L44 25 April 2021

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Area of the region $R = \frac{1}{2}\oint_C x\,dy - y\,dx = -\oint_C y\,dx = \oint_C x\,dy$

$\oint_C x\,dy - y\,dx = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = \iint_R (1+1)\,dx\,dy$

$f = -y \quad \frac{\partial f}{\partial y} = -1$

$g = x \quad \frac{\partial g}{\partial x} = 1$

$$\oint_C x\,dy - y\,dx = 2\iint_R dx\,dy$$

$$\Rightarrow \underbrace{\iint_R dx\,dy}_{\text{Area of the region } R.} = \frac{1}{2}\oint_C x\,dy - y\,dx$$

$\oint_C -y\,dx = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = \iint_R (0+1)\,dx\,dy$

$f = -y \quad \frac{\partial f}{\partial y} = -1$

$g = 0 \quad \frac{\partial g}{\partial x} = 0$

$$\underbrace{\iint_R dx\,dy}_{\text{Area}} = -\oint_C y\,dx$$

If C is a simple closed curve in the x-y plane not including origin. Find

$$\oint_C \vec{F}\cdot d\vec{s} \quad \text{where} \quad \vec{F} = \frac{y\hat{i} - x\hat{j}}{x^2+y^2}$$

$\vec{s} = x\hat{i} + y\hat{j}$

$d\vec{s} = dx\,\hat{i} + dy\,\hat{j}$

$$\vec{F}\cdot d\vec{s} = \frac{y\,dx - x\,dy}{x^2+y^2}$$

$$\bar{F}\cdot d\bar{r} = \frac{y\,dx - x\,dy}{x^2+y^2}$$

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C \frac{y\,dx - x\,dy}{x^2+y^2} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = \iint 0\,dx\,dy$$

$$= 0$$

$$f = \frac{y}{x^2+y^2} \qquad g = \frac{-x}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = y\cdot\left[\frac{-2y}{(x^2+y^2)^2}\right] + \frac{1}{x^2+y^2} \qquad \frac{\partial g}{\partial x} = -x\cdot\left[\frac{-2x}{(x^2+y^2)}\right] + \frac{1(-1)}{x^2+y^2}$$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = \frac{+2x^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2}$$

$$= 2(x^2+y^2$$

$$g = \frac{-x}{x^2+y^2} = \left[x\cdot\frac{-2x}{(x^2+y^2)^2} + \frac{1}{x^2+y^2}\right]$$

$$\frac{1}{x} = -\frac{1}{x^2}$$

$$= \frac{2x^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} = \frac{2x^2-(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$f = \frac{y}{x^2+y^2} = y\cdot\left[-\frac{1}{(x^2+y^2)^2}\cdot 2y\right] + \frac{1}{(x^2+y^2)}$$

$$= \frac{-2y^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} = \frac{-2y^2+x^2+y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

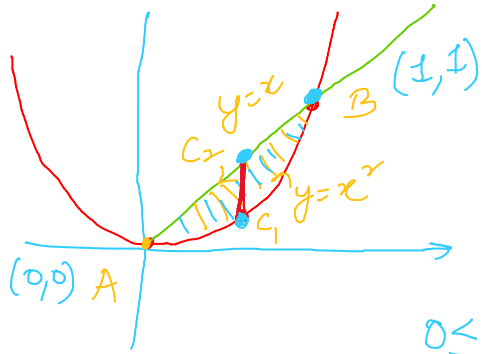
$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = \cancel{\frac{x^2-y^2}{(x^2+y^2)^2}} - \cancel{\frac{(x^2-y^2)}{(x^2+y^2)^2}} = 0$$

~~verify~~ Green's Theorem

$\oint_C (xy+y^2)\,dx + x^2\,dy$ over the closed curve C bounded by $y=x$ and $y=x^2$

$$\oint_C (xy+y^2)\,dx + x^2\,dy$$

$$= \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$



$y=x$
$y=x^2$

$x^2=x$
$x^2-x=0$
$x(x-1)=0$
$x=0,1$

$0 \le x \le 1$

$x^2 \le y \le x$

$f = xy+y^2 \qquad \frac{\partial f}{\partial y} = x+2y$

$g = x^2 \qquad \frac{\partial g}{\partial x} = 2x$

$$\int_0^1 \int_{x^2}^{x} (2x-x-2y)\,dy\,dx = \int_0^1 \int_{x^2}^{x} (x-2y)\,dy\,dx$$

$$= \int_0^1 \left| xy - \frac{\cancel{2}y^2}{\cancel{2}} \right|_{x^2}^{x} dx = \int_0^1 \left[(\cancel{x^2} - \cancel{x^2}) - (x^3 - x^4) \right] dx$$

$$= \int_0^1 (x^4 - x^3)\,dx = \left| \frac{x^5}{5} - \frac{x^4}{4} \right|_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{-1}{20} \quad \text{Ans}$$

$\oint_C y^2\,dx + x^2\,dy$, C is the circle $(x+1)^2 + (y-2)^2 = 16$

centre $(-1,2)$

$f = y^2 \quad \frac{\partial f}{\partial y} = 2y$

$g = x^2 \quad \frac{\partial g}{\partial x} = 2x$

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$$= \iint_R (2x - 2y)\, dx\,dy$$

$$\iint_R \left[2(X-1) - 2(Y+2)\right] dX\,dY$$

$$\iint_R \left[2X - 2 - 2Y - 4\right] dX\,dY$$

$$\iint_R \left[2(X-Y) - 6\right] dX\,dY$$

$\vdots \quad ?$

Centre $(-1, 2)$

rad $= 4$

$x+1 = r\cos\theta$

$y-2 = r\sin\theta$?

$x+1 = X$

$y-2 = Y$

$dx = dX$

$dy = dY$

$X = r\cos\theta$

$Y = r\sin\theta$

$0 \le r \le 4$

$0 \le \theta \le 2\pi$