# Engineering Physics

Second Edition

# About the Authors

**Hitendra K Malik** is currently Professor of Physics at the Indian Institute of Technology Delhi, from where he received his PhD degree in the field of Plasma Physics, in 1995 at the age of 24. He has been a merit scholarship holder throughout his academic career. He is the recipient of Career Award from AICTE, Government of India, for his teaching and research, Outstanding Scientist Award from VIF, India, for his contributions to Science, and 2017 Albert Nelson Marquis Lifetime Achievement Award from USA. In addition, he received the prestigious Erasmus Mundus Visiting Fellowship from European Union (Germany and France), JSPS Fellowship (two times) from Japan, FRD Fellowship from South Africa and DAAD Fellowship from Germany. Owing to his worldwide recognition, his name has been included in 'Marquis Who's Who' in 2011, published from USA. Based on the survey conducted by ResearchGate (RG), his scientific score has been found within top 5% of the scientists and researchers all over the world.

Professor Malik is highly cited in India and abroad for his research work and books with h-index of 24 and i10-index of 70. Governments of India, Germany and France, through DST, CSIR, DRDO, AICTE, DAAD, CEFIPRA, etc., have provided him funding to accomplish 12 sponsored research projects. He is on the editorial board of 5 reputed research journals (including Springer). In recognition of his outstanding research and teaching contributions, he has been asked to deliver more than 50 keynote and invited talks in India, Japan, South Korea, USA, France, Germany, South Africa, and Turkey. Also, he has been chief guest in various universities, mentor of faculty colleagues of engineering institutions, and member of organizing and advisory committees of national and international conferences held in India and abroad.

He has guided 80 PhD, postgraduate and undergraduate theses, including 22 PhD theses in the area of laser/microwave plasma interactions, particle acceleration, solitons, Terahertz radiation, Hall thrusters, plasma material interaction, and nanotechnology. He has published more than 330 scientific papers in high impact factor journals and conferences, including 19 independent articles. He has been reviewer for 72 Journals of international repute, several sponsored research projects (Indian and Foreign agencies), and 18 PhD theses. He is an expert member of academic and administrative bodies of 14 different universities and institutions from 8 states of India including UGC.

Apart from this book, he has also authored another textbook on *Laser-Matter Interaction,* CRC Press, 3 Chapters in the Books *Wave Propagation*, InTechOpen Science, Croatia (featured as highly downloaded chapter), *Society, Sustainability and Environment*, Shivalik Prakashan, New Delhi, and *Plasma Science and Nanotechnology*, Apple Academic Press, exclusive worldwide distribution by CRC Press, a Taylor & Francis Group.

**Ajay Kumar Singh** has almost two decades of teaching experience in several engineering institutions across North India. Currently, he is Professor of Physics at the Department of Applied Sciences, Maharaja Surajmal Institute of Technology (MSIT), Janakpuri, New Delhi. He has also served as the Head of Department at MSIT. Earlier, he was associated as Professor (2003–2012) at the Department of Applied Science and Humanities, Dronacharya College of Engineering, Haryana.

Dr. Singh completed his PhD from Aligarh Muslim University in the year 1999. During his PhD, his work specifically focused on Uranium concentration in rock samples, soil samples and fly ash samples. He also investigated radon levels in low and high background areas. He has published more than 20 research papers and several articles in national and international journals and conferences. He has edited and co-authored several books on Environment, Water Resources, Nuclear Physics, and Engineering Physics. His book on Engineering Physics Practical and Tutorials has been highly appreciated by students. He is a life member of Plasma Science Society of India (PSSI).

Dr. Singh was the 'B.Tech First Year Syllabus Revision Committee' coordinator representing all the affiliated engineering colleges of Guru Gobind Singh Indraprastha University (GGSIPU). He is academic coordinator of PhD scholars enrolled at MSIT under University School of Information, Communication and Technology, GGSIPU, Dwarka, New Delhi. He is also supervisor of PhD students under USICT, which is a premier constituent institute of GGSIPU. Dr. Singh has also been teaching a special course on Nanotechnology for the USICT PhD students. He is also the teacher representative in the governing board of Maharaja Surajmal Institute of Technology.

# Engineering Physics

## Second Edition

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*Dedicated to*

*OMENDRA Bhaiya and all those moments that remain with me as a source of inspiration and help me to move ahead with great success, satisfaction and optimistic approach*

# Brief Contents

# Contents

# Foreword

It gives me immense pleasure to see the present textbook on "Engineering Physics" which covers almost the entire syllabus taught at undergraduate level at different engineering colleges and institutions throughout India. I complement the authors and appreciate their efforts in bringing out this book written in a very simple language. The text is comprehensive and the explanation of topics is commendable. I understand that this book carries all the elements required for a good presentation.

I have been a student of IIT Kharagpur and later on taught at IIT Delhi. Being a part of the IIT system, I recognise that the rigorous and enriching teaching experience at IITs originating from the interaction with the best engineering students and their strong feedback results in continuous evolution and refinement of the teachers. This spirit is reflected in the comprehensive and in-depth handling of important topics in a very simple manner in this book. I am happy to note that this textbook has been penned down by IITian and hope that it would serve to be a good textbook on the subject. Since this book also covers advanced topics, it will be an important learning resource for the teachers, and those students who wish to develop research skills and pursue higher studies. I hope that the book is well received in the academic world.

Prem Vrat

Professor Prem Vrat<br>
Vice-Chancellor, U.P. Technical University, Lucknow<br>
Founder Director, IIT Roorkee

# Preface to the Second Edition

The first edition of the textbook was appreciated by the teachers and students of many universities, engineering colleges and institutes, including IIT's throughout India. Words of appreciation were also received from faculty colleagues from Japan, China, Taiwan, Russia, Canada, South Korea, Pakistan, Bangladesh, Turkey, Iran, South Africa, Germany, France, United Kingdom, and United States of America. Students preparing for GATE/CSIR competitive examinations also suggested for more examples in the book and inclusion of topics of postgraduate level. The students very enthusiastically informed us about the utility of the book for the preparation of interviews for admission in PhD programmes at IITs and other universities (including foreign universities) or to get government jobs in India.

In view of all the above points, we have come up with the second edition of the book, where we have used simple language for explaining each and every topic. We have included more physical insight, wherever required. Some chapters are thoroughly revised in terms of new topics and solved problems. We have also updated advanced topics keeping in mind the research going on in these fields. The solutions to the Objective-Type Questions are also provided at the end of the book.

In particular, **Chapter 4** includes details of the topic Population Inversion which covers various schemes for the same, i.e., two-level, three-level and four-level systems. In **Chapter 5**, a topic on Optical Fibres as a Dielectric Waveguide is included. After **Chapter 7** on Waves and Oscillations, a new **Chapter 8** on Simple Harmonic Motion and Sound Waves has been included that discusses standing waves, supersonic and shock waves, in addition to sound waves, Doppler effect and Lissajous figures. **Chapter 9** on Sound Waves and Acoustics of Buildings has been thoroughly revised. In this chapter, Recording and Reproduction of Sound has been withdrawn and other topics are revisited. New topics on ultrasonics have been included which talk about production of ultrasonic waves and their absorption, dispersion, detection and applications. In **Chapter 10** on Dielectrics, a topic Energy Stored in an Electrostatic Field is withdrawn as its concept is discussed in **Chapter 11** on Electromagnetism. Moreover, details of Clausius-Mosotti equation are revised with the inclusion of physical insight of this equation. The chapter on Electromagnetism has been thoroughly revised. For example, Section 11.21 has been rewritten in order to make the readers understand which form of the Maxwell's equations is appropriate for free space, dielectric medium and conducting medium and how are these equations modified in these media. Bound charges and bound currents are also discussed. The solution to wave equation in conducting medium is included as Section 11.28.1, where dispersion relation, skin depth and phase relationship of the electric and magnetic field vectors are discussed. New solved problems, objective-type questions and other practice problems are also included in order to provide an indepth knowledge on the electromagnetic fields and their propagation in different media.

In **Chapter 12** on Theory of Relativity, physical insight to two interesting topics, viz. Length Contraction and Time Dilation is provided. Several new solved problems on various topics are also provided for the readers. **Chapter 13** on Applied Nuclear Physics has been thoroughly revised and new topics are included on basic properties of nucleus, nuclear forces, binding energy of nucleus, nuclear stability and various nuclear models, in addition to more equations and problems, both solved and unsolved. Introduction part of **Chapter 16** on Quantum Mechanics has been revised. The topic on Thermionic Emission (Section 17.7) has been shortened but significance of Richardson's equation is included. The earlier **Chapter 21** on Photoconductivity and Photovoltaics has been withdrawn but its important topics, viz. photoconductivity, simple model of photoconductor and effect of traps, are included in **Chapter 18** on Bond Theory of Solids and Photoconductivity.

The much important **Chapter 22** on Nanophysics has been rewritten in view of recent advances in the field. Now, it is renamed as Nanoscience and Nanotechnology. Certain new topics are included to clarify how nanomaterials are different from bulk materials and to know the differences between nanoscience and nanotechnology. The chapter very systematically discusses the nanoscales in 1D, 2D, 3D and OD. Particularly, nanowires, carbon nanotubes, inorganic nanotubes, biopolymers, nanoparticles, buckyballs/fullerenes and quantum dots are discussed in detail along with the methods of their synthesis, properties and their applications. Finally, the applications, limitations and disadvantages of nanotechnology are also discussed.

The exhaustive OLC supplements of the book can be accessed at http://www.mhhe.com/malik/ep and contain the following:

**For Instructors**

- Solution Manual
- Chapter-wise Power Point slides with diagrams and notes for effective lecture presentations

**For Students**

- A sample chapter
- A Solved Question Paper
- An e-guide to aid last minute revision need

We believe the readers shall find the second edition of the book more beneficial in terms of syllabus covered, quality of topics, large number of solved problems aimed at providing physical insight to various topics, and teaching various methods of solving difficult problems. The systematic approach adopted in the present book shall certainly help the teachers and students providing for crystal clear understanding of the topics and carrying out research in the related fields. This edition will be vital in enhancing the self confidence of our UG and PG students which will help them in advancing their careers.

Finally, we look forward to receive feedback from the teachers and students on the recent edition of the book.

**H K Malik**
**Ajay K Singh**

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Piracy-related issues may also be reported.

# Preface to the First Edition

Physics is a mandatory subject for all engineering students, where almost all the important elements of the subject are covered. Finally, these evolve as different branches of the engineering course. The book entitled Engineering Physics has been written keeping in mind the need of undergraduate students from various engineering and science colleges of all Indian universities. It caters to the complete syllabus for both–Physics-I and Physics-II papers in the first year Engineering Physics course.

The aim of writing this book has been to present the material in a concise and very simple way so that even weak students can grasp the fundamentals. In view of this, every chapter starts with a simple introduction and then related topics are covered with a detailed description along with the help of figures. Particularly the solved problems (compiled from University Question Papers) are at the end of each chapter. These problems are not merely numerical; many of them focus on reasoning and require thoughtful analysis. Finally, the chapters carry unsolved questions based on which the students would be able to test their knowledge as to what they have acquired after going through various chapters. *A chapter-end summary and list of important formulae will be helpful to students for a quick review during examinations. The rich pedagogy consists of solved examples (450), objective-type questions (230), short-answer questions (224) and practice problems (617).* The manuscript has been formulated in such a way that students shall grasp the subject easily and save their time as well. Since the complete syllabus is covered in a single book, it would be highly convenient to both.

The manuscript contains 22 chapters which have been prepared as per the syllabus taught in various colleges and institutions. In particular, the manuscript discusses optics, lasers, holography, fibre optics, waves, acoustics of buildings, electromagnetism, theory of relativity, nuclear physics, solid state physics, quantum physics, magnetic properties of solids, superconductivity, photoconductivity and photovoltaic, X-rays and nanophysics in a systematic manner. We have discussed advanced topics such as laser cooling, Bose-Einstein condensation, scanning electron microscope (SEM), scanning tunnelling microscope (STM), controlled fusion including plasma, Lawson criterion, inertial confinement fusion (ICF), plasma based accelerators, namely, plasma wake field accelerator, plasma beat wave accelerator, laser wake field accelerator and self-modulated laser wake field accelerator, and nanophysics with special emphasis on properties of nanoparticles, carbon nanotubes, synthesis of nanoparticles and applications of nanotechnology. These will be of interest to the teachers who are involved in teaching postgraduate courses at the universities and the students who opt for higher studies and research as their career. Moreover, a series of review questions and problems at the end of each chapter together with the solved questions would serve as a question bank for the students preparing for various competitive examinations. They will get an opportunity to learn the subject and test their knowledge on the same platform.

The structuring of the book provides in-depth coverage of all topics. **Chapter 1** discusses Interference. **Chapter 2** is on Diffraction. **Chapter 3** is devoted to Polarization. Coherence and Lasers are described in

**Chapter 4**. **Chapter 5** discusses Fibre Optics and its Applications, while Electron Optics is dealt with in **Chapter 6**. **Chapter 7** describes Waves and Oscillations. **Chapter 8** is on Sound Waves and Acoustics. **Chapter 9** is on Dielectrics. Electromagnetic Wave Propagation is described in **Chapter 10**. **Chapter 11** discusses the Theory of Relativity.

**Chapter 12** is devoted to Nuclear Physics. Crystal Structure is described in **Chapter 13**. **Chapter 14** deals with the Development of Quantum Physics, while **Chapter 15** is on Quantum Mechanics. **Chapter 16** discusses Free Electron Theory. Band Theory of Solids is explained in **Chapter 17**. **Chapter 18** describes the Magnetic Properties of Solids. **Chapter 19** is on Superconductivity. **Chapter 20** explains X-rays in detail while **Chapter 21** is on Photoconductivity and Photovoltaics. Finally, **Chapter 22** discusses Nanophysics in great detail. The manuscript has been organised such that it provides a link between different topics of a chapter. In order to make it simpler, all the necessary mathematical steps have been given and the physical feature of the mathematical expressions is discussed as and when required.

The exhaustive OLC supplements of the book can be accessed at http://www.mhhe.com/malik/ep and contain the following:

**For Instructors**
- Solution Manual
- Chapter-wise Power Point slides with diagrams and notes for effective lecture presentations

**For Students**
- A sample chapter
- Link to reference material
- Solved Model Question Paper
- Answers to objective type questions given in the book.

We would like to thank the entire team of Tata McGrawHill Education *specifically Vibha Mahajan, Shalini Jha, Tina Jajoriya, Dipika Dey, Sohini Mukherji, Priyanka Negi and Baldev Raj* for bringing out this book in a very short time span. The reviewers of the book also deserve a special mention for taking out time to review the book. Their names are given below.

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**H K Malik**

**Ajay K Singh**

# Electromagnetism

# 11

**LEARNING OBJECTIVES**

*After reading this chapter you will be able to*

**LO 1** Understand charge density, del operator, gradient, divergence and curl

**LO 2** Explain fundamental theorem of calculus and for gradient

**LO 3** Know about Gauss's theorem and its functional correlations

**LO 4** Discuss Stokes theorem and its use in calculating electric field and electric potential

**LO 5** Know how to derive Poisson's and Laplace's equations

**LO 6** Know about capacitor and their important configurations

**LO 7** Understand the concepts of magnetic flux density, magnetic field strength and Ampere's circuital law

**LO 8** Explain electrostatic boundary conditions

**LO 9** Discuss scalar and vector potential, continuity equation, maxwell's equation in differential and integral form

**LO 10** Evaluate significance of Maxwell's equation, Maxwell's displacement current and correction in Ampere's Law

**LO 11** Learn about electromagnetic wave propagation, Transverse nature EMW, Maxwell's equation in Isotropic dielectric medium/conducting medium, electromagnetic energy density

**LO 12** Elaborate on Poynting vector and Poynting theorem, waveguide, coaxial cables

## Introduction

Ordinary matter is made up of atoms which have positively charged nuclei and negatively charged electrons surrounding them. Charge is quantized in terms of the electronic charge –e. One electronic charge e is equal to $1.602\times10^{-19}$ Coulombs. One Coulomb of charge is the charge which would flow through a 220 W light bulb (220 Vac) in one second. Do you know two charges of one Coulomb each separated by one meter would repel each other with a force of about a million tons! The separation of charges produces electric field, whereas the motion of charges generates current and hence the magnetic field. When these fields are time varying they are coupled with each other through the Maxwell's equations.

With the help of the Maxwell's equations, we can derive wave equation, based on which the propagation of electromagnetic waves can be investigated in different media.

## 11.1 CHARGE DENSITY LO1

If a charge is distributed continuously in a medium, it can be expressed in terms of a physical quantity known as charge density. There are three types of charge densities, namely linear charge density, surface charge density and volume charge density.

### 11.1.1 Linear Charge Density ($\lambda$)

If a charge is distributed continuously on a linear conductor, the charge on its unit length is called the *linear charge density*. It is generally represented by $\lambda$ and is measured in the units of C/m. If $l$ be the total length of a conductor and $\lambda$ the linear charge density, then the total charge on the conductor will be

$$q = \int_0^l \lambda dl$$

For a uniform charge distribution, $\lambda$ is constant, and is given as $\lambda = q/l$. Here $q$ is the total charge, given by $q = \lambda l$.

### 11.1.2 Surface Charge Density ($\sigma$)

If the charge is distributed over a surface, the charge on the unit area of the conductor is called the *surface charge density* ($\sigma$) and its unit is C/m$^2$.

If the surface charge density at a point of the conductor is $\sigma$, then the charge contained in a small element of area $d$S will be $\sigma\, d$S. Therefore, the total charge on the surface of the conductor,

$$q = \int_s \sigma dS$$

If the charge distribution is uniform, then the value of $\sigma$ will be constant, and is given by

$$\sigma = q/S$$

where S is total area of the surface and $q$ is the total charge, given by $q = \sigma S$.

### 11.1.3 Volume Charge Density ($\rho$)

If the charge is distributed in the volume of a conductor, the charge contained in a unit volume of the conductor is called the *volume charge density* ($\rho$) and its unit is C/m$^3$.

If the volume charge density at a point of the volume of a conductor is $\rho$, then the charge contained in a small element of the volume $d$V will be $\rho\, d$V. Therefore, the total charge contained in the conductor,

$$q = \int_v \rho dV$$

For the uniform distribution of the charge, $\rho$ will be constant, and is given by

$$\rho = q/V$$

or

$$q = \rho V$$

where $q$ is the total charge and $V$ is the total volume, which is equal to $\frac{4}{3}\pi r^3$ for a spherical shape of the conductor of radius $r$.

## 11.2 DEL OPERATOR LO1

The del operator is the differential operator, which is represented by $\vec{\nabla}$ and is given by $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$.

It is not a vector in itself, but when it operates on a scalar function it provides the resultant as a vector. For example, when $\vec{\nabla}$ is operated on a scalar function $F(x, y, z)$, we get

$$\vec{\nabla}F = \hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z}$$

$\vec{\nabla}F$ does not mean a multiplication of $\vec{\nabla}$ with $F$ rather it is an instruction to differentiate. Here we should say that $\vec{\nabla}$ is a vector operator that acts upon $F$.

The del operator $\vec{\nabla}$ can act in different ways. For example, when it acts on a scalar function $F$, the resultant $\vec{\nabla}F$ is called the gradient of a scalar function $F$. When it acts on a vector function $\vec{A}$ via the dot product, the resultant is $\vec{\nabla}\cdot\vec{A}$, which is called the divergence of a vector $\vec{A}$. When it acts on a vector function $\vec{A}$ via the cross product, the resultant is $\vec{\nabla}\times\vec{A}$, which is called the curl of a vector $\vec{A}$. Finally, the Laplacian of a scalar function $F$ is written as $\nabla^2 F$ together with $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

The del operator can be written in spherical polar coordinate system (Fig. 11.1a) as

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

In cylindrical coordinate system (Fig. 11.1b), it is given by

$$\vec{\nabla} = \hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

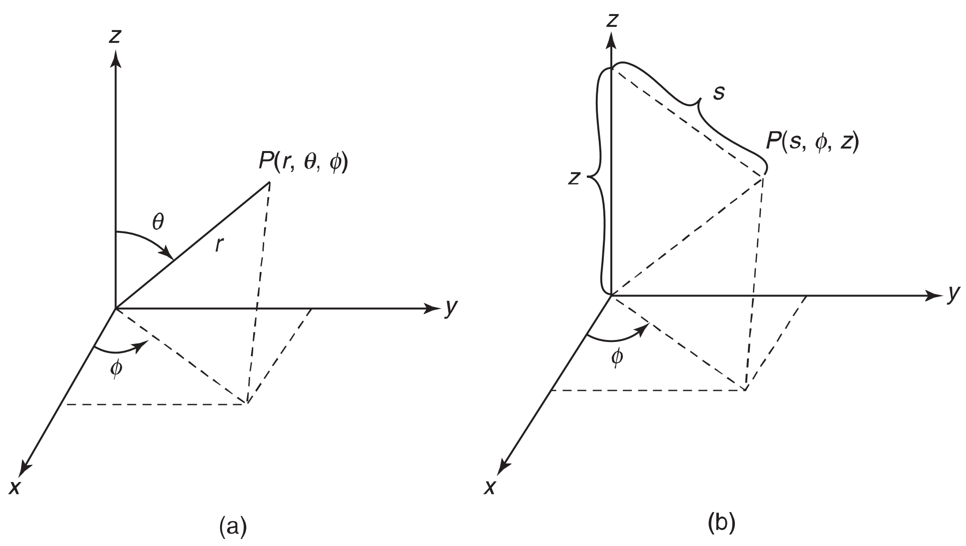


**FIGURE 11.1**

## 11.3 GRADIENT LO1

If we think of the derivative of a function of one variable we notice that it simply tells us how fast the function varies if we move a small distance. It means the gradient is the rate of change of a quantity with distance. For example, temperature gradient in a metal bar is the rate of change of temperature along the bar. However, for a function of three variables the situation is more complicated, as it will depend on what direction we choose to move. For a function $F(x, y, z)$ of three variables, we obtain from a theorem on partial derivative

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$$

Here $dF$ is a measure of changes in $F$ that occurs when we alter all three variables by small amounts $dx$, $dy$ and $dz$.

The above expression for $dF$ can be written in terms of a dot product of vectors as

$$dF = \vec{\nabla}F \cdot d\vec{l}$$

where $\vec{\nabla}F = \hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z}$ is nothing but gradient of $F$. Clearly the gradient is a vector quantity, i.e., it has both magnitude and direction. The meaning of gradient becomes clearer when we write

$$dF = \vec{\nabla}F \cdot d\vec{l}$$

or $$dF = |\vec{\nabla}F||d\vec{l}|\cos\alpha$$

where $\alpha$ is the angle between $\vec{\nabla}F$ and $d\vec{l}$. Now we fix the distance $dl$, i.e., magnitude $|d\vec{l}|$, and look around in various directions for the maximum change in $F$. Clearly the maximum change in $F$ takes place in the direction $\alpha = 0$. It means $dF$ is largest when we move in the direction of $\vec{\nabla}F$. This can also be said that the gradient $\vec{\nabla}F$ points in the direction of maximum increase of the function $F$. Hence, the gradient of a scalar field $F$ is a vector quantity which represents both the magnitude and the direction of the maximum space rate of increase of $F$.

The gradient of $F$ in Cylindrical coordinate system is written as

$$\vec{\nabla}F = \frac{\partial F}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial F}{\partial \phi}\hat{\phi} + \frac{\partial F}{\partial z}\hat{z}$$

In spherical polar coordinate system, the gradient of $F$ is written as

$$\vec{\nabla}F = \frac{\partial F}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}\hat{\phi}$$

## 11.4 DIVERGENCE LO1

As mentioned earlier, the divergence of a vector field $\vec{A}$ is represented by $\vec{\nabla}\cdot\vec{A}$. So it is given by

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Clearly the divergence of a vector field is a scalar. Also, the divergence of a scalar cannot be obtained, as the dot product of $\vec{\nabla}$ with a scalar is not possible.

In order to make clear the meaning of the divergence, we consider the net flux $\oint \vec{A}\cdot d\vec{S}$ of a vector field $\vec{A}$ from a closed surface $S$. The divergence of $\vec{A}$ is defined as the net outward flux per unit volume over a closed surface. Actually the divergence of $\vec{A}$ at a given point is a measure of how much the vector $\vec{A}$ spreads out, i.e., diverges, from that point. Fig. 11.2a shows that the divergence of a vector field at point $O$ is positive, as the vector spreads out. However, in Fig. 11.2b, the vector converges and hence the divergence at $O$ is negative. In Fig. 11.2c, we can notice that the divergence of a vector field is zero, as the magnitude of vectors remains the same.

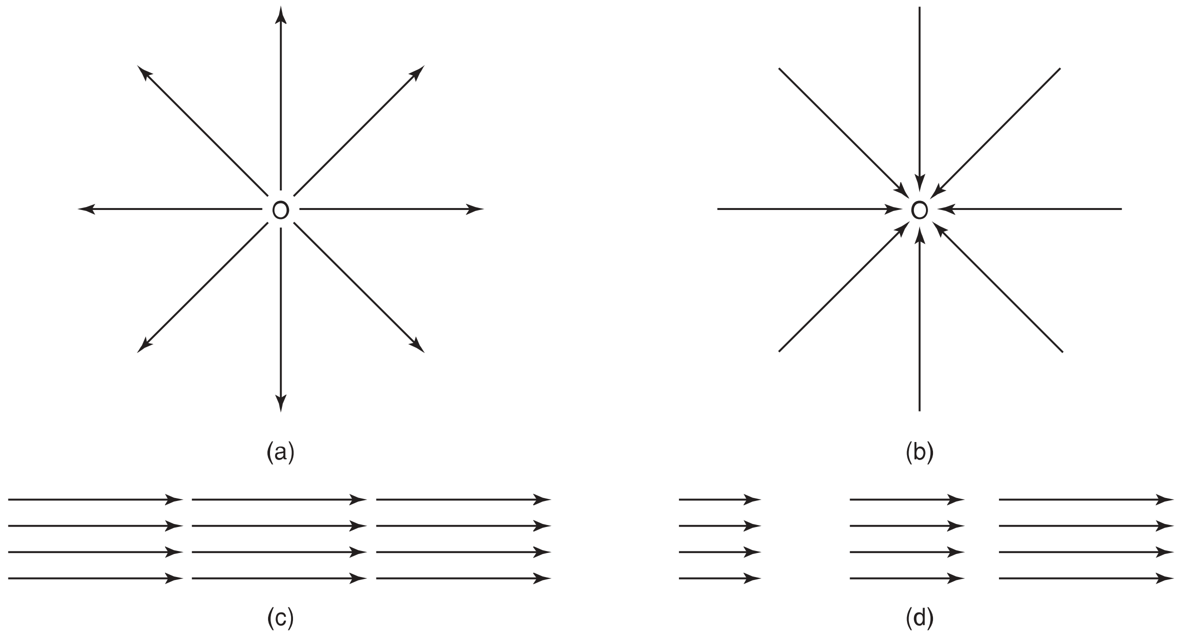


**FIGURE 11.2**

By now the difference between vector field and vector would have been clear to you. We call $\vec{A}$ as vector field because its values at different points are different, which are vectors with different magnitudes. For example, in Fig. 11.2d the magnitudes of the vectors get increased as we move towards right. Hence, the divergence of such a vector field is not zero but positive.

The divergence of a vector $\vec{A}$ in cylindrical coordinate system can be written as

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{s}\frac{\partial(sA_s)}{\partial s} + \frac{1}{s}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

In spherical polar coordinate system, it is written as

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Finally, we mention that the vector field $\vec{A}$ is said to be solenoidal or divergenceless if $\vec{\nabla}\cdot\vec{A} = 0$.

## 11.5 CURL — LO1

As mentioned earlier, the curl of vector field $\vec{A}$ is represented by $\vec{\nabla}\times\vec{A}$. So it is given by

$$\vec{\nabla}\times\vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Clearly the curl is a vector quantity, i.e., it has both direction and magnitude. Also, it is evident that we cannot have the curl of a scalar as the cross product of $\vec{\nabla}$ with a scalar is meaningless.

In order to make clear the meaning of curl, we consider the circulation of a vector field $\vec{A}$ around a closed path, i.e., $\oint \vec{A}\cdot d\vec{l}$. It is evident that the curl of $\vec{A}$ is a rotational vector. Its magnitude would be the maximum circulation of $\vec{A}$ per unit area. Its direction is the normal direction of the area when the area is oriented so as to make the circulation maximum. Actually curl of $\vec{A}$ at some point $O$ is a measure of how much the vector $\vec{A}$ curls around the point $O$. So, you may notice that the curl of vector field $\vec{A}$ in Fig. 11.2 is zero. However, in Fig. 11.3, the curl is finite and is pointing in the $x$-direction as per right hand rule.

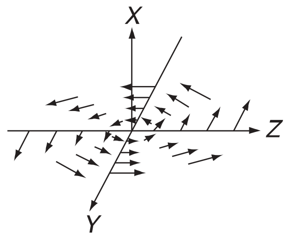

**Figure 11.3**

The curl of a vector $\vec{A}$ in cylindrical coordinate system can be written as

$$\vec{\nabla}\times\vec{A} = \left(\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left(\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial\varphi}\right)\hat{z}$$

In spherical polar coordinate system, it is written as

$$\vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right)\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

Finally, we mention that the vector field $\vec{A}$ is said to be irrotational or potential if $\vec{\nabla}\times\vec{A} = 0$.

## 11.6 FUNDAMENTAL THEOREM OF CALCULUS — LO2

If g($x$) be a function of one variable, then $dg$ would represent infinitesimal change in $g$ ($x$) when we move from $x$ to $x + dx$. This change in $g(x)$ is given by $dg = \left(\frac{dg}{dx}\right)dx$. If we move from point $a_1$ to point $a_2$, then the total change in $g(x)$ can be obtained by using the fundamental theorem of calculus. This theorem states that

$$\int_{a_1}^{a_2}\left(\frac{dg}{dx}\right)dx = g(a_2) - g(a_1)$$

It means the total change in the function can be obtained by simply subtracting the values of the function at the points $a_2$ and $a_1$. For the movement from point $a_1$ to point $a_2$, these points can be treated as the end points. In view of this, the fundamental theorem says that the integral of a derivative $\left(\frac{dg}{dx}\right)$ over an interval $a_1 \to a_2$ is given by the value of the function at the end points. The end points represent the boundaries.

Since we have three types of derivatives, namely gradient, divergence and curl in vector calculus, there are three fundamental theorems for these derivatives. The fundamental theorem for divergence is also known as Gauss's theorem or Green's theorem. Similarly, the fundamental theorem for curl is also known as Stokes' theorem.

## 11.7 FUNDAMENTAL THEOREM FOR GRADIENT — LO2

We know that the gradient is defined only for a scalar function. So we consider a scalar function of three variables $F(x, y, z)$. As discussed earlier, the change in this function is defined as $dF = \vec{\nabla}F \cdot d\vec{l}$. If we move from point $a_1$ to point $a_2$ then the total change in $F$ can be calculated using fundamental theorem of calculus. Therefore, the total change in $F$ in moving from $a_1$ to $a_2$ is given by

$$\int_{a_1}^{a_2} \vec{\nabla} F \cdot d\vec{l} = F(a_2) - F(a_1)$$

This is called the fundamental theorem for gradient, according to which the integral of a derivative over end points $a_1$ and $a_2$ is given by the value of the function at the boundaries, i.e., points $a_1$ and $a_2$. Here it can be noticed that the integral is line integral, the derivative is the gradient and the boundaries are the points $a_1$ and $a_2$. Moreover, it can be seen that the integral or the total change in function $F$ is independent of path taken from $a_1$ to $a_2$. Also, if we take both the end points same, i.e., we evaluate the close integral, then the total change in the function $F$ comes out to be zero (as the beginning and end points are identical).

## 11.8 GAUSS'S OR GREEN'S THEOREM — LO3

According to this theorem,

$$\int_v (\vec{\nabla} \cdot \vec{F}) dV = \oint_s \vec{F} \cdot d\vec{S}$$

Here $dV$ is the volume element and $d\vec{S}$ is the surface element. In the same manner as we stated fundamental theorem for gradient, the Green's theorem states that the integral of a derivative (here the divergence) over a region (here the volume) is equal to the value of the function at the boundary (here the surface). Since the boundary of a volume is always a closed surface, the R.H.S. is the integral over closed surface.

Evidently this theorem converts the volume integral into the surface integral. Therefore, this theorem is very useful in the situations where it is difficult to calculate the volume integral.

The net outward electric flux through any closed surface drawn in an electric field is equal to $\frac{1}{\varepsilon_0}$ times the total charge enclosed within the surface. It is expressed as

$$\phi = \oint_s \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \sum q = \frac{1}{\varepsilon_0} Q$$

where $Q$ is the sum of all the charges present within the surface. The charge outside of the surface is not counted, as the lines entering and leaving the surface due to this charge are the same in number. Therefore the flux $\phi$ due to the charge $q$ sitting outside the surface is expressed as

$$\phi = \oint_s \vec{E} \cdot d\vec{S} = 0$$

### 11.8.1 Differential Form of Gauss's Theorem

When a charge is distributed over a volume such that $\rho$ is the volume charge density, then the charge enclosed by the surface enclosing the volume is given by

$$q = \int_v \rho dV \tag{i}$$

Substituting this expression of $q$ in $\oint_s \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$, we get

$$\varepsilon_0 \oint_s \vec{E} \cdot d\vec{S} = \int_v \rho dV \tag{ii}$$

According to Gauss's divergence theorem, we can convert the surface integral into the volume integral as

$$\oint_s \vec{E} \cdot d\vec{S} = \int_v \text{div}\, \vec{E}\, dV \tag{iii}$$

Therefore, from Eqs (ii) and (iii), we have

$$\varepsilon_0 \int_s \text{div}\, \vec{E} dV = \int_v \rho\, dV$$

Since the above equality is true for every volume, the integrands of left and right sides should be equal, i.e.,

$$\varepsilon_0 \text{div} \vec{E} = \rho$$

$$\vec{\nabla} \cdot (\varepsilon_0 \vec{E}) = \rho \tag{iv}$$

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{v}$$

where, $\vec{D}$ is the electric flux density, given by $\vec{D} = \varepsilon_0 \vec{E}$. Eq. (v) is the differential form of Gauss's theorem.

## 11.9 STOKES' THEOREM LO4

Stokes' theorem states that the integral of the curl of a vector function over a patch of surface is equal to the value of the function at the perimeter of the patch. So here the derivative is the curl, region is the surface and the boundary is the perimeter of the patch of the surface. Therefore,

$$\int_s (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \oint_c \vec{F} \cdot d\vec{l}$$

Clearly, this theorem converts the surface integral into the line integral. Here the L.H.S. is the surface integral whereas the R.H.S. is the closed line integral. So a point of confusion is that which way we should go around, *i.e.,* clockwise or anticlockwise when we integrate the line integral. Moreover, we should know about the direction of the surface element $d\vec{S}$. For example, for a closed surface $d\vec{S}$ points outwards normal but for an open surface which way is out? In order to overcome this confusion, we apply right hand rule. So if our fingers point in the direction of line integral, then the thumb gives the direction of $d\vec{S}$.

Based on the statement of Stokes' theorem, we can make some more observations. For example, $\int_s (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$ does not depend on the shape of the surface rather it depends on the boundary line. Also for a closed surface, $\int_s (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = 0$ as the boundary line shrinks down to a point.

## 11.10 ELECTRIC FIELD AND ELECTRIC POTENTIAL LO4

Electric field is the region around a charge in which another charge experiences a force. The electrostatic field $\vec{E}$ is a special kind of field whose curl is always zero, i.e., $\vec{\nabla} \times \vec{E} = 0$. Since any vector whose curl is zero is equal to the gradient of some scalar quantity, we make use of this property to introduce the scalar quantity as the electric potential V. In order to find this, we use the Stokes' theorem $\int(\vec{\nabla}\times\vec{E})\cdot d\vec{S} = \oint \vec{E}\cdot d\vec{l}$. This gives

$$\oint \vec{E}\cdot d\vec{l} = 0 \tag{i}$$

It means the line integral of $\vec{E}$ from point $a_1$ to point $a_2$ will be the same for all the paths between these points. Hence, the line integral of $\vec{E}$ is independent of path. Since changing the path would not alter the value of integral, we can define a function, say $V$, such that

$$V = -\int_a^r \vec{E}\cdot d\vec{l} \tag{ii}$$

The differential form of the above equation is written as

$$\vec{E} = -\vec{\nabla}V \tag{iii}$$

Here $V$ is called the electric potential. Actually all the potentials are relative and there is no absolute zero potential. However, convention is that the potential is zero at infinite distance from the charge. In view of this, the lower limit $a$ in Eq. (ii) is called as a standard reference point where $V$ is zero. The upper limit is nothing but the point where $V$ is to be calculated. So $V$ depends only on the point $\vec{r}$.

### 11.10.1 Superposition Principle

According to the original principle of superposition of electrodynamics, the total force $\vec{F}$ on a charge $q$ (test charge) is equal to the vector sum of the forces due to all the source charges (considering them individually). It means

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots \tag{iv}$$

Since $\vec{F} = q\vec{E}$, from Eq.(iv) we find the following for the electric field $\vec{E}$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots \tag{v}$$

If we write $a$ for the common reference point, the above equation can be written as

$$-\int_a^r \vec{E}\cdot d\vec{l} = -\int_a^r \vec{E}_1\cdot d\vec{l} - \int_a^r \vec{E}_2\cdot d\vec{l} - \int_a^r \vec{E}_3\cdot d\vec{l} - \ldots \tag{vi}$$

or

$$\vec{\nabla}V = \vec{\nabla}V_1 + \vec{\nabla}V_2 + \vec{\nabla}V_3 + \ldots \tag{vii}$$

Now it is clear from Eq. (vii) that

$$V = V_1 + V_2 + V_3 + \ldots \tag{viii}$$

The above equation reveals that the potential $V$ at a given point $\vec{r}$ is the sum of the potentials due to all the charges. It means the electric potential also satisfies the principle of superposition and the sum is simply an ordinary sum. However, from Eq. (v) it is clear that in case of the electric field $\vec{E}$ this sum is the vector sum.

## 11.11 POISSON'S AND LAPLACE'S EQUATIONS LO5

For deriving these equations, we start with the following Gauss's law for a linear medium

$$\vec{\nabla}\cdot\vec{D} = \rho$$

or

$$\vec{\nabla}\cdot\varepsilon\vec{E} = \rho \tag{i}$$

Here $e$ is basically the free charge density (volume) and $\vec{D}$ is the electric displacement (talked in detail later).

Since $\vec{E} = -\vec{\nabla}V$, the above equation for a homogeneous medium (where $\varepsilon$ is constant) can be written as

$$\nabla^2 V = -\rho/\varepsilon \tag{ii}$$

This equation is called as Poisson's equation. For a charge free region, i.e., where $\rho = 0$, the Poisson's equation takes the form

$$\nabla^2 V = 0. \tag{iii}$$

This equation is known as Laplace's equation. This equation is much useful in solving electrostatic problems where a set of conductors are maintained at different potentials; for example, capacitors and vacuum tube diodes.

Using the expressions for Laplacian operator $\nabla^2$ in Cartesian, cylindrical and spherical coordinate systems, we can write Laplace's Eq. (iii) in these coordinate systems, respectively, as

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

## 11.12 CAPACITOR LO6

In order to construct a capacitor, we need at least two conductors that carry equal but opposite charges. If it is so, then flux lines leaving one conductor terminate at the surface of the other conductor. These conductors are also known as the plates of the capacitor, which may be separated by a dielectric or simply by free space. The ratio of the magnitude of the charge on one of the plates (say $q$) and the potential difference (say $V$) between the plates is known as the capacitance of the capacitor (represented by $C$). So

$$C = \frac{q}{V}$$

Some important capacitor configurations with two conductors are parallel plate capacitor, coaxial capacitor and spherical capacitor. These are discussed below.

### 11.12.1 Parallel Plate Capacitor

This type of capacitor has two plane conductors, which are parallel to each other. If the area of each plate be $S$ and they are separated by a distance $d$, then the capacitance of this capacitor is given as

$$C = \varepsilon\frac{S}{d}$$

Here $\varepsilon$ is the permittivity of the dielectric, which is filled between the plates. If a potential difference $V$ is applied between the plates then the energy stored in the capacitor would be

$$U = \frac{1}{2}CV^2 = \frac{q^2}{2C}$$

### 11.12.2 Coaxial Capacitor

Coaxial capacitor is simply a coaxial cable. This is also referred to as coaxial cylindrical capacitor. If $L$ be the length of the coaxial conductors, and the radii of inner and outer conductors be $r_{in}$ and $r_{out}$, then the capacitance of the capacitor is obtained as

$$C = \frac{2\pi\varepsilon L}{\ln\frac{r_{out}}{r_{in}}}$$

Here also $\varepsilon$ is the permittivity of the dielectric filling the space between the two conductors.

### 11.12.3 Spherical Capacitor

As name suggests, in this case the two conductors are in the form of spheres and these are concentric. Let the radius of inner sphere be $r_{in}$ and of the outer sphere be $r_{out}$. Also these spheres are separated by a dielectric medium of permittivity $\varepsilon$. Then the capacitance of this type of capacitor is obtained as

$$C = \frac{4\pi\varepsilon}{\frac{1}{r_{in}} - \frac{1}{r_{out}}}$$

## 11.13 MAGNETIC FLUX DENSITY ($\vec{B}$) LO7

When a magnetic material is placed in an external magnetic field, it gets magnetised. The magnetism thus produced in the material is known as induced magnetism and this phenomenon is referred to as magnetic induction. The magnetic lines of force inside such magnetised materials are called magnetic lines of induction. The number of magnetic lines of induction crossing unit area at right angles to the flux is called the magnetic flux density $\vec{B}$. Its unit is the Tesla which is equal to 1 Wb/m$^2$.

## 11.14 MAGNETIC FIELD STRENGTH ($\vec{H}$) LO7

As mentioned earlier, a magnetic material becomes magnetised when placed in a magnetic field. The actual magnetic field inside the material is the sum of external field and the field due to its magnetisation.

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \quad \text{or} \quad \vec{B} = \mu_0(\vec{H} + \vec{M})$$

Magnetic field strength at a point in a magnetic field is the magnitude of the force experienced by a unit pole situated at that point. The SI unit, corresponding to force of 1 Newton, is the A/m. The CGS unit, corresponding to a force of 1 dyne is the Oersted which is equal to 79.6 A/m.

## 11.15 AMPERE'S CIRCUITAL LAW LO7

Ampere's circuital law in magnetostatics is analogous to the Gauss's law used in electrostatics. This law says that the line integral of magnetic field $\vec{B}$ around any closed loop is equal to $\mu_0$ times the net current I flowing through the area enclosed by the loop i.e.,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

Here, $\mu_0$ is the permeability of the free space.

**Proof:** Consider a long straight conductor carrying a current $I$. By Biot-Savart law, the magnitude of the magnetic field at a point $O$, at a distance $r$ from the conductor, is given by

$$B = \frac{\mu_0}{4\pi}\frac{2I}{r} \tag{i}$$

Let us draw a circle with a radius $r$ taking $C$ as centre around the current carrying conductor Fig. 11.4. $\vec{B}$ will be the same in magnitude at all points on this circle. Again we consider a circle element of length $dl$ at the point $O$. From the figure it is clear that $d\vec{l}$ and $\vec{B}$ are in the same direction.

$$\therefore \qquad \oint \vec{B} \cdot d\vec{l} = \oint B dl \cos\theta$$

$$= B\oint dl \qquad [\because \theta = 0°]$$

$$= \frac{\mu_0}{4\pi}\frac{2I}{r} 2\pi r = \mu_0 I$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

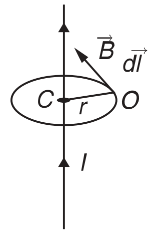


**FIGURE 11.4**

This is the required Ampere's circuital law. This law can be written in terms of volume current density $\vec{J}$, if we apply Stokes' theorem $\oint \vec{B} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{S}$.

Since $I = \int \vec{J} \cdot d\vec{S}$, we get $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ from $\oint \vec{B} \cdot d\vec{l} = \mu_0 I$. This is the another form of Ampere's law.

## 11.16 ELECTROSTATIC BOUNDARY CONDITIONS LO8

Consider the situation where the electric field exists in a region, which has two different media with permittivities as $\varepsilon_1$ (in region 1) and $\varepsilon_2$ (in region 2). Then the conditions, which should be satisfied by the field at the interface separating the two media or at the common boundary of these media, are called boundary conditions. It is seen that when we cross a boundary surface charge $\sigma$, the electric field does not remain continuous and it always undergoes a discontinuity.

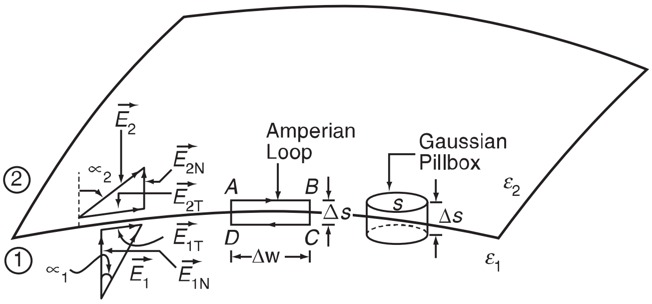


**FIGURE 11.5**

We can calculate the amount by which the electric field $\vec{E}$ changes at such a boundary, shown in Fig. 11.5. $\vec{E}_1$ is the field in the region 1 and $\vec{E}_2$ is the field in the region 2. These fields can be decomposed into two components, out of which one is tangential to the boundary (say $\vec{E}_T$) and the other is perpendicular to the boundary (say $\vec{E}_N$). So we can write

$$\vec{E}_1 = \vec{E}_{1N} + \vec{E}_{1T} \quad \text{and} \quad \vec{E}_2 = \vec{E}_{2N} + \vec{E}_{2T}$$

Now we can apply Ampere's and Gauss's laws for calculating the amount of discontinuity. For example, Ampere's law $\oint \vec{E}\cdot d\vec{l} = 0$ for the closed path *ABCDA*, whose length is $\Delta w$ and the width is $\Delta s$, follows

$$0 = E_{2T}\Delta w - E_{2N}\frac{\Delta s}{2} - E_{1N}\frac{\Delta s}{2} - E_{1T}\Delta w + E_{1N}\frac{\Delta s}{2} + E_{2N}\frac{\Delta s}{2} \quad \text{(i)}$$

In the limit $\Delta s \to 0$, i.e., when the width of the loop is small so that we are well close to the boundary, the above equation says

$$E_{2T}\Delta w = E_{1T}\Delta w$$

or

$$E_{2T} = E_{1T} \quad \text{(ii)}$$

Since $\vec{D} = \varepsilon\vec{E}$, the above equation reads for the displacement vector $\vec{D}_T$

$$\frac{D_{2T}}{\varepsilon_2} = \frac{D_{1T}}{\varepsilon_1} \quad \text{(iii)}$$

Eq. (ii) says that the tangential component of the electric field remains continuous across the boundary, as its values just below and just above the boundary are equal. It means the electric field component $\vec{E}_T$ undergoes no change on the boundary. However, you can see from Eq. (iii) that the field component $\vec{D}_T$ undergoes some change across the boundary. It means $\vec{D}_T$ is discontinuous across the boundary.

In order to check the continuity of the normal component $\vec{E}_N$ of the field $\vec{E}$, we select Gaussian pillbox of area $\vec{S}$ (upper and lower surfaces) and the thickness $\Delta s$. Now we apply Gauss's law $\oint \vec{D}\cdot d\vec{S} = q$ and obtain the following under the limit of $\Delta s \to 0$

$$D_{2N}S - D_{1N}S = \sigma S$$

or

$$D_{2N} - D_{1N} = \sigma \quad \text{(iv)}$$

Here $\sigma$ is the free charge density placed at the boundary. It is clear from Eq. (iv) that the normal component of $\vec{D}$ is discontinuous and this discontinuity amounts to the free charge density $\sigma$. If there is no free charge, the normal component of the field $\vec{D}$ will be continuous at the boundary, as in the absence of $\sigma$ Eq. (iv) follows

$$D_{2N} - D_{1N} = 0 \quad \text{(v)}$$

From the above equation, we can find the condition for the electric field component $\vec{E}_N$ as

$$\varepsilon_2 E_{2N} - \varepsilon_1 E_{1N} = 0 \quad \text{(vi)}$$

The above equation shows that the normal component $\vec{E}_N$ of the field $\vec{E}$ will be discontinuous at the boundary. If we write Eqs (ii), (iii), (iv) and (vi) together, these equations represent the boundary conditions, named dielectric – dielectric boundary conditions.

The boundary conditions are useful in finding the electric field on one side of the boundary if the field on the other side is given. In addition to this, we can determine the refraction of the electric field across the boundary. If the field $\vec{E}_1$ (or $\vec{D}_1$) in the region 1 makes an angle $\alpha_1$ and the field $\vec{E}_2$ (or $\vec{D}_2$) in the region 2 makes an angle $\alpha_2$ with the normal to the boundary, then from Eq. (ii) we get

$$E_1 \sin\alpha_1 = E_2 \sin\alpha_2 \quad \text{(vii)}$$

Similarly, Eq. (vi) yields

$$\varepsilon_1 E_1 \cos\alpha_1 = \varepsilon_2 E_2 \cos\alpha_2 \quad \text{(viii)}$$

From Eqs (vii) and (viii), we finally get

$$\frac{\tan\alpha_1}{\varepsilon_1} = \frac{\tan\alpha_2}{\varepsilon_2}$$

or
$$\frac{\tan\alpha_1}{\varepsilon_{r_1}} = \frac{\tan\alpha_2}{\varepsilon_{r_2}} \qquad [\because \varepsilon_1 = \varepsilon_0\varepsilon_{r1}, \varepsilon_2 = \varepsilon_0\varepsilon_{r2}] \tag{ix}$$

Here $\varepsilon_{r1}$ is the dielectric constant of medium 1 and $\varepsilon_{r2}$ is the dielectric constant of medium 2. The above equation is known as law of refraction of the electric field at the interface which is free of charge. This is also clear from Eq. (ix) that in general a boundary between two dielectrics produces bending of the flux lines. This is attributed to unequal polarisation charges accumulated on the sides of the boundary.

Now we can discuss some special cases of the above boundary conditions. For example, if the dielectric in the medium 2 is replaced with a conductor whose conductivity is infinite. In this case, it is obtained that the field does not exist within the conductor, i.e., $\vec{E} = 0$. Since $\vec{E} = -\vec{\nabla}V$, a potential difference between any two points in the conductor cannot exist. In other words, we say that a conductor is an equipotential body. Under this situation, only the normal component of the electric field survives, i.e., $D_T = \varepsilon_0\varepsilon_r E_T = 0$, $D_N = \varepsilon_0\varepsilon_r E_N = \sigma$. Similarly, when the dielectric in medium 1 is replaced with a conductor and the dielectric in medium 2 is replaced with free space ($\varepsilon_r = 1$), then the boundary conditions take the form $D_T = \varepsilon_0 E_T = 0$, $D_N = \varepsilon_0 E_N = \sigma$.

## 11.17 SCALAR AND VECTOR POTENTIALS — LO9

As mentioned earlier, the zero curl of electrostatic field $\vec{E}$, i.e., $\vec{\nabla}\times\vec{E} = 0$, introduces a scalar potential $V$ such that $\vec{E} = -\vec{\nabla}V$. When we analyze $\vec{\nabla}\cdot\vec{B} = 0$, we find that the field $\vec{B}$ can be written as a curl of another vector quantity (say $\vec{A}$), i.e.,

$$\vec{B} = \vec{\nabla}\times\vec{A} \tag{i}$$

Since a field can be completely determined if we know its divergence as well as its curl, the divergence of $\vec{A}$ remains to be explored. In this context, with the use of Eq. (i), Ampere's law reads

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} \tag{ii}$$

It is clear that Eq.(ii) will resemble Poisson's equation, if $\vec{\nabla}\cdot\vec{A} = 0$. This condition is known as Coulomb gauge. With the application of this condition, the Ampere's law simply yields

$$\nabla^2\vec{A} = -\mu_0\vec{J} \tag{iii}$$

The solution of the above equation can be obtained if the current density $\vec{J}$ vanishes at infinity. Then the solution comes out to be

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}}{r}\,dX \tag{iv}$$

Here $dX$ is the volume element and the vector $\vec{A}$ is called magnetic vector potential. Like the electric scalar potential $V$, the magnetic vector potential $\vec{A}$ cannot be uniquely defined as we can add to it another vector whose curl is zero. This addition does not change the field $\vec{B}$. On the other side, it is a point of observation that we cannot introduce a magnetic scalar potential $U$ such that $\vec{B} = -\vec{\nabla}U$. The reason is that it is incompatible with Ampere's law, since the curl of a gradient is always zero.

## 11.18 CONTINUITY EQUATION LO9

The continuity equation says that the total current flowing out of some volume must be equal to the rate of decrease of the charge within that volume, if the charge is neither being created nor lost. Since the charge is flowing, we consider that the charge density $\rho$ is a function of time. The transportation of the charge constitutes the current, i.e.,

$$I = \frac{dq}{dt} = \frac{d}{dt}\int_v \rho dV \tag{i}$$

Here, it is assumed that the current is extended in space of volume $V$ closed by a surface $S$. The net amount of charge which crosses a unit area normal to the directed surface in unit time is defined as the *current density* $\vec{J}$. This current density $\vec{J}$ is related to the total current $I$ flowing through the surface $S$ as

$$I = \oint_s \vec{J}\cdot d\vec{S} \tag{ii}$$

Here the integral is over closed surface, as the surface bounding the volume is closed surface. From Eqs (i) and (ii), we have

$$\oint_s \vec{J}\cdot d\vec{S} = -\frac{dq}{dt} = \frac{-d}{dt}\int_v \rho dV \tag{iii}$$

The minus sign above is needed in view of decreasing charge $\rho$ in the volume $V$. So

$$\oint_s \vec{J}\cdot d\vec{S} = -\int_v \frac{\partial \rho}{\partial t} dV \tag{iv}$$

From Gauss's divergence theorem, we have

$$\oint_s \vec{J}\cdot d\vec{S} = \int_v (\text{div}\,\vec{J}) dV$$

or

$$\int_v (\text{div}\,\vec{J})\, dV = -\int_v \frac{\partial \rho}{\partial t} dV \tag{v}$$

Since the Eq. (v) holds good for any arbitrary volume, we can put the integrands to be equal. Then

$$\text{div}\,\vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{vi}$$

This is the continuity equation.

In case of stationary currents, i.e., when the charge density at any point within the region remains constant, but the charges are moving.

$$\frac{\partial \rho}{\partial t} = 0, \tag{vii}$$

so that div $\vec{J} = 0$ or $\vec{\nabla}\cdot\vec{J} = 0$

which expresses the fact that there is no net outward flux of current density $\vec{J}$. Here the situation is the same as shown in Fig. 11.2c for zero divergence.

## 11.19 MAXWELL'S EQUATIONS: DIFFERENTIAL FORM LO10

When the charges are in motion, the electric and magnetic fields are associated with this motion which will have variations in both the space and the time. These electric and magnetic fields are inter related. This phenomenon is called *electromagnetism* which is summarised by the set of equations, known as Maxwell's equations. The Maxwell's equations are nothing but are the representation of the basic laws of electromagnetism.

First Maxwell's equation is the Gauss's law of electrostatics, i.e., $\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$. Second Maxwell's equation is the Gauss's law of magnetostatics, i.e., $\vec{\nabla} \cdot \vec{B} = 0$. Faraday's law of electromagnetic induction, i.e., $\vec{\nabla} \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$, is called Maxwell's third equation. Fourth Maxwell's equation is the modified Ampere's circuital law, i.e., $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$.

Now we look for the convenient form of Maxwell's equations while we are working with materials (having permittivity $\varepsilon$ and permeability $\mu$) that are subject to electric and magnetic polarizations. Electric polarization $\vec{P}$ provides bound charges with volume density $e_b$, given by

$$\rho_b = -\vec{\nabla} \cdot \vec{P} \tag{i}$$

Similarly, a magnetic polarization or magnetization $\vec{M}$ results in a volume bound current density, given by

$$\vec{J}_b = \vec{\nabla} \times \vec{M} \tag{ii}$$

Equations (i) and (ii) represent the static case of uniforms polarization $\vec{P}$ and uniform magnetization $\vec{M}$. However, any change in polarization $\vec{P}$ gives rise to the polarization current density, given by

$$\vec{J}_P = \frac{\partial \vec{P}}{\partial t} \tag{iii}$$

Since $\vec{J}_P$ satisfies the continuity equation, it is evident that $\vec{J}_P$ is essential to account for the conservation of bound charge. On the other hand, a changing magnetization $\vec{M}$ does not lead to any analogous accumulation of charge or current. We do not have direct control on the bound charge and current. Hence, it is advisable to reformulate Maxwell's equations such that these make explicit reference only to those sources which we control directly. These are the free charges ($\rho_f$) and currents ($\vec{J}_f$). The total volume charge density can be written as

$$\begin{aligned} \rho &= \rho_f + \rho_b \\ &= \rho_f - \vec{\nabla} \cdot \vec{P}, \end{aligned} \tag{iv}$$

whereas the total volume current density is written as

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_P = \vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \tag{v}$$

In view of total charge density $\rho$, Gauss's law reads

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0}(\rho_f - \vec{\nabla} \cdot \vec{P})$$

$$\vec{\nabla} \cdot (\varepsilon_0 \vec{E} + \vec{P}) = \rho_f$$

Recall that $\varepsilon_0 \vec{E} + \vec{P} = \vec{D}$. Hence, the first Maxwell's equation in materials takes the form

$$\vec{\nabla} \cdot \vec{D} = \rho_f \tag{vi}$$

Similarly, the modified Ampere's circuital law yields

$$\vec{\nabla} \times \vec{B} = \mu_0 \left( \vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \right) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \left( \frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_f + \frac{\partial}{\partial t}(\varepsilon_0 \vec{E} + \vec{P})$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \qquad \left( \because \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \right) \tag{vii}$$

The four Maxwell's equations in terms of free charges (density $\rho_f$) and currents are written as

$$\vec{\nabla} \cdot \vec{D} = \rho_f \qquad \text{First equation}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \text{Second equation}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{Third equation}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \qquad \text{Fourth equation}$$

However, for the sake of simplicity we shall write $\rho$ for $\rho_f$ and $\vec{J}$ for $\vec{J}_f$ in the derivation of Maxwell's equations, unless specified.

In free space, dielectric medium or conducting medium, the first and fourth Maxwell's equations assume different forms. For example, in free space and dielectric medium, free charge $\rho_f = 0$ and free current density $\vec{J}_f = 0$. Hence, the

First equation in free space yields $\vec{\nabla} \cdot \vec{D} = 0$

$\Rightarrow$ $$\vec{\nabla} \cdot (\varepsilon_0 \vec{E}) = 0 \qquad (\because \vec{P} = 0 \text{ in free space})$$

or $$\vec{\nabla} \cdot \vec{E} = 0$$

However, the first equation in dielectric medium gives

$$\vec{\nabla} \cdot \vec{D} = 0 \quad \Rightarrow \quad \varepsilon \vec{\nabla} \cdot \vec{E} = 0 \qquad (\because \vec{D} = \varepsilon \vec{E})$$

or $$\vec{\nabla} \cdot \vec{E} = 0$$

In conducting medium, any free charge resides on its surface, i.e., $\rho_f = 0$ in the medium. Hence, the first equation again gives $\vec{\nabla} \cdot \vec{E} = 0$. This can also be understand as follows.
The continuity equations for free charges reads

$$\frac{\partial \rho_f}{\partial t} + \vec{\nabla} \cdot \vec{J}_f = 0$$

or $$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla} \cdot \vec{J}_f \tag{viii}$$

Since we want to see what happens when a free charge is given to a conductor, we find Eq. (viii) in terms of $\rho_f$ by using Ohm's law $\vec{J}_f = \sigma \vec{E}$ ($\sigma$ is conductivity) and Gauss's law $\vec{\nabla} \cdot \vec{D} = \rho_f$. Hence

$$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla} \cdot (\sigma \vec{E})$$

$$\Rightarrow \qquad \frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\varepsilon} \vec{\nabla} \cdot (\varepsilon \vec{E})$$

$$\Rightarrow \qquad \frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\varepsilon} \rho_f \tag{ix}$$

Eq. (ix) is written as $\frac{1}{\rho_f} \frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\varepsilon}$ and is integrated to get

$$\rho_f(t) = \rho_f(0) e^{-\frac{\sigma}{\varepsilon} t} \tag{x}$$

Here $\rho_f(0)$ is the initial charge given to the conductor. For good conductors $\sigma \approx \infty$; means $\rho_f \to 0$ very quickly. This proves that the charge will flow out to the edges of conductor within very less time. This characteristic time is given by

$$\tau = \frac{\varepsilon}{\sigma} \tag{xi}$$

So this is clear that the Maxwell's first equation reads $\vec{\nabla} \cdot \vec{E} = 0$ in free space, dielectric and conducting medium.

Now we discuss different forms of Maxwell's fourth equation $\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$. In free space $\vec{J}_f = 0$ and $\vec{D} = \varepsilon_0 \vec{E}$ ($\because$ $\vec{P} = 0$), Hence,

$$\vec{\nabla} \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\text{or} \qquad \mu_0 \vec{\nabla} \times \vec{H} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\text{or} \qquad \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{xii}$$

In dielectric medium $\vec{J}_f = 0$ and $\vec{D} = \varepsilon \vec{E}$. Hence

$$\vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\text{or} \qquad \vec{\nabla} \times (\mu \vec{H}) = \mu \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\text{or} \qquad \vec{\nabla} \times \vec{B} = \mu \varepsilon \frac{\partial \vec{E}}{\partial t} \tag{xiii}$$

In conducting medium, $\vec{J}_f = \sigma \vec{E}$ and $\vec{D} = \varepsilon \vec{E}$.

Hence,

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \tag{xiv}$$

In view of the above discussion, this is clear that second and third Maxwell's equations remain unchanged in all types of the media.

### 11.19.1 Derivation of Maxwell's First Equation

Let us consider a surface $S$ bounding a volume $V$ in a dielectric medium, which is kept in the $\vec{E}$ field. The application of external field $\vec{E}$ polarises the dielectric medium and charges are induced, called bound charges or charges due to polarisation. The total charge density at a point in a small volume element $dV$ would then be $(\rho + \rho_p)$, where $\rho_p$ is the polarisation charge density (the same as $\rho_{\hat{b}}$), given by $\rho_P = -\text{div}\,\vec{P}$ and $\rho$ is the free charge density at that point in the small volume element $dV$.

Thus, the total charge density at that point will be $\rho - (\text{div}\vec{P})$. Then Gauss's theorem can be expressed as

$$\oint_s \vec{E}\cdot d\vec{S} = \int_v (\text{div}\,\vec{E})dV = \frac{1}{\varepsilon_0}\int_v (\rho - \text{div}\,\vec{P})\,dV$$

or
$$\varepsilon_0\int_v (\text{div}\,\vec{E})dV = \int_v (\rho - \text{div}\,\vec{P})\,dV$$

or
$$\int_v \text{div}(\varepsilon_0\vec{E} + \vec{P})\,dV = \int_v \rho dV$$

The quantity $(\varepsilon_0\vec{E} + \vec{P})$ is denoted by a quantity $\vec{D}$, called the electric displacement. Therefore,

$$\int_v (\text{div}\,\vec{D})dV = \int_v \rho dV$$

Since this equation is true for all the arbitrary volumes, the integrands in this equation must be equal, i.e.,

$$\text{div}\vec{D} = \rho$$

or
$$\vec{\nabla}\cdot\vec{D} = \rho$$

This is the Maxwell's first equation.

When the medium is isotropic, the three vectors $\vec{D}$, $\vec{E}$ and $\vec{P}$ are in the same direction and for small field $\vec{E}$, $\vec{D}$ is proportional to $\vec{E}$, i.e.,

$$\vec{D} = \varepsilon\vec{E}$$

where $\varepsilon$ is called the permittivity of the dielectric medium. The ratio $\varepsilon/\varepsilon_0$ is called the dielectric constant of the medium.

### 11.19.2 Derivation of Maxwell's Second Equation

Since the magnetic lines of force are either closed or go off to infinity, the number of magnetic lines of force entering any arbitrary surface is exactly the same as leaving it. It means the flux of magnetic induction $\vec{B}$ across any closed surface is always zero, i.e.,

$$\oint_s \vec{B}\cdot d\vec{S} = 0$$

Transforming the surface integral to volume integral using Gauss's divergence theorem, we have

$$\oint_s \vec{B}\cdot d\vec{S} = \int_v (\text{div}\,\vec{B})dV = 0$$

The integrand in the above equation should vanish for the surface boundary as the volume is arbitrary. Therefore

$$\text{div}\,\vec{B} = 0 \quad \text{or} \quad \vec{\nabla}\cdot\vec{B} = 0$$

This is the Maxwell's second equation.

### 11.19.3 Derivation of Maxwell's Third Equation

According to Faraday's law, the emf induced in a closed loop is given by

$$E_{\text{emf}} = -\frac{\partial \phi}{\partial t} = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} = -\frac{\partial}{\partial t}\int_s \vec{B} \cdot d\vec{S}$$

Here the flux $\phi = \int_s \vec{B} \cdot d\vec{S}$ where $S$ is any surface having the loop as boundary. The emf ($E_{\text{emf}}$) can also be found by calculating the work done in carrying a unit charge completely around the loop. Thus,

$$E_{\text{emf}} = \oint_c \vec{E} \cdot d\vec{l}$$

Here $\vec{E}$ is the intensity of the field associated with the induced emf. On equating the above two equations, we get

$$\oint_c \vec{E} \cdot d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

According to Stokes' theorem, the line integral can be transformed into surface integral with the help of $\oint_c \vec{E} \cdot d\vec{l} = \int_s (\vec{\nabla} \times \vec{E}) \cdot d\vec{S}$. Therefore

$$\int_s (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

This equation must be true for any surface whether small or large in the field. So the two vectors in the integrands must be equal at every point, i.e.,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This is the Maxwell's third equation.

### 11.19.4 Derivation of Maxwell's Fourth Equation

According to the Ampere's law, the work done in carrying a unit magnetic pole once around a closed arbitrary path linked with the current is expressed by

$$\oint_c \vec{H} \cdot d\vec{l} = I$$

or

$$\oint_c \vec{H} \cdot d\vec{l} = \int_c \vec{J} \cdot d\vec{S}$$

As per Stokes' theorem,

$$\oint_c \vec{H} \cdot d\vec{l} = \int_s (\vec{\nabla} \times \vec{H}) \cdot d\vec{S}$$

Therefore,

$$\int_s (\vec{\nabla} \times \vec{H}) \cdot d\vec{S} = \int_s \vec{J} \cdot d\vec{S}$$

This gives

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

The above relation is derived on the basis of Ampere's law, which holds good only for the steady current. However, for the changing electric fields, the current density should be modified. The difficulty with the above equation is that, if we take divergence of this equation, then

$$\text{div}(\vec{\nabla} \times \vec{H}) = \text{div}\, \vec{J}$$

$$\Rightarrow \quad 0 = \text{div}\, \vec{J} \qquad \text{[Since divergence of a curl = 0]}$$

$$\Rightarrow \quad \text{div}\, \vec{J} = 0$$

which conflicts with the continuity equation, as

$$\text{div}\, \vec{J} = -\frac{\partial \rho}{\partial t}$$

Therefore, Maxwell realised that the definition of the current density is incomplete and suggested to add another density $\vec{J}'$. Therefore

$$\text{curl}\, \vec{H} = \vec{J} + \vec{J}'$$

Now, taking divergence of the above equation, we get

$$\text{div (curl}\, \vec{H}) = \text{div}\, \vec{J} + \text{div}\, \vec{J}'$$

or

$$0 = \text{div}\, \vec{J} + \text{div}\, \vec{J}'$$

$$\text{div}\, \vec{J}' = -\,\text{div}\, \vec{J} = \frac{\partial \rho}{\partial t}$$

using continuity equation

Since,

$$\rho = \vec{\nabla} \cdot \vec{D}$$

$$\text{div}\, \vec{J}' = \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{D})$$

$$\vec{\nabla} \cdot \vec{J}' = \vec{\nabla} \cdot \frac{\partial \vec{D}}{\partial t}$$

Hence $\vec{J}' = \dfrac{\partial \vec{D}}{\partial t}$

Therefore, the Maxwell's fourth equation can be written as

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

The last term of R.H.S. of this equation is called Maxwell's correction and is known as *displacement current density*. The above equation is called modified Ampere's law for unsteady or changing current which is responsible for the electromagnetic fields.

## 11.20 MAXWELL'S EQUATIONS: INTEGRAL FORM LO10

There are situations where the integral form of Maxwell's equations is useful. Therefore, now we derive these equations in integral form.

### 11.20.1 Maxwell's First Equation

Differential form of the Maxwell's first equation is

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{i}$$

On integrating Eq. (i) over a volume $V$, we have

$$\int_v (\vec{\nabla} \cdot \vec{D}) dV = \int_v \rho dV$$

Using Gauss's divergence theorem, the above equation reads

$$\oint_s \vec{D} \cdot d\vec{S} = \int_v \rho dV = q$$

or

$$\oint_s \vec{D} \cdot d\vec{S} = q$$

Here $q$ is the net charge contained in the volume $V$ and $S$ is the surface bounding the volume $V$. This integral form of the Maxwell's first equation says that the total electric displacement through the surface $S$ enclosing a volume $V$ is equal to the total charge contained within this volume.

This statement can also be put in the following form: The total outward flux corresponding to the displacement vector $\vec{D}$ through a closed surface $\vec{S}$ is equal to the total charge $q$ within the volume $V$ enclosed by the surface $\vec{S}$.

### 11.20.2 Maxwell's Second Equation

Differential form of the Maxwell's second equation is

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{ii}$$

Exactly in a manner adopted above, we can show that

$$\oint_s \vec{B} \cdot d\vec{S} = 0$$

which signifies that the total outward flux of magnetic induction $\vec{B}$ through any closed surface $\vec{S}$ is equal to zero.

### 11.20.3 Maxwell's Third Equation

Differential form of the Maxwell's third equation is

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{iii}$$

On integrating Eq. (iii) over a surface $\vec{S}$ bounded by a closed path, we have

$$\int_s (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = \int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

Converting surface integral into line integral by Stokes' theorem, we get

$$\oint_c \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_s \vec{B} \cdot d\vec{S}$$

which signifies that the electromotive force around a closed path is equal to the time derivative of the magnetic flux through any closed surface bounded by that path.

### 11.20.4 Maxwell's Fourth Equation

Differential form of the Maxwell's fourth equation is

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{iv}$$

Exactly, in a manner adopted above, we can have this equation in the following form

$$\oint_c \vec{H} \cdot d\vec{l} = \int_s \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S}$$

The above equation signifies that the magnetomotive force around a closed path is a measure of the conduction current plus the time derivative of the electric flux through any surface bounded by that path.

## 11.21 SIGNIFICANCE OF MAXWELL'S EQUATIONS LO10

Maxwell's equations represent concisely the fundamentals of electricity and magnetism. From them one can develop most of the working relationships in the field.

Maxwell's first equation represents the Gauss's law for electricity which says that the electric flux out of any closed surface is proportional to the total charge enclosed within the surface. The integral form of this equation finds applications in calculating electric fields around charge objects. It is consistent with Coulomb's law when applied to the electric field of a point charge. The area integral of the electric field gives a measure of the net charge enclosed. However, the divergence of the electric field gives a measure of the density of sources.

As mentioned, the area integral of a vector field determines the net source of the field (function). The integral form $\oint_s \vec{B} \cdot d\vec{S} = 0$ of the Maxwell's second equation says that the net magnetic flux out of any closed surface is zero. This is because the magnetic flux directed inward toward the south pole, of a magnetic dipole kept in any closed surface, will be equal to the flux outward the north pole. Therefore, the net flux is zero for dipole sources. If we imagine a magnetic monopole source, the area integral $\oint_s \vec{B} \cdot d\vec{S}$ would have some finite value. Because of this and since the divergence of a vector field is proportional to the density of point source, this form of the Gauss's law for magnetic field simply says that there are no magnetic monopoles.

The Maxwell's third equation when written in the integral form states that the line integral of the electric field around a closed loop is equal to the negative of the rate of change of the magnetic flux through the area enclosed by the loop. The line integral basically is the generated voltage or emf in the loop. Therefore, the physical interpretation of Maxwell's third equation is that the changing magnetic field induces electric field.

For static electric field $\vec{E}$, the second term of the R.H.S. of the Maxwell's fourth equation vanishes and then the integral form of this equation says that the line integral of the magnetic field around a closed loop is proportional to the electric current flowing through the loop. This form of the Maxwell's equation is useful for calculating the magnetic field for simple geometries. However, this equation more specifically reveals that the changing electric field induces magnetic field. This seems complimentary to the meaning of the Maxwell's third equation. Therefore, they together yield the formulation of electromagnetic fields or electromagnetic waves, where both electric and magnetic fields propagate together and the change in one field induces the other field.

## 11.22 MAXWELL'S DISPLACEMENT CURRENT AND CORRECTION IN AMPERE'S LAW

LO10

When a current flows in a conductor, magnetic field is produced around it. A relation between the conduction current ($I$) and the magnetic field vector ($\vec{B}$) was given by Ampere, according to which the line integral of the field $\vec{B}$ along a closed curve in the magnetic field is equal to $\mu_0$ times the current I flowing in the conductor.

$$\oint_c \vec{B}\cdot d\vec{l} = \mu_0 I = \mu_0 \int_s \vec{J}\cdot d\vec{S} \qquad \text{(i)}$$

or $$\int_s (\text{curl}\,\vec{B})\cdot d\vec{S} = \mu_0 \int_s \vec{J}\cdot d\vec{S} \qquad \text{(Using Stokes' theorem)}$$

or $$\int_s (\text{curl}\,\vec{B} - \mu_0\vec{J})\cdot d\vec{S} = 0$$

For arbitrary surfaces, the above integral is true, so we get

$$\text{curl}\,\vec{B} - \mu_0\vec{J} = 0$$

or $$\text{curl}\,\vec{B} = \mu_0\vec{J} \qquad \text{(ii)}$$

or $$\text{div}(\text{curl}\,\vec{B}) = \text{div}(\mu_0\vec{J})$$

or $$0 = \mu_0\text{div}(\vec{J}) \qquad [\because \text{divergence of curl of a vector field is always zero}]$$

or $$\text{div}\,\vec{J} = 0 \qquad \text{(iii)}$$

Using continuity equation $\text{div}\,\vec{J} + \dfrac{\partial\rho}{\partial t} = 0$ or $\text{div}\,\vec{J} = -\dfrac{\partial\rho}{\partial t}$, we get $\text{div}\,\vec{J} = 0$ only if $\dfrac{\partial\rho}{\partial t} = 0$, i.e., for static charge density.

Thus, the Ampere's law given by Eq. (i) is valid only for the steady state conditions, i.e., for the fields that do not vary with time. This law is not valid for the time varying fields, such as charging and discharging of a capacitor, where $\dfrac{\partial\rho}{\partial t} \neq 0$.

Ampere's law was modified by Maxwell for the time varying fields. His concept was based on the Faraday's law of electromagnetic induction, according to which a changing magnetic field produces an electric field. On the basis of the fact that the magnetic field around a conductor is produced by the current flowing in it, Maxwell hypothesized that changing electric field should also induce a magnetic field. A changing electric field is equivalent to a current called displacement current ($I_d$) which flows as long as the electric field is changing. The displacement current produces the magnetic field the same way as the conduction current ($I$).

Thus, the total magnetic field ($\vec{B}$) will be the sum of the two terms (a) due to current $I(\vec{B}_1)$ and (b) due to current $I_d(\vec{B}_2)$, i.e.,

$$\vec{B} = \vec{B}_1 + \vec{B}_2$$

$\therefore$ $$\text{curl}\,\vec{B} = \text{curl}\,\vec{B}_1 + \text{curl}\,\vec{B}_2$$

or $$\text{curl}\,\vec{B} = \mu_0\vec{J} + \mu_0\vec{J}_d$$

or $$\text{curl}\,\vec{B} = \mu_0(\vec{J} + \vec{J}_d) \qquad \text{(iv)}$$

where $\vec{J}_d$ is the displacement current density. Also, in the integral form

$$\oint_c \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0 I_d$$

or
$$\oint_c \vec{B}\cdot d\vec{l} = \mu_0 (I + I_d) \tag{v}$$

In analogy to the Faraday's law of induction $\oint_c \vec{E}\cdot d\vec{l} = -\frac{d\phi_B}{dt}$, $I_d$ should correspond to $\varepsilon_0 \frac{d\phi_E}{dt}$. With this Eq. (v) can be written as

$$\oint_c \vec{B}\cdot d\vec{l} = \mu_0 \left( I + \varepsilon_0 \frac{d\phi_E}{dt} \right) \tag{vi}$$

Thus,
$$I_d = \varepsilon_0 \frac{d\phi_E}{dt} = \varepsilon_0 S \frac{d\vec{E}}{dt} = S\frac{d\vec{D}}{dt} = S\vec{J}_d \tag{vii}$$

where $\vec{D}$ is the electric displacement vector and $S$ is the area.

Value of $J_d$ can also be determined by taking divergence of Eq. (iv), i.e.,

$$\text{div curl } \vec{B} = \text{div } (\mu_0 \vec{J} + \mu_0 \vec{J}_d) = 0 \qquad [\because \text{div curl } \vec{B} = 0]$$

or
$$\text{div } \vec{J}_d = -\text{div } \vec{J} = \frac{\partial \rho}{\partial t} \qquad \left[\because \text{div } \vec{J} + \frac{\partial \rho}{\partial t} = 0\right]$$

or
$$\text{div } \vec{J}_d = \frac{\partial}{\partial t}(\text{div } \vec{D}) = \text{div}\left(\frac{\partial \vec{D}}{\partial t}\right) \qquad [\because \text{div } \vec{D} = \rho]$$

$\therefore$
$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t} \tag{viii}$$

Therefore, the modified form of Ampere's law is

$$\text{curl}\,\vec{B} = \mu_0 \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \tag{ix}$$

or
$$\text{curl}\,\vec{H} = \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \qquad (\because \vec{B} = \mu_0 \vec{H})$$

or
$$\text{curl}\,\vec{H} = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{x}$$

The Eqs (v), (ix) or (x) represent the Maxwell's fourth equation in different form which is nothing but the modified form of Ampere's law.

## 11.23 ELETROMAGNETIC (EM) WAVE PROPAGATION IN FREE SPACE LO11

Maxwell's equations for free space are given as follows

$$\text{div } \vec{E} = 0 \tag{i}$$

$$\text{div } \vec{H} = 0 \tag{ii}$$

$$\text{curl}\,\vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \tag{iii}$$

$$\text{curl}\,\vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{iv}$$

Taking curl of Eq. (iii), we get $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$ or $\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\mu_0 \frac{\partial}{\partial t}\left[\varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right]$

Here we have used Eq. (iv) for the value of $\vec{\nabla} \times \vec{H}$. Now from Eq. (i) $\vec{\nabla} \cdot \vec{E} = 0$. Hence

$$\nabla^2\vec{E} - \mu_0\varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

This is the wave equation governing the field $\vec{E}$. In view of the dimensions of $(\mu_0\varepsilon_0)^{-1/2}$ as of velocity (say, $v$), we can write this as

$$\nabla^2\vec{E} - \frac{1}{v^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{v}$$

Similarly the curl of Eq. (iv) gives rise to the wave equation for the field $\vec{H}$ as

$$\nabla^2\vec{H} - \mu_0\varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

or
$$\nabla^2\vec{H} - \frac{1}{v^2}\frac{\partial^2 \vec{H}}{\partial t^2} = 0 \tag{vi}$$

The plane wave solutions of Eqs. (v) and (vi) may be written as

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{and} \quad \vec{H}(\vec{r}, t) = \vec{H}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

where $\omega$ is the angular frequency of the variation of the fields $\vec{E}$ and $\vec{H}$ and $\vec{k}$ is the wave vector which tells the direction of propagation of the fields or wave. The ratio $\omega/k$ gives the phase velocity of the wave.

Now, $\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$, $\vec{E}_0 = (E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k})$

$$\vec{k} = (k_x\hat{i} + k_y\hat{j} + k_z\hat{k}),\ \vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$$

$\Rightarrow$
$$\vec{k}\cdot\vec{r} = (k_x x + k_y y + k_z z)$$

$\therefore$
$$\begin{aligned}\text{curl}\,\vec{E} &= \vec{\nabla} \times [\vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}] \\ &= \vec{\nabla} \times \{(\vec{E}_0 e^{i\vec{k}\cdot\vec{r}})e^{-i\omega t}\} \\ &= \vec{\nabla} \times \{[(E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k})e^{i(k_x x + k_y y + k_z z)}]e^{-i\omega t}\}\end{aligned}$$

Here note that $i$ appeared in exponential term is such that $i = \sqrt{-1}$, whereas $\hat{i}$ is the unit vector along the $x$-direction. When we solve the above equation with the help of expansion of curl, we obtain

$$\text{curl } \vec{E} = i\{\hat{i}[E_{0z}k_y - E_{0y}k_z] + \hat{j}[E_{0x}k_z - E_{0z}k_x] + \hat{k}[E_{0y}k_x - E_{0x}k_y]\}e^{i(k_x x + k_y y + k_z z)}e^{-i\omega t}$$

$$= i[\vec{k} \times \vec{E}_0]e^{i(\hat{k}\cdot\vec{r} - \omega t)}$$

or $$\text{curl } \vec{E} = i[\vec{k} \times \vec{E}] \quad \text{(vii)}$$

Here we have used

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Using Eqs. (iii) and (vii), we obtain

$$-\mu_0 \frac{\partial \vec{H}}{\partial t} = i[\vec{k} \times \vec{E}]$$

L.H.S. of the above equation is written as

$$-\mu_0 \frac{\partial}{\partial t}[\vec{H}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}] = i\omega\mu_0\vec{H}$$

Hence $$\vec{k} \times \vec{E} = \omega\mu_0\vec{H} \quad \text{(viii)}$$

Similarly, it can be shown using Eq. (iv) that

$$\vec{k} \times \vec{H} = -\omega\varepsilon_0\vec{E} \quad \text{(ix)}$$

From Eq. (viii) it is obvious that the magnetic field vector $\vec{H}$ is perpendicular to both the propagation vector $\vec{k}$ and the electric field vector $\vec{E}$ and according to Eq. (ix) $\vec{E}$ is perpendicular to both $\vec{k}$ and $\vec{H}$. Therefore, it may be concluded that the electric and magnetic vectors are normal to each other as well as to the direction of propagation of the wave or $\vec{E}$, $\vec{H}$ and direction of wave propagation $\vec{k}$ form a set of orthogonal vectors. Further, we can prove that the electromagnetic field or wave travels at the speed of light $c$ in free space. For this, the cross product of $\vec{k}$ with Eq. (viii) gives

$$\vec{k} \times (\vec{k} \times \vec{E}) = \omega\mu_0(\vec{k} \times \vec{H})$$

$$\vec{k}(\vec{k} \cdot \vec{E}) - k^2\vec{E} = \omega\mu_0[-\omega\varepsilon_0\vec{E}]$$ [Putting the value of $\vec{k} \times \vec{H}$ from Eq. (ix)]

Since $\vec{k}$ and $\vec{E}$ are perpendicular to each other, $\vec{k} \cdot \vec{E} = 0$ and the above equation reads

$$k^2\vec{E} - \omega^2\mu_0\varepsilon_0\vec{E} = 0$$

$$(k^2 - \omega^2\mu_0\varepsilon_0)\vec{E} = 0$$

This relation between $\omega$ and $k$ is known as dispersion relation.

Since $\vec{E}$ cannot be zero for the wave, $k^2 - \omega^2\mu_0\varepsilon_0 = 0$

$$\Rightarrow \quad \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3 \times 10^8 \text{ m/sec} = c, \text{ the speed of light.}$$

Therefore, the phase velocity $\frac{\omega}{k}$ of the electromagnetic wave is equal to the speed of light $c$ in free space or vacuum.

## 11.24 TRANSVERSE NATURE OF ELECTROMAGNETIC WAVES LO11

Transverse nature of electromagnetic waves can be proved with the help of Maxwell's equations div $\vec{E} = 0$ and div $\vec{H} = 0$ for free space. Using different relations as discussed in the previous section div $\vec{E}$ and div $\vec{H}$ can be calculated as follows.

$$\vec{\nabla}\cdot\vec{E} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot[(E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k})e^{i(k_x x + k_y y + k_z z - \omega t)}]$$

$$= \frac{\partial}{\partial x}[E_{0x}e^{i(k_x x + k_y y + k_z z - \omega t)}] + \frac{\partial}{\partial y}[E_{0y}e^{i(k_x x + k_y y + k_z z - \omega t)}] + \frac{\partial}{\partial z}[E_{0z}e^{i(k_x x + k_y y + k_z z - \omega t)}]$$

$$= (E_{0x}ik_x + E_{0y}ik_y + E_{0z}ik_z)e^{i(k_x x + k_y y + k_z z - \omega t)}$$

$$= i(k_x E_{0x} + k_y E_{0y} + k_z E_{0z})e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= i(\vec{k}\cdot\vec{E}_0)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= i\vec{k}\cdot\vec{E} \qquad (\because \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \vec{E})$$

$\therefore$ For free space, $\vec{\nabla}\cdot\vec{E} = 0$

$$i\vec{k}\cdot\vec{E} = 0 \quad \text{or} \quad \vec{k}\cdot\vec{E} = 0$$

It means the wave vector $\vec{k}$ is perpendicular to $\vec{E}$.

Similarly, from $\vec{\nabla}\cdot\vec{H} = 0$ it can be shown that $\vec{k}\cdot\vec{H} = 0$.

Hence, the wave vector $\vec{k}$ is perpendicular to $\vec{H}$. Therefore, the relations $\vec{k}\cdot\vec{E} = 0$ and $\vec{k}\cdot\vec{H} = 0$ indicate that the electromagnetic field vectors $\vec{E}$ and $\vec{H}$ (or $\vec{B}$, as $\vec{B} = \mu_0\vec{H}$) both are perpendicular to the direction of propagation vector $\vec{k}$. It means that the electromagnetic waves are transverse in nature.

## 11.25 MAXWELL'S EQUATIONS IN ISOTROPIC DIELECTRIC MEDIUM: EM WAVE PROPOGATION LO11

In an isotropic dielectric medium, the conduction or free current density $\vec{J}$ and volume charge density $\rho$ are zero. Further, the displacement vector $\vec{D}$ and the magnetic field $\vec{B}$ are defined as $\vec{D} = \varepsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$. In fact $\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\vec{E} + \varepsilon_0\chi_e\vec{E} \equiv \varepsilon\vec{E}$ together with $\varepsilon = \varepsilon_0(1 + \chi_e)$ and $\vec{B} = \mu_0\vec{H} + \mu_0\vec{M} = \mu_0\vec{H} + \mu_0\chi_m\vec{H} \equiv \mu\vec{H}$ together with $\mu = \mu_0(1 + \chi_m)$ for the isotropic linear dielectric (polarizable and magnetic) medium. Here, the vectors $\vec{P}$ and $\vec{M}$ give respectively the measure of polarization and magnetization of the medium. However, for the dielectric medium, it would be sufficient to remember that $\varepsilon_0$ and $\mu_0$ of free space have been simply replaced with $\varepsilon$ and $\mu$. Hence, for dielectric medium

$$J = 0 \quad (\text{or } \sigma = 0, f = 0, D = \varepsilon E \text{ and } \vec{B} = \mu\vec{H})$$

where $\varepsilon$ and $\mu$, which are respectively the absolute permittivity and permeability of the medium. Under this situation, we can express the Maxwell's equation as

$$\vec{\nabla}\cdot\vec{E} = 0 \tag{i}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \tag{ii}$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{iii}$$

$$\vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \tag{iv}$$

Taking curl of Eq. (iii), we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left[-\mu \frac{\partial \vec{H}}{\partial t}\right]$$

or

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$$

or

$$0 - \nabla^2 E = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t} \qquad \text{[Using Eqs (i) and (iv)]}$$

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \tag{v}$$

Similarly, taking curl of Eq. (iv) and using Eqs. (ii) and (iii), we get

$$\nabla^2 \vec{H} = \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} \tag{vi}$$

As discussed earlier, $\frac{1}{\sqrt{\mu\varepsilon}}$ gives the phase velocity of the wave in the medium. If we represent this as $v$, we obtain from Eqs. (v) and (vi)

$$\nabla^2 \vec{E} - \frac{1}{v^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

and

$$\nabla^2 \vec{H} - \frac{1}{v^2} \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

Eqs. (v) and (vi) are the wave equations in an isotropic linear dielectric medium.

Now, $v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0 \mu_r \varepsilon_0 \varepsilon_r}}$ $\qquad (\because\ \mu = \mu_0\mu_r \text{ and } \varepsilon = \varepsilon_0\varepsilon_r)$

or

$$v = \frac{c}{\sqrt{\mu_r \varepsilon_r}} \qquad \left[\because\ c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}\right] \tag{vii}$$

Eq. (vii) shows that the propagation velocity of an electromagnetic wave in a dielectric medium is less than that in free space.

Also, refractive index $= \frac{c}{v} = \sqrt{\mu_r \varepsilon_r}$ (viii)

For non-magnetic dielectric medium $\mu_r \approx 1$. Hence, refractive index $= \sqrt{\varepsilon_r}$ or Refractive index $= \sqrt{\text{Relative permittivity}}$

## 11.26 MAXWELL'S EQUATIONS IN CONDUCTING MEDIUM: EM WAVE PROPAGATION AND SKIN DEPTH

LO11

We consider a linear and isotropic conducting medium whose permeability is $\mu$, permittivity is $\varepsilon$ and the conductivity is $\sigma$. Under this situation, we can write the Maxwell's equation as

$$\vec{\nabla}\cdot\vec{E}=0 \quad \text{(i)}$$

$$\vec{\nabla}\cdot\vec{H}=0 \quad \text{(ii)}$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}=-\mu\frac{\partial\vec{H}}{\partial t} \quad \text{(iii)}$$

$$\vec{\nabla}\times\vec{H}=\vec{J}+\varepsilon\frac{\partial\vec{E}}{\partial t}$$

or $$\vec{\nabla}\times\vec{H}=\sigma\vec{E}+\varepsilon\frac{\partial\vec{E}}{\partial t} \quad [\because \vec{J}=\sigma\vec{E}] \quad \text{(iv)}$$

Taking curl of Eq. (iii), we have

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}\times\left[-\mu\frac{\partial\vec{H}}{\partial t}\right]$$

$$=-\mu\left[\vec{\nabla}\times\frac{\partial\vec{H}}{\partial t}\right]$$

$$=-\mu\frac{\partial}{\partial t}[\vec{\nabla}\times\vec{H}]$$

$$=-\mu\frac{\partial}{\partial t}\left[\sigma\vec{E}+\varepsilon\frac{\partial\vec{E}}{\partial t}\right] \quad \text{[using Eq. (iv)]}$$

or $$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=-\mu\sigma\frac{\partial\vec{E}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

Also, $\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-(\vec{\nabla}\cdot\vec{\nabla})\vec{E}=-\nabla^2\vec{E}$ $[\therefore \vec{\nabla}\cdot\vec{E}=0$ from Eq. (i)]

$\therefore$ $$-\nabla^2\vec{E}=-\mu\sigma\frac{\partial\vec{E}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

or $$\nabla^2\vec{E}=+\mu\sigma\frac{\partial\vec{E}}{\partial t}+\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{(v)}$$

Eq. (v) is the electromagnetic wave equation for the electric field $\vec{E}$ in a conducting medium.

In case of non-conducting medium $\sigma = 0$. Hence, from Eq. (v)

$$\nabla^2\vec{E}=\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{(vi)}$$

Eqs (v) and (vi) show that the term $\mu\sigma\frac{\partial\vec{E}}{\partial t}$ is the dissipative term which allows the current to flow through the medium due to the appearance of conductivity $\sigma$.

Now, by taking curl of Eq. (iv), we obtain

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H})=\vec{\nabla}\times\left[\sigma\vec{E}+\varepsilon\frac{\partial\vec{E}}{\partial t}\right]$$

$$=\sigma(\vec{\nabla}\times\vec{E})+\varepsilon\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

$$=\sigma\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)+\varepsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) \qquad \text{[Using Eq. (iii)]}$$

$$=-\mu\sigma\frac{\partial\vec{H}}{\partial t}-\varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2}$$

$\because$ $$\vec{\nabla}\times(\vec{\nabla}\times\vec{H})=\vec{\nabla}(\vec{\nabla}\cdot\vec{H})-(\vec{\nabla}\cdot\vec{\nabla})\vec{H}$$

$$=0-\nabla^2\vec{H} \qquad [\therefore\ \vec{\nabla}\cdot\vec{H}=0 \text{ from Eq. (ii)}]$$

$\therefore$ $$-\nabla^2\vec{H}=-\mu\sigma\frac{\partial\vec{H}}{\partial t}-\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}$$

or $$\nabla^2\vec{H}=\mu\sigma\frac{\partial\vec{H}}{\partial t}+\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} \qquad \text{(vii)}$$

Equations (vii) represents the electromagnetic wave equation for magnetic field ($\vec{H}$) in conducting medium.

In case of non-conducting medium $\sigma = 0$, the wave equation takes the form

$$\nabla^2\vec{H}=\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} \qquad \text{(viii)}$$

Equations (vii) show that the term $\mu\sigma\frac{\partial\vec{H}}{\partial t}$ is the dissipative term which allows the current to flow through the conducting medium.

### 11.26.1 Solution of Wave Equation

Equations (v) and (vii) are called inhomogeneous wave equation due to the presence of dissipative term. These equations in one-dimension (along $z$-axis) are written as

$$\frac{\partial^2\vec{E}}{\partial z^2}=\mu\sigma\frac{\partial\vec{E}}{\partial t}+\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \qquad \text{(ix)}$$

$$\frac{\partial^2\vec{H}}{\partial z^2}=\mu\sigma\frac{\partial\vec{H}}{\partial t}+\mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} \qquad \text{(x)}$$

We assume the following plane wave solutions to the above equations

$$\vec{E}\,(z,t)=\vec{E}_0\,e^{i(kz-\omega t)} \qquad \text{(xi)}$$

and $$\vec{H}\,(z,t)=\vec{H}_0\,e^{i(kz-\omega t)} \qquad \text{(xii)}$$

The use of Eq. (xi) in Eq. (ix) and Eq. (xii) in Eq. (x) leads

$$k^2=\mu\varepsilon\omega^2+i\mu\sigma\omega \qquad \text{(xiii)}$$

This relation is called dispersion relation that governs the electromagnetic wave propagation in a conducting medium. Equation (xiii) suggests that the quantity $k$, i.e., wave number, will be a complex quantity. So, we assume

$$k=k_r+ik_i \qquad \text{(xiv)}$$

With this the fields $\vec{E}$ and $\vec{H}$ become

$$\vec{E}(z,t)=\vec{E}_0 e^{-k_i z}\cdot e^{i(k_r z-\omega t)} \tag{xv}$$

and
$$\vec{H}(z,t)=\vec{H}_0 e^{-k_i z}\, e^{i(k_r z-\omega t)} \tag{xvi}$$

### 11.26.2 Skin Depth

The expressions (xv) and (xvi) follow that the amplitude of the electric field $\vec{E}$ is $E_0 e^{-k_i z}$ and that of the magnetic field $\vec{H}$ is $H_0 e^{-k_i z}$. Hence the amplitude of electromagnetic wave will decrease exponentially as it propagates through the conductor. This is called the attenuation of the wave and the distance through which the amplitude is reduced by a factor of $1/e$ is called the skin depth or penetration depth $\delta$. At $z = \delta$, the amplitude is $E_0/e$. Hence

$$E_0 e^{-k_i\delta}=E_0/e \tag{i}$$

This gives the skin depth as

$$\delta=\frac{1}{k_i} \tag{ii}$$

Equation (ii) shows that the imaginary part of the wave number $k$ is the measure of the skin depth. However, the real part $k_r$ of $k$ determines the wave propagation characteristics in the following manner.

Wavelength $\lambda = 2\pi/k_r$ (iii)

Phase velocity $v = \omega/k_r$ (iv)

Refractive index $n=\dfrac{c}{v}=\dfrac{ck_r}{\omega}$ (v)

By putting $k = k_r + i\, k_i$ in Eq. (xiii) we obtain

$$k_r=\omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]} \tag{vi}$$

and
$$k_i=\omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} \tag{vii}$$

For good conductors, $\sigma >> \omega\varepsilon$. This condition when put in Eqs. (vi) and (vii) gives

$$k_r=k_i=\omega\sqrt{\frac{\mu\varepsilon}{2}\frac{\sigma}{\omega\varepsilon}}$$

$$\Rightarrow \qquad k_r=k_i=\sqrt{\frac{\omega\mu\sigma}{2}}=\sqrt{\pi f\mu\sigma} \tag{viii}$$

Hence, the skin depth is given by

$$\delta=\frac{1}{\sqrt{\pi f\mu\sigma}} \tag{ix}$$

Since $\delta$ is inversely proportional to $f$, which is the frequency of electromagnetic wave, high frequency waves are found to penetrate less into the conductor. Also, the penetration will be less in the medium having high conductivity $\sigma$. Ideally an electromagnetic wave will not penetrate into a perfect conductor as $\sigma = \infty$.

### 11.26.3 Phase Relationship of $\vec{E}$ and $\vec{B}$ Fields

In view of the imaginary wave number $k_i$, we can also make another observation with regard to the phase difference between $\vec{E}$ and $\vec{H}$ vectors. If we take the direction of $\vec{E}$ field along the $x$-axis, then $\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$ gives

$$\vec{H}(z,t) = \hat{j}\frac{kE_0}{\omega\mu} e^{-k_i z} e^{i(k_r z - \omega t)} \quad \text{(x)}$$

Since $k$ is the complex quantity, it can be represented as $k = k' e^{i\theta_k}$. Here $k' = \sqrt{k_r^2 + k_i^2}$ and $\theta_k = \tan^{-1}\left(\frac{k_i'}{k_r}\right)$. Then the expression of $\vec{H}$ becomes

$$\vec{H}(z,t) = \hat{j}\frac{k'E_0}{\omega\mu} e^{-k_i z} e^{i(k_r z - \omega t + \theta_k)} \quad \text{(xi)}$$

A comparison of Eq. (xi) with $\vec{E}(z,t) = \hat{i}E_0 e^{-k_i z} e^{i(k_r z - \omega t)}$ reveals that the electric field and magnetic field vectors do not remain in phase when electromagnetic wave propagates in a conducing medium. This is in contrast to the cases of vacuum and dielectrics.

## 11.27 ELECTROMAGNETIC ENERGY DENSITY — LO11

It can be proved that the work done in assembling a static charge distribution (number $n$) against the Coulomb repulsion of like charges is

$$W_E = \frac{1}{2}\sum_{j=1}^{n} q_j V(\vec{r}_j) \quad \text{(i)}$$

For a volume charge density $\rho$, this equation takes the form

$$W_E = \frac{1}{2}\int \rho V dX \quad \text{(ii)}$$

Here $dX$ is the volume element. Now the above equation can be written in terms of the resulting electric field $\vec{E}$ if we apply Gauss's law $\varepsilon_0 \vec{\nabla} \cdot \vec{E} = \rho$ and mention the potential $V$ in terms of $\vec{E}$. This yields the following relation where the integration is over all the space containing the whole charge distribution.

$$W_E = \frac{\varepsilon_0}{2}\int E^2 dX \quad \text{(iii)}$$

The same way we can derive an expression for the work done on a unit charge against the back emf in one trip around the circuit, as follows

$$W_B = \frac{1}{2\mu_0}\int B^2 dX \quad \text{(iv)}$$

Here $B$ is the resulting magnetic field. Eqs. (iii) and (iv) suggest that the total energy stored in electromagnetic field would be

$$W_{EM} = \frac{1}{2}\int\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)dX \quad \text{(v)}$$

Therefore, the electromagnetic energy density can be obtained as

$$U_{EM} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) \tag{vi}$$

For a monochromatic plane electromagnetic wave, $B = \frac{E}{c}$, where $c \left(\equiv \frac{1}{\sqrt{\mu_0 \varepsilon_0}}\right)$ is the speed of light. Hence, it can be seen that the contribution of magnetic field $\vec{B}$ to the energy density is the same as that of the electric field $\vec{E}$.

## 11.28 POYNTING VECTOR AND POYNTING THEOREM LO12

The electromagnetic waves carry energy when they propagate and there is an energy density associated with both the electric and magnetic fields.

The amount of energy flowing through unit area, perpendicular to the direction of energy propagation per unit time, i.e., the rate of energy transport per unit area, is called the *poynting vector*. It is also termed as instantaneous energy flux density and is represented by $\vec{S}$ (or $\vec{P}$, sometimes). Mathematically it is defined

$$\vec{S} = \vec{E} \times \vec{H}$$

where $\vec{E}$ and $\vec{H}$ represent the instantaneous values of the electric and magnetic field vectors. This is clear that the rate of energy transport $\vec{S}$ is perpendicular to both $\vec{E}$ and $\vec{H}$ and is in the direction of propagation of the wave, as $\vec{E} \times \vec{H}$ is in the direction of $\vec{k}$. Since the poynting vector represents the rate of energy transport per unit area, its units are W/m$^2$.

**Derivation:** We can calculate the energy density carried by electromagnetic waves with the help of Maxwell's equations given below.

$$\text{div}\,\vec{D} = 0 \tag{i}$$

$$\text{div}\,\vec{B} = 0 \tag{ii}$$

$$\text{curl}\,\vec{E} = \frac{-\partial \vec{B}}{\partial t} \tag{iii}$$

$$\text{curl}\,\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{iv}$$

Take scalar (dot) product of Eq. (iii) and Eq. (iv) with $\vec{H}$ and $\vec{E}$ respectively, i.e.,

$$\vec{H}\cdot \text{curl}\,\vec{E} = -\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} \tag{v}$$

and

$$\vec{E}\cdot \text{curl}\,\vec{H} = \vec{E}\cdot\vec{J} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t} \tag{vi}$$

Subtract Eq. (vi) from Eq. (v), i.e.,

$$\begin{aligned}\vec{H}\cdot \text{curl}\,\vec{E} - \vec{E}\cdot \text{curl}\,\vec{H} &= -\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} - \vec{E}\cdot\vec{J} - \vec{E}\cdot\frac{\partial \vec{D}}{\partial t} \\ &= -\left(\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t}\right) - \vec{E}\cdot\vec{J}\end{aligned}$$

or $$\operatorname{div}(\vec{E}\times\vec{H}) = -\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}+\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right)-\vec{E}\cdot\vec{J} \tag{vii}$$

$[\because \operatorname{div}(\vec{A}\times\vec{B}) = \vec{B}\cdot\operatorname{curl}\vec{A} - \vec{A}\cdot\operatorname{curl}\vec{B}]$

Using the relations $\vec{B}=\mu\vec{H}$ and $\vec{D}=\varepsilon\vec{E}$, we can get

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}=\vec{E}\cdot\frac{\partial}{\partial t}(\varepsilon\vec{E})=\frac{1}{2}\varepsilon\frac{\partial}{\partial t}(E)^2=\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{E}\cdot\vec{D}\right) \qquad [\because E^2=\vec{E}\cdot\vec{E}]$$

$$\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}=\vec{H}\cdot\frac{\partial}{\partial t}(\mu\vec{H})=\frac{1}{2}\mu\frac{\partial}{\partial t}(H)^2=\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{H}\cdot\vec{B}\right) \qquad [\because H^2=\vec{H}\cdot\vec{H}]$$

Now Eq. (vii) can be written as

$$\operatorname{div}(\vec{E}\times\vec{H})=\frac{\partial}{\partial t}\left[\frac{1}{2}(\vec{H}\cdot\vec{B}+\vec{E}\cdot\vec{D})\right]-\vec{E}\cdot\vec{J}$$

or $$\vec{E}\cdot\vec{J}=\frac{\partial}{\partial t}\left[\frac{1}{2}(\vec{H}\cdot\vec{B}+\vec{E}\cdot\vec{D})\right]-\operatorname{div}(\vec{E}\times\vec{H}) \tag{viii}$$

Integrating Eq. (viii) over a volume $V$ enclosed by a surface $S$, we get

$$\int_v \vec{E}\cdot\vec{J}dV=-\int_v\left[\frac{\partial}{\partial t}\left\{\frac{1}{2}(\vec{H}\cdot\vec{B}+\vec{E}\cdot\vec{D})\right\}\right]dV-\int_v \operatorname{div}(\vec{E}\times\vec{H})dV$$

or $$\int_v \vec{E}\cdot\vec{J}dV=-\frac{\partial}{\partial t}\int_v\left[\left(\frac{1}{2}\mu H^2+\frac{1}{2}\varepsilon E^2\right)\right]dV-\oint_s(\vec{E}\times\vec{H})\cdot d\vec{S} \tag{ix}$$

$[\because \quad \vec{B}=\mu\vec{H}, \vec{D}=\varepsilon\vec{E} \quad \text{and} \quad \int_v \operatorname{div}(\vec{E}\times\vec{H})dV=\oint_s(\vec{E}\times\vec{H})\cdot d\vec{S}]$

Eq. (ix) can also be written as

$$\int_v \vec{E}\cdot\vec{J}\,dV=-\frac{\partial}{\partial t}\int_v\left[\frac{1}{2}\mu H^2+\frac{1}{2}\varepsilon E^2\right]dV-\oint_s(\vec{E}\times\vec{H})\cdot d\vec{S} \tag{x}$$

### *Interpretation*

(a) $\int_v(\vec{E}\cdot\vec{J})dV$ : This term represents the rate of energy transferred into the electromagnetic field through the motion of charges in the volume $V$, i.e., the total power dissipated in a volume $V$.

(b) $\frac{\partial}{\partial t}\int_v\left[\frac{1}{2}\mu H^2+\frac{1}{2}\varepsilon E^2\right]dV$ : The terms $\frac{1}{2}\mu H^2$ and $\frac{1}{2}\varepsilon E^2$ represent the energy stored in electric and magnetic fields respectively and their sum will be equal to the total energy stored in electromagnetic field. Therefore, this total expression represents the rate of decrease of energy stored in volume $V$ due to electric and magnetic fields.

(c) $\int_s(\vec{E}\times\vec{H})\cdot d\vec{S}$ : This term represents the amount of electromagnetic energy crossing the closed surface per second or the rate of flow of outward energy through the surface $S$ enclosing volume $V$. The vector $(\vec{E}\times\vec{H})$ is known as the *poynting vector* $\vec{S}$ or $\vec{S}=(\vec{E}\times\vec{H})$.

Equation ($x$) is also known as *poynting theorem* or *work-energy theorem*, according to which the power transferred into the electromagnetic field is equal to the sum of the time rate of change of electromagnetic energy within a certain volume and the time rate of the energy flowing out through the boundary surface. This is also called as the *energy conservation* law in electromagnetism.

## 11.29 WAVE PROPAGATION IN BOUNDED SYSTEM: WAVEGUIDE LO12

The first waveguide was proposed by *Thomson* in 1893 and was experimentally verified by *Lodge* in 1894. A structure that can guide waves like electromagnetic waves, light waves or sound waves is called *waveguide*. For each type of the wave there are different types of waveguides. For example, depending on the frequency of electromagnetic wave the waveguide can be constructed from either conductive or dielectric material. Such electromagnetic waveguides are especially useful in the microwave and optical frequency ranges. The waveguides used at optical frequencies (optical waveguides) are typically dielectric waveguides. In such waveguides, a dielectric material with high permittivity (i.e., high index of refraction) is surrounded by a material with lower permittivity. This type of structure guides optical waves by the process of total internal reflection. The most common optical waveguide is optical fibre. On the other hand, a structure that guides sound waves is called an *acoustic waveguide*. A duct for sound propagation also behaves as a transmission line. The duct contains some medium like air which supports the propagation of sound waves.

### 11.29.1 Electromagnetic Waveguides

The original and the most common meaning of waveguide is a hollow metal pipe used for guiding the waves. The electromagnetic waves in such waveguides may be imagined as waves travelling down the guide in a zig zag path as these waves are repeatedly reflected between opposite walls of the guide (for example, rectangular waveguide). The first mathematical analysis of the propagating modes (waves) within a hollow metal cylinder was performed by *Rayleigh* in 1897.

To function properly, a waveguide must have a certain minimum diameter relative to the wavelength of the signal. If the waveguide is too narrow or the frequency is too low (the wavelength is too long), the electromagnetic field cannot propagate. There is a minimum frequency, known as *cutoff frequency* for the propagation of the wave, i.e., a wave can propagate only if its frequency is larger than the cutoff frequency. The cutoff frequency is decided by the dimensions of the waveguide.

### 11.29.2 Modes in Waveguides

In order to analyse the mode (wave) propagation in the waveguide, we solve the Maxwell's equations along with appropriate bounding conditions determined by the properties of the materials and their interfaces. These equations admit multiple solutions, or modes, which are origin functions of the equation system. The propagation of the waveguide modes depends on the operating wavelength and polarization, and shape and size of the waveguide. The longitudinal mode can be realised in a cavity (closed end waveguide), the longitudinal mode is particularly standing wave pattern formed by the waves confined in the cavity. However, a number of transverse modes can be excited in the waveguide, which are classified below.

(1) ***Transverse Electric (TE) Modes:*** These modes do not have electric field in the direction of propagation. So electric field vector is in transverse direction.

(2) ***Transverse Magnetic (TM) Modes:*** These modes have no magnetic field in the direction of propagation. So magnetic field vector is in transverse direction.

(3) ***Transverse Electromagnetic (TEM) Modes:*** These modes have no electric and magnetic fields in the direction of mode propagation. In hollow waveguides, TEM modes are not possible because as per Maxwell's equation the electric field then must have zero divergence, zero curl and be zero at the boundaries. This will result in a zero field or $\nabla^2\vec{E} = 0$. However, TEM modes can propagate in a coaxial cable.

(4) ***Hybrid Modes:*** These modes have both electric and magnetic field components in the direction of propagation. The mode for which the cutoff frequency is the minimum is called the *fundamental mode*. For example, Transverse Electric $TE_{10}$ mode is the fundamental mode for rectangular waveguide whereas $TE_{11}$ mode is the fundamental mode for circular waveguide.

## 11.30 COAXIAL CABLE LO12

A transmission line is the material medium that forms a path from one place to another for directing the transmission of energy like electromagnetic waves, acoustic waves or electric power. Open wire transmission lines have the property that the electromagnetic wave propagating down the line extends into the space surrounding the parallel wires. The transmission lines cannot be bent, twisted or otherwise shaped without changing their characteristic impedance. They also cannot be attached to anything conductive, as the extended fields will induce currents in the nearby conductors. This will cause unwanted radiation and detuning of the line. However, this problem is solved by coaxial lines that confine the electromagnetic wave to the area inside the cable, as the coaxial cable consists of a round conducting wire surrounded by an insulating spaces (called the dielectric), surrounded by a cylindrical conducting sheath that is usually surrounded by a final insulating layer (called jacket). Here the transmission of energy occurs totally through the dielectric inside the cable between the conductors. Coaxial lines (cables) can therefore be bent and moderately twisted without negative effects. Also, they can be strapped to conductive supports without inducing unwanted currents in them. Coaxial cable is used as a high-frequency transmission line to carry a high frequency or broadband signals. Coaxial cables may be flexible or rigid depending on the type of sheath. Rigid types have a solid sheath, whereas flexible types have an interwoven sheath usually of thin copper wire. The inner insulator has a significant effect on the cable's properties such as its characteristic impedance and its attenuation. The characteristic impedance in ohms is calculated from the ratio of the inner diameter $d$ and outer diameter $D$ of the dielectric and the dielectric constant $\varepsilon_r$ as below

$$Z_0 = \frac{138}{\sqrt{\varepsilon_r}} \log\left(\frac{D}{d}\right)$$

The most common impedances that are widely used are 50 or 52 Ω for industrial and commercial radio frequency applications, and 75 Ω for domestic television and radio, although other impedances are available for specified applications.

### 11.30.1 Signal Propagation in Coaxial Cables

In radio-frequency applications up to a few GHz ($10^9$Hz) the wave propagates only in the transverse electromagnetic (TEM) mode which means the electric and magnetic fields are both perpendicular to the direction of propagation. However, above a certain cutoff frequency, transverse electric ($TE$) and/or transverse magnetic ($TM$) modes can also propagate, as they do in a waveguide. It is usually undesirable to transmit signals above the cutoff frequency, since it may cause multiple modes with different phase velocities to propagate, interfering with each other. This cutoff frequency is roughly inversely proportional to the outer diameter.