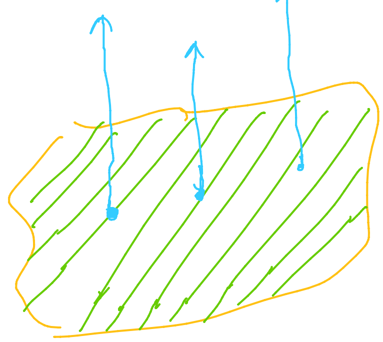
L45 28 April 2021

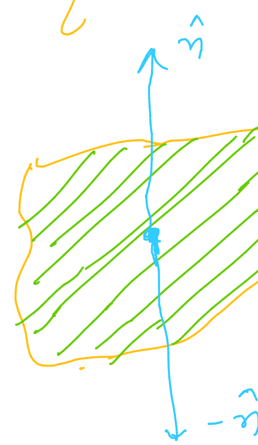
28 April 2021 12:09

Orientable Surface

Mobius strip

A smooth surface S is s.t.b orientable if there exists a continuous unit normal vector field $\hat{n}$ defined at each point (x,y,z) on the surface.





Let S be the orientable surface.
We choose a unit normal $\hat{n}$ over the surface S

$$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

where α, β, γ are the angles which the normal vector $\hat{n}$ makes with the positive direction of x-axis, y-axis and z-axis respectively.

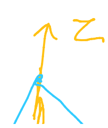
$$\hat{n}\cdot\hat{i} = \cos\alpha, \qquad \hat{n}\cdot\hat{j} = \cos\beta \qquad \hat{n}\cdot\hat{k} = \cos\gamma$$

Let $\vec{v}(x,y,z) = v_1(x,y,z)\,\hat{i} + v_2(x,y,z)\,\hat{j} + v_3(x,y,z)\,\hat{k}$ be a vector field

$$\iint_S (\vec{v}\cdot\hat{n})\,dA = \iint_S (v_1\cos\alpha + v_2\cos\beta + v_3\cos\gamma)\,dA$$

$$= \iint_S v_1\cos\alpha\,dA + v_2\cos\beta\,dA + v_3\cos\gamma\,dA$$

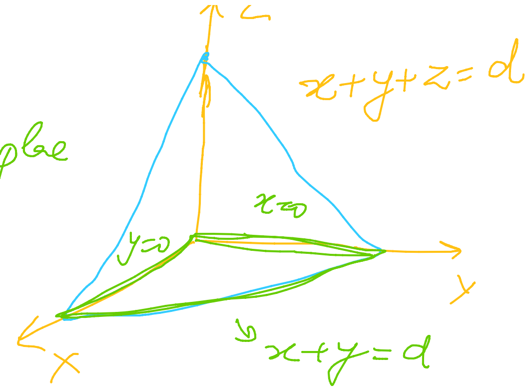
$\cos\alpha\,dA = dy\,dz$



$x+y+z=d$

$\cos\alpha \, dA = dy\,dz$

$dA = \dfrac{dy\,dz}{\hat{n}\cdot\hat{i}}$] → projection on y-z plane



or

$\cos\beta \, dA = dz\,dx$

$dA = \dfrac{dz\,dx}{\hat{n}\cdot\hat{j}}$ → projection on z-x plane

or

$\cos\gamma \, dA = dx\,dy$

$dA = \dfrac{dx\,dy}{\hat{n}\cdot\hat{k}}$ → projection of xy plane

Flux of a vector field $\vec{v}$ through a surface S

Let $\vec{v} = v_1(x,y,z)\,\hat{i} + v_2(x,y,z)\,\hat{j} + v_3(x,y,z)\,\hat{k}$ be a vector field representing the velocity of a fluid.

$$\text{flux} = \iint_S (\vec{v}\cdot\hat{n})\, dA$$ → Volume of the fluid passing through S per unit time.

Surface Area $\iint_S dA$

Evaluate the surface integral $\iint \vec{F}\cdot\hat{n}\, dA$ where $\vec{F} = 6z\,\hat{i} + 6\,\hat{j} + 3y\,\hat{k}$ and S is the portion of the plane $2x+3y+4z=12$, which is in the first octant.

$f(x,y,z) = 2x+3y+4z-12 = 0$

grad f.

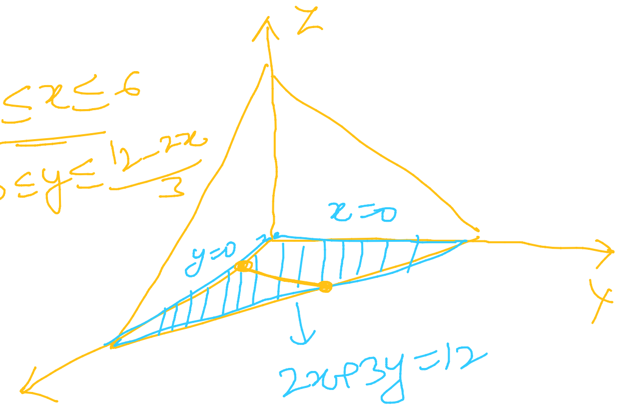
$f(x,y,z) = 2x+3y+4z-12=0$

grad f.

$\vec{n} = \text{grad } f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = 2\hat{i}+3\hat{j}+4\hat{k}$

$$\hat{n} = \frac{\text{grad} f}{|\text{grad} f|} = \frac{2\hat{i}+3\hat{j}+4\hat{k}}{\sqrt{4+9+16}} = \frac{2\hat{i}+3\hat{j}+4\hat{k}}{\sqrt{29}}$$

$$\vec{F}\cdot\hat{n} = (6z\hat{i}+6\hat{j}+3y\hat{k})\cdot\left(\frac{2\hat{i}+3\hat{j}+4\hat{k}}{\sqrt{29}}\right) = \frac{12z+18+12y}{\sqrt{29}}$$



projection on the xy plane

$0 \le x \le 6$

$0 \le y \le \frac{12-2x}{3}$

$$dA = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dx\,dy}{4/\sqrt{29}}$$

$$\iint_S \vec{F}\cdot\hat{n}\,dA = \iint_R \left(\frac{12z+18+12y}{\cancel{\sqrt{29}}}\right)\frac{dx\,dy}{4/\cancel{\sqrt{29}}}$$

$3y = 12-2x$

$y = \frac{12-2x}{3}$

$2x=12$

$x=6$

$4z = 12-2x-3y$

$z = \frac{12-2x-3y}{4}$

$$= \frac{1}{4}\iint_R (12z+18+12y)\,dx\,dy$$

$$= \frac{1}{4}\iint_R \left[3(12-2x-3y)+18+12y\right]dx\,dy$$

$\begin{array}{r} 36 \\ 18 \\ \hline \end{array}$

$\begin{array}{r} 12 \\ -9 \end{array}$

$$= \frac{1}{4}\iint_R (54-6x+3y)\,dx\,dy$$

$$= \frac{1}{4}\int_0^6\int_0^{\frac{12-2x}{3}} (54-6x+3y)\,dy\,dx$$

$$= \frac{1}{4}\int_0^6 \int_0 (54 - 6x + 3y)\, dy\, dx$$

$$= \frac{1}{4}\int_0^6 \left[[54-6x]\left[\frac{12-2x}{3}\right] + \frac{3}{2}\left[\frac{12-2x}{3}\right]^2 \right] dx$$

?

The Surface Area of the sphere $x^2+y^2+z^2 = a^2$

$$\iint_S dA = ? \qquad = 4\pi a^2$$

$$\iint_S dA = \text{Surface Area}$$

$$\iint_S$$

$$\iint_S (\vec{F}\cdot\hat{n})\, dA = \text{flux}$$

$$dA = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} \qquad dA = \frac{dy\,dz}{\hat{n}\cdot\hat{i}} \qquad dA = \frac{dz\,dx}{\hat{n}\cdot\hat{j}}$$