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Classification of Partial Diff. Equations

Let us consider a 2nd order partial diff. equation

$$A\cdot\frac{\partial^2 u}{\partial x^2} + B\cdot\frac{\partial^2 u}{\partial x\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$$

where $u = f(x,y)$

or

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$$

or

$$Ar + Bs + Ct + Dp + Eq + Fu = 0$$

$$Ar + Bs + Ct + f(x,y,u,p,q) = 0 \quad \text{——— (1)}$$

Equation (1) will be

1. Hyperbolic iff $B^2 - 4AC > 0$
2. Parabolic iff $B^2 - 4AC = 0$
3. Elliptic iff $B^2 - 4AC < 0$

Note :- Coefficient of 2nd order partial derivatives will decide the nature of PDEs.

Problem 1. $\frac{\partial^2 u}{\partial x\partial y} = 3\frac{\partial u}{\partial y}$

$$\frac{\partial^2 u}{\partial x\partial y} - 3\frac{\partial u}{\partial y} = 0 \Rightarrow s - 3q = 0$$

$A = 0 \quad B = 1 \quad C = 0$

$$B^2 - 4AC = (1)^2 - 4(0)(0) = 1 > 0$$

$\boxed{\text{Hyperbolic}}$

$r = \frac{\partial^2 u}{\partial x^2}$

$s = \frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}$

$t = \frac{\partial^2 u}{\partial y^2}$

$p = \frac{\partial u}{\partial x}$

$q = \frac{\partial u}{\partial y}$

$\delta y.$

Problem 2. $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$u_{xx} = r \rightarrow A$

$u_{xy} = s \rightarrow B$

$u_{yy} = t \rightarrow C$

$u_{xx} + 2u_{xy} + u_{yy} = 0$

$r + 2s + t = 0$

$A = 1, \quad B = 2, \quad C = 1$

$B^2 - 4AC = (2)^2 - 4(1)(1) = 4 - 4 = 0$

Parabolic

Problem 3. $\dfrac{\partial^2 u}{\partial x^2} + 3\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial u}{\partial x}$

$u_{xx} + 3u_{yy} - u_x = 0$

$r + 3t - p = 0$

$A = 1, \quad B = 0, \quad C = 3.$

$B^2 - 4AC = (0)^2 - 4(1)(3) = -12 < 0$

Elliptic

Problem 4. $y\dfrac{\partial^2 u}{\partial x^2} + 2x\dfrac{\partial^2 u}{\partial x \partial y} + y\dfrac{\partial^2 u}{\partial y^2} = 0$

$y\,u_{xx} + 2x\,u_{xy} + y\,u_{yy} = 0$

$A = y \quad B = 2x \quad C = y$

$B^2 - 4AC = 4x^2 - 4y^2 = 4(x^2 - y^2)$

Case (1)

If $x^2 - y^2 > 0$ then Hyperbolic

$x^2 - y^2 = 0$ then parabolic

$x^2 - y^2 < 0$ then Elliptic

Problem 5. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(Laplace Equation)

$u_{xx} + u_{yy} = 0$

$A = 1, \quad B = 0, \quad C = 1$

$B^2 - 4AC = 0 - 4(1)(1) = -4 < 0$ Elliptic

Problem 6. $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(1D Wave Equation) $u(x,t)$

$c^2 u_{xx} - u_{tt} = 0$

$A = c^2, \quad B = 0, \quad C = -1$

$B^2 - 4AC = 0 - 4(c^2)(-1) = 4c^2 > 0$

Hyperbolic

Problem 7. $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(1D Heat Equation)

$c^2 u_{xx} - u_t = 0$

$A = c^2, \quad B = 0, \quad C = 0$

$B^2 - 4AC = 0 - 4(c^2)(0) = 0$ (Parabolic)

The nature of PDE: $(1-y)\frac{\partial^2 u}{\partial x^2} + 2x\frac{\partial^2 u}{\partial x \partial y} + (1+y)\frac{\partial^2 u}{\partial y^2} = 0$ is:

(A) Hyperbolic

(B) Parabolic

(C) Elliptic

(D) All of above is possible

$A = (1-y) \quad B = 2x \quad C = (1+y)$

$B^2 - 4AC = 4x^2 - 4(1-y)(1+y) = 4x^2 - 4(1-y^2)$

$= 4(x^2+y^2-1)$

if $x^2+y^2-1 > 0 \rightarrow$ Hyperbolic

$x^2+y^2-1 = 0$ Parabolic

$x^2+y^2-1 < 0$ Elliptic

Q42. The partial differential equation $\frac{\partial^2 u}{\partial x \partial y} = 3\frac{\partial u}{\partial x}$ is classified as

(a) Elliptic (b) Hyperbolic (c) Parabolic (d) None of these

$\frac{\partial^2 u}{\partial x \partial y} = 3\frac{\partial u}{\partial x}$

$s - 3p = 0$

$A = 0, B = 1, C = 0$ $\quad B^2 - 4AC = (1)^2 - 0 = 1 > 0$

Q43. If $z = f\left(\frac{x}{y}\right)$, a is any fixed constant and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. Then the partial differential equation of minimum order satisfied by z is

(a) $p = aq$ (b) $z = pq$ (c) $z = ap + qb$ (d) $px + qy = 0$

$z = f\left(\frac{x}{y}\right)$

$\frac{\partial z}{\partial x} = f'\left(\frac{x}{y}\right) \cdot \frac{1}{y} \Rightarrow yp = f'(x/y)$

$\frac{\partial z}{\partial y} = f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) \Rightarrow \frac{y^2 q}{-x} = f'(x/y)$

$$\frac{\partial z}{\partial y} = f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) \Rightarrow \frac{y^2 q}{-x} = f'(x/y)$$

$$yp = \frac{y^2 q}{-x} \Rightarrow -x\cancel{y}p = y^{\cancel{2}} q$$

$$\boxed{xp + yq = 0}$$