L35 09 April 2021

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Geometrical representation of the gradient

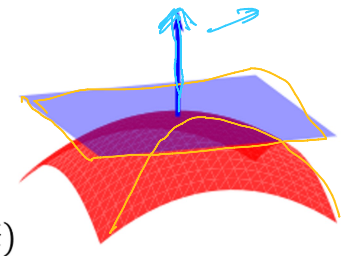
Geometrically, Gradient of a scalar field represents a vector normal to the surface,

Let $f(x, y, z)$ be a given scalar surface.

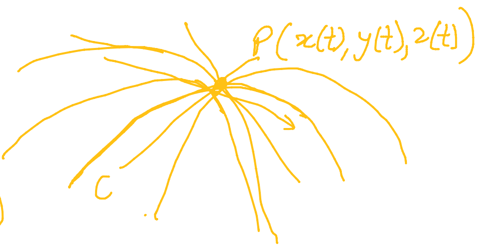
Let $\vec{\nabla} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$

Then vector normal to the surface f is given by:

Normal vector, $\vec{n} = \text{grad}(f) = \vec{\nabla}f = \left(f_x\hat{\imath} + f_y\hat{\jmath} + f_z\,\hat{k}\right)$



$\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$



$f(x(t), y(t), z(t)) = K$ (Level Surface)

$\frac{d}{dt} f(x(t), y(t), z(t)) = 0$

$\Rightarrow \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt} = 0$

$\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)\cdot\left(\hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt}\right) = 0$

$\boxed{\nabla f \cdot \vec{r}\,'(t) = 0}$ $\rightarrow$ gradient is $\perp$ to the tangent vector ($\vec{r}\,'(t)$)

$\nabla f(P)$ is orthogonal (normal) to every tangent vector at P.

$\Rightarrow \nabla f$ is normal to the surface $f(x, y, z) = K$

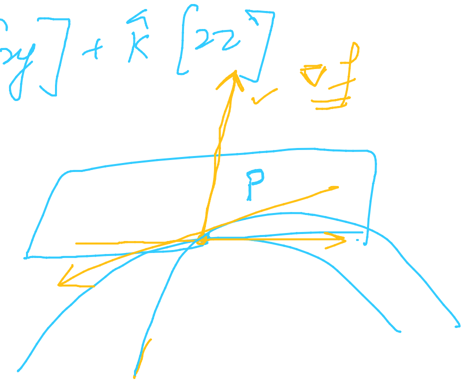
$f(x, y, z) = x^2 + y^2 + z^2$ at $(1, 1, 1)$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)f$ $\quad \hat{i}[2x] + \hat{j}[2y] + \hat{k}[2z]$

$f(x,y,z) = x+y+z$ $\quad (\;,\;,\;)$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f = i[2x] + \hat{j}[2y] + \hat{k}[2z]$$

$$\nabla f\Big|_{(1,1,1)} = 2\hat{i} + 2\hat{j} + 2\hat{k}$$



The unit vector normal to the surface

$$\hat{n} = \frac{\text{grad} f}{|\text{grad} f|}$$

Find the normal vector and the unit normal vector to the given surface/curve

① $f = x^2 + y^2 = 25$, $(3,4)$

$$\nabla f = \left[\hat{i}\left(\frac{\partial}{\partial x}\right) + \hat{j}\frac{\partial}{\partial y}\right] f = \hat{i}[2x] + \hat{j}[2y]$$

$$\nabla f_{(3,4)} = 6\hat{i} + 8\hat{j} \longrightarrow \text{normal vector}$$

unit normal vector $\hat{n} = \dfrac{6\hat{i}+8\hat{j}}{\sqrt{36+64}} = \dfrac{6\hat{i}+8\hat{j}}{10}$

② $z = xy$ $\quad (-1,-2,2)$

$$f(x,y,z) = z - xy = 0$$

$$\nabla f = \hat{i}(-y) + \hat{j}(-x) + \hat{k}(1)$$

$$\nabla f_{(-1,-2,2)} = 2\hat{i} + \hat{j} + \hat{k} \longrightarrow \text{normal vector}$$

$\nabla f_{(-1,-2,2)} = 2\hat{i}+\hat{j}+\hat{k}$ → normal vector

Unit normal $= \dfrac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{4+1+1}} = \dfrac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}$

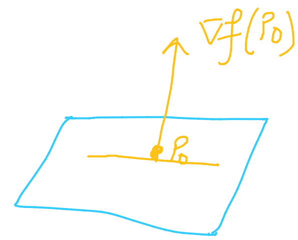
Q12. Unit normal vector to the surface $A = x^2 + y^2 - 3e^z$ at (2, -1, 0)
(a) $(i + j + k)/\sqrt{3}$ (b) $(i + j + 1k)/\sqrt{2}$ (c) $(4i - 2j - 3k)/\sqrt{23}$ (d) None of these

$\nabla f \quad \hat{i}(2x)+\hat{j}(2y)+\hat{k}(-3e^z)$

$\nabla f\big|_{(2,-1,0)} = 4\hat{i}-2\hat{j}-3\hat{k}$

$\hat{n} = \dfrac{4\hat{i}-2\hat{j}-3\hat{k}}{\sqrt{16+4+9}} = \dfrac{4\hat{i}-2\hat{j}-3\hat{k}}{\sqrt{29}}$

The eqn of the tangent plane at point $P(x_0,y_0,z_0)$ on the surface $f(x,y,z)=K$



$(x-x_0)\hat{i}+(y-y_0)\hat{j}+(z-z_0)\hat{k}$ and $\nabla f(P_0)$ are orthogonal

$$\nabla f(P_0)\cdot\left[(x-x_0)\hat{i}+(y-y_0)\hat{j}+(z-z_0)\hat{k}\right]=0$$

$$\left[\hat{i}\frac{\partial f(P_0)}{\partial x}+\hat{j}\frac{\partial f(P_0)}{\partial y}+\hat{k}\frac{\partial f(P_0)}{\partial z}\right]\cdot\left[(x-x_0)\hat{i}+(y-y_0)\hat{j}+(z-z_0)\hat{k}\right]=0$$

$$\left[(x-x_0)\frac{\partial f(P_0)}{\partial x}+(y-y_0)\frac{\partial f(P_0)}{\partial y}+(z-z_0)\frac{\partial f(P_0)}{\partial z}\right]=0$$

↓ eqn of the tangent plane passing

$\Rightarrow$ eqn of the tangent plane passing through P_0

Find the eqn of the tangent plane.

① $x^2-3y^2-z^2=2$ at $(3,1,2)$

$(x-x_0)\frac{\partial f}{\partial x}+(y-y_0)\frac{\partial f}{\partial y}+(z-z_0)\frac{\partial f}{\partial z}=0$

$\left.\frac{\partial f}{\partial x}\right|_{(3,1,2)}=2x=6$

$\left.\frac{\partial f}{\partial y}\right|_{(3,1,2)}=-6y=-6$

$\left.\frac{\partial f}{\partial z}\right|_{(3,1,2)}=-2z=-4$

$6(x-3)-6(y-1)-4(z-2)=0$

$6x-6y-4z-18+6+8=0$

$6x-6y-4z=4$ Ans

Q26. If A is constant vector and $R = 3xI + 3yJ + 3zK$ then $grad(3A.R) =$
(a) A **(b)** 27A **(c)** 9A **(d)** 3A

grad( )

$3\vec{A}\cdot\vec{R}$

$\vec{A}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$\vec{R}=3x\hat{i}+3y\hat{j}+3z\hat{k}$

$f=3\vec{A}\cdot\vec{R}=(3xa_1+3ya_2+3za_3)$

$\nabla f=\hat{i}(3a_1)+\hat{j}(3a_2)+\hat{k}(3a_3)$

$=3(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})$

$=3\vec{A}$