L40 19 April 2021

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Unit-6 Vector Calculus-II

Line Integral w.r.t arc length

Let C represents the curve

$x = x(t), \; y = y(t), \; z = z(t), \quad a \le t \le b$

Let $f(x,y,z)$ continuous on C. Then

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$$

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(t)\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt}$$

$$ds = \frac{ds}{dt}\,dt$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

Problem 1. Evaluate $\int_C f(x,y,z)ds$ where $f(x,y,z) = 2x + 3y$ and curve C is:

$C: x = t, y = 2t, z = 3t, 0 \le t \le 3.$

$\int_C f(x,y,z)\,ds$ $\qquad f(x,y,z) = 2x+3y \qquad 0 \le t \le 3$

$C: \quad x = t, \quad y = 2t, \quad z = 3t$

$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2, \quad \frac{dz}{dt} = 3$

$f(x,y,z) = 2(t) + 3(2t) = 2t + 6t$

$= 8t$

$$\int_C f(x,y,z)\,ds = \int_a^b f(t)\cdot\frac{ds}{dt}\cdot dt$$

$$= \int_0^3 8t\cdot\left(\sqrt{(1)^2+(2)^2+(3)^2}\right)dt$$

$$= \int_0^3 8\sqrt{14}\, t\, dt = 8\sqrt{14}\left[\frac{t^{\cancel{3}2}}{2}\right]_0^3 = \overset{4}{\cancel{8}}\sqrt{14}\cdot\left[\frac{9}{\cancel{2}}\right]$$

$$= 36\sqrt{14}$$

② $f(x,y,z) = xy^2z$ C is the line segment from $(1,2,2)$ to $(2,3,5)$

$$\frac{x-1}{2-1} = \frac{y-2}{3-2} = \frac{z-2}{5-2}$$

$$\frac{x-1}{1} = \frac{y-2}{1} = \frac{z-2}{3} = t$$

$x = t+1$, $y = 2+t$, $z = 3t+2$

$\frac{dx}{dt} = 1$ $\frac{dy}{dt} = 1$ $\frac{dz}{dt} = 3$

$1 \le x \le 2$

$1 \le t+1 \le 2$

$\boxed{0 \le t \le 1}$

$f(x,y,z) = xy^2z$

$f(t) = (t+1)(2+t)^2(3t+2)$

$= (t+1)(4+t^2+4t)(3t+2)$

$= (4t+4+t^3+t^2+4t^2+4t)(3t+2)$

$= (t^3+5t^2+8t+4)(3t+2)$

$= 3t^4+15t^3+24t^2+12t+2t^3+10t^2+16t+8$

$f(t) = 3t^4+17t^3+34t^2+28t+8$

$$\int_C f(x,y,z)\,ds = \int_0^1 (3t^4+17t^3+34t^2+28t+8)\sqrt{1+1+9}\,.dt$$

$$= \sqrt{11}\left[\frac{3t^5}{5}+17\frac{t^4}{4}+34\frac{t^3}{3}+28\frac{t^2}{2}+8t\right]_0^1$$

$$= \sqrt{11}\left[\frac{3}{5}+\frac{17}{4}+\frac{34}{3}+\frac{28}{2}+8\right]$$

Q34. Evaluate line integral $\int (x^2 + yz)dz$, over C given by x=t, y=t², z=3t and $1 \le t \le 2$.

a. 163/4 b. 163 c. 4 d. none of these

$\int (x^2+yz)\,dz$

$x = t$

$y = t^2$

$\int (x^2+yz)\,dz$

$x=t$
$y=t^2$
$z=3t$
$dz=3\,dt$

16×3

$\int (t^2+t^2(3t))\,3\,dt$

$= 3\int_1^2 (t^2+3t^3)\,dt = 3\left[\frac{t^3}{3}+\frac{3t^4}{4}\right]_1^2$

$=3\left[\left(\frac{8}{3}+\frac{48}{4}\right)-\left(\frac{1}{3}+\frac{3}{4}\right)\right]$

$=3\left[\frac{7}{3}+\frac{45}{4}\right]$

Line Integrals of vector field:

Let us consider a vector field

$\vec{v} = v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$

$v_1(x,y,z)$

$\vec{v}$ is continuous on C $\quad \vec{s} = x\hat{i}+y\hat{j}+z\hat{k}$

$\vec{s}(t) = x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}, \quad a\le t\le b$

Then the line integral of $\vec{v}$ over C is

$$\int_C \vec{v}\cdot d\vec{s} = \int_a^b (v_1\hat{i}+v_2\hat{j}+v_3\hat{k})\cdot\frac{d\vec{s}}{dt}\,dt$$

Work done

$$= \int_a^b (v_1\hat{i}+v_2\hat{j}+v_3\hat{k})\cdot\left(\frac{dx}{dt}\hat{i}+\frac{dy}{dt}\hat{j}+\frac{dz}{dt}\hat{k}\right)dt$$

$$= \int_a^b \left(v_1\frac{dx}{dt}+v_2\frac{dy}{dt}+v_3\frac{dz}{dt}\right)\cdot dt$$

Problem 1. Evaluate $\int_C \vec{v}.\overrightarrow{dr}$ where $\vec{v} = xy\hat{\imath}+y^2\hat{\jmath}+e^z\hat{k}$ and curve c is given by:

$C: x=t^2, y=2t, z=t, 0\le t\le 1.$

$C: x = t^2, y = 2t, z = t, 0 \le t \le 1.$

$\int_C \bar{v} \cdot d\bar{s} =$

$\bar{v} = (t^2)(2t)\hat{i} + 4t^2\hat{j} + e^t\hat{k}$

$\bar{v} = 2t^3\hat{i} + 4t^2\hat{j} + e^t\hat{k}$

$\vec{s} = t^2\hat{i} + 2t\hat{j} + t\hat{k}$

$\frac{d\vec{s}}{dt} = (2t\hat{i} + 2\hat{j} + \hat{k})$

$= \int_0^1 (2t^3\hat{i} + 4t^2\hat{j} + e^t\hat{k}) \cdot (2t\hat{i} + 2\hat{j} + \hat{k}) \, dt$

$= \int_0^1 (4t^4 + 8t^2 + e^t) \, dt = \left(4\frac{t^5}{5} + 8\frac{t^3}{3} + e^t\right)_0^1$

$= \frac{4}{5} + \frac{8}{3} + e^1 - 1$

$= \left(\frac{12+40}{15} - 1\right) + e$

Find work done by force $\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$ in moving a particle from (1,2,2) to (3,6,6).

$\int_C \bar{v} \cdot d\bar{s}$

(1,2,2) (3,6,6)

$\frac{x-1}{3-1} = \frac{y-2}{6-2} = \frac{z-2}{6-2}$

$\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-2}{4} = t$

$x = 2t+1, \; y = 4t+2, \; z = 4t+2$

$1 \le x \le 3$

$1 \le 2t+1 \le 3$

$0 \le 2t \le 2$

$\boxed{0 \le t \le 1}$

$\int_C \bar{v} \cdot d\bar{s} = \int_0^1 \left((2t+1)\hat{i} + (4t+2)\hat{j} + (4t+2)\hat{k}\right) \cdot (2\hat{i} + 4\hat{j} + 4\hat{k}) \, dt$

$= \int_0^1 \left((4t+2) + 16t+8 + 16t+8\right) dt$

$$= \int_0^1 \left[(4t+2) + 16t+8 + 16t+8\right] dt$$

$$= \int_0^1 (36t + 18)\, dt = \left[36 \cdot \frac{t^2}{2} + 18t\right]_0^1$$

$$= (18+18) - 0 = \underline{\underline{36}}$$