Rough paths programming test

Tuesday 23rd October, 2018

We will work with 2-dimensional paths $X_t = \left(X_t^{(1)}, X_t^{(2)}\right) \in \mathbb{R}^2$, and use the notation $X_{a,b} := X_b - X_a$. For fixed times $t_0 < t_1 < \ldots < t_n$, and $(n+1)$ data points

$$X_{t_0}, X_{t_1}, \ldots, X_{t_n},$$

write code in Python to

(i) Plot the path X_t obtained from the piecewise linear interpolation of the data points.

(ii) Compute members of the level 2 signature of X_t, i.e. $S(X)^{1,1}_{t_i,t_j}, S(X)^{1,2}_{t_i,t_j}, S(X)^{2,1}_{t_i,t_j}$, or $S(X)^{2,2}_{t_i,t_j}$, for any $t_i < t_j$.

(iii) Apply your code to plot the path X_t and compute $S(X)^{1,2}_{t_0,t_8} - S(X)^{2,1}_{t_0,t_8}$ for the following data

t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
(1, 0)	(0.707, 0.707)	(0, 1)	(-0.707, 0.707)	(-1, 0)	(-0.707, -0.707)	(0, -1)	(0.707, -0.707)	(1, 0)

Hint: Use the following formulas

$$\int_{t_i}^{t_j} X^{(k)}_{t_i,r} \, \mathrm{d}X^{(l)}_r = \sum_{m=i}^{j-1} \int_{t_m}^{t_{m+1}} X^{(k)}_{t_i,r} \, \mathrm{d}X^{(l)}_r, \quad k, l \in \{1, 2\},$$

and

$$\begin{aligned} \int_{t_m}^{t_{m+1}} X^{(k)}_{t_i,r} \, \mathrm{d}X^{(l)}_r &= \int_{t_m}^{t_{m+1}} \left[X^{(k)}_{t_m} - X^{(k)}_{t_i} + \frac{r - t_m}{t_{m+1} - t_m} (X^{(k)}_{t_{m+1}} - X^{(k)}_{t_m}) \right] \left(\frac{X^{(l)}_{t_{m+1}} - X^{(l)}_{t_m}}{t_{m+1} - t_m} \right) \mathrm{d}r \\ &= \left(X^{(l)}_{t_{m+1}} - X^{(l)}_{t_m} \right) \left[\left(X^{(k)}_{t_m} - X^{(k)}_{t_i} \right) + \frac{1}{2} \left(X^{(k)}_{t_{m+1}} - X^{(k)}_{t_m} \right) \right]. \end{aligned}$$