Chapter 5

Linear Differential Equations

5.1 Introduction

Linear differential equations occur in the study of many practical problems in science and engineering. Constant coefficient equations arise in the theory of electric circuits, vibrations etc. Variable coefficient equations arise in many areas of physics, electric circuits, mathematical modelling of physical problems etc. Some of the important variable coefficient differential equations are Bessel equation, Legendre equation, Chebyshev equation etc. The solution of constant coefficient equations can be obtained in terms of known standard functions. However, no such solution procedure exists for variable coefficient equations. Often, we attempt their solution in the form of an infinite series. These solutions may sometimes reduce to known standard functions.

A linear ordinary differential equation of order n, is written as

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = r(x)$$

or

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \ldots + a_{n-1}(x)y'(x) + a_n(x)y(x) = r(x) \tag{5.1}$$

where y is the dependent variable and x is the independent variable and $a_0(x) \neq 0$. If $r(x) = 0$, then it is called a *homogeneous equation*, otherwise it is called a *non-homogeneous equation*. For example, a second order homogeneous equation is of the form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \quad a_0(x) \neq 0 \tag{5.2}$$

and a non-homogeneous second order equation is of the form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x), \quad a_0(x) \neq 0. \tag{5.3}$$

If $a_i(x)$, $i = 0, 1, 2$ are constants then the equations are linear second order constant coefficient equations. A few examples of linear second order equations are

$$y'' + 4y' + 3y = x^2e^x, \tag{5.4}$$

$$y'' + 2y' + y = \sin x, \tag{5.5}$$

$$x^2y'' + xy' + (x^2 - 4)y = 0, \tag{5.6}$$

$$(1 - x^2)y'' - 2xy' + 20y = 0. \tag{5.7}$$

Equations (5.4), (5.5) are constant coefficient second order equations and Eqs. (5.6), (5.7) are variable coefficient second order equations.

5.2 Solutions of Linear Differential Equations

We assume that x in Eq. (5.1) varies on some interval I, where the interval may be open, closed, semi-open or infinite. For example, the differential equation may be valid for all $x \in (0, \infty)$ or $x \in (-\infty, \infty)$. If $y_1(x)$ is a solution of the Eq. (5.1), then it must identically satisfy the equation. Hence, $y_1(x)$ must be continuously differentiable $n - 1$ times and $y_1^{(n)}(x)$ must be continuous on I.

We now state an important result regarding the uniqueness of solutions.

Theorem 5.1 If the functions $a_0(x), a_1(x), \ldots, a_n(x)$ and $r(x)$ are continuous over I and $a_0(x) \neq 0$ on I, then there exists a unique solution to the initial value problem

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = r(x) \tag{5.8}$$

$$y(x_0) = c_1,\ y'(x_0) = c_2,\ \ldots,\ y^{(n-1)}(x_0) = c_n \tag{5.9}$$

where $x_0 \in I$, and $c_1, c_2, \ldots, c_n$ are n known constants.

This theorem does not give us a procedure to find the solutions but guarantees that there exists a unique solution if the conditions stated in the theorem are satisfied.

If the conditions of the Theorem 5.1 are satisfied, then the differential Equation (5.8) is said to be *normal* on I (these conditions are both necessary and sufficient for the differential equation to be normal).

A point $x_0 \in I$, for which $a_0(x) \neq 0$, is called an *ordinary point* or a *regular point* of the differential equation.

Example 5.1 Find the intervals on which the following differential equations are normal.

(a) $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$, n an interger.

(b) $x^2y'' + xy' + (n^2 - x^2)y = 0$, n real.

(c) $\sqrt{x}\, y'' + 6xy' + 15y = \ln (x^4 - 256)$.

Solution

(a) Here, $a_0(x) = (1 - x^2)$, $a_1(x) = -2x$, and $a_2(x) = n(n + 1)$. Now, a_0, a_1 and a_2 are continuous everywhere in $(-\infty, \infty)$. Also, $a_0(x) = 1 - x^2 \neq 0$ for all $x \in (-\infty, \infty)$, except at the points $x = -1, 1$. Hence, the differential equation is normal on every subinterval I of the open intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$.

(b) Here, $a_0(x) = x^2$, $a_1(x) = x$, $a_2(x) = n^2 - x^2$. We find that a_0, a_1 and a_2 are continuous everywhere in $(-\infty, \infty)$. Also, $a_0(x) = x^2 \neq 0$ for all $x \in (-\infty, \infty)$ except at $x = 0$. Hence, the differential equation is normal on every subinterval I of the open intervals $(-\infty, 0)$, $(0, \infty)$.

(c) Here, $a_0(x) = \sqrt{x}$, $a_1(x) = 6x$, $a_2(x) = 15$, and $r(x) = \ln (x^4 - 256)$. Now, a_0, a_1, a_2 and $r(x)$ are continuous for all x satisfying $x > 4$. Hence, the differential equation is normal on every subinterval I of the open interval $(4, \infty)$.

Remark 1

If the functions $a_0(x), a_1(x), \ldots, a_n(x)$ are continuous over I and $a_0(x) \neq 0$ on I, then the only solution of the homogeneous initial value problem

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = 0 \tag{5.10}$$

$$y(x_0) = 0,\ y'(x_0) = 0, \ldots, y^{(n-1)}(x_0) = 0 \tag{5.11}$$

where $x_0 \in I$, is the trivial solution $y \equiv 0$ on I.

Linear combination of functions Let $f_1(x), f_2(x), \ldots, f_n(x)$ be n functions. Then

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x), \text{ where } c_1, c_2, \ldots, c_n \text{ are constants}$$

is called a linear combination of the given functions.

We had earlier defined that a function $y_1(x)$ is a solution of a non-homogeneous or a homogeneous equation, if the equation reduces to an identity when $y_1(x)$ is substituted into it. Let $y_1(x), y_2(x), \ldots, y_m(x)$ be m solutions of the linear homogeneous equation (5.10). Then, we show in the following that the *superposition principle* or *linearity principle* holds.

Theorem 5.2 If $y_1(x), y_2(x), \ldots, y_m(x)$ are m solutions of the linear homogeneous equation (5.10) on I, then a linear combination of the solutions $c_1y_1(x) + c_2y_2(x) + \ldots + c_my_m(x)$, where $c_1, c_2, \ldots, c_m$ are constants is also a solution of Eq. (5.10) on I.

Proof Substituting the linear combination $y = c_1y_1(x) + c_2y_2(x) + \ldots + c_my_m(x)$ into Eq. (5.10), we get

$$a_0(x)\frac{d^n}{dx^n}[c_1y_1(x) + c_2y_2(x) + \ldots + c_my_m(x)]$$

$$+ a_1(x)\frac{d^{n-1}}{dx^{n-1}}[c_1y_1(x) + c_2y_2(x) + \ldots + c_my_m(x)]$$

$$+ \ldots + a_{n-1}(x)\frac{d}{dx}[c_1y_1(x) + c_2y_2(x) + \ldots + c_my_m(x)]$$

$$+ a_n(x)[c_1y_1(x) + c_2y_2(x) + \ldots + c_my_m(x)]$$

$$= c_1\left[a_0(x)\frac{d^ny_1}{dx^n} + a_1(x)\frac{d^{n-1}y_1}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy_1}{dx} + a_n(x)y_1\right]$$

$$+ c_2\left[a_0(x)\frac{d^ny_2}{dx^n} + a_1(x)\frac{d^{n-1}y_2}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy_2}{dx} + a_n(x)y_2\right]$$

$$+ \ldots + c_m\left[a_0(x)\frac{d^ny_m}{dx^n} + a_1(x)\frac{d^{n-1}y_m}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy_m}{dx} + a_n(x)y_m\right]$$

$$= c_1[0] + c_2[0] + \ldots + c_m[0] = 0$$

since $y_1(x), y_2(x), \ldots, y_m(x)$ are solutions of the linear homogeneous equation.

Remark 2

Superposition principle does not hold for a non-homogeneous equation or a nonlinear equation.

Example 5.2 Show that e^{-x}, e^{x} and their linear combination $c_1e^{-x} + c_2e^{x}$ are solutions of the homogeneous equation $y'' - y = 0$.

Solution For $y_1 = e^{-x}$, we have $y_1' = -e^{-x}$, $y_1'' = e^{-x}$, $y_1'' - y_1 = 0$.

For $y_2 = e^{x}$, we have $y_2' = e^{x}$, $y_2'' = e^{x}$, $y_2'' - y_2 = 0$.

Hence, e^{-x} and e^{x} are solutions of $y'' - y = 0$.

Substituting $y = c_1e^{-x} + c_2e^{x} = c_1 y_1 + c_2 y_2$, we obtain

$$y'' - y = (c_1y_1 + c_2 y_2)'' - (c_1y_1 + c_2y_2) = c_1(y_1'' - y_1) + c_2(y_2'' - y_2) = c_1(0) + c_2(0) = 0.$$

5.2.1 Linear Independence and Dependence

Let $f_1(x), f_2(x), \ldots, f_n(x)$ be n functions. Then, these functions are said to be *linearly independent* on some interval I (where they are defined), if the equation

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0 \tag{5.12}$$

implies $c_1 = 0 = c_2 = \ldots = c_n$.

These functions are said to be *linearly dependent* on I, if Eq. (5.12) holds for $c_1, c_2, \ldots, c_n$ not all zero. In this case, one or more functions can be expressed as a linear combination of the remaining functions. For example, if $c_1 \neq 0$, then

$$f_1(x) = -\frac{1}{c_1}[c_2 f_2(x) + \ldots + c_n f_n(x)].$$

Conversely, if any function $f_i(x)$ can be expressed as a linear combination of the functions $f_1, f_2, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n$ then the given set of functions are linearly dependent.

Example 5.3 Show that the functions $f_1(x) = x^2, f_2(x) = x^3, f_3(x) = 6x^2 - x^3$ are linearly dependent on any interval I.

Solution We have $f_3(x) = 6x^2 - x^3 = 6f_1(x) - f_2(x)$. Hence, the given functions are linearly dependent on any interval I.

Example 5.4 Show that the functions $x^2 - 1$, $3x^2$ and $2 - 5x^2$ are linearly dependent.

Solution The given functions are linearly dependent if the equation

$$c_1(x^2 - 1) + c_2(3x^2) + c_3(2 - 5x^2) = 0 \tag{5.13}$$

holds for c_1, c_2, c_3 not all zero. We have from Eq. (5.13)

$$(c_1 + 3c_2 - 5c_3)x^2 + (2c_3 - c_1) = 0, \quad \text{for all } x.$$

We have, $c_1 + 3c_2 - 5c_3 = 0$, and $2c_3 - c_1 = 0$. The solution of these equations is $c_1 = 2c_3, c_2 = c_3$ where c_3 is arbitrary. For example, if $c_3 = 1$, then $c_1 = 2, c_2 = 1$ and $f_3(x) = -2f_1(x) - f_2(x)$. The given functions are linearly dependent.

Example 5.5 Show that the functions x, x^2, x^3 are linearly independent on any interval I.

Solution We have $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = x^3$. Substituting in equation (5.12), we get $c_1x + c_2x^2 + c_3x^3 = 0$. For finding the values of the three constants, take three distinct arbitrary points $x_0, x_1, x_2 (\neq 0)$ on I. Hence

$$c_1x_0 + c_2x_0^2 + c_3x_0^3 = 0, \; c_1x_1 + c_2x_1^2 + c_3x_1^3 = 0, \; c_1x_2 + c_2x_2^2 + c_3x_2^3 = 0.$$

This system of homogeneous algebraic equations has a non-trivial solution if the determinant of the coefficient matrix vanishes, that is

$$\det = \begin{vmatrix} x_0 & x_0^2 & x_0^3 \\ x_1 & x_1^2 & x_1^3 \\ x_2 & x_2^2 & x_2^3 \end{vmatrix} = 0.$$

Evaluating the determinant, we have

$$\det = (x_1 - x_0)(x_2 - x_0)(x_2 - x_1)x_0x_1x_2.$$

Since x_0, x_1, x_2 are distinct, $\det \neq 0$. Therefore, the only solution is $c_1 = 0, c_2 = 0, c_3 = 0$. Hence, the given functions are linearly independent.

The procedure used in Example 5.5 is lengthy and a difficult one. It is not always possible to examine the linear dependence or independence in this way. A very elegant procedure to test the linear independence or dependence of a given set of functions is the application of *Wronskians*. Let $f_1(x),\ f_2(x), \ldots,\ f_n(x)$ be n functions. The Wronskian of these functions is denoted by $W(f_1, f_2, \ldots, f_n)$ and is defined by

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} = W(x). \tag{5.14}$$

The Wronskian of the n functions exists if all the functions $f_1, f_2, \ldots, f_n$ are differentiable $n - 1$ times on the interval I. If any one or more functions are not differentiable then the Wronskian does not exist.

We have the following result for testing the linear dependence or independence of the solutions of the linear homogeneous differential equation (5.10).

Theorem 5.3 If the coefficients $a_0(x), a_1(x), \ldots, a_n(x)$ in the linear homogeneous equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}y' + a_n y = 0, \quad a_0 \neq 0 \tag{5.15}$$

are continuous on I and $y_1(x), y_2(x), \ldots, y_n(x)$ are n solutions of this equation, then

(i) $W(x) = W(y_1, y_2, \ldots, y_n) \neq 0$ for all $x \in I \Leftrightarrow y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent on I,

(ii) $W(x_0) = 0$ where $x_0 \in I$ is any fixed point, implies $W(x) = 0$ for all x in I and the functions $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly dependent.

Proof Let $y_1(x), y_2(x), \ldots, y_n(x)$ be linearly dependent on I. By definition, there exist constants $c_1, c_2, \ldots, c_n$ not all zero, such that

$$c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) = 0, \text{ for all } x \in I. \tag{5.16}$$

Differentiating Eq. (5.16) successively, $n - 1$ times, we get

$$c_1 y_1'(x) + c_2 y_2'(x) + \ldots + c_n y_n'(x) = 0$$

$$c_1 y_1''(x) + c_2 y_2''(x) + \ldots + c_n y_n''(x) = 0$$

$$\cdots \qquad \cdots$$

$$c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)}(x) + \ldots + c_n y_n^{(n-1)}(x) = 0. \tag{5.17}$$

Eqs. (5.16), (5.17) form a homogeneous, linear system of algebraic equations. Non-trivial solutions of the system exist if and only if the determinant of the coefficient matrix is zero for all $x \in I$. But this determinant is the Wronskian $W(x)$ of the solutions. Hence, if the solutions are dependent then $W(x) = 0$ for all $x \in I$.

Let now $W(x_0) = 0$ for some fixed point $x_0 \in I$. Then, the system of equations given by (5.16), (5.17) has a nontrivial solution $c_1 = c_1^*, c_2 = c_2^*, \ldots, c_n = c_n^*$ not all zero. Hence, $y^*(x) = c_1^* y_1(x) + c_2^* y_2(x) + \ldots + c_n^* y_n(x)$ is a solution of the linear homogeneous Equation (5.15). Using Eqs. (5.16) and (5.17), we find that $y^*(x)$ also satisfies the initial conditions $y^*(x_0) = 0, (y^*)'(x_0) = 0, \ldots, (y^*)^{(n-1)}(x_0) = 0$. Now, the differential equation (5.15) and these conditions form a homogeneous initial value problem. Hence, $y^*(x) \equiv 0$ is the solution of the initial value problem. Since the solution of the initial value problem is unique, we obtain $y^*(x) = y(x) = 0$, or, for all x, $c_1^* y_1(x) + c_2^* y_2(x) + \ldots + c_n^* y_n(x) = 0$, where not all c_i are zero. Hence, the solutions are dependent. Since x_0 is arbitrary, $W(x_0) = 0$ for some $x_0 \in I$ implies $W(x) = 0$ for all $x \in I$.

We now define the general solution of the homogeneous Eq. (5.15).

Theorem 5.4 If the coefficients $a_0(x), a_1(x), \ldots, a_n(x), a_0(x) \neq 0$, in the linear homogeneous equation (5.15) are continuous on I, then the equation (5.15) has n linearly independent solutions. If $y_1(x)$, $y_2(x), \ldots, y_n(x)$ are n linearly independent solutions, then the general solution is given by $y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$, that is, their linear combination.

The n linearly independent solutions $y_1(x), y_2(x), \ldots, y_n(x)$ are also called the *fundamental solutions* of Eq. (5.15) on I. This set of fundamental solutions forms a *basis* of the nth order linear homogeneous equation.

Example 5.6 Show that the functions x, x^2, x^3 are linearly independent on any interval I, not containing zero (see Example 5.5).

Solution The Wronskian of the functions is

$$W(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = x(12x^2 - 6x^2) - (6x^3 - 2x^3) = 2x^3.$$

Therefore, $W(x) \neq 0$ on any interval not containing zero. Hence, the functions are linearly independent in $(-\infty, 0), (0, \infty)$.

Example 5.7 Show that the functions $1, \sin x, \cos x$ are linearly independent.

Solution The Wronskian of the functions is

$$W(x) = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = -1.$$

Hence, the given functions are linearly independent on any interval I.

Example 5.8 Show that e^x, e^{2x}, e^{3x} are the fundamental solutions of $y''' - 6y'' + 11y' - 6y = 0$, on any interval I.

Solution Substituting $y = e^x, e^{2x}, e^{3x}$, we find that they satisfy the differential equation

$$y''' - 6y'' + 11y' - 6y = 0.$$

The Wronskian of these functions is

$$W(x)=\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{6x}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2e^{6x} \neq 0 \cdot$$

Therefore, the solutions are linearly independent and they form a set of fundamental solutions on any interval I.

Example 5.9 Show that the set of functions $\{x, 1/x\}$ forms a basis of the equation $x^2y'' + xy' - y = 0$. Obtain a particular solution when $y(1) = 1$, $y'(1) = 2$.

Solution We have

$$y_1(x) = x,\ y_1' = 1,\ y_1'' = 0, \quad \text{and} \quad x^2 y_1'' + xy_1' - y_1 = x - x = 0$$

$$y_2(x) = 1/x,\ y_2' = -1/x^2,\ y_2'' = 2/x^3$$

and $$x^2 y_2'' + xy_2' - y_2 = x^2\left(\frac{2}{x^3}\right) + x\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right) = 0.$$

Hence $y_1(x)$ and $y_2(x)$ are solutions of the given equation. The Wronskian is given by

$$W(y_1, y_2) = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -\frac{2}{x} \neq 0, \quad \text{for } x \geq 1.$$

Therefore, the set $\{y_1(x), y_2(x)\}$ forms a basis of the equation. The general solution is

$$y(x) = c_1y_1(x) + c_2y_2(x) = c_1x + \frac{c_2}{x}.$$

Substituting in the given conditions, we get

$$y(1) = 1 = c_1 + c_2,\ y'(1) = 2 = c_1 - c_2.$$

Solving, we obtain $c_1 = 3/2$, $c_2 = -1/2$.

The particular solution is $y(x) = \dfrac{1}{2}\left(3x - \dfrac{1}{x}\right)$.

Exercise 5.1

From the following linear differential equations, find the constant coefficient and variable coefficient equations.

1. $y'' - a^2y = 0$.
2. $y' = y/x$.
3. $y''' + 3y'' + 6y' + 12y = x^2$.
4. $x^3y''' + 9x^2y'' + 18xy' + 6y = 0$.
5. $(1 - x)y'' + xy' - y = 0$.
6. $y'' - (1 + x^2)y = 0$.

Find the intervals on which the following differential equations are normal.

7. $y' = 3y/x$.
8. $(1 + x^2)y'' + 2xy' + y = 0$.
9. $x^2y'' - 4xy' + 6y = x$.
10. $y'' + 3y' + \sqrt{x}\, y = \sin x$.
11. $y''' + 9y' + y = \log (x^2 - 9)$.
12. $y'' + |x|y' + y = x \ln x$.

13. $x(1-x)y'' - 3xy' - y = 0$.

14. $y'' + xy' + 6y = \ln \sin(\pi x/4)$.

15. Verify that $y = x^2$ is a solution of $x^2y'' + xy' - 4y = 0$, $x \in (0, \infty)$ and satisfies the conditions $y(0) = 0$, $y'(0) = 0$. Does Theorem 5.1 guarantee the existence and uniqueness of such a solution? Is the Remark 1 applicable in this case?

16. By inspection find a solution of $x^2y'' + xy' - y = 0$, $x \in (-\infty, \infty)$ which satisfies the conditions $y(0) = 0$, $y'(0) = 2$. Does Theorem 5.1 guarantee the existence and uniqueness of such a solution?

17. Show that

$$y_1(x) = x^3 - x^2, \ -3 \le x \le 3, \quad \text{and} \quad y_2(x) = \begin{cases} x^2 - x^3, & -3 \le x \le 0, \\ x^3 - x^2, & 0 \le x \le 3 \end{cases}$$

both satisfy the differential equation $x^2y'' - 4xy' + 6y = 0$ and the conditions $y(2) = 4$, $y'(2) = 8$. But $y_1(x)$, $y_2(x)$ are different. Does this contradict Theorem 5.1?

Verify that the given functions are solutions of the associated differential equation. Verify also that a linear combination of these functions is also a solution.

18. $1, x, e^x$; $y''' - y'' = 0$.

19. e^x, e^{-2x}; $y'' + y' - 2y = 0$.

20. $e^{-x}\cos 2x, e^{-x}\sin 2x$; $y'' + 2y' + 5y = 0$.

Examine whether the following functions are linearly independent for $x \in (0, \infty)$.

21. $2x, 6x + 3, 3x + 2$.

22. $x^2 - x, 3x^2 + x + 1, 9x^2 - x + 2$.

23. $x^2 - 2x, 3x^2 + x + 2, 4x^2 - x + 1$.

24. $\sin x, \sin 2x, \sin 3x$.

25. $1, \cos x, \sin x$.

26. $e^x, \sinh x, \cosh x$.

27. $x^2, 1/x^2$.

28. $\ln x, \ln x^2, \ln x^3$.

29. $x - 1, x + 1, (x-1)^2$.

30. $e^{-x}, \sinh x, \cosh x$.

31. Find the intervals on which the three functions $1, \cos x, \sec x$, $x > 0$ are linearly independent.

32. Determine how many of the given functions are linearly independent on $[0, 1]$.

(i) $1, 1 + x, x^2, x(1-x), x$; (ii) $1 + x, 1 - x, 1, x^2, 1 + x^2$.

33. Show that $y_1(x) = \sin x$, and $y_2(x) = 4\sin x - 2\cos x$ are linearly independent solutions of $y'' + y = 0$. Write the solution $y_3(x) = \cos x$ as a linear combination of y_1 and y_2.

34. Let $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ be a second order differential equation. Let $a_0(x), a_1(x), a_2(x)$ be continuous and $a_0(x) \ne 0$ on I and $y_1(x), y_2(x)$ be two linearly independent solutions. Show that the Wronskian of $y_1(x), y_2(x)$ satisfies the differential equation $a_0(x)W'(x) + a_1(x)W(x) = 0$. Also, show that the Wronskian is given by

$$W(x) = c\,e^{-\int [a_1(x)/a_0(x)]dx}.$$

(This is called the *Abel's formula*).

35. Show that $\cos at$, $\sin at$ are solutions of the equation $y'' + a^2y = 0$, $a \ne 0$ on any interval. Show that they are independent. Use the result (Abel's formula) given in Problem 34 and find the Wronskian. Are the two Wronskians same?

36. Show that e^{2x} and xe^{2x} are solutions of the equation $y'' - 4y' + 4y = 0$ on any interval. Show that they are independent. Use the result given in problem 34 and find the Wronskian. Are the two Wronskians same?

Show that in the following problems, $\{y_i(x)\}$ forms a set of fundamental solutions (basis) to the corresponding differential equation.

37. $x^{1/4}, x^{5/4}$; $16x^2y'' - 8xy' + 5y = 0$, $x > 0$.

38. $e^{2x}\cos 3x$, $e^{2x}\sin 3x$; $2y'' - 8y' + 26y = 0$.
39. $1, x^2$; $x^2y'' - xy' = 0$, $x > 0$.
40. e^x, e^{2x}, e^{-3x}; $y''' - 7y' + 6y = 0$.
41. $e^x, e^x\cos x, e^x\sin x$; $y''' - 3y'' + 4y' - 2y = 0$.
42. $e^{2x}, e^{-x}\cos(\sqrt{3}x), e^{-x}\sin(\sqrt{3}x)$; $y''' - 8y = 0$.
43. $\sin(\ln x^2)$, $\cos(\ln x^2)$; $x^2y'' + xy' + 4y = 0$, $x > 0$.
44. Let the coefficients $a_0(x), a_1(x), a_2(x)$ in the equation $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ be continuous and $a_0(x) \neq 0$ on I. Let $\{y_1(x), y_2(x)\}$ be the basis (set of fundamental solutions) of the equation. Show that the set $\{u(x), v(x)\}$ such that $u = ay_1(x) + by_2(x)$, $v = cy_1(x) + dy_2(x)$, is also a basis of the equation if $ad - bc \neq 0$. If $y_1(x) = \cosh kx$, $y_2 = \sinh kx$, obtain a simple form of u and v.
45. Let $y_1(x), y_2(x)$ be the linearly independent solutions of the equation $y'' + a(x)y' + b(x)y = 0$ on I. Show that there is no point $x_0 \in I$ at which (i) both $y_1(x), y_2(x)$ vanish, (ii) both $y_1(x), y_2(x)$ take extreme values.
46. Let $\{y_1(x), y_2(x)\}$ be the basis of the equation $y'' + a(x)y' + b(x)y = 0$. Show that the equation can be written as the Wronskian $W(y, y_1, y_2) = 0$.
47. Let $y_1(x)$ be a solution of the homogeneous equation $y'' + a(x)y' + b(x)y = 0$, on the interval $I: \alpha \le x \le \beta$. The coefficients $a(x)$ and $b(x)$ are continuous on I. If the curve $y = y_1(x)$ is tangential to the x-axis at a point x_1 in I, then prove that $y_1(x) \equiv 0$.

Using the problem 46, find a differential equation of the form $y'' + a(x)y' + b(x)y = 0$ for which the following functions are solutions.

48. e^{3x}, e^{-2x}.
49. $e^{-(\alpha + i\omega)x}, e^{-(\alpha - i\omega)x}$.
50. e^{5x}, xe^{5x}.

5.3 Methods for Solution of Linear Equations

In this section, we shall discuss various methods of finding solution of linear equations. We first define the differential operator D.

5.3.1 Differential Operator D

Sometimes, it is convenient to write the given linear differential equation in a simple form using the differential operator $D = d/dx$. We define an operator T as a transformation $T: V \to W$ that transforms a function f in V into another function $T(V)$ in W. Let the operator D be defined, over the set V_1 of all differentiable functions f on I, by $D = d/dx$.

Then, we write
$$Df(x) = Df = (Df)(x) = \frac{df}{dx} = f'. \tag{5.18}$$

We have, for example $D(x^n) = \dfrac{d}{dx}(x^n) = nx^{n-1}$, n constant; $D(\cos x) = -\sin x$, etc.

Let $f(x)$ and $g(x)$ be differentiable functions. Since D is a linear operator, we have
$$D(af + bg) = aDf + bDg, \quad a, b \text{ constants.}$$

We also have for $f \in V_2$, the set of functions having a second derivative on I
$$D(Df) = D(f') = \frac{d}{dx}(f') = f''.$$

We simply write $D(Df) = D(D)f = D^2f$ so that

$$D^3(f) = D(D^2f) = D(f'') = f''', \ldots, D^k(f) = f^{(k)}.$$

where f is sufficiently differentiable.

We define $D^0 \equiv 1$, so that if 1 is the operator defined by $1(f) = f$, we have $D^0(f) = 1(f) = f$. We, now define the operator L by

$$L = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{d}{dx} + a_n(x)$$
$$= a_0(x)D^n + a_1(x)D^{n-1} + \ldots + a_{n-1}(x)D + a_n(x) = P(D) \quad (5.19)$$

which is a polynomial in D, so that

$$Ly = a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y$$
$$= a_0(x)D^ny + a_1(x)D^{n-1}y + \ldots + a_{n-1}(x)Dy + a_n(x)y$$
$$= [a_0(x)D^n + a_1(x)D^{n-1} + \ldots + a_{n-1}(x)D + a_n(x)]y = P(D)y. \quad (5.20)$$

For example, the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

can be written as

$$Ly = (D^2 + 5D + 6)y = 0 \quad (5.21)$$

where the operator L is given by $L = P(D) = D^2 + 5D + 6$.

Similarly, the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = x^2$$

can be written as

$$Ly = (D^2 + 2D + 2)y = x^2 \quad (5.22)$$

where the operator L is defined by $L = P(D) = D^2 + 2D + 2$.

Suppose $y = e^{mx}$. Then, $D(y) = D(e^{mx}) = me^{mx}$ and $D^2(y) = m^2e^{mx}$. Substituting in Eq. (5.21), we obtain

$$Ly = (D^2 + 5D + 6)y = (m^2 + 5m + 6)y = P(m)y$$

using Eq. (5.19). Therefore,

$$P(D)y = (D^2 + 5D + 6)e^{mx} = P(m)y.$$

In general, substituting $y = e^{mx}$ in the equation (5.20), we get

$$P(D)y = (a_0D^n + a_1D^{n-1} + \ldots + a_{n-1}D + a_n)e^{mx}$$
$$= (a_0m^n + a_1m^{n-1} + \ldots + a_{n-1}m + a_n)e^{mx} = P(m)e^{mx}. \quad (5.23)$$

When a_i, $i = 0, 1, \ldots, n$ are constants, the operator $L = P(D)$ can often be factorised. For example, we have

(i) $D^2 + 5D + 6 = (D + 2)(D + 3)$

(ii) $D^3 - 6D^2 + 11D - 6 = (D - 1)(D - 2)(D - 3)$.

When a_i are functions of x, factorisation is often not possible. For example, $x^2Dy \neq D(x^2y)$, or in general $a(x)Dy \neq D[a(x)y]$, since the right hand side is $D[a(x)y] = a(x)y' + a'(x)y$.

5.3.2 Solution of Second Order Linear Homogeneous Equations with Constant Coefficients

Consider the linear homogeneous second order equation

$$ay'' + by' + cy = 0, \quad a, b, c \text{ are constants.} \tag{5.24}$$

In the operator notation, we write the equation as

$$Ly = P(D)y = aD^2y + bDy + cy = (aD^2 + bD + c)y = 0. \tag{5.25}$$

In the previous chapter, we have shown that the solution of the first order equation $y' + my = 0$ is $y = e^{-mx} + c$; and the solution of the equation $y' - my = 0$ is $y = e^{mx} + c$. Therefore, it is natural to try for a particular solution of the form $y = e^{mx}$, for Eq. (5.25), where m is an unknown constant to be determined. Since $y' = me^{mx}$, $y'' = m^2e^{mx}$, we obtain from Eq. (5.24)

$$(am^2 + bm + c)e^{mx} = 0.$$

Since $e^{mx} \neq 0$, we obtain

$$am^2 + bm + c = 0. \tag{5.26}$$

This is an algebraic equation in m. It is called the *characteristic equation* or the *auxiliary equation* of the linear homogeneous equation (5.24) (we can write the characteristic equation by replacing y'' by m^2, y' by m and y by 1 in Eq. (5.24) implicitly noting that solutions of the form e^{mx} are being determined). The roots of this equation are called the *characteristic roots*. The quadratic equation (5.26) has the roots

$$m = [-b \pm \sqrt{b^2 - 4ac}\,]/2a.$$

We have the following three cases.

(i) The roots are real and distinct, say $m = m_1, m_2$; $m_1 \neq m_2$ if $b^2 - 4ac > 0$.

(ii) The roots are real and equal, say $m = m_1, m_1$ if $b^2 - 4ac = 0$.

(iii) The roots are complex if $b^2 - 4ac < 0$.

To find the complete solution in the above three cases, we proceed as follows.

Real and distinct roots

Let the distinct roots be $m = m_1$ and $m = m_2$. Then, we obtain two solutions of the equation (5.24) as e^{m_1x} and e^{m_2x}. The two solutions are linearly independent on any interval I, since the Wronskian,

$$W(y_1, y_2) = y_1y_2' - y_2y_1' = m_2e^{m_1x}e^{m_2x} - m_1e^{m_1x}e^{m_2x}$$

$$= (m_2 - m_1)e^{(m_1+m_2)x} \neq 0.$$

Hence, the general solution of Eq. (5.24) is

$$y(x) = c_1e^{m_1x} + c_2e^{m_2x}. \tag{5.27}$$

Example 5.10 Find the solution of the differential equation $y'' - y' - 6y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^2 - m - 6 = 0, \quad \text{or} \quad (m-3)(m+2) = 0, \quad \text{or} \quad m = -2, 3.$$

The two linearly independent solutions are e^{3x} and e^{-2x}. The general solution is

$$y(x) = Ae^{3x} + Be^{-2x}.$$

Example 5.11 Solve the initial value problem

$$4y'' - 8y' + 3y = 0, \; y(0) = 1, \; y'(0) = 3.$$

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$4m^2 - 8m + 3 = 0, \quad \text{or} \quad m = 1/2, 3/2.$$

Hence, the linearly independent solutions are $e^{x/2}$ and $e^{(3x)/2}$. The general solution is

$$y(x) = Ae^{(3x)/2} + Be^{x/2}.$$

Substituting the initial conditions, we get

$$y(0) = 1 = A + B, \quad y'(0) = 3 = \frac{3A}{2} + \frac{B}{2}.$$

Solving the above equations, we get $A = 5/2$ and $B = -3/2$. The solution of the initial value problem is $y(x) = [5e^{(3x)/2} - 3e^{x/2}]/2$.

Real and equal roots

Real and equal roots are obtained for the characteristic equation (5.26) when $b^2 - 4ac = 0$. In this case the repeated root is $m = -b/(2a)$. This value gives one solution as $y_1(x) = e^{mx} = e^{-(bx)/(2a)}$. We need to determine another linearly independent solution $y_2(x)$, so that $\{y_1(x), y_2(x)\}$ forms a basis for the equation. The second solution $y_2(x)$ can be determined in a number of ways. We shall show that if m is a repeated root then e^{mx} and xe^{mx} are the two linearly independent solutions.

For $y_2(x) = xe^{mx}$, $m = -b/(2a)$, we have

$$y_2 = xe^{mx}, \; y_2' = (1 + mx)e^{mx}, \; y_2'' = (2 + mx)me^{mx}.$$

Substituting in Eq. (5.24), we get

$$[ma(2 + mx) + b(1 + mx) + cx]e^{mx} = 0$$

or

$$[(2ma + b) + (am^2 + bm + c)x]e^{mx} = 0.$$

Since $2ma + b = -b + b = 0$ and $am^2 + bm + c = 0$, this equation is automatically satisfied. Therefore, xe^{mx} is also a solution. Since e^{mx} and xe^{mx} are linearly independent, they form a set of the two fundamental solutions. Hence, the general solution is

$$y(x) = Ae^{mx} + Bxe^{mx} = (A + Bx)e^{mx}, \quad m = -b/(2a). \tag{5.28}$$

Alternative We can use the following method (*reduction of order*) to find the second linearly independent solution.

Let

$$y_2(x) = u(x)\,y_1(x)$$

where $y_1(x) = e^{mx}$, $m = -b/(2a)$ be a solution of Eq. (5.24). We have

$$y_2' = uy_1' + u'y_1, \quad y_2'' = uy_1'' + 2u'y_1' + u''y_1.$$

Substituting in the differential equation, we obtain

$$a(uy_1'' + 2u'y_1' + u''y_1) + b(uy_1' + u'y_1) + cuy_1$$
$$= ay_1u'' + (2ay_1' + by_1)u' + (ay_1'' + by_1' + cy_1)u = 0. \tag{5.29}$$

Since $y_1(x)$ is a solution, we have $ay_1'' + by_1' + cy_1 = 0$.

Also, $$2ay_1' + by_1 = 2a\left(-\frac{b}{2a}\right)e^{-(bx)/(2a)} + be^{-(bx)/(2a)} = 0.$$

Hence, Eq. (5.29) reduces to $ay_1u'' = 0$. Since $a \neq 0$, $y_1 \neq 0$, we get $u'' = 0$, whose solution is $u = c_1x + c_2$. Therefore, $y_2(x) = (c_1x + c_2)y_1(x) = c_1xy_1(x) + c_2y_1(x)$. Since $y_1(x)$ is a solution, the second linearly independent solution is $xy_1(x)$, (note that a linear combination of the two linearly independent solutions is also a solution). The general solution is

$$y(x) = Ay_1(x) + Bxy_1(x) = (A + Bx)e^{mx}, \; m = -b/(2a)$$

which is same as Eq. (5.28).

Alternative The second linearly independent solution can be determined by factorising the differential operator and reducing the given second order equation to a first order equation. We have

$$(aD^2 + bD + c)y = a\left[D^2 + \frac{b}{a}D + \frac{c}{a}\right]y = 0, \quad a \neq 0.$$

Since $b^2 - 4ac = 0$ and $m = m_1 = -b/(2a)$ is a repeated root, the operator is factorisable so that we can write the equation as

$$(D - m_1)(D - m_1)y = 0. \tag{5.30}$$

Set $(D - m_1)y = u$. Then, Eq. (5.30) reduces to $(D - m_1)u = 0$ or $u' - m_1u = 0$ whose solution is $u = c_1e^{m_1x}$. Substituting in the equation $(D - m_1)y = u$, we obtain

$$(D - m_1)y = y' - m_1y = u = c_1e^{m_1x}.$$

The integrating factor of this equation is e^{-m_1x}. Therefore, the solution of this equation is

$$ye^{-m_1x} = \int c_1e^{m_1x}e^{-m_1x}\,dx + c_2 = c_1x + c_2$$

or $$y = (c_1x + c_2)c^{m_1x}$$

which is same as $y_2(x)$ obtained in the previous case.

Example 5.12 Find the solution of the differential equation $4y'' + 4y' + y = 0$.

Solution Assume a solution of the form $y = e^{mx}$. The characteristic equation is given by

$$4m^2 + 4m + 1 = 0, \text{ or } (2m + 1)^2 = 0, \text{ or } m = -1/2, -1/2,$$

which is a repeated root. Hence, the general solution is $y(x) = (A + Bx)e^{-x/2}$.

Example 5.13 Solve the initial value problem

$$y'' + 6y' + 9y = 0,\ y(0) = 2,\ y'(0) = 3.$$

Solution Assume a solution of the form $y = e^{mx}$. The characteristic equation is given by

$$m^2 + 6m + 9 = 0, \text{ or } (m + 3)^2 = 0, \text{ or } m = -3, -3,$$

which is a repeated root. The general solution is $y(x) = (A + Bx)e^{-3x}$. Substituting in the initial conditions, we get

$$y(0) = 2 = A,\ y' = Be^{-3x} - 3(A + Bx)e^{-3x},\ y'(0) = 3 = B - 3A.$$

The solution is $A = 2$, $B = 9$. The solution of the given initial value problem is $y(x) = (2 + 9x)e^{-3x}$.

Example 5.14 Factorising the differential operator and reducing it into first order equations, solve the differential equation $y'' - 4y' - 5y = 0$.

Solution In the operator notation, the differential equation can be written as

$$(D^2 - 4D - 5)y = 0, \quad \text{or} \quad (D - 5)(D + 1)y = 0. \tag{5.31}$$

Set $(D + 1)y = u$. Then, we obtain from Eq. (5.31), $(D - 5)u = 0$. This is a first order equation whose solution is $u = Ae^{5x}$. Hence,

$$(D + 1)y = Ae^{5x}.$$

This is a first order linear equation, whose integrating factor is e^x. Hence, we have the solution as

$$e^x y = \int Ae^{6x}dx + B = \frac{A}{6}e^{6x} + B,$$

or

$$y = \frac{A}{6}e^{5x} + Be^{-x} = Ce^{5x} + Be^{-x}$$

where $C = A/6$ is an arbitrary constant.

We could have written Eq. (5.31) as $(D + 1)(D - 5)y = 0$ and obtain the same answer.

Example 5.15 Factorising the differential operator and reducing it to first order equations, solve the differential equation $4y'' + 12y' + 9y = 0$

Solution In the operator notation, the differential equation can be written as

$$(4D^2 + 12D + 9)y = (2D + 3)^2 y = 0. \tag{5.32}$$

Set $(2D + 3)y = u$. Then, we obtain from Eq. (5.32),

$$(2D + 3)u = 0.$$

The solution of this equation is $u = Ae^{-(3x)/2}$. Therefore,

$$(2D + 3)y = Ae^{-(3x)/2}, \quad \text{or} \quad \left(D + \frac{3}{2}\right)y = \frac{A}{2}e^{-(3x)/2}.$$

This is a linear first order equation whose integrating factor is $e^{(3x)/2}$. The solution is given by

$$ye^{(3x)/2} = \int \frac{A}{2}dx + B = \frac{Ax}{2} + B, \quad \text{or} \quad y = (Cx + B)e^{-(3x)/2},$$

where $C = A/2$.

Complex roots

When $b^2 - 4ac < 0$, then the roots of the characteristic equation (5.26) are complex. We have

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a} = p \pm iq$$

where $p = -b/(2a)$ and $q = \sqrt{4ac - b^2}/(2a)$. Since the characteristic equation (5.26) has real coefficients, the complex roots occur in conjugate pairs and are of the form $p \pm iq$. Then, the solution of the equation can be written as

$$y(x) = Ae^{(p+iq)x} + Be^{(p-iq)x} = Ae^{px}e^{iqx} + Be^{px}e^{-iqx} = (Ae^{iqx} + Be^{-iqx})e^{px}$$
$$= [A(\cos qx + i \sin qx) + B(\cos qx - i \sin qx)]e^{px}$$

by the Euler formula. Simplifying, we obtain

$$y(x) = [c_1 \cos qx + c_2 \sin qx]e^{px} \tag{5.33}$$

where $c_1 = A + B$ and $c_2 = i(A - B)$. Therefore, the two linearly independent solutions are $y_1 = e^{px} \cos qx$ and $y_2 = e^{px} \sin qx$. The Wronskian is given by

$$W(y_1, y_2) = \begin{vmatrix} e^{px} \cos qx & e^{px} \sin qx \\ e^{px}(p \cos qx - q \sin qx) & e^{px}(p \sin qx + q \cos qx) \end{vmatrix} = qe^{2px} \neq 0$$

showing that $y_1(x)$ and $y_2(x)$ are linearly independent.

Example 5.16 Find the solution of the differential equation $y'' + 2y' + 2y = 0$.

Solution Assume a solution of the form $y = e^{mx}$. The characteristic equation is given by

$$m^2 + 2m + 2 = 0, \quad \text{or} \quad m = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i = p \pm iq.$$

The general solution is

$$y(x) = (A \cos qx + B \sin qx)e^{px} = (A \cos x + B \sin x)e^{-x}.$$

Example 5.17 Find the solution of the initial value problem

$$y'' + 4y' + 13y = 0, \ y(0) = 0, \ y'(0) = 1.$$

Solution Assume a solution of the form $y = e^{mx}$. The characteristic equation is given by

$$m^2 + 4m + 13 = 0, \quad \text{or} \quad m = \frac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i = p \pm iq.$$

The general solution is given by

$$y(x) = [A \cos qx + B \sin qx]e^{px} = [A \cos 3x + B \sin 3x]e^{-2x}.$$

Substituting in the initial conditions, we obtain

$$y(0) = 0 = A,$$
$$y'(x) = Be^{-2x}(3 \cos 3x - 2 \sin 3x), \quad y'(0) = 1 = 3B, \quad \text{or} \quad B = 1/3.$$

The solution of the initial value problem is

$$y(x) = (e^{-2x} \sin 3x)/3.$$

Example 5.18 Find all the non-trivial solutions, if any, of the boundary value problem

$$y'' + \omega^2 y = 0, \; y(0) = 0, \; y(l) = 0.$$

Solution Assume a solution of the form $y = e^{mx}$. The characteristic equation is given by

$$m^2 + \omega^2 = 0, \quad \text{or} \quad m = \pm i\omega.$$

The general solution is

$$y(x) = A \cos \omega x + B \sin \omega x. \tag{5.34}$$

Substituting in the boundary conditions, we obtain

$$y(0) = 0 = A, \; y(l) = 0 = B \sin (\omega l).$$

If $B = 0$, then we obtain the trivial solution $y = 0$.

For $B \neq 0$, we get $\sin \omega l = 0 = \sin n\pi$, $n = 1, 2, \ldots$

Therefore, $\omega = n\pi/l$. The general solution is

$$y_n(x) = B_n \sin [(n\pi x)/l], \; n = 1, 2, \ldots$$

where B_i's are arbitrary. There are infinite number of solutions. Since the boundary value problem is homogenous, by the superposition principle, the sum of these solutions is also a solution. Therefore, the general solution is given by

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right).$$

(The convergence of such an infinite series called the Fourier series, is discussed in chapter 9.)

5.3.3 Method of Reduction of Order for Variable Coefficient Linear Homogeneous Second Order Equations

Suppose that we know one of the solutions of the second order equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \; a_0(x) \neq 0 \text{ on } I. \tag{5.35}$$

Then, we can obtain the second linearly independent solution by the method of reduction of order. Let $y = y_1(x)$ be a non-trivial solution of Eq. (5.35), that is

$$a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0. \tag{5.36}$$

Then, we write the second solution as $y_2(x) = u(x)y_1(x)$. Since $u(x) = y_2(x)/y_1(x)$ is not a constant, y_1 and y_2 are two linearly independent solutions of Eq. (5.35). Now,

$$y_2' = u'y_1 + uy_1', \quad \text{and} \quad y_2'' = u''y_1 + 2u'y_1' + uy_1''.$$

Substituting in Eq. (5.35) and collecting the terms, we get

$$a_0(x)y_1u'' + [2a_0(x)\, y_1' + a_1(x)y_1]u' + [a_0(x)\, y_1'' + a_1(x)\, y_1' + a_2(x)y_1]u = 0.$$

Using Eq. (5.36), we obtain

$$a_0(x)y_1u'' + [2a_0(x)y_1' + a_1(x)y_1]u' = 0.$$

Now, let $v = u'$. Then, we have

$$a_0(x)y_1v' + [2a_0(x)y_1' + a_1(x)y_1]\,v = 0 \tag{5.37}$$

which is a first order equation in v.

Separating the variables, we obtain

$$\frac{v'}{v} = -\frac{(2a_0y_1' + a_1y_1)}{a_0y_1} = -\left[\frac{2y_1'}{y_1} + \frac{a_1}{a_0}\right].$$

Integrating, we obtain

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & uy_1 \\ y_1' & uy_1' + y_1u' \end{vmatrix} = \begin{vmatrix} y_1 & uy_1 \\ y_1' & uy_1' + \dfrac{1}{y_1}e^{-\int p\,dx} \end{vmatrix} = e^{-\int p\,dx}.$$

where $p(x) = a_1(x)/a_0(x)$. Integrating $u' = v$, we obtain $u = \int v(x)\,dx$. The second linearly independent solution is given by $y_2(x) = u(x)y_1(x)$. It can be verified that the Wronskian of y_1, y_2 is equal to

$$W(y_1, y_2) = e^{-\int p(x)dx} \neq 0$$

showing that $y_1(x)$ and $y_2(x)$ are linearly independent.

Example 5.19 It is known that $1/x$ is a solution of the differential equation $x^2y'' + 4xy' + 2y = 0$. Find the second linearly independent solution and write the general solution.

Solution Write $y_2(x) = u(x)y_1(x) = u(x)/x$. Here, $p(x) = a_1(x)/a_0(x) = 4/x$. Hence,

$$v(x) = \frac{1}{y_1^2}e^{-\int p(x)dx} = x^2e^{-\int (4/x)dx} = x^2\left(\frac{1}{x^4}\right) = \frac{1}{x^2}.$$

$$u(x) = \int v(x)\,dx = \int \frac{dx}{x^2} = -\frac{1}{x}, \text{ and } y_2(x) = u(x)y_1(x) = -\frac{1}{x^2}.$$

The general solution is $y(x) = Ay_1(x) + By_2(x) = \dfrac{A}{x} + \dfrac{B}{x^2}$.

Exercise 5.2

Show that the given set of functions $\{y_1(x), y_2(x)\}$ forms a basis of the equation and hence solve the initial value problem.

1. e^x, e^{4x}, $y'' - 5y' + 4y = 0$, $y(0) = 2$, $y'(0) = 1$.
2. e^{2x}, e^{-2x}, $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = 4$.
3. e^{-3x}, xe^{-3x}, $y'' + 6y' + 9y = 0$, $y(0) = 1$, $y'(0) = 2$.
4. $x^2, 1/x^2$, $x^2y'' + xy' - 4y = 0$, $y(1) = 2$, $y'(1) = 6$.
5. $x, x\ln x$, $x^2y'' - xy' + y = 0$, $y(1) = 3$, $y'(1) = 4$.

Find a general solution of the following differential equations.

6. $y'' - 4y = 0$.
7. $y'' - y' - 2y = 0$.
8. $y'' + y' - 2y = 0$.
9. $y'' - 4y' - 12y = 0$.

10. $y'' + 4y' + y = 0$.
11. $4y'' - 9y' + 2y = 0$.
12. $4y'' + 8y' - 5y = 0$.
13. $y'' + 2y' + y = 0$.
14. $y'' + 2\pi y' + \pi^2 y = 0$.
15. $9y'' - 12y' + 4y = 0$.
16. $4y'' + 4y' + y = 0$.
17. $25y'' - 20y' + 4y = 0$.
18. $y'' + 25y = 0$.
19. $y'' + 4y' + 5y = 0$.
20. $y'' - 2y' + 2y = 0$.
21. $(4D^2 - 4D + 17)y = 0$.
22. $(D^2 - 6D + 18)y = 0$.
23. $(D^2 + 9D)y = 0$.
24. $[D^2 - 2aD + (a^2 + b^2)]y = 0$.

Find a differential equation of the form $ay'' + by' + cy = 0$, for which the following functions are solutions.

25. e^{3x}, e^{-2x}.
26. $e^{x/4}, e^{-(3x)/4}$.
27. $1, e^{-2x}$.
28. e^{2x}, xe^{2x}.
29. e^{-x}, xe^{-x}.
30. e^{-3ix}, e^{3ix}.
31. $e^{-(a+ib)x}, e^{-(a-ib)x}$.
32. $e^{(5+3i)x}, e^{(5-3i)x}$.

Solve the following initial value problems.

33. $y'' - y = 0$, $y(0) = 0$, $y'(0) = 2$.
34. $y'' - y' - 12y = 0$, $y(0) = 4$, $y'(0) = -5$.
35. $y'' + y' - 2y = 0$, $y(0) = 0$, $y'(0) = 3$.
36. $\dfrac{d^2\theta}{dt^2} + g\theta = 0$, g constant, $\theta(0) = a$, constant, $\dfrac{d\theta}{dt}(0) = 0$.
37. $y'' - 4y' + 5y = 0$, $y(0) = 2$, $y'(0) = -1$.
38. $25y'' - 10y' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$.
39. $4y'' + 12y' + 9y = 0$, $y(0) = -1$, $y'(0) = 2$.
40. $9y'' + 6y' + y = 0$, $y(0) = 0$, $y'(0) = 1$.

Solve the following boundary value problems.

41. $y'' + 25y = 0$, $y(0) = 1$, $y(\pi) = -1$.
42. $y'' - 36y = 0$, $y(0) = 2$, $y(1/6) = 1/e$.
43. $y'' + 2y' + 2y = 0$, $y(0) = 1$, $y(\pi/2) = e^{-\pi/2}$.
44. $9y'' - 6y' + y = 0$, $y(1) = e^{1/3}$, $y(2) = 1$.
45. $y'' - 4y' + 3y = 0$, $y(0) = 1$, $y(1) = 0$.
46. Verify that $(D - 2)(D + 3) \sin x = (D + 3)(D - 2) \sin x = (D^2 + D - 6) \sin x$.
47. Show that $x^2Dy \neq D(x^2y)$.
48. Find the conditions under which the following equations hold.
    (i) $(D + a)[D + b(x)]f(x) = [D + b(x)][D + a]f(x)$, a constant.
    (ii) $[D + a(x)][D + b(x)]f(x) = [D + b(x)][D + a(x)]f(x)$.

Factorize the operator and find the solution of the following differential equations using the method of reduction of order or by the direct method.

49. $(D^2 + 5D + 4)y = 0$.
50. $(4D^2 + 8D + 3)y = 0$.

51. $(4D^2 + 12D + 9)y = 0.$

52. $(D^2 + 6D + 9)y = 0.$

53. $(D^2 - 4)y = 0.$

54. $(9D^2 + 6D + 1)y = 0.$

55. The displacement $x(t)$ of a particle is governed by the differential equation $\ddot{x} + \dot{x} + bx = c\dot{x}$, $b > 0$. For what values of b and c is the motion of the particle oscillatory?

56. Find all non-trivial solutions of the boundary value problem
$$y'' + \omega^2 y = 0,\ y(0) = 0,\ y(\pi) = 0.$$

57. Find all the non-trivial solutions of the boundary value problem
$$y'' + \omega^2 y = 0,\ y'(0) = 0,\ y'(\pi) = 0.$$

58. Find all non-trivial solutions of the boundary value problem
$$y'' + \omega^2 y = 0,\ y(0) = 0,\ y'(\pi) = 0.$$

59. If $a^2 > 4b$, then show that the solution of the differential equation $y'' + ay' + by = 0$ can be expressed as $y(x) = e^{px}(A \cosh qx + B \sinh qx)$ where $p = -a/2$ and $q = \sqrt{a^2 - 4b}/2$.

60. The motion of a damped mechanical system is governed by the linear differential equation $m\ddot{y} + c\dot{y} + ky = 0$ in which m (mass), k (spring modulus), c (damping factor) are positive constants and dot denotes derivative with respect to time t. Discuss the behaviour of the general solution when $t \to \infty$ in the following three cases: (i) $c^2 > 4mk$ (over damping), (ii) $c^2 < 4mk$ (under damping), (iii) $c^2 = 4mk$ (critical damping). In each case, obtain the solution subject to the initial conditions $y(0) = 0$, $\dot{y}(0) = v_0$.

Find the solution of the following differential equations, if one of its solutions is known.

61. $y'' - y' - 6y = 0$, $y_1 = e^{-2x}$.

62. $y'' + 3y' - 4y = 0$, $y_1 = e^x$.

63. $(x^2 - 1)y'' - 2xy' + 2y = 0$, $y_1 = x$, $x \neq \pm 1$.

64. $x^2y'' + xy' + (x^2 - 1/4)y = 0$, $x > 0$, $y_1 = x^{-1/2} \sin x$.

65. $(x - 2)y'' - xy' + 2y = 0$, $x \neq 2$, $y_1 = e^x$.

5.3.4 Solution of Higher Order Homogeneous Linear Equations with Constant Coefficients

In this section, we shall extend the methods discussed in section 5.3.2, for the solution of higher order linear homogeneous equations with constant coefficients.

Consider the nth order homogeneous linear equation with constant coefficients
$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = 0. \tag{5.38}$$
We attempt to find a solution of the form $y = e^{mx}$, as in the case of second order equations. Substituting $y = e^{mx}$, $y^{(k)} = m^k e^{mx}$, $k = 1, 2, \ldots, n$ in Eq. (5.38) and cancelling e^{mx}, we obtain the characteristic equation as
$$a_0 m^n + a_1 m^{n-1} + \ldots + a_{n-1} m + a_n = 0. \tag{5.39}$$
The degree of this algebraic equation is same as the order of the differential equation. This equation has n roots. All the roots may be real and distinct, all or some of the roots may be equal, all or some of the roots may be complex. Consider the following cases.

Real and distinct roots

Let the polynomial equation (5.39) have all real and distinct roots as $m_1, m_2, \ldots m_n$. Then the n solutions

$$y_1(x) = e^{m_1 x}, y_2(x) = e^{m_2 x}, \ldots, y_n(x) = e^{m_n x} \tag{5.40}$$

are the linearly independent solutions of the differential equation (5.38). Since $m_1 \neq m_2 \neq \ldots \neq m_n$, it can be easily shown that the Wronskian of the solutions $y_1, y_2, \ldots, y_n$ given in Eq. (5.40) does not vanish and therefore they are linearly independent solutions.

Hence, the set of the solutions forms a basis and the general solution is given by

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}. \tag{5.41}$$

Example 5.20 Find the general solution of the differential equation

$$y''' - 2y'' - 5y' + 6y = 0.$$

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^3 - 2m^2 - 5m + 6 = 0.$$

The roots of this equation are $m = 1, -2, 3$. Since the roots are real and distinct, the general solution of the equation is given by

$$y(x) = Ae^x + Be^{-2x} + Ce^{3x}.$$

Example 5.21 Solve the differential equation $y''' - y'' - 4y' + 4y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^3 - m^2 - 4m + 4 = 0 \text{ or } (m - 1)(m^2 - 4) = 0.$$

The roots of this equation are $m = 1, -2, 2$ which are real and distinct. The general solution of the equation is given by

$$y(x) = Ae^x + Be^{-2x} + Ce^{2x}.$$

Example 5.22 Solve the differential equation $y^{iv} - 5y'' + 4y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as.

$$m^4 - 5m^2 + 4 = 0 \quad \text{or} \quad (m^2 - 4)(m^2 - 1) = 0.$$

The roots of this equation are $m = -1, 1, -2, 2$. The general solution is

$$y(x) = Ae^{-x} + Be^x + Ce^{-2x} + De^{2x}.$$

Example 5.23 Solve the differential equation $4y^{iv} - 12y''' - y'' + 27y' - 18y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$4m^4 - 12m^3 - m^2 + 27m - 18 = 0.$$

We find that $m = 1$ is a root. We write the equation as

$$(m - 1)(4m^3 - 8m^2 - 9m + 18) = 0, (m - 1)(m - 2)(4m^2 - 9) = 0.$$

The roots of the characteristic equation are $m = 1, 2, 3/2, -3/2$. The general solution is

$$y(x) = Ae^x + Be^{2x} + Ce^{-3x/2} + De^{3x/2}.$$

Example 5.24 Solve the initial value problem

$$y''' - 6y'' + 11y' - 6y = 0,\ y(0) = 0,\ y'(0) = -4,\ y''(0) = -18.$$

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^3 - 6m^2 + 11m - 6 = 0, \quad \text{or} \quad (m-1)(m-2)(m-3) = 0.$$

The roots of this equation are $m = 1, 2, 3$ and the general solution is

$$y(x) = Ae^x + Be^{2x} + Ce^{3x}.$$

Substituting the initial conditions, we get

$$y(0) = 0 = A + B + C,\ y'(0) = -4 = A + 2B + 3C,\ y''(0) = -18 = A + 4B + 9C.$$

Solving, we obtain $A = 1$, $B = 2$ and $C = -3$. Hence, the particular solution is $y(x) = e^x + 2e^{2x} - 3e^{3x}$.

Real multiple roots

The characteristic equation (5.39) may have some multiple roots. Let r be the multiplicity of the root m_1, that is the root $m = m_1$ is repeated r times. Let the remaining $n - r$ roots be real and distinct. Substituting $m = m_1$ we obtain $y_1(x) = e^{m_1 x}$ as one of the solutions. We shall now show that the remaining $r - 1$ linearly independent solutions corresponding to the multiple root $m = m_1$ are given by

$$xy_1, x^2y_1, \ldots, x^{r-1}y_1.$$

That is, the linearly independent solutions in this case are

$$e^{m_1x}, xe^{m_1x}, x^2e^{m_1x}, \ldots, x^{r-1}e^{m_1x} \tag{5.42}$$

since the Wronskian of these solutions $W \neq 0$.

If
$$L[y] = a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_{n-1}y' + a_ny$$

then, substituting $y = e^{mx}$ in this equation, we get

$$\begin{aligned} L[e^{mx}] &= [a_0m^n + a_1m^{n-1} + \ldots + a_n]e^{mx} \\ &= (m - m_1)^r g(m)e^{mx},\ g(m_1) \neq 0 \end{aligned} \tag{5.43}$$

since $m = m_1$ is a multiple root of multiplicity r. Consider now m as a parameter. Differentiating Eq. (5.43) with respect to m, we get

$$\frac{d}{dm} L[e^{mx}] = r(m - m_1)^{r-1} g(m)e^{mx} + (m - m_1)^r \frac{d}{dm}[g(m)e^{mx}].$$

Now, L is a linear differentiable operator with respect to the independent variable x. Since m and x are independent, we obtain

$$\begin{aligned} \frac{d}{dm} L[e^{mx}] &= L\left[\frac{d}{dm} e^{mx}\right] = L[xe^{mx}] \\ &= r(m - m_1)^{r-1} g(m)e^{mx} + (m - m_1)^r \frac{d}{dm}[g(m)e^{mx}]. \end{aligned} \tag{5.44}$$

Since the right hand side of Eq. (5.44) vanishes at $m = m_1$, xe^{m_1x} is also a solution of the differential equation. Differentiating Eq. (5.44) with respect to m, we get

$$\frac{d}{dm} L[xe^{mx}] = L\left[\frac{d}{dm}(xe^{mx})\right] = L[x^2 e^{mx}]$$

$$= r(r-1)(m-m_1)^{r-2} g(m)e^{mx} + 2r(m-m_1)^{r-1} \frac{d}{dm}[g(m)e^{mx}]$$

$$+ (m-m_1)^r \frac{d^2}{dm^2}[g(m)e^{mx}]. \tag{5.45}$$

The right hand side of Eq. (5.45) vanishes at $m = m_1$ again. Hence, $x^2 e^{m_1 x}$ is also a solution. After $r-1$ differentiations, the first term on the right hand side is obtained as $r!(m-m_1)g(m)e^{mx}$ which vanishes for $m = m_1$. The other terms also vanish for $m = m_1$. Therefore, $x^{r-1}e^{m_1 x}$ is also a solution. If we differentiate one more time, that is r times, the first term on the right hand side becomes $r!g(m)e^{mx}$ which does not vanish at $m = m_1$, showing that $x^r e^{m_1 x}$ is not a solution. Hence, we find that $e^{m_1 x}, xe^{m_1 x}, x^2 e^{m_1 x}, \ldots, x^{r-1}e^{m_1 x}$ are the linearly independent solutions corresponding to the multiple root $m = m_1$. For example, if $m = m_1$ is a multiple root of order 3, then $e^{m_1 x}$, $xe^{m_1 x}$ and $x^2 e^{m_1 x}$ are the linearly independent solutions.

Example 5.25 Solve the differential equation $y''' - 3y' - 2y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^3 - 3m - 2 = 0, \quad \text{or} \quad (m+1)(m^2 - m - 2) = 0$$

or

$$(m+1)^2 (m-2) = 0, \quad \text{or} \quad m = -1, -1, 2.$$

Corresponding to the double root $m = -1$, the linearly independent solutions are e^{-x} and xe^{-x}. Hence, the general solution is

$$y(x) = Ae^{2x} + (Bx + C)e^{-x}.$$

Example 5.26 Solve the differentiable equation $8y''' - 12y'' + 6y' - y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$8m^3 - 12m^2 + 6m - 1 = 0, \quad \text{or} \quad (2m-1)^3 = 0, \quad \text{or} \quad m = 1/2, 1/2, 1/2.$$

The general solution is $\quad y(x) = (A + Bx + Cx^2)e^{x/2}$.

Example 5.27 Solve the initial value problem

$$y''' + 3y'' - 4y = 0, \; y(0) = 1, \; y'(0) = 0, \; y''(0) = 1/2.$$

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^3 + 3m^2 - 4 = 0, \quad \text{or} \quad (m-1)((m^2 + 4m + 4) = 0, \quad \text{or} \quad (m-1)(m+2)^2 = 0.$$

The roots of this equation are $m = 1, -2, -2$. The general solution is

$$y(x) = Ae^x + (Bx + C)e^{-2x}.$$

Substituting in the initial conditions, we get

$$y(0) = 1 = A + C,$$

$$y'(x) = Ae^x + Be^{-2x} - 2(Bx + C)e^{-2x}, \; y'(0) = 0 = A + B - 2C,$$

$$y''(x) = Ae^x - 4Be^{-2x} + 4(Bx + C)e^{-2x}, \; y''(0) = \frac{1}{2} = A - 4B + 4C.$$

The solution of the system is $A = 1/2$, $B = 1/2$ and $C = 1/2$. The particular solution is

$$y(x) = [e^x + (x+1)e^{-2x}]/2.$$

Simple complex roots

Since the coefficients in the characteristic equation (5.39) are real, complex roots occur in conjugate pairs. That is, if $p + iq$ is a root, then $p - iq$ is also a root. In this case, the linearly independent solutions are given by $e^{px}\cos qx$ and $e^{px}\sin qx$. If the characteristic equation (5.39) has r complex conjugate pairs of roots $p_k \pm i q_k$, $k = 1, 2, \ldots r$, then the corresponding linearly independent solutions are $e^{p_1 x}\cos q_1 x$, $e^{p_1 x}\sin q_1 x$, $e^{p_2 x}\cos q_2 x$, $e^{p_2 x}\sin q_2 x, \ldots, e^{p_r x}\cos q_r x$ and $e^{p_r x}\sin q_r x$.

Example 5.28 Solve the differential equation $y^{iv} + 5y'' + 4y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^4 + 5m^2 + 4 = 0, \quad \text{or} \quad (m^2+4)(m^2+1) = 0.$$

The roots are $m = \pm i, \pm 2i$. The general solution is

$$y(x) = A\cos x + B\sin x + C\cos 2x + D\sin 2x.$$

Example 5.29 Solve the initial value problem

$$y^{iv} + 2y''' + 11y'' + 18y' + 18 = 0,\ y(0) = 2,\ y'(0) = 3,\ y''(0) = -11,\ y'''(0) = -23.$$

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^4 + 2m^3 + 11m^2 + 18m + 18 = 0 \quad \text{or} \quad (m^2+9)(m^2+2m+2) = 0.$$

The roots are $m = \pm 3i, -1 \pm i$. The general solution is

$$y(x) = A\cos 3x + B\sin 3x + e^{-x}(C\cos x + D\sin x).$$

Substituting in the initial conditions, we get

$$\begin{aligned}
y(0) &= 2 = A + C,\\
y'(x) &= -3A\sin 3x + 3B\cos 3x + e^{-x}(-C\sin x + D\cos x - C\cos x - D\sin x),\\
y'(0) &= 3 = 3B + D - C,\\
y''(x) &= -9A\cos 3x - 9B\sin 3x\\
&\quad + e^{-x}[-(C+D)\cos x + (C-D)\sin x + (C+D)\sin x + (C-D)\cos x]\\
&= -9A\cos 3x - 9B\sin 3x + 2e^{-x}[C\sin x - D\cos x],\\
y''(0) &= -11 = -9A - 2D,\\
y'''(x) &= 27A\sin 3x - 27B\cos 3x + 2e^{-x}[C\cos x + D\sin x - C\sin x + D\cos x],\\
y'''(0) &= -23 = -27B + 2C + 2D.
\end{aligned}$$

Therefore, we have the system of equations

$$A + C = 2, \quad 3B - C + D = 3,$$
$$-9A - 2D = -11, \quad -27B + 2C + 2D = -23.$$

The solution of this system is $A = 1$, $B = 1$, $C = 1$, $D = 1$. The particular solution is

$$y(x) = \cos 3x + \sin 3x + e^{-x}(\cos x + \sin x).$$

Example 5.30 Find the non trivial solutions of the boundary value problem

$$y^{\text{iv}} - \omega^4 y = 0,\ y(0) = 0,\ y''(0) = 0,\ y(l) = 0,\ y''(l) = 0.$$

Solution Assume the solution to be of the form $y = e^{mx}$. The characteristic equation is given by

$$m^4 - \omega^4 = 0, \quad \text{or} \quad m^2 = \pm\,\omega^2, \quad \text{or} \quad m = \pm\,\omega, \pm\, i\omega.$$

The general solution is given by

$$\begin{aligned} y(x) &= A_1 e^{\omega x} + B_1 e^{-\omega x} + C\cos\omega x + D\sin\omega x \\ &= A\cosh\omega x + B\sinh\omega x + C\cos\omega x + D\sin\omega x \end{aligned}$$

Substituting in the initial conditions, we get

$$y(0) = A + C = 0.$$

$$y'' = \omega^2\,[A\cosh\omega x + B\sinh\omega x - C\cos\omega x - D\sin\omega x];$$

$$y''(0) = \omega^2(A - C) = 0, \quad \text{or} \quad A - C = 0.$$

Solving the two equations, we get $A = 0$, $C = 0$. We also have

$$y(l) = 0 = B\sinh\omega l + D\sin\omega l,\ y''(l) = 0 = B\sinh\omega l - D\sin\omega l.$$

Adding, we obtain $2B\sinh\omega l = 0$, or $B = 0$. Therefore, we obtain $D\sin\omega l = 0$. Since, we require non-trivial solutions, we have $D \neq 0$. Hence, $\sin\omega l = 0 = \sin n\pi$, $n = 1, 2, \ldots$.

Therefore, $\omega = n\pi/l$, $n = 1, 2, \ldots$

The solution of the boundary value problem is

$$y_n(x) = D_n \sin(n\pi x/l),\ n = 1, 2, \ldots,$$

By superposition principle, the solution can be written as

$$y(x) = \sum_{n=1}^{\infty} D_n \sin(n\pi x/l).$$

Multiple complex roots

This case is a combination of the two earlier cases of real multiple roots and simple complex roots. Now, if $p + iq$ is a multiple root of order m, then $p - iq$ is also a multiple root of order m. For example, if $p_1 + iq_1$ is a double root, then $p_1 - iq_1$ is also a double root. The corresponding linearly independent solutions are

$$e^{p_1 x}\cos q_1 x,\ e^{p_1 x}\sin q_1 x,\ xe^{p_1 x}\cos q_1 x,\ xe^{p_1 x}\sin q_1 x.$$

Example 5.31 Solve the differential equation $y^{\text{iv}} + 32y'' + 256y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^4 + 32m^2 + 256 = 0, \text{ or } (m^2 + 16)^2 = 0.$$

The roots of this equation are the double roots $m = \pm\, 4i$. Therefore, the general solution is

$$y(x) = (Ax + B)\cos 4x + (Cx + D)\sin 4x.$$

Exercise 5.3

Find the general solution of the following differential equations.

1. $y''' - 9y' = 0$.
2. $2y''' + y'' - 13y' + 6y = 0$.
3. $3y''' - 2y'' - 3y' + 2y = 0$.
4. $y^{iv} - 13y'' + 36y = 0$.
5. $4y^{iv} - 12y''' + 7y'' + 3y' - 2y = 0$.
6. $y^{iv} + y''' - 4y'' - 4y' = 0$.
7. $8y^{iv} - 6y''' - 7y'' + 6y' - y = 0$.
8. $144y^{iv} - 25y'' + y = 0$.
9. $y''' - 2y'' + y' = 0$.
10. $y''' + 4y'' + 5y' + 2y = 0$.
11. $y''' - 2y'' - 4y' + 8y = 0$.
12. $27y''' - 27y'' + 9y' - y = 0$.
13. $y^{iv} - 11y''' + 35y'' - 25y' = 0$.
14. $y^{iv} - 3y''' + 3y'' - y' = 0$.
15. $4y^{iv} + 4y''' - 3y'' - 2y' + y = 0$.
16. $9y^{iv} - 66y''' + 157y'' - 132y' + 36y = 0$.
17. $y''' + y' = 0$.
18. $y''' - 2y'' + 4y' - 8y = 0$.
19. $y''' + 5y'' + 8y' + 6y = 0$.
20. $y''' - 7y'' + 19y' - 13y = 0$.
21. $y^{iv} + 8y'' - 9y = 0$.
22. $y^{iv} + y''' + 14y'' + 16y' - 32y = 0$.
23. $4y^{iv} + 101y'' + 25y = 0$.
24. $y^{iv} + 2y''' - 9y'' - 10y' + 50y = 0$.
25. $y^{iv} + 50y'' + 625y = 0$.
26. $y^{iv} + 2y'' + y = 0$.

Find a homogeneous linear differential equation with real constant coefficients of lowest order which has the following particular solution.

27. $5 + e^x + 2e^{3x}$.
28. $e^{-x} + \cos 5x + 3 \sin 5x$.
29. $xe^{-x} + e^{2x}$.
30. $1 + x + e^x - 3e^{3x}$.
31. $x^2e^{2x} + 2e^{-2x}$.
32. $3 \cos 2x + 5 \sinh 3x$.

Solve the following initial value problems.

33. $y''' - 2y'' - 5y' + 6y = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$.
34. $4y''' - 4y'' - 9y' + 9y = 0$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$.
35. $y''' - 5y'' + 7y' - 3y = 0$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -5$.
36. $y^{iv} - 2y''' - 3y'' + 4y' + 4y = 0$, $y(0) = 3$, $y'(0) = 3$, $y''(0) = 3$, $y'''(0) = 6$.
37. $y^{iv} + y'' = 0$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = -1$, $y'''(0) = -1$.
38. $y''' - y'' + 4y' - 4y = 0$, $y(0) = 0$, $y'(0) = 3$, $y''(0) = -5$.
39. $y''' + y'' - 2y = 0$, $y(0) = 2$, $y'(0) = 2$, $y''(0) = -3$.
40. $y^{iv} - 3y''' = 0$, $y(0) = 2$, $y'(0) = 5$, $y''(0) = 15$, $y'''(0) = 27$.

Find the solution of the following differential equations satisfying the given conditions.

41. $y''' + \pi^2 y' = 0$, $y(0) = 0$, $y(1) = 0$, $y'(0) + y'(1) = 0$.
42. $y''' - 36y' = 0$, $y(0) = 2$, $y'(0) = 12$, $y'(1) = 6 \sinh (6) + 12 \cosh (6)$.
43. $y^{iv} + 13y'' + 36y = 0$, $y(0) = 0$, $y''(0) = 0$, $y(\pi/2) = -1$, $y'(\pi/2) = -4$.
44. $y^{iv} - \omega^4 y = 0$, $\omega \neq 0$, $y(0) = 0$, $y''(0) = 0$, $y(\pi) = 0$, $y''(\pi) = 0$.
45. $y^{iv} + 10y'' + 9y = 0$, $y'(0) = 0$, $y'''(0) = 0$, $y'(\pi/2) = 5$, $y'''(\pi/2) = -53$.

5.4 Solution of Non-Homogeneous Linear Equations

In the previous section, we have discussed methods for finding the general and particular solutions