A SHORT NOTE ON NON-DIAGONALIZABLE VERTEX-PRIMITIVE DIGRAPHS

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ABSTRACT. In the 1980's, P. J. Cameron asked whether the adjacency matrix of any arc-transitive digraph is diagonalizable. Babai gave a negative answer to this question and posed an open problem regarding the existence of vertex-primitive digraphs whose adjacency matrices are non-diagonalizable. Recently, Li, Xia, Zhou and Zhu constructed an infinite family of such digraphs and gave some finite examples which are not Cayley digraphs. Among these finite examples was a digraph arising from the action of $\mathrm{PSL}_2(17)$ on 2-subsets of $\mathrm{PG}_1(17)$. In this note, we show that the smallest vertex-primitive digraphs whose adjacency matrices are non-diagonalizable arise from the action of $\mathrm{PSL}_2(17)$ on 2-subsets of $\mathrm{PG}_1(17)$. We also show that the automorphism group of a non-diagonalizable vertex-primitive digraph is either of almost simple type, diagonal type or product type.

1. INTRODUCTION

All groups and (di)-graphs considered in this paper are finite. A ***digraph*** is a pair $X = (V,E)$, where V is a non-empty set and E is a relation on V which is disjoint from the diagonal relation $\{(v,v) : v \in V\}$. As usual, if the relation E is symmetric (i.e., $(u,v) \in E$ implies $(v,u) \in E$), then we simply call $X = (V,E)$ a graph. The ***adjacency matrix*** of the digraph $X = (V,E)$ is the 01-matrix $A(X)$ indexed in its rows and columns by vertices in V, and for any $u,v \in V$, $A(X)_{uv} = 1$ if $(u,v) \in E$ and 0, otherwise. Whenever $X = (V,E)$ is a graph, then its adjacency matrix $A(X)$ is symmetric and thus diagonalizable over the reals. This is however not always the case for any digraph (for example, a digraph whose adjacency matrix is a Jordan block of a nilpotent matrix). We will call the digraph $X = (V,E)$ ***diagonalizable*** if its adjacency matrix is diagonalizable over $\mathbb{C}$, otherwise it is called ***non-diagonalizable***.

One of the main objectives of spectral graph theory is to study digraphs using the linear algebraic properties of matrices associated to them. In the case of graphs, for instance, one can further ask for the structures of graphs whose adjacency matrix or Laplacian matrix can be diagonalized using matrices with nice properties such as Hadamard matrices [4]. For (non-symmetric) digraphs however, even the fact of being diagonalizable is not guaranteed. One can construct many examples of (non-symmetric) digraphs that are diagonalizable, and some that are non-diagonalizable. However, when more conditions such as regularity or symmetry are imposed on the structure of the digraph, then non-diagonalizable digraphs become rare. The purpose of this paper is to shed a light on some important properties regarding the diagonalizability of vertex-primitive digraphs using some well-known results in the theory of association schemes.

A digraph $X = (V,E)$ is called ***regular*** if there exists a positive integer d, called the ***valency*** of X, such that

$$d = |\{v \in V : (u,v) \in E\}| = |\{v \in V : (v,u) \in E\}|,$$

for all $u \in V$. A digraph $X = (V,E)$ is called ***arc-transitive*** if its automorphism group $\mathrm{Aut}(X)$ acts transitively on the set of arcs E. The digraph $X = (V,E)$ is called ***vertex-primitive*** if $\mathrm{Aut}(X)$ in its

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action on V is primitive, that is, the only $\mathrm{Aut}(X)$-invariant partitions of V are those with one part or where each part is a vertex.

In [5], Cameron asked whether all arc-transitive digraphs are diagonalizable. In [8], Godsil showed that there are non-diagonalizable vertex-transitive digraphs. Improving this result, Babai gave a negative answer to Cameron's question in [1]. The following problem was also posed in [1, Problem 2.4].

Problem 1.1. *Construct vertex-primitive digraphs that are not diagonalizable.*

Recently, Li et al. [10] constructed an infinite family of digraphs answering Problem 1.1. Their construction boils down to finding a non-diagonalizable vertex-primitive digraph and using a graph product to obtain an infinite family. In addition, they asked a question [10, Question 5.2] regarding the smallest vertex-primitive digraphs (in terms of the number of vertices) that are not diagonalizable. They also conjecture the following.

Conjecture 1.2. [10] *Every arc-transitive vertex-primitive digraph is diagonalizable.*

The main objectives of this note is to map out the classes of primitive groups from which one would expect to find more non-diagonalizable vertex-primitive digraphs and to determine the smallest such digraphs. The results presented in this note follow from a well-known result in the theory of ***association schemes***. The set of all orbitals of a transitive group forms what is known as an ***orbital scheme***. These combinatorial objects are well studied and some of their algebraic properties are also well understood. For instance, the adjacency matrices of the orbital corresponding to a permutation group $G \leq \mathrm{Sym}(\Omega)$ commute if and only if the permutation character $\mathbf{1}_{G_\omega}^G$ of the transitive group $G \leq \mathrm{Sym}(\Omega)$ is multiplicity-free, that is, its irreducible constituents have multiplicity equal to 1. Using the theory of commutative association, one can show the following (see Section 2 as well as [3]).

Lemma 1.3. *If $X = (V,E)$ is a vertex-transitive graph whose vertex-stabilizer is multiplicity-free, then it is diagonalizable.*

Using this fact, one can deduce that all vertex-primitive graphs of rank up to 4 are diagonalizable. The latter also holds for the rank 5 case since it is not hard to show that the point-stabilizer of a primitive group of rank 5 is also multiplicity-free. Hence, a necessary condition for a vertex-primitive digraph to be non-diagonalizable is for it to arise from some primitive group of rank at least 6. We would like to note here that surprisingly there are no known primitive groups of rank 6 whose point stabilizer is not multiplicity-free (see [11] for details).

Using the fact that any vertex-transitive digraph is a union of orbital digraphs in the orbital scheme arising from the full automorphism group, we are able to prove many corollaries of Lemma 1.3. In particular, we determine the smallest non-diagonalizable vertex-primitive digraphs, and we give some necessary conditions for a vertex-primitive digraph to be non-diagonalizable in terms of the O'Nan-Scott types of its automorphism group.

This note is organized as follows. In Section 2, we recall some results on commutative association schemes. In Section 3.1 and Section 3.2, we respectively determine the smallest non-diagonalizable vertex-primitive digraphs and the O'Nan-Scott types of non-diagonalizable vertex-primitive digraphs.

2. Association schemes

Let Ω be a finite set and $\mathcal{R} = \{O_1, O_2, \ldots, O_k\}$ a collection of relations on Ω. The pair $(\Omega, \mathcal{R})$ is called an ***association scheme*** if

(1) $\mathcal{R}$ is a partition of $\Omega \times \Omega$.
(2) $\{(\omega,\omega) : \omega \in \Omega\} \in \mathcal{R}$.
(3) If $O \in \mathcal{R}$, then $O^* = \{(\omega',\omega) : (\omega,\omega') \in O\} \in \mathcal{R}$.

(4) For any $i, j, k \in \{1, 2, \ldots, k\}$, there exists a positive integer p_{ij}^k such that, given $(\omega, \omega') \in O_k$,

$$p_{ij}^k := \left|\{\delta \in \Omega : (\omega, \delta) \in O_i \text{ and } (\delta, \omega') \in O_j\}\right|.$$

The numbers (p_{ij}^k) are called the intersection numbers of the association scheme $(\Omega, \mathcal{R})$. Moreover, if $p_{ij}^k = p_{ji}^k$ for any $i, j, k \in \{1, 2, \ldots, k\}$, then $(\Omega, \mathcal{R})$ is called ***commutative***. It is worth noting that for any $i \in \{1, 2, \ldots, k\}$, the pair (Ω, O_i) determines a regular digraph. For any $O \in \mathcal{R}$, the digraph (Ω, O) is referred as the digraph corresponding to the relation O. If O is symmetric (i.e., $(\omega, \omega') \in O$ implies that $(\omega', \omega) \in O$), then (Ω, O) is an undirected graph. The ***Bose-Mesner algebra*** or the ***adjacency algebra*** corresponding to the association scheme $(\Omega, \mathcal{R})$ is the complex matrix algebra generated by the adjacency matrices of the digraphs of its relations.

An important class of association schemes arises from the orbitals (i.e., orbits on pairs) of a transitive permutation group[1]. Indeed, if $G \leq \mathrm{Sym}(\Omega)$ is a transitive group and $\mathcal{R}$ is the collection of all orbitals of G, then $(\Omega, \mathcal{R})$ is an association scheme. Any association scheme arising in this way from a transitive group is called ***Schurian.*** The digraphs corresponding to the classes in a Schurian association schemes are its ***orbital digraphs***.

Obviously, certain structures of the Schurian association scheme depend on the transitive group it arises from. Henceforth, we let $G \leq \mathrm{Sym}(\Omega)$ be a transitive group and $(\Omega, \mathcal{R})$ be the corresponding association scheme, where $\mathcal{R}$ is the collection of all orbitals of G, and recall that a point stabilizer of G is a subgroup of the form $G_\omega = \{g \in G : \omega^g = \omega\}$, where $\omega \in \Omega$. Then, $(\Omega, \mathcal{R})$ is a commutative association scheme if and only if G_ω is a ***multiplicity-free subgroup*** of G (see [3]); that is, if $\mathbf{1}_{G_\omega}^G$ is the permutation character of $G \leq \mathrm{Sym}(\Omega)$, then any irreducible constituent of $\mathbf{1}_{G_\omega}^G$ has multiplicity 1. In the case where G_ω is a multiplicity-free subgroup of G, the eigenvalues of the digraphs in the association scheme can be determined using the irreducible characters of the group G. Further, the eigenspaces of these graphs are direct sums of certain irreducible $\mathbb{C}G$-modules appearing the decomposition of the permutation module of the action of G (see [3] for details on these).

Assume henceforth that $G_\omega \leq G$ is multiplicity-free. Then, the association scheme $(\Omega, \mathcal{R})$ arising from the transitive group $G \leq \mathrm{Sym}(\Omega)$, where $\mathcal{R}$ is the collection of all orbitals of G, is a commutative association scheme. In this case, if all digraphs in the Schurian association scheme are undirected (i.e., they are all graphs), then the adjacency matrices of the orbital graphs in $(\Omega, \mathcal{R})$ are simultaneously diagonalizable. Since any vertex-transitive digraph is a union of orbital digraphs of its full automorphism group, it must be diagonalizable. If there is at least one orbital digraph in $(\Omega, \mathcal{R})$ that is not undirected, then we lose the property of being symmetric, however, the adjacency matrices of the orbital digraphs remain simultaneously diagonalizable. This follows from the multiplicity-freeness of the point stabilizer. Indeed, as the association scheme $(\Omega, \mathcal{R})$ is commutative, the adjacency matrices of the orbital digraphs are normal matrices (i.e., they commute with their conjugate transpose) that are also pairwise commuting. Therefore, the adjacency matrix of any orbital digraph (or any matrix in the Bose-Mesner algebra) is diagonalizable over $\mathbb{C}$ through a common set of orthogonal eigenvectors, and the proof of the next lemma follows.

Lemma 2.1. *If $X = (V, E)$ is a vertex-transitive graph whose vertex-stabilizer is multiplicity-free, then it is diagonalizable.*

An immediate consequence of this is that any vertex-transitive digraph whose automorphism group is of rank at most 5 is diagonalizable. In the next two sections, we will give other consequences of the above-mentioned lemma.

[1]These are also known as homogeneous coherent configurations.

3. Some consequences of Lemma 1.3

3.1. **The smallest non-diagonalizable vertex-primitive graphs.** In [10], four examples of non-diagonalizable and non-Cayley vertex-primitive digraphs were given; they were obtained through computer search. Among these was a vertex-primitive digraph of size 153 arising from the action of $\mathrm{PSL}_2(17)$ on 2-subsets of the projective line $\mathrm{PG}_1(17)$. In the next theorem, we prove that this digraph is one of the two smallest non-diagonalizable vertex-primitive digraphs.

Theorem 3.1. *The smallest non-diagonalizable vertex-primitive digraphs arise from the action of* $\mathrm{PSL}_2(17)$ *on* 2*-subsets of* $\mathrm{PG}_1(17)$*, with point stabilizer isomorphic to* D_{16}*. In particular, there are two non-diagonalizable vertex-primtive digraphs on* 153 *vertices, up to isomorphism, both of which are of degree* 24.

Proof. By Lemma 2.1, we can eliminate the vertex-primitive digraphs arising from primitive groups whose vertex-stabilizers are multiplicity-free. Using `Sagemath` [12], the only primitive groups of degree less or equal to 153 whose point stabilizers are not multiplicity-free are one of the pairs (G, G_ω) given by

(1) $(\mathrm{PSL}_2(11), \mathrm{D}_{12})$ with rank 9,
(2) $(\mathrm{PSL}_2(13), \mathrm{D}_{14})$ with rank 9,
(3) $(\mathrm{PSL}_2(13), \mathrm{D}_{12})$ with rank 12,
(4) $(\mathrm{PSL}_2(13), \mathrm{Alt}(4))$ with rank 11,
(5) $(\mathrm{PSL}_2(17), \mathrm{D}_{18})$ with rank 12,
(6) $(\mathrm{PSL}_2(17), \mathrm{D}_{16})$ with rank 15.

As any vertex-primitive digraph whose automorphism group contains $G \leq \mathrm{Sym}(\Omega)$ must be a union of orbital digraphs of G, it is enough to check all the possible unions of orbital digraphs giving a non-symmetric digraph. We can show using `Sagemath` that all vertex-primitive graphs whose automorphism groups and vertex stabilizers are one of the pair (1)-(5) are diagonalizable.

Now, consider the pair in (6). This primitive group is $\mathrm{PSL}_2(17)$ in its action on the 2-subsets of the projective line $\mathrm{PG}_1(17)$. The point stabilizer of this action is isomorphic to the dihedral group D_{16}. There exists a non-symmetric orbital O and two symmetric orbitals O_1 and O_2 such that the digraphs whose set of arcs are respectively $O \cup O_1$ and $O \cup O_2$ both have valency 24. The minimal polynomials of these graphs are respectively

$$(x-24)(x-4)(x+8)x^2(x^3-3x^2-9x+3)$$

and

$$(x-24)(x-6)x^2(x^2+4x+8)(x^3+3x^2-36x+51).$$

Consequently, these digraphs are both non-diagonalizable and vertex primitive. Using `Sagemath`, it can be verified that any non-diagonalizable vertex-primitive digraph with 153 vertices is isomorphic to one of these graphs. □

3.2. **O'Nan-Scott types of non-diagonalizable vertex-primitive digrahps.** Let $G \leq \mathrm{Sym}(\Omega)$ be a primitive group. Recall that the ***socle*** $\mathrm{Soc}(G)$ of G is the subgroup of G generated all its minimal normal subgroups. The O'Nan-Scott theorem [6, Theorem 4.1.A] gives a comprehensive description of the socle of primitive groups and their embeddings. In particular, primitive groups can be categorized as one of the following types (see [6] for the definitions):

- if $\mathrm{Soc}(G)$ is regular, then G is either of
  - ***affine type***: $\mathrm{Soc}(G)$ is an elementary abelian p-group and $G \leq \mathrm{AGL}_n(p)$ whose point stabilizer is irreducible,
  - ***twisted wreath product type***,
- if $\mathrm{Soc}(G)$ is not regular, then $\mathrm{Soc}(G)$ is non-abelian and G is one of the following types

– ***almost simple type:*** $\mathrm{Soc}(G)$ is a non-abelian simple group and G is contained in the automorphism of $\mathrm{Soc}(G)$,
– ***diagonal type***,
– ***product type***.

In [2], Baddeley extensively studied and categorized the multiplicity-free subgroups of primitive permutation groups. Using this result, another corollary of Lemma 1.3 is the following.

Theorem 3.2. *Let $X = (V, E)$ be a vertex-primitive graph and let $G = \mathrm{Aut}(X)$. If X is non-diagonalizable, then one of the following occurs*

(1) G is of almost simple type,
(2) G is of diagonal-type and $\mathrm{Soc}(G) = T^k$, where T is a non-abelian simple group, and $k \geq 4$ or $k = 3$ and $T \notin \{\mathrm{Alt}(5), \mathrm{Sym}(5)\}$,
(3) G is of product-type not arising from some transitive subgroup H such that $K^n \leq G \leq H \wr \mathrm{Sym}(n)$ and $K \leq H$ is a multiplicity-free.

An immediate consequence of Theorem 3.2 is that vertex-primitive digraphs whose automorphism groups admit a regular socle are diagonalizable. We also note that the infinite family constructed in [10] is of product type. Therefore, it is of interest to ask whether there is an infinite family of non-diagonalizable vertex-primitive digraphs with the property that the socle of its automorphism group is almost simple. In [10], the authors gave other examples of non-diagonalizable vertex-primitive digraphs arising from the action of $\mathrm{PSL}_2(25)$ on 2-subsets of $\mathrm{PG}_1(25)$. We have verified the following through `Sagemath` by searching equitable partitions from subgroups, whose corresponding quotient matrices are not diagonalizable.

Example 3.3. *The action of $\mathrm{PSL}_2(49)$ on 2-subsets of $\mathrm{PG}_1(49)$ gives rise to a number of non-diagonalizable vertex-primitive digraphs. These graphs are a union of two orbital digraphs, exactly one of which is undirected. In particular, there exists a non-diagonalizable vertex-primitive digraph whose minimal polynomial admits a factor equal to $(x-6)^2$.*

Example 3.4. *The action of $\mathrm{PSL}_2(81)$ on 2-subsets of $\mathrm{PG}_1(81)$ gives rise to a non-diagonalizable vertex-primitive digraph whose minimal polynomial is $(x-6)^2$.*

We are not able to determine whether there is an infinite family of non-diagonalizable vertex-primitive digraphs arising from almost simple groups at the moment. Example 3.3 and Example 3.4 both arise from a simple linear group of degree 2, but we have not been able to show whether these are part of a larger family. Therefore, we ask the following question.

Question 3.5. *When does the group $\mathrm{PSL}_2(q)$ acting on 2-subsets of $\mathrm{PG}_1(q)$ give rise to a non-diagonalizable vertex-primitive digraph?*

4. A REMARK ABOUT CONJECTURE 1.2

Let $X = (V, E)$ be a vertex-primitive digraph whose automorphism group is G. It is well-known that X is arc-transitive if and only if it is an orbital digraph of G. Hence, in order to prove Conjecture 1.2, one needs to show that all orbital digraphs arising from a primitive group are diagonalizable. Of course, one should only consider the primitive groups whose point-stabilizers are not multiplicity-free.

Since the use of association schemes is one of the few methods to globally study the eigenvalues of orbital graphs, Conjecture 1.2 seems to be intractable so far. We will show that even in the case where X is a Cayley digraph, the problem remains highly non-trivial.

From now on, we let $G \leq \mathrm{Sym}(\Omega)$ be a primitive group such that G_ω is not multiplicity-free, and assume that $R \leq G$ is a regular subgroup. Since G contains a regular subgroup, all of its orbital digraphs are Cayley digraphs. Fix $\omega \in \Omega$. By regularity of R, for any $\omega' \in \Omega$, there is a unique $r \in R$ such that $\omega' = \omega^r$. Therefore, there is a one-to-one correspondence between Ω and R which identifies ω' with $r \in R$ such that $\omega^r = \omega'$. The action of G on R can be understood as follows. For any $g \in G$ and $r \in R$, r^g is the unique element of R such that $\omega^{r^g} = \omega^{rg}$. Using the action of G_ω on R, it is not hard to determine the connection sets of the orbital digraphs.

It is well known that the orbitals of G are in one-to-one correspondence with the suborbits of G (i.e., the orbits of G_ω). Assume that G has rank $d+1$, and let $O_0(\omega), O_1(\omega), \ldots, O_d(\omega)$ be its suborbits of G. Let $S_0, S_1, \ldots, S_d$ be the orbits of G_ω on R. Without loss of generality, we may assume that for any $i \in \{0,1,\ldots,d\}$, there exists $r_i \in S_i$ such that $\omega^{r_i} \in O_i(\omega)$. Then, it is not hard to show that for any $i \in \{0,1,\ldots,d\}$, we have

$$S_i = \{r \in R : \omega^r \in O_i(\omega)\}.$$

Using [3, Theorem 6.1], the orbital digraphs of G are the Cayley digraphs $\mathsf{Cay}(R, S_i)$, for all $i \in \{0,1,\ldots,d\}$. Moreover, the orbital scheme of $G \leq \mathrm{Sym}(\Omega)$ is isomorphic to a ***Schur ring*** (see [3, Theorem 6.1]).

Let $\Phi : G \to \mathrm{GL}_{|G|}(\mathbb{C})$ be the regular representation of the group G. For each $i \in \{1,2,\ldots,d\}$, by [9, Proposition 7.1] the adjacency matrix of $\mathsf{Cay}(R, S_i)$ is similar to the matrix

$$\Phi(S_i) = \bigoplus_{\mathfrak{X} \in \mathrm{IRR}(G)} \mathrm{trace}(\mathfrak{X}(1))\mathfrak{X}(S_i)$$

where $\mathrm{IRR}(G)$ is a complete set of non-equivalent irreducible representations of G. Consequently, $\mathsf{Cay}(R, S_i)$ is diagonalizable if and only if $\Phi(S_i)$ is diagonalizable, or equivalently $\mathfrak{X}(S_i)$ for any $\mathfrak{X} \in \mathrm{IRR}(G)$. Since S_i is an orbit of G_ω on R, there exists $h_1, h_2, \ldots, h_{t-1} \in G_\omega$, where $t = |S_i|$, such that

$$S_i = \{s, s^{h_1}, s^{h_2}, \ldots, s^{h_{t-1}}\}.$$

In addition, as G is a finite group, $\Phi(g)$ is diagonalizable for any $g \in G$. Hence, $\Phi(S_i)$ is a sum of diagonalizable matrices. Consequently, to prove Conjecture 1.2 for Cayley digraphs, one needs to study when a sum of matrices from the regular representation is diagonalizable. We finish this section by providing some observations.

Observation 4.1. *Note that $S_i \not\subset C_G(S_i)$, otherwise by primitivity of G, all of its orbital digraphs are strongly connected and so $\langle S_i \rangle = G$. Consequently, G is abelian and G_ω is multiplicity-free.*

Observation 4.2. *If every S_i is a union of conjugacy classes for $i \in \{1,2,\ldots,d\}$, then $\mathsf{Cay}(R, S_i)$ is diagonalizable. This is due to the fact that is a digraph in a commutative association scheme, namely the* ***conjugacy class scheme*** *of G (see [7] for details).*

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