L29 27 March 2021

27 March 2021 11:05

D'Alembert's Solution of Infinitely long wave (string)

Let given wave equation be: $\boxed{\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}}$ (1)

such that $-\infty < x < \infty, t > 0$

with initial displacement $= f(x)$ and initial velocity $= g(x)$

Then. D' Alembert's solution of equation (1) is given by:

$$u(x,t) = \frac{1}{2}[f(x+Ct) + f(x-Ct)] + \frac{1}{2C}\int_{x-Ct}^{x+Ct} g(s)ds$$

$u(x,0) = f(x)$

$\frac{\partial u}{\partial t}(x,0) = g(x)$

$u(x,t)$

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

The problem by change of variable

Consider $\boxed{\begin{aligned}\xi &= x+ct \\ \eta &= x-ct\end{aligned}}$ so that u becomes a function of ξ and η

$\frac{\partial \xi}{\partial x} = 1$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi}\cdot 1 + \frac{\partial u}{\partial \eta}\cdot 1$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial \eta}{\partial x} =$$

$$\boxed{\frac{\partial^2 u}{\partial x^2}} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\cdot 1 + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) = \boxed{\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial t} = \frac{\partial u}{\partial \xi}c + \frac{\partial u}{\partial \eta}(-c) = c\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)$$

$$\frac{\partial^2 u}{\partial t^2} = c\left[\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\cdot\underbrace{\frac{\partial \xi}{\partial t}}_{c} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\cdot\underbrace{\frac{\partial \eta}{\partial t}}_{-c}\right]$$

$$= c^2\left[\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2}\right]$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right]$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right]}$$

Equation ①

$$\cancel{c^2}\left[\cancel{\frac{\partial^2 u}{\partial \xi^2}} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \cancel{\frac{\partial^2 u}{\partial \eta^2}}\right] = \cancel{c^2}\left[\cancel{\frac{\partial^2 u}{\partial \xi^2}} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \cancel{\frac{\partial^2 u}{\partial \eta^2}}\right]$$

$$4\cdot\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial \xi \partial \eta} = 0} \quad \text{or} \quad \frac{\partial^2 u}{\partial \eta \partial \xi} = 0$$

Integrate w.r.t. x

$$\int \frac{\partial^2 u}{\partial x \partial y} = \int 0$$

$$\frac{\partial u}{\partial y} = f(y)$$

Integrate w.r.t. η $\qquad \frac{\partial u}{\partial \xi} = h(\xi)$

Integrate w.r.t. ξ

$$u(\xi,\eta) = \underbrace{\int h(\xi)\cdot d\xi}_{\phi(\xi)} + \psi(\eta)$$

$$\boxed{u(\xi,\eta) = \phi(\xi) + \psi(\eta)}$$

$$\boxed{u(x,t) = \phi(x+ct) + \psi(x-ct)}$$

→ General solution 7 (1e wave eqn.)

$u(x,0) = f(x)$	$\Big\}$
$\frac{\partial u}{\partial t}(x,0) = g(x)$	

$$u(x,0) = \phi(x) + \psi(x)$$

$$\boxed{f(x) = \phi(x) + \psi(x)} \longrightarrow ②$$

$$\frac{\partial u}{\partial t} = c\phi'(x+ct) + \psi'(x-ct)(-c)$$

$\frac{\partial u}{\partial t}(x,0) = g(x)$;

$$\frac{\partial u}{\partial t} = c\,\phi'(x+ct) + \psi'(x-ct)(-c)$$

$$\frac{\partial u}{\partial t} = c\left[\phi'(x+ct) - \psi'(x-ct)\right]$$

$$\boxed{g(x) = c\left[\phi'(x) - \psi'(x)\right]} \quad \text{— } ③$$

② $\Rightarrow$ $\qquad f'(x) = \phi'(x) + \psi'(x)$

③ $\Rightarrow$ $\qquad \frac{1}{c}\,g(x) = \phi'(x) - \psi'(x)$

Add. $\qquad f'(x) + \frac{1}{c}\,g(x) = 2\,\phi'(x)$

$$\boxed{\phi'(x) = \frac{1}{2}f'(x) + \frac{1}{2c}\,g(x)} \quad \text{— } ④$$

Ily $\qquad f'(x) - \frac{1}{c}\,g(x) = 2\,\psi'(x)$

$$\boxed{\psi'(x) = \frac{1}{2}f'(x) - \frac{1}{2c}\,g(x)} \quad \text{— } ⑤$$

Integrate ④

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds + K$$

Integr ⑤

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds - K$$

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

$$= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds + \cancel{K} + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)\,ds \cancel{-K}$$

$x+ct$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$

Find the De Alembert sol. of wave eqn with conditions

$f(x) = \sin x, \quad g(x) = a$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

$$= \frac{1}{2}\left[\sin(x+ct) + \sin(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} a\,ds$$

$$= \frac{1}{2}\left[\sin x\cos ct + \cancel{\cos x\sin ct} + \sin x\cos ct - \cancel{\cos x\sin ct}\right] + \frac{1}{2c}a\left[\cancel{x}+ct-\cancel{x}+ct\right]$$

$$= \frac{1}{\cancel{2}}\left[\cancel{2}\sin x\cos ct\right] + \frac{1}{\cancel{2c}}a\,\cancel{2c}t$$

$$\boxed{u(x,t) = \sin x\cos ct + at}$$

$f(x) = 0, g(x) = \cos x$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \cos s\,ds = \frac{1}{2c}\left[\sin s\right]_{x-ct}^{x+ct}$$

$$u(x,t) = \frac{1}{2c}\int_{x-ct} \cos s\, ds = \frac{1}{2c}\Big[\sin s\Big]_{x-ct}$$

$$= \frac{1}{2c}\Big[\sin(x+ct) - \sin(x-ct)\Big]$$

$$= \frac{1}{2c}\Big[\cancel{\sin x\cos ct} + \cos x\sin ct - \cancel{\sin x\cos ct} + \cos x\sin ct\Big]$$

$$= \frac{1}{\cancel{2}c}\Big[\cancel{2}\cos x\sin ct\Big] = \frac{1}{c}\cos x\sin ct$$