MTH165

Unit 2
Differentiation and integration

L10-General rules of differentiation

Revision

Consider the two statements

(i) a particular eigen value may be zero. (ii) a particular eigen vector may be zero.

which of the above correct.

A) only (i)   B) only (ii)   C) Both (i) and (ii)   D) Both are incorrect.

Cayley Hamilton Theorem is

A) Every symmetric matrix satisfies its own characteristic equation

B) Every square matrix satisfies its own characteristic equation.

C) Every orthogonal matrix satisfies its own characteristic equation

D) Every real symmetric matrix satisfies its own characteristic equation

The History of Differentiation

Differentiation is part of the science of Calculus, and was first developed in the 17th century by two different Mathematicians.

Gottfried Leibniz
(1646-1716)

Germany 

Sir Isaac Newton
(1642-1727)

England 

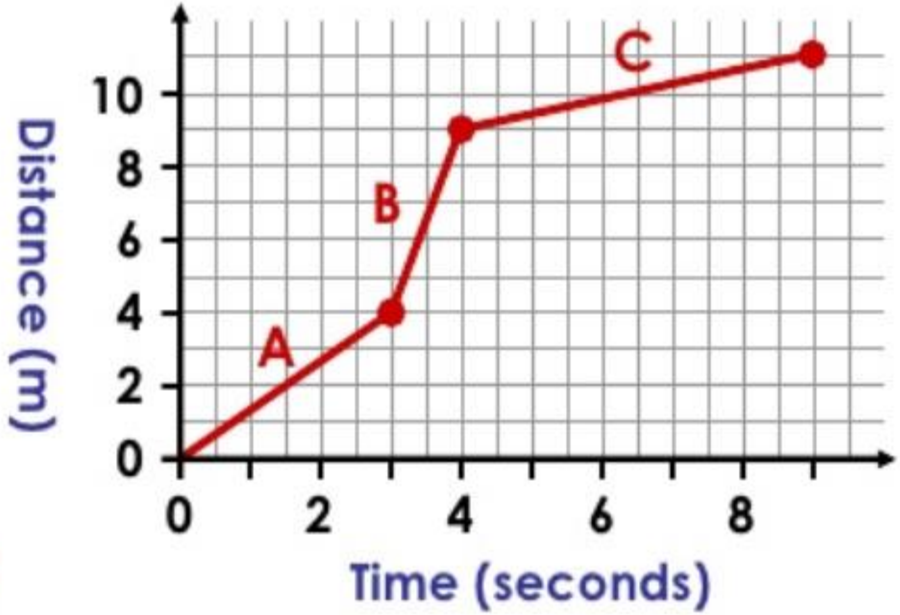
Differentiation, or finding the instantaneous rate of change, is an essential part of:

- Mathematics and Physics
- Chemistry
- Biology
- Computer Science
- Engineering
- Navigation and Astronomy

Calculating Speed

Example

Calculate the speed for each section of the journey opposite.



$$\text{speed in A} = \frac{4}{3} \approx 1.33 \text{ m/s}$$

$$\text{speed in B} = \frac{5}{1} = 5 \text{ m/s}$$

$$\text{speed in C} = \frac{2}{5} = 0.4 \text{ m/s}$$

$$\text{average speed} = \frac{11}{9} \approx 1.22 \text{ m/s}$$

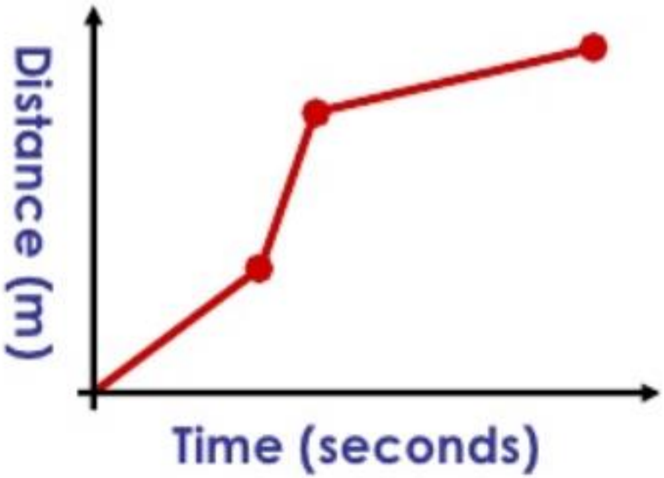
Notice the following things:

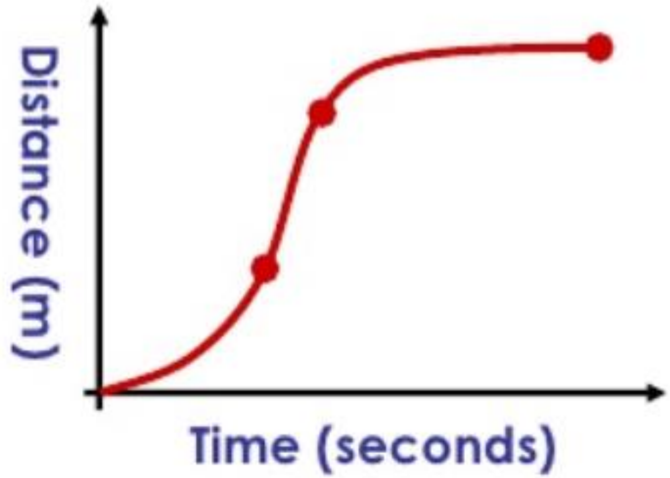
- the speed at each instant is not the same as the average

$$S = \frac{D}{T} = \frac{\Delta y}{\Delta x} = m$$

Instantaneous Speed

$$S = \frac{D}{T} = \frac{\Delta y}{\Delta x} = m$$





In reality speed does not often change instantly. The graph on the right is more realistic as it shows a gradually changing curve.

The journey has the same average speed, but the instantaneous speed is different at each point because the gradient of the curve is constantly changing. How can we find the instantaneous speed?

Introduction to Differentiation

Differentiate means

'find out how fast something is changing in comparison with something else at any one instant'.

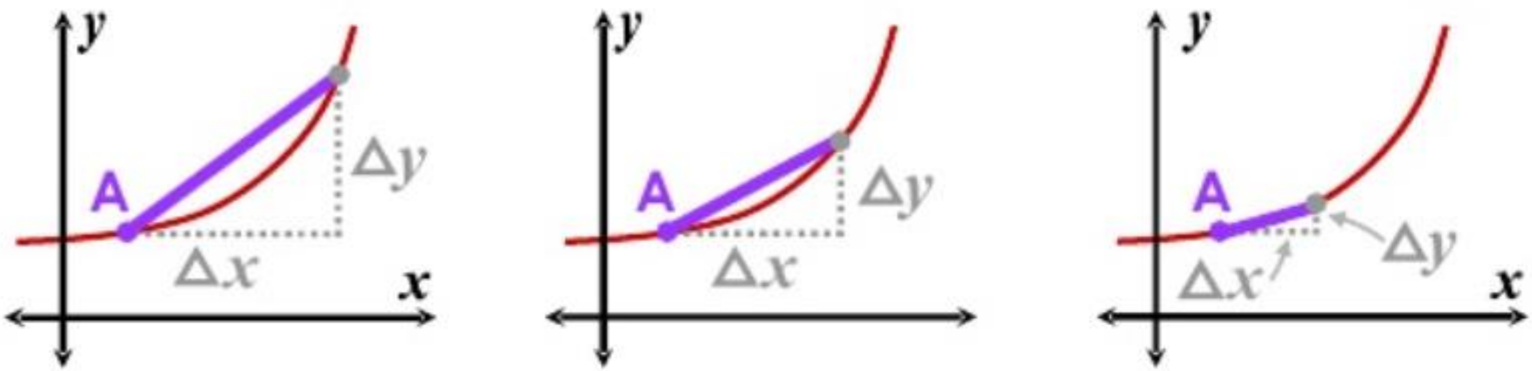
speed = $\frac{\Delta D}{\Delta T}$ 'rate of change of distance with respect to time'

acceleration = $\frac{\Delta S}{\Delta T}$ 'rate of change of speed with respect to time'

gradient = $\frac{\Delta y}{\Delta x}$ 'rate of change of y-coordinate with respect to x-coordinate'

Estimating the Instantaneous Rate of Change

The diagrams below show attempts to estimate the instantaneous gradient (the rate of change of y with respect to x) at the point A.



Notice that the accuracy improves as Δx gets closer to zero.

The instantaneous rate of change is written as:

$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x} \text{ as } \Delta x \text{ approaches 0.}$$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ is called the derivative of f at a.

We write: $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

"The derivative of f with respect to x is …"

There are many ways to write the derivative of $y = f(x)$

$$y = x^2 - 3$$

$$y' = \lim_{h \to 0} \frac{(x+h)^2 - 3 - (x^2 - 3)}{h}$$

$$y' = \lim_{h \to 0} \frac{\cancel{x^2} + 2x\cancel{h} + h^{\cancel{2}} - \cancel{x^2}}{\cancel{h}}$$

$$y' = \lim_{h \to 0} 2x + \cancel{h}^{\,0}$$

$$y' = 2x$$

Use the definition of derivative to find the derivative of $f(x) = x^2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative at a Point

The derivative of the function f at the point x = a is the limit

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Provided the limit exists.

1. Find $\frac{dy}{dx}$ using the four-step rule given $y = 1 - x^2$.

MCQ

The definition of the first derivative of a function $f(x)$ is

(A) $f'(x) = \dfrac{f(x+\Delta x)+f(x)}{\Delta x}$

(B) $f'(x) = \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

(C) $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x)+f(x)}{\Delta x}$

(D) $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

Notation

There are lots of ways to denote the derivative of a function y = f(x).

$f'(x)$ the derivative of f

y' y prime

$\frac{df}{dx}$ the derivative of f with respect to x.

$\frac{dy}{dx}$ the derivative of y with respect to x.

$\frac{d}{dx}f(x)$ the derivative of f at x

Basic Differentiation

The instant rate of change of y with respect to x is written as $\frac{dy}{dx}$.

By long experimentation, it is possible to prove the following:

If $y = x^n$

then $\boxed{\frac{dy}{dx} = nx^{n-1}}$

How to Differentiate:

- multiply by the power
- reduce the power by one

Note that $\frac{dy}{dx}$ describes both the rate of change and the gradient.

Differentiation of Expressions with Multiple Terms

The basic process of differentiation can be applied to every x-term in an algebraic expession.

$$y = ax^{m} + bx^{n} + \ldots$$

$$\frac{dy}{dx} = amx^{m-1} + bnx^{n-1} + \ldots$$

Important

Expressions **must** be written as the sum of individual terms before differentiating.

How to Differentiate:

- **multiply** every x-term by the power
- **reduce** the power of every x-term by one

Examples of Basic Differentiation

Example 1

Find $\frac{dy}{dx}$ for $y = 3x^4 - 5x^3 + \frac{7}{x^2} + 9$

$$y = 3x^4 - 5x^3 + 7x^{-2} + 9$$

$$\therefore \frac{dy}{dx} = 12x^3 - 15x^2 - 14x^{-3}$$

$$= \underline{12x^3 - 15x^2 - \frac{14}{x^3}}$$

this disappears because

$9 = 9x^0$

(multiply by zero)

Example 2

Find the **gradient** of the curve $y = \dfrac{(x+3)(x-5)}{x^2}$ at the point (5,0).

$$y = \frac{x^2 - 2x - 15}{x^2}$$

$$= 1 - \frac{2}{x} - \frac{15}{x^2}$$

$$= 1 - 2x^{-1} - 15x^{-2}$$

disappears (multiply by zero)

$$\therefore \frac{dy}{dx} = 2x^{-2} + 30x^{-3}$$

$$= \frac{2}{x^2} + \frac{30}{x^3}$$

At $x = 5$, $\dfrac{dy}{dx} = \dfrac{2}{25} + \dfrac{30}{125}$

$$= \frac{8}{25}$$

The Derived Function

It is also possible to express differentiation using function notation.

If	$f(x)$	=	x^n
then	$f'(x)$	=	nx^{n-1}

Newton

Leibniz

$f'(x)$ and $\dfrac{dy}{dx}$

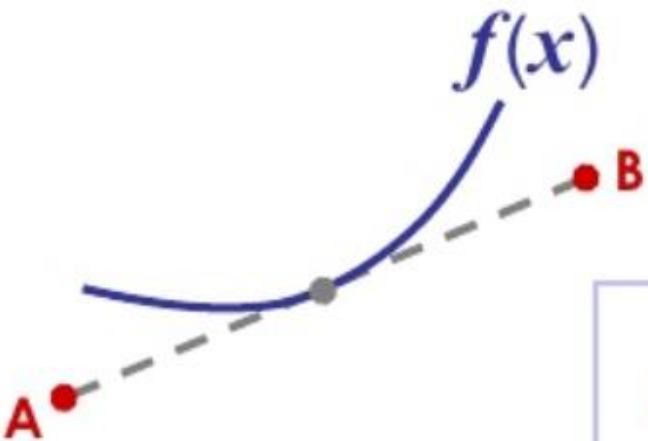
mean exactly the same thing written in different ways.

The word 'derived' means 'produced from', for example orange juice is derived from oranges.

The derived function $f'(x)$ is the rate of change of the function $f(x)$ with respect to x .

Tangents to Functions

A tangent to a function is a straight line which intersects the function in only one place, with the same gradient as the function.



$$\boxed{m_{AB} = f'(x)}$$

The gradient of any tangent to the function $f(x)$ can be found by substituting the x-coordinate of intersection into $f'(x)$.

Equations of Tangents

REMEMBER

$$y - b = m(x - a)$$

Straight Line Equation

To find the equation of a tangent:

- differentiate
- substitute x-coordinate to find gradient at point of intersection
- substitute gradient and point of intersection into $y - b = m(x - a)$

Example

Find the equation of the tangent to the function

$$f(x) = \frac{1}{2}x^3$$

at the point (2,4).

$$f'(x) = \frac{3}{2}x^2$$

$$\therefore \quad m = f'(2) = \frac{3}{2} \times (2)^2 = 6$$

substitute: $y - 4 = 6(x - 2)$

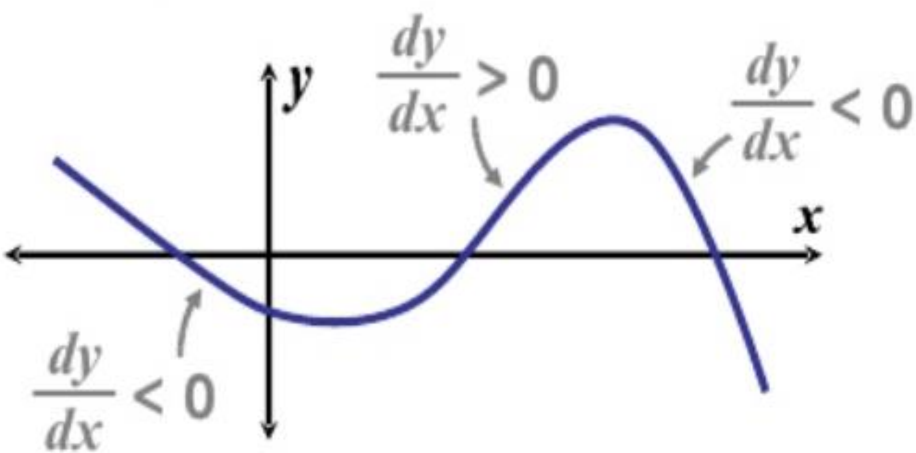
$$\therefore \quad \underline{\underline{6x - y - 8 = 0}}$$

Increasing and Decreasing Curves

The gradient at any point on a curve can be found by differentiating.

If $\dfrac{dy}{dx} > 0$ then y is increasing.

If $\dfrac{dy}{dx} < 0$ then y is decreasing.



Gradient **REMEMBER**

Positive
uphill slope

Negative
downhill slope

Alternatively,

If $f'(x) > 0$ then $f(x)$ is increasing.

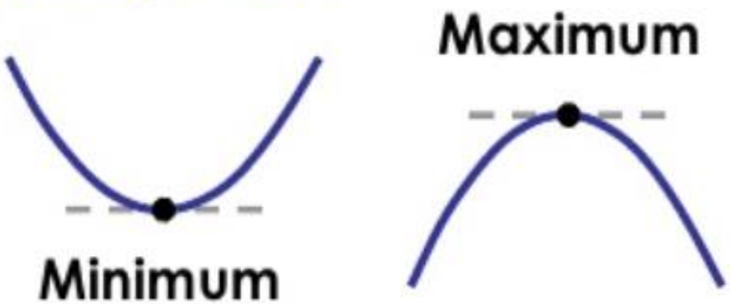
If $f'(x) < 0$ then $f(x)$ is decreasing.

Stationary Points

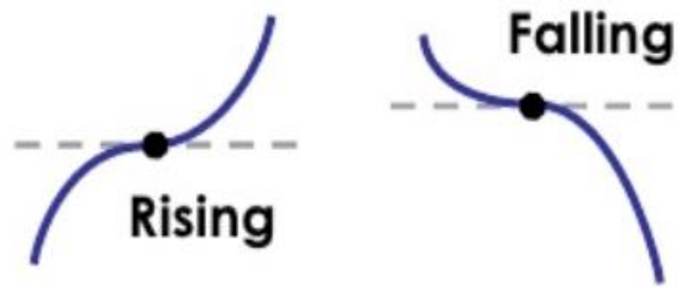
If a function is neither increasing or decreasing, the gradient is zero and the function can be described as stationary.

There are two main types of stationary point.

Turning Points



Points of Inflection



At any stationary point, $\frac{dy}{dx} = 0$ or alternatively $f'(x) = 0$

Investigating Stationary Points

Example

Find the stationary point of

$$f(x) = x^2 - 8x + 3$$

and determine its nature.

> Use a **nature table** to reduce the amount of working.

Stationary point given by

$$f'(x) = 0$$

$$f'(x) = 2x - 8$$

$$\therefore\ 2x - 8 = 0$$

$$\underline{x = 4}$$

'slightly less than four'

'slightly more'

x	4^-	4	4^+
$f'(x)$	$-$	0	$+$
slope	\	—	/

gradient is positive

$\therefore$ The stationary point at $x = 4$ is a minimum turning point.

Investigating Stationary Points

Example 2

Investigate the stationary points of

$$y = 4x^3 - x^4$$

$$\therefore \frac{dy}{dx} = 12x^2 - 4x^3 = 0$$

$$4x^2(3 - x) = 0$$

$4x^2 = 0$ or $3 - x = 0$

$\underline{x = 0}$ $\qquad$ $\underline{x = 3}$

$\therefore y = 0$ $\qquad$ $\therefore y = 27$

stationary point at $(0,0)$:

x	0^-	0	0^+
$\frac{dy}{dx}$	+	0	+
slope	/	—	/

$\therefore$ rising point of inflection

stationary point at $(3, 27)$:

x	3^-	3	3^+
$\frac{dy}{dx}$	+	0	–
slope	/	—	\

$\therefore$ maximum turning point

Differentiation Formulas:

1. $\frac{d}{dx}(x) = 1$
2. $\frac{d}{dx}(ax) = a$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\tan x) = \sec^2 x$
7. $\frac{d}{dx}(\cot x) = -\csc^2 x$
8. $\frac{d}{dx}(\sec x) = \sec x \tan x$
9. $\frac{d}{dx}(\csc x) = -\csc x(\cot x)$
10. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(a^x) = (\ln a)a^x$
13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

Integration Formulas:

1. $\int 1\,dx = x + C$
2. $\int a\,dx = ax + C$
3. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
4. $\int \sin x\,dx = -\cos x + C$
5. $\int \cos x\,dx = \sin x + C$
6. $\int \sec^2 x\,dx = \tan x + C$
7. $\int \csc^2 x\,dx = -\cot x + C$
8. $\int \sec x(\tan x)\,dx = \sec x + C$
9. $\int \csc x(\cot x)\,dx = -\csc x + C$
10. $\int \frac{1}{x}\,dx = \ln|x| + C$
11. $\int e^x\,dx = e^x + C$
12. $\int a^x\,dx = \frac{a^x}{\ln a} + C\ a > 0,\ a \neq 1$
13. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
14. $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
15. $\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$

Differentiation of Log and Exponential Function

$$\frac{d(e^x)}{dx} = e^x$$

$$\frac{d(\ln(x))}{dx} = \frac{1}{x}$$

$$\frac{d(a^x)}{dx} = a^x \log a$$

$$\frac{d(x^x)}{dx} = x^x(1 + \ln x)$$

$$\frac{d(\log_a x)}{dx} = \frac{1}{x} \times \frac{1}{\ln a}$$

Given $y = 5e^{3x} + \sin x$, $\dfrac{dy}{dx}$ is

(A) $5e^{3x} + \cos x$

(B) $15e^{3x} + \cos x$

(C) $15e^{3x} - \cos x$

(D) $2.666e^{3x} - \cos x$

Given $y = x^3 \ln x$, $\dfrac{dy}{dx}$ is

(A) $3x^2 \ln x$

(B) $3x^2 \ln x + x^2$

(C) x^2

(D) $3x$

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Unit 2
Differentiation and integration

L11- Logarithmic differentiation

Derivatives of Logarithmic Functions

In this section we use implicit differentiation to find the derivatives of the logarithmic functions $y = \log_a x$ and, in particular, the natural logarithmic function $y = \ln x$.

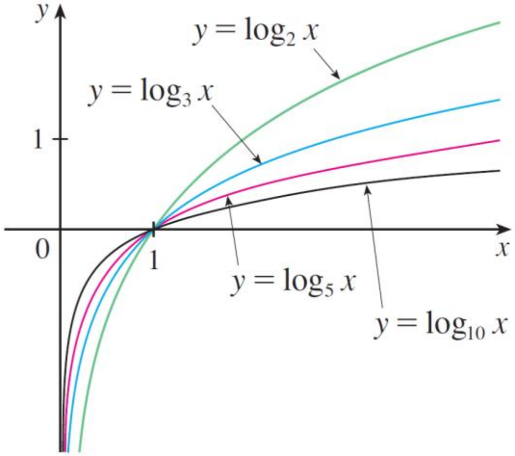


Figure 12

Derivatives of Logarithmic Functions

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

In general, if we combine Formula 2 with the Chain Rule, we get

$$\boxed{\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}} \quad \text{or} \quad \boxed{\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}}$$

Example 2

Find $\frac{d}{dx}\ln(\sin x)$.

Solution:

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x$$

$$= \cot x$$

$$\boxed{\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}}$$

Logarithmic Differentiation

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms.

The method used in the next example is called **logarithmic differentiation**.

Example 15

Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}$.

Solution:

We take logarithms of both sides of the equation and use the properties of logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2 + 1) - 5\ln(3x + 2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2 + 1} - 5\cdot\frac{3}{3x + 2}$$

Example 15 – *Solution*

cont'd

Solving for *dy*/*dx*, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

Because we have an explicit expression for *y*, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

Logarithmic Differentiation

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the properties of logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

The Power Rule If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

MTH165

Unit 2
Differentiation and integration

L12-Integration by parts and Partial fractions

The derivative of $y = \sqrt{e^{\sqrt{x}}}$ w.r.t x is

(a) $\frac{1}{\sqrt{x}} \sqrt{e^{\sqrt{x}}}$

(b) $\frac{1}{4\sqrt{x}} \sqrt{e^{\sqrt{x}}}$

(c) $\frac{1}{2} \sqrt{e^{\sqrt{x}}}$

(d) None of these

if $y = \sin(\tan^{-1} e^{-x})$ then $\frac{dy}{dx} =$

(a) $\dfrac{e^{-x}}{1+e^{-2x}}$

(b) $\dfrac{e^{-x}}{1+e^{2x}} \sin(\cot^{-1} e^{-x})$

(c) $-\dfrac{e^{-x}}{1+e^{-2x}} \cos(\tan^{-1} e^{-x})$

(d) None of these

- ... finding an Integral is the **reverse** of finding a Derivative.
- (So you should really know about Derivatives)
- Like here:
- Example: What is an integral of 2x?
- We know that the derivative of x^2 is 2x ...
- ... so an integral of 2x is x^2

Remember the three General rules for Integration

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

2. $\int k\, f(x)dx = k \int f(x)dx$

3. $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

Indefinite Integrals

1. $\int \cos x \, dx = \sin x + C$

2. $\int \sin x \, dx = -\cos x + C$

3. $\int \sec^2 x \, dx = \tan x + C$

4. $\int \csc^2 x \, dx = -\cot x + C$

5. $\int \csc^2 x \, dx = -\cot x + C$

6. $\int \sec x \tan x \, dx = \sec x + C$

7. $\int \csc x \cot x \, dx = -\csc x + C$

Indefinite integrals result from trigonometric identities and u-substitution

8. $\int \tan x \, dx = ln|\sec x| + C$

9. $\int \cot x \, dx = ln|\sin x| + C$

10. $\int \sec x \, dx = ln|\sec x + \tan x| + C$

11. $\int \csc x \, dx = ln|\csc x - \cot x| + C$

Examples

A. Integrate $\int \sin 3x\, dx$

u-substitution

u = 3x

du = 3dx

(1/3) du = dx

$$\int \sin 3x\, dx = \int \sin u \left(\frac{1}{3}\right) du$$

$$= (1/3) \int \sin u\, du$$

$$= \left(\frac{1}{3}\right)(-\cos u) + C$$

$$= -\frac{1}{3}\cos 3x + C$$

Recall

$$\int \sin x\, dx = -\cos x + C$$

Examples

C. Integrate $\int 5 \sec 4x \tan 4x \; dx$

u-substitution

u = 4x

du = 4dx

(1/4) du = dx

$$\int 5 \sec 4x \tan 4x \; dx = 5 \int \sec u \tan u \left(\frac{1}{4}\right) du$$

$$= (5/4) \int \sec u \tan u \, du$$

$$= \left(\frac{5}{4}\right) sec\, u + C$$

$$= \left(\frac{5}{4}\right) sec\, 4x + C$$

Recall

$$\int \sec x \tan x \; dx = \sec x + C$$

Trigonometric Functions

A. $\tan x = \dfrac{\sin x}{\cos x}$

B. $\sec x = \dfrac{1}{\cos x}$

C. $\cot x = \dfrac{\cos x}{\sin x} = \dfrac{1}{\tan x}$

D. $\csc x = \dfrac{1}{\sin x}$

C. $cos^2 x + \ sin^2 x = 1$

D. $\sin 2x = 2 \sin x \cos x$

E. $\cos 2x = 2\cos^2 x - 1$ *so that* $cos^2 x = \dfrac{1 + \cos 2x}{2}$

F. $\cos 2x = 1 - 2\sin^2 x$ *so that* $sin^2 x = \dfrac{1 - \cos 2x}{2}$

G. $\cos 2x = \ cos^2 x - \ sin^2 x$

H. $1 + tan^2 x = \ sec^2 x$ *so that* $tan^2 x = sec^2 x - 1$

I. $1 + cot^2 x = \ csc^2 x$ *so that* $cot^2 x = csc^2 x - 1$

Examples

E. Integrate $\int 3\cos^2 5x\,dx$

$$\int 3\cos^2 5x\,dx = 3\int \cos^2 5x\,dx$$

$$= 3\int \frac{1+\cos 2(5x)}{2}\,dx$$

$$= (3/2)\int (1+\cos 10x)\,dx$$

$$= (3/2)\left(\int 1\,dx + \int \cos 10x\,dx\right)$$

$$= (3/2)\left(x + \int \cos 10x\,dx\right)$$

u-substitution

u = 10x

du = 10dx

(1/10)du = dx

$$(3/2)\left(x + \int \cos 10x\,dx\right) = (3/2)\left(x + \int \cos u\,(1/10)\,du\right)$$

$$= (3/2)\left(x + (1/10)\int \cos u\,du\right)$$

$$= (3/2)\left(x + (1/10)(\sin u + C)\right)$$

$$= (3/2)x + (3/20)\sin u + C$$

$$= (3/2)x + (3/20)\sin 10x + C$$

INTEGRATE THE FOLLOWING

1. $\int x\, e^{-x^2}\, dx$

u-substitution

$$u = -x^2$$
$$du = -2x\, dx$$
$$-\frac{1}{2}\, du = x\, dx$$

$$\int x\, e^{-x^2}\, dx = \int e^u \left(-\frac{1}{2}\right) du$$
$$= -\frac{1}{2}\int e^u\, du$$
$$= -\frac{1}{2}\, x^u + C$$
$$\boxed{= -\frac{e^{-x^2}}{2} + C}$$

2. $\int 2sec^2 x \tan x\, dx$

u-substitution

$$u = \tan x$$
$$du = \sec^2 x\, dx$$

$$\int 2sec^2 x \tan x\, dx = \int 2\tan x\, sec^2 x\, dx$$
$$= \int 2u\, du$$
$$= u^2 + C$$
$$\boxed{= tan^2 x + C}$$

The value of $\int \sqrt{1+\cos 2x}\,dx$ is

(a) $\sin 2x + c$

(b) $\sqrt{2}\sin x + c$

(c) $\sqrt{2}\cos x + c$

(d) None of these

INTEGRATION BY PARTS

$$\int \underline{uv}\,dx = u\int v\,dx - \int \frac{d}{dx}(u)\int v\,dx$$

I L A T E

inverse Log. alg. Trig. Expo.

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

$u' = \frac{d}{dx}(u)$, $u'' = \frac{d}{dx}(u')$, $u''' = \frac{d}{dx}(u'')$ ---

$v_1 = \int v\,dx$, $v_2 = \int v_1\,dx$, $v_3 = \int v_2\,dx$ ---

$$\int x^2 \sin^{-1}x \, dx = \_\_\_\_$$

(a) $\frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\sqrt{1-x^2} + C$

(b) $\frac{x^3}{3}\sin^{-1}x + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{3}(1-x^2) + C$

(c) $\frac{x^3}{3}\sin^{-1}x + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{9}(1-x^2)^{3/2} + C$

(d) None of these

$$\int x^3 \sin x^2 \, dx = ?$$

(a) $\frac{1}{2}\sin x^2 - \frac{x^2}{2}\sin x^2 + c$

(b) $\frac{1}{2}\sin x^2 - \frac{x^2}{2}\cos x^2 + c$

(c) $\sin x^2 - x^2 \sin x^2 + c$

(d) None of these.

1) $\int e^{ax} \sin bx \, dx = \dfrac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C$

2) $\int e^{ax} \cos bx \, dx = \dfrac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C$

3) $\int e^{x}[f(x) + f'(x)] \, dx = e^{x} f(x) + C$

$\int e^{ax}[a f(x) + f'(x)] \, dx = e^{ax} f(x) + C$

The value of $\int \log x \, dx$ is

(a) $\log x - x + c$

(b) $x \log x + x + c$

(c) $x \log x - x + c$

(d) None of these

1. $\int x \log(1+x)\,dx$

2. $\int \sin\sqrt{x}\,dx$

3. $\int x^2 \cos x\,dx$

4. $\int x \tan^{-1} x\,dx$

5. $\int \frac{x\cos^{-1} x}{\sqrt{1-x^2}}\,dx$

PARTIAL FRACTIONS

we want to evaluate ∫ [P(x)/Q(x)] dx, where P(x)/Q(x) is a proper rational fraction. In such cases, it is possible to write the integrand as a sum of simpler rational functions by using partial fraction decomposition. Post this, integration can be carried out easily. The following image indicates some simple partial fractions which can be associated with various rational functions:

PARTIAL FRACTIONS

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where $x^2 + bx + c$ cannot be factorised further	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$,

Partial Fractions

$$\int \frac{dx}{x^2 - 5x + 6}$$

$$\frac{1}{x^2 - 5x + 6} = \frac{A}{x-3} + \frac{B}{x-2}$$

$$\frac{1}{x^2 - 5x + 6} = \frac{A(x-2)}{x-3} + \frac{B(x-3)}{x-2}$$

$$1 = A(x-2) + B(x-3)$$

If x = 2: 1=-B so B = -1

If x =3: 1=A

$$\int \frac{dx}{x^2 - 5x + 6} = \int \left(\frac{1}{x-3} - \frac{1}{x-2} \right) dx$$

$$\ln|x-3| - \ln|x-2| + C \qquad or \qquad \ln\left|\frac{x-3}{x-2}\right| + C$$

MCQ: Partial fractions of $f(x^2 + 1)/(x^3 + 1)$ will be of the form

A. $A/(x-1) - B/(x^2 - x + 1)$
B. $A/(x+1) - B/(x^2 - x + 1)$
C. $A/(x+1) - bx + c/(x^2 - x + 1)$
D. None of Above

MCQ: $X/(x+2)(x-3) =$

A. $2/5(x+2) -3/5(x-3)$
B. $2/5(x+2) +3/5(x-3)$
C. $2/5(x-2) +3/5(x+3)$
D. None of Above

Partial Fractions-Repeated linear factors

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx \qquad \frac{5x^2+20x+6}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$5x^2+20x+6 = A(x+1)^2 + Bx(x+1) + Cx$$

If x =0: 6= A

If x = -1: -9 = -C, so C = 9

If x = 1: 31=6(4)+2B+9, B = - 1

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx = \int \left(\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2} \right) dx$$

$$6\ln|x| - \ln|x+1| - 9(x+1)^{-1} + C$$

$$\ln\left|\frac{x^6}{x+1}\right| - \frac{9}{x+1} + C$$

Quadratic Factors

$$\int \frac{2x^3 - 4x - 8}{x(x-1)(x^2+4)}dx \qquad \frac{2x^3 - 4x - 8}{x(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}$$

$$2x^3 - 4x - 8 = A(x-1)(x^2+4) + Bx(x^2+4) + (Cx+D)x(x-1)$$

If $x = 0$ then A = 2

If $x = 1$ then B = -2

If x = -1 2 = -C +D

If x = 2 8 = 2C+D

Solving the system of equations you find C = 2 and D = 4.

$$\int \frac{2x^3 - 4x - 8}{x(x-1)(x^2+4)}dx = \int \left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4} \right) dx$$

$$2\ln|x| - 2\ln|x-1| + \ln(x^2+4) + 2\arctan\frac{x}{2} + C$$

Repeated quadratic Factors

$$\int \frac{8x^3+13x}{(x^2+2)^2}dx$$

$$\frac{8x^3+13x}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}$$

$$8x^3+13x = (Ax+B)(x^2+2)+Cx+D$$

$$8x^3+13x = Ax^3+2Ax+Bx^2+2B+Cx+D$$

$$8x^3+0x^2+13x+0 = Ax^3+Bx^2+2Ax+Cx+D+2B$$

For third degree: A=8 For second degree: B=0

For first degree: 13=2A+C

For constant: D+2B=0

Repeated quadratic Factors

$$\int \frac{8x^3+13x}{(x^2+2)^2}dx$$

A=8 B=0

13=2A+C

D+2B=0

So, D=0 and C = -3

$$\int \frac{8x^3+13x}{(x^2+2)^2}dx = \int \left(\frac{8x}{x^2+2} + \frac{-3x}{(x^2+2)^2} \right) dx$$

$$\int \frac{8x^3+13x}{(x^2+2)^2}dx = 4\ln(x^2+2) + \frac{3}{2(x^2+2)} + C$$

Partial Fractions - Practice Problems

Evaluate each of the following integrals.

1. $\displaystyle\int \frac{4}{x^2 + 5x - 14}\,dx$

2. $\displaystyle\int \frac{8 - 3t}{10t^2 + 13t - 3}\,dt$

3. $\displaystyle\int_{-1}^{0} \frac{w^2 + 7w}{(w+2)(w-1)(w-4)}\,dw$

4. $\displaystyle\int \frac{8}{3x^3 + 7x^2 + 4x}\,dx$

5. $\displaystyle\int_{2}^{4} \frac{3z^2 + 1}{(z+1)(z-5)^2}\,dz$

MTH165

Unit 2
Differentiation and integration

L14-Properties of definite integral

Revision

$$\int e^x \sin e^x \, dx =$$

(a) $\cos e^x + c$

(b) $-\cos e^x + c$

(c) $\sin e^x + c$

(d) $e^x \cos e^x + c$

Revision

Partial Fractions of $\frac{x^2}{x^4 - x^2 - 12}$ are

(a) $\frac{2/7}{x^2-4} + \frac{3/7}{x^2+2}$

(b) $\frac{2/7}{x^2+4} + \frac{5/7}{x^2-3}$

(c) $\frac{2/7}{x^2-4} + \frac{3/7}{x^2+3}$

(d) None of these

DEFINITION

Let f(x) be a continuous function defined on [a, b],

$\int f(x)\,dx = F(x) + c$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$ is called definite integral. This formula is known as Newton-Leibnitz formula.

Note :

- The indefinite integral $\int f(x)\,dx$ is a function of x, where as definite integral $\int_a^b f(x)\,dx$ is a number.
- Given $\int f(x)\,dx$ we can find $\int_a^b f(x)\,dx$, but given $\int_a^b f(x)\,dx$ we cannot find $\int f(x)\,dx$

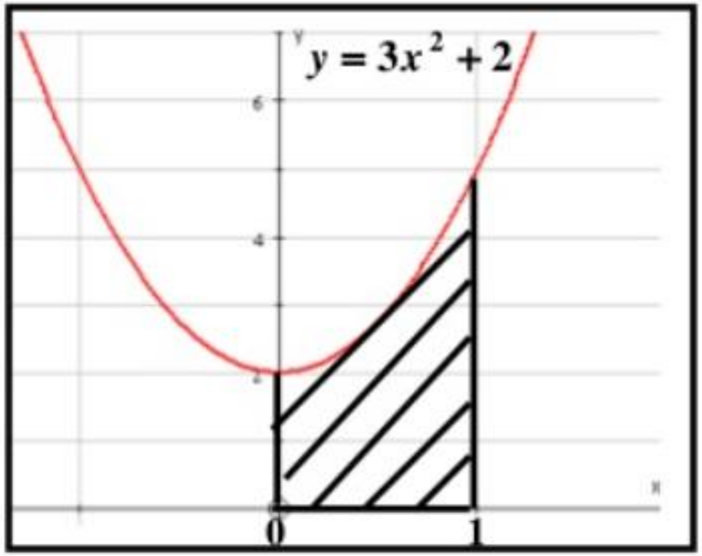
Areas

Definite integration results in a value.

It can be used to find an area bounded, in part, by a curve

e.g. $\int_0^1 3x^2 + 2\,dx$ gives the area shaded on the graph

The limits of integration . . .



Definite integration results in a value.

It can be used to find an area bounded, in part, by a curve

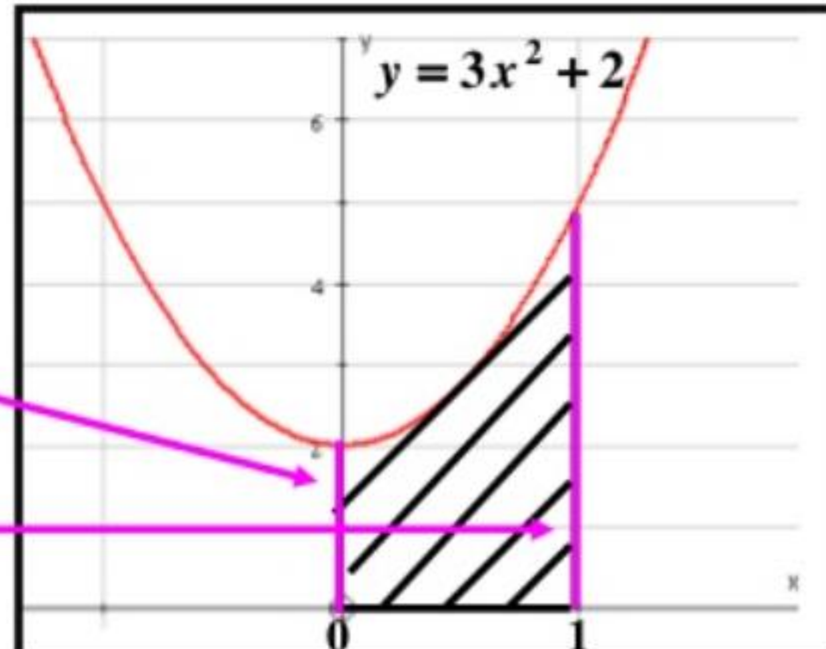
e.g. $\int_0^1 (3x^2+2)\,dx$ gives the area shaded on the graph

The limits of integration . . .

. . . give the boundaries of the area.

$x = 0$ is the lower limit
(the left hand boundary)

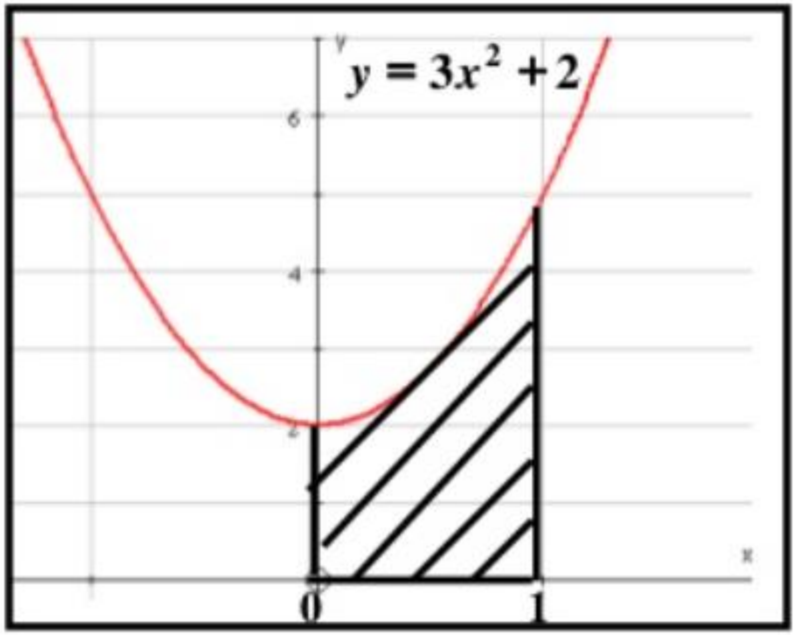
$x = 1$ is the upper limit
(the right hand boundary)



Since $$\int_0^1 \left(3x^2 + 2\right)dx$$

$$= \left[x^3 + 2x \right]_0^1 = 3$$

the shaded area equals 3



The units are usually unknown in this type of question

PROPERTIES OF DEFINITE INTEGRAL

1. $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

i.e. definite integral is independent of variable of integration.

2. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

3. $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

where c may lie inside or outside the interval [a, b].

4. $$\int_a^a f(x)\,dx = \int_0^a (f(x) + f(-x))\,dx$$

$$= 2\int_0^a f(x)\,dx$$, if f(–x) = f(x) i.e. f(x) is even.

$$= 0$$, if f(–x) = –f (x) i.e. f(x) is odd.

5. $$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Further $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

6. $$\int_0^{2a} f(x)\,dx = \int_0^{a} (f(x) + f(2a-x))\,dx$$

$$= 2\int_0^{a} f(x)\,dx \text{ , if } f(2a-x) = f(x)$$

$$= 0 \text{ , if } f(2a-x) = -f(x)$$

INTEGRATION OF ABSOLUTE VALUE FUNCTION

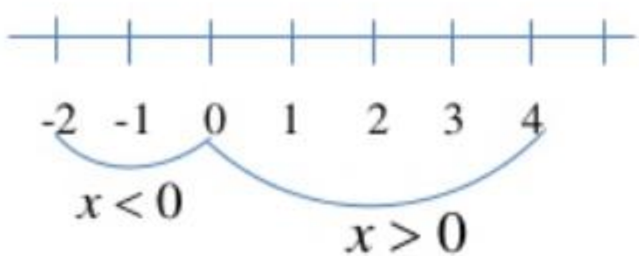
EXAMPLE 1. $\int_{-2}^{4} |x|\,dx$ $\qquad |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

1st solution

$$\int_{-2}^{4} |x|\,dx = \int_{-2}^{0} -x\,dx + \int_{0}^{4} x\,dx$$

$$= -\frac{x^2}{2}\Bigg]_{-2}^{0} + \frac{x^2}{2}\Bigg]_{0}^{4} = 0 - \left[-\frac{(-2)^2}{2}\right] + \frac{(4)^2}{2} - 0$$

$$= 2 + 8 = 10$$



2. $\int_{-3}^{3} |1-x| dx$

$$|1-x| = \begin{cases} 1-x & \text{if } 1\text{-x} \geq 0, \quad \text{x} \leq 1 \\ -(1-x) & \text{if } 1\text{-x} < 0, \quad \text{x} > 1 \end{cases}$$

$$\int_{-3}^{3} |1-x| dx = \int_{-3}^{1} (1-x) dx + \int_{1}^{3} -(1-x) dx$$

$$= \left. \text{x} - \frac{\text{x}^2}{2} \right]_{-3}^{1} + \left. \frac{x^2}{2} - x \right]_{1}^{3}$$

$$= \frac{1}{2} + \frac{15}{2} + \frac{3}{2} + \frac{1}{2} = \frac{20}{2} = 10$$

EXERCISES

$$\int_0^5 |2x - 5|\, dx$$

$$\int_1^4 (3 - |x - 3|)\, dx$$

$$\int_0^4 |x^2 - 9|\, dx$$

$$\int_0^4 |x^2 - 4x + 3|\, dx$$

INTEGRATION OF PIECEWISE FUNCTION

EXAMPLE

1. $\int_{-2}^{4} f(x)dx$; $\quad f(x) = \begin{cases} 2 + x^2, & -2 \le x < 0 \\ \dfrac{1}{2}x + 2, & 0 \le x \le 4 \end{cases}$

solution

$$\int_{-2}^{4} f(x)dx = \int_{-2}^{0} \left(2 + x^2\right)dx + \int_{0}^{4} \left(\frac{1}{2}x + 2\right)dx$$

$$= 2x + \frac{x^3}{3}\Bigg]_{-2}^{0} + \frac{x^2}{4} + 2x\Bigg]_{0}^{4} = \frac{56}{3}$$

EXERCISES

1. *In each part, evaluate the integral given that*

$$f(x)=\begin{cases}|x+2|, & if\ x\geq 0\\ x+2, & if\ x<0\end{cases}$$

a. $\int_{-2}^{0} f(x)dx$

b. $\int_{-2}^{2} f(x)dx$

c. $\int_{0}^{6} f(x)dx$

d. $\int_{-4}^{6} f(x)dx$

$$\int_0^{\pi/2} \cos x \, e^{\sin x} \, dx =$$

(a) 0

(b) 1

(c) -1

(d) e-1

$$\int_0^{\pi/4} (\cos^2 x - \sin^2 x)\,dx =$$

(a) 0

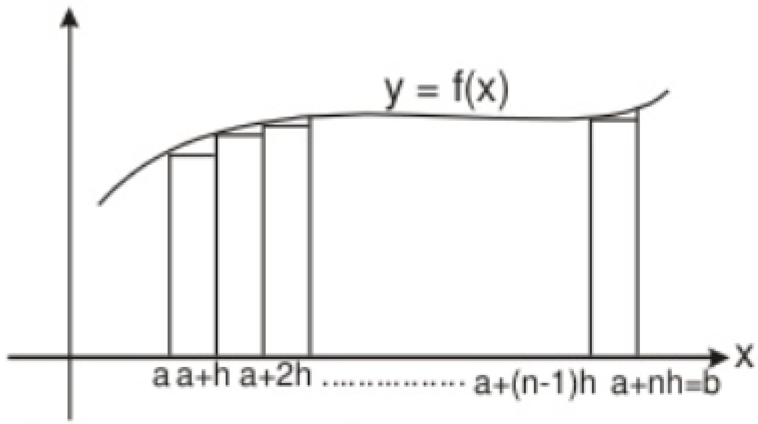
(b) $\frac{1}{2}$

(c) $-\frac{1}{2}$

(d) None of these

DEFINITE INTEGRAL AS A LIMIT OF SUM

Let f(x) be a continuous real valued function defined on the closed interval [a, b] which is divided into n parts as shown in figure.



The point of division on x-axis are

$a, a + h, a + 2h \ldots\ldots\ldots\ldots a + (n - 1)h, a + nh$,

where $\frac{b-a}{n} = h$.

Let S_n denotes the area of these n rectangles. Then,

$S_n = hf(a) + hf(a + h) + hf(a + 2h) + \ldots\ldots\ldots + hf(a + (n - 1)h)$

Clearly, S_n is area very close to the area of the region bounded by curve $y = f(x)$, x–axis and the ordinates x = a, x = b.

Hence $\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=0}^{n-1} h\,f(a+rh) = \lim_{n\to\infty} \sum_{r=0}^{n-1} \left(\frac{b-a}{n}\right) f\left(a+\frac{(b-a)r}{n}\right)$$

Definite integral as a Limit of a Sum

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \cdots f(a+\overline{n-1}\,h)\right]$$

where $h = \dfrac{b-a}{n}$

Tutorial

MTH165

MCQ

$\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta} dx$ is equal to

(a) $2(\sin x + x \cos \theta) + C$

(b) $2(\sin x - x \cos \theta) + C$

(c) $2(\sin x + 2x \cos \theta) + C$

(d) $2(\sin x - 2x \cos \theta) + C$

$\int \frac{1}{\sin^2 x \cos^2 x} dx$ is equal to

(a) $\sin^2 x - \cos^2 x + C$

(b) -1

(c) $\tan x + \cot x + C$

(d) $\tan x - \cot x + C$

MCQ

$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$ equals to

(*a*) 0 (*b*) $\frac{1}{2}$

(*c*) 1 (*d*) $\frac{3}{2}$

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx$$

(a) $\frac{\pi}{4}$

(b) $\frac{\pi}{2}$

(c) π

(d) None of these

$$\int_0^1 |3x-1|\,dx$$

(a) $\frac{1}{6}$

(b) $\frac{3}{4}$

(c) $\frac{5}{6}$

(d) None of these

$$\int \frac{xe^x}{(1+x)^2}\,dx$$

(a) $\dfrac{e^x}{1+x}$

(b) $\dfrac{1+xe^x}{(1+x)^2}$

(c) $-\dfrac{e^x}{1+x}$

(d) None of these

Definite integral as a Limit of a Sum

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \cdots f(a+\overline{n-1}\,h)\right]$$

where $h = \dfrac{b-a}{n}$