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A Generalization of the Classical Kelly Betting Formula to the Case of Temporal Correlation

Joseph D. O'Brien, Kevin Burke, Mark E. Burke, and B. Ross Barmish

Abstract—For sequential betting games, Kelly’s theory, aimed at maximization of the logarithmic growth of one’s account value, involves optimization of the so-called betting fraction K . In this Letter, we extend the classical formulation to allow for temporal correlation among bets. To demonstrate the potential of this new paradigm, for simplicity of exposition, we mainly address the case of a coin-flipping game with even-money payoff. To this end, we solve a problem with *memory depth* m . By this, we mean that the outcomes of coin flips are no longer assumed to be i.i.d. random variables. Instead, the probability of heads on flip k depends on previous flips $k-1, k-2, \dots, k-m$. For the simplest case of n flips, with $m=1$, we obtain a closed form solution K_n for the optimal betting fraction. This generalizes the classical result for the memoryless case. That is, instead of fraction $K^* = 2p-1$ which pervades the literature for a coin with probability of heads $p \geq 1/2$, our new fraction K_n depends on both n and the parameters associated with the temporal correlation. Generalizations of these results for $m > 1$ and numerical simulations are also included. Finally, we indicate how the theory extends to time-varying feedback and alternative payoff distributions.

Index Terms—Stochastic systems, Markov processes, Finance, Control applications.

I. INTRODUCTION

IN Kelly’s 1956 seminal paper [1], the notion of *Expected Logarithmic Growth* (ELG) was introduced as the performance criterion for a memoryless repeated betting game. For a sequence of i.i.d. gambles, for example a coin flip with the probability of heads being $p > 1/2$, the theory leads to an *optimal betting fraction* K^* , which, owing to its constant nature from bet to bet is viewed as a time-invariant feedback gain. That is, with V_k being the account value after k plays, the optimal $(k+1)$ -th bet size is K^*V_k , where, for classical coin flipping with even-money payoff, $K^* = 2p-1$.

The ELG approach has resulted in a voluminous body of literature extending and applying the theory to other well-known gambling games such as Blackjack and sports betting considered in [2] and [3], asset management and stock trading as in [4]–[10]; see also the extensive bibliography in the textbook [11]. Papers such as [12]–[17] have also

covered related issues including asymptotics, problems related to aggressiveness of wagers and alternative risk metrics.

The main feature which differentiates this paper from existing work is our emphasis on the issue of *temporal correlation* among games. While it is standard to assume correlation among components of a multi-dimensional bet, for example, in modern portfolio theory [5], temporal correlation is an entirely different matter. Interestingly, although temporal effects are studied in the context of prediction for financial time-series, as in [18] and [19], this issue has received little attention in the context of bet sizing in Kelly’s ELG framework; e.g., see [20] where only one numerical example is considered.

Motivated by the fact that a bettor may gain an “edge” by taking advantage of temporal correlation, we generalize ELG theory to the case of a *history-cognizant* coin where each outcome is no longer i.i.d. but dependent on the previous m results. Our analysis of this new framework lays the groundwork for its use in financial applications with temporally correlated returns, in particular by relating the binary lattice model proposed in the sequel to stock price movements (up or down) over a sequence of time points. With this setting in mind, our primary analysis considers two-outcome, even-money random variables $X_k \in \{-1, 1\}$ with a time-invariant feedback gain governing the bet size which takes temporal autocorrelation into account. Although we also provide further extensions to accommodate ℓ possible outcomes given by $X_k \in \{x_1, x_2, \dots, x_\ell\}$ and time-varying feedback gains, our main focus is developing a new ELG theory in the presence of memory with arbitrary depth, $m \geq 1$.

The remainder of the paper is organized as follows: After formalizing the notion of *autocorrelated betting* in Section II, we consider the *history-cognizant coin* in Section III. Then, our main result and extensions are provided in Section IV. This includes, for n bets and memory depth $m=1$, a closed-form solution for the optimal betting fraction K_n , generalizing the classical $K^* = 2p-1$ result, and steady state analysis; Section V is devoted to proof of this main theorem. Section VI provides results for arbitrary memory depth $m \geq 1$ and model estimation, followed by numerical simulations and conclusions in the remaining two sections.

II. AUTOCORRELATED KELLY BETTING

For the sake of self-containment, before introducing autocorrelation, we review Kelly’s classical solution. Indeed, we start by considering a discrete-time even-money coin-flipping game with repeat i.i.d. bets and initial account value $V_0 > 0$. Letting

Joseph D. O’Brien is a doctoral candidate in MACSI, Department of Mathematics and Statistics, University of Limerick, Ireland, and, Kevin Burke is a faculty member and Mark E. Burke an emeritus faculty member in the same department. B. Ross Barmish is a faculty member in ECE at Boston University and emeritus, also in ECE, at the University of Wisconsin. E-mail: Joseph.O’Brien@ul.ie, Kevin.Burke@ul.ie, Mark.Burke@ul.ie, bob.barmish@gmail.com.

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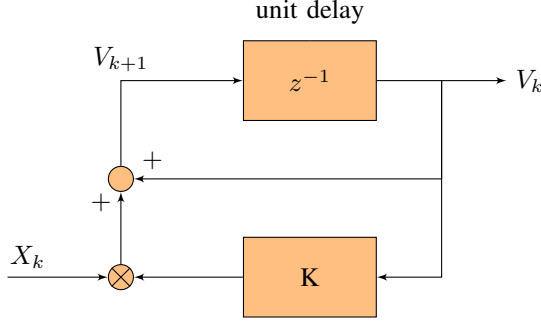


Fig. 1: Feedback control configuration

V_n denote the bettor's account value after n plays, the classical *Kelly* strategy is aimed at maximizing the expected value of the *logarithm* of V_n rather than simply its expected value. Letting $X_k \in \{-1, 1\}$ be a random variable which represents the pay-off from the k th coin toss where, $X_k = 1$ corresponds to a head and $X_k = -1$ corresponds to a tail, the $(k+1)$ -th bet is KV_k with $-1 \leq K \leq 1$. The quantity $|K|$ is referred to as the betting fraction with $K < 0$ representing a bet on tails rather than heads. Viewing V_k as a state, as noted for example in [14], this defines a linear time-invariant feedback control $u_k = KV_k$ for the nonlinear system as depicted in Figure 1, leading to the update

$$\begin{aligned} V_{k+1} &= V_k + X_k u_k \\ &= (1 + KX_k)V_k. \end{aligned}$$

We proceed to consider a game of n bets with outcomes given by the sample path

$$\mathbf{X} \doteq (X_0, X_1, X_2, \dots, X_{n-1}) \in \mathcal{X} \doteq \{-1, 1\}^n.$$

This being the case, the corresponding account value at terminal stage n , as a function of the pair (K, \mathbf{X}) , resulting from this sample path is given by

$$V_n(K, \mathbf{X}) = V_0 \prod_{k=0}^{n-1} [1 + KX_k],$$

and an optimal betting fraction is obtained by maximizing the Expected Logarithm Growth given by

$$\text{ELG}(K) = \frac{1}{n} \mathbb{E} \left\{ \log \left[\frac{V_n(K, \mathbf{X})}{V_0} \right] \right\}.$$

Since $\text{ELG}(K)$ above is independent of V_0 , in the sequel, without loss of generality, whenever convenient, it is assumed that $V_0 = 1$. In the standard i.i.d. setup, where p is the probability of a head, the ELG is maximized at $K^* = 2p - 1$.

We now proceed to generalize the standard approach by assuming a probability distribution over \mathcal{X} is available, and let $P_{\mathbf{X}}$ denote the probability of a sample path \mathbf{X} . This is a joint distribution over the components X_i of \mathbf{X} , and, at this high level of generality, this probability distribution is arbitrary. In the analysis to follow, we first provide a result for this general case which is abstract and then specialize to a scenario frequently encountered in practice. That is, we consider the case when the outcome of a given coin toss is correlated with

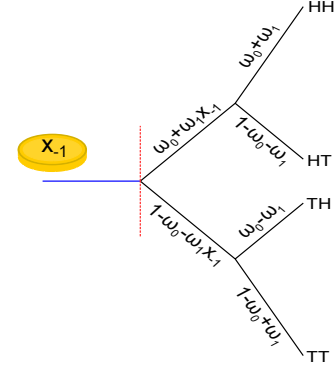


Fig. 2: Sample paths with $n = 2$ and $X_{-1} = x_{-1} \in \{-1, 1\}$

the previous m outcomes. We henceforth refer to m as the *memory depth* noting that a small value of m means that the k th outcome is only related to the recent history. In this case, it is straightforward to see that the probability $P_{\mathbf{X}}$ of a sequence \mathbf{X} reduces to

$$P_{\mathbf{X}} = \prod_{k=0}^{n-1} \Pr(X_k | X_{k-1}, X_{k-2}, \dots, X_{k-m})$$

which is initialized by the m events $X_{-m}, X_{-m+1}, \dots, X_{-1}$ prior the first outcome X_0 at stage $k = 0$.

III. THE HISTORY-COGNIZANT COIN

Building on the above, we consider the simplest case of an autocorrelated bet: a coin whose current flip is affected by the previous flip. This is a coin with Markov memory, i.e., the probability of a head on the k th flip is

$$\begin{aligned} \Pr(X_k = 1 | X_{k-1}, X_{k-2}, \dots) &= \Pr(X_k = 1 | X_{k-1}) \\ &= \omega_0 + \omega_1 X_{k-1} \end{aligned}$$

with ω_0 and ω_1 assumed to be known; see Section VI for considerations of how these parameters may be estimated. For the parameterized linear function above, it is readily verified that the conditions $|\omega_1| < 0.5$, $|\omega_1| < \omega_0 < 1 - |\omega_1|$ guarantee that $\Pr(X_k = 1 | X_{k-1}) \in (0, 1)$. Now, via a straightforward calculation, these requirements reduce to

$$\left| \omega_0 - \frac{1}{2} \right| + |\omega_1| < \frac{1}{2}$$

which we recognize as describing the interior of an ℓ^1 sphere, the so-called “diamond” with center $(1/2, 0)$ and radius $(1/2)$. In this setting, we have memory depth $m = 1$, and we assume that we have observed one coin toss prior to betting, i.e., $X_{-1} = x_{-1} \in \{1, -1\}$. Figure 2 shows some illustrative sample paths, consistent with the formulation presented above. As mentioned in Section I, this binary lattice can serve as a model for stock price movements over time categorized as “up” ($X_k = 1$) or “down” ($X_k = -1$).

IV. MAIN RESULT

In this section, we provide our main result whose proof is relegated to the next section. The first part of the theorem

below provides an abstract characterization of the optimal betting fraction K_n in terms of the expected number of heads $\mathbb{E}(H_n(\mathbf{X}))$ in the sample path \mathbf{X} of length n , i.e., it holds for arbitrary sample path distributions $P_{\mathbf{X}}$ including and beyond those considered here. It also points the way to the second part of the theorem which addresses the case of a history-cognizant coin and makes use of notation

$$p_0 \doteq \omega_0 + \omega_1 x_{-1}$$

corresponding to the unconditional probability that $X_0 = 1$,

$$p_\infty \doteq \frac{\omega_0 - \omega_1}{1 - 2\omega_1}$$

which is later seen to be the steady state unconditional probability of heads, and

$$\lambda_n \doteq \frac{1}{n} \left[\frac{1 - (2\omega_1)^n}{1 - 2\omega_1} \right]$$

which satisfies the condition

$$0 < \lambda_n < 1$$

since $|2\omega_1| < 1$ and tells us the relative weights of p_0 and p_∞ in the optimal solution.

Theorem: For n flips of the history-cognizant coin with memory-depth $m = 1$ and conditional probability of heads given by $\Pr(X_k = 1 | X_{k-1}) = \omega_0 + \omega_1 X_{k-1}$, the expected logarithmic growth $\text{ELG}(K)$ is maximized by

$$K_n = 2 \left\{ \frac{\mathbb{E}(H_n(\mathbf{X}))}{n} \right\} - 1,$$

where the expected value above is obtained as the convex combination

$$\frac{\mathbb{E}(H_n(\mathbf{X}))}{n} = \lambda_n p_0 + (1 - \lambda_n) p_\infty.$$

A. Special Cases, Generalizability and Remarks

The remainder of this section focusses on finer details of our theory including its reduction to the classical Kelly formula for the memoryless case, results regarding the limiting values of the parameters used in the theorem, and generalizations of the theory beyond the simple case of time-invariant even-money two-outcome bets.

When Coin Flips are Independent: For the special case with all payoffs X_k being i.i.d., we note that $\mathbb{E}(H_n) = np$ where $p = p_0 = p_\infty = \omega_0$ is the probability of a head. In this case, $K_n = K^* = 2p - 1$, which is the classical result obtained in the absence of autocorrelation among bets as described in Section II.

Long Run Steady State Considerations: Of general interest are the limiting values of the quantities $\mathbb{E}(H_n)$ and K_n described in the theorem as $n \rightarrow \infty$. The first point to note is that $\lambda_n(\omega_1) \rightarrow 0$ which in turn implies that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(H_n(\mathbf{X}))}{n} = p_\infty$$

and immediately leads to optimal betting fraction

$$K_\infty = 2p_\infty - 1.$$

The interpretation of this limit is quite simple: If we are playing forever, the long-run probability of a head, p_∞ , leading to K_∞ is the same betting fraction as that which one would obtain by ignoring correlation among the X_k and treating p_∞ as if it is the unconditional probability of a heads in the classical i.i.d. case. On the other hand, if we are betting for a fixed time horizon n , the difference between $\mathbb{E}(H_n)/n$ and p_∞ is important. In particular, the optimal betting fraction K_n depends on n and the startup probability $p_0 = \omega_0 + \omega_1 x_{-1}$, whereas K_∞ does not. In practice, the importance of x_{-1} depends on the size of n , and magnitude of autocorrelation coefficients ω_0 and ω_1 .

Multiple Outcomes: As indicated in Section I, our theory may be modified to address more general scenarios. For example, consider the case where there are ℓ possible outcomes $x_1, x_2, \dots, x_\ell \in (-1, \infty)$ for X_k . Let $P_{\mathbf{X}}$ be an arbitrary probability mass function over sample paths and let $H_{n,i}(\mathbf{X})$ be a random variable denoting the number of times, in a path of length n , that $X_k = x_i$ for $i = 1, 2, \dots, \ell$. Then, using an argument quite similar to the one used in the proof of the theorem, we obtain

$$\text{ELG}(K) = \sum_{i=1}^{\ell} \frac{\mathbb{E}(H_{n,i}(\mathbf{X}))}{n} \log(1 + K x_i),$$

which is straightforward to maximize numerically since it is concave in K .

Time-Varying Feedback: A second generalization which may also be considered, involves the use of time-varying feedback gains rather than the time-invariant K synonymous with previous literature. The most straightforward extension in this direction is where, *prior* to the start of the game, the bettor decides on a vector of betting fractions

$$\mathbf{K} = [\tilde{K}_0, \tilde{K}_1, \dots, \tilde{K}_{n-1}]^T.$$

Note that the tilde notation distinguishes these time-varying gains from the time-invariant K_n of our main theorem. Thus, defining $p_k = \Pr(X_k = 1)$ to be the unconditional probability of a head on the k th coin toss,

$$\text{ELG}(\mathbf{K}) = \frac{1}{n} \sum_{k=0}^{n-1} \left\{ p_k \log(1 + \tilde{K}_k) + (1 - p_k) \log(1 - \tilde{K}_k) \right\},$$

and this is maximized at

$$\tilde{K}_k = 2p_k - 1.$$

Interestingly, it is straightforward to show that

$$K_n = \frac{1}{n} \sum_{k=0}^{n-1} \tilde{K}_k,$$

i.e., the time-invariant gain over n bets is the average of the time-varying gains.

V. PROOF OF THE THEOREM

This section can be skipped by those readers solely interested in the application aspects of this work. Indeed, to determine the optimal betting fraction, we maximize the Expected Logarithm Growth. For simplicity of notation, we suppress the dependence of H_n on the sample path \mathbf{X} and calculate

$$\begin{aligned} \text{ELG}(K) &= \frac{1}{n} \mathbb{E} \left\{ \sum_{k=0}^{n-1} \log(1 + K X_k) \right\} \\ &= \frac{1}{n} \sum_{\mathbf{X} \in \mathcal{X}} P_{\mathbf{X}} \left\{ \sum_{k=0}^{n-1} \log(1 + K X_k) \right\} \\ &= \frac{1}{n} \sum_{\mathbf{X} \in \mathcal{X}} P_{\mathbf{X}} \{ H_n \log(1 + K) \\ &\quad + (n - H_n) \log(1 - K) \} \\ &= \frac{\mathbb{E}(H_n)}{n} \log(1 + K) \\ &\quad + \left\{ 1 - \frac{\mathbb{E}(H_n)}{n} \right\} \log(1 - K). \end{aligned}$$

Now, noting that $K = K_n = 2 \{ \mathbb{E}(H_n)/n \} - 1$ is the unique point of zero derivative and that $\text{ELG}(K)$ is a concave function, it follows that K_n is the unique maximizer.

It remains to derive an explicit formula for $\mathbb{E}(H_n)/n$ for the case of the history-cognizant coin. First, since the expected number of heads on the k th coin toss is $p_k = \Pr(X_k = 1)$, the expected number of heads in n coin tosses is

$$\mathbb{E}(H_n) = \sum_{k=0}^{n-1} p_k.$$

Now, to obtain a formula for the sum above, beginning with conditional probability $\Pr(X_k = 1 | X_{k-1}) = \omega_0 + \omega_1 X_{k-1}$, using the law of total expectation, we obtain a recursion

$$\begin{aligned} p_k &= \mathbb{E} \{ \Pr(X_k = 1 | X_{k-1}) \} \\ &= \omega_0 + \omega_1 \mathbb{E}(X_{k-1}) \\ &= 2\omega_1 p_{k-1} + \omega_0 - \omega_1, \end{aligned}$$

where the last line follows since $\mathbb{E}(X_{k-1}) = 2p_{k-1} - 1$. Initializing with $p_0 = \omega_0 + \omega_1 x_{-1}$, the solution to this linear equation is, since $|2\omega_1| < 1$,

$$p_k = (2\omega_1)^k p_0 + [1 - (2\omega_1)^k] p_{\infty}$$

where $p_{\infty} = (\omega_0 - \omega_1)/(1 - 2\omega_1)$. Thus,

$$\begin{aligned} \frac{\mathbb{E}(H_n)}{n} &= \frac{1}{n} \sum_{k=0}^{n-1} (2\omega_1)^k p_0 + \frac{1}{n} \sum_{k=0}^{n-1} [1 - (2\omega_1)^k] p_{\infty} \\ &= \lambda_n p_0 + (1 - \lambda_n) p_{\infty}. \quad \square \end{aligned}$$

VI. DEEPER MEMORY

We consider the general case of memory depth $m > 1$ and show how analytic expressions for $\mathbb{E}(H_n)$ can be efficiently obtained. Indeed, beginning with parameterization of the conditional probability of heads

$$\Pr(X_k = 1 | X_{k-1}, X_{k-2}, \dots, X_{k-m}) = \omega_0 + \sum_{i=1}^m \omega_i X_{k-i}$$

with assumed initial conditions

$$X_{-i} = x_{-i} \text{ for } i = 1, 2, \dots, m.$$

To ensure that $\Pr(X_k = 1 | X_{k-1}, X_{k-2}, \dots, X_{k-m}) \in (0, 1)$ we assume the ω parameters to lie in the “hyperdiamond”

$$\left| \omega_0 - \frac{1}{2} \right| + \sum_{i=1}^m |\omega_i| < \frac{1}{2}.$$

Then, the unconditional probability p_k that $X_k = 1$ is

$$\begin{aligned} p_k &= \mathbb{E} \{ \Pr(X_k = 1 | X_{k-1}, X_{k-2}, \dots, X_{k-m}) \}, \\ &= \omega_0 + \sum_{i=1}^m \omega_i \mathbb{E}(X_{k-i}) \end{aligned}$$

for $k = 0, 1, \dots, n-1$. Substituting $\mathbb{E}(X_{k-i}) = 2p_{k-i} - 1$, above and taking note of the “induced” initial conditions

$$p_{-i} = \frac{(x_{-i} + 1)}{2} \text{ for } i = 1, 2, \dots, m,$$

we arrive at the recursion

$$p_k = \omega_0 - \sum_{i=1}^m \omega_i + 2 \sum_{i=1}^m \omega_i p_{k-i}$$

which holds for $k = 0, 1, \dots, n-1$, and from which $E(H_n)$ and hence K_n may be calculated. To illustrate a specific case, for memory depth $m = 3$ and $n = 2$ flips, we find that

$$\begin{aligned} \mathbb{E}(H_2) &= x_{-1}(2\omega_1^2 + \omega_1 + \omega_2) + x_{-2}(2\omega_1\omega_2 + \omega_2 + \omega_3) \\ &\quad + x_{-3}(2\omega_1\omega_3 + \omega_3) + 2\omega_0\omega_1 + 2\omega_0 - \omega_1. \end{aligned}$$

State-Space Formulation: As an alternative to the above, which may perhaps prove useful in future research, we consider a standard state-space realization of the “delay system” to represent the scalar recursion for p_k . That is, by introducing the m -dimensional state vector which is given by $v_k = [p_{k-m+1}, p_{k-m+2}, \dots, p_k]^T$, we readily obtain a classical companion form realization $v_{k+1} = A v_k + b u_k$ with triple (A, b, c) , input $u(k) \equiv 1$ and output being p_k . To illustrate, for memory depth $m = 3$, we obtain

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2\omega_3 & 2\omega_2 & 2\omega_1 \end{bmatrix}; \quad b = \begin{bmatrix} 0 \\ 0 \\ \omega_0 - \omega_1 - \omega_2 - \omega_3 \end{bmatrix};$$

and $c = [0 \ 0 \ 1]$ with solution of the recursion given by

$$\begin{aligned} p_k &= c \left(A^k v_0 + \sum_{i=0}^{k-1} A^{k-1-i} b \right) \\ &= c (A^k v_0 + (I - A)^{-1} (I - A^k) b) \end{aligned}$$

and the matrix $I - A$ is guaranteed to be invertible since $\det(I - A) = 1 - 2 \sum_{i=1}^m \omega_i$ must be non-zero due to the hyperdiamond constraint on the ω_i . In addition, from Gerschgorin’s circle theorem [21] and the hyperdiamond constraint, each eigenvalue of A has magnitude less than 1, and so $A^k \rightarrow 0$ as $k \rightarrow \infty$. Using this fact, beginning with p_k above, this leads to the generalization of our steady-state unconditional probability

$$p_{\infty} = c(I - A)^{-1} b.$$

Now recognizing that this corresponds to the transfer function $H(z)$ for the triple (A, b, c) evaluated at $z = 1$, we immediately arrive at

$$p_\infty = \frac{\omega_0 - (\omega_1 + \dots + \omega_m)}{1 - 2(\omega_1 + \dots + \omega_m)}.$$

Estimation: In practice, prior to betting, it is necessary to obtain values for the ω_i parameters. First, define the “response” variable $Y_k = (X_k + 1)/2$ such that

$$\mathbb{E}(Y_k | X_{k-1}, X_{k-2}, \dots, X_{k-m}) = \omega_0 + \sum_{i=1}^m \omega_i X_{k-i}.$$

Then, having observed data $x_{-\ell}, \dots, x_{-1}$, we compute the $(\ell - m) \times 1$ response vector

$$y \doteq \frac{1}{2} \{ [x_{-\ell+m}, x_{-\ell+m+1}, \dots, x_{-1}]^T + 1 \},$$

and minimize the residual sum of squares

$$\text{RSS}(\omega) = \sum_{k=-\ell+m}^{-1} \left(y_k - \omega_0 - \sum_{i=1}^m \omega_i x_{k-i} \right)^2,$$

with respect to $\omega = [\omega_0, \omega_1, \dots, \omega_m]^T$. Whereas classical estimation theory leads to the *least squares* solution

$$\hat{\omega} = \underset{\omega}{\text{argmin}} \text{RSS}(\omega) = (X^T X)^{-1} X^T y$$

with X being the $(\ell - m) \times (m + 1)$ matrix whose i th row is given by $[1, x_{i-\ell+m-2}, x_{i-\ell+m-3}, \dots, x_{i-\ell-1}]$, enforcement of the hyperdiamond constraints leads to a positive-definite convex program to be solved.

VII. NUMERICAL SIMULATION

We present results for simulations with returns driven by a process with $\Pr(X_k = 1 | X_{k-1}) = \omega_0 + \omega_1 X_{k-1}$, comparing ELG performance for the classical Kelly K^* -bettor, the K_n -bettor, and the \mathbf{K} -bettor. Accordingly, from Section IV, the classical K^* -bettor, disregarding temporal correlation, works with the probability of heads being p_∞ and uses the time-invariant betting fraction $K^* = 2p_\infty - 1$. On the other hand, assuming ω_0 and ω_1 are perfectly estimated, the K_n -bettor exploits temporal correlation and uses fraction $K_n = 2\{\lambda_n p_0 + (1 - \lambda_n)p_\infty\} - 1$, while the \mathbf{K} -bettor, who is also aware of the correlation, uses separate K values for each stage via $\tilde{K}_k = 2p_k - 1$.

To provide a flavor of our findings, we first consider the following scenario: We initialize the game by supposing that the prior event was $X_{-1} = 1$, and take $\omega_0 = 0.55$, $\omega_1 = 0.20$. In this specific situation, a straightforward calculation using our theory directly leads to values of $p_0 = 0.75$, $p_\infty = 0.583$, and $\lambda_n \approx 0.556(1 - 0.4^n)/n$; hence, $K^* \approx 0.167$ and $K_n \approx 0.556(1 - 0.4^n)/n + 0.167$ from which it is immediately clear that $K_n > K^*$ (but tends to K^* in the long run per Section IV). In this setting, due to the positive correlation ($\omega_1 > 0$), and fact that $X_{-1} = 1$, earlier coin tosses are more likely to be heads than later ones. Therefore, when betting for a fixed time horizon n , the K_n -bettor takes advantage

of the temporal correlation by placing larger bets than those suggested by the correlation-ignoring K^* value. However, although accounting for autocorrelation, K_n is time-invariant. For $n = 2$, $K_n = 0.4$, whereas $\mathbf{K} = (0.5, 0.3)$ (and recall from Section IV that K_n is the average of the elements of \mathbf{K}). Thus, the \mathbf{K} -bettor, recognizing that X_0 is most likely to be a head, bets more heavily on the first bet than on the second. Of course, both the K_n - and \mathbf{K} -bettors bet more heavily than the K^* -bettor, and, indeed, for the $n = 2$ case, we find that $\text{ELG}(K^*) \approx 0.053$, $\text{ELG}(K_n) \approx 0.082$, and $\text{ELG}(\mathbf{K}) \approx 0.088$.

Beyond $n = 2$, Figure 3(a) shows ELG values for a range of n over which the \mathbf{K} -bettor outperforms the K_n -bettor who in turn outperforms the K^* -bettor. This scenario is analogous to one which arises for a financial asset on an upward trend (since $p_k \geq p_\infty \approx 0.5833$). In such a setting, the majority of strategies will do well, e.g., all three here have positive ELG, but, importantly, incorporating temporal correlation boosts performance. Figure 3(b) displays the results for a similar simulation but with $\omega_1 = -0.2$. As with the first scenario, this represents long-run upward trend since $p_\infty \approx 0.5357$, but the negative autocorrelation means that the process fluctuates more; in particular, X_0 is most likely to be a tail since X_{-1} was a head. In this setting, neither the K^* - nor the K_n -bettors do very well, albeit the latter at least has non-negative ELG, whereas the \mathbf{K} -bettor has significantly improved performance. To see why this is, consider the $n = 2$ case where $K^* \approx 0.071$, $K_n = -0.04$, and $\mathbf{K} = (-0.3, 0.22)$. Thus, the \mathbf{K} -bettor makes use of the fluctuation by betting on tails first and on heads second but bets less heavily in the second due to the increased uncertainty; the K_n -bettor averages over these fluctuations, slightly favouring tails but ultimately betting very little, whereas the K^* -bettor wrongly favours heads.

Lastly, in Figure 3(c) we briefly consider another interesting scenario shown which corresponds to $X_{-1} = 1$, $\omega_0 = 0.35$ and $\omega_1 = 0.33$. The important feature in this case is that it corresponds to a long-run downward trend (with $p_\infty = 0.058$) but where the positive temporal correlation and the fact that $X_{-1} = 1$ mean that $X_0 = 1$ is most likely. In this case, the K^* -bettor suboptimally takes a heavy short position (i.e., bets on tails) with $K^* \approx -0.883$ leading to large negative ELG value early on (only becoming positive for $n > 10$). In contrast, both K_n - and \mathbf{K} -bettors achieve positive growth over all n . Note that, for all three scenarios, and over all n values, $\text{ELG}(K^*) \leq \text{ELG}(K_n) \leq \text{ELG}(\mathbf{K})$ which is consistent with our exploitation of temporal correlation to obtain improved Kelly-type betting strategies which have not been considered in the existing literature.

VIII. CONCLUSION

In this paper, we formulated a class of Kelly optimal ELG problems which account for temporal correlation over the sequence of gambles. In the main theorem, for memory depth $m = 1$ and n flips, a closed form solution for the optimal betting fraction K_n was obtained. The paper also includes analysis for the case when $n \rightarrow \infty$ and solutions for deeper

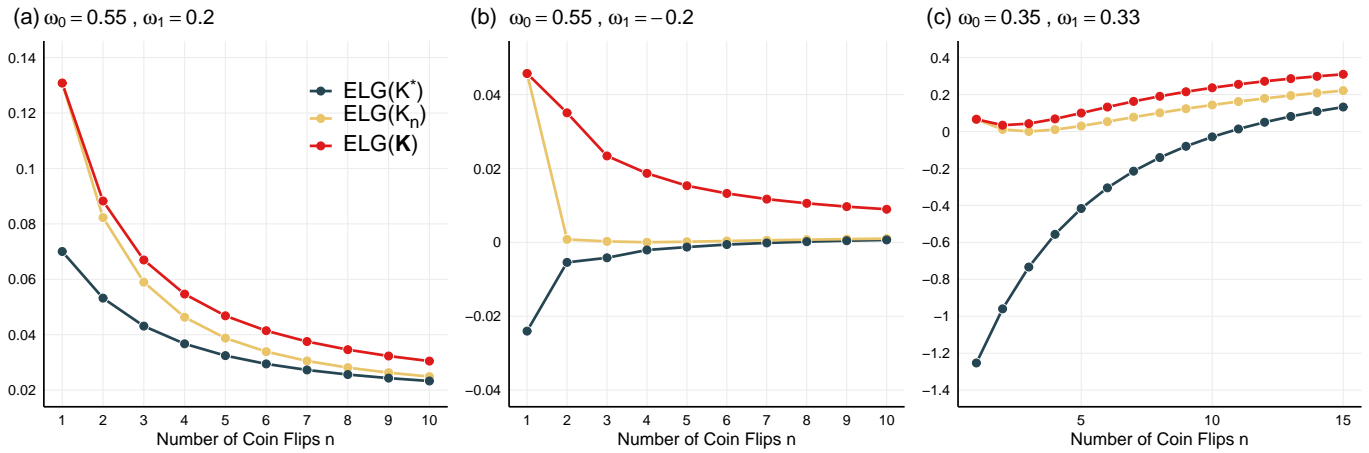


Fig. 3: Simulation demonstrating the ELG of account value for the three bettors with $X_{-1} = 1$, and different combinations of ω_0 and ω_1 .

memory $m > 1$ which can be obtained by either propagation of the recursive formula for p_k or use of the state space realization for the associated delay system. While our primary focus has been the development of ELG theory in the presence of autocorrelation, we have also provided extensions to multiple-payoffs and time-varying feedback gains. Numerical simulations which included comparison with classical Kelly betting results on games with temporal correlation were also shown in order to demonstrate the potential advantages offered by our framework.

In future work, we envision our theory to be especially applicable to scenarios in which an investor wishes to incorporate temporal correlations into algorithmic trading strategies over the course of time. In particular, the conceptual framework introduced within this Letter has potential to provide the base upon which multiple extensions beyond those proposed above can be built; one such avenue is the so-called *portfolio case* with correlation both temporally and across components. In this case it is felt that concave programming will play an important role in computation; e.g., see [22]. Finally, a further direction of research involves a study of ELG performance as a function of betting frequency in the context of the temporal correlation framework introduced here; e.g., see [23] for initial work along these lines and [24] for analysis of the memoryless case.

REFERENCES

- [1] J. L. Kelly, "A New Interpretation of Information Rate," *Bell System Technical Journal*, vol. 35.4, pp. 917–926, 1956.
- [2] E. O. Thorp, "The Kelly Criterion in Blackjack Sports Betting and the Stock Market," *Handbook of Asset and Liability Management: Theory and Methodology*, vol. 1, pp. 385–428, Elsevier Science, 2006.
- [3] J. Haigh, "The Kelly Criterion and Bet Comparisons in Spread Betting," *Journal of the Royal Statistical Society: Series D (The Statistician)*, vol. 49, pp. 531–539, 2000.
- [4] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, John Wiley and Sons, 2006.
- [5] D. G. Luenberger, *Investment Science*, Oxford University Press, 2013.
- [6] L. M. Rotando and E. O. Thorp, "The Kelly Criterion and the Stock Market," *The American Mathematical Monthly*, vol. 99, pp. 922–931, 1992.
- [7] S. Maslov, and Y. C. Zhang, "Optimal Investment Strategy for Risky Assets," *International Journal of Theoretical and Applied Finance*, vol. 1, pp. 377–387, 1998.
- [8] N. H. Hakansson, "On Optimal Myopic Portfolio Policies With and Without Serial Correlation of Yields," *Journal of Business*, vol. 44, pp. 324–334, 1971.
- [9] A. W. Lo, H. A. Orr, and R. Zhang, "The Growth of Relative Wealth and the Kelly Criterion," *Journal of Bioeconomics*, vol. 20, pp. 49–67, 2018.
- [10] S. Browne, and W. Whitt, "Portfolio Choice and the Bayesian Kelly Criterion," *Advances in Applied Probability*, vol. 28, pp. 1145–1176, 1996.
- [11] L. C. MacLean, E. O. Thorp, and W. T. Ziemba, *The Kelly Capital Growth Investment Criterion: Theory and Practice*, World Sci., 2010.
- [12] L. C. MacLean, E. O. Thorp, and W. T. Ziemba, "Long-term Capital Growth: The Good and Bad Properties of The Kelly and Fractional Kelly Capital Growth Criteria," *Quantitative Finance*, vol. 10, pp. 681–687, 2010.
- [13] L. C. Maclean, W. T. Ziemba and G. Blazenko, "Growth Versus Security in Dynamic Investment Analysis," *Management Science*, vol. 38, pp. 1562–1585, 1992.
- [14] C. H. Hsieh, B. R. Barmish and J. A. Gubner, "Kelly Betting Can be Too Conservative," *Proceedings of IEEE Conference on Decision and Control*, pp. 3695–3701, 2016.
- [15] N. Rujeerapaiboon, B. R. Barmish and D. Kuhn, "On Risk Reduction in Kelly Betting Using the Conservative Expected Value," *Proceedings of the IEEE Conference on Decision and Control*, pp. 58015806, 2018.
- [16] L. C. Maclean, R. Sanegre, Y. Zhao, and W. T. Ziemba, "Capital Growth with Security," *Journal of Economic Dynamics and Control*, vol. 28, pp. 937–954, 2004.
- [17] E. Busseti, E. K. Ryu, and S. Boyd, "Risk-constrained Kelly Gambling," *The Journal of Investing*, vol. 25, pp. 118–134, 2016.
- [18] J. Y. Campbell, J. G. Sanford, and W. Jiang, "Trading Volume and Serial Correlation in Stock Returns," *The Quarterly Journal of Economics*, vol. 108, pp. 905–939, 1993.
- [19] J. Lewellen, "Momentum and Autocorrelation in Stock Returns," *The Review of Financial Studies*, vol. 15, pp. 533–564, 2002.
- [20] Y. Hirono, and H. Yoshimasa, "Jarzynski-type Equalities in Gambling: Role of Information in Capital Growth," *Journal of Statistical Physics*, vol. 161, pp. 721–742, 2015.
- [21] G. H. Golub, and C. F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, 2012.
- [22] S. Boyd, and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.
- [23] M. E. Wu, H. H. Tsai, R. Tso, and C. Y. Weng, "An Adaptive Kelly Betting Strategy for Finite Repeated Games," *IGEC Conference*, pp. 39–46, 2015.
- [24] C. H. Hsieh, B. R. Barmish and J. A. Gubner, "At What Frequency Should the Kelly Bettor Bet?" *Proceedings of the American Control Conference*, pp. 5485–5490, 2018.