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Stoke's Theorem :- (Relation between the line and the surface integral)

If S be an open surface bounded by a closed curve C and $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be any continuously differentiable vector point function then.

$$\oint_C \vec{F}\cdot d\vec{s} = \iint_S (\text{curl}\,\vec{F})\cdot\hat{n}\,dA$$

$$\oint_C (f_1\hat{i}+f_2\hat{j}+f_3\hat{k})\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$= \oint_C f_1dx + f_2dy + f_3dz = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dA$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right) - \hat{j}\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_1}{\partial z}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)$$

Projection on xy plane. $\hat{n} = \hat{k}$

$$dA = \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

$$dA = \frac{dx\,dy}{1}$$

Problem 3: Evaluate $\oint_C \vec{V}.\vec{dr}$ by Stokes' theorem where $\vec{V} = (3x-y)\hat{\imath} - 2yz^2\hat{\jmath} - 2y^2z\hat{k}$ and S is the surface bounded by sphere $x^2+y^2+z^2 = 16, z > 0$.

$$\vec{v} = (3x-y)\hat{i} - 2yz^2\hat{j} - 2y^2z\hat{k}$$

$S:$ $x^2+y^2+z^2=16,\ z>0$

$$\oint_C \vec{v}\cdot d\vec{s} = \iint_S (\nabla\times\vec{v})\cdot\hat{n}\,dA$$

$$\nabla\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x-y & -2yz^2 & -2y^2z \end{vmatrix}$$

$$\begin{vmatrix} & & \\ 3x-y & -2yz^2 & -2y^2z \end{vmatrix}$$

$$\nabla \times \bar{u} = \hat{i}\left(\cancel{-4yz} + \cancel{4yz}\right) - \hat{j}\left(0-0\right) + \hat{k}\left(0+1\right)$$

$$= 0\hat{i} - 0\hat{j} + \hat{k} = \hat{k}$$

$$\vec{n} = \text{grad}(s) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\text{grad}(s)}{|\text{grad}(s)|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{4(x^2+y^2+z^2)}} = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{4(16}}$$

$$= \frac{(x\hat{i}+y\hat{j}+z\hat{k})}{4}.$$

Projection of the sphere $x^2+y^2+z^2=16$ on the xy plane $(z=0)$ will be a circle $x^2+y^2=16$

$$dA = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dx\,dy}{\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{4}\cdot\hat{k}} = \frac{dx\,dy}{z/4}$$

Stoke's Thm

$$\oint_C \bar{u}\cdot d\bar{r} = \iint_S (\text{curl}\,\bar{u})\cdot\hat{n}\,dA = \iint_R (\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{4}\right)\cdot\frac{dx\,dy}{z/4}$$

$$= \iint_R \cancel{z/4}\cdot\frac{dx\,dy}{\cancel{z/4}}$$

$$= \iint_R dx\,dy = \text{Area of the circle } x^2+y^2=16$$

$$= \pi(4)^2$$

$$= 16\pi$$

If S is the surface $x^2+y^2+z^2=1$ then

$\cancel{\oint_C \vec{F}\cdot d\vec{r}} - \cancel{\oint_C \vec{F}\cdot d\vec{r}}$ $\longleftarrow$ $\boxed{\iint_S (\text{Curl}\,\vec{F})\cdot d\vec{S} = 0}$

$= 0$