MTH165

Mathematics for Engineers

#Zero Lecture

Books Required

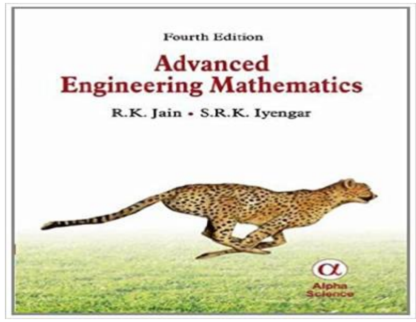
Text Book:

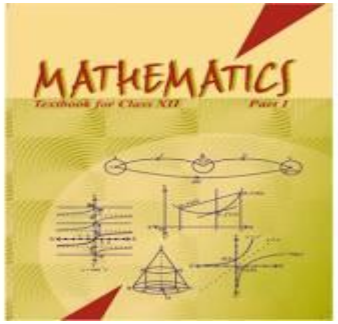
ADVANCED ENGINEERING MATHEMATICS BY JAIN AND IYENGAR

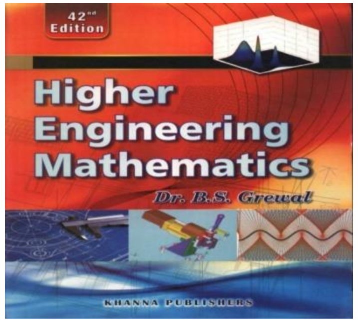
References Books:

MATHEMATICS FOR CLASS 12 PART 1-2 BY NCERT

HIGHER ENGINERING MATHEMATICS BY B.S GREWAL







Course Assessment Model

Teaching Model:

L-T-P: 3-1-0 (3 Lectures, 1 Tutorial, 0 Practical)

Marks Breakup:

Attendance	5
CA (2 best out of 3 online Tests)	30
MTE (MCQ/Viva based)	25
ETE (MCQ)	40
Total	**100**

Course Outcomes

Through this course students should be able to

1. Recall the concepts of matrices and its application to solve the system of linear equations.

2. Review the basic concept of calculus of one variable.

3. Apply the concept of calculus to evaluate extreme values of functions.

4. Describe calculus of multivariate functions and their applications.

5. Evaluate surface and volume integral using multiple integral.

6. Describe the concept of Fourier series and its application.

Syllabus Distribution: Unit-1

Linear Algebra

- Review of matrices
- Elementary operations of matrices
- Rank of a matrix
- Linear dependence and independence of vectors
- Solution of Linear system of equations
- Inverse of matrices
- Eigen values and Eigen vectors
- Properties of Eigen values
- Cayley-Hamilton theorem

Unit-2

Differential and Integral Calculus

- Differential coefficient
- General rules of differentiation
- Derivatives of standard functions
- Derivatives of Parametric forms
- Derivatives of implicit functions
- Logarithmic differentiation

- Integration as an inverse process of differentiation
- Methods of integration-By Substitution
- Methods of integration-By Parts
- Methods of integration-By Partial fractions
- Definite integral as Limit of sum
- Properties of definite integral

Unit-3

Applications of Derivatives

- Rolles' Theorem
- Mean Value Theorems
- Taylors' theorem with remainders
- Maclaurins' theorem with remainders
- Indeterminate Forms
- L-Hospital rule
- Maxima and Minima

Unit-4

Multivariate Functions

- Functions of two variables
- Limits and Continuity
- Partial derivatives
- Total derivative and differentiability
- Chain rule
- Euler’s theorem for Homogeneous functions
- Maxima and Minima
- Lagrange method of multiplier

Unit-5

Multivariate Integrals

- Double integrals
- Change of order of integration
- Triple integrals
- Change of variables
- Application of double integrals to calculate area and volume
- Application of triple integrals to calculate volume

Unit-6

Fourier Series

- Euler Coefficients
- Fourier Series
- Fourier Series for Even and Odd functions
- Half range Fourier Series
- Parseval’s Formula
- Complex form of Fourier Series

What Do You Think?

What could be considered the greatest achievements of the human mind ?

It's the Greatest!

- Consider that all these things emerged because of technological advances
- Those advances relied on CALCULUS !
- Calculus has made it possible to:
  - Build giant bridges
  - Travel to the moon
  - Predict patterns of population change

Use of Matrices:

Use in Cryptography

The key **matrix** is used to encrypt the messages, and its inverse is used to decrypt the encoded messages. It is important that the key **matrix** be kept secret between the message senders and intended recipients. If the key **matrix** or its inverse is discovered, then all intercepted messages can be easily decoded.

Use in Geology

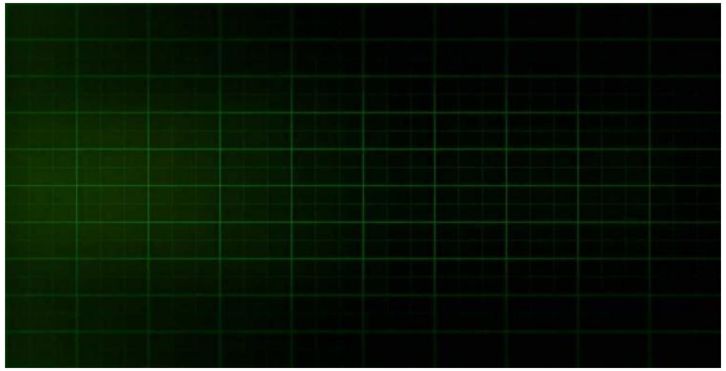
In **Geology**, **Matrices** are used for taking seismic surveys. They are used for plotting graphs, statistics and also to do scientific studies in almost different fields. ... **Matrices** are used in calculating the Gross domestic products in economics which eventually helps in calculating the goods production efficiently.

Use in Robotics

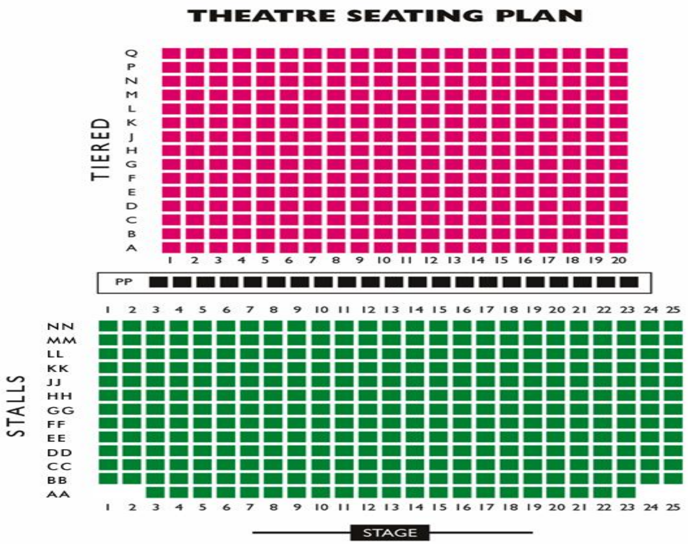
Application of Homogeneous Transformation **Matrices** to the Simulation of VLP Systems. In **robotics**, Homogeneous Transformation **Matrices** (HTM) have been used as a tool for describing both the position and orientation of an object and, in particular, of a **robot** or a **robot** component

You might have observed in use of matrices in routine:

Grid of Computer Screen



Online Booking of Cinema Hall



<u>You might have observed in use of matrices in routine:</u>

Republic Day Parade

Matrix Movie

Uses of Matrices in Various Fields:

Encryption

Games especially 3D

Economics and business

Construction

Dance – contra dance

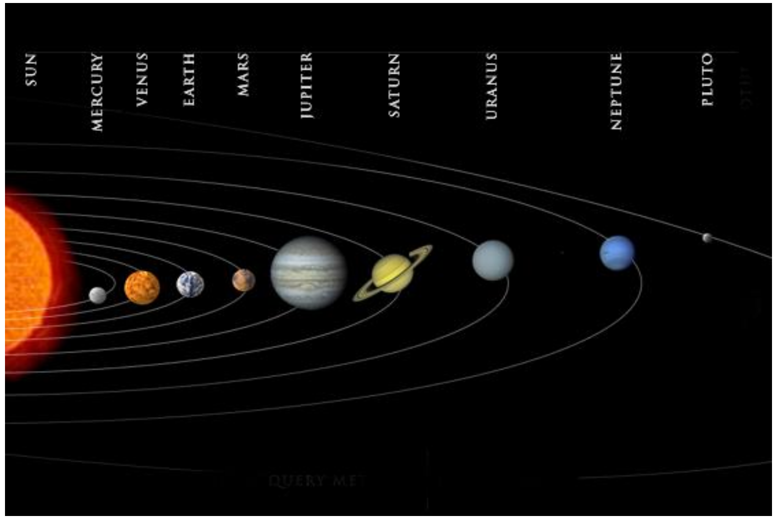
Animation

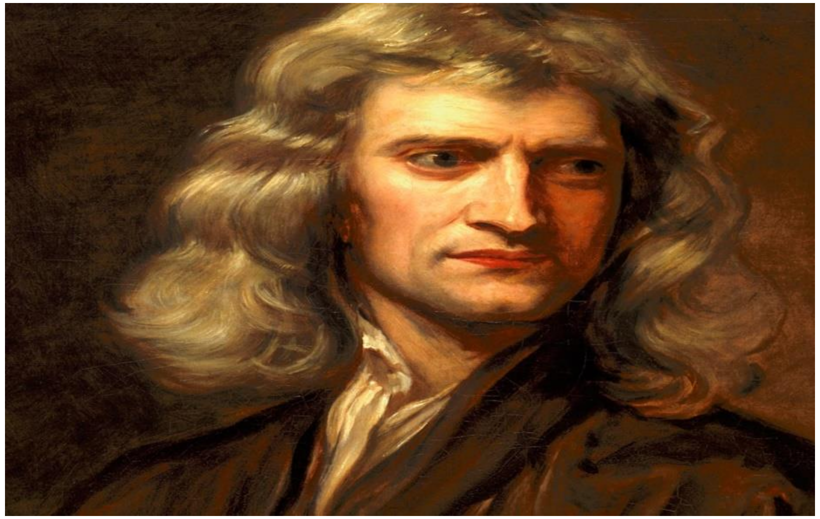
Physics

Geology

Uses of Calculus:

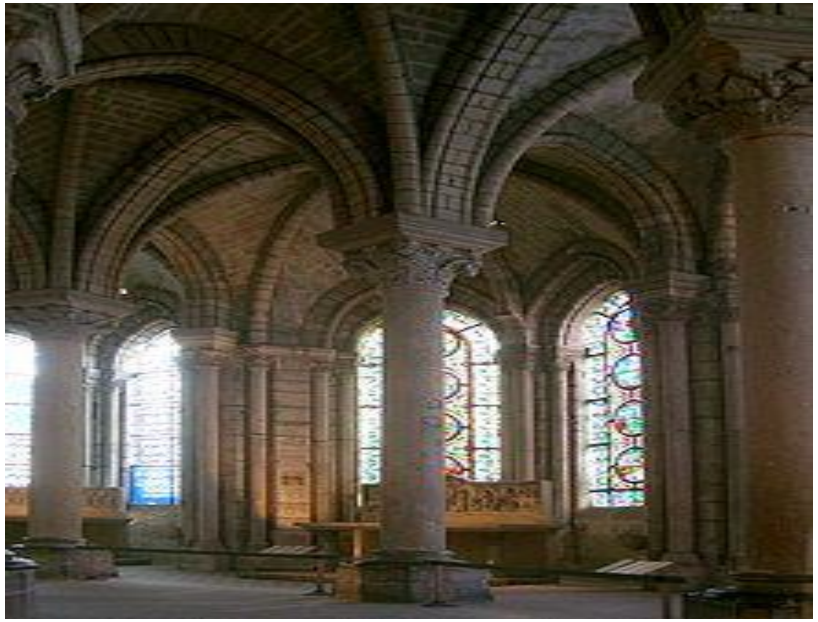
Sir Isaac Newton used calculus to solve many physics problems such as the problem of planetary motion, shape of the surface of a rotating fluid etc. – recorded in Principia Mathematica





Uses of Calculus:

Gottfried Leibniz developed calculus to find area under curves



Rate of change is everywhere….

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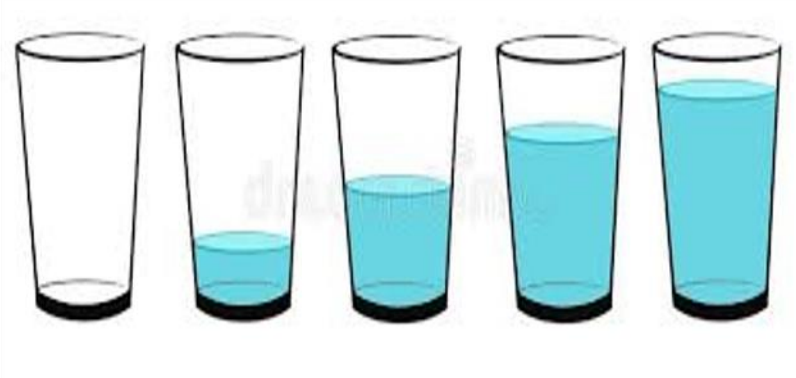
Time Taken to Fill

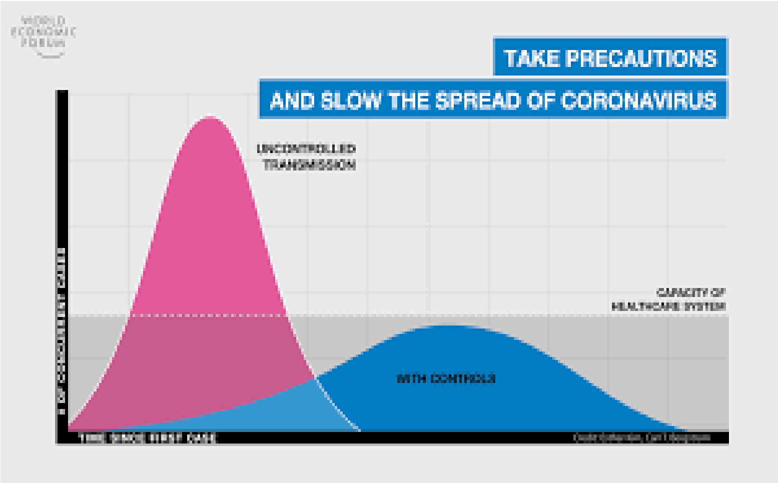


Time Taken to Grow into

Rate of change is everywhere….

Time taken to Reach

Rate of change is everywhere….



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Differentiation and Integration are Inverse of each other…

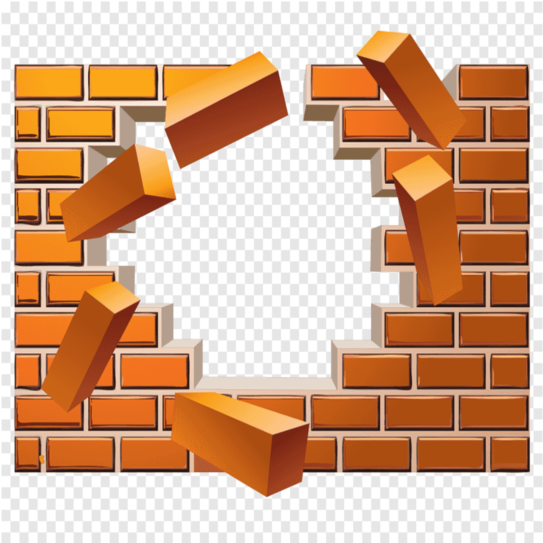
$$\frac{d}{d(Pizza)} = Pizza\ Slices$$

Differentiation and Integration are Inverse of each other…

$$\int_{Slices=1}^{8}$$

$$=$$

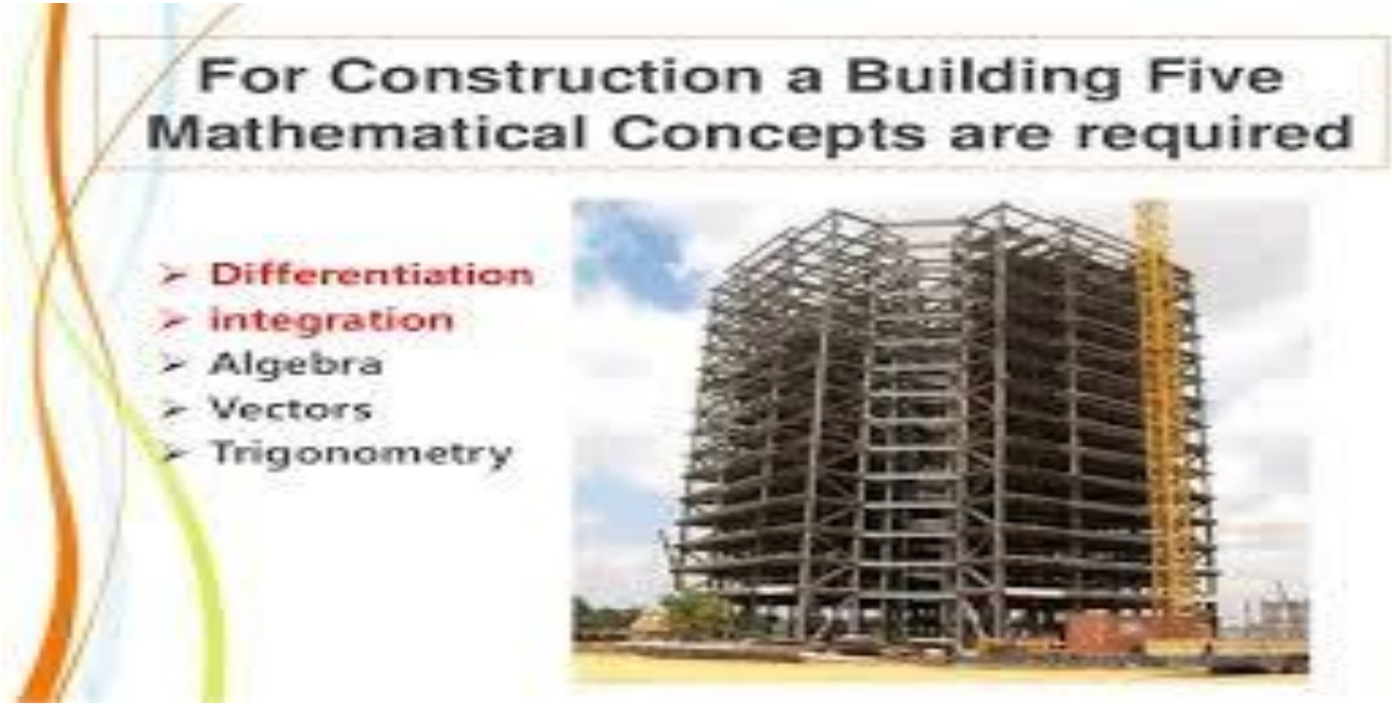
Differentiation and Integration are Inverse of each other…



Differentiation Integration

Calculus in Civil Engineering

Derivatives are used to determine the maximum and minimum values of the functions like cost, strength, amount of material used in a building, profit loss etc.



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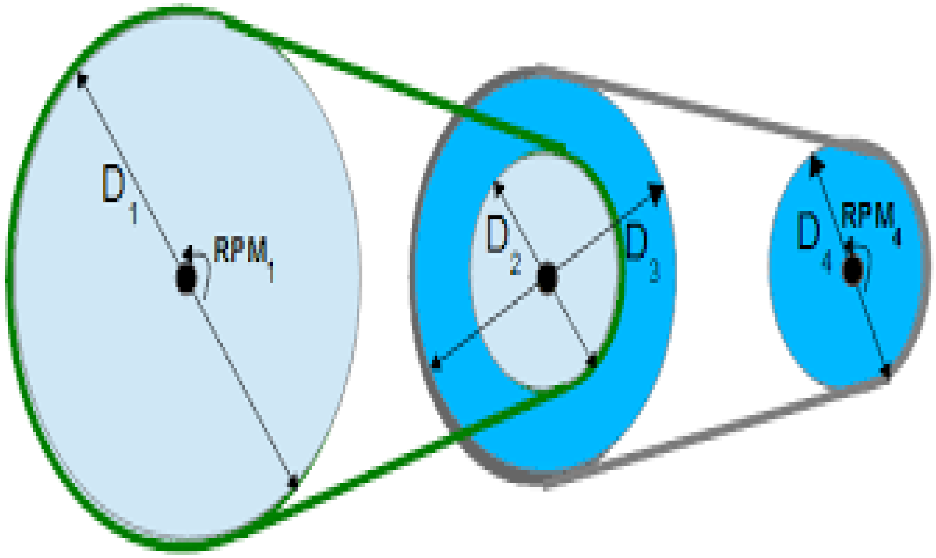
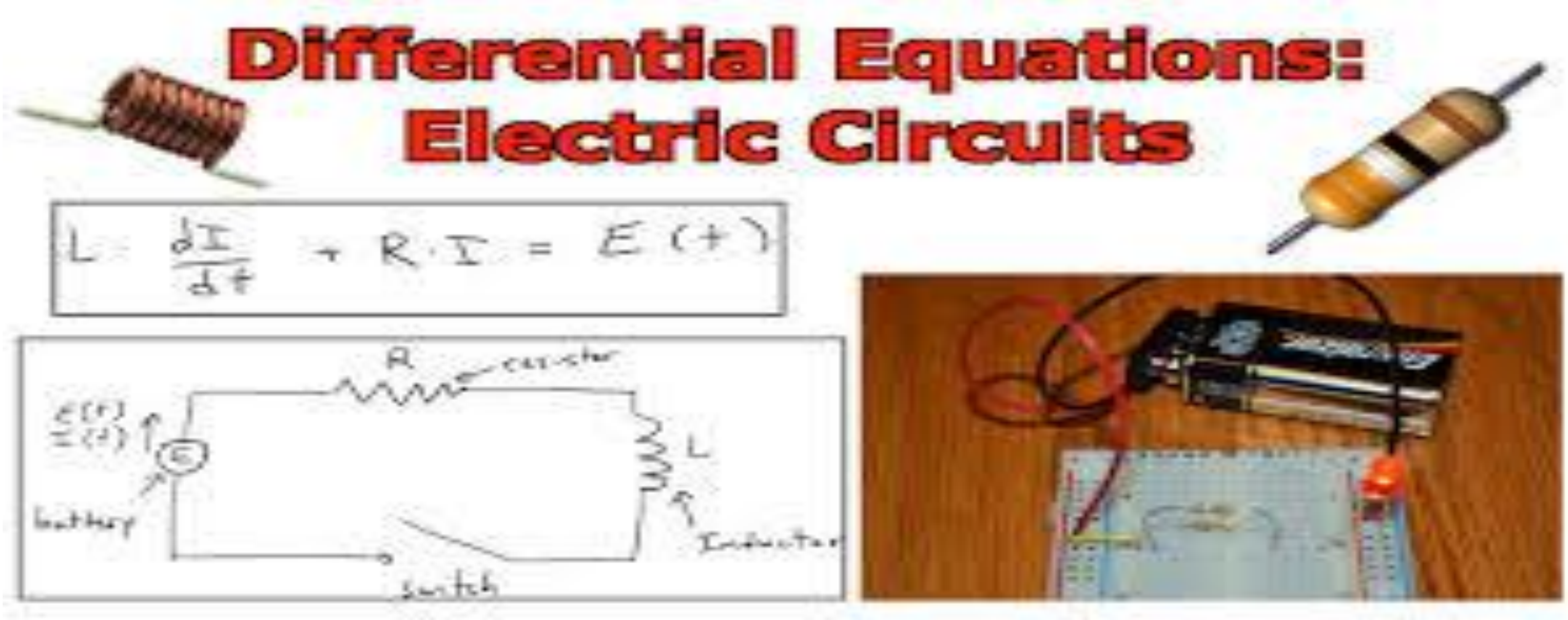
CALCULUS AND MECHANICAL ENGINEERING TECHNOLOGY

The best example of use of chain rule in differentiation, is the working of pulleys of different sizes with same belt to reduce the effort and optimize the output.



Calculus and Electronics Engineering

The various kinds of LCR circuits can not be solved without differential equations and Ohm’s law which is dependent on derivative of voltage.



Uses of Calculus in Computer Science/Programming

By Amber Barnett

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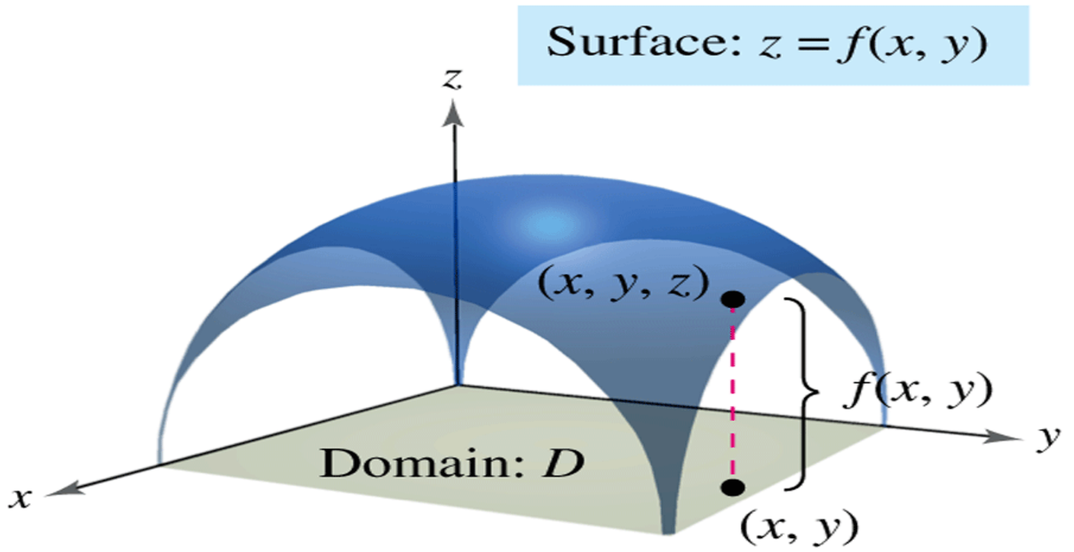
Development of different kinds of computer languages such as C, C++, Java, Linux, Python and development of various mobile apps has a great reliance on Calculus.

Multivariate Calculus

Definition of a Function of Two Variables

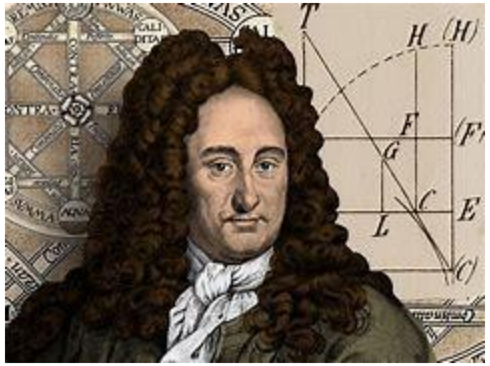
Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is called a **function of x and y.** The set D is the **domain** of f, and the corresponding set of values for $f(x, y)$ is the **range** of f.



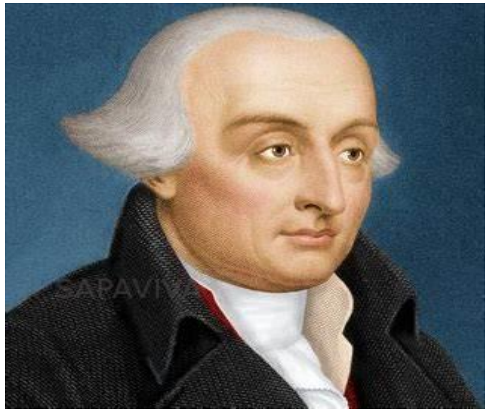
Multivariate Calculus

Major Contributors are:

Leibnitz



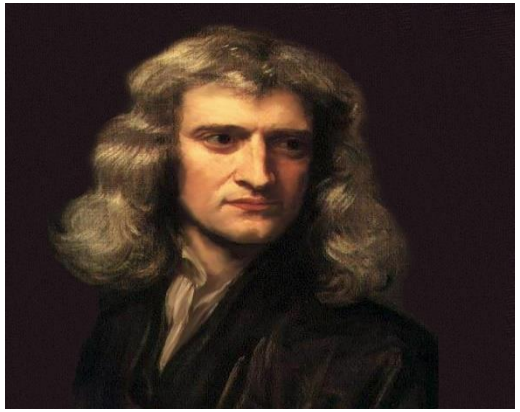
Lagrange



Multivariate Calculus

Major Contributors are:

Newton

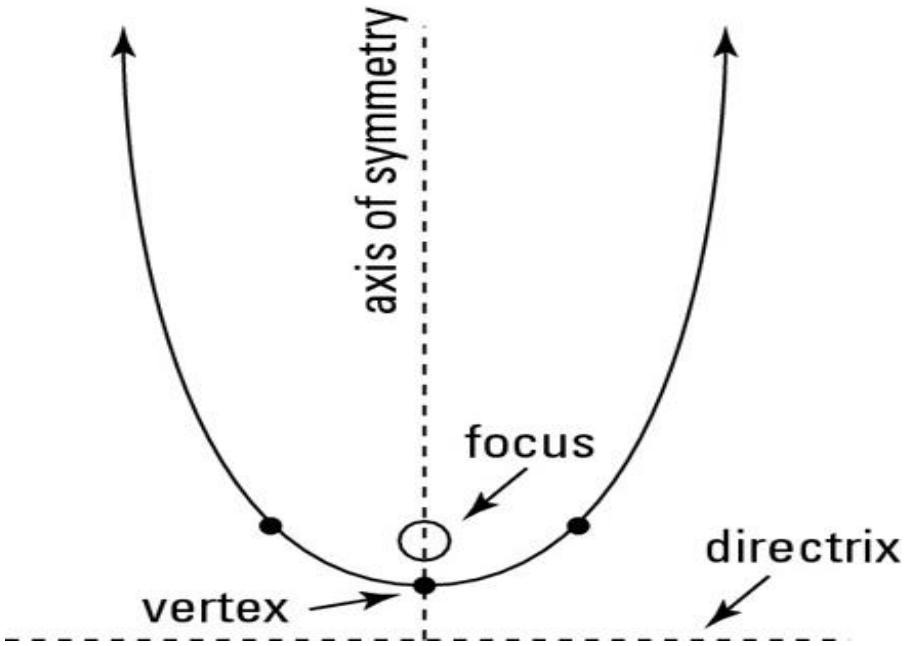


Leonhard Euler

Difference:

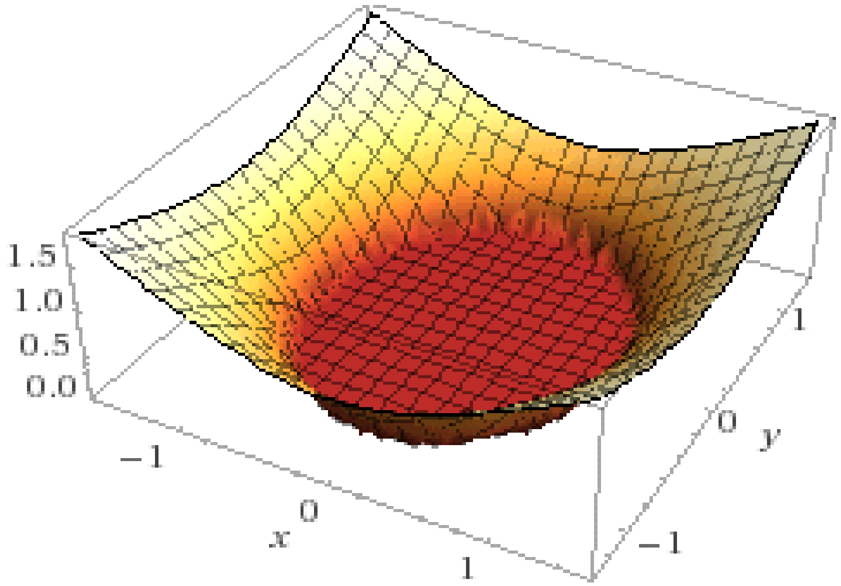
Single variable calculus

$$y = x^2$$



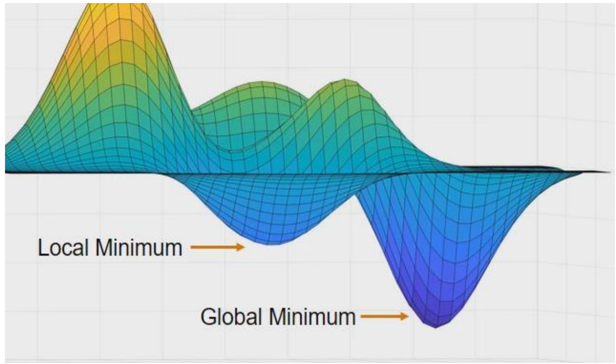
Multivariable Calculus

$$z = x^2 + y^2$$

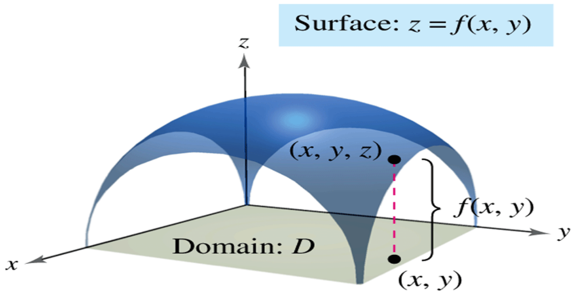


Uses of Multivariate Calculus:

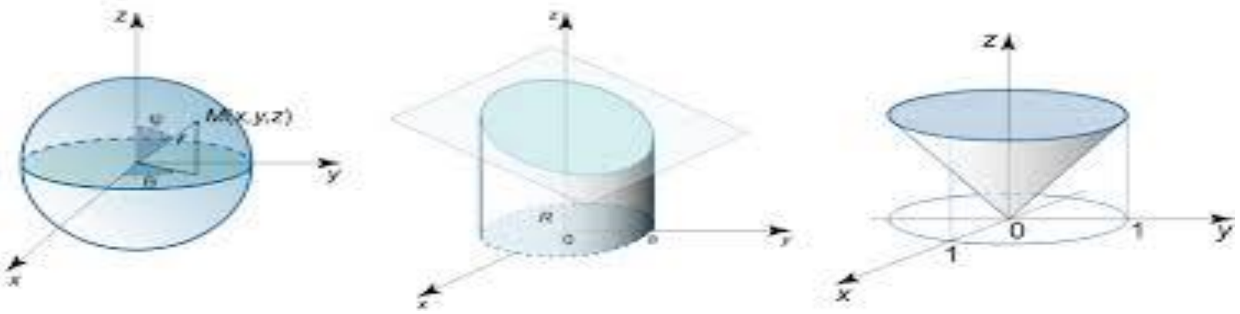
Maxima and Minima



Area under curve

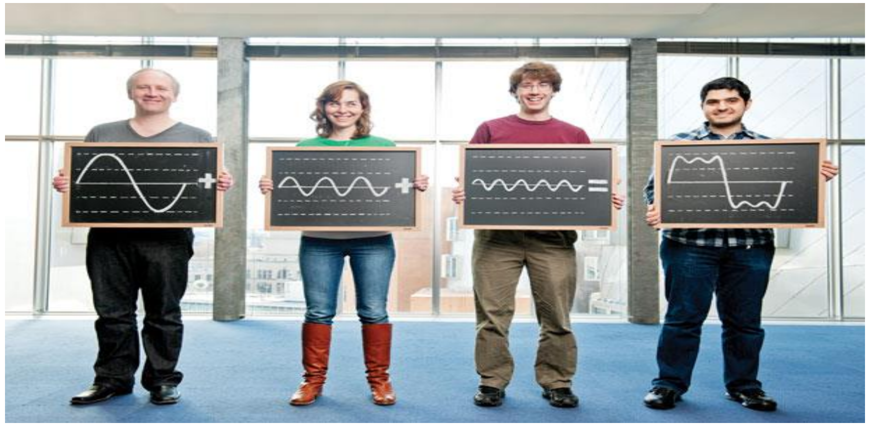


Volume

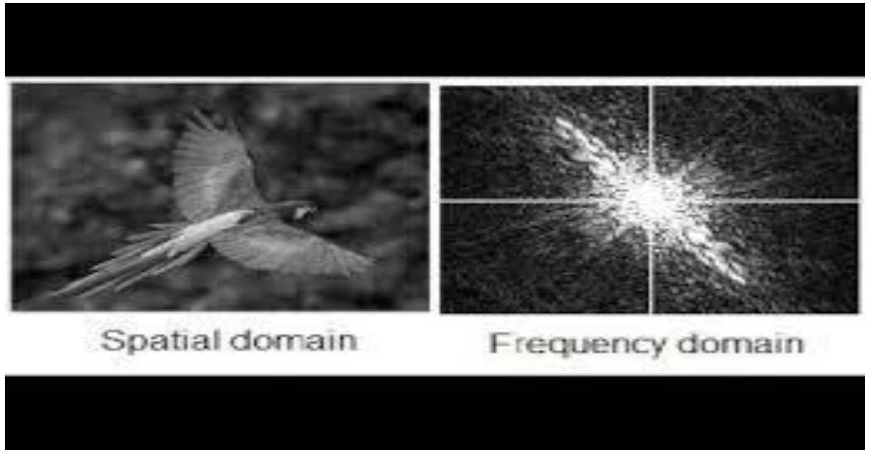


Uses of Fourier Series:

In Signal Processing



In Image Processing



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MTH165

Unit 1
Linear Algebra
L1-Review Of Matrices

Matrices - Introduction

Matrix algebra has at least two advantages:

•Reduces complicated systems of equations to simple expressions

•Adaptable to systematic method of mathematical treatment and well suited to computers

Definition:

A matrix is a set or group of numbers arranged in a square or rectangular array enclosed by two brackets

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 2 \\ -3 & 0 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Matrices - Introduction

Properties:

•A specified number of rows and a specified number of columns

•Two numbers (rows x columns) describe the dimensions or size of the matrix.

Examples:

3x3 matrix

2x4 matrix

1x2 matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & -1 & 5 \\ 3 & 3 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 3 & -3 \\ 0 & 0 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Matrices - Introduction

A matrix is denoted by a bold capital letter and the elements within the matrix are denoted by lower case letters

e.g. matrix [**A**] with elements a_{ij}

$$\mathbf{A}_{mxn} = {}_m\mathbf{A}^n \quad \begin{bmatrix} a_{11} & a_{12}\ldots & a_{ij} & a_{in} \\ a_{21} & a_{22}\ldots & a_{ij} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{ij} & a_{mn} \end{bmatrix}$$

i goes from 1 to m

j goes from 1 to n

MCQ

If $A = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 5 & 3 \end{bmatrix}$, the order of matrix A is

a) 3×2

b) 2×3

c) 1×3

d) 3×1

Matrices - Introduction

TYPES OF MATRICES

1. Column matrix or vector:

The number of rows may be any integer but the number of columns is always 1

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

2. Row matrix or vector

Any number of columns but only one row

$$\begin{bmatrix} 1 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

3. Rectangular matrix

Contains more than one element and number of rows is not equal to the number of columns

$$\begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix}$$

$$m \neq n$$

Matrices - Introduction

TYPES OF MATRICES

4. Square matrix

The number of rows is equal to the number of columns

(a square matrix $\mathbf{A}_{m \times m}$ has an order of m)

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

The principal or main diagonal of a square matrix is composed of all elements a_{ij} for which $i=j$

Matrices - Introduction

TYPES OF MATRICES

5. Diagonal matrix

A square matrix where all the elements are zero except those on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} \neq 0$ for some or all $i = j$

Matrices - Introduction

TYPES OF MATRICES

6. Unit or Identity matrix - I

A diagonal matrix with ones on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} = 1$ for some or all $i = j$

Matrices - Introduction

TYPES OF MATRICES

7. Null (zero) matrix - 0

All elements in the matrix are zero

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$a_{ij} = 0$$ For all i,j

Matrices - Introduction

TYPES OF MATRICES

8. Triangular matrix

A square matrix whose elements above or below the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

8a. Upper triangular matrix

A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ 0 & a_{ij} & a_{ij} \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 7 & 4 & 4 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i > j$

Matrices - Introduction

TYPES OF MATRICES

8b. Lower triangular matrix

A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i < j$

Matrices – Introduction

TYPES OF MATRICES

9. Scalar matrix

A diagonal matrix whose main diagonal elements are equal to the same scalar

A scalar is defined as a single number or constant

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ 0 & a_{ij} & 0 \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} = a$ for all $i = j$

MCQ

If A =$[1 \quad 2 \quad 3]$, which type of the given matrix B?

a) Unit matrix
b) Row matrix
c) Column matrix
d) Square matrix

MTH165

Unit 1
Linear Algebra

L2-Operations Of Matrices

MCQ

1. If a matrix has 6 elements, then number of possible orders of the matrix can be

(a) 2

(b) 4

(c) 3

(d) 6

2. If A = $[a_{ij}]$ is a 2 × 3 matrix, such that $a_{ij} = \frac{(-i+2j)^2}{5}$.then a_{23} is

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$

(c) $\frac{9}{5}$ (d) $\frac{16}{5}$

Matrices - Operations

EQUALITY OF MATRICES

Two matrices are said to be equal only when all corresponding elements are equal Therefore their size or dimensions are equal as well

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \mathbf{A} = \mathbf{B}$$

Matrices - Operations

Some properties of equality:

•If **A** = **B**, then **B** = **A** for all **A** and **B**

•If **A** = **B**, and **B** = **C**, then **A** = **C** for all **A**, **B** and **C**

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

If **A** = **B** then $a_{ij} = b_{ij}$

Matrices - Operations

ADDITION AND SUBTRACTION OF MATRICES

The sum or difference of two matrices, **A** and **B** of the same size yields a matrix **C** of the same size

$$c_{ij} = a_{ij} + b_{ij}$$

Matrices of different sizes cannot be added or subtracted

Matrices - Operations

Commutative Law:

A + **B** = **B** + **A**

Associative Law:

A + (**B** + **C**) = (**A** + **B**) + **C** = **A** + **B** + **C**

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix}$$

A
2x3

B
2x3

C
2x3

Matrices - Operations

A + **0** = **0** + **A** = **A**

A + (**-A**) = **0** (where –**A** is the matrix composed of $-a_{ij}$ as elements)

$$\begin{bmatrix} 6 & 4 & 2 \\ 3 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

Matrices - Operations

SCALAR MULTIPLICATION OF MATRICES

Matrices can be multiplied by a scalar (constant or single element)

Let k be a scalar quantity; then

kA = Ak

Ex. If k=4 and

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix}$$

Matrices - Operations

$$4 \times \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} \times 4 = \begin{bmatrix} 12 & -4 \\ 8 & 4 \\ 8 & -12 \\ 16 & 4 \end{bmatrix}$$

Properties:

- k (**A** + **B**) = k**A** + k**B**
- (k + g)**A** = k**A** + g**A**
- k(**AB**) = (k**A**)**B** = **A**(k)**B**
- k(g**A**) = (kg)**A**

Matrices - Operations

MULTIPLICATION OF MATRICES

The product of two matrices is another matrix

Two matrices **A** and **B** must be **conformable** for multiplication to be possible

i.e. the number of columns of **A** must equal the number of rows of **B**

Example.

$$\mathbf{A} \quad \text{x} \quad \mathbf{B} \quad = \quad \mathbf{C}$$

$$(1 \text{x} 3) \quad (3 \text{x} 1) \quad (1 \text{x} 1)$$

Matrices - Operations

B x **A** = Not possible!

(2x1) (4x2)

A x **B** = Not possible!

(6x2) (6x3)

Example

A x **B** = **C**

(2x3) (3x2) (2x2)

Matrices - Operations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$(a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) = c_{11}$$
$$(a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) = c_{12}$$
$$(a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) = c_{21}$$
$$(a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) = c_{22}$$

Successive multiplication of row *i* of **A** with column *j* of **B** – row by column multiplication

Matrices - Operations

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} (1\times 4)+(2\times 6)+(3\times 5) & (1\times 8)+(2\times 2)+(3\times 3) \\ (4\times 4)+(2\times 6)+(7\times 5) & (4\times 8)+(2\times 2)+(7\times 3) \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Remember also:

IA = **A**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix} = \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

MCQ

If $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, then A^2 is

(a) $\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 0 \\ 4 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

Matrices - Operations

Assuming that matrices **A**, **B** and **C** are conformable for the operations indicated, the following are true:

1. **AI** = **IA** = **A**
2. **A**(**BC**) = (**AB**)**C** = **ABC** - (associative law)
3. **A**(**B**+**C**) = **AB** + **AC** - (first distributive law)
4. (**A**+**B**)**C** = **AC** + **BC** - (second distributive law)

Caution!

1. **AB** not generally equal to **BA**, **BA** may not be conformable
2. If **AB** = **0**, neither **A** nor **B** necessarily = **0**
3. If **AB** = **AC**, **B** not necessarily = **C**

Matrices - Operations

AB not generally equal to **BA**, **BA** may not be conformable

$$T = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

$$TS = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 15 & 20 \end{bmatrix}$$

$$ST = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 23 & 6 \\ 10 & 0 \end{bmatrix}$$

Matrices - Operations

If **AB** = **0**, neither **A** nor **B** necessarily = **0**

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrices - Operations

TRANSPOSE OF A MATRIX

If :

$$\underset{2x3}{A} = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

Then transpose of A, denoted A^T is:

$$A^T = {}_2A^{3^T} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

$$a_{ij} = a_{ji}^T$$ For all *i* and *j*

Matrices - Operations

To transpose:

Interchange rows and columns

The dimensions of $\mathbf{A}^T$ are the reverse of the dimensions of $\mathbf{A}$

$$A = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix} \qquad 2 \text{ x } 3$$

$$A^T = {}_3A^{T\,2} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix} \qquad 3 \text{ x } 2$$

Matrices - Operations

Properties of transposed matrices:

1. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
2. $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$
3. $(k\mathbf{A})^T = k\mathbf{A}^T$
4. $(\mathbf{A}^T)^T = \mathbf{A}$

Matrices - Operations

1. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 2 \\ 3 & -5 \\ -1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 5 & -2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

Matrices - Operations

$(\mathbf{AB})^{T} = \mathbf{B}^{T}\ \mathbf{A}^{T}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 8 \end{bmatrix}$$

Matrices - Operations

SYMMETRIC MATRICES

A Square matrix is symmetric if it is equal to its transpose:

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}}$$

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

SKEW SYMMETRIC MATRICES

A Square matrix is skew symmetric if it is equal to negative of its transpose:

For

$$B = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -9 \\ 3 & 9 & 0 \end{bmatrix}$$

$$B' = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 9 \\ -3 & -9 & 0 \end{bmatrix}$$

(a) In a skew-symmetric matrix $\mathbf{A} = (a_{ij})$, all its diagonal elements are zero.

(b) The matrix which is both symmetric and skew-symmetric must be a null matrix.

(c) For any real square matrix $\mathbf{A}$, the matrix $\mathbf{A} + \mathbf{A}^T$ is always symmetric and the matrix $\mathbf{A} - \mathbf{A}^T$ is always skew-symmetric. Therefore, a real square matrix $\mathbf{A}$ can be written as the sum of a symmetric matrix and a skew-symmetric matrix. That is

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T).$$

MCQ

If a matrix A is both symmetric and skew symmetric then matrix A is

(a) a scalar matrix

(b) a diagonal matrix

(c) a zero matrix of order $n \times n$

(d) a rectangular matrix.

MCQ

The diagonal elements of a skew symmetric matrix are

(a) all zeroes

(b) are all equal to some scalar $k (\neq 0)$

(c) can be any number

(d) none of these

Conjugate matrix Let $\mathbf{A} = (a_{ij})$ be a complex matrix. Let $\bar{a}_{ij}$ denote the complex conjugate of a_{ij}. Then, the matrix $\bar{\mathbf{A}} = (\bar{a}_{ij})$ is called the *conjugate matrix* of A.

Hermitian and skew-Hermitian matrices A complex matrix $\mathbf{A}$ is called an *Hermitian matrix* if $\bar{\mathbf{A}} = \mathbf{A}^T$ or $\mathbf{A} = (\bar{\mathbf{A}})^T$ and a *skew-Hermitian* matrix if $\bar{\mathbf{A}} = -\mathbf{A}^T$ or $\mathbf{A} = -(\bar{\mathbf{A}})^T$. Sometimes, a Hermitian matrix is denoted by $\mathbf{A}^H$ or $\mathbf{A}^*$.

Remark 4

(a) If $\mathbf{A}$ is a real matrix, then an Hermitian matrix is same as a symmetric matrix and a skew-Hermitian matrix is same as a skew-symmetric matrix.

(b) In an Hermitian matrix, all the diagonal elements are real (let $a_{jj} = x_j + iy_j$; then $a_{jj} = \bar{a}_{jj}$ gives $x_j + iy_j = x_j - iy_j$ or $y_j = 0$ for all j).

(c) In a skew-Hermitian matrix, all the diagonal elements are either 0 or pure imaginary (let $a_{jj} = x_j + iy_j$; then $a_{jj} = -\bar{a}_{jj}$ gives $x_j + iy_j = -(x_j - iy_j)$ or $x_j = 0$ for all j).

(d) For any complex square matrix $\mathbf{A}$, the matrix $\mathbf{A} + \bar{\mathbf{A}}^T$ is always an Hermitian matrix and the matrix $\mathbf{A} - \bar{\mathbf{A}}^T$ is always a skew-Hermitian matrix. Therefore, a complex square matrix $\mathbf{A}$ can be written as the sum of an Hermitian matrix and a skew-Hermitian matrix, that is

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \bar{\mathbf{A}}^T) + \frac{1}{2}(\mathbf{A} - \bar{\mathbf{A}}^T).$$

Matrices - Operations

INVERSE OF A MATRIX

Consider a scalar k. The inverse is the reciprocal or division of 1 by the scalar.

Example:

$k=7$ the inverse of k or $k^{-1} = 1/k = 1/7$

Division of matrices is not defined since there may be $\mathbf{AB} = \mathbf{AC}$ while $\mathbf{B} \neq \mathbf{C}$

Instead matrix inversion is used.

The inverse of a square matrix, **A**, if it exists, is the unique matrix $\mathbf{A}^{-1}$ where:

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Matrices - Operations

Example:

$$A{=}_2A^2 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Because:

$$\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrices - Operations

Properties of the inverse:

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A^{-1})^{-1} = A$$

$$(A^{T})^{-1} = (A^{-1})^{T}$$

$$(kA)^{-1} = \frac{1}{k}A^{-1}$$

A square matrix that has an inverse is called a nonsingular matrix

A matrix that does not have an inverse is called a singular matrix

Square matrices have inverses except when the determinant is zero

When the determinant of a matrix is zero the matrix is singular

Matrices - Operations

DETERMINANT OF A MATRIX

To compute the inverse of a matrix, the determinant is required

Each square matrix **A** has a unit scalar value called the determinant of **A**, denoted by det **A** or |**A**|

If $$A = \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}$$

then $$|A| = \begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix}$$

Matrices - Operations

If **A** = [**A**] is a single element (1x1), then the determinant is defined as the value of the element

Then |**A**| =det **A** = a_{11}

If **A** is (n x n), its determinant may be defined in terms of order (n-1) or less.

Matrices - Operations

MINORS

If **A** is an n x n matrix and one row and one column are deleted, the resulting matrix is an (n-1) x (n-1) submatrix of **A**.

The determinant of such a submatrix is called a minor of **A** and is designated by m_{ij} , where *i* and *j* correspond to the deleted row and column, respectively.

m_{ij} is the minor of the element a_{ij} in **A**.

Matrices - Operations

eg.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Each element in **A** has a minor

Delete first row and column from **A** .

The determinant of the remaining 2 x 2 submatrix is the minor of a_{11}

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Matrices - Operations

Therefore the minor of a_{12} is:

$$m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

And the minor for a_{13} is:

$$m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Matrices - Operations

COFACTORS

The cofactor C_{ij} of an element a_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} m_{ij}$$

When the sum of a row number *i* and column *j* is even, $c_{ij} = m_{ij}$ and when *i*+*j* is odd, $c_{ij} = -m_{ij}$

$$c_{11}(i=1, j=1) = (-1)^{1+1} m_{11} = +m_{11}$$

$$c_{12}(i=1, j=2) = (-1)^{1+2} m_{12} = -m_{12}$$

$$c_{13}(i=1, j=3) = (-1)^{1+3} m_{13} = +m_{13}$$

Matrices - Operations

DETERMINANTS CONTINUED

The determinant of an n x n matrix **A** can now be defined as

$$|A| = \det A = a_{11}c_{11} + a_{12}c_{12} + \ldots + a_{1n}c_{1n}$$

The determinant of **A** is therefore the sum of the products of the elements of the first row of **A** and their corresponding cofactors.

(It is possible to define |**A**| in terms of any other row or column but for simplicity, the first row only is used)

Matrices - Operations

Therefore the 2 x 2 matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Has cofactors :

$$c_{11} = m_{11} = \left|a_{22}\right| = a_{22}$$

And:

$$c_{12} = -m_{12} = -\left|a_{21}\right| = -a_{21}$$

And the determinant of **A** is:

$$\left|A\right| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Matrices - Operations

Example 1:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$|A| = (3)(2) - (1)(1) = 5$$

Matrices - Operations

For a 3 x 3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The cofactors of the first row are:

$$c_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$c_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{23}a_{31})$$

$$c_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$$

Matrices - Operations

The determinant of a matrix A is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Which by substituting for the cofactors in this case is:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Matrices - Operations

Example 2:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$|A| = (1)(2-0) - (0)(0+3) + (1)(0+2) = 4$$

Matrices - Operations

ADJOINT MATRICES

A cofactor matrix **C** of a matrix **A** is the square matrix of the same order as **A** in which each element a_{ij} is replaced by its cofactor c_{ij} .

Example:

If $$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

The cofactor C of A is $$C = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$$

Matrices - Operations

The adjoint matrix of **A**, denoted by adj **A**, is the transpose of its cofactor matrix

$$adjA = C^T$$

It can be shown that:

A(adj **A**) = (adj**A**) **A** = |**A**| **I**

Example:

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$|A| = (1)(4) - (2)(-3) = 10$$

$$adjA = C^T = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

Matrices - Operations

$$A(adjA)=\begin{bmatrix}1 & 2\\ -3 & 4\end{bmatrix}\begin{bmatrix}4 & -2\\ 3 & 1\end{bmatrix}=\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}=10I$$

$$(adjA)A=\begin{bmatrix}4 & -2\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ -3 & 4\end{bmatrix}=\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}=10I$$

Matrices - Operations

USING THE ADJOINT MATRIX IN MATRIX INVERSION

Since

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

and

$$\mathbf{A}(\text{adj }\mathbf{A}) = (\text{adj}\mathbf{A})\,\mathbf{A} = |\mathbf{A}|\,\mathbf{I}$$

then

$$A^{-1} = \frac{adjA}{|A|}$$

Matrices - Operations

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}$$

To check

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^{-1}A = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Matrices - Operations

Example 2

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

The determinant of **A** is

$$|\mathbf{A}| = (3)(-1-0)-(-1)(-2-0)+(1)(4-1) = -2$$

The elements of the cofactor matrix are

$$c_{11} = +(-1), \qquad c_{12} = -(-2), \qquad c_{13} = +(3),$$
$$c_{21} = -(-1), \qquad c_{22} = +(-4), \qquad c_{23} = -(7),$$
$$c_{31} = +(-1), \qquad c_{32} = -(-2), \qquad c_{33} = +(5),$$

Matrices - Operations

The cofactor matrix is therefore

$$C = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}$$

so

$$adjA = C^T = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix}$$

and

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}$$

Matrices - Operations

The result can be checked using

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

The determinant of a matrix must not be zero for the inverse to exist as there will not be a solution

Nonsingular matrices have non-zero determinants

Singular matrices have zero determinants

MTH165

Unit 1
Linear Algebra

Rank of Matrix

Rank of Matrix

The rank of a matrix is the order of the largest non-zero square submatrix.

Rank of Matrix: A matrix A is s.t.b of rank r if

(i) it has at least one non-zero Minor of order r.

(ii) and every minor of order higher than r vanishes

REVISION MCQ

Rank of the matrix A =

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 \end{bmatrix}$$

A. 0

B. 1

C. 2

D. 3

Rank of Matrix Using Elementary Transformation

- "row-rank of a matrix" counts the max. number of linearly independent rows.
- "column-rank of a matrix" counts the max. number of linearly independent columns.
- One application: Given a large system of linear equations, count the number of essentially different equations.
  - The number of essentially different equations is just the row-rank of the augmented matrix.

Evaluating the row-rank by definition

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly dependent

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly dependent

$\Rightarrow$ Row-Rank = 2

Calculation of row-rank via RREF

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow[\text{Row reductions}]{} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row-rank = 2

Row-rank = 2

Because row reductions
do not affect the number
of linearly independent rows

Calculation of column-rank by definition

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

List all combinations of columns

$\Rightarrow$ Column-Rank = 2

Linearly independent??

$$\begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 6 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 & 1 \\ 1 & 4 \\ 2 & 8 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 4 & 6 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 4 & 8 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{Y} \\ \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 6 & 8 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \text{N} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{N} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{N} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \end{array} \quad \begin{array}{c} \text{N} \\ \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{bmatrix} \end{array} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 6 & 8 \end{bmatrix} \ \text{N}$$

Theorem

Given any matrix, its row-rank and column-rank are equal.

In view of this property, we can just say the "rank of a matrix". It means either the row-rank or column-rank.

Why row-rank = column-rank?

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Apply row reductions.
row-rank and column-rank
do not change.

$$\downarrow$$

Apply column reductions.
row-rank and column-rank
do not change.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The top-left corner is
an identity matrix.

The row-rank and column-rank of this
“normal form” is certainly
the size of this identity submatrix,
and are therefore equal.

For each of the following matrices, find a row-equivalent matrix which is in reduced row echelon form. Then determine the rank of each matrix.

(a) $A = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}.$

(b) $B = \begin{bmatrix} 2 & 6 & -2 \\ 3 & -2 & 8 \end{bmatrix}.$

(c) $C = \begin{bmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{bmatrix}.$

(d) $D = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}.$

(e) $E = \begin{bmatrix} -2 & 3 & 1 \end{bmatrix}.$

(c) $C = \begin{bmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & -1 & 2 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{bmatrix} \xrightarrow[R_3-6R_1]{R_2-4R_1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -10 \\ 0 & 5 & -10 \end{bmatrix}$$

$$\xrightarrow{R_3-R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix C has rank 2

MCQ

Find the rank of the matrix A= $\begin{bmatrix} 1 & 3 & 5 \\ 4 & 6 & 7 \\ 1 & 2 & 2 \end{bmatrix}$.

a) 3

b) 2

c) 1

d) 0

LINEAR INDEPENDENT AND DEPENDENT OF VECTORS

In the theory of vector spaces, a set of vectors is said to be linearly dependent if at least one of the vectors in the set can be defined as a linear combination of the others; if no vector in the set can be written in this way, then the vectors are said to be linearly independent

Linear Dependence and Independence :-

- A finite set of vector of a vector space is said to be **Linearly Dependent(LD)** if there exists a set of scalars k_1 , k_2 , ……, k_n **not all zero** such that,

$$k_1u_1 + k_2u_2 + \ldots\ldots + k_nu_n = \bar{o}$$

- A finite set of vector of a vector space is said to be **Linearly Independent(LI)** if there exists scalars k_1 , k_2 , ……, k_n such that,

$$k_1u_1 + k_2u_2 + \ldots\ldots + k_nu_n = \bar{o} \quad \Rightarrow k_1 = k_2 = \ldots\ldots = k_n = 0$$

- **<u>Properties For LI - LD</u>**

- Property 1: Any subset of a vector space is either L.D. or L.I.
- Property 2: A set of containing only ō vector that is {ō} is L.D.
- Property 3: A set is containing the single non zero vector is L.I.
- Property 4: A set having one of the vector as zero vector is L.D.

EXAMPLES

- **Consider the set of vectors to check LI or LD $\{(1,0,0),(0,1,0),(0,0,1)\}$ in R^3.**

  - Solution :-

  Let k_1,k_2,k_3 belongs to R such that,

  $k_1(1,0,0)+k_2(0,1,0)+k_3(0,0,1)=(0,0,0)$

  $(k_1,k_2,k_3)=(0,0,0)$

  $\Rightarrow k_1=0,k_2=0,k_3=0$

Therefore, the set $\{i,j,k\}$ is LI.

Determine whether the vectors are LI in R^3

(1-,2,1),(2,1,-1),(7,-4,1).

- Solution :-

Let k_1,k_2,k_3 belongs to R such that,

$k_1(1,-2,1)+k_2(2,1,-1)+k_3(7,-4,1)=(0,0,0)$

$k_1+2k_2+7k_3=0$

$-2k_1+1k_2-4k_3=0$

$k_1-k_2+k_3=0$

MCQ

What is the determinant of the equivalent matrix if you have the following two equations:

$x + y = 0$

$2x - 3y = 0$

a. -5
b. -1
c. 0
d. 2

$$|A| = \begin{vmatrix} 1 & 2 & 7 \\ -2 & 1 & -4 \\ 1 & -1 & 1 \end{vmatrix}$$

$$|A| = 1[(1)(1)-(-4)(-1)]$$
$$-2[(-2)(1)-(-4)(1)]$$
$$+7[(-2)(-1)-(1)(1)]$$
$$|A| = -3-4+7$$
$$|A| = 0$$

Since the determinant of the system is zero, the system of these equations has a nontrivial solution. That is at least one of k_1, k_2, k_3 is nonzero. Thus the vectors are LD.

1.) Which of the following sets of polynomials in P_2 are dependent?

i. $2-x+4x^2$, $3+6x+2x^2$, $2+10x-4x^2$.

ii. $2+x+x^2$, $x+2x^2$, $2+2x+3x^2$.

Solution:-

i.) $2\text{-}x+4x^2$, $3+6x+2x^2$, $2+10x\text{-}4x^2$

Let,

$k_1p_1+k_2p_2+k_3p_3=0$

$\Rightarrow k_1(2\text{-}x+4x^2)+k_2(3+6x+2x^2)+k_3(2+10x-4x^2)=0$

$2k_1+3k_2+2k_3=0$

$-k_1+6k_2+10k_3=0$

$4k_1+2k_2-4k_3=0$

$$\sim |A| = \begin{bmatrix} 2 & 3 & 2 \\ -1 & 6 & 10 \\ 4 & 2 & -4 \end{bmatrix}$$

$$\sim \text{A} = \begin{vmatrix} 2 & 3 & 2 \\ -1 & 6 & 10 \\ 4 & 2 & -4 \end{vmatrix}$$

$$= 2(-24-20)-3(4-40)+2(-2-24)$$

$$= -88+108-52$$

$|A|$= -32 $\neq$ 0

Therefore the system has unique solution

The given vectors are not L.D(i.e they are L.I).

MTH165

Unit 1
Linear Algebra

L4-Solution of linear system of equations

Revision

Compute the rank

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix}$$

a) **1**

b) **2**

c) **3**

d) **None of these**

Revision

Compute the rank:

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 0 \\ 1 & 3 & -2 \end{pmatrix}$$

a) **1**

b) **2**

c) **3**

d) **None of these**

What does it mean if three equations are linearly independent?

a. Two of the equations can be combined to come up with the third equation.

b. There is no way to combine any two equations to come up with the third equation.

c. The graphical representations of the equations are lines that do not intersect.

d. The graphical representations of the equations are lines that do intersect.

Solution of linear system of equations by using rank of matrix

Given the linear system $Ax = B$ and the augmented matrix $(A|B)$.

1. If $rank(A) = rank(A|B) =$ the number of rows in x, then the system has a unique solution.
2. If $rank(A) = rank(A|B) <$ the number of rows in x, then the system has ∞-many solutions.
3. If $rank(A) < rank(A|B)$, then the system is inconsistent.

Solve

$$\begin{aligned} x+2y-\ \ z &= 3 \\ 2x+2y\qquad &= 4 \\ x+3y-2z &= 4 \end{aligned}$$

Solution

Since

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 1 & 2 & 0 & 4 \\ 1 & 3 & -2 & 4 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The system is equivalent to $x = 1 - z$, $y = 1 + z$, *where* z *is free.*

MCQ

A set of linear equations is represented by the matrix equation $Ax = b$. The necessary condition for the existence of a solution for this system is

A. A must be invertible

B. b must be linearly depended on the columns of A

C. b must be linearly independent of the columns of A

D. None of these

Example Solve

$$\begin{aligned} x+2y-3z &= 1 \\ 2x+4y-6z &= 1 \\ 3+6y-9z &= 1 \end{aligned}$$

Solution

Since

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & 4 & -6 & 1 \\ 3 & 6 & -9 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

there is no solution.

Example Solve

$$\begin{aligned} x+2y+z &= 1 \\ 2x+2y &= 1 \\ x+3y+z &= 1 \end{aligned}$$

Solution

Since

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{array}\right)$$

we have $x = 1/2$, $y = 0$, $z = 1/2$.

Solve
$$\begin{aligned} x+2y-\ z &= 1 \\ 2x+2y\qquad &= 1 \\ x+3y-2z &= 1 \end{aligned}$$

a) **Unique Solution**

b) **No solution**

c) **Infinite many solution**

d) **None of these**

The system of linear equations

$(4d - 1)x + y + z = 0$

$-y + z = 0$

$(4d - 1)z = 0$

has a non-trivial solution, if d equals

A. 1/2

B. 1/4

C. 3/4

Inverse of Matrix by Gauss Jordan Method

- When a matrix A has an inverse, A is called **invertible** (or **nonsingular**); otherwise, A is called **singular.** A nonsquare matrix cannot have an inverse.

- To see this, note that if A is of dimension $m \times n$ and B is of dimension $n \times m$ (where $m \neq n$), then the products AB and BA are of different dimensions and so cannot be equal to each other.

- Not all square matrices have inverses, as you will see later in this section. When a matrix does have an inverse, however, that inverse is unique. Example 2 shows how to use systems of equations to find the inverse of a matrix.

Finding an Inverse Matrix

Let A be a square matrix of dimension $n \times n$.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain

   $$[A \ \vdots \ I].$$

2. If possible, row reduce A to I using elementary row operations on the *entire* matrix

   $$[A \ \vdots \ I].$$

   The result will be the matrix

   $$[I \ \vdots \ A^{-1}].$$

   If this is not possible, then A is not invertible.

3. Check your work by multiplying to see that

   $$AA^{-1} = I = A^{-1}A.$$

Find the inverse of the matrix A using Gauss-Jordan elimination.

$$A = \begin{bmatrix} 2 & 8 & 13 \\ 4 & 14 & 9 \\ 10 & 15 & 7 \end{bmatrix}$$

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Unit 1
Linear Algebra

L5- INVERSE OF MATRIX

Inverse of Matrix by Gauss Jordan Method

- When a matrix *A* has an inverse, *A* is called **invertible** (or **nonsingular**); otherwise, *A* is called **singular.** A nonsquare matrix cannot have an inverse.

- To see this, note that if *A* is of dimension $m \times n$ and *B* is of dimension $n \times m$ (where $m \neq n$), then the products *AB* and *BA* are of different dimensions and so cannot be equal to each other.

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The result will be the matrix

$$[I \vdots A^{-1}].$$

If this is not possible, then A is not invertible.

3. Check your work by multiplying to see that

$$AA^{-1} = I = A^{-1}A.$$

Example: find the Inverse of "A":

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

We start with the matrix A, and write it down with an Identity Matrix I next to it:

$$\begin{array}{cc} A & I \\ \left[\begin{matrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{matrix}\right. & \left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right] \end{array}$$

(This is called the "Augmented Matrix")

Now we do our best to turn "A" (the Matrix on the left) into an Identity Matrix. The goal is to make Matrix A have **1**s on the diagonal and **0**s elsewhere (an Identity Matrix) ... and the right hand side comes along for the ride, with every operation being done on it as well.

But we can only do these "**Elementary Row Operations**":

- **swap** rows
- **multiply** or divide each element in a a row by a constant
- replace a row by **adding** or subtracting a multiple of another row to it

And we must do it to the **whole row**, like this:

Start with **A** next to **I**

$$\begin{array}{c}
\qquad\qquad A \qquad\qquad\qquad\qquad I \\
\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]
\end{array}$$

Add row 2 to row 1,

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Add

then divide row 1 by 5,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Divide by 5

Then take 2 times the first row, and subtract it from the second row,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & -2 & -0.4 & 0.6 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Subtract x 2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Multiply by $-\frac{1}{2}$

Multiply second row by -1/2,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array}\right]$$

Swap

Now swap the second and third row,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 0 & -0.2 & 0.3 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array}\right]$$

Subtract

I $\quad$ A^{-1}

Last, subtract the third row from the second row,

And we are done!

And matrix **A** has been made into an Identity Matrix ...

... and at the same time an Identity Matrix got made into $\mathbf{A}^{-1}$

Find the inverse of the matrix A using Gauss-Jordan elimination.

$$A = \begin{bmatrix} 2 & 8 & 13 \\ 4 & 14 & 9 \\ 10 & 15 & 7 \end{bmatrix}$$

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Unit 1
Linear Algebra

L5-Eigen Values and Eigen Vectors

Some Applications of the Eigenvalues and Eigenvectors of a square matrix

1. **Communication systems**:

Eigenvalues were used by Claude Shannon to determine the theoretical limit to how much information can be transmitted through a communication medium like your telephone line or through the air. This is done by calculating the eigenvectors and eigenvalues of the communication channel (expressed a matrix), and then waterfilling on the eigenvalues. The eigenvalues are then, in essence, the gains of the fundamental modes of the channel, which themselves are captured by the eigenvectors.

2. **Designing bridges**:

The natural frequency of the bridge is the eigenvalue of smallest magnitude of a system that models the bridge. The engineers exploit this knowledge to ensure the stability of their constructions.

[Watch the video on the collapse of the Tacoma Narrow Bridge which was built in 1940]

3. **Designing car stereo system**:

Eigenvalue analysis is also used in the design of the car stereo systems, where it helps to reproduce the vibration of the car due to the music.

4. **Electrical Engineering**:

The application of eigenvalues and eigenvectors is useful for decoupling three-phase systems through symmetrical component transformation.

5. **Mechanical Engineering**:

Eigenvalues and eigenvectors allow us to "reduce" a linear operation to separate, simpler, problems. For example, if a stress is applied to a "plastic" solid, the deformation can be dissected into "principle directions"- those directions in which the deformation is greatest. Vectors in the principle directions are the eigenvectors and the percentage deformation in each principle direction is the corresponding eigenvalue.

Oil companies frequently use eigenvalue analysis to explore land for oil. Oil, dirt, and other substances all give rise to linear systems which have different eigenvalues, so eigenvalue analysis can give a good indication of where oil reserves are located. Oil companies place probes around a site to pick up the waves that result from a huge truck used to vibrate the ground. The waves are changed as they pass through the different substances in the ground. The analysis of these waves directs the oil companies to possible drilling sites.

Very (very, very) roughly then, the eigenvalues of a linear mapping is a measure of the distortion induced by the transformation and the eigenvectors tell you about how the distortion is oriented. It is precisely this rough picture which makes **PCA** (Principal Component Analysis = A statistical procedure) very useful.

Geometric interpretation of Eigenvalues and Eigenvectors

An $n \times n$ matrix **A** multiplied by $n \times 1$ vector **x** results in another $n \times 1$ vector **y=Ax**. Thus **A** can be considered as a transformation matrix.

In general, a matrix acts on a vector by changing both its magnitude and its direction. However, a matrix may act on certain vectors by changing only their magnitude, and leaving their direction unchanged (or possibly reversing it). These vectors are the **eigenvectors** of the matrix.

A matrix acts on an eigenvector by multiplying its magnitude by a factor, which is positive if its direction is unchanged and negative if its direction is reversed. This factor is the **eigenvalue** associated with that eigenvector.

Let x be an eigenvector of the matrix **A.** Then there must exist an eigenvalue λ such that $Ax = \lambda x$ or, equivalently,

$$Ax - \lambda x = 0 \quad \text{or}$$

$$(A - \lambda I)x = 0$$

If we define a new matrix $B = A - \lambda I$, then

$$Bx = 0$$

If **B** has an inverse then $x = B^{-1}0 = 0$. But an eigenvector cannot be zero.

Thus, it follows that x will be an eigenvector of **A** if and only if **B** does not have an inverse, or equivalently $\det(B)=0$, or

$$\det(A - \lambda I) = 0$$

This is called the **characteristic equation** of **A**. Its roots determine the eigenvalues of **A**.

Eigen Vector-

In linear algebra , an eigenvector or characteristic vector of a square matrix is a vector that does not changes its direction under the associated linear transformation.

In other words – If V is a vector that is not zero, than it is an eigenvector of a square matrix A if Av is a scalar multiple of v. This condition should be written as the equation:

$$AV = \lambda v$$

Eigen Value-

In above equation λ is a scalar known as the **eigenvalue** or **characteristic value** associated with eigenvector **v.**

We can find the eigenvalues by determining the roots of the characteristic equation-

$$|A - \lambda I| = 0$$

Example 1: Find the eigenvalues of $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = (\lambda - 2)(\lambda + 5) + 12$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$$

two eigenvalues: $-1, -2$

Note: The roots of the characteristic equation can be repeated. That is, $\lambda_1 = \lambda_2 = \ldots = \lambda_k$. If that happens, the eigenvalue is said to be of multiplicity k.

Example 2: Find the eigenvalues of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

$\lambda = 2$ is an eigenvector of multiplicity 3.

Eigenvectors

To each distinct eigenvalue of a matrix **A** there will correspond at least one eigenvector which can be found by solving the appropriate set of homogenous equations. If $\boldsymbol{\lambda}_i$ is an eigenvalue then the corresponding eigenvector $\mathbf{x}_i$ is the solution of $(\mathbf{A} - \boldsymbol{\lambda}_i\mathbf{I})\mathbf{x}_i = \mathbf{0}$

Example 1 (cont.):

$$\lambda = -1 : (-1)I - A = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

$$x_1 - 4x_2 = 0 \Rightarrow x_1 = 4t,\ x_2 = t$$

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix},\ t \neq 0$$

$$\lambda = -2 : (-2)I - A = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \end{bmatrix},\ s \neq 0$$

Ex.1 Find the eigenvalues and eigenvectors of matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Taking the determinant to find characteristic polynomial A-

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 3 - 4\lambda + \lambda^2 = 0$$

It has roots at $\lambda = 1$ and $\lambda = 3$, which are the two eigenvalues of A.

Eigenvectors v of this transformation satisfy the equation,

$$Av = \lambda v$$

Rearrange this equation to obtain-

$$(A - \lambda I)v = 0$$

For $\lambda = 1$, Equation becomes,

$$(A - I)v = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has the solution, $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For λ = 3, Equation becomes,

$$(A-3I)u=0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has the solution- $u=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus, the vectors $v_{\lambda=1}$ and $v_{\lambda=3}$ are eigenvectors of A associated with the eigenvalues $\lambda = 1$ and $\lambda = 3$, respectively.

Ex.2 Find the eigenvalue and eigenvector of matrix A.

$$A = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

the matrix has the characteristics equation-

$$|\lambda I - A| = \begin{vmatrix} \lambda+4 & -1 & 0 \\ 0 & \lambda+3 & -1 \\ 0 & 0 & \lambda+2 \end{vmatrix}$$

$$= (\lambda+4)(\lambda+3)(\lambda+2) = 0$$

therefore the eigen values of A are-

$$\lambda_1 = -2, \lambda_2 = -3, \lambda_3 = -4$$

For λ = -2, Equation becomes,

$$(\lambda_1 I - A)v_1 = 0$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which has the solution-

$$v = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Similarly for λ = -3 and λ = -4 the corresponding eigenvectors u and x are-

$$u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

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Unit 1
Linear Algebra

L7- Properties of Eigen Values and Eigen Vectors

1. The eigenvalues of

$$\begin{bmatrix} 5 & 6 & 17 \\ 0 & -19 & 23 \\ 0 & 0 & 37 \end{bmatrix}$$

are

(A) $-19, 5, 37$

(B) $19, -5, -37$

(C) $2, -3, 7$

(D) $3, -5, 37$

If $\begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 8 & -4 & 2 \\ 4 & 0 & 2 \\ 0 & -2 & -4 \end{bmatrix}$, the eigenvalue corresponding to the eigenvector is

(A) 1
(B) 4
(C) -4.5
(D) 6

If $[A]$ is a $n \times n$ matrix and λ is one of the eigenvalues and $[X]$ is a $n \times 1$ corresponding eigenvector, then

$$[A][X] = \lambda[X]$$

$$\begin{bmatrix} 8 & -4 & 2 \\ 4 & 0 & 2 \\ 0 & -2 & -4 \end{bmatrix}\begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} = \lambda\begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -18 \\ -16 \\ 4 \end{bmatrix} = \lambda\begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$$

$$4\begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} = \lambda\begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$$

$$\lambda = 4$$

Properties

Definition: The trace of a matrix A, designated by tr(A), is the sum of the elements on the main diagonal.

Property 1: The sum of the eigenvalues of a matrix equals the trace of the matrix.

Property 2: A matrix is singular if and only if it has a zero eigenvalue.

Property 3: The eigenvalues of an upper (or lower) triangular matrix are the elements on the main diagonal.

Properties

Property 5: If λ is an eigenvalue of **A** then $k\lambda$ is an eigenvalue of **kA** where **k** is any arbitrary scalar.

Property 6: If λ is an eigenvalue of **A** then λ^k is an eigenvalue of A^k for any positive integer **k**.

Property 7: If λ is an eigenvalue of **A** then λ is an eigenvalue of A^T.

Property 8 : The product of the eigenvalues (counting multiplicity) of a matrix equals the determinant of the matrix.

■ REVIEW OF THE KEY IDEAS ■

1. $A\boldsymbol{x} = \lambda\boldsymbol{x}$ says that eigenvectors $\boldsymbol{x}$ keep the same direction when multiplied by A.

2. $A\boldsymbol{x} = \lambda\boldsymbol{x}$ also says that $\det(A - \lambda I) = 0$. This determines n eigenvalues.

3. The eigenvalues of A^2 and A^{-1} are λ^2 and λ^{-1}, with the same eigenvectors.

4. The sum of the λ's equals the sum down the main diagonal of A (*the trace*). The product of the λ's equals the determinant.

5. Singular matrices have $\lambda = 0$. Triangular matrices have λ's on their diagonal.

Find the eigenvalues and eigenvectors of A and A^2 and A^{-1} and $A + 4I$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Check the trace $\lambda_1 + \lambda_2$ and the determinant $\lambda_1 \lambda_2$ for A and also A^2.

Solution The eigenvalues of A come from $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0.$$

This factors into $(\lambda-1)(\lambda-3) = 0$ so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. For the trace, the sum $2+2$ agrees with $1+3$. The determinant 3 agrees with the product $\lambda_1\lambda_2 = 3$. The eigenvectors come separately by solving $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ which is $A\boldsymbol{x} = \lambda\boldsymbol{x}$:

$$\boldsymbol{\lambda = 1:} \quad (A - I)\boldsymbol{x} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{\lambda = 3:} \quad (A - 3I)\boldsymbol{x} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A^2 and A^{-1} and $A + 4I$ keep the *same eigenvectors as* A. Their eigenvalues are λ^2 and λ^{-1} and $\lambda + 4$:

$$A^2 \text{ has eigenvalues } 1^2 = 1 \text{ and } 3^2 = 9 \qquad A^{-1} \text{ has } \frac{1}{1} \text{ and } \frac{1}{3} \qquad A + 4I \text{ has } \begin{matrix} 1 + 4 = 5 \\ 3 + 4 = 7 \end{matrix}$$

The trace of A^2 is $5 + 5$ which agrees with $1 + 9$. The determinant is $25 - 16 = 9$.

Find the eigenvalues and eigenvectors of this 3 by 3 matrix A:

Symmetric matrix
Singular matrix
Trace $1 + 2 + 1 = 4$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the _____ eigenvectors as A. Its eigenvalues are _____ by 1.

Compute the eigenvalues and eigenvectors of A and A^{-1}. Check the trace !

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the _____ eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues _____.

Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

MTH165

Unit 1
Linear Algebra

L8- Cayley Hamilton Theorem

Determinant of square matrix is equal to

A) Sum of all elements
B) Product of diagonal elements
C) Product of its eigen values
D) Sum of its eigen values .

If 1,2,3 are eigen values of matrix A then eigen values of matrix A^3 are

A)1,8,27
B) 1,4,9,
C) 2,3,4,
D) 4,5,6

If λ is eigen value of matrix A then eigen values of matrix A^{-1} is

A) λ
B) $-\lambda$
C) $\frac{1}{\lambda}$
D)1.

The Cayley Hamilton Theorem

A square matrix satisfies its own characteristic equation.

- If the characteristic equation is

$$(-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0$$

then

$$(-1)^n \mathbf{A}^n + c_{n-1}\mathbf{A}^{n-1} + \cdots + c_1\mathbf{A} + c_0\mathbf{I} = \mathbf{0} \qquad (1)$$

- Suppose

$$\mathbf{A} = \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}$$

then $\lambda^2 - \lambda - 2 = 0$.

$$\mathbf{A}^2 - \mathbf{A} - 2\mathbf{I} = 0 \quad \text{or} \quad \mathbf{A}^2 = \mathbf{A} + 2\mathbf{I} \tag{2}$$

and also

$$\mathbf{A}^3 = \mathbf{A}^2 + 2\mathbf{A} = 2\mathbf{I} + 3\mathbf{A}$$

$$\mathbf{A}^4 = \mathbf{A}^3 + 2\mathbf{A}^2 = 6\mathbf{I} + 5\mathbf{A}$$

$$\mathbf{A}^5 = 10\mathbf{I} + 11\mathbf{A}$$

$$\mathbf{A}^6 = 22\mathbf{I} + 21\mathbf{A} \tag{3}$$

- From the above discussions, we can write

$$\mathbf{A}^m = c_0\mathbf{I} + c_1\mathbf{A} \quad \text{and} \quad \lambda^m = c_0 + c_1\lambda \tag{5}$$

Using $\lambda_1 = -1$, $\lambda_2 = -2$, we have

$$(-1)^m = c_0 + c_1(-1)$$

$$2^m = c_0 + c_1(2)$$

$$c_0 = 1/3[2^m + 2(-1)^m], c_1 = 1/3[2^m - (-1)^m]$$

then

$$\mathbf{A}^m = \begin{pmatrix} \frac{1}{3}[-2^m + 4(-1)^m] & \frac{4}{3}[2^m - (-1)^m] \\ -\frac{1}{3}[2^m - (-1)^m] & \frac{1}{3}[2^{m+2} - (-1)^m] \end{pmatrix} \tag{6}$$

- Similar to the previous discussions, we have

$$\mathbf{A}^m = c_0\mathbf{I} + c_1\mathbf{A} + c_2\mathbf{A}^2 + \ldots + c_{n-1}\mathbf{A}^{n-1}$$

where c_k, $k = 0, 1, \ldots, n-1$, depend on m.

Use the Cayley-Hamilton theorem to find $\mathbf{M}^6$ if $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

Characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$

$\mathbf{M}^2 - 5\mathbf{M} + 6\mathbf{I} = 0$

$$\Rightarrow \quad \mathbf{M}^2 = 5\mathbf{M} - 6\mathbf{I}$$

$$\Rightarrow \quad \mathbf{M}^4 = (5\mathbf{M} - 6\mathbf{I})^2$$

$$= 25\mathbf{M}^2 - 60\mathbf{M} + 36\mathbf{I}$$

$$= 25(5\mathbf{M} - 6\mathbf{I}) - 60\mathbf{M} + 36\mathbf{I}$$

$$= 65\mathbf{M} - 114\mathbf{I}$$

$$\mathrm{M}^6 = \mathrm{M}^4 \times \mathrm{M}^2$$

$$= (65\mathbf{M} - 114\mathbf{I})(5\mathbf{M} - 6\mathbf{I})$$

$$= 325\mathbf{M}^2 - 960\mathbf{M} + 684\mathbf{I}$$

$$= 325(5\mathbf{M} - 6\mathbf{I}) - 960\mathbf{M} + 684\mathbf{I}$$

$$= 665\mathbf{M} - 1266\mathbf{I}$$

$$= 665\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} - 1266\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -601 & 665 \\ -1330 & 1394 \end{bmatrix}$$