Tutorial

MTH165

if $f(x) = x(x-2)^2$ in $[0,2]$ then By Roll's thm value of c is

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) None of these if $f(x) = e^x(\sin x - \cos x)$ on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ then By Roll's thm value of c is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) π (d) None of these

$f(x) = |x-1|$ in $[1,2]$

(a) $c=1$ (b) $c=0$ (c) $c=\frac{1}{2}$ (d) Not applicable if $f(x) = \sin x - \sin 2x$ on $[0, \pi]$.

then by Roll's thm value of c is

(a) $c = \frac{\pi}{4}$ (b) $c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$

(c) $c = \cos^{-1}\left(\frac{1 \pm \sqrt{32}}{7}\right)$ (d) Not applicable if $f(x) = \frac{1}{4x-1}$ in the interval $[1, 4]$

then value of c by MVT is

(a) $c = \frac{1+\sqrt{5}}{2}$ (b) $= \frac{1+\sqrt{45}}{2}$

(c) $c = \frac{1}{2}$ (d) Not applicable

The value 'c' of Lagrange's Mean value theorem when $f(x) = x(x-2)$ in $[1, 2]$ is

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) Not applicable

The expansion of $f(x) = \frac{1}{1+x}$ about Point $x=1$ is

(a) $\sum_{n=0}^{\infty} \frac{1}{2}(x-1)^n$

(b) $\sum_{n=0}^{\infty} (-1)^n \left(\frac{x-1}{2}\right)^n$

(c) $\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{2^{n+1}}$

(d) None of these

The Taylor's expansion of $f(x) = e^{\cos x}$ about $x = \frac{\pi}{2}$ is

(a) $1 - \left(x - \frac{\pi}{2}\right) + \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 - \frac{1}{8}\left(x - \frac{\pi}{2}\right)^4 + \cdots$

(b) $1 + \left(x - \frac{\pi}{2}\right) + \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4}\left(x - \frac{\pi}{2}\right)^4 + \cdots$

(c) $1 - \left(x - \frac{\pi}{2}\right) + \frac{1}{2}\left(x - \frac{\pi}{2}\right)^3 - \frac{1}{4}\left(x - \frac{\pi}{2}\right)^4 + \cdots$

(d) None of these

1. $e^{\sin x}$ about $a=0$

2. $f(x) = x^4 - 7x^3 + 5x^2 - 6x - 1$ about $a=2$

3. $\sin x$ about $a = \frac{\pi}{2}$

4. $\frac{1}{x+3}$ about $a = 1$

5. 3^x about $a = 1$ $\cancel{x = t}$

MTH165

Unit 3
Applications of Derivatives

L16-Roll’s Theorem and Mean Value Theorem

Revision

(1) The value of $\int_{-1}^{1} \frac{1+\sin x}{1+x^2}\,dx$ is

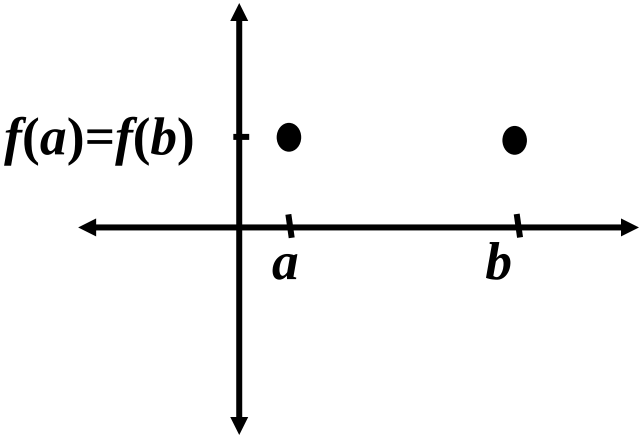
(a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $-\frac{\pi}{2}$

(2) The value of $\int_{0}^{1} \sqrt{\frac{1-x}{1+x}}\,dx$ is

(a) $\frac{\pi}{2}+1$ (b) $\frac{\pi}{2}-1$ (c) -1 (d) 1

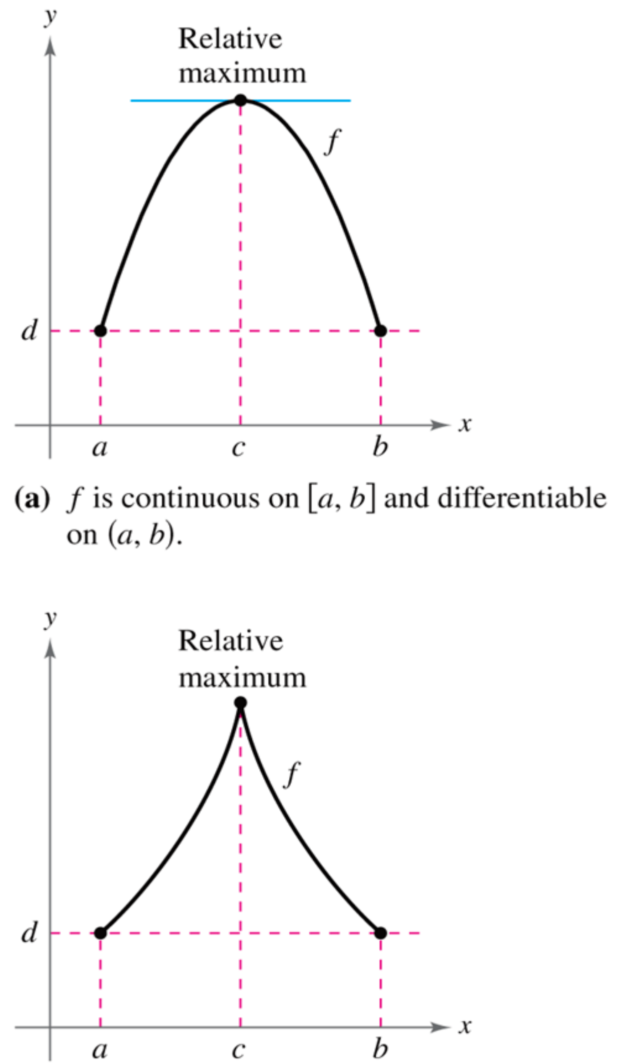
Rolle's Theorem

If you connect from $f\,(a)$ to $f\,(b)$ with a smooth curve, there will be at least one place where $f\,'(c) = 0$



Rolle's Theorem

Rolle's theorem is an important basic result about differentiable functions. Like many basic results in the calculus it seems very obvious. It just says that between any two points where the graph of the differentiable function *f* (*x*) cuts the horizontal line there must be a point where *f* ′(*x*) = 0. The following picture illustrates the theorem.



(a) f is continuous on $[a, b]$ and differentiable on (a, b).

(b) f is continuous on $[a, b]$.

Rolle's Theorem

If two points at the same height **are connected by a continuous, differentiable function, then there has to be** at least one **place between those two points where the derivative, or slope, is** zero**.**

Rolle's Theorem

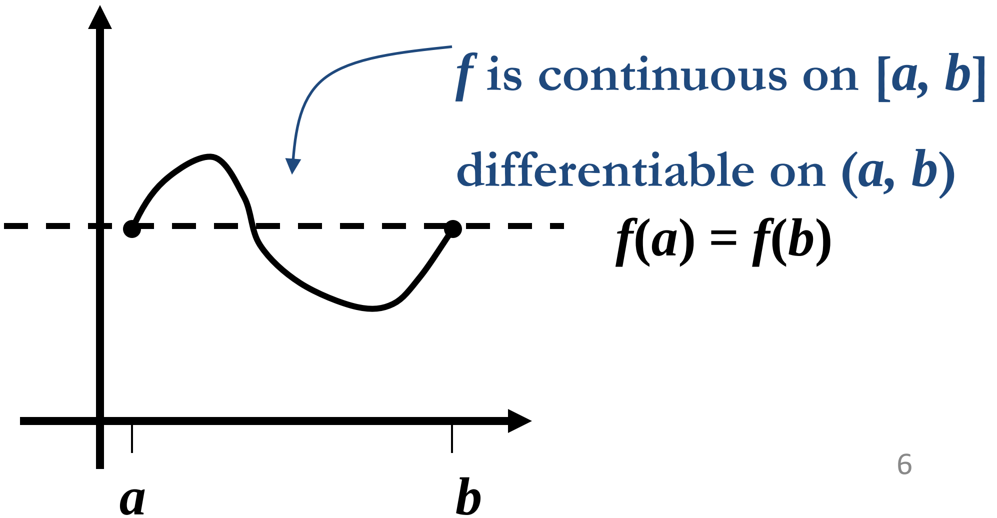
If **1) $f(x)$ is continuous on $[a, b]$,**

2) $f(x)$ is differentiable on (a, b), and

3) $f(a) = f(b)$

then there is at least one value of x on (a, b), call it c, such that

$f'(c) = 0$.



Example

<u>Example 1</u> $f(x) = x^4 - 2x^2$ on $[-2, 2]$

(*f* is continuous and differentiable)

$f(-2) = 8 = f(2)$

Since , then Rolle's Theorem applies…

$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 0$

then, $x = -1$, $x = 0$, and $x = 1$

Rolle’s Theorem

Does Rolle’s Theorem apply?

If not, why not?

If so, find the value of c.

Example 2 $f(x) = 4 - x^2$ $[-2, 2]$

Rolle's Theorem

Does Rolle's Theorem apply?

If not, why not?

If so, find the value of c.

<u>Example 3</u> $f(x) = x^3 - x$ $[-1, 1]$

Example

<u>Example 4</u> $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ on $[-1, 1]$

(Graph the function over the interval on your calculator)

continuous on [-1, 1]

<u>not</u> differentiable at 0

<u>not</u> differentiable on (-1, 1)

$f(-1) = 1 = f(1)$

Rolle's Theorem Does NOT apply since

Rolle's Theorem

Does Rolle's Theorem apply?

If not, why not?

If so, find the value of c.

Example 5 $$f(x)=\frac{x^2+4}{x^2} \qquad [-2, 2]$$

Note

When working with Rolle's make sure you

1. State $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b).
2. Show that $f(a) = f(b)$.
3. State that there exists at least one $x = c$ in (a, b) such that $f'(c) = 0$.

This theorem only guarantees the existence of an extrema in an open interval. It does not tell you how to find them or how many to expect. If YOU can not find such extrema, it does not mean that it can not be found. In most of cases, it is enough to know the existence of such extrema.

Mean Value Theorem- MVT

The Mean Value Theorem is one of the most important theoretical tools in Calculus. It states that if *f*(*x*) is defined and continuous on the interval [*a*,*b*] and differentiable on (*a*,*b*), then there is at least one number *c* in the interval (*a*,*b*) (that is *a*<*c*<*b*) such that

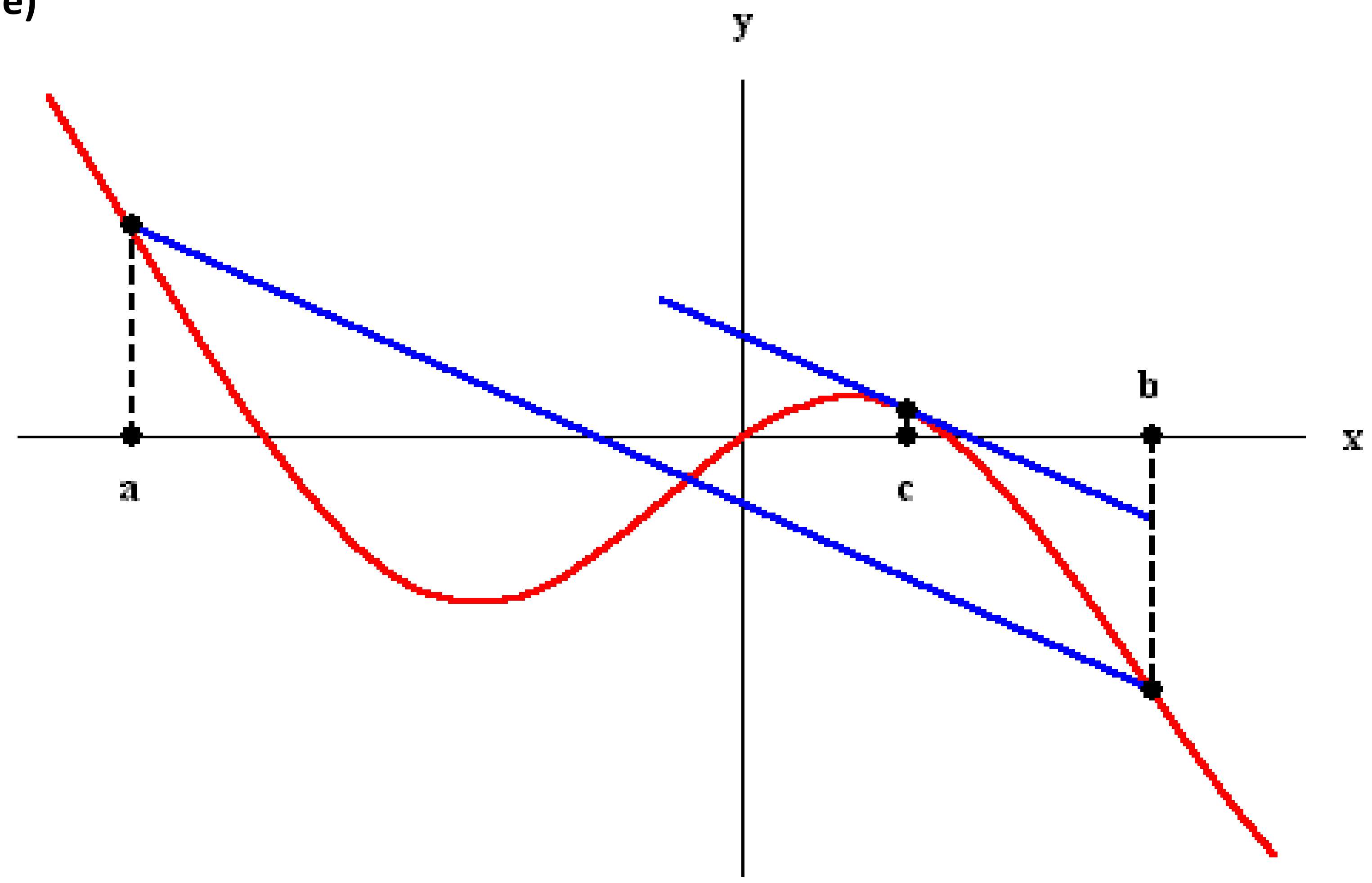
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In other words, there exists a point in the interval (*a*,*b*) which has a horizontal tangent. In fact, the Mean Value Theorem can be stated also in terms of slopes. Indeed, the number

$$\frac{f(b) - f(a)}{b - a}$$

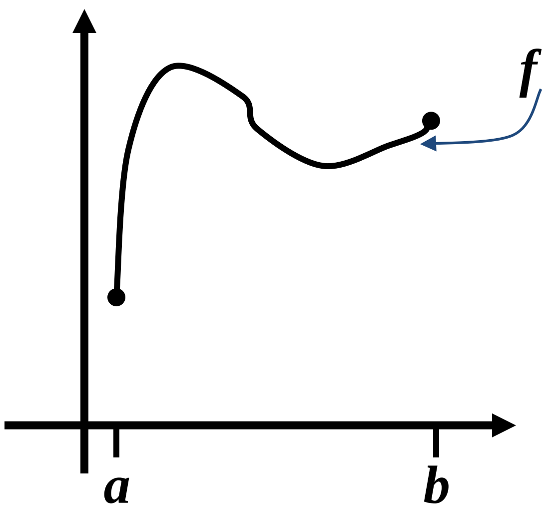
is the slope of the line passing through (*a*, *f*(*a*)) and (*b*, *f*(*b*)). So the conclusion of the Mean Value Theorem states that there exists a point such that the tangent line is parallel to the line passing through (*a*, *f*(*a*)) and (*b*, *f*(*b*)).

(see Picture)



The special case, when $f(a) = f(b)$ is known as <u>Rolle's Theorem</u>. In this case, we have $f\,'(c) = 0$.

Mean Value Theorem- MVT



If: ***f*** **is continuous on [*a*, *b*],**

differentiable on (*a*, *b*)

Then: there is a *c* in (*a*, *b*) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

if

(i) $f(x)$ is continuous in the closed interval $[a, a+h]$

(ii) $f(x)$ is diff. in the open interval $(a, a+h)$.

then there exist at least one number θ $(0<\theta<1)$ s.t

$$\boxed{f(a+h) = f(a) + h f'(a+\theta h)}$$

Verify Roll's th^m for $f(x) = \dfrac{\sin x}{e^x}$ in $(0, \pi)$

Sol (i) $f(x)$ is continuous as

$\sin x$ and e^x are continuous in $[0, \pi]$.

(ii) $f'(x) = \dfrac{e^x \cos x - e^x \sin x}{e^{2x}}$

$\Rightarrow f(x)$ is diff in $(0, \pi)$

(iii) $f(\pi) = 0 = f(0)$

$\therefore$ Roll's th^m is applicable

Example

Example 6 $f(x) = x^3 - x^2 - 2x$ on [-1,1]

(f is continuous and differentiable)

MVT applies

$$f'(x) = 3x^2 - 2x - 2$$

$$f'(c) = \frac{-2-0}{1-(-1)} = -1$$

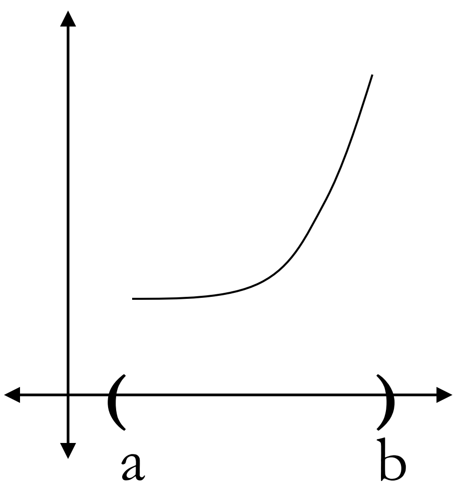
$$3c^2 - 2c - 2 = -1$$

$$(3c+1)(c-1) = 0$$

$$c = -\frac{1}{3}, \quad c = 1$$

Mean Value Theorem- MVT

Note:



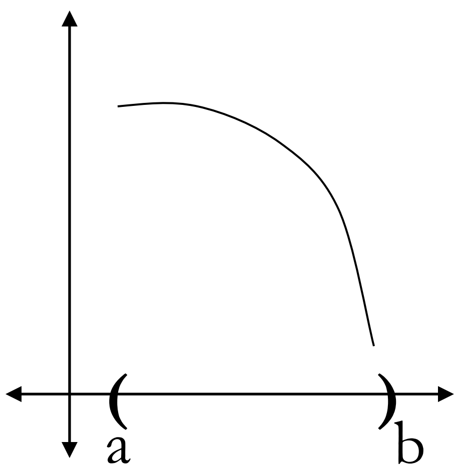
$$f'(x) > 0 \text{ on } (a,b) \quad \Rightarrow$$

$$f \text{ is increasing on } (a,b)$$

The graph of f is rising

Mean Value Theorem- MVT

Note:



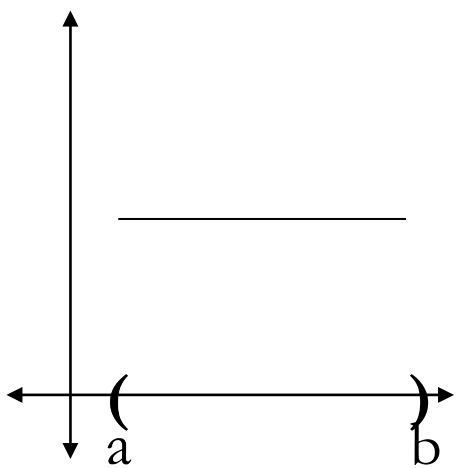
$$f'(x) < 0 \text{ on } (a,b) \quad \Rightarrow$$

$$f \text{ is decreasing on } (a,b)$$

The graph of f is falling

Mean Value Theorem- MVT

Note:



f is constant on (a,b)

The graph of f is level

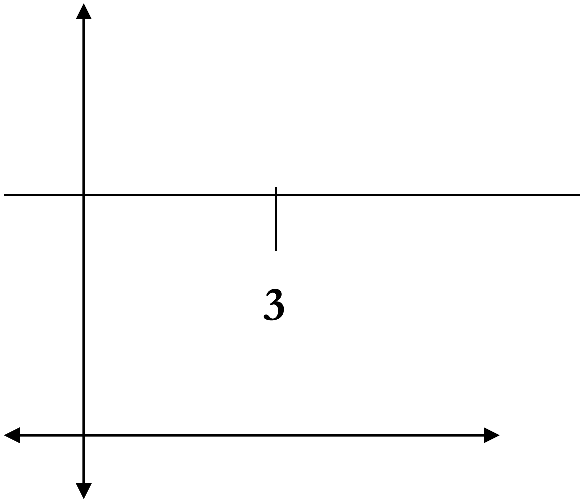
Example

Example 7 $f(x) = x^2 - 6x + 12$

$$f'(x) = 2x - 6$$

$$= 2(x-3)$$

$$= 0 \text{ iff } x = 3$$



Finding a Tangent Line

<u>Example 8</u> Find all values of c in the open interval (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

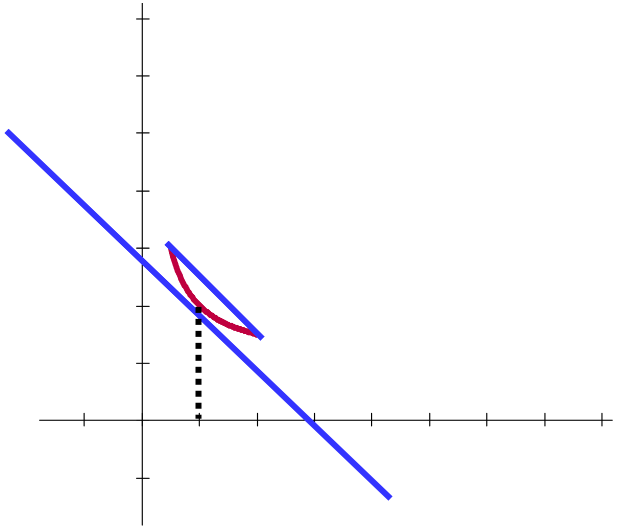
$$f(x) = \frac{x+1}{x},\ [\tfrac{1}{2}, 2]$$

$$f'(x) = \frac{d}{dx}\left(1 + \frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\frac{f(2) - f(1/2)}{2 - 1/2} = \frac{3/2 - 3}{3/2} = -1$$

$$f'(c) = -\frac{1}{c^2} = -1$$

$\boldsymbol{c = 1}$



$f(x) = \cos x$ in $\left[0, \frac{\pi}{2}\right]$

① $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$.

② $f'(x) = -\sin x$

$\Rightarrow f'(x)$ exist in $\left(0, \frac{\pi}{2}\right)$

then by MVT, we can find $c \in \left(0, \frac{\pi}{2}\right)$

s.t $\frac{f(b)-f(a)}{b-a} = f'(c) \Rightarrow \frac{f\left(\frac{\pi}{2}\right) - f(0)}{\pi/2 - 0} = -\sin c$

Application of MVT

<u>Example 9</u> When an object is removed from a furnace and placed in an environment with a constant temperature of 90^{o} F, its core temperature is 1500^{o} F. Five hours later the core temperature is 390^{o} F. Explain why there must exist a time in the interval when the temperature is decreasing at a rate of 222^{o} F per hour.

<u>Solution</u>

Let $g(t)$ be the temperature of the object.

Then $g(0) = 1500$, $g(5) = 390$

$$\text{Avg. Temp.} = \frac{g(5) - g(0)}{5 - 0} = \frac{390 - 1500}{5} = -222$$

By MVT, there exists a time $0 < t_o < 5$, such that $g'(t_o) = -222^{o}$ F

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Unit 3
Applications of Derivatives

L17- Taylor’s theorem and Maclaurin theorems

Revision

if $f(x) = x^2 - x - 12$ in $[-3, 4]$. then acc. to roll's th^m the value of $c \in (-3, 4)$ is

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) 0

Revision

If $f(x) = \sqrt{x-2}$ in $[2, 3]$, then according to Lagrange's Mean Value th^m, the point $c \in (2, 3)$ is

(a) 1 (b) $\frac{9}{2}$ (c) $\frac{9}{4}$ (d) None of these

Taylor and Maclaurin Series

We start by supposing that f is any function that can be represented by a power series

$$\boxed{1} \quad f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 + \cdots \quad | x - a | < \boldsymbol{R}$$

Let's try to determine what the coefficients c_n must be in terms of f.

To begin, notice that if we put $x = a$ in Equation 1, then all terms after the first one are 0 and we get

$$f(a) = c_0$$

Taylor and Maclaurin Series

We can differentiate the series in Equation 1 term by term:

$$\boxed{2} \quad f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4c_4(x - a)^3 + \cdots \qquad |x - a| < \boldsymbol{R}$$

and substitution of $x = a$ in Equation 2 gives

$$f'(a) = c_1$$

Now we differentiate both sides of Equation 2 and obtain

$$\boxed{3} \quad f''(x) = 2c_2 + 2 \cdot 3c_3(x - a) + 3 \cdot 4c_4(x - a)^2 + \cdots \qquad |x - a| < \boldsymbol{R}$$

Again we put $x = a$ in Equation 3. The result is

$$f''(a) = 2c_2$$

Taylor and Maclaurin Series

Let's apply the procedure one more time. Differentiation of the series in Equation 3 gives

$$\boxed{4}\quad f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x - a) + 3 \cdot 4 \cdot 5c_5(x - a)^2 + \cdots \quad |x - a| < \boldsymbol{R}$$

and substitution of $x = a$ in Equation 4 gives

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

By now you can see the pattern. If we continue to differentiate and substitute $x = a$, we obtain

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \cdots \cdot nc_n = n!c_n$$

Taylor and Maclaurin Series

Solving this equation for the *n*th coefficient c_n, we get

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This formula remains valid even for $n = 0$ if we adopt the conventions that $0! = 1$ and $f^{(0)} = f$. Thus we have proved the following theorem.

5 Theorem If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n \qquad |x - a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Taylor and Maclaurin Series

Substituting this formula for c_n back into the series, we see that *if f* has a power series expansion at *a*, then it must be of the following form.

$$\boxed{6} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

The series in Equation 6 is called the **Taylor series of the function *f* at *a*** (or **about *a*** or **centered at *a***).

Taylor and Maclaurin Series

For the special case *a* = 0 the Taylor series becomes

$$\boxed{7} \qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

Example 1

Find the Maclaurin series of the function $f(x) = e^{x}$.

Solution:

If $f(x) = e^x$, then $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ for all n. Therefore the Taylor series for f at 0 (that is, the Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Taylor and Maclaurin Series

Let’s investigate the more general question: Under what circumstances is a function equal to the sum of its Taylor series?

In other words, if *f* has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

As with any convergent series, this means that *f*(*x*) is the limit of the sequence of partial sums.

Taylor and Maclaurin Series

In the case of the Taylor series, the partial sums are

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x - a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Notice that T_n is a polynomial of degree n called the ***n*th-degree Taylor polynomial of *f* at *a***.

Taylor and Maclaurin Series

For instance, for the exponential function $f(x) = e^x$, the result of Example 1 shows that the Taylor polynomials at 0 (or Maclaurin polynomials) with $n = 1, 2,$ and 3 are

$$T_1(x) = 1 + x$$

$$T_2(x) = 1 + x + \frac{x^2}{2!}$$

$$T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

Taylor and Maclaurin Series

The graphs of the exponential function and these three Taylor polynomials are drawn in Figure 1.

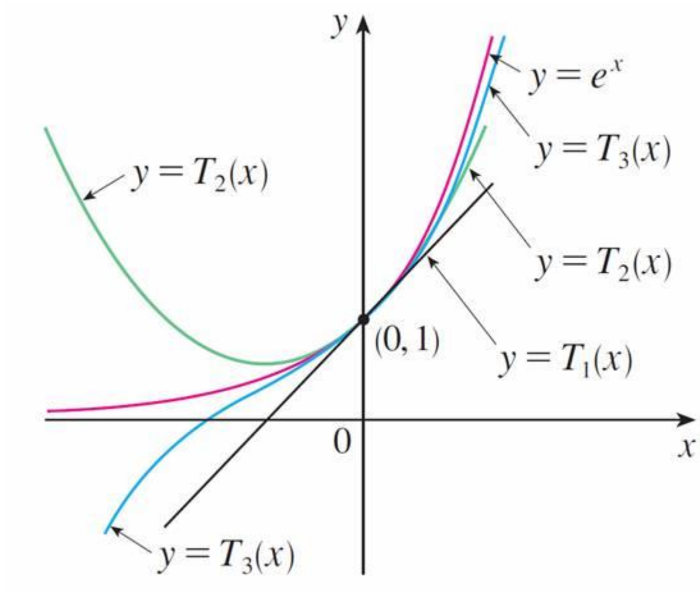

Figure 1

As n increases, $T_n(x)$ appears to approach e^x in Figure 1. This suggests that e^x is equal to the sum of its Taylor series.

Taylor and Maclaurin Series

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n \to \infty} T_n(x)$$

If we let

$$R_n(x) = f(x) - T_n(x) \qquad \text{so that} \qquad f(x) = T_n(x) + R_n(x)$$

then $R_n(x)$ is called the **remainder** of the Taylor series. If we can somehow show that $\lim_{n \to \infty} R_n(x) = 0$, then it follows that

$$\lim_{n \to \infty} T_n(x) = \lim_{n \to \infty} [f(x) - R_n(x)] = f(x) - \lim_{n \to \infty} R_n(x) = f(x)$$

Example 8

Find the Maclaurin series for $f(x) = (1 + x)^k$, where k is any real number.

Solution:

Arranging our work in columns, we have

$$
\begin{array}{ll}
f(x) = (1 + x)^k & f(0) = 1 \\
f'(x) = k(1 + x)^{k-1} & f'(0) = k \\
f''(x) = k(k-1)(1 + x)^{k-2} & f''(0) = k(k-1) \\
f'''(x) = k(k-1)(k-2)(1 + x)^{k-3} & f'''(0) = k(k-1)(k-2) \\
\vdots & \vdots \\
f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1 + x)^{k-n} & f^{(n)}(0) = k(k-1)\cdots(k-n+1)
\end{array}
$$

Example 8 – *Solution*

cont'd

Therefore the Maclaurin series of $f(x) = (1 + x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

This series is called the **binomial series**.

Notice that if k is a nonnegative integer, then the terms are eventually 0 and so the series is finite. For other values of k none of the terms is 0 and so we can try the Ratio Test.

Example 8 – *Solution*

cont'd

If its nth term is a_n, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!}\cdot\frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right|$$

$$= \frac{|k-n|}{n+1}|x| = \frac{\left|1-\dfrac{k}{n}\right|}{1+\dfrac{1}{n}}|x| \to |x| \qquad \text{as } n \to \infty$$

Thus, by the Ratio Test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$.

Taylor and Maclaurin Series

The traditional notation for the coefficients in the binomial series is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

and these numbers are called the **binomial coefficients**.

The following theorem states that $(1 + x)^k$ is equal to the sum of its Maclaurin series.

Taylor and Maclaurin Series

It is possible to prove this by showing that the remainder term $R_n(x)$ approaches 0, but that turns out to be quite difficult.

17 The Binomial Series If k is any real number and $|x| < 1$, then

$$(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Taylor and Maclaurin Series

We collect in the following table, for future reference, some important Maclaurin series that we have derived in this section and the preceding one.

$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$	$R = 1$
$(1+x)^k = \sum_{n=0}^{\infty} \dbinom{k}{n} x^n = 1 + kx + \dfrac{k(k-1)}{2!}x^2 + \dfrac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$R = 1$

Important Maclaurin Series and their Radii of Convergence

Table 1

The expansion of $f(x) = \frac{1}{1+x}$ about Point $x=1$ is

(a) $\sum_{n=0}^{\infty} \frac{1}{2}(x-1)^n$

(b) $\sum_{n=0}^{\infty} (-1)^n \left(\frac{x-1}{2}\right)^n$

(c) $\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{2^{n+1}}$

(d) None of these

Multiplication and Division of Power Series

Example 13

Find the first three nonzero terms in the Maclaurin series for (a) $e^x \sin x$ and (b) $\tan x$.

Solution:

(a) Using the Maclaurin series for e^x and $\sin x$ in Table 1, we have

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right)$$

Example 13 – *Solution*

cont'd

We multiply these expressions, collecting like terms just as for polynomials:

$$
\begin{array}{rccccccccc}
 & 1 + x & + & \frac{1}{2}x^2 & + & \frac{1}{6}x^3 & + & \cdots & & \\
\times & x & & & - & \frac{1}{6}x^3 & + & \cdots & & \\
\hline
 & x & + & x^2 & + & \frac{1}{2}x^3 & + & \frac{1}{6}x^4 & + & \cdots \\
+ & & & & - & \frac{1}{6}x^3 & - & \frac{1}{6}x^4 & - & \cdots \\
\hline
 & x & + & x^2 & + & \frac{1}{3}x^3 & + & \cdots & &
\end{array}
$$

Example 13 – *Solution*

cont'd

Thus $$e^x \sin x = x + x^2 + \tfrac{1}{3}x^3 + \cdots$$

(b) Using the Maclaurin series in Table 1, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}$$

Example 13 – *Solution*

cont'd

We use a procedure like long division:

$$
\begin{array}{r|l}
 & x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots \\
\hline
1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \cdots & x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \cdots \\
 & x - \tfrac{1}{2}x^3 + \tfrac{1}{24}x^5 - \cdots \\
\cline{2-2}
 & \qquad \tfrac{1}{3}x^3 - \tfrac{1}{30}x^5 + \cdots \\
 & \qquad \tfrac{1}{3}x^3 - \tfrac{1}{6}x^5 + \cdots \\
\cline{2-2}
 & \qquad\qquad \tfrac{2}{15}x^5 + \cdots
\end{array}
$$

Thus
$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots$$

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Unit 3
Applications of Derivatives

L17- Taylor's theorem and Maclaurin theorems

Taylor's theorem

① $f(x), f'(x), f''(x), ---, f^{n-1}(x)$ be continuous in $[a,b]$

② $f^{n}(x)$ should exist in (a,b) |

there exist at least one real number c s.t

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{\lfloor 2}f''(a) + --- + \frac{(b-a)^n}{\lfloor n} f^{n}(c) \quad -①$$

Then $\Rightarrow$ ① is called Taylor's series with Lagrange's form of remainder.

if $n=1$, this theorem becomes Lagrange's Mean value theorem

So, this theorem is also called Higher mean value theorem or Generalised Mean value theorem.

Another form

Put $b-a=h$ and $c=a+\theta h$, $0<\theta<1$

then $\Rightarrow$ (1) can be written as

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\cdots+\frac{h^n}{\lfloor n}f^n(a+\theta h) \quad \text{--- (2)}$$

Here $R_n=\dfrac{h^n}{\lfloor n}f^n(a+\theta h)$, $0<\theta<1$ is called Lagrange's form of Remainder.

Put $a=0$, $h=x$ in $=n$ ②, we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots + \frac{x^n}{\lfloor n} f^n(\theta x) \quad \text{—③}$$

where $0<\theta<1$

$=n$ ③ is called Maclaurin's theorem with Lag. form of Remainder

$$R_n = \frac{x^n}{\lfloor n} f^n(\theta x)$$

Example

if $f(x) = \log(1+x)$, $x > 0$

using Maclaurin's theorem, show that for $0 < \theta < 1$, $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$

By Mac. thm with Remainder R_3, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(\theta x)$$

$$f(x) = \log(1+x)$$

$$f'(x) = \frac{1}{1+x} \longrightarrow f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \longrightarrow f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \longrightarrow f'''(\theta x) = \frac{2}{(1+\theta x)^3}$$

Taylor's Series:

If $f(x+h)$ can be expanded as an infinite series, then

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x) + \cdots \infty$$

$$= \sum_{n=0}^{\infty} \frac{f^n(x)}{\lfloor n} h^n \qquad \text{——— (1)}$$

if $f(x)$ possesses all derivatives of all orders and remainder $R_n \to 0$ as $n \to \infty$ then the Taylor's theorem becomes the Taylor's Series (1).

Replacing x by a and h by $(x-a)$ in ①, we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor\underline{2}} f''(a) + \cdots \infty$$

$$= \sum_{n=0}^{\infty} \frac{f^n(a)}{\lfloor\underline{n}} (x-a)^n \quad \text{—— ②}$$

Taking $a=0$, we get Maclaurin's Series

$$f(x) = f(a) + x f'(a) + \frac{x^2}{\lfloor\underline{2}} f''(a) + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^n(a)}{\lfloor\underline{n}} x^n \quad \text{—— ③}$$

1. $e^{\sin x}$ about $a=0$
2. $f(x) = x^4 - 7x^3 + 5x^2 - 6x - 1$ about $a=2$
3. $\sin x$ about $a = \frac{\pi}{2}$
4. $\frac{1}{x+3}$ about $a=1$
5. 3^x about $a=1$ $\cancel{x = t}$

Taylor and Maclaurin Series

- It is possible to prove this by showing that the remainder term $R_n(x)$ approaches 0, but that turns out to be quite difficult.

$\boxed{17}$ **The Binomial Series** If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Taylor and Maclaurin Series

- We collect in the following table, for future reference, some important Maclaurin series that we have derived in this section and the preceding one.

$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$	$R = 1$
$(1+x)^k = \sum_{n=0}^{\infty} \dbinom{k}{n} x^n = 1 + kx + \dfrac{k(k-1)}{2!}x^2 + \dfrac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$R = 1$

Important Maclaurin Series and their Radii of Convergence

Table 1

EXAMPLE

The Taylor's expansion of $f(x) = \log x$ about $x=1$ is

$$\log(x-1) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

The expansion of $f(x) = \frac{1}{1+x}$ about Point $x=1$ is

(a) $\sum_{n=0}^{\infty} \frac{1}{2}(x-1)^n$

(b) $\sum_{n=0}^{\infty} (-1)^n \left(\frac{x-1}{2}\right)^n$

(c) $\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{2^{n+1}}$

(d) None of these

The Taylor's expansion of $f(x) = e^{\cos x}$ about $x = \frac{\pi}{2}$ is

(a) $1 - \left(x - \frac{\pi}{2}\right) + \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 - \frac{1}{8}\left(x - \frac{\pi}{2}\right)^4 + \cdots$

(b) $1 + \left(x - \frac{\pi}{2}\right) + \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4}\left(x - \frac{\pi}{2}\right)^4 + \cdots$

(c) $1 - \left(x - \frac{\pi}{2}\right) + \frac{1}{2}\left(x - \frac{\pi}{2}\right)^3 - \frac{1}{4}\left(x - \frac{\pi}{2}\right)^4 + \cdots$

(d) None of these

Multiplication and Division of Power Series

Example 13

- Find the first three nonzero terms in the Maclaurin series for (a) $e^x \sin x$ and (b) $\tan x$.
- Solution:
- (a) Using the Maclaurin series for e^x and $\sin x$ in Table 1,

  we have

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right)$$

Example 13 – *Solution*

cont'd

- We multiply these expressions, collecting like terms just as for polynomials:

$$
\begin{array}{rccccccccc}
 & 1 & + & x & + & \frac{1}{2}x^2 & + & \frac{1}{6}x^3 & + & \cdots \\
\times & & & x & & & - & \frac{1}{6}x^3 & + & \cdots \\
\hline
 & & & x & + & x^2 & + & \frac{1}{2}x^3 & + & \frac{1}{6}x^4 + \cdots \\
+ & & & & & & - & \frac{1}{6}x^3 & - & \frac{1}{6}x^4 - \cdots \\
\hline
 & & & x & + & x^2 & + & \frac{1}{3}x^3 & + & \cdots
\end{array}
$$

Example 13 – *Solution*

cont'd

- Thus

$$e^x \sin x = x + x^2 + \tfrac{1}{3}x^3 + \cdots$$

- (b) Using the Maclaurin series in Table 1, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}$$

Example 13 – *Solution*

cont'd

- We use a procedure like long division:

$$
\begin{array}{r|l}
 & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \\
\cline{2-2}
1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots \\
 & x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \cdots \\
\cline{2-2}
 & \qquad \frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots \\
 & \qquad \frac{1}{3}x^3 - \frac{1}{6}x^5 + \cdots \\
\cline{2-2}
 & \qquad\qquad \frac{2}{15}x^5 + \cdots
\end{array}
$$

-

- Thus

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

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Unit 3
Applications of Derivatives

L18-Indeterminant Forms

Revision

1. Expansion of function f(x) is?

a) $f(0) + {}^{x}/_{1!}\, f'(0) + {}^{x^2}/_{2!}\, f''(0) \ldots\ldots\ldots + {}^{x^n}/_{n!}\, f^n(0)$

b) $1 + {}^{x}/_{1!}\, f'(0) + {}^{x^2}/_{2!}\, f''(0) \ldots\ldots\ldots + {}^{x^n}/_{n!}\, f^n(0)$

c) $f(0) - {}^{x}/_{1!}\, f'(0) + {}^{x^2}/_{2!}\, f''(0) \ldots\ldots\ldots + (-1)^\wedge n\ {}^{x^n}/_{n!}\, f^n(0)$

d) $f(1) + {}^{x}/_{1!}\, f'(1) + {}^{x^2}/_{2!}\, f''(1) \ldots\ldots\ldots + {}^{x^n}/_{n!}\, f^n(1)$

Revision

The expansion of $f(x) = \cos x$ in powers of $\left(x - \frac{\pi}{2}\right)$ is

(a) $\left(x - \pi/2\right) + \frac{1}{\underline{|2}}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{\underline{|3}}\left(x - \frac{\pi}{2}\right)^2 + - - - - -$

(b) $-\left(x - \frac{\pi}{2}\right) + \frac{1}{\underline{|3}}\left(x - \frac{\pi}{2}\right)^3 - \frac{1}{\underline{|5}}\left(x - \frac{\pi}{2}\right)^5 + - - -$

(c) $1 - \frac{1}{\underline{|2}}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{\underline{|4}}\left(x - \frac{\pi}{2}\right)^4 - - - -$

(d) None of above

Indeterminate Forms

- What are indeterminate forms?
  - In calculus and other branches of mathematical analysis, limits involving an algebraic combination of functions in an independent variable may often be evaluated by replacing these functions by their limits.
  - If the expression obtained after this substitution does not give enough information to determine the original limit, it is said to take on an *indeterminate form.*

Types of Indeterminate forms

- There are seven types of indeterminate forms :

1. $0/0$
2. ∞/∞
3. $0 \times \infty$
4. $\infty - \infty$
5. 0^0
6. 1^∞
7. ∞^0

0/0 Form

- Limit of the form $\frac{f(x)}{g(x)}$, where $\lim f(x) = \lim g(x) = 0$ are called indeterminate form of the type 0/0.

Consider: $\lim\limits_{x\to 2} \frac{x^2-4}{x-2}$

If we try to evaluate this by direct substitution, we get: $\frac{0}{0}$

Zero divided by zero can not be evaluated, and is an example of **indeterminate form**.

In this case, we can evaluate this limit by factoring and canceling:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x\to 2}(x+2) = 4$$

Indeterminate form 0/0

Fig. 1: $y = \frac{x}{x}$

Fig. 2: $y = \frac{x^2}{x}$

Fig. 3: $y = \frac{\sin x}{x}$

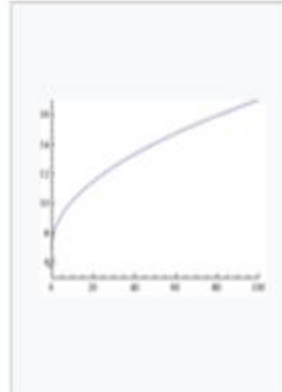


Fig. 4: $y = \frac{x - 49}{\sqrt{x} - 7}$

Fig. 5: $y = \frac{ax}{x}$ where $a = 2$

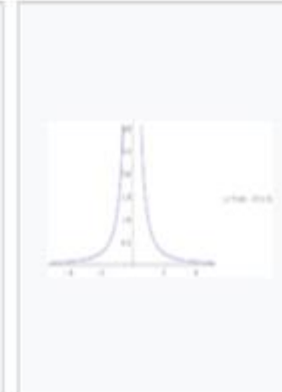


Fig. 6: $y = \frac{x}{x^3}$

The indeterminate form $0/0$ is particularly common in calculus, because it often arises in the evaluation of derivatives using their definition in terms of limit.

As mentioned above,

$$\lim_{x \to 0} \frac{x}{x} = 1, \qquad \text{(see fig. 1)}$$

while

$$\lim_{x \to 0} \frac{x^2}{x} = 0, \qquad \text{(see fig. 2)}$$

This is enough to show that $0/0$ is an indeterminate form. Other examples with this indeterminate form include

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1, \qquad \text{(see fig. 3)}$$

and

$$\lim_{x \to 49} \frac{x - 49}{\sqrt{x} - 7} = 14, \qquad \text{(see fig. 4)}$$

$$\lim_{x \to 0} \frac{ax}{x} = a. \qquad \text{(see fig. 5)}$$

The value ∞ can also be obtained (in the sense of divergence to infinity):

$$\lim_{x \to 0} \frac{x}{x^3} = \infty. \qquad \text{(see fig. 6)}$$

Indeterminate form 0^0

Main article: Zero to the power of zero

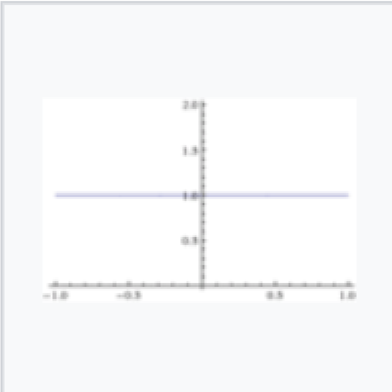


Fig. 7: $y = x^0$

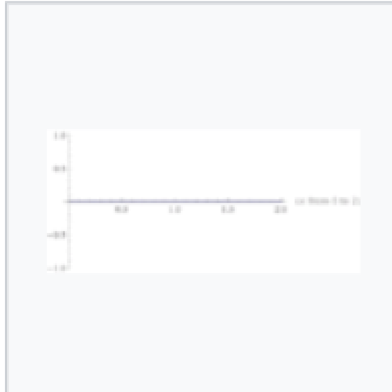


Fig. 8: $y = 0^x$

The following limits illustrate that the expression 0^0 is an indeterminate form:

$$\lim_{x \to 0^+} x^0 = 1, \qquad \text{(see fig. 7)}$$

$$\lim_{x \to 0^+} 0^x = 0. \qquad \text{(see fig. 8)}$$

Thus, in general, knowing that $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$ is not sufficient to evaluate the limit

$$\lim_{x \to c} f(x)^{g(x)}.$$

L' Hopital's Rule

- L'Hopital's rule is a general method for evaluating the indeterminate forms 0/0 and ∞/∞. This rule states that (under appropriate conditions)

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

where f' and g' are the derivatives of f and g.

- Note that this rule does *not* apply to expressions ∞/0, 1/0, and so on.
- These derivatives will allow one to perform algebraic simplification and eventually evaluate the limit.

Indeterminate forms

1) $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$

2) $0 \cdot (\pm\infty)$

$\infty - \infty$

Rearrange

3) $\infty^0, 0^0, 1^{\pm\infty}$

ln both sides

L'Hospital Rule **Only** Applies

$$\frac{0}{0} \quad or \quad \frac{\pm\infty}{\pm\infty}$$

L' Hopital's Rule

- Rules to evaluate 0/0 form :
  1. Check whether the limit is an indeterminate form. If it is not, then we cannot apply L' Hopital's rule.
  2. Differentiate f(x) and g(x) separately.
  3. If $g'(a) \neq 0$, then the limit will exist. It may be finite, $+\infty$ or $-\infty$. If $g'(a)=0$ then follow rule 4.
  4. Differentiate f'(x) & g'(x) separately.
  5. Continue the process till required value is reached.

Example:

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2} = \lim_{x\to 0}\frac{\sin x}{1+2x} = 0$$

If it's no longer indeterminate, then **STOP**!

If we try to continue with L'Hôpital's rule:

$$= \lim_{x\to 0}\frac{\sin x}{1+2x} = \lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}$$

which is wrong, wrong, wrong!

On the other hand, you can apply L'Hôpital's rule as many times as necessary as long as the fraction is still indeterminate:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} \longleftarrow \frac{0}{0}$$

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{2}}-1-\frac{1}{2}x}{x^2}$$

(Rewritten in exponential form.)

$$=\lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{2x} \longleftarrow \frac{0}{0}$$

$$=\lim_{x\to 0}\frac{-\frac{1}{4}(1+x)^{-\frac{3}{2}}}{2} \longleftarrow \text{not } \frac{0}{0}$$

$$=\frac{-\frac{1}{4}}{2}$$

$$=-\frac{1}{8}$$

Example 1: $\lim_{x\to 2}\dfrac{x^2-4}{x-2}=\lim_{x\to 2}\dfrac{2x}{1}=2(2)=4$

Example 2: $\lim_{x\to 0}\dfrac{\tan 3x}{\sin 2x}=\lim_{x\to 0}\dfrac{3\sec^2 3x}{2\cos 2x}=\dfrac{3(1)}{2(1)}=\dfrac{3}{2}$

Example 3: $\lim_{h\to 0}\dfrac{\sqrt[3]{8+h}-2}{h}=\lim_{h\to 0}\dfrac{\frac{1}{3}(8+h)^{-2/3}(1)}{1}=\lim_{h\to 0}\dfrac{1}{3(8+h)^{2/3}}=\dfrac{1}{3(8)^{2/3}}=\dfrac{1}{12}$

Example 4: $\lim_{x\to \pi/3}\dfrac{\cos x-1/2}{x-\pi/3}=\lim_{x\to \pi/3}\dfrac{-\sin x}{1}=-\sin\left(\pi/3\right)=-\dfrac{\sqrt{3}}{2}$

∞ / ∞ Form

- If $\lim_{x \to c} f(x) = \infty, \lim_{x \to c} g(x) = \infty$, then it is indeterminate form of type 0/0.

EXAMPLES:

1. Find $\lim_{x \to \infty} \frac{5x - 2}{7x + 3}$.

Solution 1: We have

$$\lim_{x \to \infty} \frac{5x - 2}{7x + 3} = \left[\frac{\infty}{\infty}\right] = \lim_{x \to \infty} \frac{\frac{5x-2}{x}}{\frac{7x+3}{x}} = \lim_{x \to \infty} \frac{\frac{5x}{x} - \frac{2}{x}}{\frac{7x}{x} + \frac{3}{x}} = \lim_{x \to \infty} \frac{5 - \frac{2}{x}}{7 + \frac{3}{x}} = \frac{5 - 0}{7 + 0} = \frac{5}{7}$$

0x∞ Form

- Limit of the form $\lim_{x \to c} f(x) = 0,\ \lim_{x \to c} g(x) = \infty$ are called indeterminate form of the type 0x∞.
- If we write f(x) g(x) = f(x)/[1/g(x)], then the limit becomes of the form (0/0).
- This can be evaluated by using L' Hopital's rule.

Example

Example 3.1. Consider $\lim_{x\to\infty}\left(x\cdot\sin\left(\frac{2}{x}\right)\right)$. This has the form $\infty\cdot 0$ if you try to evaluate directly. But if you flip the x to the denominator

$$\lim_{x\to\infty}\left(\frac{\sin\left(\frac{2}{x}\right)}{1/x}\right)$$

then this limit has the form $\frac{0}{0}$. We have simply taken the ∞, and transformed it into a 0 in the denominator. This limit can be done with L'Hôpital's rule.

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{\sin\left(\frac{2}{x}\right)}{1/x}\right) &= \lim_{x\to\infty}\frac{\cos\left(\frac{2}{x}\right)\left(-\frac{2}{x^2}\right)}{-1/x^2} \\
&= \lim_{x\to\infty} 2\cos\left(\frac{2}{x}\right) \\
&= 2\cos(2/\infty) \\
&= 2\cos(0) \\
&= 2
\end{aligned}$$

So in this case, we could evaluate the limit by flipping it to $\frac{0}{0}$ and using L'Hôpital's rule.

MCQ

Find $lt_{x\to 0}\dfrac{(3e^{x}-2e^{2x}-e^{3x})}{(e^{x}+e^{2x}-2e^{3x})}$

a) 3⁄2

b) 0

c) 4⁄3

d) –4⁄3

Answer: c

Explanation: Form is 0 / 0

Applying L hospitals rule we have

$$= lt_{x \to 0} \frac{3e^{x} - 4e^{2x} - 3e^{3x}}{e^{x} + 2e^{2x} - 6e^{3x}}$$

$$= \frac{3 - 4 - 3}{1 + 2 - 6}$$

$$= \frac{4}{3}$$

$\infty - \infty$ Form

- Limit of the form $\lim_{x \to c} f(x) = \infty, \lim_{x \to c} g(x) = \infty$ are called indeterminate form of the type ∞-∞.
- If we write $\lim_{x \to c}(f(x) - g(x)) = \lim_{x \to c} \frac{1/g(x) - 1/f(x)}{1/(f(x)g(x))}$, then the limit becomes of the form (0/0) and can be evaluated by using the L' Hopital's rule.

Example

11. Find $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$.

Solution: We have

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=[\infty-\infty]=\lim_{x\to 1}\left(\frac{1\cdot(x-1)}{\ln x\cdot(x-1)}-\frac{\ln x\cdot 1}{\ln x\cdot(x-1)}\right)$$

$$=\lim_{x\to 1}\frac{x-1-\ln x}{\ln x(x-1)}=\left[\frac{0}{0}\right]=\lim_{x\to 1}\frac{(x-1-\ln x)'}{(\ln x(x-1))'}$$

$$=\lim_{x\to 1}\frac{x'-1'-(\ln x)'}{(\ln x)'\cdot(x-1)+\ln x\cdot(x-1)'}=\lim_{x\to 1}\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln x}$$

$$=\lim_{x\to 1}\frac{\left(1-\frac{1}{x}\right)x}{\left(\frac{x-1}{x}+\ln x\right)x}=\lim_{x\to 1}\frac{1\cdot x-\frac{1}{x}\cdot x}{\frac{x-1}{x}\cdot x+\ln x\cdot x}$$

$$=\lim_{x\to 1}\frac{x-1}{x-1+x\ln x}=\left[\frac{0}{0}\right]=\lim_{x\to 1}\frac{(x-1)'}{(x-1+x\ln x)'}=\lim_{x\to 1}\frac{x'-1'}{x'-1'+x'\ln x+x(\ln x)'}$$

$$=\lim_{x\to 1}\frac{1-0}{1-0+1\cdot\ln x+x\cdot\frac{1}{x}}=\lim_{x\to 1}\frac{1}{2+\ln x}=\frac{1}{2+0}=\frac{1}{2}$$

MCQ

$$\underset{x \to \frac{\pi}{2}}{\mathrm{Lt}} \frac{\log\left(x - \frac{\pi}{2}\right)}{\tan x} =$$

(a) 1 (b) 0 (c) -1 (d) does not exist

0^0 Form

- Limit of the form $\lim_{x \to c} f(x) = 0^+, \lim_{x \to c} g(x) = 0$ are called indeterminate form of the type 0^0.

- If we write $\lim_{x \to c} f(x)^{g(x)} = \exp \lim_{x \to c} \frac{g(x)}{1/\ln f(x)}$, then the limit becomes of the form (0/0) and can be evaluated by using the L' Hopital's rule.

Example

Example 1: Find $\lim_{x \to 0^+} x^x$.

This is an indeterminate form of the type 0^0. Let $y = x^x \Rightarrow \ln y = \ln x^x = x \ln x$. $\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+}(-x) = 0.$

Thus, $\lim_{x \to 0^+} x^x = e^0 = 1$.

MCQ

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{\cot x}{x}\right) =$$

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) does not exist

MCQ

$$\underset{x \to 0}{\text{Lt}} \sin x \log x^2 =$$

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) does not exist

1^∞ Form

- Limit of the form $\lim_{x\to c} f(x) = 1,\ \lim_{x\to c} g(x) = \infty$ are called indeterminate form of the type 1^∞ .
- If we write $\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \dfrac{\ln f(x)}{1/g(x)}$, then the limit becomes of the form (0/0) and can be evaluated by using the L' Hopital's rule.

Example

Example 3: Find $\lim_{x\to 0^+}(\cos x)^{1/x}$.

This is an indeterminate form of the type 1^∞. Let $y=(\cos x)^{1/x} \Rightarrow$

$\ln y = \ln\left[(\cos x)^{1/x}\right] = \dfrac{\ln(\cos x)}{x}$. $\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\dfrac{\ln(\cos x)}{x} =$

$\lim_{x\to 0^+}(-\tan x) = 0$. Thus, $\lim_{x\to 0^+}(\cos x)^{1/x} = e^0 = 1$.

∞^0 Form

- Limit of the form $\lim_{x\to c} f(x) = \infty,\ \lim_{x\to c} g(x) = 0$ are called indeterminate form of the type ∞^0.
- If we write $\lim_{x\to c} f(x)^{g(x)} = \exp \lim_{x\to c} \frac{\ln f(x)}{1/g(x)}$, then the limit becomes of the form (0/0) and can be evaluated by using the L' Hopital's rule.

Example

Example 2: Find $\lim\limits_{x\to+\infty}(e^x+1)^{-2/x}$.

This is an indeterminate form of the type ∞^0. Let $y=(e^x+1)^{-2/x}\Rightarrow$

$\ln y=\ln\left[(e^x+1)^{-2/x}\right]=\dfrac{-2\ln(e^x+1)}{x}$. $\lim\limits_{x\to+\infty}\ln y=\lim\limits_{x\to+\infty}\dfrac{-2\ln(e^x+1)}{x}=$

$\lim\limits_{x\to+\infty}\dfrac{-2\left(\dfrac{e^x}{e^x+1}\right)}{1}=\lim\limits_{x\to+\infty}\dfrac{-2e^x}{e^x+1}=\lim\limits_{x\to+\infty}\dfrac{-2e^x}{e^x}=-2$. Thus, $\lim\limits_{x\to+\infty}(e^x+1)^{-2/x}=$

e^{-2}.

The following table lists the most common indeterminate forms and the transformations for applying l'Hôpital's rule.

Indeterminate form	Conditions	Transformation to 0/0	Transformation to ∞/∞
0/0	$\lim_{x\to c} f(x)=0,\ \lim_{x\to c} g(x)=0$	–	$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{1/g(x)}{1/f(x)}$
∞/∞	$\lim_{x\to c} f(x)=\infty,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{1/g(x)}{1/f(x)}$	–
0 × ∞	$\lim_{x\to c} f(x)=0,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c} f(x)g(x)=\lim_{x\to c}\frac{f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)g(x)=\lim_{x\to c}\frac{g(x)}{1/f(x)}$
∞ - ∞	$\lim_{x\to c} f(x)=\infty,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c}(f(x)-g(x))=\lim_{x\to c}\frac{1/g(x)-1/f(x)}{1/(f(x)g(x))}$	$\lim_{x\to c}(f(x)-g(x))=\ln\lim_{x\to c}\frac{e^{f(x)}}{e^{g(x)}}$
0^0	$\lim_{x\to c} f(x)=0^+, \lim_{x\to c} g(x)=0$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$
1^∞	$\lim_{x\to c} f(x)=1,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$
∞^0	$\lim_{x\to c} f(x)=\infty,\ \lim_{x\to c} g(x)=0$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$

① $\underset{x \to 0}{\mathrm{Lt}} \dfrac{1-\cos^2 x}{\sin x^2} =$

(a) 0 (b) 1 (c) −1 (d) does not exist

② $\underset{x \to 0}{\mathrm{Lt}} \dfrac{x - \tan^{-1} x}{x - \sin x} =$

(a) 0 (b) 1 (c) 2 (d) does not exist

③

$$\underset{x \to 1}{Lt} \frac{x^2 \sin \frac{1}{x}}{\sin x} =$$

(a) 0 (b) 1 (c) 2 (d) does not exist

④

$$\underset{x \to 1}{Lt} (1-x) \tan \frac{\pi x}{2} =$$

(a) $\frac{\pi}{2}$ (b) $\frac{2}{\pi}$ (c) π (d) does not exist

MTH165

Unit 3
Applications of Derivatives
L19-Maxima Minima

Revision

$$\underset{x \to a}{Lt} \left(2 - \frac{x}{a} \right)^{\tan \frac{\pi x}{2a}} =$$

(a) $e^{\frac{1}{\pi}}$ (b) $\frac{2}{\pi}$ (c) $e^{\frac{2}{\pi}}$ (d) none of these

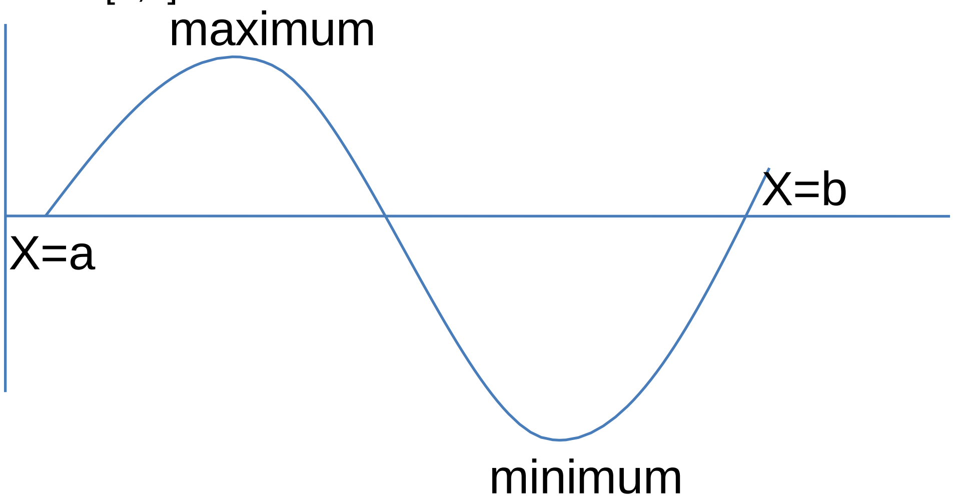
Revision

$$\underset{x \to 0}{Lt}\ x \tan\left(\frac{\pi}{2} - x\right) =$$

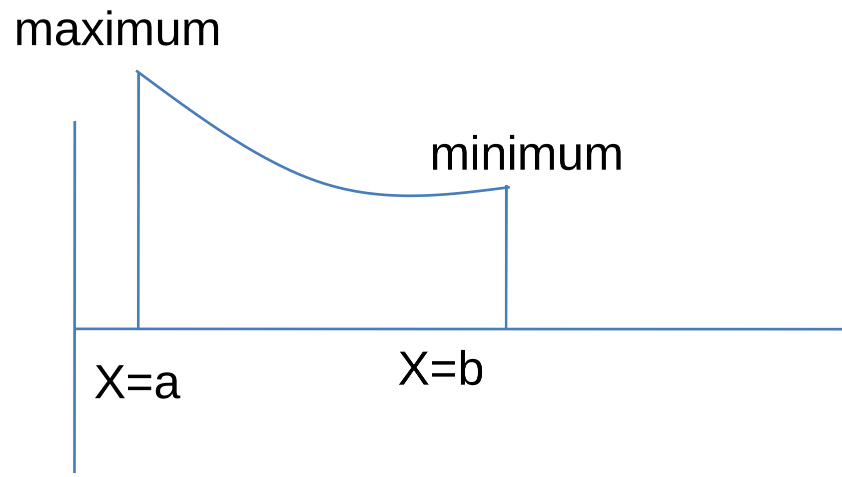
(a) 0 (b) 1 (c) e (d) none of these

Observe these graphs

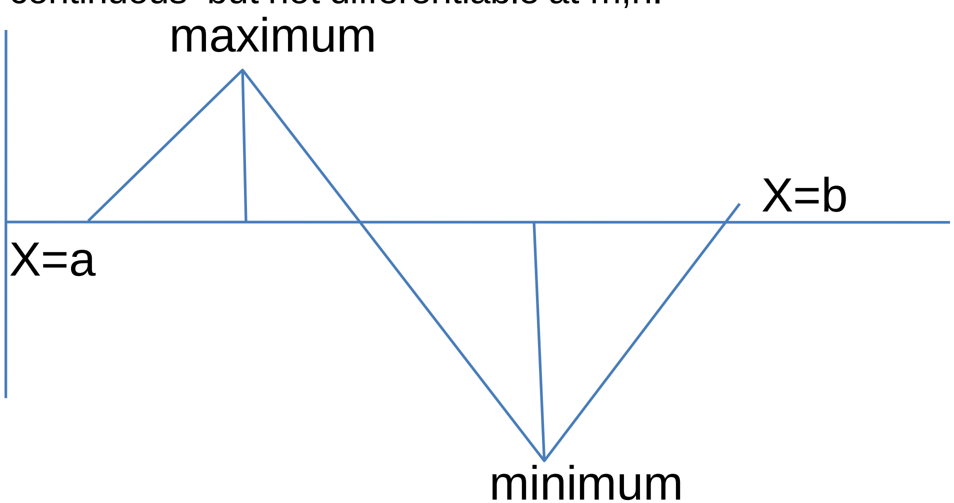
Maximum and minimum are in the interior point of interval [a,b]



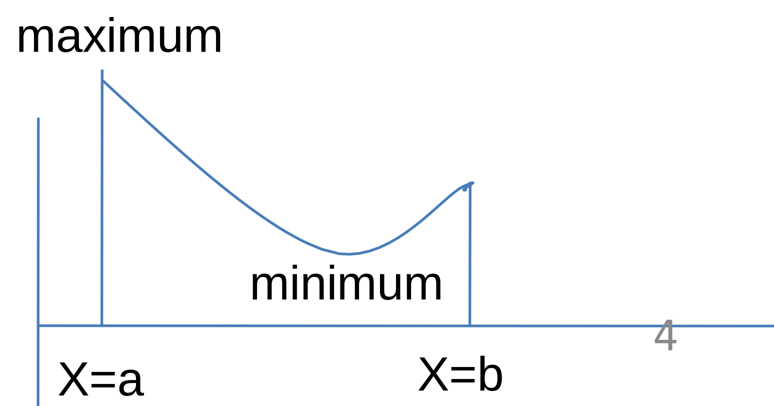
Maximum and minimum are in the end points of interval [a,b]



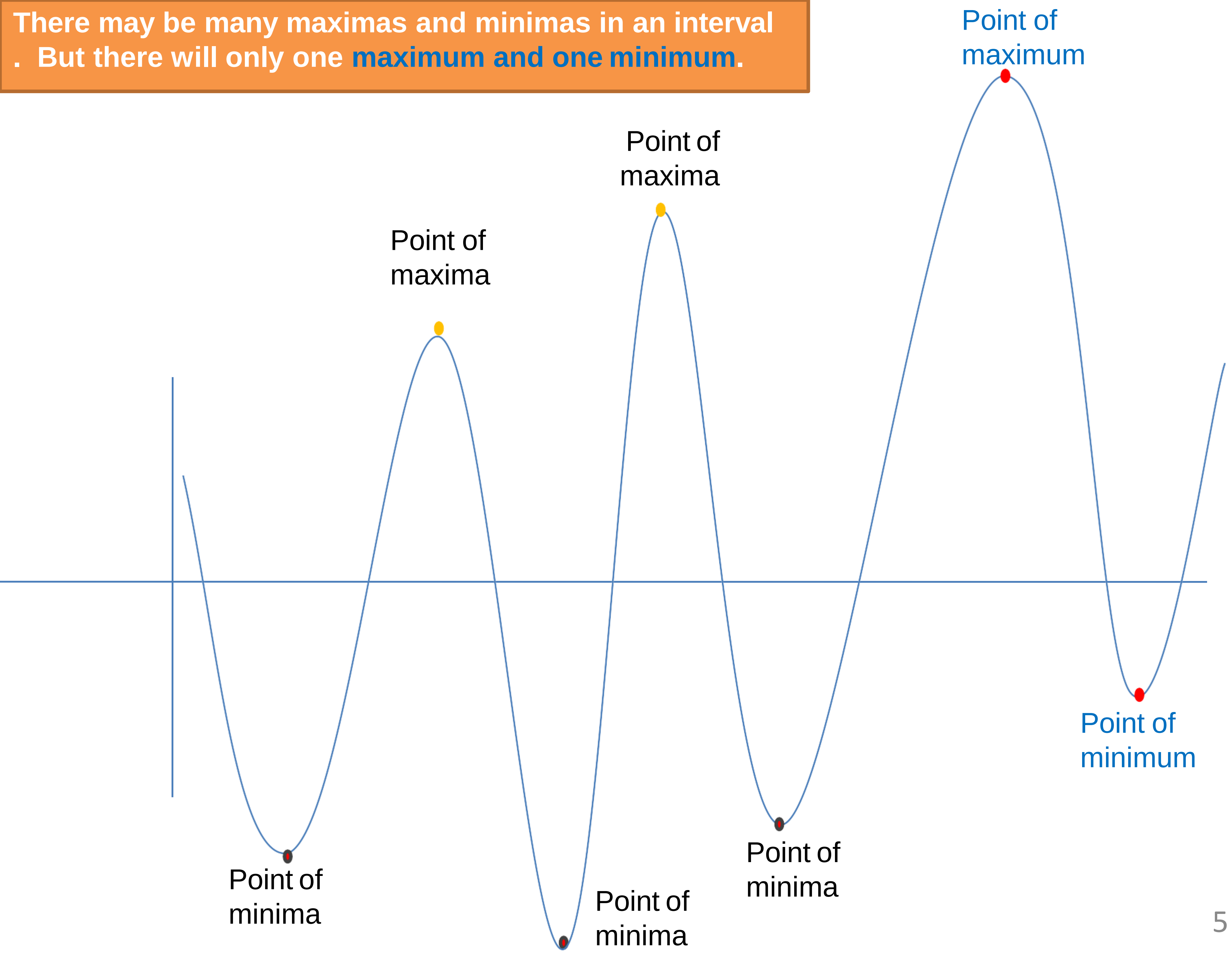
Maximum and minimum are in the interior point of interval [a,b] . At x and y slope is not zero. This function is continuous but not differentiable at m,n.



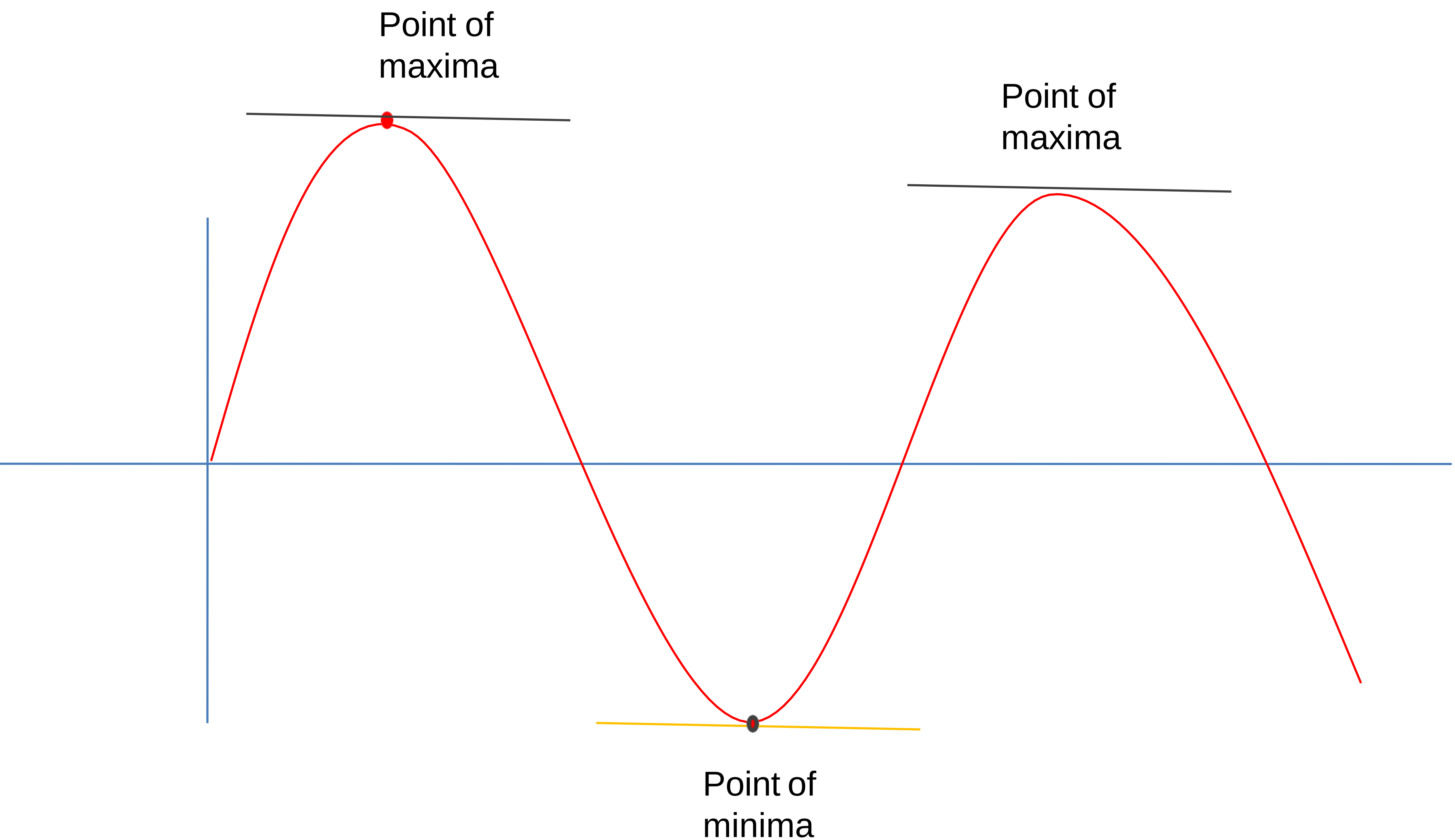
Maximum at end point, minimum at interior point of interval [a,b]



There may be many maximas and minimas in an interval . But there will only one maximum and one minimum.



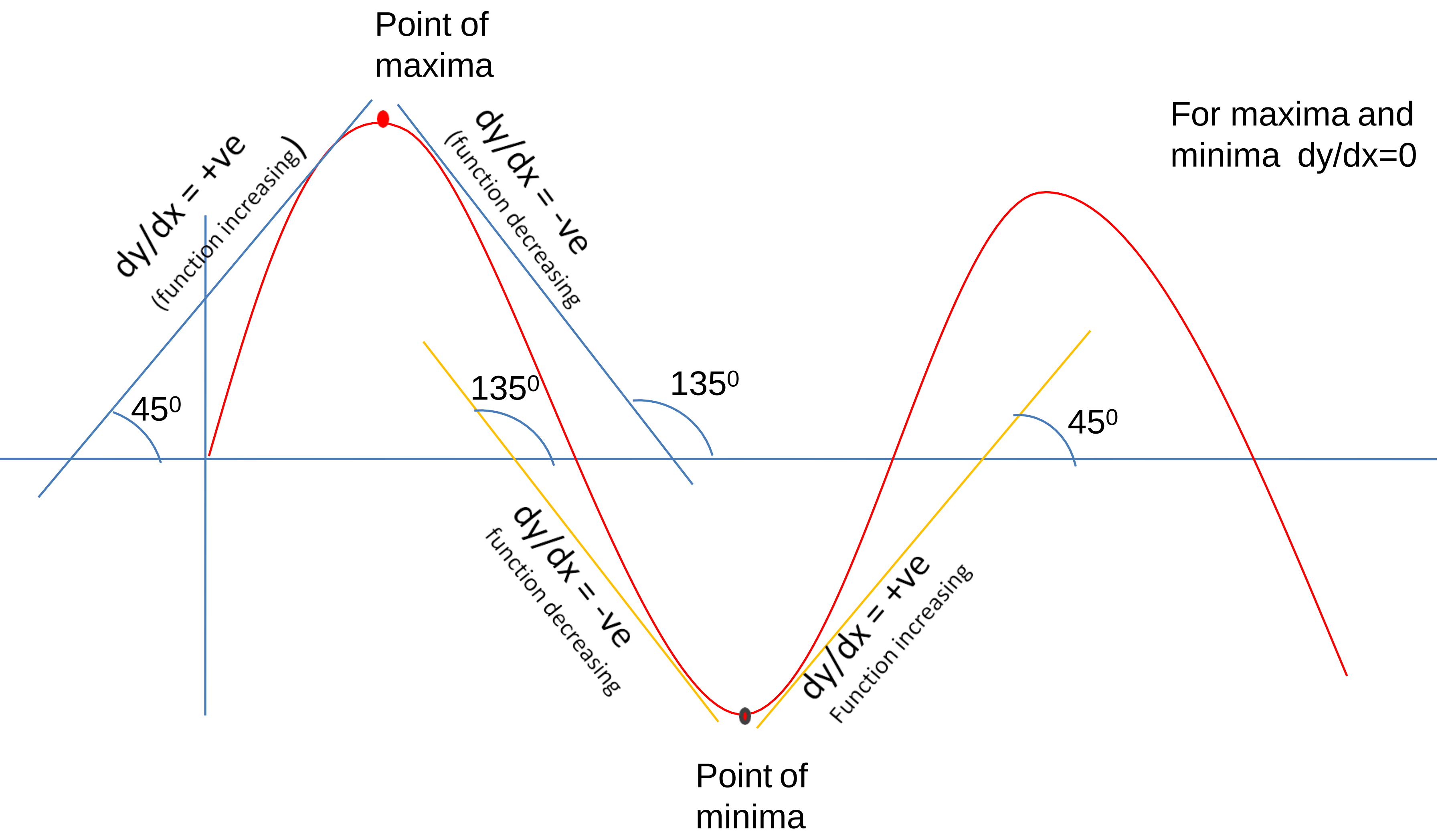
- For maxima and minima $m = dy/dx = \tan 0^0 = 0$
- $dy/dx = 0$ means tangent is parralel to X –axis.



Point of maxima and minima

Gradient m =dy/dx (angle formed from positive direction of X axis.)

see m =dy/dx = tan45^0 = +1 (positive)
and m =dy/dx = tan135^0 = -1 (negative)

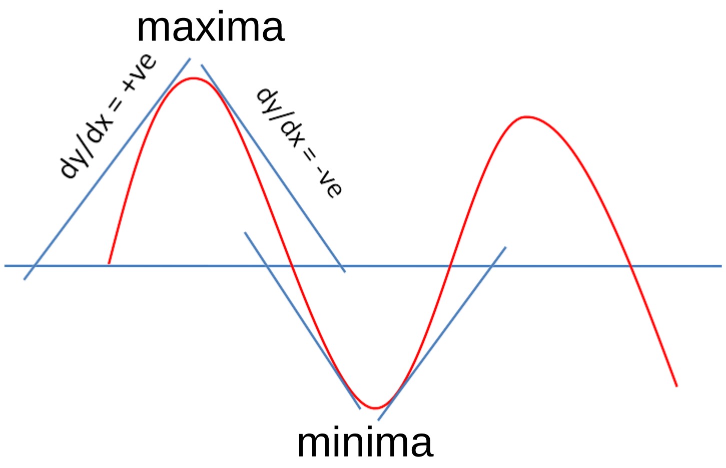


For maxima and minima dy/dx=0

First derivative test for maxima and minima

- If at left of any point dy/dx = +ve and right of this point dy/dx = -ve then the point will be point of maxima.

- If at left of any point dy/dx = -ve and right of this point dy/dx=+ve then the point will be point of minima.



The First Derivative Test Suppose that c is a critical number of a continuous function f.

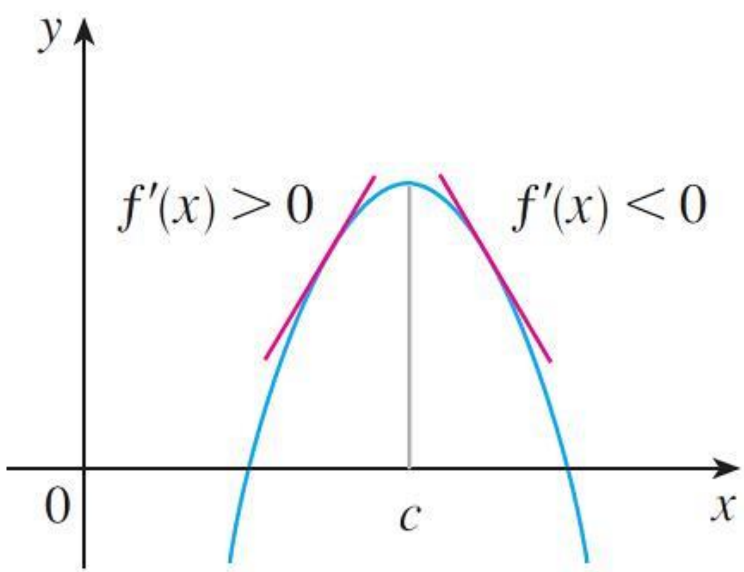
(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

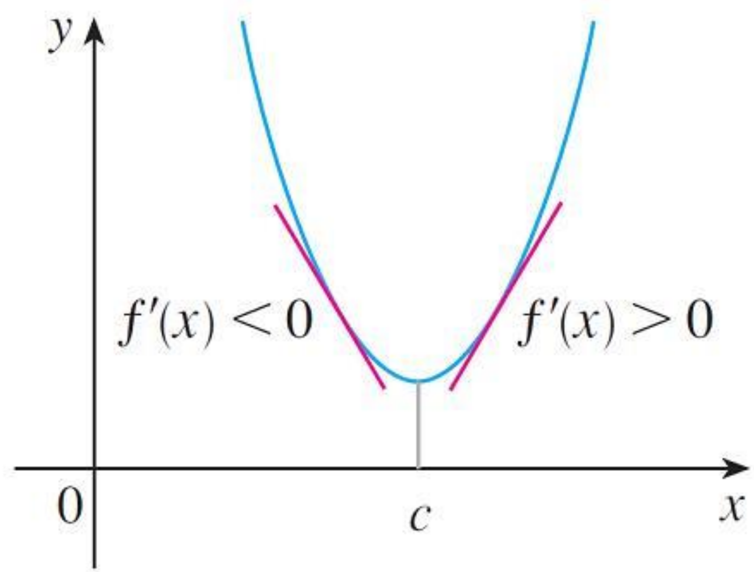
The First Derivative Test is a consequence of the I/D Test. In part (a), for instance, since the sign of $f'(x)$ changes from positive to negative at c, f is increasing to the left of c and decreasing to the right of c. It follows that f has a local maximum at c.

It is easy to remember the First Derivative Test by visualizing diagrams such as those in Figure 3.



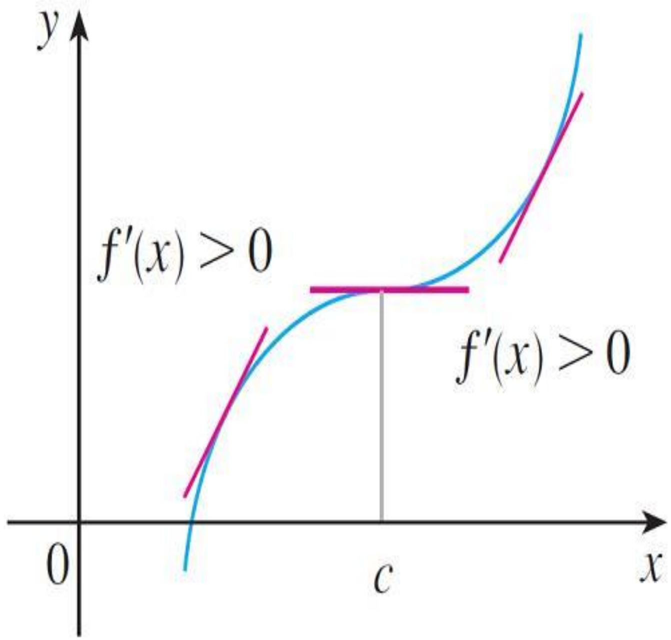
Local maximum

Figure 3(a)



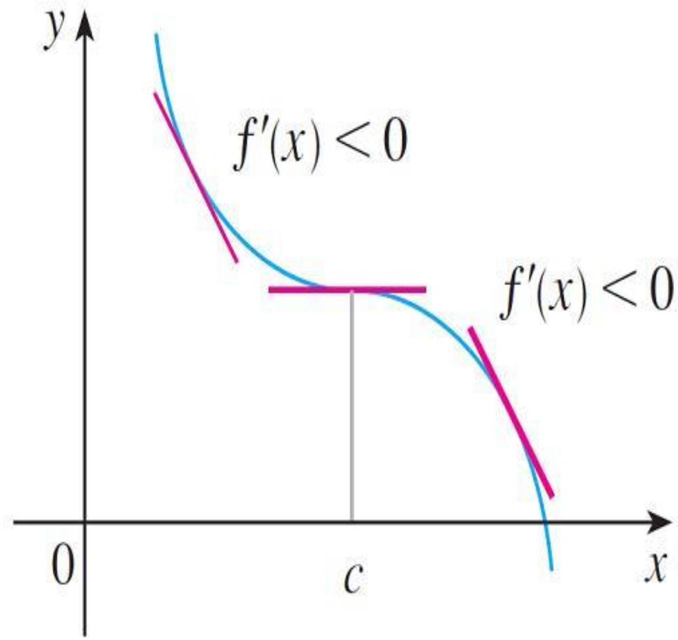
Local minimum

Figure 3(b)



No maximum or minimum

Figure 3(c)



No maximum or minimum

Figure 3(d)

Procedure to find point of maxima and minima by Second derivative test

For knowing point of maxima and minima

a. Find first derivative

b. put dy/dx=0 and find the points for which dy/dx=0

c. now calculate d^2y/dx^2. If for above any value dy/dx=-ve then it will be the point of maxima. If it is positive then it will point of minima.

d. If $d^2y/dx^2 = 0$,then apply first derivative test .

Example

Find the maximum and minimum value of $(2x-1)^2+3$.

MCQ

The stationary points of $x^3 - 6x^2 + 9x + 1$ on the interval $[0,5]$ are

(a) $x = 1,\ x = 2$ (b) $x = 1,\ x = 3$ (c) $x = 2,\ x = 3$ (d) none of these

Find the points of absolute maximum and minimum of $y = (x-1)^{\frac{1}{3}}(x-2);\ 1 \le x \le 9.$

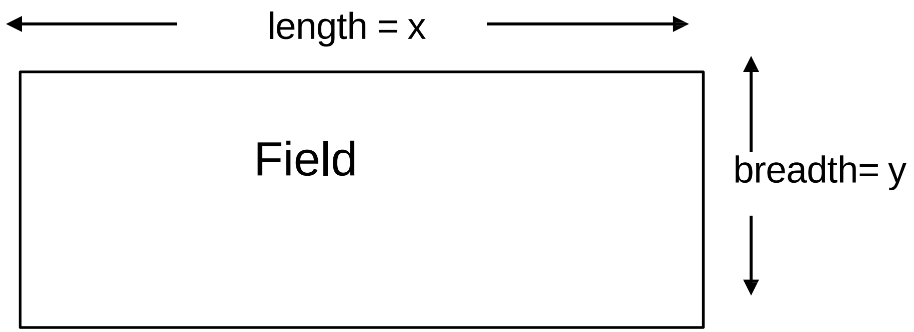
MCQ

The maximum value of $f(x) = |x|, x \in R$ is

(a) 0 (b) 1 (c) 2 (d) no maximum value

Problem

- Find the dimensions of the rectangular field of maximum area which can be fenced by 36 m fence.



Given $2(x+y) = 36$

So $x+y = 18$

Area $A = x.y$

$A = x(18-x) = 18x-x^2$

So $dA/dx = d/dx(18x - x^2) = 18- 2x$

Put $dA/dx = 0$

So $18-2x =0$

Or $x= 9$

Now find d^2A/dx^2

$d^2A/dx^2 = d/dx(18-2x) = 0-2 = -2$ (-ve)

So $x= 9$ will be the point for which area of field will be

maximum. So maximum area $= x(18-x) = 9(18-9) = 81\ m^2$

Observe it

X+y= 18

Find x.y = maximum

Factors for which x+y = 18 may be

1x17 = 17 (product is minimum here) 2x16 = 32

3x15= 45

4x14 = 56

5x13= 65

6x12= 32

7x11= 77

8x10= 80

9x9= 81 (product is maximum here)

MCQ

The maximum value of xy subject to $x+y=8$ is

(a) 8 (b) 16 (c) 20 (d) 24

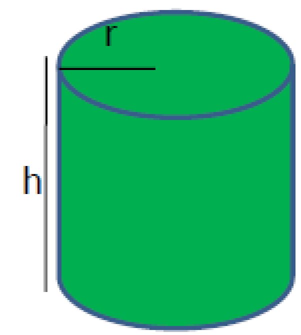
Problem 2

- A cylinder has a fixed surface area .Establish a relation between radius and height of a cylinder for which it's volume is maximum.

$S = 2\pi rh + 2\pi r^2$ (given)

$V = \pi r^2 h$

(We have to maximise volume. So first reduce variables r and h in either r or h)



$V = \pi r^2 ((s - 2\pi r^2)/2\pi r) = r(s - 2\pi r^2)/2$

$V = rs - 2\pi r^3$

$dv/dr = d/dr(rs - 2\pi r^3)$

$dv/dr = s - 6\pi r^2$

For maxima and minima $dv/dr = 0$

So $s - 6\pi r^2 = 0$ or $s = 6\pi r^2$

Or $2\pi rh + 2\pi r^2 = 6\pi r^2$ or $h = 2r$

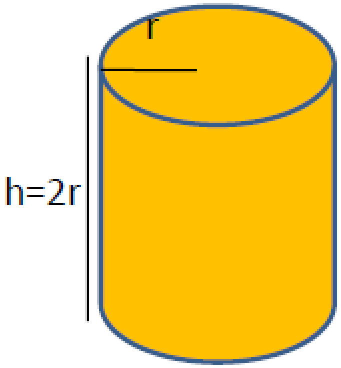
Now for knowing whether for $h=2r$ volume of cylinder is maximum or minimum, we calculate d^2v/dr^2

- $dv/dr = s - 6\pi r^2$

- $d^2v/dr^2 = -12\pi r = -ve$

- So volume will maximum at h= 2r

Cylinder of maximum volume for given surface area



Uses of maxima and minima

- For marketing purposes we require vessels of different shapes for which fabrication cost is less but they could contain more material e.g. 1 litre container of ghee.
- For getting more rectangular land area when total perimeter of land is given.
- In factories using resources so that the fabrication cost of commodity become less.