Thursday, October 21, 2021 12:09 PM

Theorems: Undirected Graphs

Theorem 1

The Handshaking theorem:

$$2e = \sum_{v \in V} \deg(v)$$

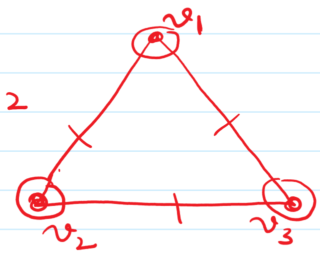
(why?) Every edge connects 2 vertices

Hand-shaking theorem.

$$\sum_{i=1}^{n} \deg(v_i) = 2e.$$

$\deg(v_1) = 2$, $\deg(v_2) = 2$

$\deg(v_3) = 2$

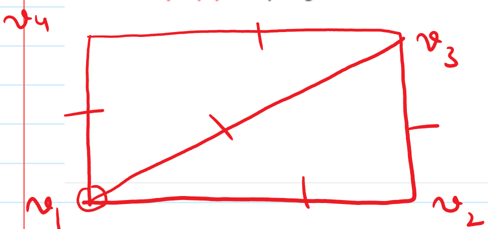


$\deg(v_1) + \deg(v_2) + \deg(v_3)$

$= 2+2+2 = 6 = 2 \times 3$

$\sum_{i=1}^{3} \deg(v_i) = 2 \times$ no. of edges

$= 2 \times e$



Verify Hand-shaking for the above graph

$\deg(v_1) = 3$, $\deg(v_2) = 2$, $\deg(v_3) = 3$, $\deg(v_4) = 2$

$\deg(v_1) + \deg(v_2) + \deg(v_3) + \deg(v_4)$

$= 3 + 2 + 3 + 2$

$= 10 = 2 \times 5 = 2e$

Theorems: Undirected Graphs

Theorem 2:

An undirected graph has even number of vertices with odd degree

Proof V1 is the set of even degree vertices and V2 refers to odd degree vertice

$2e = \sum_{v \in V} \deg(v) = \sum_{u \in V_1} \deg(u) + \sum_{v \in V_2} \deg(v)$

$\Rightarrow \deg(v)$ is even for $v \in V_1$,

$\Rightarrow$ The first term in the right hand side of the last inequality is even.

$\Rightarrow$ The sum of the last two terms on the right hand side of the last inequality is even since sum is 2e.

Hence second term is also even

$\Rightarrow$ second term $\sum_{v \in V_2} \deg(u) = even$

$2+4+6 = $ even

Q No. of odd degree vertices in any graph are always even.

Sol^n Suppose that in any graph G there are n vertices and e edges.

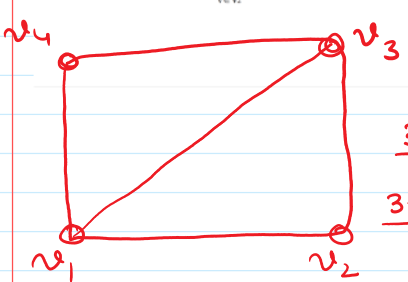
$\therefore$ By Hand-shaking theorem.

$$\sum_{i=1}^{n} \deg(v_i) = 2e$$

$$\sum_{v_i \in odd} \deg(v_i) + \sum_{v_j \in even} \deg(v_j) = 2e.$$

$$\sum_{v_i \in odd} \deg(v_i) + even = even$$

$$\sum_{v_i \in odd} \deg(v_i) = even - even = even.$$



$3+5+7 = 15$

$3+5+7+9 = 24$

$V_1 = \{v_1, v_3\}$ ✓

$V_2 = \{v_2, v_4\}$ ✓

This situation is possible only when no of odd degree ~~vertices are even~~.

Is it possible to have a graph with the following degree of vertices: 1, 2, 3, 4, 5.

1 odd ✓, 3 odd ✓, 5 odd ✓

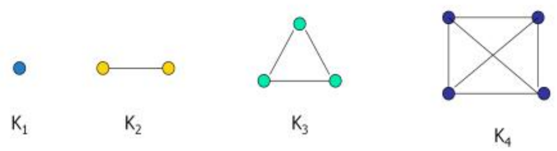
~~This is not possible.~~

of vertices: odd odd odd — This is not possible.

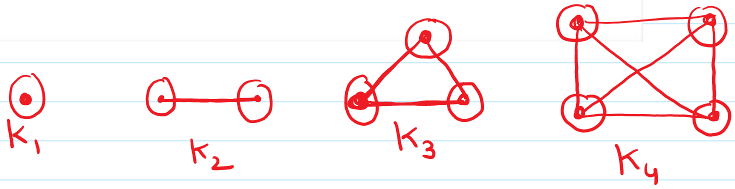
Simple graphs – special cases

- **Complete graph:** K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

Representation Example: K_1, K_2, K_3, K_4



Complete Graph: If $\exists$ an edge between every pair of vertices then that graph is called a complete graph. A complete graph is denoted by $K_{\boxed{n}}$. Where n represents no. of vertices

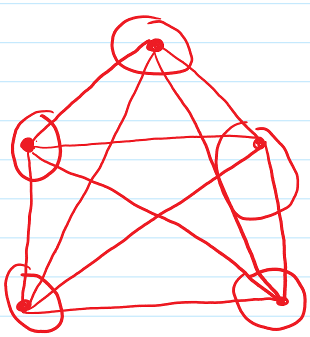


In K_n graph degree of Each vertex $= \underline{n-1}$

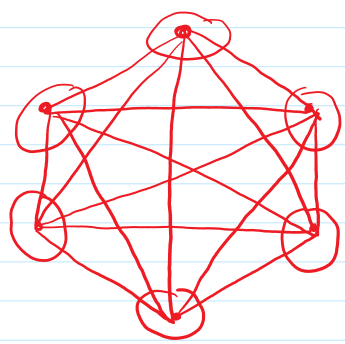
In K_{100} graph what is degree of Each vertex?

(a) 98 (b) 99 (c) 100 (d) 101

Q Draw K_5 and K_6 graph.



K_5



K_6

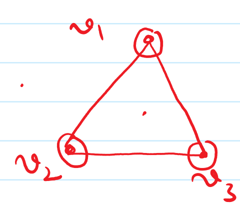
Simple graphs – special cases

- **Cycle:** C_n, $n \geq 3$ consists of n vertices $v_1, v_2, v_3 \ldots v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\} \ldots \{v_{n-1}, v_n\}, \{v_n, v_1\}$

Representation Example: C_3, C_4

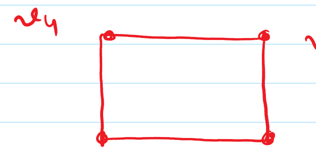
Cycle C_n, $n \geq 3$

$\{v_1, v_2\}, \{v_2, v_3\}, \ldots \{v_{n-1}, v_n\}, \{v_n, v_1\}$



C_3

$deg(v_i) = 2$



C_4

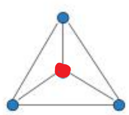
$deg(v_i) = 2$

Draw C_5, C_6, C_7 & C_8 cycle.

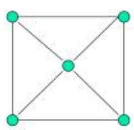
Simple graphs – special cases

- **Wheels:** W_n, obtained by adding additional vertex to Cn and connecting all vertices to this new vertex by new edges.

Representation Example: W_3, W_4

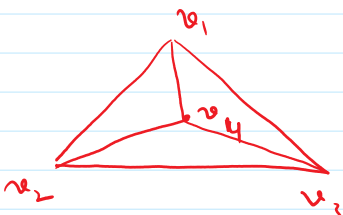


W_3

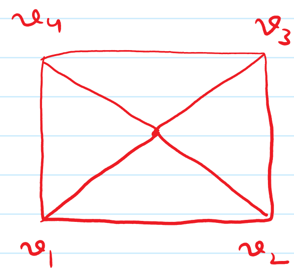


W_4

Wheel : W_n



W_3



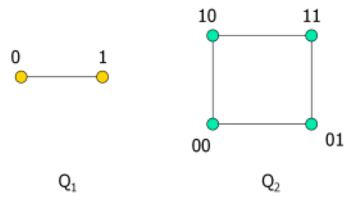
W_4

Draw W_5, W_6, W_7, W_8.

Simple graphs – special cases

- **N-cubes:** Q_n, vertices represented by 2n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ by exactly one bit positions

  Representation Example: Q_1, Q_2

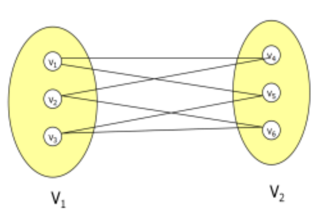


Bipartite graphs

- In a simple graph G, if V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2)

  Application example: Representing Relations

  Representation example: $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6\}$,



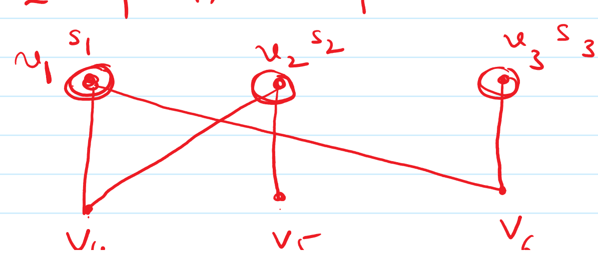
Bi-partite graph.

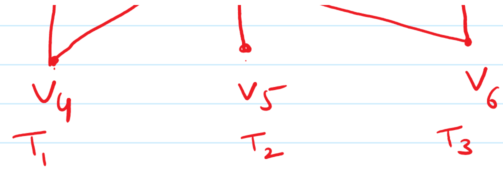
Two partitions of the vertex set

$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

$V_1 = \{v_1, v_2, v_3\}$ $\quad V_1 \cap V_2 = \varphi$

$V_2 = \{v_4, v_5, v_6\}$ $\quad V_1 \cup V_2 = V$





Complete Bipartite graphs

- $K_{m,n}$ is the graph that has its vertex set portioned into two subsets of m and n vertices, respectively There is an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.

Representation example: $K_{2,3}$, $K_{3,3}$

$K_{2,3}$ $\qquad$ $K_{3,3}$

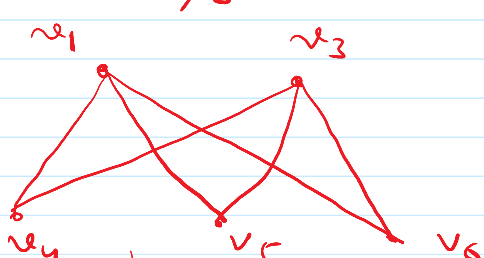
$K_{2,3}$

$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

$V_1 = \{v_1, v_2, v_3\}$ ✓

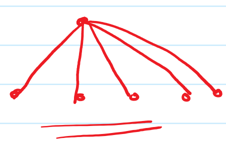
$V_2 = \{v_4, v_5, v_6\}$ ✓

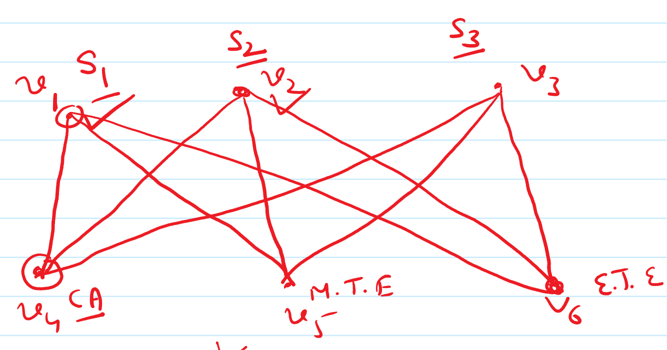
$V_1 \cap V_2 = \phi$, $\quad V_1 \cup V_2 = V$



$K_{2,3}$

$K_{1,5}$





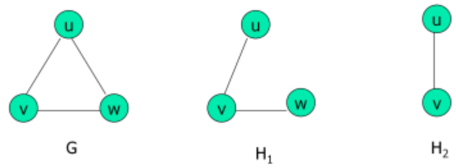
$K_{3,3}$

Subgraphs

- A subgraph of a graph G = (V, E) is a graph H =(V', E') where V' is a subset of V and E' is a subset of E

  Application example: solving sub-problems within a graph

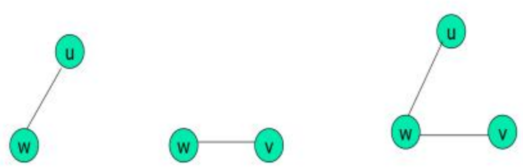
  Representation example: V = {u, v, w}, E = ({u, v}, {v, w}, {w, u}), H_1, H_2



Subgraphs

- G = G1 U G2 wherein E = E1 U E2 and V = V1 U V2, G, G1 and G2 are simple graphs of G

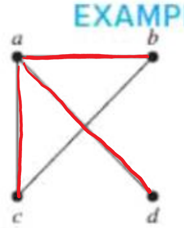
Representation example: V1 = {u, w}, E1 = {{u, w}}, V2 = {w, v}, E1 = {{w, v}}, V = {u, v ,w}, E = {{{u, w}, {{w, v}}



G1 G2 G

AdJacency matrix

EXAMPLE Use an adjacency matrix to represent the graph shown in Figure .



A simple graph.

Solution: We order the vertices as a, b, c, d. The matrix representing this graph is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

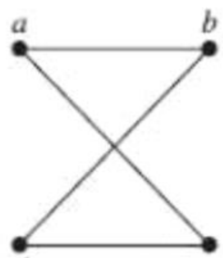
$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

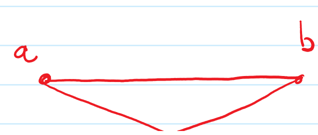
	a	b	c	d
a	0	1	1	1
b	1	0	1	0
c	1	1	0	0
d	1	0	0	0

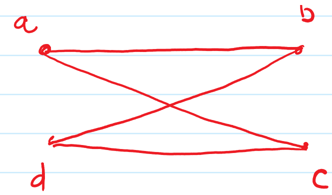
EXAMPLE Draw a graph with the adjacency matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

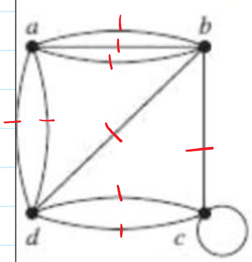
with respect to the ordering of vertices a, b, c, d.







EXAMPLE Use an adjacency matrix to represent the pseudograph shown.



Solution: The adjacency matrix using the ordering of vertices a, b, c, d is

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}.$$

	a	b	c	d
a	0	3	0	2
b	3	0	1	1
c	0	1	1	2
d	2	1	2	0

EXAMPLE Represent the graph shown in Figure with an incidence matrix.

Solution: The incidence matrix is

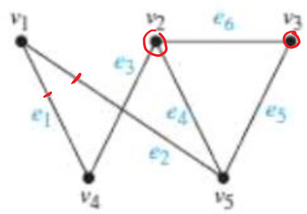


FIGURE **An undirected graph.**

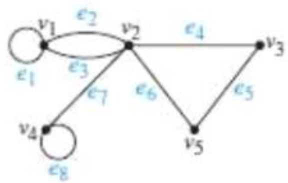
$$
\begin{array}{c}
\\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\left[\begin{array}{cccccc}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0
\end{array}\right].
\end{array}
$$

Incidence matrix may or may not be a square matrix.

	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	0	0	1	1	0	1
v_3	0	0	0	0	1	1
v_4	1	0	1	0	0	0
v_5	0	1	0	1	1	0

EXAMPLE Represent the pseudograph shown in Figure using an incidence matrix.

Solution: The incidence matrix for this graph is

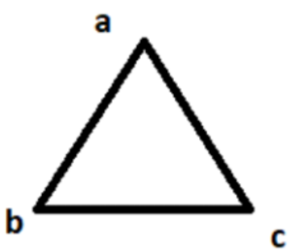


$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

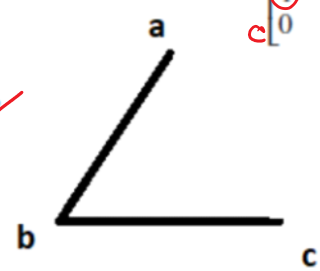
A pseudograph.

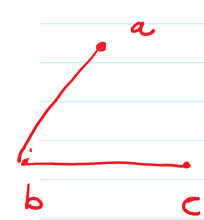
Quiz 2 : Which of the following graph has the adjacency matrix $\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$ **?**

A.



B. ✓

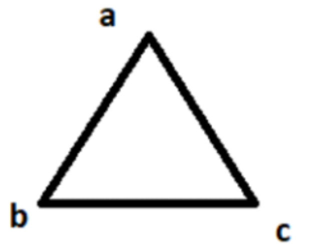




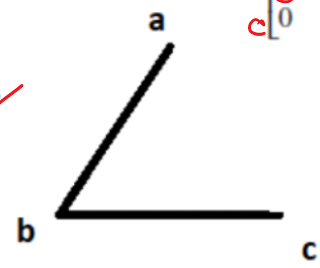
a

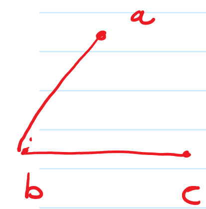
Quiz 2 : Which of the following graph has the adjacency matrix $\begin{array}{c} a \\ b \\ c \end{array}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **?**

A.

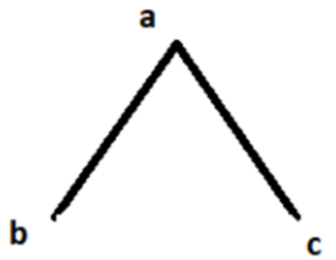


B. ✓

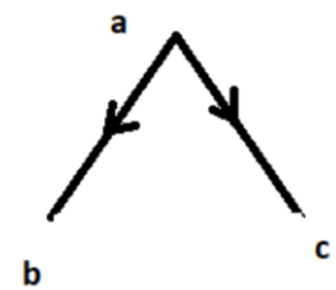


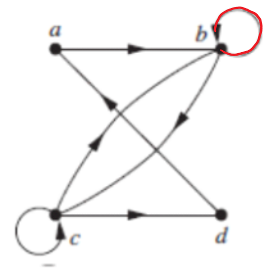


C.



D.



Quiz 3 : Which of the following adjacency matrix represents  **?**

A. $\begin{pmatrix} 1 & 1 & 1 & 0 \\ \boxed{0 \;\; 1 \;\; 0 \;\; 1} \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ ×

C. $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ ✓

B. $\begin{pmatrix} 1 & 1 & 1 & 0 \\ \boxed{0 \;\; 1 \;\; 0 \;\; 1} \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ ×

D. $\begin{pmatrix} 0 & 1 & 0 & 0 \\ \boxed{0 \;\; 1 \;\; 0 \;\; 0} \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ ×

	a	b	c	d
a	0	1	0	0
b	0	1	1	0
c				
d				

$$\alpha_n + \alpha_{n-1} + \alpha_{n-2} + \alpha_{n-3} = 0$$

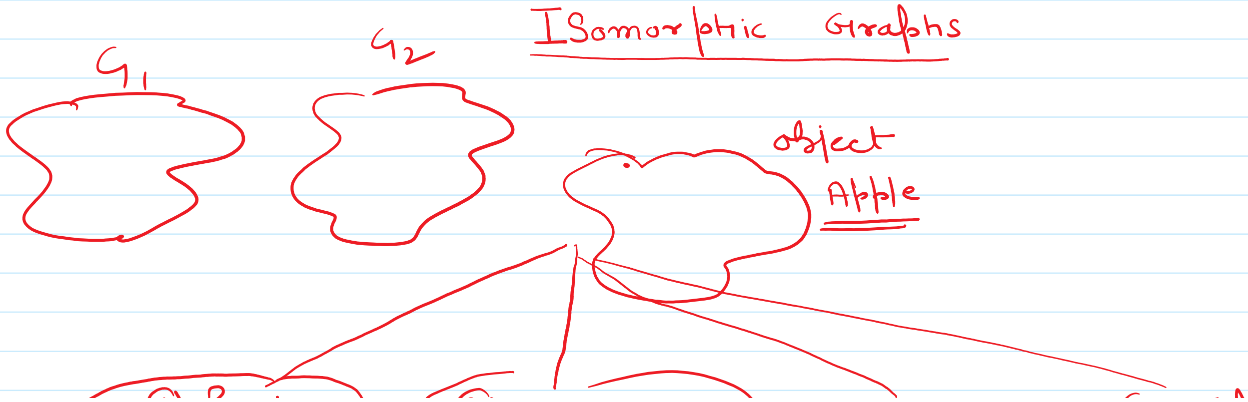
$$\alpha^3 + \alpha^2 + \alpha + 1 = 0$$

$$\alpha_n - \alpha_{n-(k-1)} - \alpha_{n-k} = 0$$

Its order $= n - (n-k) = k$

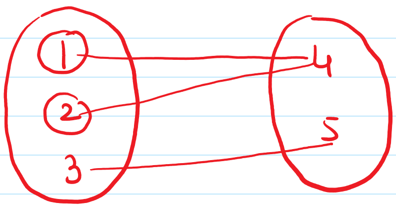
$$\boxed{\alpha^k - \alpha - 1 = 0}$$

Isomorphic Graphs



① Punjab ② Tamilnadu ③ China ④ Japan.

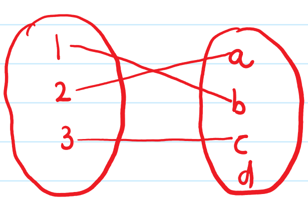
One-one mapping: Let X and Y be two non-empty sets. Then a function $f: X \to Y$ is called 1-1 function if $x_1, x_2 \in D_f$ s.t $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$



$1 \neq 2$

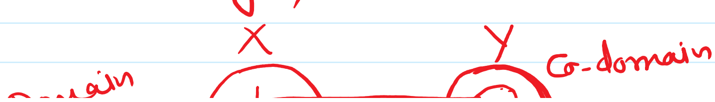
$\boxed{f(1) = f(2) = 4}$

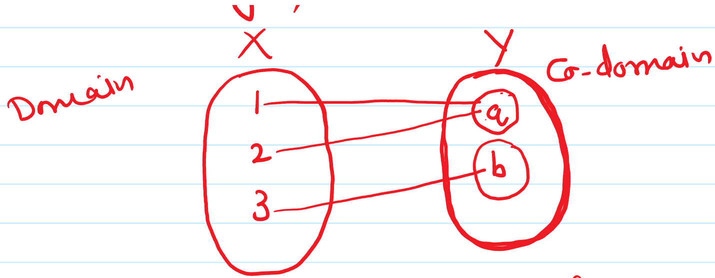
This function is not a 1-1 function / injective function.

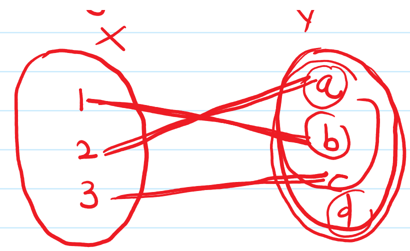


This function is a 1-1 function.

Onto function: Let X and Y be two non-empty sets. Then a function $f: X \to Y$ is called an onto function if for each $y \in Y$ if $x \in X$ s.t $f(x) = y$







This function is an onto function.

Range = {a, b}, Codomain = {a, b}

$\boxed{\text{Range} = \text{Codomain}}$

This function is not onto function.

Range = {a, b, c}

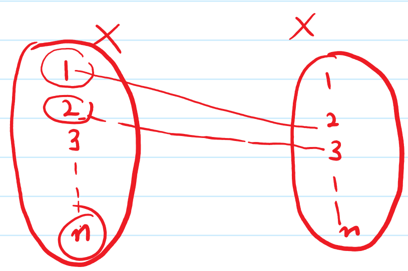
Codomain = {a, b, c, d}

$\boxed{\text{Range} \subseteq \text{Codomain}}$

Bijective function: A function $f: X \to X$ is called bijective function if it is both 1-1 and onto.

Total no. of bijective functions:

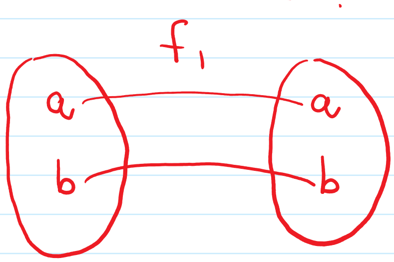
If X contains n elements then total no of bijections $= n!$

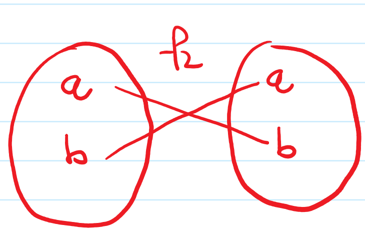


$n \times (n-1) \times (n-2) \times \cdots \times 1$

$= n!$

How many bijections are possible for a set X having two elements {a, b}. $2! = 2 \times 1 = 2$





$3! = 6$