L38 14 April 2021

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Divergence and Curl of a vector field :-

Let $\vec{v} = v_1(x,y,z)\hat{i} + v_2(x,y,z)\hat{j} + v_3(x,y,z)\hat{k}$ defines a vector field

Divergence of vector field $\vec{v}$

It is denoted by div $\vec{v}$, is defined as the scalar field

$$\text{div}\,\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

$$\text{div}\,\vec{v} = \nabla\cdot\vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

But

$$\vec{v}\cdot\nabla = \left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$\nabla\cdot\vec{v} \neq \vec{v}\cdot\nabla \qquad = \left(v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} + v_3\frac{\partial}{\partial z}\right)$$

\# Find the divergence of the vector field

$$\vec{v} = xy\hat{i} + yz\hat{j} + zx\hat{k}$$

$$\text{div}\,\vec{v} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zx)$$

$$= y + z + x$$

$$\text{div}\,\vec{v} = x + y + z$$

Curl of vector field $\vec{v}$

Curl of a vector field $\vec{v}$, denoted by Curl $\vec{v}$ is defined

Curl of a vector field $\vec{v}$, denoted by curl v is defined as the vector field.

$$\text{Curl } \vec{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

$$\text{Curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$$

But $\nabla \times \vec{v} \neq -(\vec{v} \times \nabla)$

Find the curl of the vector field

$$\vec{v} = xe^{-y}\hat{i} + 2ze^{-y}\hat{j} + xy^2\hat{k}$$

$$\text{Curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^{-y} & 2ze^{-y} & xy^2 \end{vmatrix} = \hat{i}\left(2xy - 2e^{-y}\right) - \hat{j}\left(y^2 - 0\right) + \hat{k}\left(0 + xe^{-y}\right)$$

$$= \hat{i}\left(2xy - 2e^{-y}\right) - y^2\hat{j} + xe^{-y}\hat{k}$$

Curl of a gradient

∇f

$$\boxed{\text{Curl}(\text{grad} f) = \nabla \times (\nabla f) = 0}$$

$$\nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} =$$

$$\left| \quad \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right|$$

$$= \hat{i}\left[\frac{\partial^2 f}{\cancel{\partial y}\partial z} - \frac{\partial^2 f}{\cancel{\partial y}\partial z}\right] - \hat{j}\left[\frac{\partial^2 f}{\cancel{\partial x}\partial z} - \frac{\partial^2 f}{\cancel{\partial x}\partial z}\right] + \hat{k}\left[\frac{\partial^2 f}{\cancel{\partial x}\partial y} - \frac{\partial^2 f}{\cancel{\partial x}\partial y}\right] = 0$$

$$= 0 \checkmark$$

#

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\operatorname{div}(\vec{r}) = \nabla \cdot \vec{r}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= 1 + 1 + 1 = \underline{\underline{3}}$$

$$\text{Curl } \vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}[0-0] - \hat{j}[0-0] + \hat{k}[0-0] = 0$$