Efficient Tropical SVMs via Mean-Payoff Games

Xavier Allamigeon
Inria and CMAP
École polytechnique
Palaiseau, France

Samuel Boïté
Inria and CMAP
École polytechnique
Palaiseau, France

Stéphane Gaubert
Inria and CMAP
École polytechnique
Palaiseau, France

Théo Molfessis
Inria and CMAP
École polytechnique
Palaiseau, France

Abstract

In 2006, Gärtner and Jaggi introduced a tropical analogue of support vector machines, using a single tropical hyperplane in dimension n to separate n classes of points. Efficiently computing tropical separators has remained an open problem. We introduce an algorithm for Tropical Support Vector Machines that overcomes the combinatorial explosion of previous approaches. Our main result shows that the spectral radius of a specially constructed Shapley operator fully characterizes separability and margin, and gives a sense for data overlap in the non-separable case. This provides a reduction to mean-payoff games, a well studied class of problems in algorithmic game theory. This approach enables computing of an optimal separating hyperplane via scalable iterative algorithms – with a complexity linear in the size of the data set and pseudo-polynomial in the desired precision. Finally, we combine tropical classifiers with linear feature maps to construct piecewise-linear classifiers.

1 Introduction

Classification is a fundamental task in machine learning, and Support Vector Machines (SVMs) have been a cornerstone method for decades. Traditional SVMs create decision boundaries using affine hyperplanes, which provide maximum-margin separation with strong generalization guarantees [28]. However, these linear boundaries become limiting when faced with complex, nonlinear data patterns, typically requiring kernel methods or feature engineering [25].

Motivation: Beyond Linear Boundaries. We build on max-plus algebra, a framework where standard addition becomes the maximum operation, and multiplication becomes addition [18]. This leads to different geometric structures with attractive properties for machine learning: (1) instead of creating binary partitions, tropical hyperplanes divide space into multiple sectors, making them naturally suited for multi-class problems; (2) their piecewise-linear nature captures more complex patterns while maintaining interpretability; (3) the resulting decision boundaries coincide with those created by modern deep learning models with ReLU activations [31]. These properties provide richer, yet interpretable, decision boundaries that can capture nonlinear patterns in data while maintaining computational tractability [19].

Tropical geometry has emerged as a powerful tool for modeling piecewise-linear phenomena in machine learning. Together with polyhedral geometry, it has been used to bound the number of linearity regions ot functions realized by these networks [31, 22]. It has been successfully applied to linear regression [20, 3], principal component analysis [30], neural network analysis [19], and clustering [21].

Previous Work on Tropical SVMs. Gärtner and Jaggi [12] introduced tropical SVMs using linear programming formulations. Their work introduced an elegant geometric approach to multiclass problems, since a single tropical hyperplane in dimension d partitions the space in d regions. Despite

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these theoretical benefits, their method required exploring all possible sector assignment combinations, leadi ng to exponential worst-case complexity. This computational barrier has limited practical applications. Tang et al. [27] and Monot et al [21] later developed specialized algorithms for binary classification cases where data points from the same category stay in the same sector, showing promising results in computational biology for analyzing evolutionary trees. quote lavishly yoshida

Background on mean-payoff games. Our approach uses concepts from game theory–specifically, Shapley operators arising in zero-sum dynamic games. Shapley introduced an "operator approach" for discounted stochastic games [26], and Gillette later formulated the undiscounted, infinite-horizon variant now known as mean-payoff games [**?**]. The Shapley operators we employ encode the one-step payoffs and transitions in such games, whose objective is the long-run average reward per time unit [32]. The spectral properties of these operators have been extensively studied to characterize fixed points and convergence in nonlinear systems [17, 13]. In particular, the spectral radius of a Shapley operator equals the game's value. Mean-payoff games lie in NP $\cap$ coNP, but no polynomial-time algorithm is known. Nonetheless, large instances can be solved efficiently by iterative schemes such as relative Krasnoselskii-Mann iteration, which require $O(L/\varepsilon^2)$ operator evaluations, which is *linear* in input size L for a precision ε.

Contributions. We develop a new approach to tropical classification using mean-payoff games, overcoming the computational limitations of previous approaches:

1. We establish a direct connection between separability and the spectral radius $\rho(T)$ of a Shapley operator constructed from class-specific projections.
2. We prove that when data are separable, the optimal margin equals $-\rho(T)$. Moreover, in the binary non-separable case, $\rho(T)$ quantifies exactly how much the data points would need to be perturbed to achieve separability.
3. We develop an algorithm based on mean-payoff games and Krasnoselskii–Mann iteration that computes the optimal classifier in a time that is linear in the input size.
4. We extend our framework to tropical polynomial classifiers, enabling more expressive piecewise linear decision boundaries (see Figure 1) while preserving theoretical margin guarantees.

This work makes tropical SVMs tractable for real-world applications, enabling natural multi-class classification, and opening new directions for piecewise-linear methods that balance expressivity, interpretability, and computational efficiency.

The remainder of the paper is organized as follows. Section 2 introduces the essential concepts from tropical geometry. Section 3 presents our spectral framework and main theoretical results, showing how separability connects to spectral properties. Section 4 details our algorithm and implementation, explaining how we achieve pseudo-polynomial complexity. Section 5 extends the framework to polynomials for more expressive decision boundaries. Section 6 explores connections with classical SVMs, and Section 7 discusses limitations and future directions. The code to reproduce our figures and experiments is available at `https://github.com/samuelbx/tropical-svm`.

2 Tropical Geometry Preliminaries

We now introduce the key concepts from tropical geometry that form the foundation of our approach.

The Max-Plus Semiring. The tropical (or max-plus) semiring $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ replaces traditional arithmetic operations with:

$$x \oplus y = \max(x, y) \quad \text{(tropical addition)} \tag{1}$$
$$x \odot y = x + y \quad \text{(tropical multiplication)} \tag{2}$$

These operations may seem strange at first, but they naturally model systems where we care about "bottlenecks" or "critical paths." For example, in project planning, if task A takes x days and task B takes y days, the project completion time depends on the maximum ($x \oplus y$) of these durations if tasks are parallel, and their sum ($x \odot y$) if sequential.

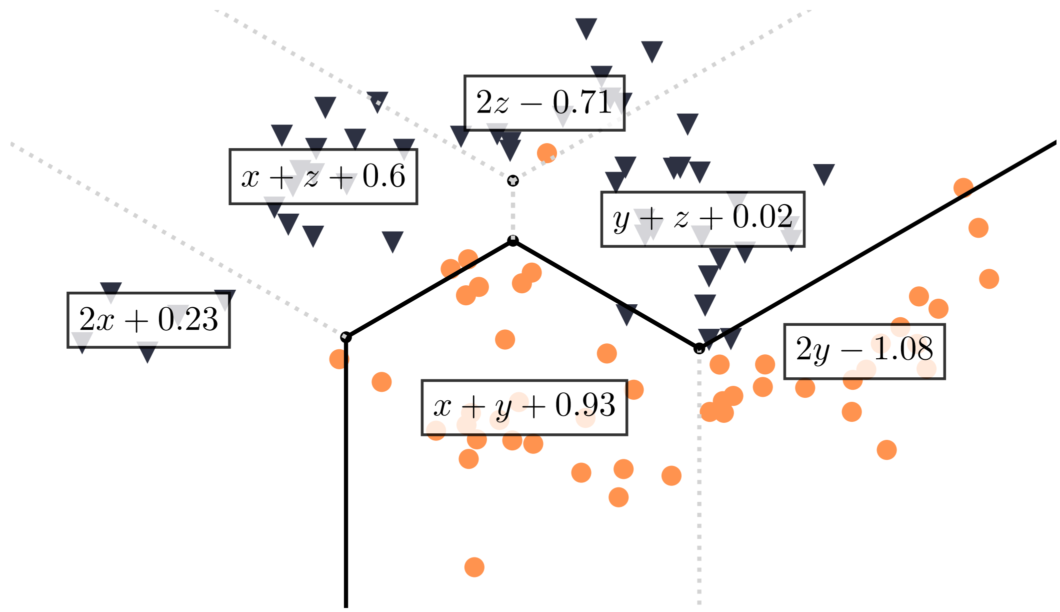

Figure 1: Visualization of a degree-2 polynomial classifier. Each region corresponds to a sector where a specific affine combination of the features dominate, creating an interpretable piecewise-linear decision boundary. Inference remains computationally efficient, requiring only the evaluation of the dominant monomial at each test point.

Projective Space. The tropical projective space identifies points that differ by adding the same constant to all coordinates. Formally, it's the quotient of $\mathbb{T}^d \setminus \{(-\infty, \ldots, -\infty)\}$ by the equivalence relation $x \sim y$ if $x = y + c \cdot \mathbf{1}$ for some constant c. In practice, we embed data from $\mathbb{R}^d$ into the projective space via:

$$x = (x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_d, -(x_1 + \cdots + x_d))$$

This transformation makes our classifier invariant to shifts—adding the same constant to all features doesn't change the classification. It's similar to how projective geometry in computer vision makes analysis invariant to camera distance.

Hyperplanes and Sectors. A tropical hyperplane with apex $a \in \mathbb{T}^d$ is defined as:

$$\mathcal{H}_a = \left\{x \in \mathbb{T}^d : \text{the maximum of } (x_i + a_i) \text{ over } 1 \leq i \leq d \text{ is attained at least twice}\right\}. \quad (3)$$

This hyperplane divides the space into at most d sectors. The i-th sector contains points where the maximum of $x + a$ occurs at the i-th coordinate:

include a picture of tropical hyperplane in the body

$$S_i(a) = \left\{x \in \mathbb{T}^d : i \in \arg\max_j (x_j + a_j)\right\}. \quad (4)$$

Unlike classical hyperplanes that create two half-spaces, tropical hyperplanes create multiple sectors—one for each dimension. They naturally support multi-class classification, where we can assign different sectors to different classes.

Hilbert Seminorm. The Hilbert seminorm measures the "spread" of coordinates:

$$\|x\|_H = \max_i x_i - \min_i x_i \quad (5)$$

This induces a projective distance $d_H(x, y) = \|x - y\|_H$ that remains invariant to adding the same constant to all coordinates [9]. We use this distance to define margins in tropical classification. For classification, it tells us how confidently a point belongs to its assigned sector rather than another sector.

Convexity and Projections. A set $C \subset \mathbb{T}^d$ is a *tropical convex cone* if for all x, y in C and coefficients λ, μ in $\mathbb{T}$, the point $(\lambda \odot x) \oplus (\mu \odot y)$ is also in C [9, 11]. The tropical convex hull of points $\{x_1, \dots, x_p\}$ is defined as:

$$\text{cone}_{\max}(X) = \left\{ \bigoplus_{i=1}^{p} \lambda_i \odot x_i : \lambda_i \in \mathbb{T} \} \right\} \tag{6}$$

The tropical projection $P_X(y)$ of a point y onto this convex hull is:

$$P_X(y) = \max\{z \in \text{cone}_{\max}(X) : z \leq y\} \tag{7}$$

SG. This is the projection on a cone, not on a convex set, perhaps we need to speak only of cones (modulo passing to the projective space), then this is ok. For a convex set, the max may be taken over a nonempty set

Tropical convexity generalizes the idea of conventional convexity to the max-plus setting. A conical tropical convex hull contains all tropical linear combinations of vectors. Moreover, the projection finds a "closest" point in the conical convex hull [2].

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These projections will play a central role in our classification framework. We will use them to build class-specific operators that characterize the separability of data.

3 Spectral Framework for Tropical SVMs

Having established the basics of tropical geometry, we now develop our spectral approach to classification. The key insight is connecting the separability of data classes to the spectral properties of a specially constructed operator.

Shapley Operators and Their Spectral Theory. A Shapley operator $T : \mathbb{R}^d \to \mathbb{R}^d$ satisfies two fundamental properties [16]:

1. *Monotonicity:* If $x \leq y$ coordinatewise, then $T(x) \leq T(y)$ coordinatewise
2. *Additive homogeneity:* For any constant $\alpha \in \mathbb{R}$, $T(\alpha + x) = \alpha + T(x)$, where $\alpha + x$ denotes the vector obtained by adding α to every entry of x.

It admits a unique continuous extension $\mathbb{T}^d \to \mathbb{T}^d$, also denoted by T [5]. Then, the spectral radius of T is defined as:

$$\rho(T) = \max \{\lambda \in \mathbb{R} : \exists u \neq -\infty \text{ with } T(u) = \lambda + u\}, \tag{8}$$

where $-\infty \in \mathbb{T}^n$ denotes the all $-\infty$ vector – the "zero" vector in max-plus algebra. The maximum is always achieved.

Equivalently, $\rho(T)$ is the smallest value of $\lambda \in \mathbb{R}$ for which there exists a vector $u \in \mathbb{R}^d$ satisfying $T(u) \leq \lambda + u$ [23, 1, 4].

Constructing the Classification Operator. Consider a classification problem with K classes, each represented by a set of points $X^1, \dots, X^K \subset \mathbb{T}^d$. We define an operator T^k for each class k by taking the diagonal-free variant of the tropical projection onto the convex hull of points in that class:

$$T^k(x) = P^{\text{DF}}_{X^k}(x)$$

This diagonal-free variant is defined in Section 4.

We then combine these operators into a single classification operator T defined coordinatewise as:

$$T(x)_i = \max\nolimits_2\{T^1(x)_i, \dots, T^K(x)_i\}$$

where $\max_2$ denotes the second-largest value among the outputs. For binary classification ($K = 2$), this simplifies to $T(x) = \min\{T^1(x), T^2(x)\}$.

Main Theorem: Spectral Separability. Our central result establishes the connection between separability and the spectral radius:

Theorem 1 (Spectral Separability Criterion)**.** *Let $X^1, \ldots, X^K \subset \mathbb{T}^d$ be labeled point sets and let T be the classification operator defined above. Then:*

(1) Separability Criterion: The data are separable by a hyperplane if and only if $\rho(T) < 0$.

(2) Margin Optimality: In the separable case, the maximum achievable margin equals $-\rho(T)$.

(3) Soft-Margin Interpretation: For binary classification with overlapping data, $\rho(T)$ is positive and quantifies the minimal perturbation needed to achieve separability.

Moreover, both $\rho(T)$ and an associated apex vector a satisfying $T(a) = \rho(T) + a$ can be computed in pseudo-polynomial time using mean-payoff game algorithms (see Section 4).

A complete proof is provided in Appendix A, where we will see that some of these results remain with more general assumptions. It is organized as follows: first, Lemma 2 states that for tropical projections (or their diagonal-free counterpart), $-\rho(T)$ is a lower bound on the margin. Then, Lemma 3 gives a construction for a separating hyperplane of margin $-\rho(T)$ in the separable case. Finally, Lemma 4 handles the binary overlapping case.

Benefits of the Spectral Approach. Our framework advances the tropical SVM foundation established by Gärtner and Jaggi [12]. Their work showed that tropical SVM could be formulated as finding a point of maximum margin within a tropical polytope defined by sector constraints, but required exhaustive exploration of sector assignments—leading to exponential complexity.

In contrast, our spectral characterization offers a complete theoretical understanding of when data are tropically separable, an exact formula for the optimal margin, an efficient algorithm to find the optimal classifier without combinatorial exploration and a natural interpretation for non-separable cases.

Limitations. The primary limitation of our spectral approach is its sensitivity to outliers. Since convex hulls encompass all points in each class, a single misplaced point can significantly alter the classification boundary and potentially render previously separable data inseparable.

Additionally, because our method is formulated as a spectral criterion rather than an optimization problem, it's challenging to incorporate relaxation mechanisms that would handle misclassified points in a controlled way. Traditional SVMs use slack variables to allow soft margins and tolerate some misclassifications, but our current approach doesn't have a direct equivalent. This suggests an important direction for future research.

4 Algorithm and Implementation

Having established the theoretical foundation, we now present our algorithm for tropical SVM based on the spectral criterion. The key insight is that we can compute the spectral radius and optimal hyperplane efficiently without exploring all possible sector assignments.

Efficient Computation of Projections. The first step is to compute the projections efficiently. The projection $P_X(y)$ of a point y onto the tropical convex hull of a set $X \subset \mathbb{T}^d$ can be computed using:

$$P_X(y)_i = \max_{1 \leq j \leq p} \left\{ X_{ij} + \min_{1 \leq k \leq d} (-X_{kj} + y_k) \right\}$$

where p is the number of points in X and X_{ij} is the i-th coordinate of the j-th point.

This formulation can be viewed as a mean-payoff game [3] and computed in $\mathcal{O}(pd)$ time—linear in both the number of points and dimensions. There are two players, "Max" and "Min" (the maximizer and the minimizer), who alternate their actions. Starting on node i, Player Max chooses to move to node j, and receives X_{ij} from Player Min. Similarily, Player Min in turn chooses a node k and has to pay $-X_{kj}$ to Player Max.

This enables us to define diagonal-free "projections" [14]:

$$P_X^{\mathrm{DF}}(y)_i = \max_{1\le j\le p} \left\{ X_{ij} + \min_{k\ne i}(-X_{kj} + y_k) \right\}. \tag{9}$$

In this modified game, the opponent is prevented from replying eye-for-an-eye.

Computing the Spectral Radius with Krasnoselskii-Mann Iterations. To compute the spectral radius $\rho(T)$ and a corresponding eigenvector a, we apply a Krasnoselskii-Mann iteration scheme [7], outlined in Algorithm 1.

Algorithm 1 Krasnoselskii–Mann Iteration for Tropical SVM

Input: Shapley operator T, convergence threshold $\varepsilon > 0$
Initialize: $x^{(0)} \in \mathbb{R}^d$ (typically set to $\mathbf{1}_d$), $\lambda^{(0)} = 0$
for $k = 0, 1, 2, \ldots$ until convergence **do**
  $z^{(k+1)} \leftarrow \frac{1}{2}\big(x^{(k)} + T(x^{(k)})\big)$
  $x^{(k+1)} \leftarrow z^{(k+1)} - \max\big(z^{(k+1)}\big)\, \mathbf{1}_d$
  $\lambda^{(k+1)} \leftarrow 2\max\big(z^{(k+1)}\big) - \max\big(x^{(k)}\big)$
  if $\left\|T(x^{(k)}) - x^{(k)}\right\|_H \le \varepsilon$ **then**
    break
  end if
end for
Return: $\rho(T) \approx \lambda^{(\text{final})}$, $a \approx x^{(\text{final})} - \frac{1}{d}\sum_{i=1}^{d} x_i^{(\text{final})} \cdot \mathbf{1}_d$

Recent results by Allamigeon et al. [6] show that this iteration achieves an ε-approximation in $O(1/\varepsilon^2)$ iterations. By avoiding the combinatorial exploration of sector assignments, our method runs in pseudo-polynomial time—specifically $O\big(\frac{nd}{\varepsilon^2}\big)$ for n data points in d dimensions.

[Samuel] true source?

The Complete Tropical SVM Algorithm. Algorithm 2 presents our complete procedure for tropical SVM classification.

Algorithm 2 Tropical SVM

Input: Labeled point sets $X^1, \ldots, X^K \subset \mathbb{T}^d$
Construct Operators:
  For each class k, compute the tropical projection operator $T^k(x) = P_{X^k}(x)$
  Combine via $T(x)_i = \max_2\{T^1(x)_i, \ldots, T^K(x)_i\}$ for each coordinate i
Compute Eigenpair:
  Run Krasnoselskii-Mann iterations (Algorithm 1) to obtain $\rho(T)$ and apex a
Assign Sectors to Classes:
if $\rho(T) < 0$ {Separable case} **then**
  Assign coordinate i to class k if $T^k(a)_i > \rho(T) + a_i$
else
  Use a heuristic (e.g., majority vote) for non-separable data
end if
Output: Apex a, margin $-\rho(T)$ (if separable), and sector assignments for classification

Empirical Performance Analysis. We validated the computational complexity of our algorithm through empirical testing (detailed in Appendix B). Our implementation achieves tractable performance on standard benchmark datasets, with performance scaling linearly with both dataset size and feature dimension as predicted by our theoretical analysis.

For the first time, we demonstrate that tropical SVMs can be practically computed with pseudo-polynomial guarantees, removing the exponential barrier that limited previous approaches. While specialized classical SVM libraries remain faster on conventional tasks, our approach enables new applications where piecewise-linear decision boundaries offer advantages.

Future optimizations could include kernel methods, sparse representations for large datasets, and parallel implementations to further improve performance on high-dimensional problems.

5 Tropical Polynomials for Enhanced Expressivity

While hyperplanes provide effective classification boundaries, we can achieve even greater expressivity by extending our framework to tropical polynomials. These polynomials create more flexible decision boundaries while maintaining theoretical guarantees.

Tropical Polynomial Kernel. A tropical polynomial in $\mathbb{T}^d$ takes the form:

$$f(x) = \max_{\alpha \in A}(c_\alpha + \langle \alpha, x \rangle) \tag{10}$$

where $A \subset \mathbb{Z}^d$ is a finite set of integer vectors (monomials), $c_\alpha \in \mathbb{R}$ are coefficients, and $\langle \alpha, x \rangle = \alpha_1 x_1 + \cdots + \alpha_d x_d$.

To fit such polynomials, we use a feature map $\Phi_A : \mathbb{T}^d \to \mathbb{T}^{|A|}$ defined by:

$$[\Phi_A(x)]_\alpha = \langle \alpha, x \rangle \tag{11}$$

This feature map transforms the original data into a higher-dimensional space where each coordinate corresponds to a monomial term. A hyperplane in this feature space corresponds to a polynomial in the original space.

Strategic Monomial Selection. The choice of monomial set A critically affects both the expressivity and computational complexity of the resulting classifier. We explore two complementary approaches:

1. **Homogeneous Monomials:** We use all monomials of degree s, defined as $A_s = \{\alpha \in \mathbb{N}^d : \sum_i \alpha_i = s\}$. The number of such monomials is $\binom{s+d-1}{s}$, which grows polynomially with s for fixed dimension d.
2. **Adaptive Selection:** We sample pairs of points from different classes and construct monomials corresponding to the slopes of separating lines between them. This approach focuses computational resources on the most discriminative monomials.

The degree parameter s controls the trade-off between expressivity and overfitting. Higher degrees create more flexible boundaries but may overfit the training data. Cross-validation can guide this selection process.

Importantly, only a sparse subset of monomials typically becomes active in the final classifier, making the approach computationally efficient even with many candidate monomials.

Classification with Polynomials. To perform classification with polynomials:

1. Map the original data to the feature space: $\Phi_A(X^k) = \{\Phi_A(x) : x \in X^k\}$ for each class k.
2. Apply the tropical SVM algorithm in this feature space to find apex $a \in \mathbb{T}^{|A|}$ and spectral radius $\rho(T)$.
3. The classifier in the original space is the polynomial:

$$f_a(x) = \max_{\alpha \in A}(-a_\alpha + \langle \alpha, x \rangle) \tag{12}$$

4. Classify new points by identifying which sector of $f_a(x)$ they fall into.

Margin Guarantees. When extending to polynomial classifiers, we maintain theoretical guarantees on the margin. Specifically, when $\rho(T) < 0$ (indicating separability in the feature space), the data are separable in the original space with a margin of at least $-\rho(T)/\|\Phi_A\|_{\text{op},\infty}$, where $\|\Phi_A\|_{\text{op},\infty}$ is the operator norm of the feature map.

For homogeneous degree-s monomials, this simplifies to $-\rho(T)/s$, meaning the margin scales inversely with the polynomial degree. This gives a principled way to balance expressivity with generalization through the choice of polynomial degree.

This approach maintains the core theoretical guarantees of the hyperplane formulation while substantially increasing model flexibility, as illustrated in Figures 2 and 3 on the following pages.

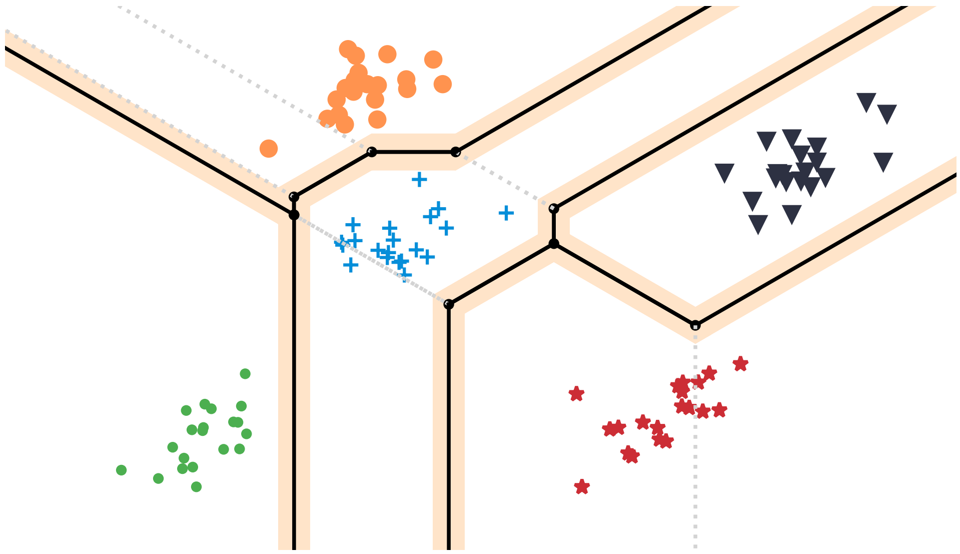

Figure 2: Multi-class classification using a cubic polynomial classifier (degree-3 monomials). Each color represents a different class, and the boundaries show where the dominant monomial changes. Note how the polynomial naturally separates the five clusters with piecewise-linear boundaries. The light orange band around the decision boundaries indicates the margin, which maintains a guaranteed lower bound of $-\rho(T)/3$ due to our theoretical results.

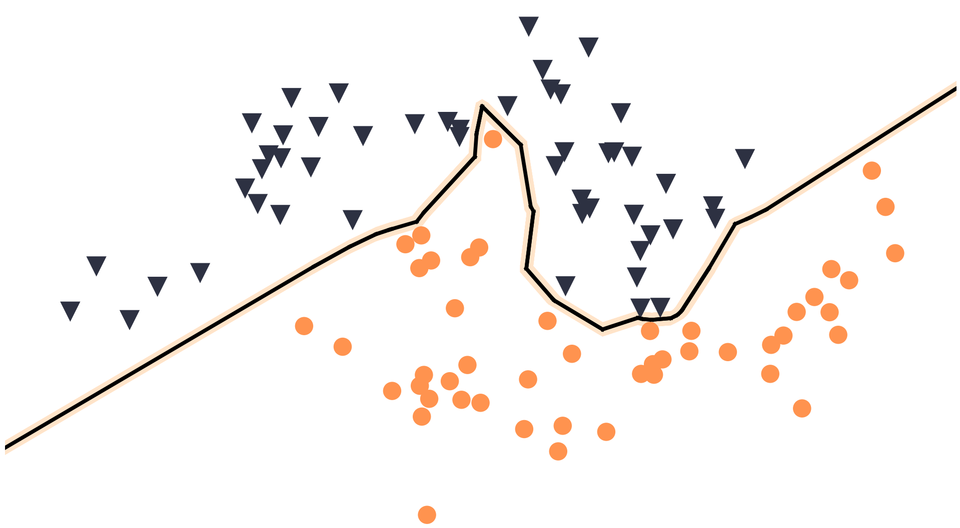

Figure 3: Visualization of a polynomial classifier using the adaptive monomial selection strategy described in Section 5. Rather than using all possible monomials, this approach selects terms based on pairs of points from different classes, focusing computational resources on the most discriminative features. The resulting boundary adapts closely to the data's structure while maintaining margin guarantees.

6 Connection to Classical SVMs

Tropical hyperplanes can be seen as limits of classical hyperplanes under logarithmic scaling, a process known as Maslov dequantization [29]. This connection suggests a naive approach to tropical SVM: raise the data to a power $\beta > 0$, apply a classical SVM in the transformed space to maximize the margin, then map the result back via a logarithm. As β tends to infinity, the resulting decision boundary converges to a tropical hyperplane.

However, this method suffers from severe numerical instability for large β and fails to guarantee a good margin in the original space. The limiting hyperplane is typically suboptimal compared to the true classifier obtained through our spectral approach, as illustrated in Figure 4.

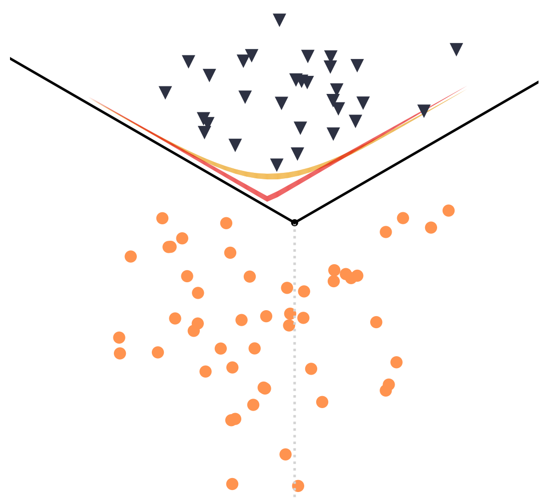

Figure 4: Comparison of hyperplanes obtained through Maslov dequantization and our spectral approach. The limiting hyperplane (red) from dequantization is suboptimal compared to the spectral classifier (black).

Interestingly, the connection could be exploited in the reverse direction: tropical SVMs may offer efficient, interpretable warm-starts for classical methods, especially in high-dimensional or large-scale settings.

7 Discussion and Future work

We addressed a key computational barrier limiting tropical SVM's practical application in machine learning. By reformulating tropical classification through spectral theory and mean-payoff games, we reduced complexity from exponential to pseudo-polynomial, making tropical SVMs feasible for real-world applications. Our framework provides natural multi-class capabilities, theoretical margin guarantees, and interpretable piecewise-linear decision boundaries.

Our work opens several promising avenues for further investigation.

[Samuel] Future work: relationship between trop poly & ReLU? Hybrid approaches between tropical & classical SVMs? Generalization bounds for tropical poly wrt degree? Addressing sentivity to outliers? Parallel, efficient algorithms?

Acknowledgements

TBD

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A Proof of Theorem 1 (Spectral Separability)

We provide a complete proof of our main result on spectral separability.

For any Shapley operator T, we define its fixed-point set as $\mathcal{S}(T) = \{x \in \mathbb{T}^d : x \leq T(x)\}$. An important property is that any tropically convex set V can be written as $V = S(P_V)$. Both P and P^{DF} have identical fixed-point sets and can serve as the class operators T^k in our framework. We first characterize the maximal possible value for the margin.

Lemma 2 (Upper bound on margin)**.** *Let T^k be tropical projections on finite point clouds or their diagonal-free equivalents, and T defined as the $\max_2$ of the T^k, as in Equation 3. If the hyperplane $\mathcal{H}_a$ separates $X^1, \ldots, X^K$ with a margin of at least $\gamma > 0$, then $T(a) \leq -\gamma + a$. Therefore, if we have an eigenpair $(\rho(T), a)$ of T, necessarily $\gamma \leq -\rho(T)$.*

Proof. Consider two different classes k and ℓ. Let I^k denote the set of coordinates assigned to class k. For any point $x \in X^k$, the margin condition implies

$$d_H(x, S^\ell) = \max_j (x_j - a_j) - \max_{i \in I^\ell} (x_i - a_i) \geq \gamma. \tag{13}$$

This can be rewritten as: for all $i \in I^\ell$,

$$x_i - a_i \leq \max_j (x_j - a_j) - \gamma. \tag{14}$$

Note that $\max_j (x_j - a_j) = \max_{j \neq i} (x_j - a_j)$ since $\gamma > 0$.

Taking the maximum over all $x \in X^k$, we get

$$\max_{x \in X^k} (x_i - a_i) \leq \max_{x \in X^k} \left(\max_{j \neq i} (x_j - a_j) \right) - \gamma. \tag{15}$$

By the definition of T^k, this implies

$$T^k(a)_i \leq -\gamma + a_i. \tag{16}$$

Since this holds for all $i \in I^\ell$ and all classes $k \neq \ell$, we have $T(a)_i \leq -\gamma + a_i$ for all i, which gives us $T(a) \leq -\gamma + a$. Finally, if $T(a) = \rho(T) + a$, then $\rho(T) \leq -\gamma$. □

Lemma 3 (Max-margin hyperplane)**.** *Let T^k be Shapley operators, and T defined as before. If $\rho(T) < 0$, then there exists a hyperplane $\mathcal{H}_a$ that separates the $\mathcal{S}(T^k)$ with a margin of at least $-\rho(T)$.*

Proof. Since $\rho(T) < 0$, there exists $a \in \mathbb{R}^d$ such that $T(a) = \rho(T) + a$. In Algorithm 2, we defined the sectors as

$$I^k = \left\{ i : T^k(a)_i > \rho(T) + a_i \right\}. \tag{17}$$

First, we show these sectors are disjoint. If $i \in I^k \cap I^\ell$ for $k \neq \ell$, then $T^k(a)_i > \rho(T) + a_i$ and $T^\ell(a)_i > \rho(T) + a_i$. But then the second-largest value among $\{T^1(a)_i, \ldots, T^K(a)_i\}$ would exceed $\rho(T) + a_i$, contradicting $T(a)_i = \rho(T) + a_i$.

Next, we show that each point belongs to its assigned sector. For $x \in \mathcal{S}(T^k)$ and $i \notin I^k$, we have

$$x_i \leq T^k(x)_i = (T^k(x) - T^k(a))_i + T^k(a)_i. \tag{18}$$

Since T^k is non-expansive, $(T^k(x) - T^k(a))_i \leq \max_j (x_j - a_j)$. And since $i \notin I^k$, we have $T^k(a)_i \leq \rho(T) + a_i$. Combining these,

$$x_i - a_i \leq \max_j (x_j - a_j) + \rho(T). \tag{19}$$

With $\rho(T) < 0$, this implies $x_i - a_i < \max_j (x_j - a_j)$, meaning i cannot be the $\arg\max$ of $x - a$. Therefore, the $\arg\max$ must lie in I^k, placing x in its correct sector.

Finally, for the margin, consider $x \in \mathcal{S}(T^k)$ and sector S^ℓ with $\ell \neq k$. The distance is

$$d_H(x, S^\ell) = \max_j (x_j - a_j) - \max_{i \in I^\ell} (x_i - a_i). \tag{20}$$

Using the inequalities derived above, we get $d_H(x, S^\ell) \geq -\rho(T)$. $\square$

In the binary case where data overlaps, we have $T = \min(T^1, T^2)$, hence $\mathcal{S}(T)$ characterizes the intersection of convex hulls of both point clouds. As shown in [3], this implies that $\rho(T)$ equals the inner radius of this intersection, measuring the extent of overlap. In the non-separable case, the apex a given by our Algorithm lies exactly at the center of the inner ball of $\mathcal{S}(T^1) \cap \mathcal{S}(T^2)$.

In the multi-class case, the corresponding Shapley operator T can be equivalently expressed as

$$T = \max_{1 \leq k < l \leq n} \min(T^k, T^l). \tag{21}$$

Intuitively, this can be thought of as representing the union of pairwise intersections between data classes, although the $\max$ operator does not strictly correspond to a union of fixed-point sets. This formulation provides an intuition on why our operator effectively characterizes separability across multiple classes.

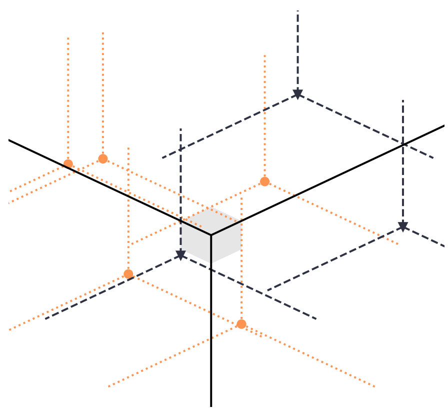


Figure 5: Non-separable case: $\rho(T) \geq 0$, indicating class overlap. The spectral radius quantifies the minimum perturbation required to achieve separability, and the inner radius of the convex hulls' intersection.

Lemma 4 (Overlap interpretation)**.** *Let T^k be the diagonal-free projections. For binary classification with $\rho(T) \geq 0$, there exists a perturbation of the point sets X^1 and X^2, with each point moved by at most $\rho(T)$ in the tropical metric, such that the tropical convex hulls of the perturbed sets have empty intersection.*

Proof. Let a be the eigenvector corresponding to $\rho(T)$, i.e., $T(a) = \rho(T) + a$. Let $\mathcal{H}_a$ be the tropical hyperplane with apex a. We construct a perturbation by projecting onto $\mathcal{H}_a$ any point whose distance to $\mathcal{H}_a$ is less than $\rho(T)$.

For a point $x \in \mathbb{T}^d$, its distance to $\mathcal{H}_a$ is

$$d_H(x, \mathcal{H}_a) = \max_i (x_i - a_i) - \max_2 (x_i - a_i), \tag{22}$$

where $\max_2$ denotes the second-largest value.

For each point $x_j \in X^1 \cup X^2$, we define a perturbed point $\tilde{x}_j$ as follows:

- If $d_H(x_j, \mathcal{H}_a) \geq \rho(T)$, then $\tilde{x}_j = x_j$ (no perturbation)
- If $d_H(x_j, \mathcal{H}_a) < \rho(T)$, let $s = \arg\max_i (x_{ji} - a_i)$ be the sector of x_j. We set:

$$\tilde{x}_{ji} = \begin{cases} x_{ji} - d_H(x_j, \mathcal{H}_a) & \text{if } i = s \\ x_{ji} & \text{otherwise} \end{cases} \tag{23}$$

This projection ensures that $\tilde{x}_j$ lies exactly on $\mathcal{H}_a$.

Therefore, for any x^c in the tropical convex hull of class $c \in \{1, 2\}$,

$$[x^c]_i \leq T^c(x^c)_i = \underbrace{(T^c(x^c) - T^c(a))_i}_{\leq \max_{k \neq i}(x^c_k - a_k)} + T^c(a)_i. \tag{24}$$

Fix a coordinate i. If $x \in X^1 \cup X^2$ is in sector $s \neq i$, and reaches it second argmax at coordinate t, then for $k \neq s$, by definition, $(\tilde{x}_j - a)_k \leq (x_j - a)_t \leq (\tilde{x}_j - a)_s$, hence $\tilde{x}_{jk} - \max_{k \neq i}(\tilde{x}_{jk} - a_k) \leq a_i$. Otherwise, x_j is in the i-th sector and $\max_{k \neq i}(\tilde{x}_{jk} - a_k) = x_{jt} - a_t$, thus

$$\tilde{x}_{ji} - \max_{k \neq i}(\tilde{x}_j - a)_k = (\tilde{x}_j - a)_i - (\tilde{x}_j - a)_t + a_i \geq a_i,$$

with equality if $d_H(x_j, \mathcal{H}_a) \leq \rho(T)$.

Suppose by symmetry that $T(a)_i = T^1(a)_i = \rho(T) + a_i$. We also have $T^2(a) \geq \rho(T) + a_i$. Then, using the proof of Theorem 22 in [3], there exists witness points $x_{j_1} \in X^1$ and $x_{j_2} \in X^2$ in sector i , with x_{j_1} being at distance $\rho(T)$ from $\mathcal{H}_a$ and x_{j_2} at distance greater than $\rho(T)$. Therefore, $\tilde{x}_{j_1 i} - \max_{k \neq i}(\tilde{x}_{j_1 k} - a_k) = a_i$ and $\tilde{x}_{j_2 i} - \max_{k \neq i}(\tilde{x}_{j_2 k} - a_k) \geq a_i$. Moreover, for any j such that $x_j \in X^1$ is in sector i, Equation 24 gives $d_H(x_j, \mathcal{H}_a) \leq \rho(T)$.

Let $\tilde{T}^1$ and $\tilde{T}^2$ be the diagonal-free projections over transformed projections, and $\tilde{T} = \min(\tilde{T}^1, \tilde{T}^2)$. We have just shown that $\tilde{T}(a)_i = \tilde{T}^1(a)_i = a_i$, and finally $\tilde{T}(a) = a$. □

We summarize the proof of the Main Theorem, which holds when T^k are diagonal-free projections:

Proof of Theorem 1. For part 1 (Separability Criterion): If we have an eigenpair $(\rho(T), a)$ such that the data are separable with margin $\gamma > 0$, then by Lemma 2, $\rho(T) \leq -\gamma < 0$. Conversely, if $\rho(T) < 0$, then by Lemma 3, the data are separable.

For part 2 (Margin Optimality): Lemma 2 shows that no hyperplane can achieve a margin larger than $-\rho(T)$, while Lemma 3 provides a construction achieving exactly this margin.

For part 3 (Soft-Margin Interpretation): Lemma 4 shows that in the binary case, $\rho(T)$ is positive and characterizes the overlap between the tropical convex hulls. □

B Empirical Evaluation

To validate our theoretical complexity analysis, we evaluated both standard tropical SVM and an enhanced tropical polynomial implementation (with $k = 4$ sampling points per class) against scikit-learn's LinearSVC on benchmark datasets [24, 8]. All experiments used 5-fold cross-validation with standardized features. We compare training times for a fixed convergence threshold. All experiments were conducted on a MacBook Air M2 with 16GB RAM using NumPy [15].

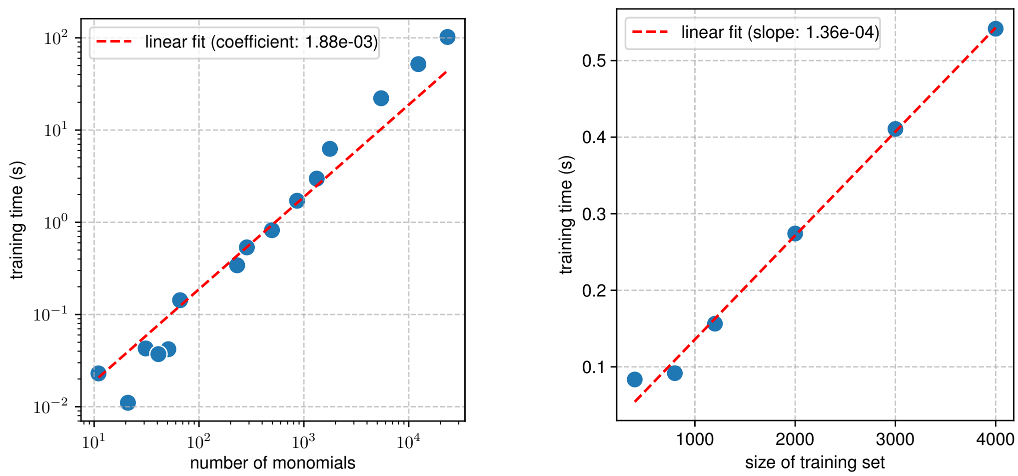
Dataset	**#C**	**#S**	**Accuracy (%) / Training Time (s) / #KM Iter**							
			Tropical SVM			**Tropical Poly**			**Linear SVC**	
			Acc	Time	#KM	Acc	Time	#KM	Acc	Time
Iris	3	150	66.7	0.0006	11.2	94.0	0.102	108.6	92.7	0.0004
Wine	3	178	74.2	0.0019	25.0	93.3	0.115	113.6	97.8	0.0006
Breast Cancer	2	569	82.1	0.0081	32.6	91.7	0.596	432.6	96.7	0.0008
Waveform	3	5000	49.9	0.0756	23.0	64.4	9.330	319.8	86.7	0.0246

Table 1: Performance comparison across benchmark datasets (5-fold cross-validation). Accuracy standard deviations range from 1.7% to 9.1%. #C: number of classes; #S: number of samples; #KM: average number of Krasnoselskii-Mann iterations. Chosen convergence threshold ε is 10^{-3} times the the characteristic variation scale of the considered data.

The standard tropical SVM implementation exhibits training times close to LinearSVC on smaller datasets, demonstrating practical computational feasibility. The tropical polynomial variant requires additional computation but substantially improves accuracy, approaching LinearSVC on datasets like Iris. Tropical SVM indeed fails to seprarate these datasets and thus performs poorly, while polynomials make the data separable.

Notably, the spectral radius values closely align with our theoretical predictions: datasets yielding negative spectral radius values (Wine: -7.53, Breast Cancer: -1.76, Iris: -0.97) exhibited the highest tropical polynomial accuracy, while the non-separable Waveform dataset (spectral radius: 0.87) proved more challenging.

As expected, the number of monomials increases from $d+1$ in standard tropical SVM to $O(d^k)$ in the polynomial variant, explaining the observed computation-accuracy tradeoff. While specialized classical SVM implementations remain faster, our results conclusively demonstrate that tropical SVMs can be practically computed with pseudo-polynomial guarantees.



(a) Training time vs. number of monomials

(b) Training time vs. size of training set

Figure 6: Performance scaling analysis of tropical SVM on MNIST data

Figure 6 reveals that the tropical SVM algorithm exhibits linear scaling behavior both with respect to the number of effective monomials and the training set size. We used a Python script that systematically varies PCA dimensions, polynomial degrees, and sample sizes on MNIST data [10].

Fix algorithm. Regen figures and tables. Add KM iterations vs. $1/\varepsilon$.

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