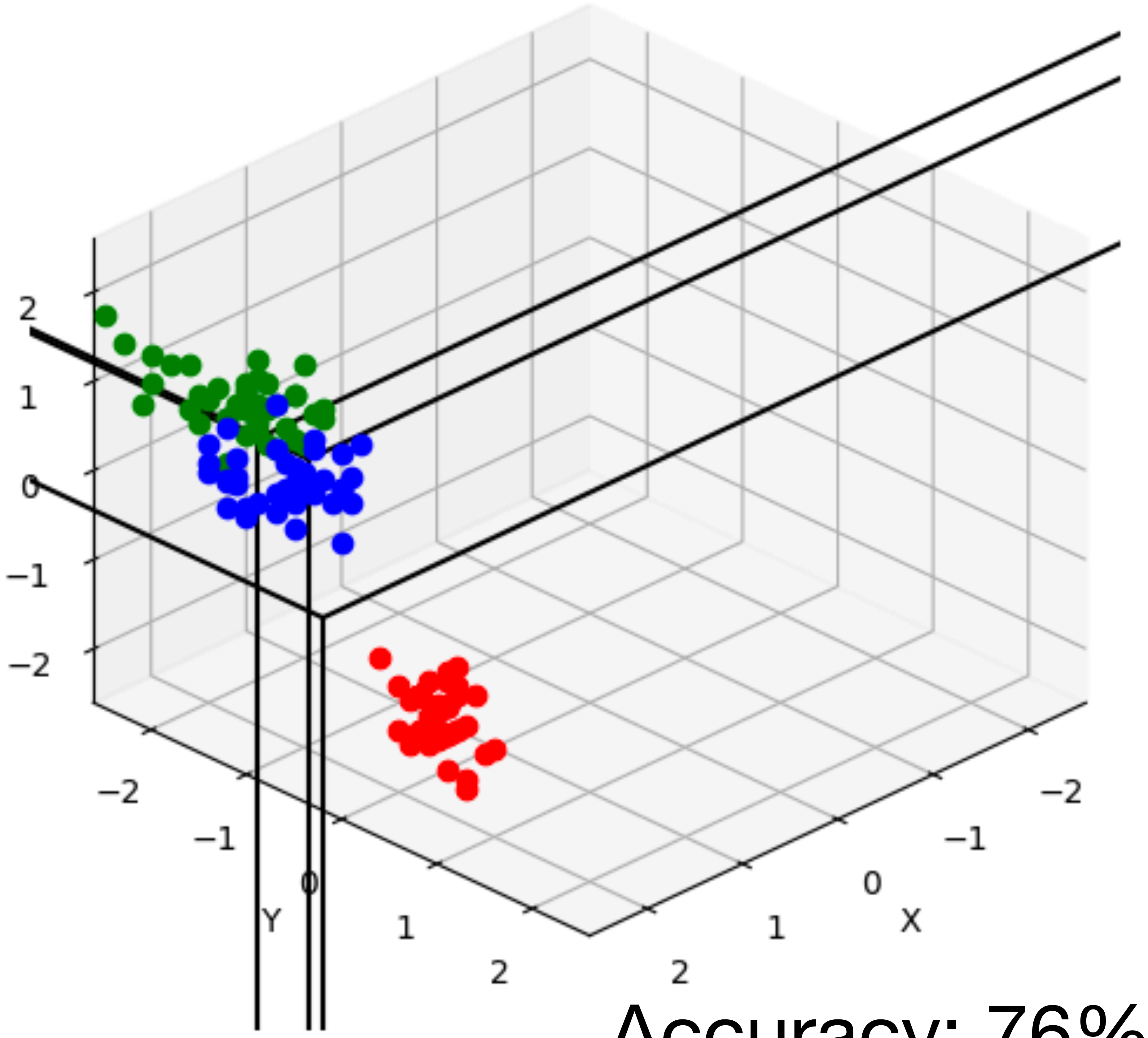
Accuracy: 80%

Topics being explored

2. Adding Features: Tropically Polynomial Decision Boundaries



One-vs-all hyperplanes ($\mathscr{A}_1$)



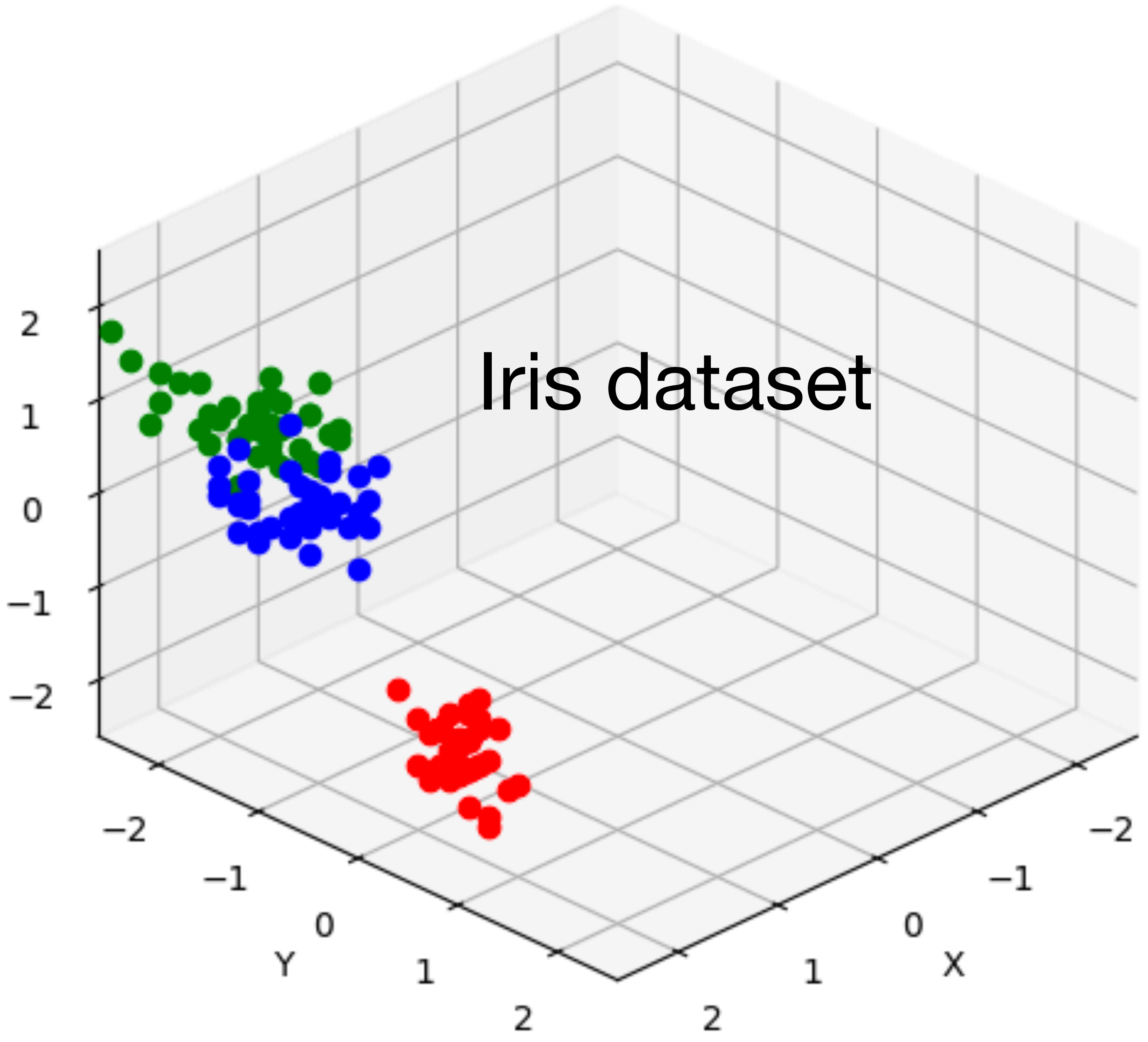
Accuracy: 76%

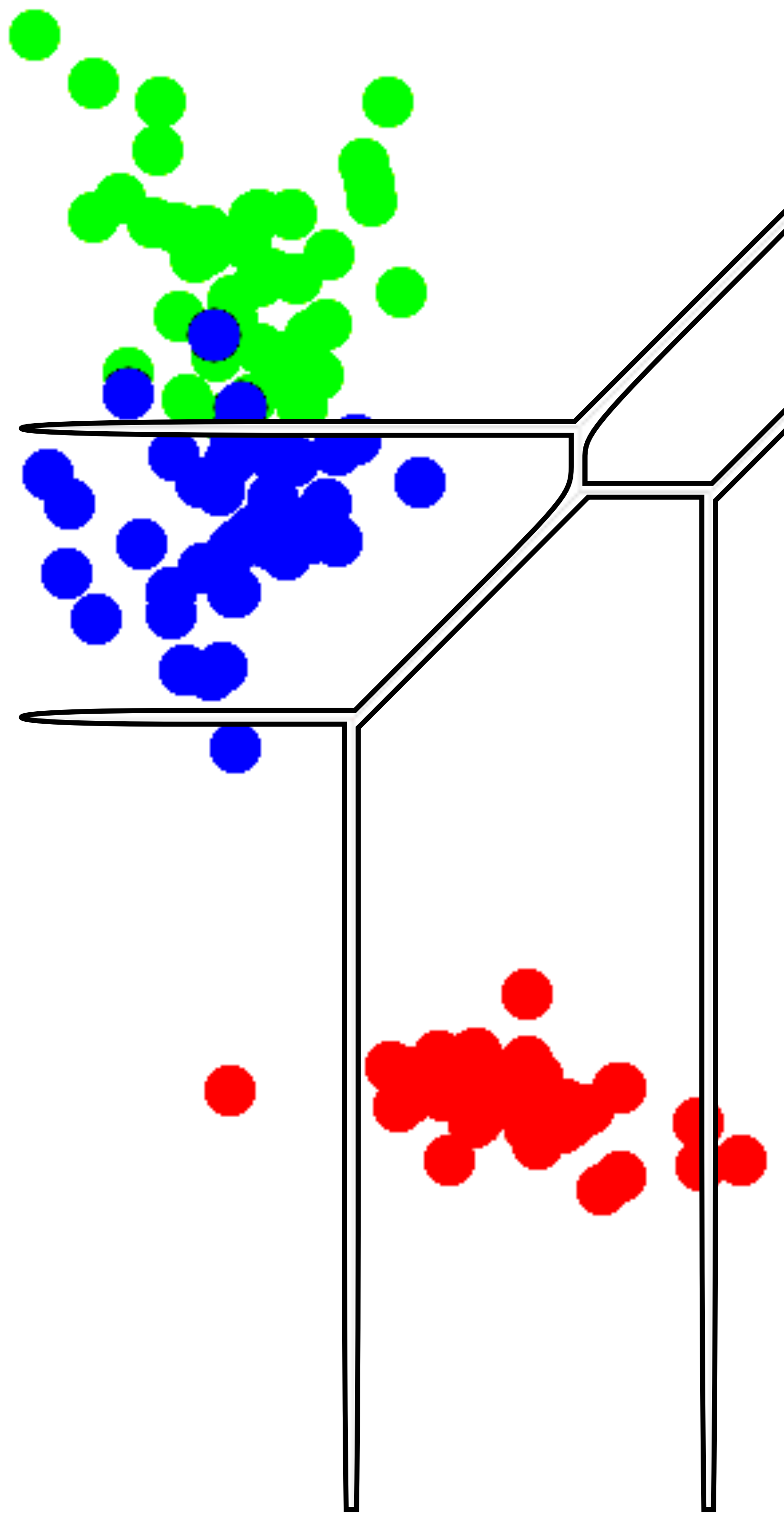
Blue class hard to separate

Topics being explored

2. Adding Features: Tropically Polynomial Decision Boundaries

Veronese trick ($\mathscr{A}_2$) : better!

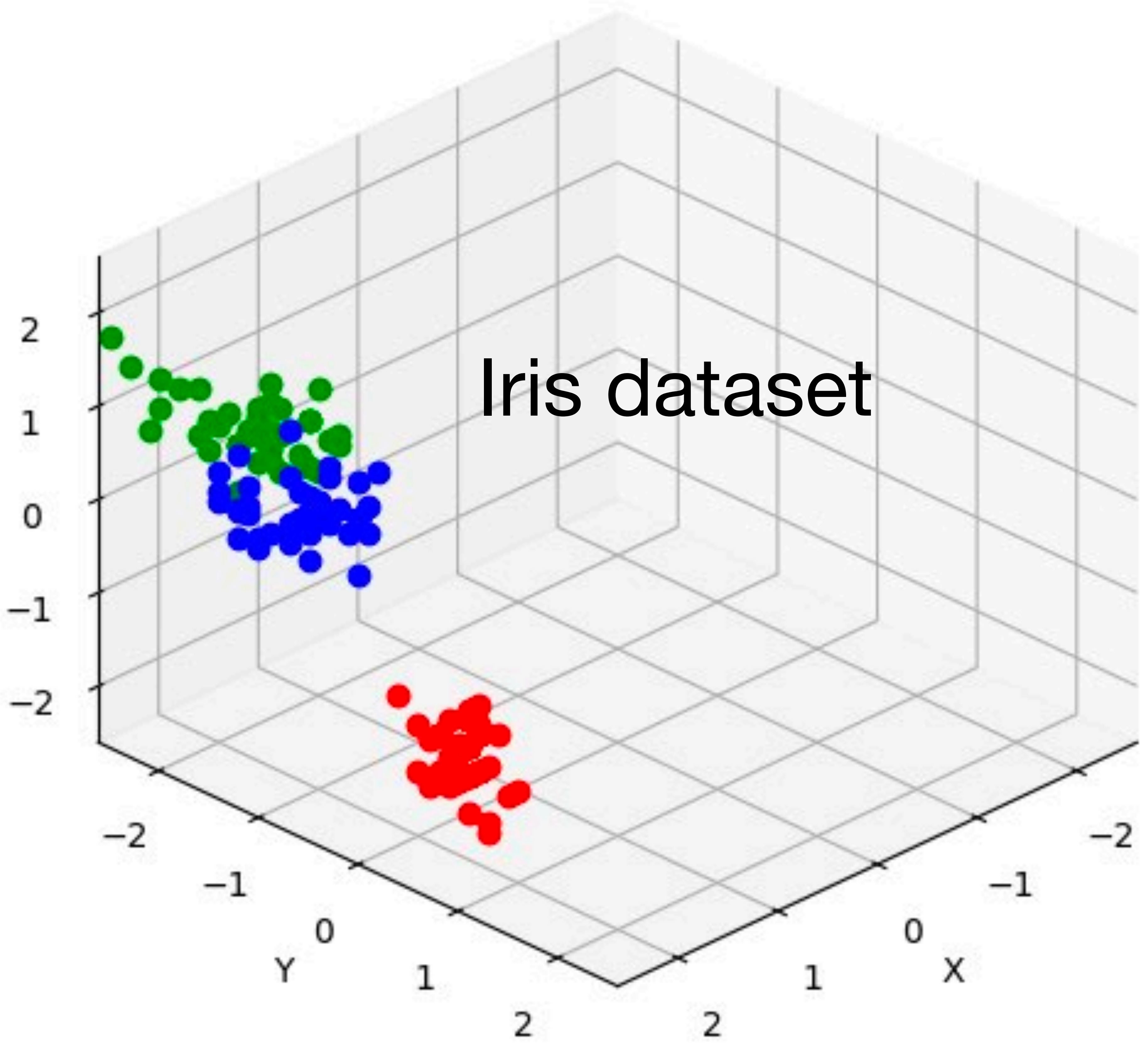




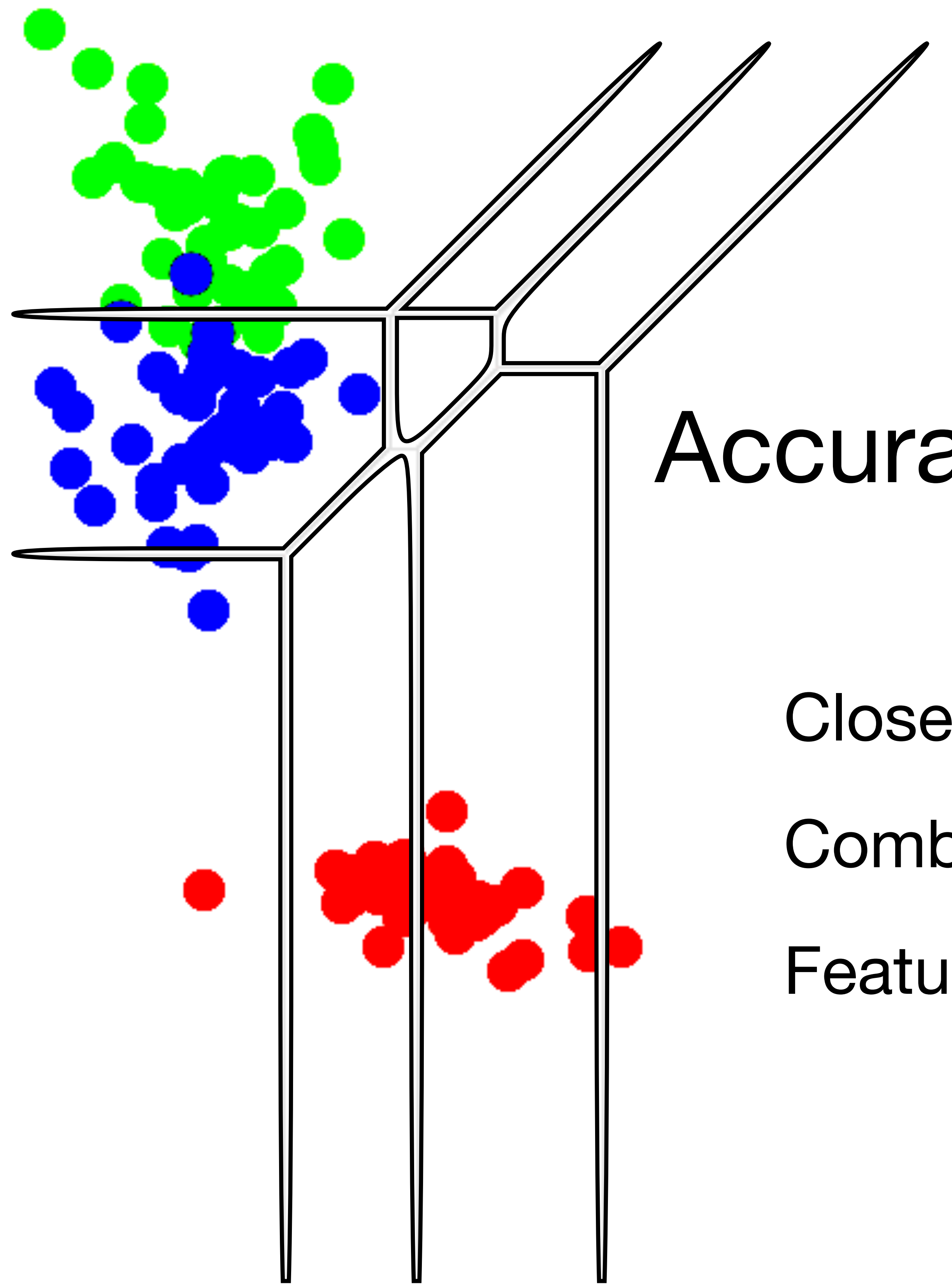
Generated with Polymake, approximate overlap

Topics being explored

2. Adding Features: Tropically Polynomial Decision Boundaries



Veronese trick ($\mathscr{A}_3$) : overfitting!



Accuracy: 98%

Closed cells

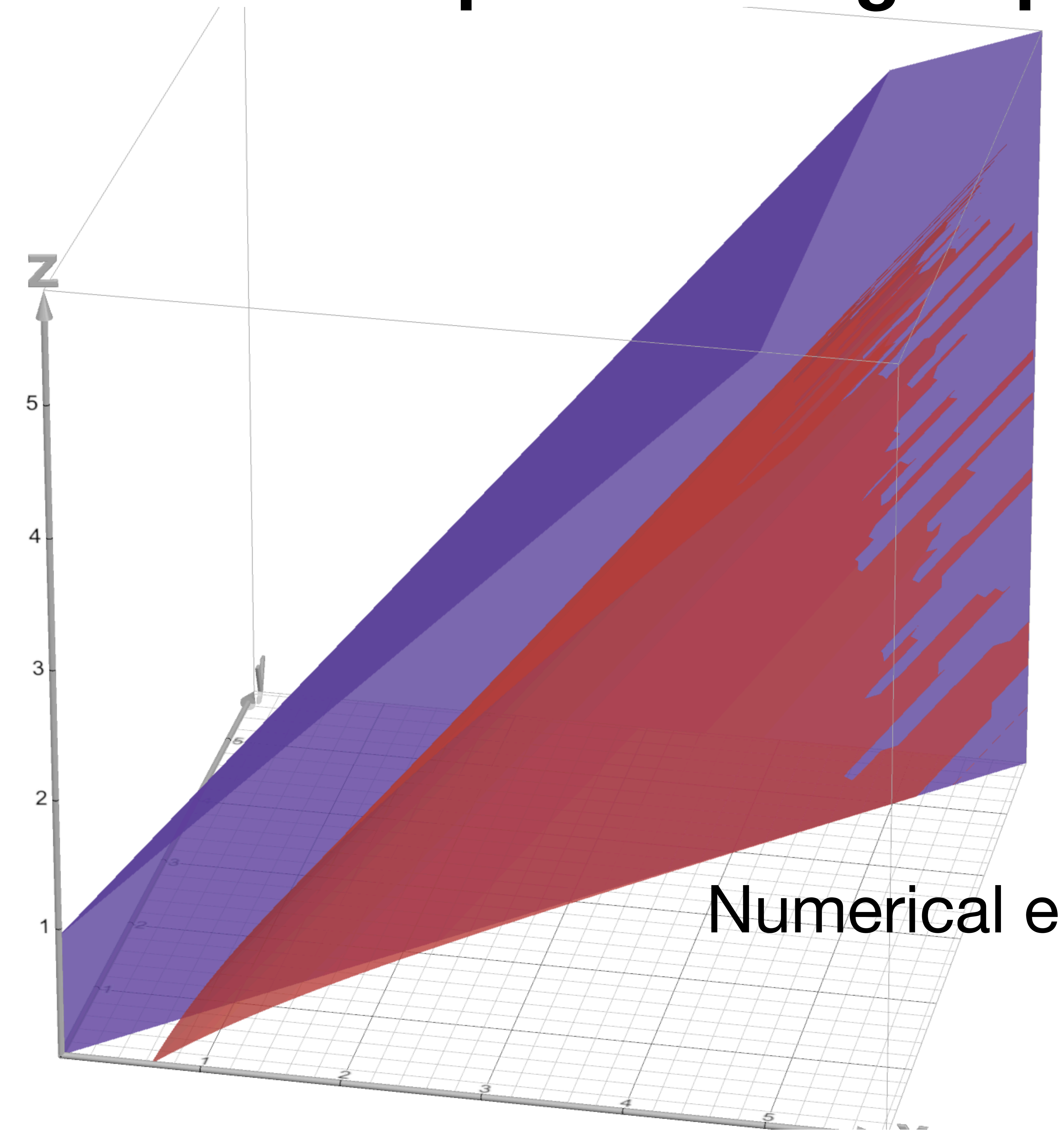
Combinatorics

Feature selection?

Generated with Polymake, approximate overlap

Topics being explored

3. Linear Hyperplanes Look Tropical on Log Paper



Numerical errors!

Topics being explored

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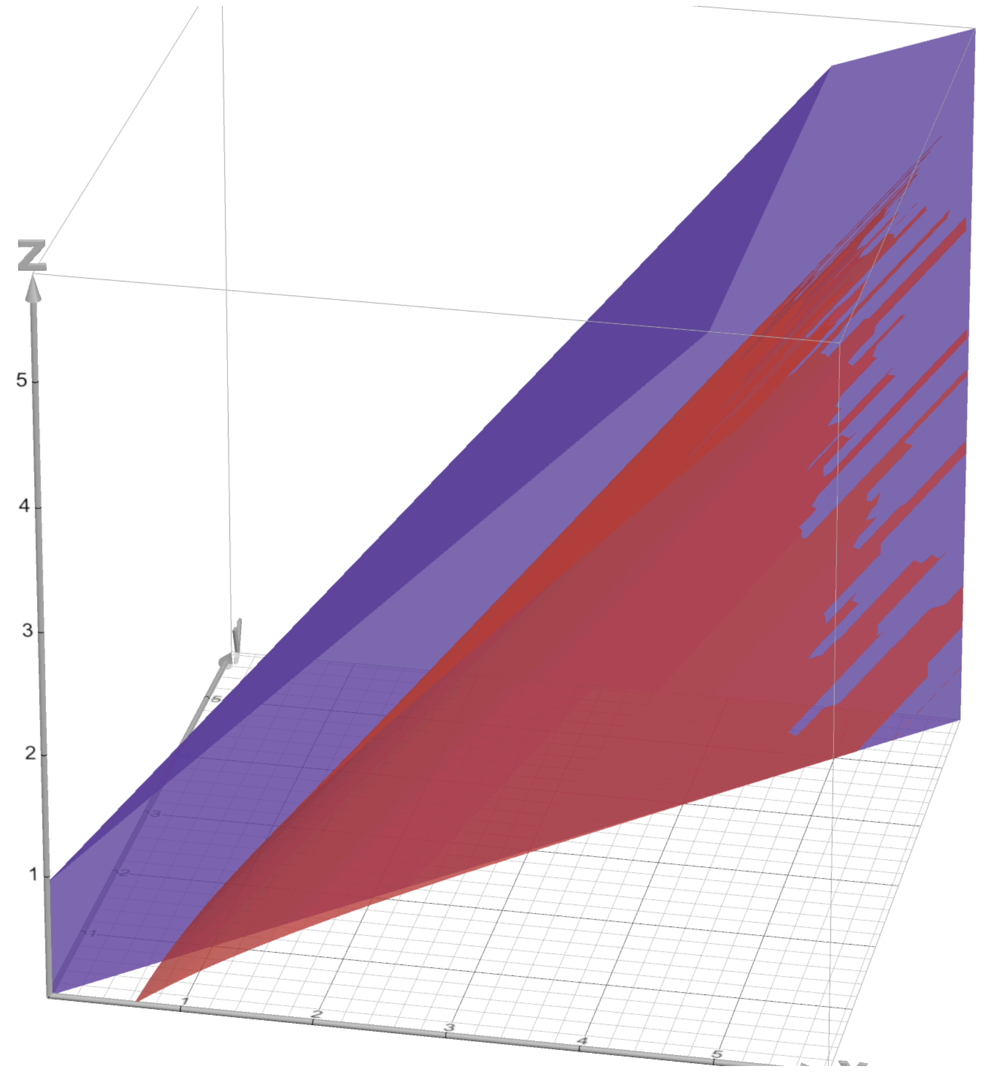
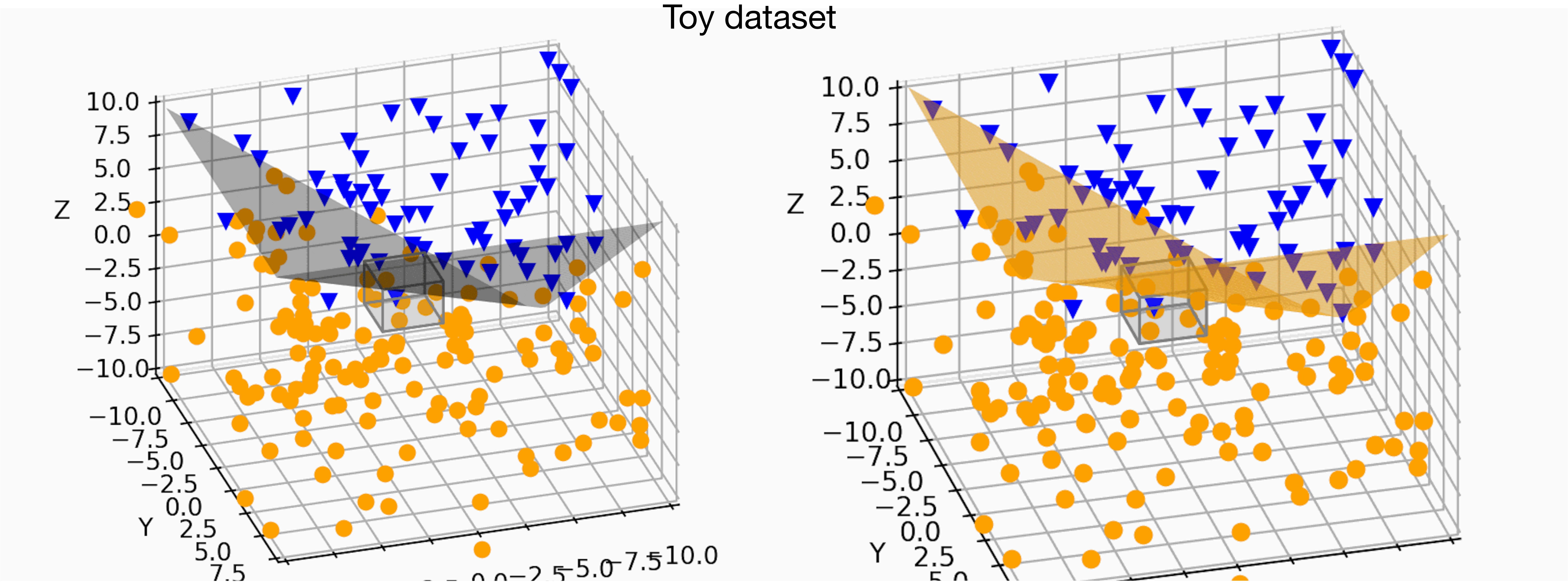


- New data: $x^\beta = (x_{ij}^\beta := e^{\beta X_{ij}})_{ij}$.
- Hypersurface converges: $d_H\left(\beta^{-1}\log H^\beta, H^{trop}\right) \leq \beta^{-1}\log d$.
- When d high, β has to compensate, however if $\beta\bar{X}$ too high, numerical error!
- Tropical method will give higher numerical accuracy on higher dimensions.
- ***Bridging linear and tropical SVM theories?***

Topics being explored

3. Linear Hyperplanes Look Tropical on Log Paper



Tropical hyperplane

Classical hyperplane on logarithmic paper

$$d = 3, \quad \beta = 10$$