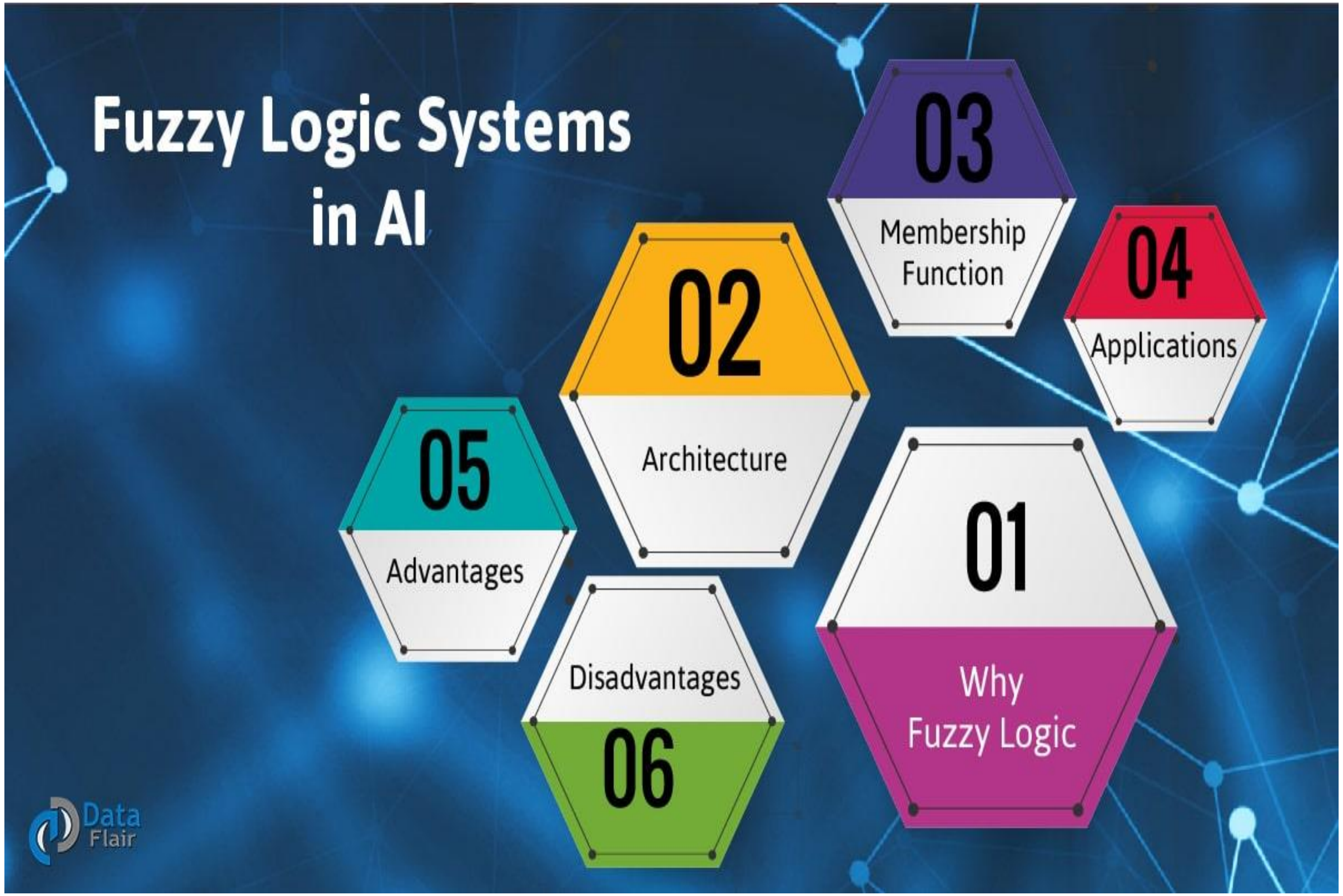


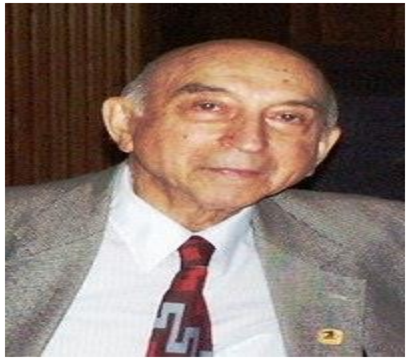
Fuzzy Logic

Unit 6

What is Fuzzy logic?

- The term *fuzzy logic* was introduced with the 1965 proposal of fuzzy set theory by Lotfi Zadeh
- Fuzzy logic is a mathematical language to express something.
  - This means it has grammar, syntax, semantic like a language for communication.
- **Fuzzy logic deals with Fuzzy set or Fuzzy algebra.**
- Dictionary meaning of **fuzzy** is not clear, noisy, etc.

  Example: Is the picture on this slide is fuzzy?



- Antonym of fuzzy is **crisp**

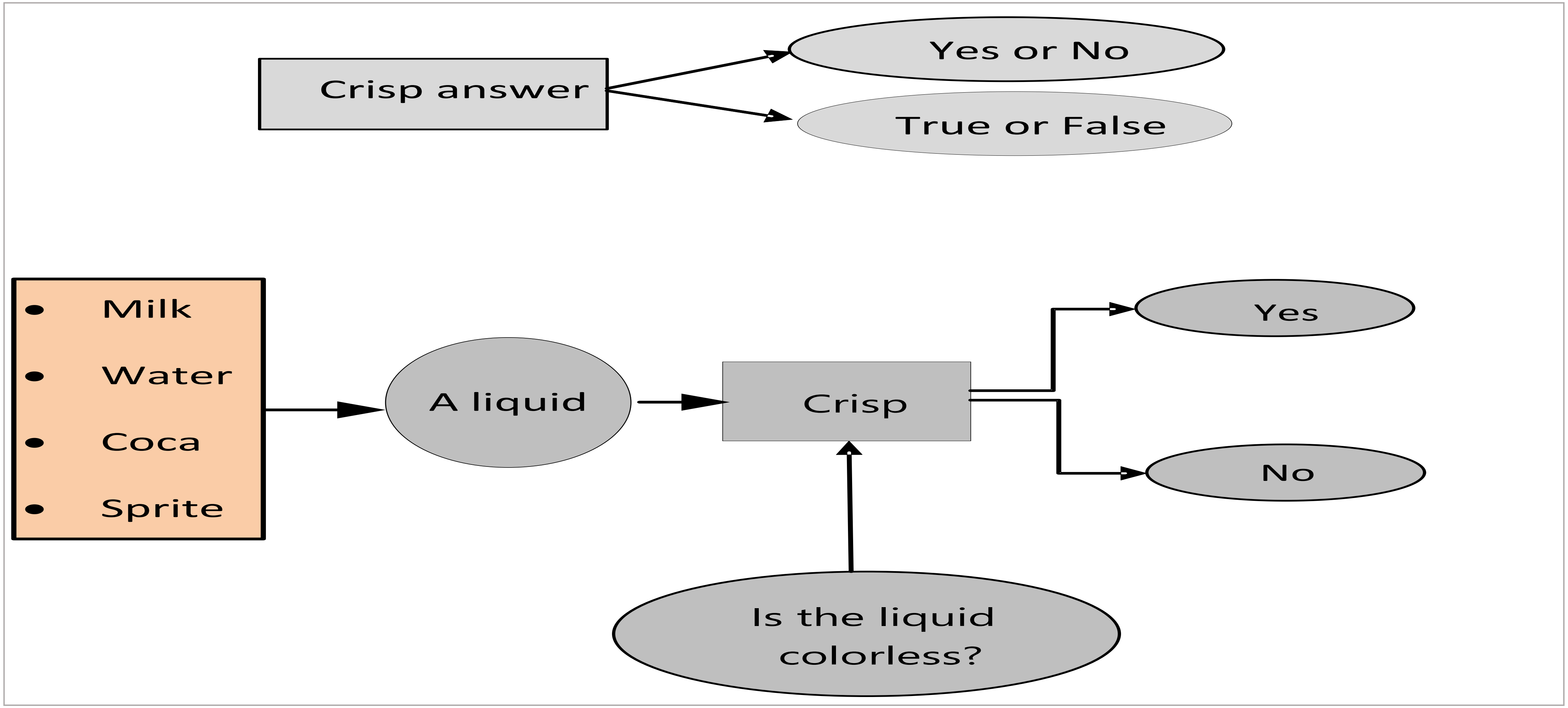
  Example: Are the chips crisp?

Fuzzy Set Vs Crisp Set

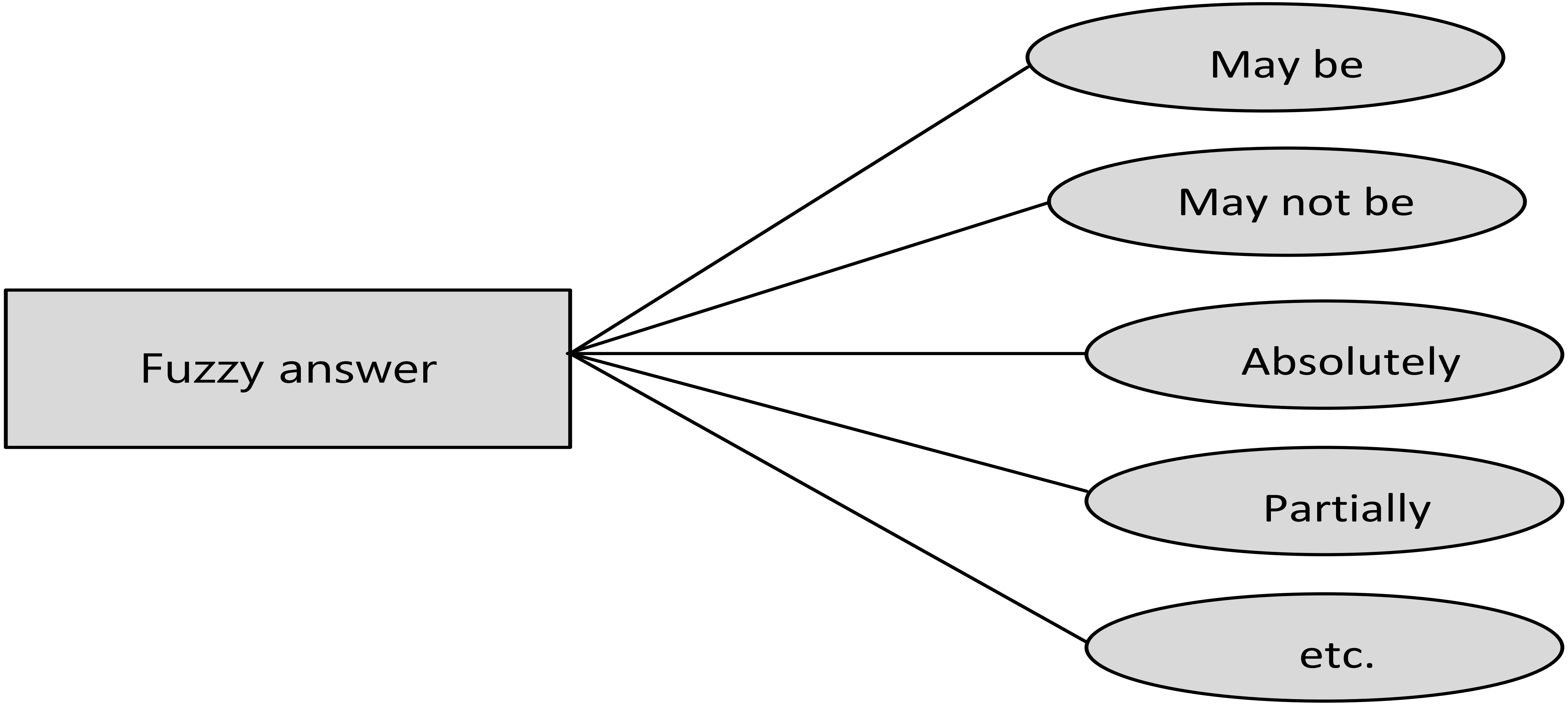
- A Fuzzy set is a set Containing elements that have varying degrees of membership on that set , Its values lie between 0 to 1.

- A Crisp set is a set containing elements that have a full membership on that set , Its values is either 0 or 1.

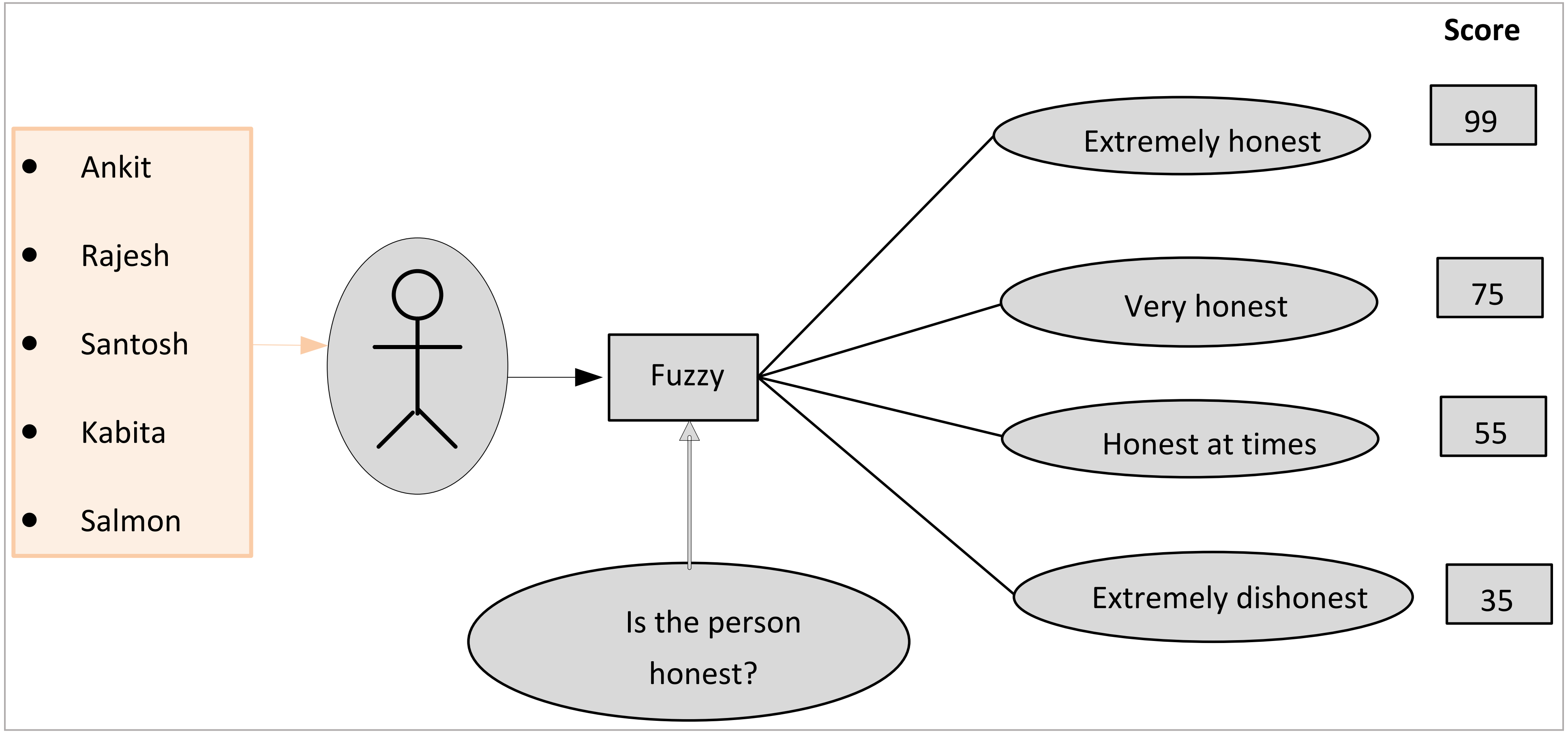
Example : Fuzzy logic vs. Crisp logic



Example : Fuzzy logic vs. Crisp logic



Example : Fuzzy logic vs. Crisp logic



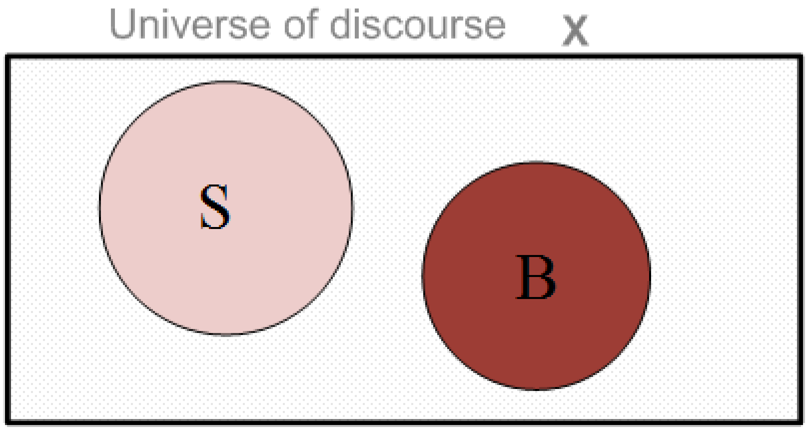
Comparison Chart

BASIS FOR COMPARISON	FUZZY SET	CRISP SET
Basic	Prescribed by vague or ambiguous properties.	Defined by precise and certain characteristics.
Property	Elements are allowed to be partially included in the set.	Element is either the member of a set or not.
Applications	Used in fuzzy controllers	Digital design
Logic	Infinite-valued	bi-valued

Applications

- Facial pattern recognition
- Air conditioners,
- Washing machines
- Vacuum cleaners
- Antiskid braking systems
- Transmission systems control of subway systems
- Unmanned helicopters
- Weather forecasting systems
- Models for new product pricing or project risk assessment
- Medical diagnosis and treatment plans
- Stock trading.

Crisp set.

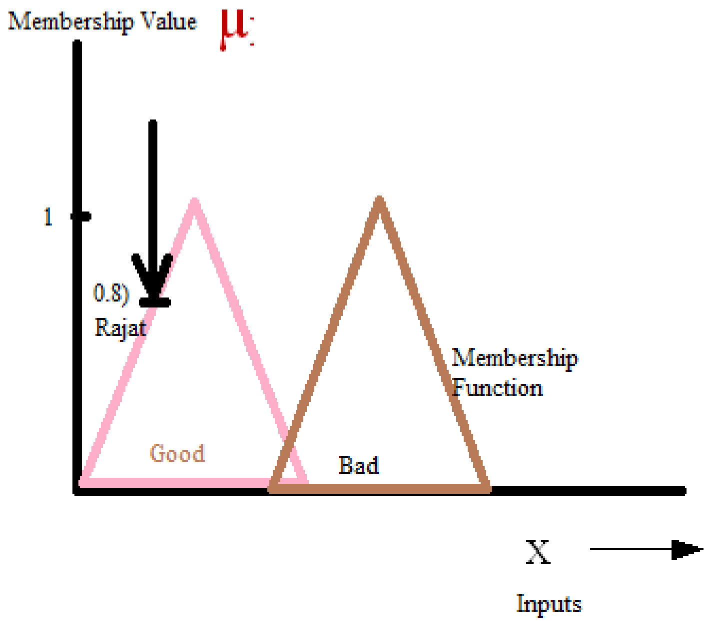


X= The entire population of India.

S= All **Good students** = $\{s_1, s_2, s_3, \ldots \ldots, s_L,\}$

B= All **Bad students**=$\{b_1, b_2, b_3, \ldots \ldots, b_N,\}$

Fuzzy set



Let us discuss about fuzzy set.

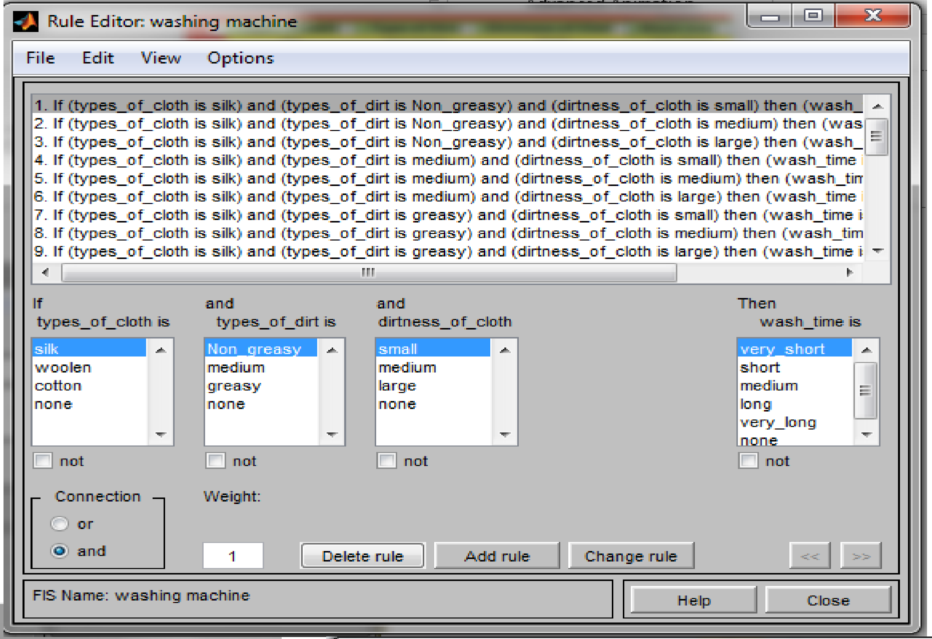
X = All students in NPTEL.

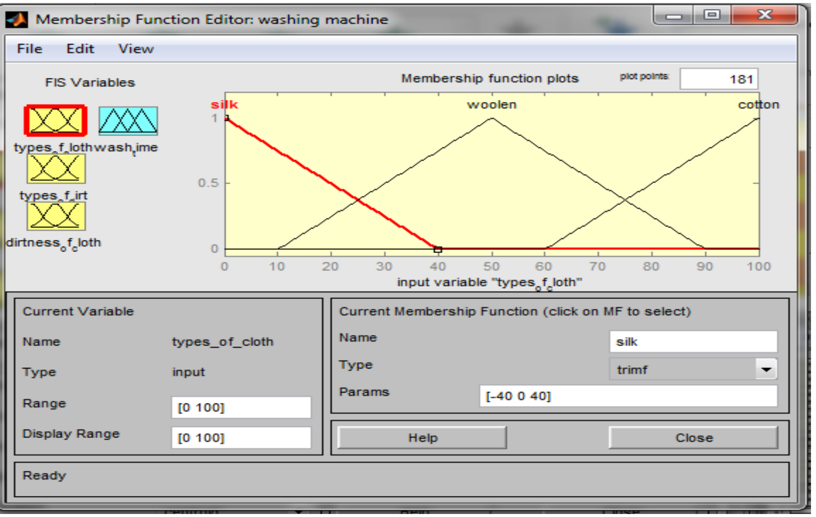
S = All **Good students**.

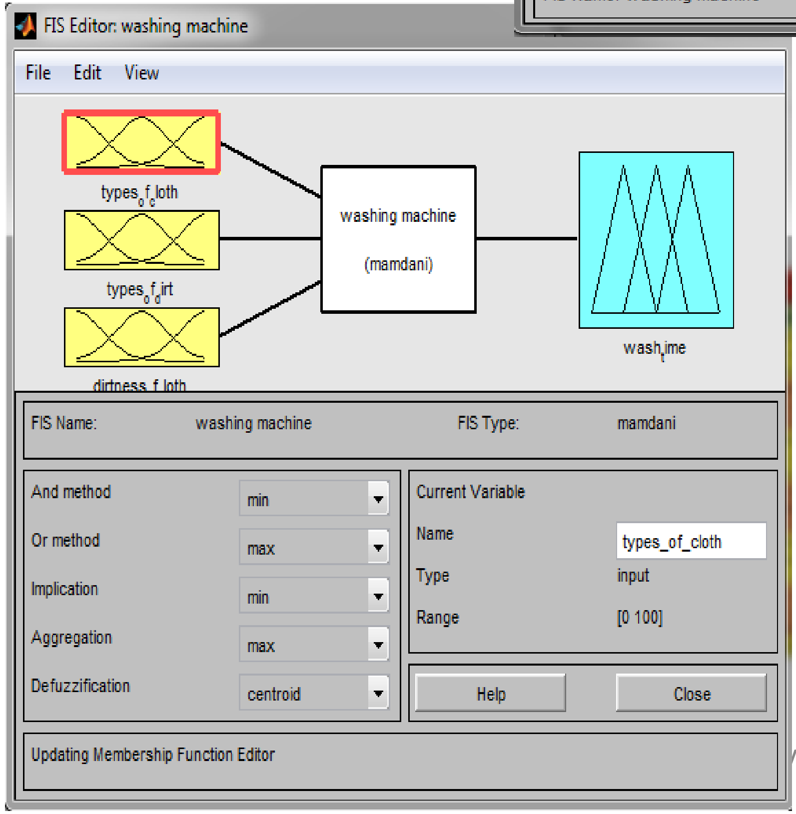
$S = \{(s, g(s)) \mid s \in X\}$ and *g(s)* is a measurement of goodness of the student *s*.

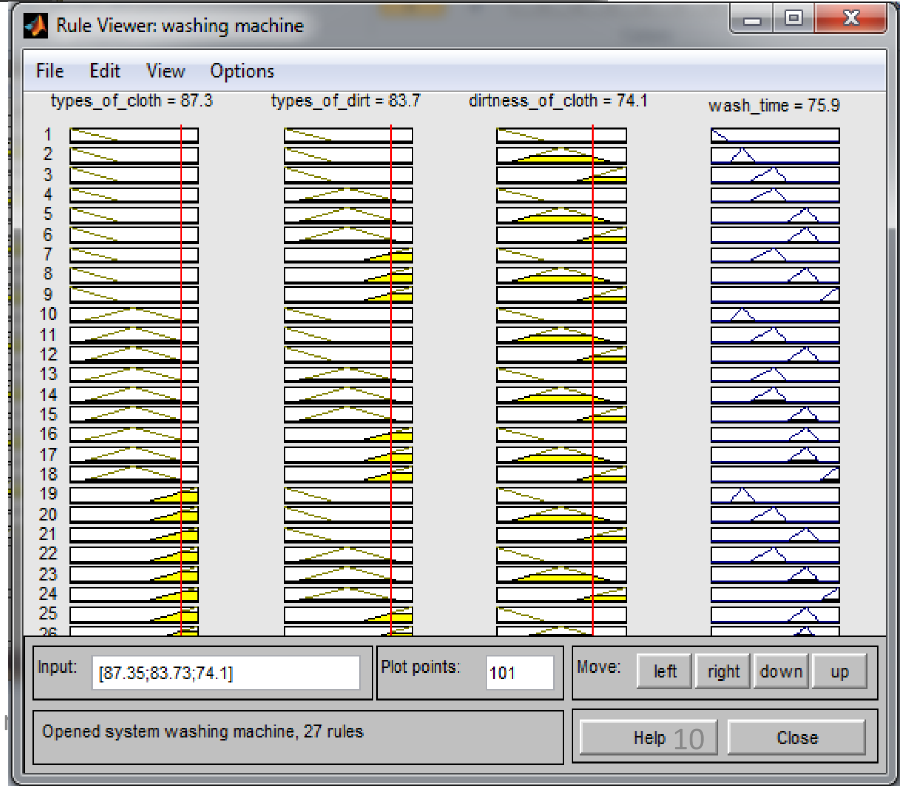
Example: $S = \{(Rajat, 0.8), (Kabita, 0.7), (Salman, 0.1), (Ankit, 0.9)\}$, etc.

Rule No.	Type of Cloth	Type of Dirt	Dirtiness of Cloth	Wash time
	Linguistic Inputs			Linguistic Output
1	Silk	Non greasy	Small	Very Short
2	Silk	Non greasy	Medium	Short
3	Silk	Non greasy	Large	Medium
4	Silk	Medium	Small	Medium
5	Silk	Medium	Medium	Long
6	Silk	Medium	Large	Long
7	Silk	Greasy	Small	Medium
8	Silk	Greasy	Medium	Long
9	Silk	Greasy	Large	Very long
10	Woolen	Non greasy	Small	Short
11	Woolen	Non greasy	Medium	Medium
12	Woolen	Non greasy	Large	Long
13	Woolen	Medium	Small	Medium
14	Woolen	Medium	Medium	Medium
15	Woolen	Medium	Large	Long
16	Woolen	Greasy	Small	Long
17	Woolen	Greasy	Medium	Long
18	Woolen	Greasy	Large	Very long
19	Cotton	Non greasy	Small	Short









Fuzzy vs. Probability

Fuzzy : When we say about certainty of a thing

Example: A patient come to the doctor and he has to diagnose so that medicine can be prescribed.

Doctor prescribed a medicine with certainty 60% that the patient is suffering from flue. So, the disease will be cured with certainty of 60% and uncertainty 40%. Here, in stead of flue, other diseases with some other certainties may be.

Probability: When we say about the chance of an event to occur

Example: India will win the T20 tournament with a chance 60% means that out of 100 matches, India own 60 matches.

Prediction vs. Forecasting

The Fuzzy vs. Probability is analogical to Prediction vs. Forecasting

Prediction : When you start guessing about things.

Forecasting : When you take the information from the past job and apply it to new job.

The main difference:

- **Prediction** is an actual act of indicating that something will happen in the future with or without prior information.

- **Forecasting** refers to a calculation or an estimation which uses data from previous events, combined with recent trends to come up a future event outcome.
- In short, all forecasts are predictions but not all predictions are forecasts. Anyone can do a prediction since it does not require any special skills but for one to do forecasting he/she requires special skills.

FORECASTING VERSUS PLANNING

Basis of Comparison	Forecasting	Prediction
Meaning	Process of creating future predictions with relevant data	Process of creating future predictions with or without relevant data
Accuracy	More accurate	Lower probability of happening
Application	Mostly applied in the meteorology, economic and financial sectors	Can be applied almost anywhere
Bias	Forecasts are generated from calculation and data assessment	Is subject to bias
Quantification	Easily Quantified	Can't be quantified
Basis	Done using scientific methods	Arrived at by arbitrary methods e.g. instincts
Application level	Aggregate level	Customer level

Fuzzy Membership Functions

Fuzzy membership functions

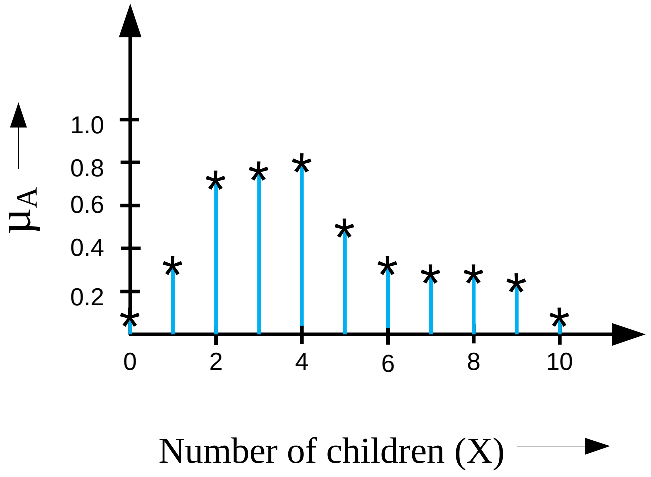
A fuzzy set is completely characterized by its membership function (sometimes abbreviated as MF and denoted as μ). So, it would be important to learn how a membership function can be expressed (mathematically or otherwise).

Note: A membership function can be on

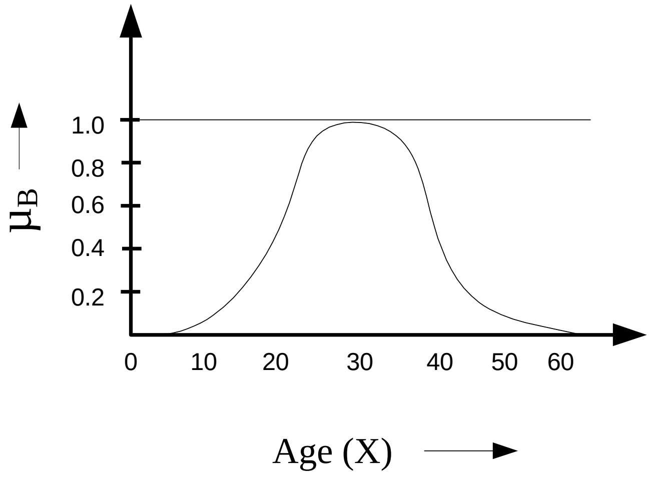
a) a discrete universe of discourse and

b) a continuous universe of discourse.

Example:



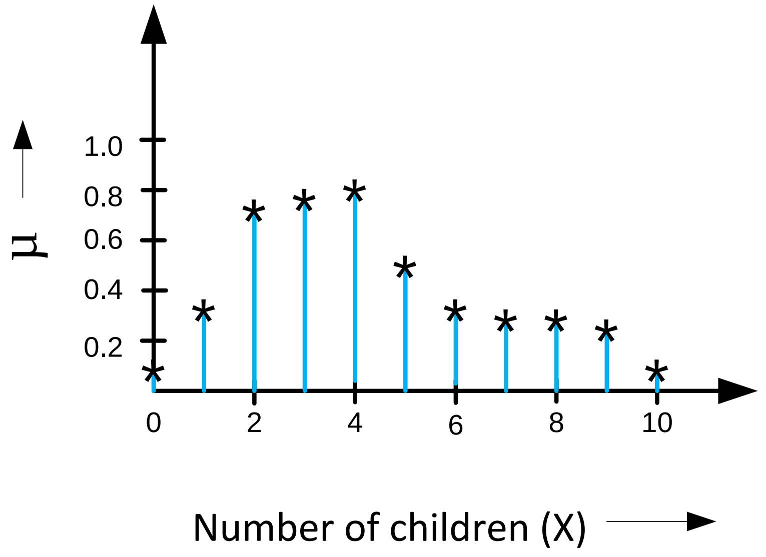
A = Fuzzy set of "Happy family"



B = "Young age"

Membership function with discrete membership values

Either elements or their membership values (or both) also may be of discrete values.



A = "Happy family"

A ={(0,0.1),(1,0.30),(2,0.78)……(10,0.1)}

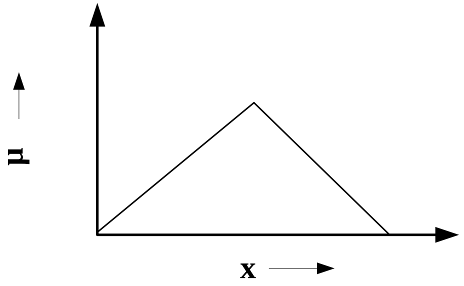
Note : X = discrete value

How you measure happiness ??

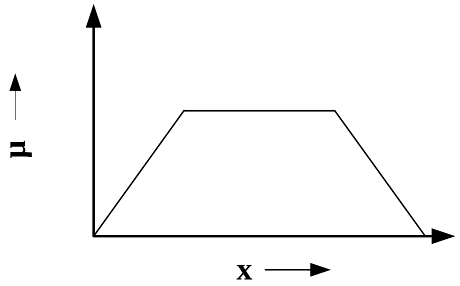
Fuzzy membership functions

So, membership function on a discrete universe of course is trivial. However, a membership function on a continuous universe of discourse needs a special attention.

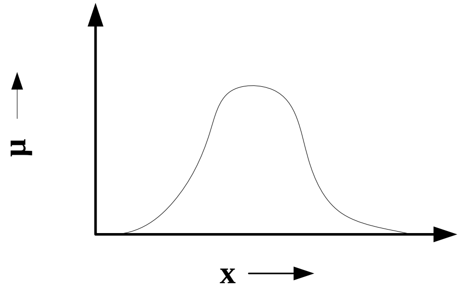
Following figures shows typical examples of membership functions.



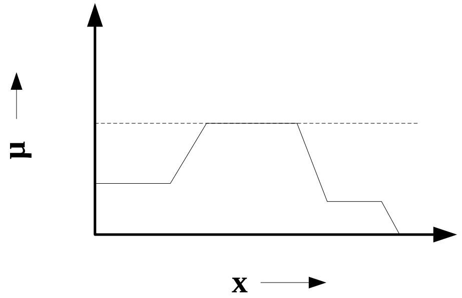
< triangular >



< trapezoidal >



< curve >



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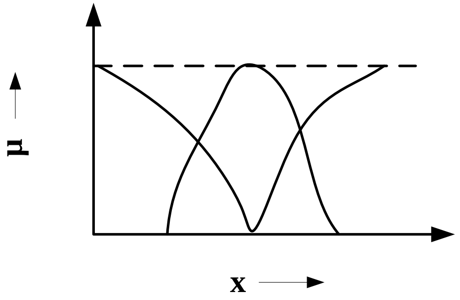
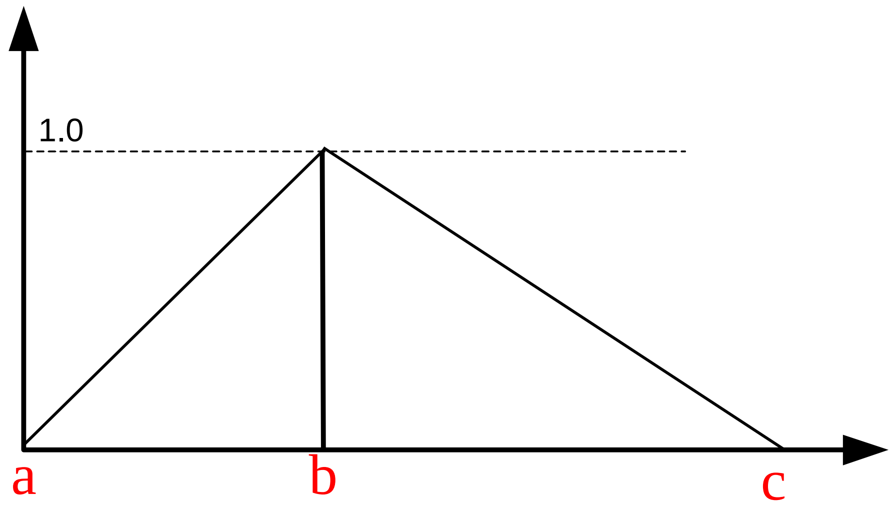


< non-uniform >

Fuzzy MFs : Formulation and parameterization

In the following, we try to parameterize the different MFs on a continuous universe of discourse.

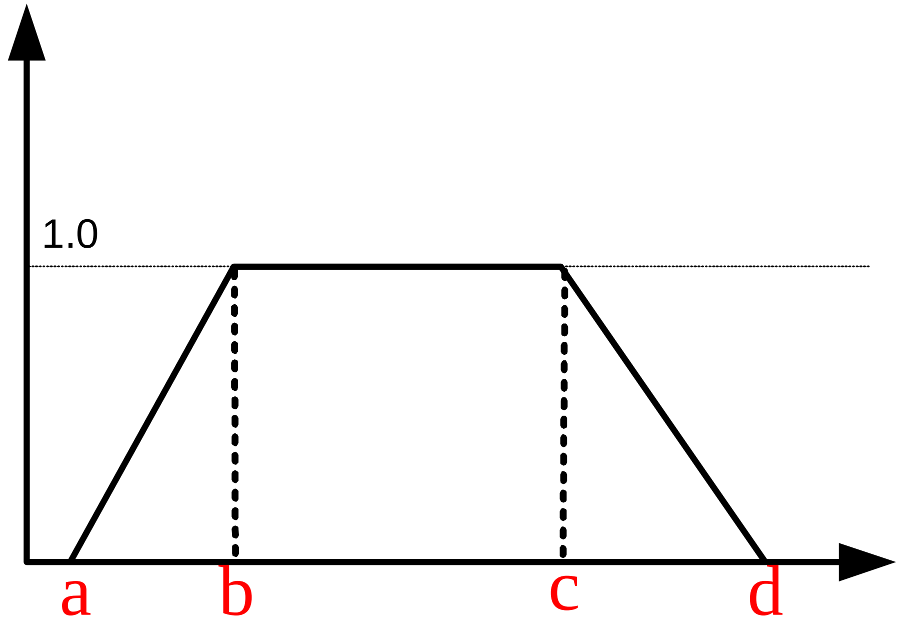
Triangular MFs : A triangular MF is specified by three parameters $\{a, b, c\}$ and can be formulated as follows.



$$\text{triangle}\,(x; a, b, c) = \left\{ \begin{array}{cc} 0 & if\ x \leq a \\ \dfrac{x-a}{b-a} & if\ a \leq x \leq b \\ \dfrac{c-x}{c-b} & if\ b \leq x \leq c \\ 0 & if\ c \leq x \end{array} \right\}$$

Fuzzy MFs: Trapezoidal

A **trapezoidal MF** is specified by four parameters $\{a, b, c, d\}$ and can be defined as follows:

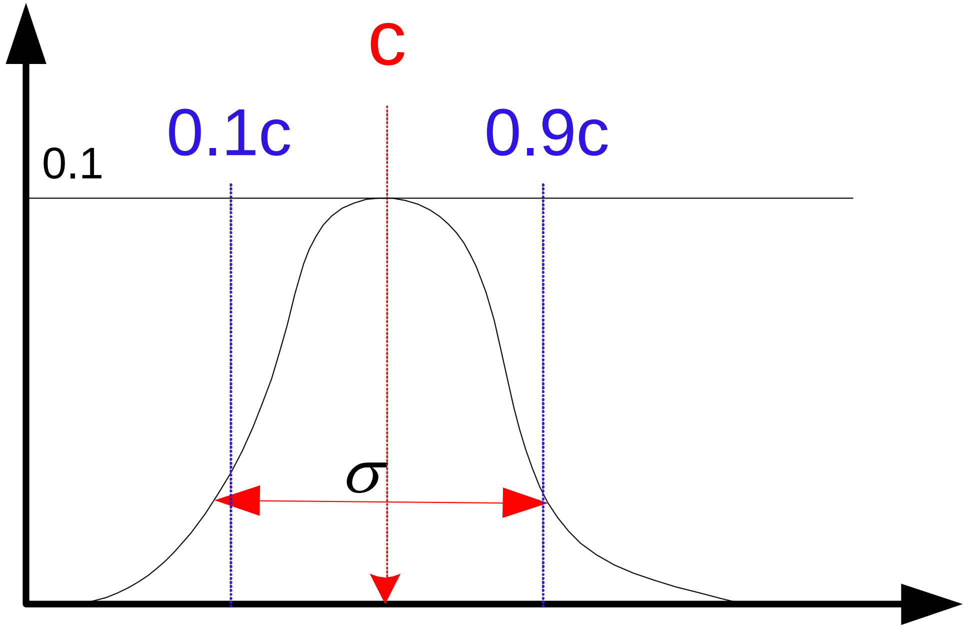


$$\text{trapozoid}\,(x; a, b, c, d) = \begin{cases} 0 & if\ x \le a \\ \dfrac{x-a}{b-a} & if\ a \le x \le b \\ 1 & if\ b \le x \le c \\ \dfrac{d-x}{d-c} & if\ c \le x \le d \\ 0 & if\ d \le x \end{cases}$$

Fuzzy MFs: Gaussian

A **Gaussian MF** is specified by two parameters $\{c, \sigma\}$ and can be defined as below:

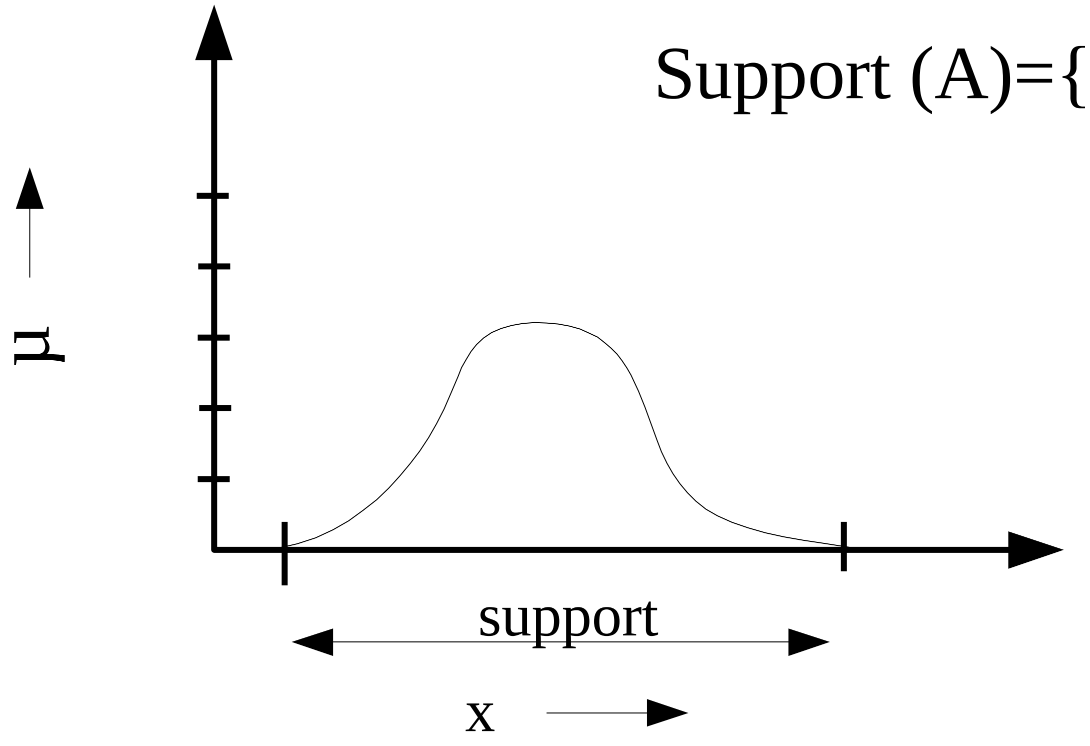
$$gaussian\ (x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$



Fuzzy terminologies: Support

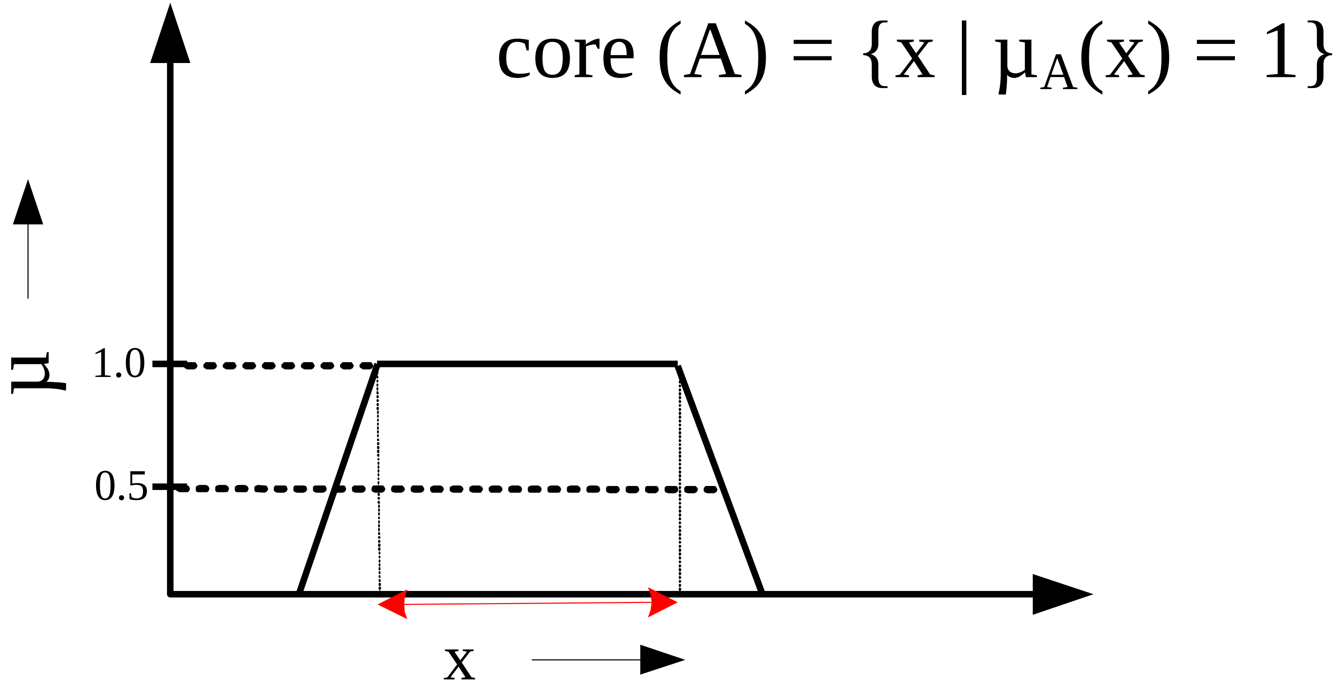
Support: The support of a fuzzy set A is the set of all points $x \in X$ such that $\mu_A(X) > \ 0$

Support (A)={x | μ_A(x) > 0}



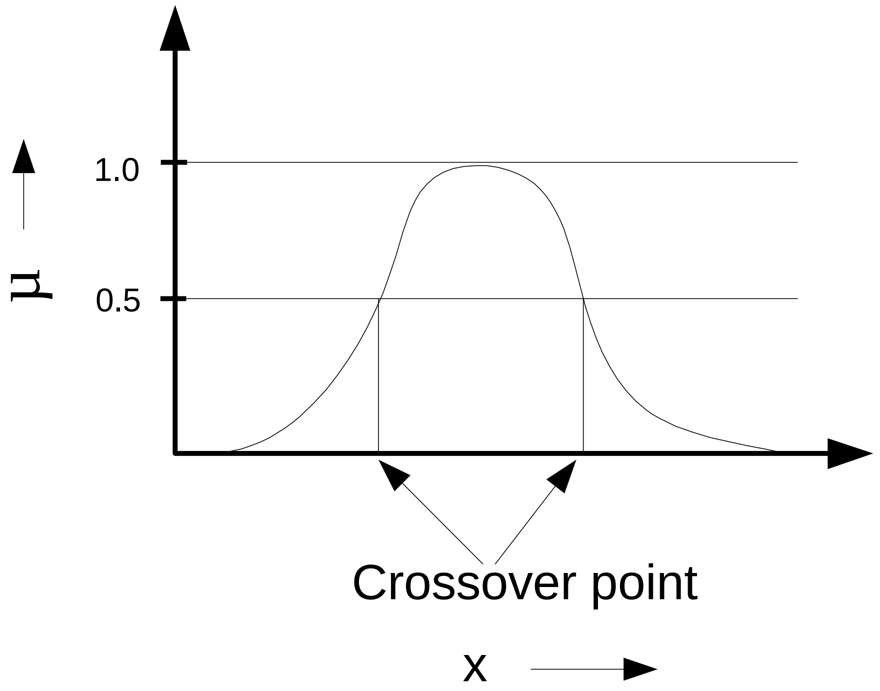
Fuzzy terminologies: Core

Core: The core of a fuzzy set A is the set of all points x in X such that $\mu_A(X) = 1$



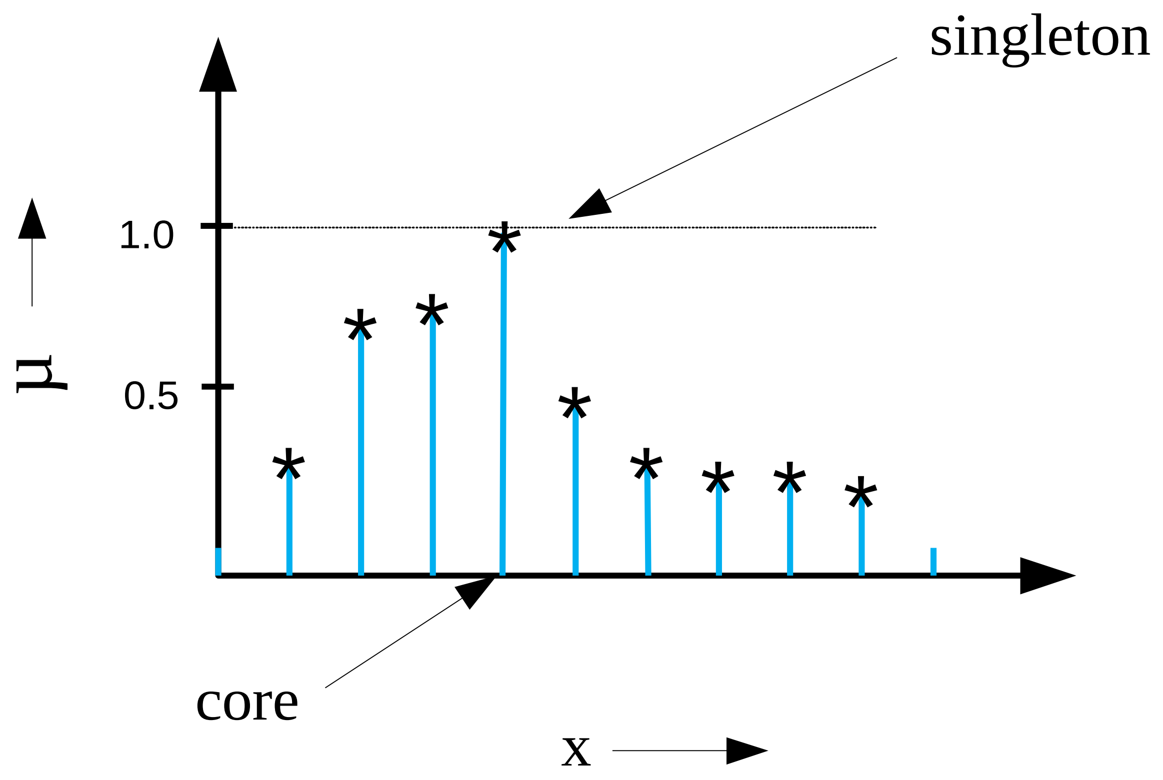
Fuzzy terminologies: Crossover points

Crossover point : A crossover point of a fuzzy set A is a point $x \in X$ at which $\mu_A(X) = 0.5$. That is Crossover (A) $=\{x|\mu_A(x) = 0.5\}$



Fuzzy terminologies: Fuzzy Singleton

Fuzzy Singleton : A fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called a fuzzy singleton. That is $|A| = \{x|\mu_A(x) = 1\}$

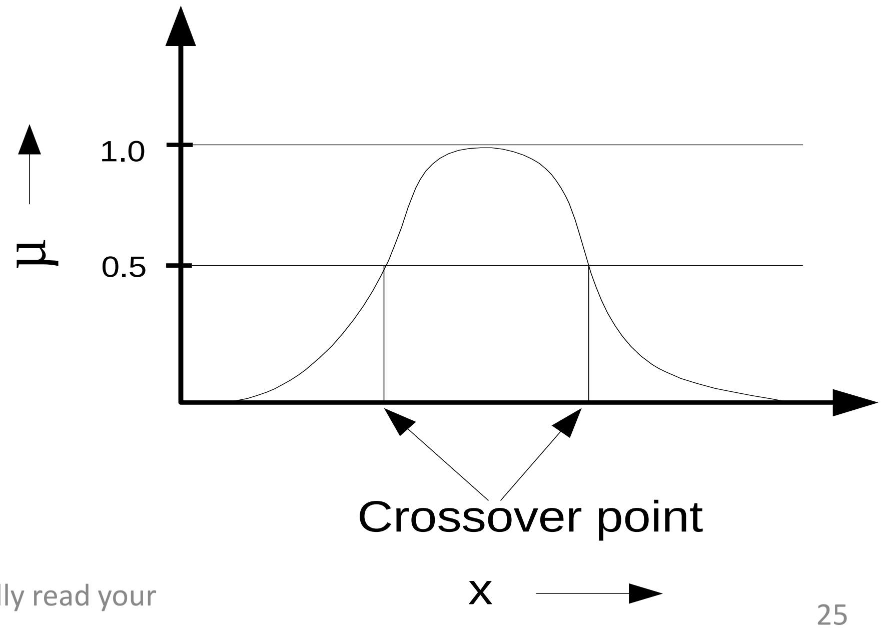


Fuzzy terminologies: Bandwidth

Bandwidth :
For a fuzzy set, the bandwidth (or width) is defined as the distance between the two unique crossover points:

$$\text{Bandwidth }(A) = |x_1 - x_2|$$

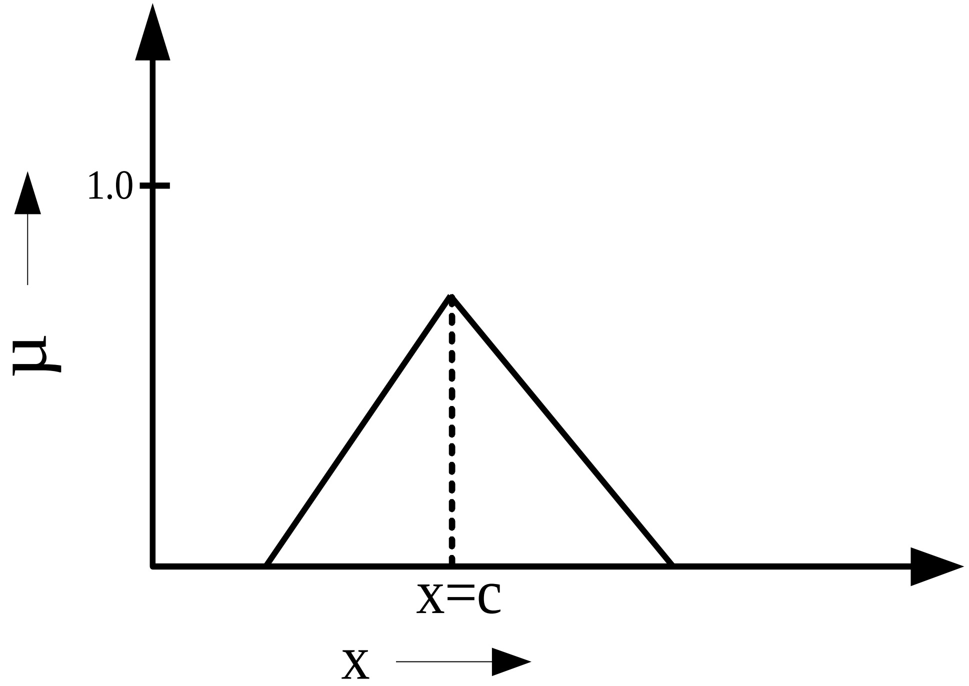
where $\mu_A(x_1) = \mu_A(x_2) = 0.5$



Fuzzy terminologies: Symmetry

Symmetry :

A fuzzy set A is symmetric if its membership function around a certain point $x = c$, namely $\mu_A(x + c) = \mu_A(x - c)$ for all $x \in X$



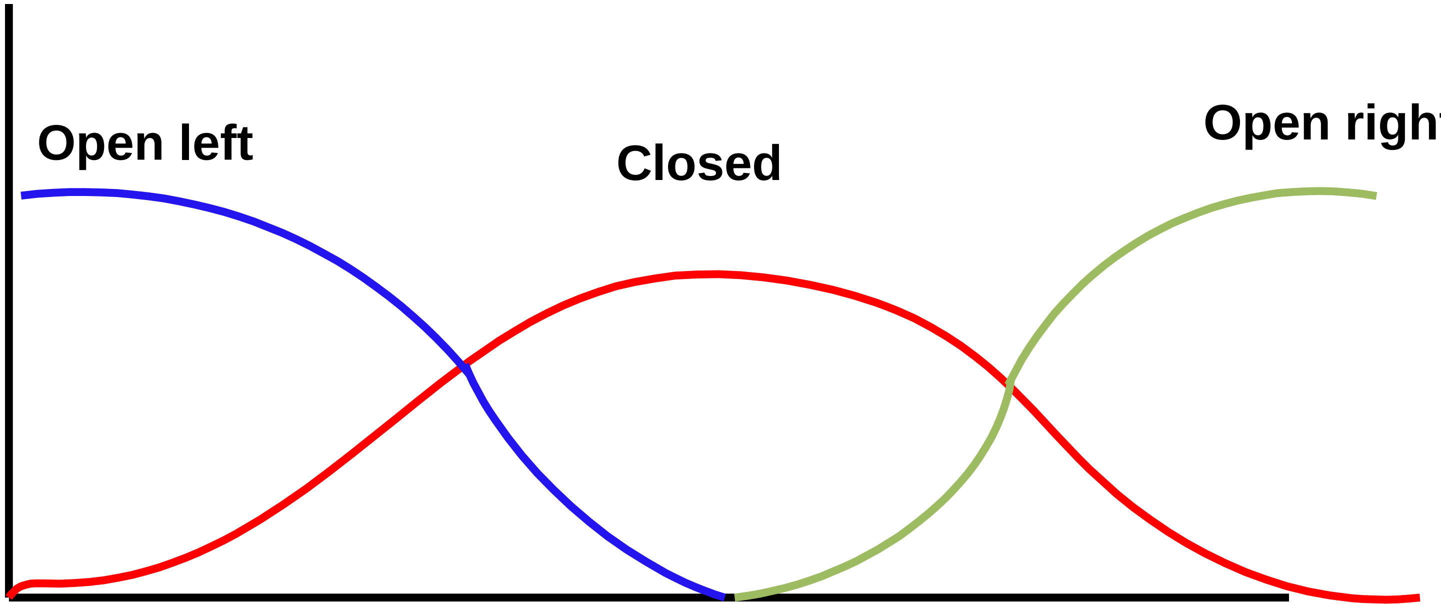
Fuzzy terminologies: Open and Closed

A fuzzy set A is

Open left : If $\lim x \to -\infty\ \mu_A(x) = 1$and $\lim x \to +\infty\ \mu_A(x) = 0$

Open right: If $\lim x \to -\infty\ \mu_A(x) = 0$and $\lim x \to +\infty\ \mu_A(x) = 1$

Closed: If $\lim x \to -\infty\ \mu_A(x) = \lim x \to +\infty\ \mu_A(x) = 0$



Operations on Fuzzy Sets

Basic fuzzy set operations: Union

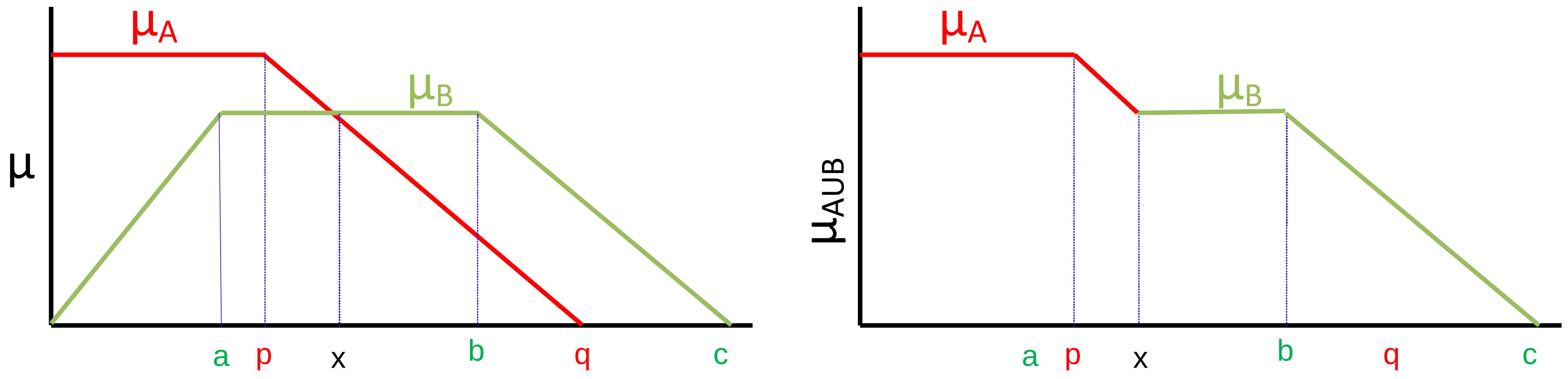
Union (A ∪ B) $\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and

$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;

$C = A \cup B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.5)\}$



Basic fuzzy set operations: Intersection

Intersection (A ∩ B

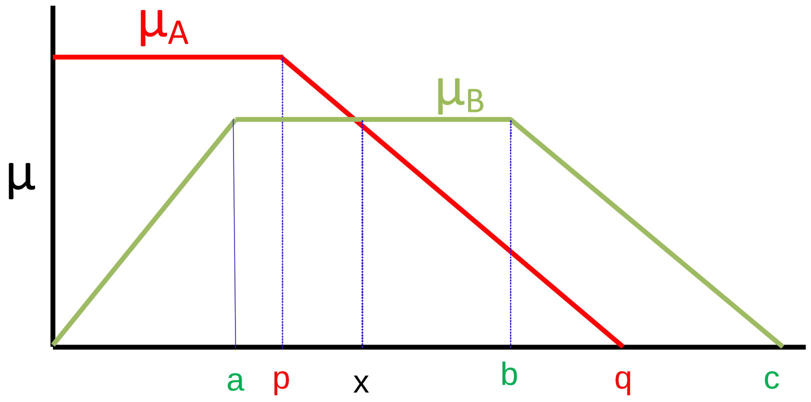
$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

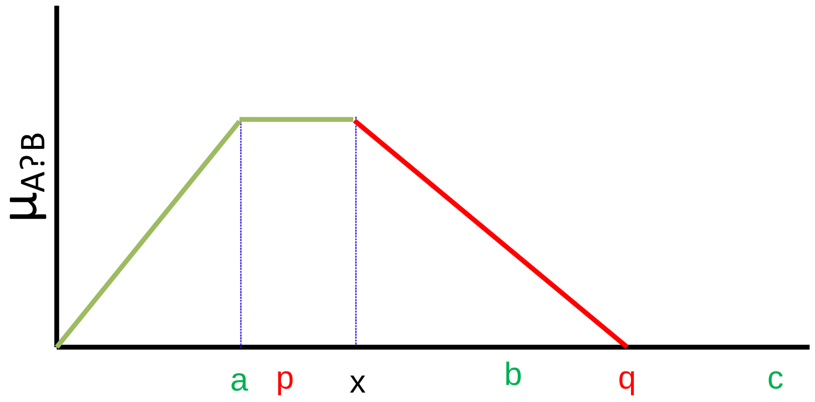
Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and

$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\};$

$C = A \cap B = \{(x_1, 0.2), (x_2, 0.1), (x_3, 0.4)\}$





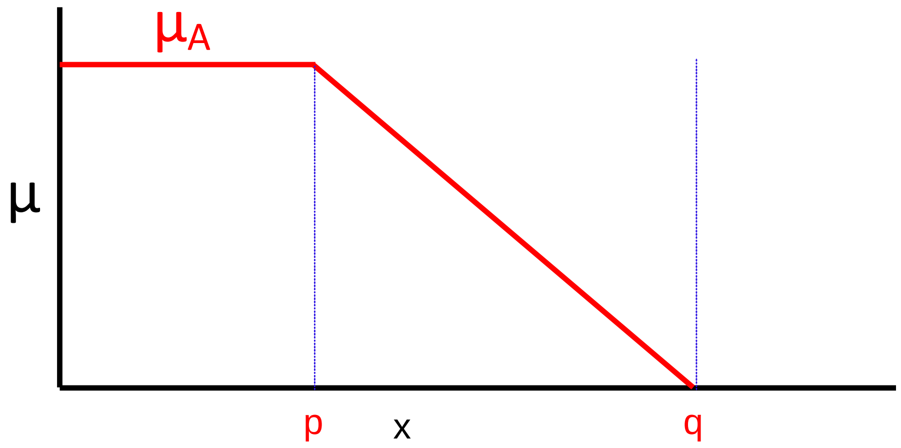
Basic fuzzy set operations: Complement

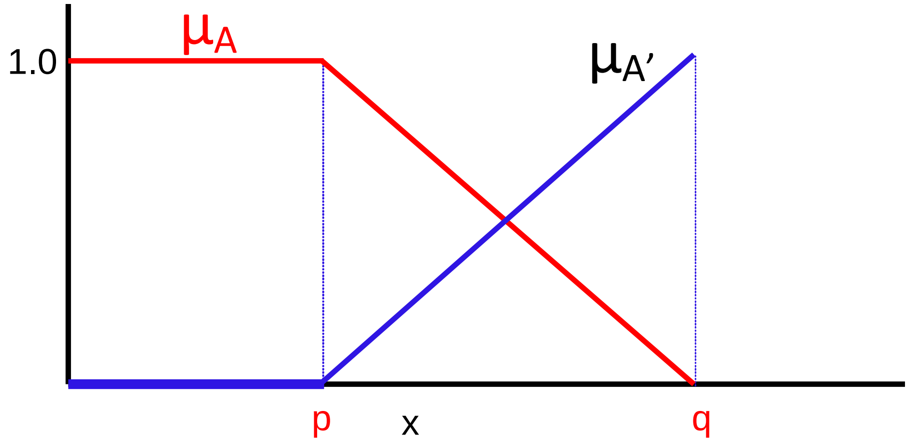
Complement (A^c) $\mu_{A^c}(x) = 1 - \mu_A(x)$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$

$C = A^c = \{(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)\}$





Basic fuzzy set operations: Products

Algebric product or Vector product ($A \cdot B$):

$$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$$

Scalar product ($\alpha \times A$):

$$\mu_{\alpha A}(x) = \alpha \times \mu_A(x)$$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and

$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;

Basic fuzzy set operations: Sum and Difference

Sum (A + B):

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x)$$

Difference ($A - B = A \cap B^C$) :

$$\mu_{A-B}(x) = \mu_{A \cap B^C}(x)$$

Disjunctive sum:

$$A \oplus B = (A^C \cap B) \cup (A \cap B^C)$$

Bounded Sum:

$$| A(x) \oplus B(x) |= \mu_{|A(x) \oplus B(x)|} = \min\{1, \mu_A(x) + \mu_B(x)\}$$

Bounded Difference:

$$| A(x) \ominus B(x) | = \mu_{|A(x) \ominus B(x)|} = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$$

Q1) $\underset{\sim}{A} = \left\{ \frac{1}{1} + \frac{0.5}{1.5} + \frac{0.3}{2.0} + \frac{0.4}{2.5} \right\}$

$\underset{\sim}{B} = \left\{ \frac{0.4}{1} + \frac{0.2}{1.5} + \frac{0.7}{2.0} + \frac{0.1}{2.5} \right\}$

Sol^n: Algebraic Sum

$$\mu_{A+B}(x) = [\mu_A(x) + \mu_B(x)] - [\mu_A(x) \cdot \mu_B(x)]$$

$$1.4 - 0.4 = 1$$
$$0.7 - 0.1 = 0.6$$
$$1.0 - 0.21 = 0.79$$
$$0.5 - 0.04 = 0.46$$

$$= \left\{ \frac{1}{1} + \frac{0.6}{1.5} + \frac{0.79}{2.0} \quad \frac{0.46}{2.5} \right\}$$

Sol^n: Algebraic Product $- \boxed{\mu_{A.B} = \mu_A(x) \cdot \mu_B(x)}$

$$\mu_{A.B} = \left\{ \frac{0.4}{1} + \frac{0.1}{1.5} + \frac{0.21}{2.0} + \frac{0.04}{2.5} \right\}$$

FUZZY SET OPERATIONS

Q1) $\underset{\sim}{A} = \left\{ \frac{1}{1} + \frac{0.5}{1.5} + \frac{0.3}{2.0} + \frac{0.4}{2.5} \right\}$

$\underset{\sim}{B} = \left\{ \frac{0.4}{1} + \frac{0.2}{1.5} + \frac{0.7}{2.0} + \frac{0.1}{2.5} \right\}$

Soln: Bounded Sum $\boxed{\mu_{A \oplus B} = \min[1, (\mu_A + \mu_B)]}$

$\mu_{A+B}(1) = \min[1, (1+0.4)] = \min(1, 1.4) = 1$

$\mu_{A+B}(2) = \min[1, (0.7)] = 0.7$

$(3) = \min[1, 1] = 1$

$(4) = \min[1, (0.5)] = 0.5$

$$\boxed{\mu_{A \oplus B} = \left\{ \frac{1}{1} + \frac{0.7}{1.5} + \frac{1}{2.0} + \frac{0.5}{2.5} \right\}}$$

$$\mu_{A\ominus B} = Max\left[0, \left(\mu_A(x) - \mu_B(x)\right)\right]$$

$$\underset{\sim}{A} = \left\{ \frac{1}{1.4} + \frac{0.5}{1.5} + \frac{0.3}{2.0} + \frac{0.4}{2.5} \right\}$$

$$\underset{\sim}{B} = \left\{ \frac{0.4}{1.4} + \frac{0.2}{1.5} + \frac{0.7}{2.0} + \frac{0.1}{2.5} \right\}$$

FOR 1st value $= Max\ [0, (1 - 0.4)]$

$= Max\ [0, 0.6] = \underline{0.6}$

2nd vale $= Max\ [0, (0.5 - 0.2)]$

$= Max\ [0, 0.3] = \underline{0.3}$

3rd vale $= Max\ [0, (0.3 - 0.7)]$

$= Max\ [0, -0.4] = \underline{0}$

4th valu $= Max\ [0, (0.4 - 0.1)]$

$= Max\ [0, 0.3] = \underline{0.3}$

$$\mu_{A\ominus B} = \left\{ \frac{0.6}{1.4} + \frac{0.3}{1.5} + \frac{0}{2.0} + \frac{0.3}{2.5} \right\}$$

Basic fuzzy set operations: Cartesian product

Caretsian Product ($A \times B$): $\mu_{A\times B}(x, y) = \min(\mu_A(x), \mu_B(y))$

Example:

$$A(x) = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5), (x_4, 0.6)\}$$

$$B(y) = \{(y_1, 0.8), (y_2, 0.6), (y_3, 0.3)\}$$

$$\mathrm{A} \times B = \min(\mu_A(x), \mu_B(y)) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.3 \end{bmatrix} \end{array}$$

Properties of fuzzy sets

Commutativity :

$$A \cap B = B \cap A$$
$$A \cup \mathrm{B} = B \cup \mathrm{A}$$

Associativity :

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributivity :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Properties of fuzzy sets

Idempotence :

$$A \cup A = A$$
$$A \cap A = \emptyset\,;$$
$$A \cup \emptyset; = A$$
$$A \cap \emptyset\,; = \emptyset;$$

Transitivity :

If $A \subseteq B; B \subseteq C$ then $A \subseteq C$

Involution :

$$(A^c)^c = A$$

De Morgan's law :

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

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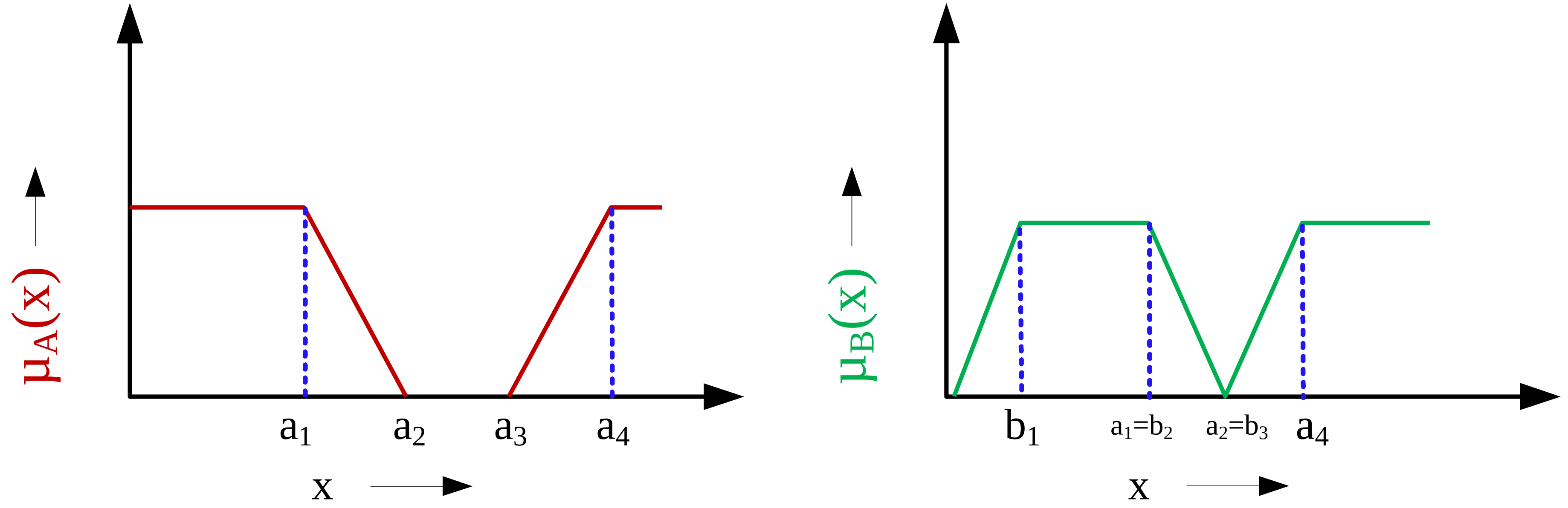
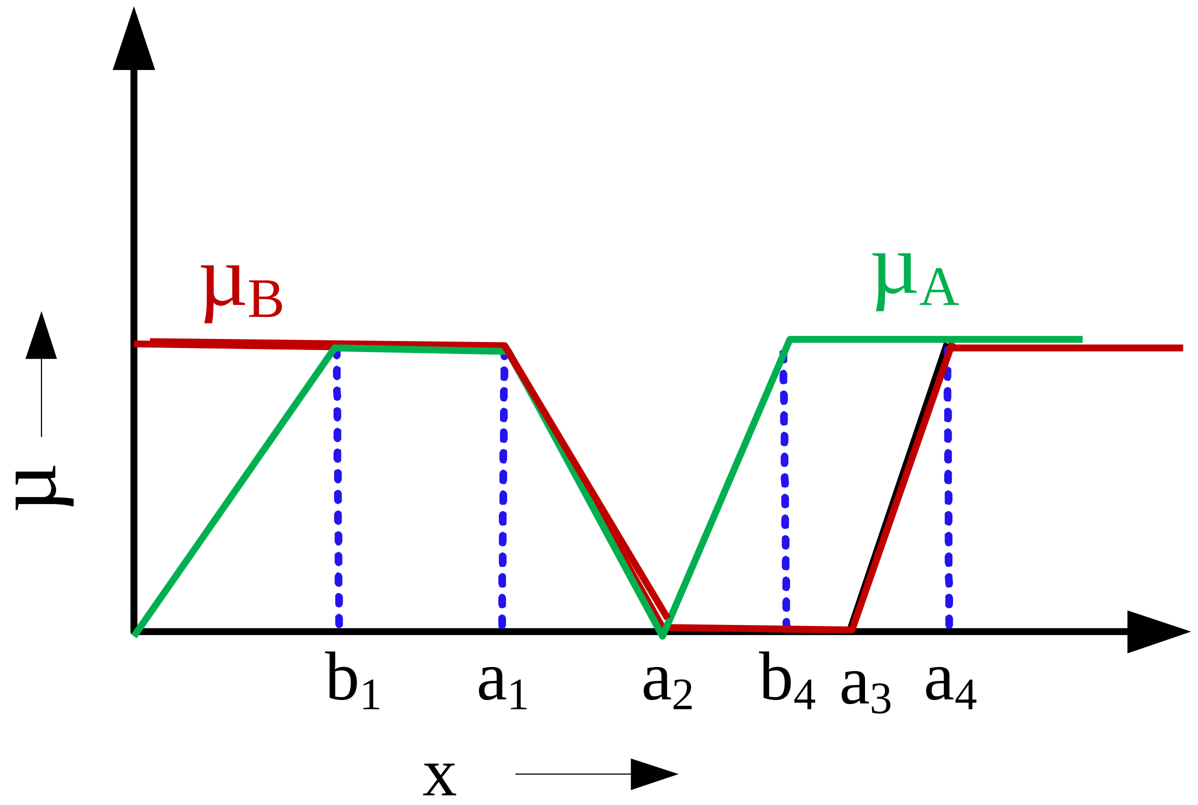
Example 1: Fuzzy Set Operations

Let A and B are two fuzzy sets defined over a universe of discourse X with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively. Two MFs $\mu_A(x)$ and $\mu_B(x)$ are shown graphically.



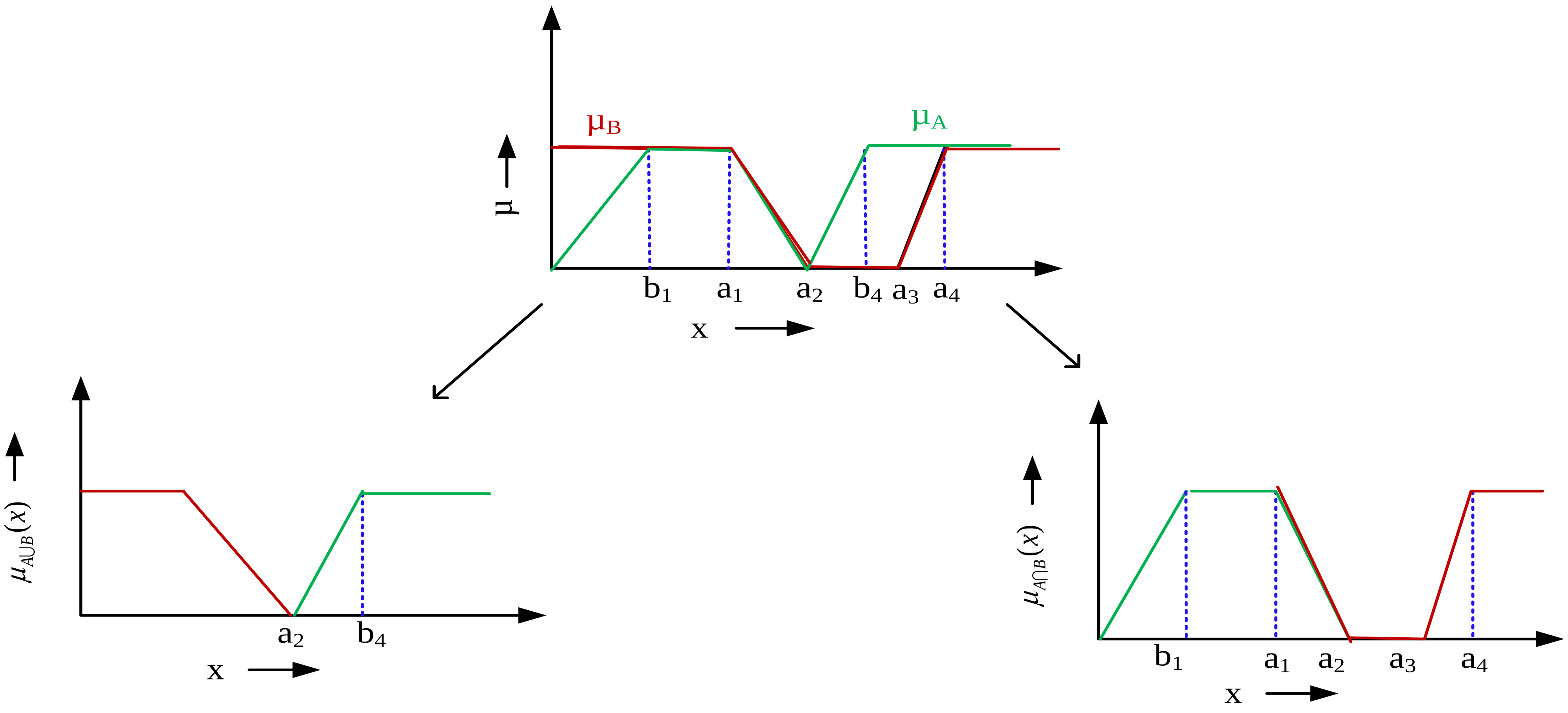
Example 1: Plotting two sets on the same graph

Let’s plot the two membership functions on the same graph



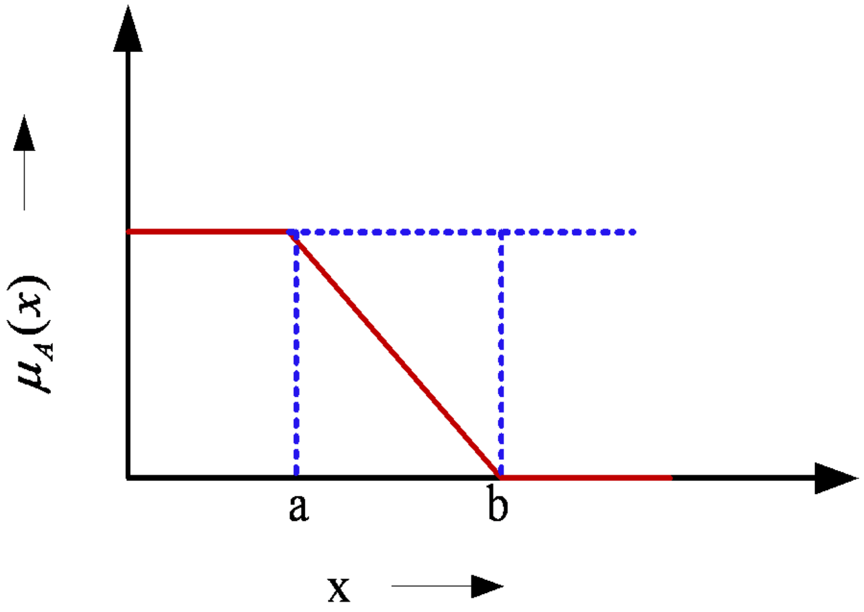
Example 1: Union and Intersection

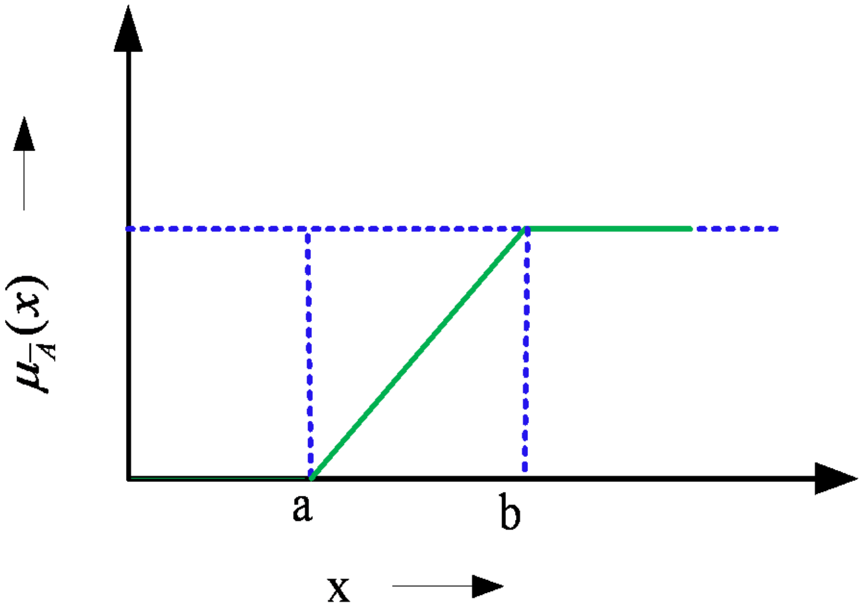
The plots of union $A \cup B$ and intersection $A \cap B$ are shown in the following.



Example 1: Complementation

The plots of union $\mu_{\bar{A}}(x)$ of the fuzzy set A is shown in the following.





Fuzzy set operations: Practice

Consider the following two fuzzy sets A and B defined over a universe of discourse [0,5] of real numbers with their membership functions

$$\mu_A(x) = \frac{x}{1+x} \text{ and } \mu_B(x) = 2^{-x}$$

Determine the membership functions of the following and draw them graphically.

I. $\bar{A}, \bar{B}$

II. $A \cup B$

III. $A \cap B$

IV. $(A \cup B)^c$

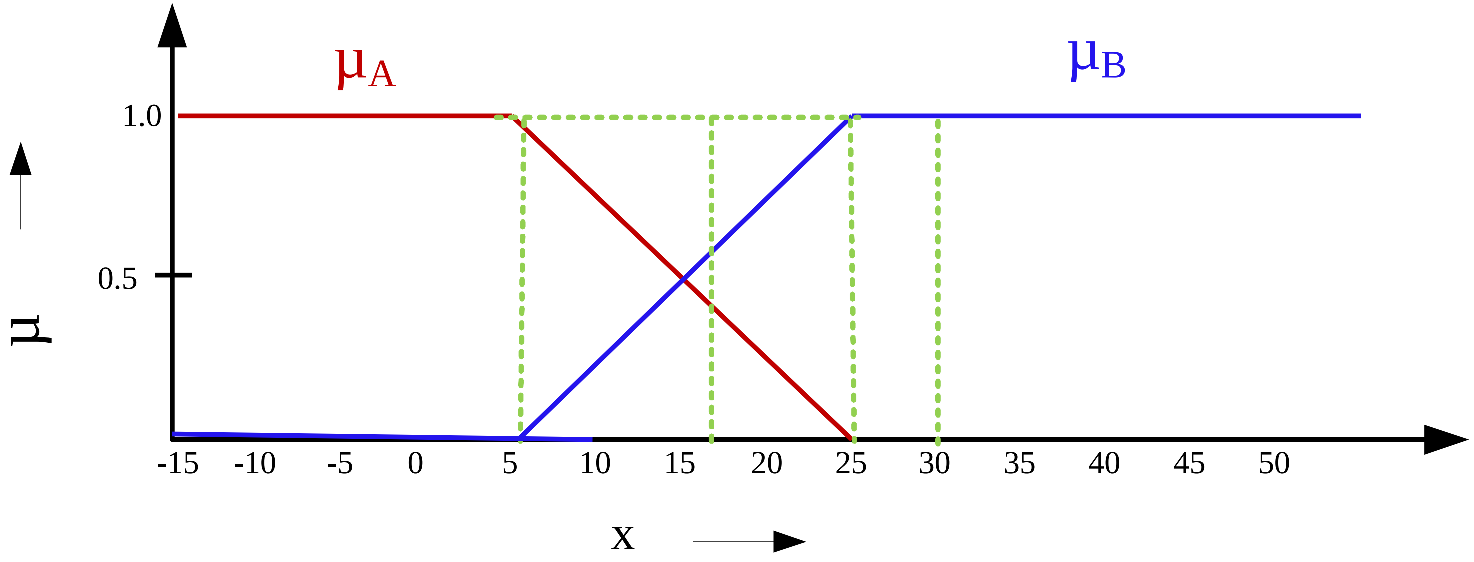
Example 2: A real-life example

Two fuzzy sets A and B with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively defined as below.

A = **Cold climate** with $\mu_A(x)$ as the MF.

B = **Hot climate** with $\mu_B(x)$ as the M.F.

Here, X being the universe of discourse representing entire range of temperatures.



Example 2: A real-life example

What are the fuzzy sets representing the following?

1. **Not cold climate**
2. **Not hot climate**
3. **Extreme climate**
4. **Pleasant climate**

Note: Note that ”Not cold climate” $\neq$ ”Hot climate” and vice-versa.

Example 2: A real-life example

Answer would be the following.

- ✓ **Not cold climate**

  $\bar{A}$ with $1 - \mu_A(x)$ as the MF.

- ✓ **Not hot climate**

  $\bar{B}$ with $1 - \mu_B(x)$ as the MF.

- ✓ **Extreme climate**

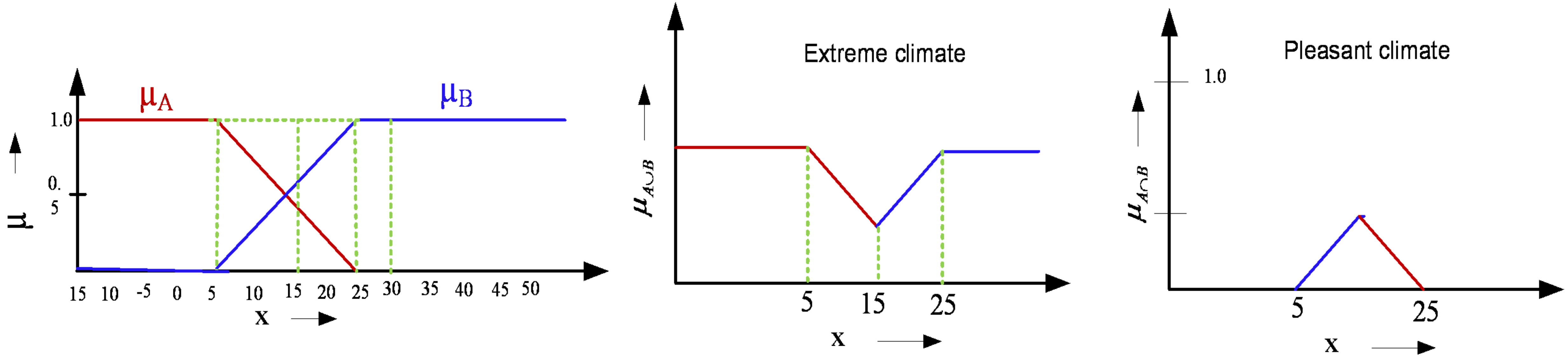
  $\mathrm{A} \cup B$ with $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$ as the MF.

- ✓ **Pleasant climate**

  $\mathrm{A} \cap B$ with $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$ as the MF.

Example 2: A real-life example

The plot of the MFs of $A \cup B$ and $A \cap B$ are shown in the following.



Fuzzy Relations

- Crisp relations
- Operations on crisp relations
- Examples on crisp relations
- Fuzzy relations
- Operations on fuzzy relations
- Examples on fuzzy relations

Crisp relations

- **Order pairs:**

  Suppose, A and B are two (crisp) sets. Then Cartesian product denoted as $A \times B$ is a collection of order pairs, such that

$$A \times B = \{(a, b) | a \in A \; and \; b \in B\}$$

Note :

(1) $A \times B \neq B \times A$ (2) $|A \times B| = |A| \times |B|$

(3) $A \times B$ provides a mapping from $a \in A$ to $b \in B$.

A particular mapping so mentioned is called a relation.

Crisp relations

Example:

Consider the two crisp sets A and B as given below.
$A = \{1, 2, 3, 4\}$ $B = \{3, 5, 7\}$.

Then, $A \times B = \{(1,3), (1,5), (1,7), (2,3), (2,5), (2,7), (3,3), (3,5), (3,7), (4,3), (4,5), (4,7)\}$

Let us define a relation as $R = \{(a, b) | b = a + 1, (a, b) \in A \times B\}$

Then, $R = \{(2,3), (4,5)\}$ in this case.

Crisp relations

We can represent the relation R in a matrix form as follows.

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 5 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Operations on crisp relations

Suppose, $R(x,y)$ and $S(x,y)$ are the two relations defined over two crisp sets $x \in A$ and $y \in B$

- **Union:** $R(x,y) \cup S(x,y) = max(R(x,y), S(x,y));$
- **Intersection:** $R(x,y) \cap S(x,y) = min(R(x,y), S(x,y));$
- **Complement:** $\overline{R(x,y)} = 1 - R(x,y)$

Example: Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations defined over two crisp sets $x \in A$ and $y \in B$

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \quad S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the following

- $R \cup S$
- $R \cap S$
- $\bar{R}$

Composition of two crisp relations

Given R is a relation on X, Y and S is another relation on Y, Z. Then, $R \circ S$ is called a composition of relation on X and Z which is defined as follows.

$$R \circ S = \{(x, z) | (x, y) \in R \; and \; (y, z) \in S \; and \; \forall y \in Y\}$$

Max-Min Composition

Given the two relation matrices R and S, the max-min composition is defined as $T = R \circ S$;

$$T(x, z) = max\{min\{R(x, y), S(y, z) \; and \; \forall y \in Y\}\}$$

Composition: Composition

Example : Given $X = \{1,3,5\}$; $Y = \{1,3,5\}$; $R = \{(x,y)|y = x + 2\}$;
$S = \{(x,y)|x < y\}$

Here, R and S is on $X \times Y$.

Thus, we have $R = \{(1,3),(3,5)\}$, $S = \{(1,3),(1,5),(3,5)\}$

$$R = \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad and \quad S = \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Using max-min composition

$$R \circ S = \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fuzzy relations

- Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp set $X_1, X_2, \dots, X_n$
- Here, n-tuples $(x_1, x_2, \dots, x_n)$ may have varying degree of memberships within the relationship.
- The membership values indicate the strength of the relation between the

Fuzzy relations

Example:

$$X = \{typhoid, viral, cold\}, Y = \{running\ nose, high\ temp, shivering\}$$

The fuzzy relation R is defined as

$$R = \begin{array}{c|ccc} & running\ nose & high\ temperature & shivering \\ \hline typhoid & 0.1 & 0.9 & 0.8 \\ viral & 0.2 & 0.9 & 0.7 \\ cold & 0.9 & 0.4 & 0.6 \end{array}$$

Fuzzy Cartesian product

Suppose

- A is a fuzzy set on the universe of discourse X with $\mu_A(x)|x \in X$
- B is a fuzzy set on the universe of discourse Y with $\mu_B(y)|y \in Y$

Then $R = A \times B \subset X \times Y$; where R has its membership function given by $\mu_R(x, y) = \mu_{A\text{x}B}(x, y) = min\{\mu_A(x), \mu_B(y)\}$

Fuzzy Cartesian product

Example :

$A = \{(a1, 0.2), (a2, 0.7), (a3, 0.4)\}$ and $B = \{(b1, 0.5), (b2, 0.6)\}$

$$R = A \times B = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cc} \boldsymbol{b_1} & \boldsymbol{b_2} \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Operations on Fuzzy relations

Let R and S be two fuzzy relations on $A \times B$.

- **Union:** $\mu_{R \cup S}(a,b) = max\{\mu_R(a,b), \mu_S(a,b)\}$
- **Intersection:** $\mu_{R \cap S}(a,b) = min\{\mu_R(a,b), \mu_S(a,b)\}$
- **Complement:** $\mu_{\bar{R}}(a,b) = 1 - \mu_R(a,b)$
- **Composition:** $T = R \circ S$

$$\mu_{R \circ S} = max_{y \in Y}\{min\big(\mu_R(x,y), \mu_S(y,z)\big)\}$$

Operations on Fuzzy relations: Example

Example : $X = (x_1, x_2, x_3),\ \ Y = (y_1, y_2),\ \ Z = (z_1, z_2, z_3),$

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{array} \qquad and \qquad S = \begin{array}{c} \\ y_1 \\ y_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix} \end{array}$$

$$R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

$$\mu_{R\circ S}(x_1, y_1) = max\{min(\mu_R(x_1, y_1), \mu_S(y_1, z_1)), min(\mu_R(x_1, y_2), \mu_S(y_2, z_1))\}$$

$$= max\{min(0.5, 0.6), min(0.1, 0.5)\} = max\{0.5, 0.1\} = 0.5 \ and\ so\ on.$$

Fuzzy relation : An example

Consider the following two sets P and D, which represent a set of paddy plants and a set of plant diseases. More precisely

$P = \{P_1, P_2, P_3, P_4\}$ a set of four varieties of paddy plants

$D = \{D_1, D_2, D_3, D_4\}$ of the four various diseases affecting the plants.

In addition to these, also consider another set $S = \{S_1, S_2, S_3, S_4\}$ be the common symptoms of the diseases.

Let, R be a relation on $P \times D$, representing which plant is susceptible to which diseases, which is stated as

$$R = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.4 & 0.2 \end{bmatrix} \end{array}$$

Fuzzy relation : An example

Also, consider T be the another relation on $D \times S$, which is given by

$$S = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1.0 & 1.0 & 1.4 & 0.6 \\ 0.0 & 0.0 & 0.5 & 0.9 \\ 0.9 & 1.0 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

Obtain the association of plants with the different symptoms of the disease using **max-min composition**.

Hint: Find $R \circ T$, and verify that

$$R \circ S = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix} \end{array}$$

2D membership function : An example

Let, $X = R^{+} = y$ (the positive real line) and
$R = X \times Y$ = "y is much greater than x"

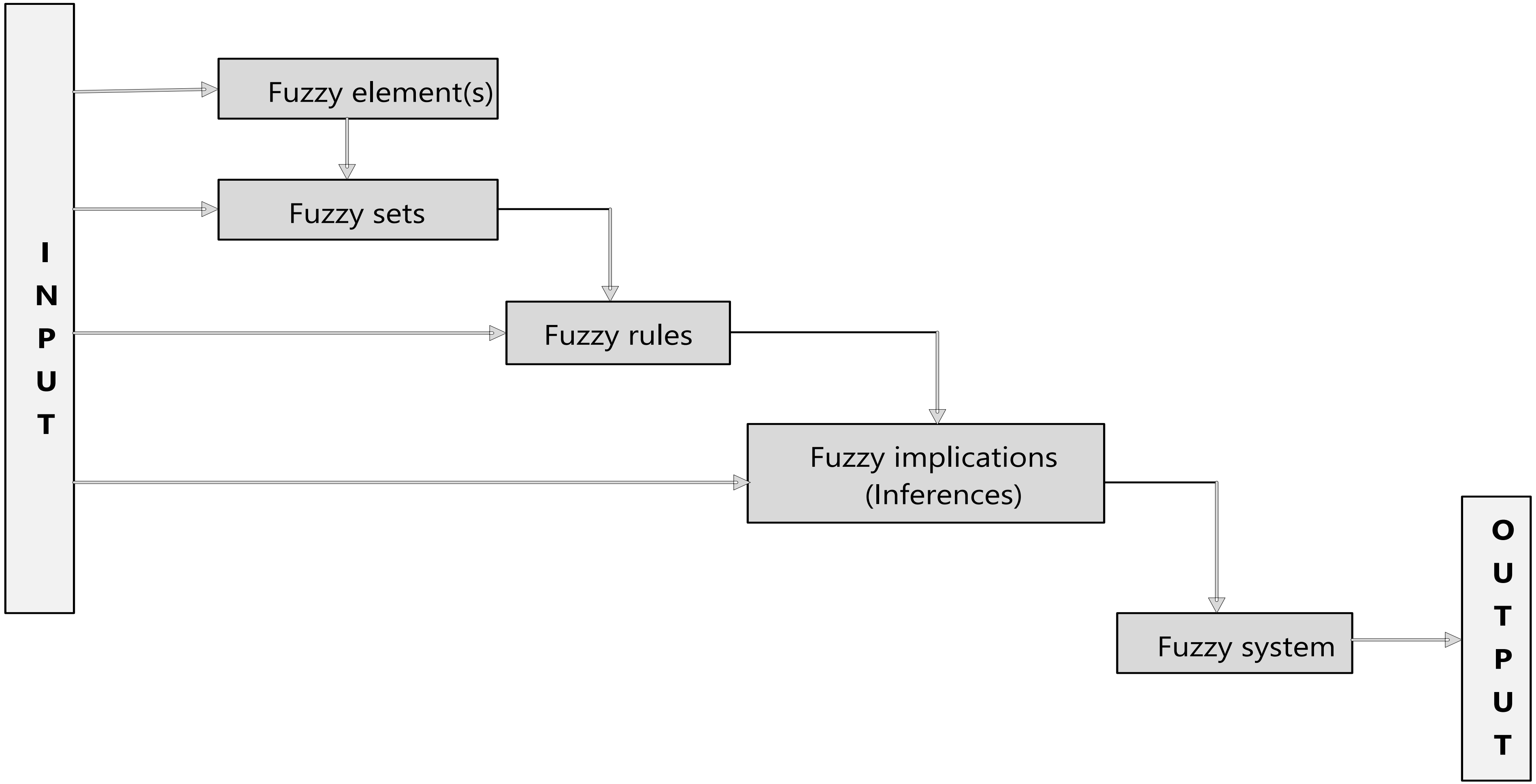
The membership function of $\mu_R(x, y)$ is defined as

$$\mu_R(x,y) = \begin{cases} \dfrac{(y-x)}{4} & if\ y > x \\ 0 & if\ y \le x \end{cases}$$

Suppose, $X = \{3,4,5\}$ and $Y = \{3,4,5,6,7\}$, then

$$R = \begin{array}{c} \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0.25 & 0.5 & 0.75 & 1.0 \\ 0 & 0 & 0.25 & 0.5 & 0.75 \\ 0 & 0 & 0 & 0.25 & 0.5 \end{bmatrix} \end{array}$$

Concept of fuzzy system



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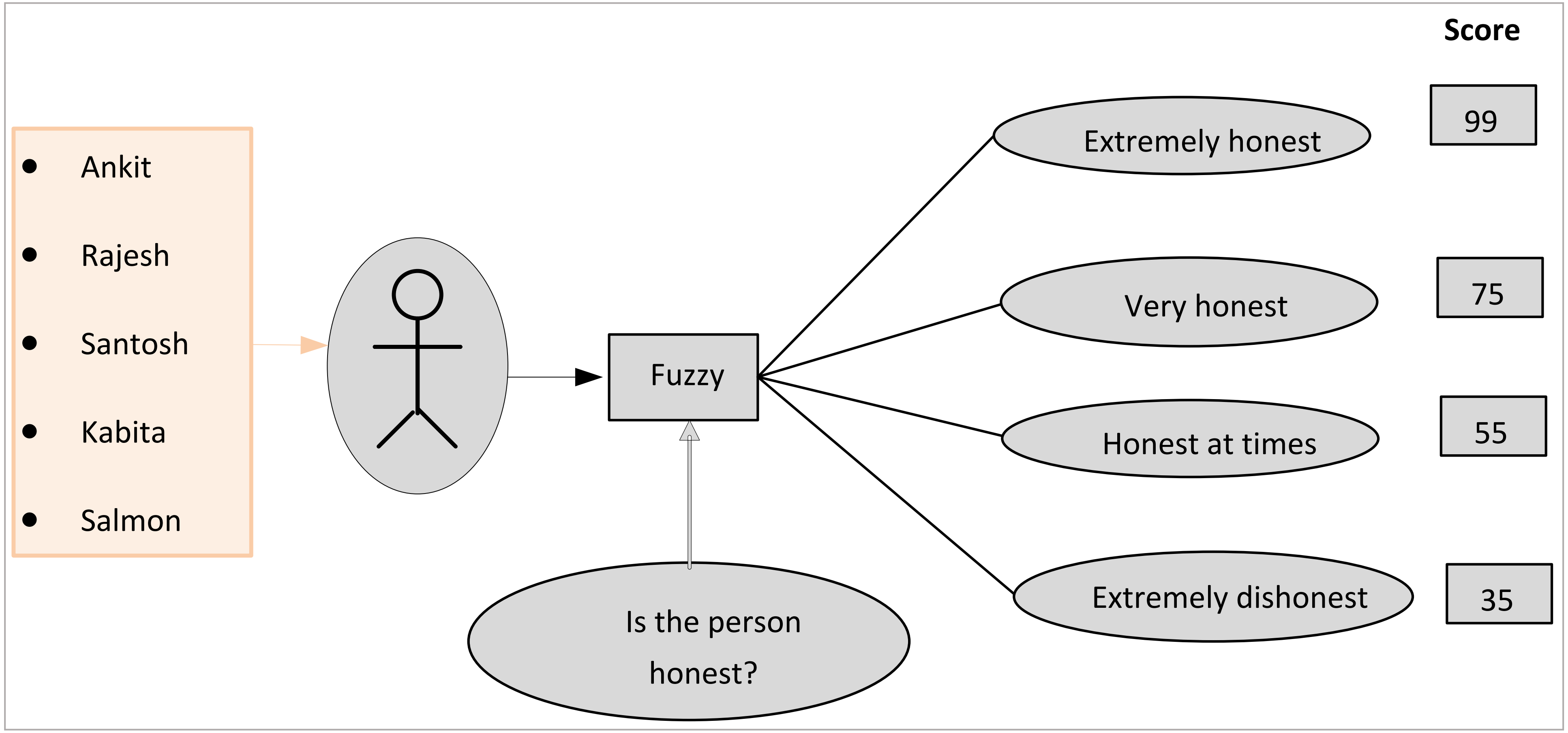
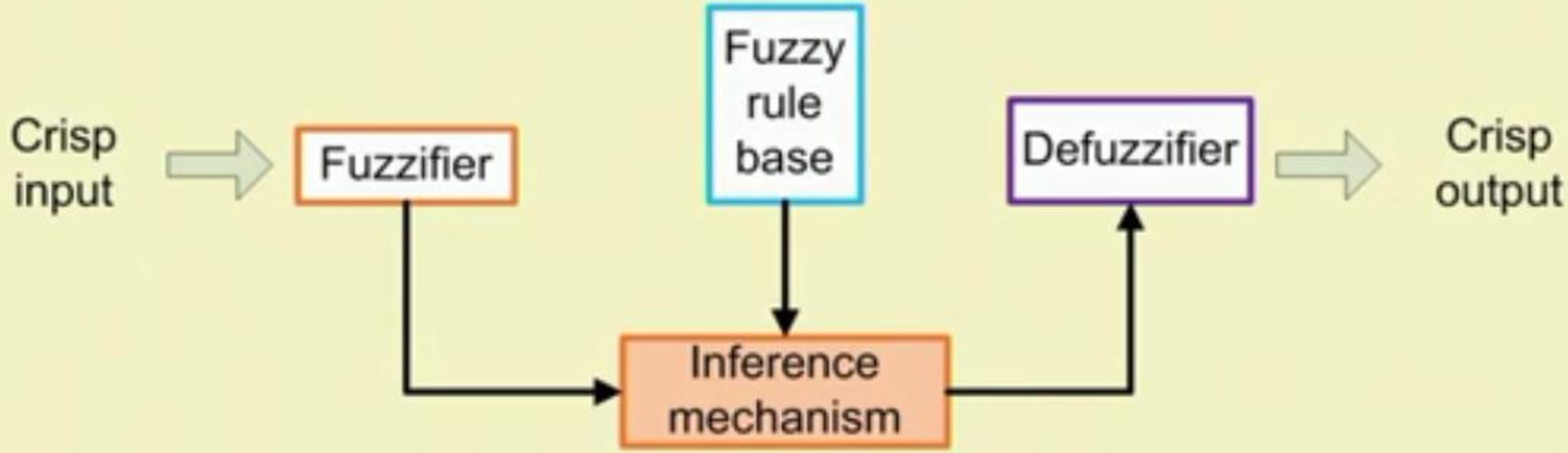
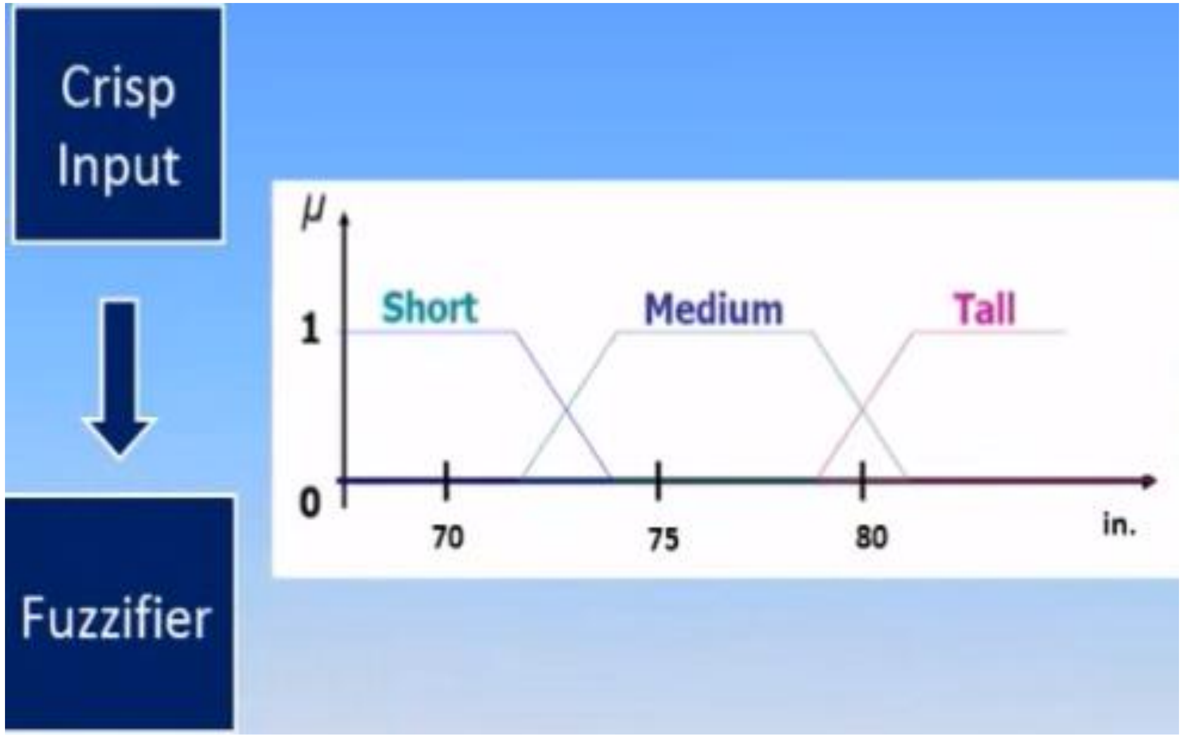
Example : Fuzzy logic vs. Crisp logic

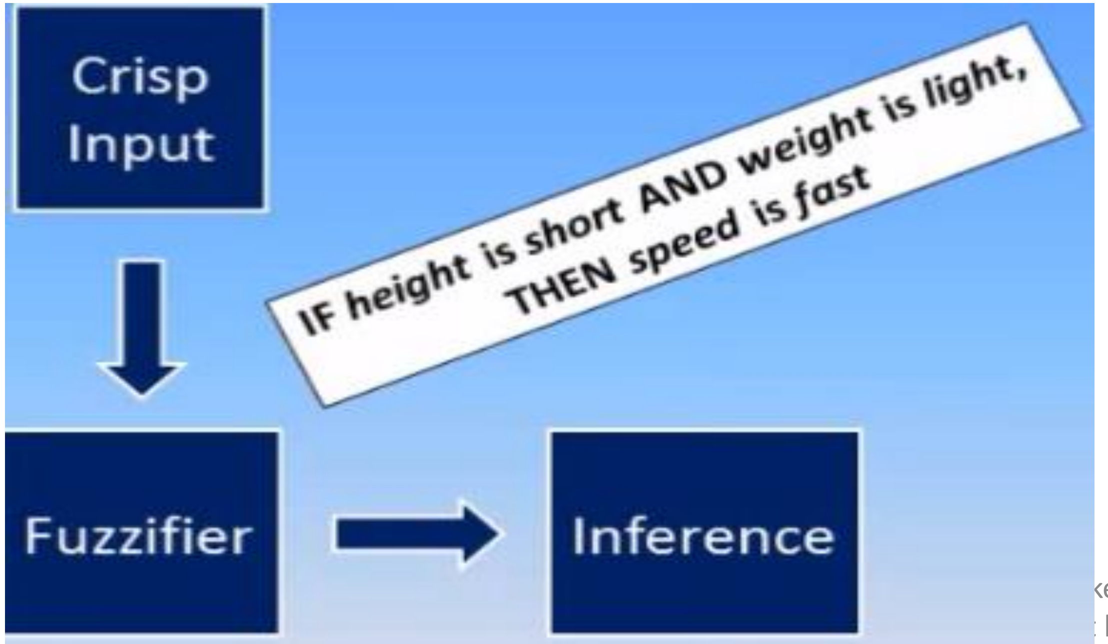


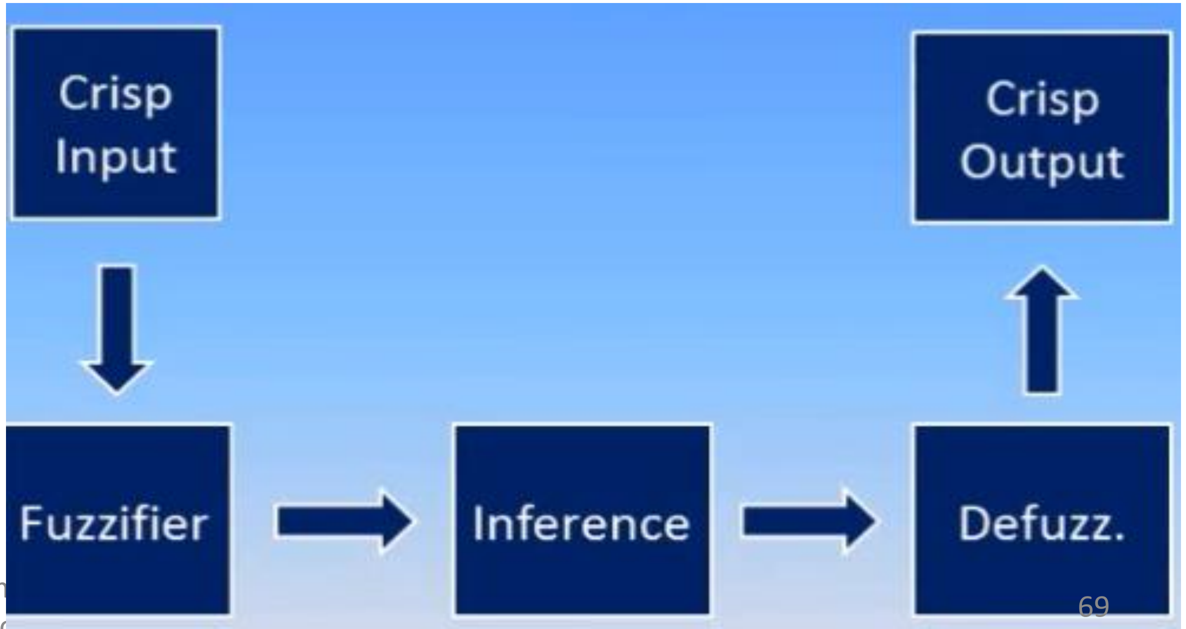
Generic structure of a Fuzzy system

Following figure shows a general framework of a fuzzy system.









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What is the Fuzzification?

Fuzzification is the process of converting a clear input to a **fuzzy** 26 value. It converts a clear point value of the process state variable to be compatible with the representation of the fuzzy set of the system state variable in the precedent of the rule.

Fuzzification is done based on the type of the inference engine or the strategy of inference like disjunction rule-based or composition based.

Fuzzifier:

Fuzzification is the process of converting a crisp input to a fuzzy value. The manipulation of data in an FLC is based on the theory of fuzzy sets, fuzzification is necessary and desirable at an early stage. Therefore, the fuzzifier can be defined as a mapping from an observed input space to fuzzy set labels in a universe of specified input universe of discourse.

Fuzzy implication : Example 1

- If pressure is High then temperature is Low
- If mango is Yellow then mango is Sweet else mango is Sour
- If road is Good then driving is Smooth else traffic is High
- The fuzzy implication is denoted as $R : A \rightarrow B$
- In essence, it represents a binary fuzzy relation R on the (Cartesian) product of $A \times B$

Fuzzy if then Rule

$$T_{HIGH} = \{(25, 0.1), (30, 0.2), (35, 0.5), (40, 0.6)\}$$

$$P_{LOW} = \{(2, 0.3), (5, 0.5), (6, 0.4)\}$$

$$\boxed{\text{If Temp is high then pressure is low}}$$

$$R : T_{HIGH} \longrightarrow P_{LOW}$$

$$R : A \longrightarrow B$$

$$A \times B$$

$$R = \begin{array}{c c} & \begin{array}{ccc} 2 & 5 & 6 \end{array} \\ \begin{array}{c} 25 \\ 30 \\ 35 \\ 40 \end{array} & \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.2 \\ 0.3 & 0.5 & 0.4 \\ 0.3 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

What is the defuzzification?

Defuzzification is the process of convert the set of controller output values into a single pointwise value and performs output renormalization that maps the pointwise value of the controller output into its physical domain.

Defuzzification methods

A number of defuzzification methods are known. Such as

1) **Lambda-cut method**
2) **Weighted average method**
3) **Maxima methods**
4) **Centroid methods**

Lambda-cut for a fuzzy set : Example

$A_1 = \{(x_1, 0.9), (x_2, 0.5), (x_3, 0.2), (x_4, 0.3)\}$

$\lambda = 0.6$

$A_{0.6} = \{(x_1, 1), (x_2, 0), (x_3, 0), (x_4, 0)\} = \{x_1\}$

$A_2 = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.8), (x_4, 0.7)\}$

$\lambda = 0.2$

$A_{0.2} = \{(x_1, 0), (x_2, 1), (x_3, 1), (x_4, 1)\} = \{x_2, x_3, x_4\}$

Lambda-cut sets : Example

Two fuzzy sets P and Q are defined on *x* as follows.

$\mu(x)$	x_1	x_2	x_3	x_4	x_5
P	0.1	0.2	0.7	0.5	0.4
Q	0.9	0.6	0.3	0.2	0.8

Find the following :

a) $P_{0.2}, Q_{0.3}$

b) $(P \cup Q)_{0.6}$

c) $(P \cup \bar{P})_{0.8}$

d) $(P \cap Q)_{0.4}$

Lambda-cut for a fuzzy relation

The Lambda-cut method for a fuzzy set can also be extended to fuzzy relation also.

Example: For a fuzzy relation R

$$R = \begin{bmatrix} 1 & 0.2 & 0.3 \\ 0.5 & 0.9 & 0.6 \\ 0.4 & 0.8 & 0.7 \end{bmatrix}$$

We are to find λ-cut relations for the following values of

$$\lambda = 0, 0.2, 0.9, 0.5$$

$$R_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } R_{0.2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and}$$

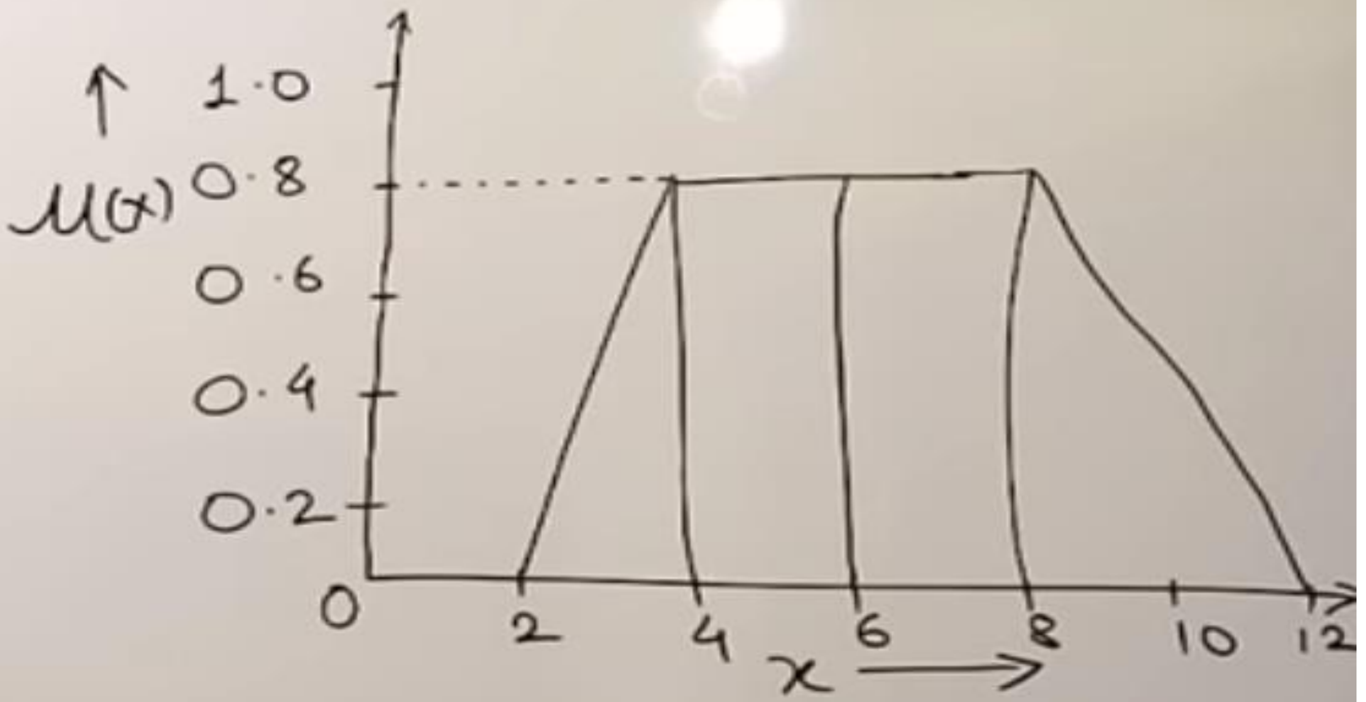
$$R_{0.9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } R_{0.5} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Maxima Methods

(I) First of Maxima (FOM)

(II) Last of Maxima (LOM)

(III) Mean of Maxima (MOM)



$$x^* = \frac{\sum_{x_i \in M} x_i}{|M|}$$

$$M = \{x \mid \mu_A(x) = \text{height of fuzzy set}\}$$

$|M|$ = Cardinality of set M

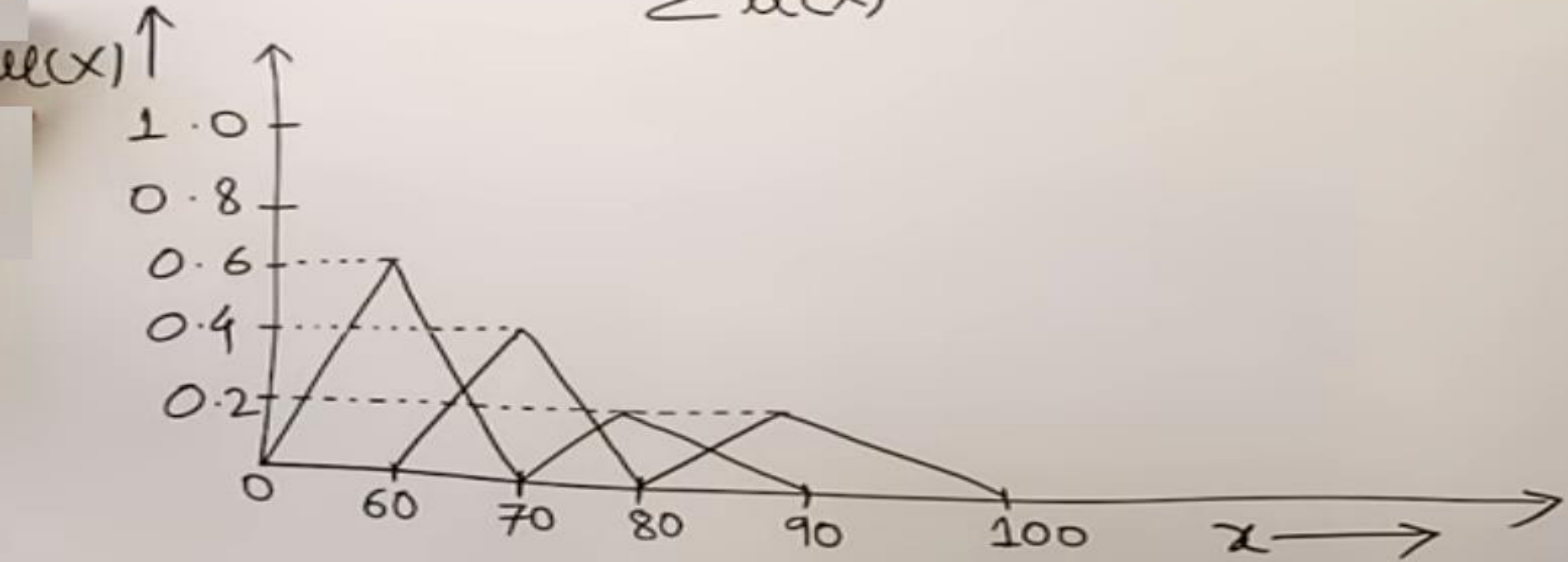
$x = 4, 6, 8$

$$x^* = \frac{4+6+8}{3}$$

$$\boxed{x^* = 6}$$

Weighted Average Method

$$x^* = \frac{\sum \mu(x) \cdot x}{\sum \mu(x)}$$

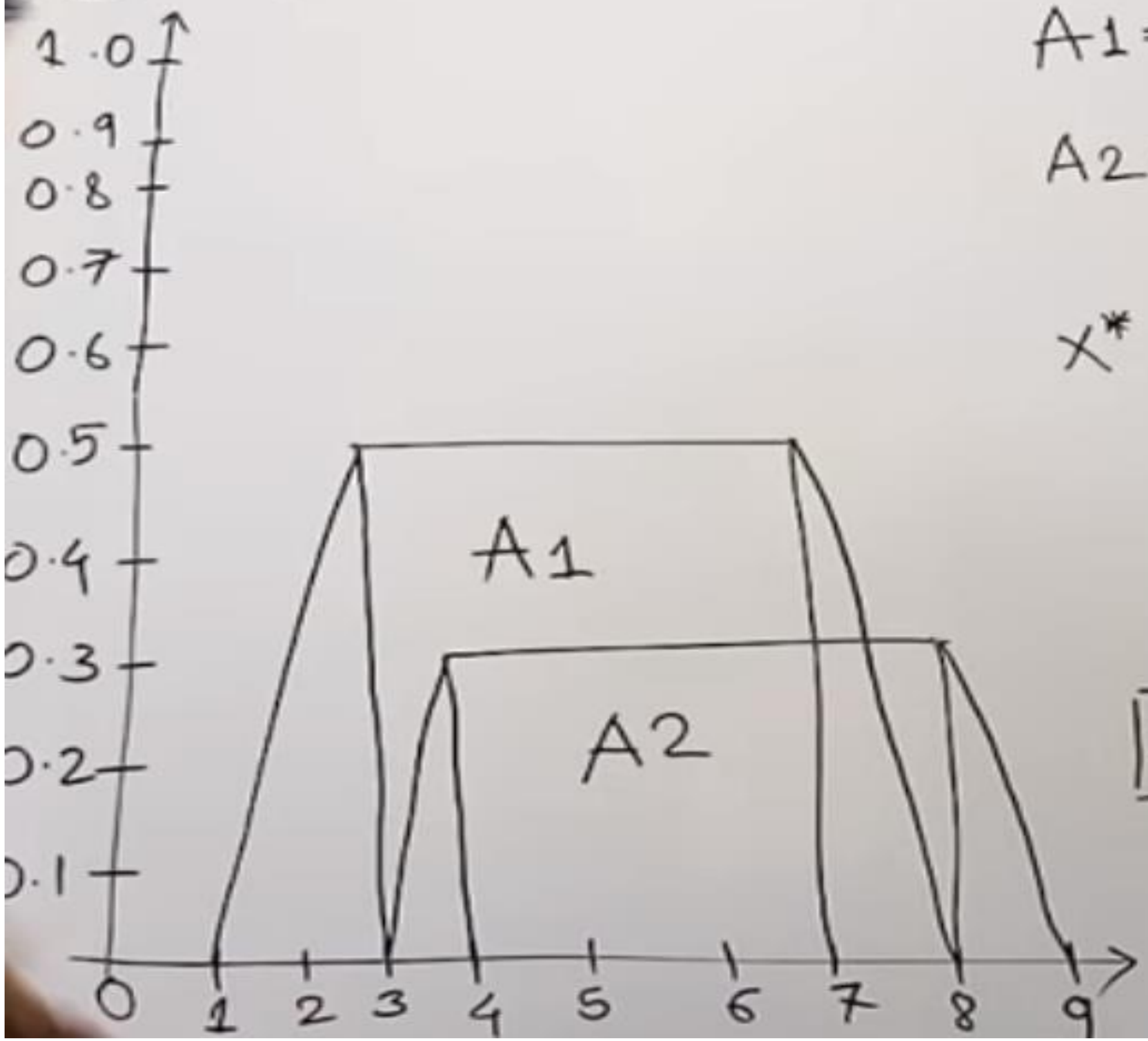


$$x^* = \frac{(60 \times 0.6 + 70 \times 0.4 + 80 \times 0.2 + 90 \times 0.2)}{0.6 + 0.4 + 0.2 + 0.2}$$

$$\boxed{x^* = 70}$$

Centroid Method

I Center of sum (COS)



$$X^* = \frac{\sum A \quad x_c}{\sum A}$$

$$A1 = \frac{1}{2}\left[(8-1)+(7-3)\right] \times 0.5 = 2.75$$

$$A2 = \frac{1}{2}\left[(9-3)+(8-4)\right] \times 0.3 = 1.50$$

$$X^* = \frac{A_1 \cdot x_{c_1} + A_2 \cdot x_{c_2}}{A_1 + A_2}$$

$$= \frac{2.75 \times 5 + 1.50 \times 6}{2.75 + 1.5}$$

$$\boxed{X^* = 5.35}$$