L33 05 April 2021

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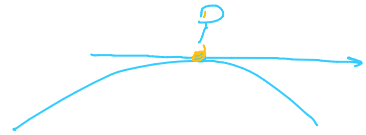
Parametric eqn of the tangent line.



$x = t^2 - 1$, $y = t+1$, $z = \frac{t}{t+1}$, $t = 2$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = (t^2-1)\hat{i} + (t+1)\hat{j} + \left(\frac{t}{t+1}\right)\hat{k}$

$\frac{t}{t+1}$

$r'(t)\ \frac{d\vec{r}}{dt} = (2t)\hat{i} + \hat{j} + \left(\frac{1}{t+1}\right)^2 \hat{k}$ $\quad t=2$

$\frac{2}{2+1}$ $\quad \frac{\cancel{(t+1)}\cdot 1 - \cancel{t(1)}}{(t+1)^2} = \frac{1}{(t+1)^2}$

$r'(2) = \left.\frac{d\vec{r}}{dt}\right|_{t=2} = 4\hat{i} + \hat{j} + \frac{1}{9}\hat{k}$

$$\boxed{\vec{r}(2) = 3\hat{i} + 3\hat{j} + \frac{2}{3}\hat{k}}$$

$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n} = t$

$$\boxed{\vec{r}'(t) = \underbrace{\left(3, 3, \frac{2}{3}\right)}_{(x_0, y_0, z_0)} + t\underbrace{\left(4, 1, \frac{1}{9}\right)}_{l,\ m,\ n}}$$

$\frac{x-3}{4} = \frac{y-3}{1} = \frac{z-2/3}{1/9} = t$

$$\boxed{\vec{r}'(t) = (3+4t)\hat{i} + (3+t)\hat{j} + \left(\frac{2}{3} + \frac{1}{9}t\right)\hat{k}}$$

$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n} = t$

$P(x_0, y_0, z_0)$

pt. from which the line passes.

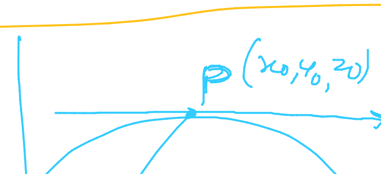
$x = x_0 + lt$, $y = y_0 + mt$, $z = z_0 + nt$

$\vec{r}(t) = (x_0 + lt)\hat{i} + (y_0 + mt)\hat{j} + (z_0 + nk)\hat{k}$

$P\left(3, 3, \frac{2}{3}\right)$

$\vec{r}(t) =$

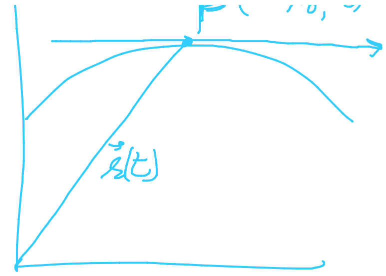
\# $x = \sin t$ $\quad y = \cos t$, $\quad z = t$, $\quad t = \pi/4$



$P(x_0, y_0, z_0)$

$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$

$x = \sin t \quad y = \cos t, \quad z = t, \quad t = \pi/4$



$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r}(t) = (\sin t)\hat{i} + (\cos t)\hat{j} + t\hat{k}$

$x_0 \quad y_0 \quad z_0$

$\vec{r}(\pi/4) = \left(\sin\frac{\pi}{4}\right)\hat{i} + \left(\cos\frac{\pi}{4}\right)\hat{j} + \frac{\pi}{4}\hat{k} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{\pi}{4}\hat{k}$

$\frac{d\vec{r}}{dt} = (\cos t)\hat{i} - (\sin t)\hat{j} + \hat{k}$

$l \quad m \quad n$

$\left.\frac{dr}{dt}\right|_{\pi/4} = \left(\cos\frac{\pi}{4}\right)\hat{i} - \left(\sin\frac{\pi}{4}\right)\hat{j} + \hat{k} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} + \hat{k}$

$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n} = t$

$\boxed{\vec{r}(t) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}t\right)\hat{i} + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}t\right)\hat{j} + \left(\frac{\pi}{4} + t\right)\hat{k}}$

$x = x_0 + lt$

$y = y_0 + mt$

$z = z_0 + nt$

$\vec{r}(t) = (x_0 + lt)\hat{i} + (y_0 + mt)\hat{j} + (z_0 + nt)\hat{k}$

$x = t, \quad y = e^t, \quad z = 1 \quad$ at $t = 1$

$P(1, e, 1)$

$\vec{r}(t) = t\hat{i} + e^t\hat{j} + \hat{k}$

$\vec{r}(1) = \hat{i} + e\hat{j} + \hat{k}$

$\frac{d\vec{r}(t)}{dt} = \hat{i} + e^t\hat{j} + 0 \Rightarrow \left.\frac{d\vec{r}}{dt}\right|_{t=1} = \hat{i} + e\hat{j} + 0\hat{k} \longrightarrow (1, e, 0)$

$\vec{r}(t) = (x_0 + lt)\hat{i} + (y_0 + mt)\hat{j} + (z_0 + nt)\hat{k}$

$= (1 + t)\hat{i} + (e + et)\hat{j} + (1 + 0t)\hat{k}$

$\boxed{\vec{r}(t) = (1 + t)\hat{i} + (e + et)\hat{j} + \hat{k}}$

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(1) $\vec{v}(t) = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

① $\vec{v}(t) = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$\vec{v}'(t) = v_1'\hat{i} + v_2'\hat{j} + v_3'\hat{k}$

$\vec{v}''(t) = v_1''\hat{i} + v_2''\hat{j} + v_3''\hat{k}$

$\vec{a}\cdot\vec{b} = |a||b|\cos\theta$

$\vec{a}\times\vec{b} = (|a|\,|b|\sin\theta)\cdot\hat{n}$

② $(\vec{u}+\vec{v})' = \vec{u}' + \vec{v}'$

③ $(f(t)\,\vec{u}(t))' = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$

④ $(\vec{u}(t)\cdot\vec{v}(t))' = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

⑤ $(\vec{u}(t)\times\vec{v}(t))' = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

Length of a Space Curve

Let the curve c be represented in the parametric form

$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, $a \le t \le b$

$$\text{Length} = l = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

$$\text{or} \quad \int_a^b \sqrt{r'(t)\cdot r'(t)}\,dt$$

Problem 1. Find the length of the following curve:

$\overrightarrow{r(t)} = (a\cos t)\hat{\imath} + (a\sin t)\hat{\jmath}, \quad 0 \le t \le 2\pi$

$x = a\cos t$

$\frac{dx}{dt} = -a\sin t$

$y = a\sin t$

$\frac{dy}{dt} = a\cos t$

$$l = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2}\,.\,dt$$

$$= \int^{2\pi} \sqrt{a^2(\sin^2 t + \cos^2 t)}\,dt = \int^{2\pi} a\,.\,dt = a[t]_0^{2\pi}$$

$$= \int_0^{2\pi} \sqrt{a^2(\sin^2 t + \cos^2 t)}\, dt = \int_0^{2\pi} a \cdot dt = a \cdot [t]_0^{2\pi}$$

$$\boxed{\ell = 2\pi a}$$

Problem 2. Find the length of the following curve:

$\overrightarrow{r(t)} = (\cos t)\hat{\imath} + (\sin t)\hat{\jmath} + (3t)\hat{k}, \quad -2\pi \leq t \leq 2\pi$

$$\ell = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \cdot dt = \int_{-2\pi}^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 9} \cdot dt$$

$$= \int_{-2\pi}^{2\pi} \sqrt{1+9} \cdot dt = \sqrt{10}\,[dt]_{-2\pi}^{2\pi} = \sqrt{10}\,[2\pi - (-2\pi)]$$

$$= 4\pi\sqrt{10}$$

$$|\vec{r}(t)| = \sqrt{x^2 + y^2 + z^2}$$

$$\vec{r}(t) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath} + \frac{d^2z}{dt^2}\hat{k}$$

Speed $\quad |\vec{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$