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Topics in
Probability Theory and Stochastic Processes
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Stirling's Formula from Simple Functions, Sequences and Series

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Rating

Mathematicians Only: prolonged scenes of intense rigor.

Section Starter Question

What is the geometric summation formula? How can you use the geometric sum formula to derive the series expansion for $\log(1+x)$? What do you need to know about the geometric sum formula to justify its use to derive the series expansion for $\log(1+x)$?

Key Concepts

1. **Stirling's Formula**, also known as Stirling's Approximation, is the asymptotic relation
$$n! \sim \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n} .$$

2. The formula is useful in estimating large factorial values, but its main mathematical value is in limits involving factorials.

3. An improved inequality version of Stirling's Formula is
$$\sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n+1/(12n+1)} < n! < \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n+1/(12n)} .$$

4. Some related inequalities and asymptotics for binomial coefficients are
$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{\mathrm{e}n}{k}\right)^k$$
and
$$\binom{n}{k} \sim \left(\frac{n^k}{k!}\right)$$
if $k = o(n^{1/2})$ as $n \to \infty$.

Vocabulary

1. **Stirling's Formula**, also known as Stirling's Approximation, is the asymptotic relation
$$n! \sim \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n}\,.$$

2. The **double factorial** notation is $n!! = n \cdot (n-2) \cdots 4 \cdot 2$ if n is even, and $n!! = n \cdot (n-2) \cdots 3 \cdot 1$ if n is odd.

Mathematical Ideas

Following usual mathematical conventions in subjects beyond calculus, all logarithms are natural logarithms with base e.

Stirling's Formula, also known as Stirling's Approximation, is the asymptotic relation
$$n! \sim \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n}\,.$$
The formula is useful in estimating large factorial values, but its main mathematical value is in limits involving factorials. Another attractive form of Stirling's Formula is:
$$n! \sim \sqrt{2\pi n} \left(\frac{n}{\mathrm{e}}\right)^n\,.$$

An improved inequality version of Stirling's Formula is
$$\sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n+1/(12n+1)} < n! < \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n+1/(12n)}\,. \tag{1}$$
See Stirling's Formula in MathWorld.com.

A consequence of the improved inequality is the simple and useful inequality about Stirling's Formula for all n
$$\sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n} < n!\,.$$

Here we rigorously derive Stirling's Formula using elementary sequences and series expansions of the logarithm function, based on the sketch in Kazarinoff [3], based on the note by Nanjundiah [4], using motivation and background from the article by Hammett [2].

A weak form of Stirling's Formula

Theorem 1.

$$\sqrt[n]{n!} \sim \frac{n}{\mathrm{e}}.$$

Remark. This form of Stirling's Formula is weaker than the usual form since it does not give direct estimates on $n!$. On the other hand,it avoids the determination of the asymptotic constant $\sqrt{2\pi}$ which usually requires Wallis's Formula or equivalent. For many purposes of estimation or limit taking this version of Stirling's Formula is enough, and the proof is elementary. The proof is taken from [6, pages 314-315].

Proof.

1. Start from the series expansion for the exponential function and then crudely estimate:

$$\begin{aligned} \mathrm{e}^n &= 1 + \frac{n}{1!} + \dots + \frac{n^{n-1}}{(n-1)!} + \\ &\quad \frac{n^n}{n!}\left(1 + \frac{n}{n+1} + \frac{n^2}{(n+1)(n+2)} + \dots\right) \\ &< n\frac{n^n}{n!} + \frac{n^n}{n!}\sum_{j=0}^{\infty}\left(\frac{n}{n+1}\right)^j \\ &= (2n+1)\frac{n^n}{n!}. \end{aligned}$$

2. On the other hand, $\mathrm{e}^n > n^n/n!$ by dropping all but the $n^n/n!$ term from the series expansion for the exponential.

3. Rearranging these two inequalities

$$\frac{n^n}{\mathrm{e}^n} < n! < \frac{(2n+1)n^n}{\mathrm{e}^n}.$$

4. Now take the nth root of each term, and use o $\sqrt[n]{2n+1} \to 1$ as $n \to \infty$.

$\square$

Some motivating functions

The basic calculus fact

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \mathrm{e}$$

motivates studying the parametrized family of functions

$$a_{\alpha}(x) = \left(1+\frac{1}{x}\right)^{x+\alpha}.$$

Actually,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \mathrm{e}$$

can be proved directly using information about the family of functions, without resorting to L'Hospital's Rule.

Graphs of $a_{\alpha}(x)$ for representative values of α are in Figure 1. The graph suggests $a_{\alpha}(x)$ is decreasing if $\alpha > 1/2$ and eventually increasing for $\alpha < 1/2$.

Lemma 2. *1. $a_{\alpha}(x)$ decreases with x for $x > 1$ and any fixed $\alpha \geq 1/2$*

2. $a_{\alpha}(x)$ increases with x for $x > \max[1, \frac{1}{3-6\alpha}]$ and any fixed $\alpha < 1/2$.

Remark. The condition $x > \max\left[1, \frac{1}{3-6\alpha}\right]$ reduces to $x > 1$ for $\alpha \leq \frac{1}{3}$ and $x > 1/(3-6\alpha)$ for $\frac{1}{3} < \alpha < \frac{1}{2}$

Remark. Essentially the same lemma is proved in a different way in Lemma 6.

Remark. The results of this lemma are equivalent to a remark due to I. Schur, see [5], problem 168, on page 38 with solution on page 215, and a generalization appears in the *College Mathematics Journal*, September 2020, Problem 1182, so this is frequently rediscovered.

Proof. 1. Introduce the parametrized family $g_{\alpha}(x) = \log a_{\alpha}(x) = (x + \alpha)\log(1+1/x)$.

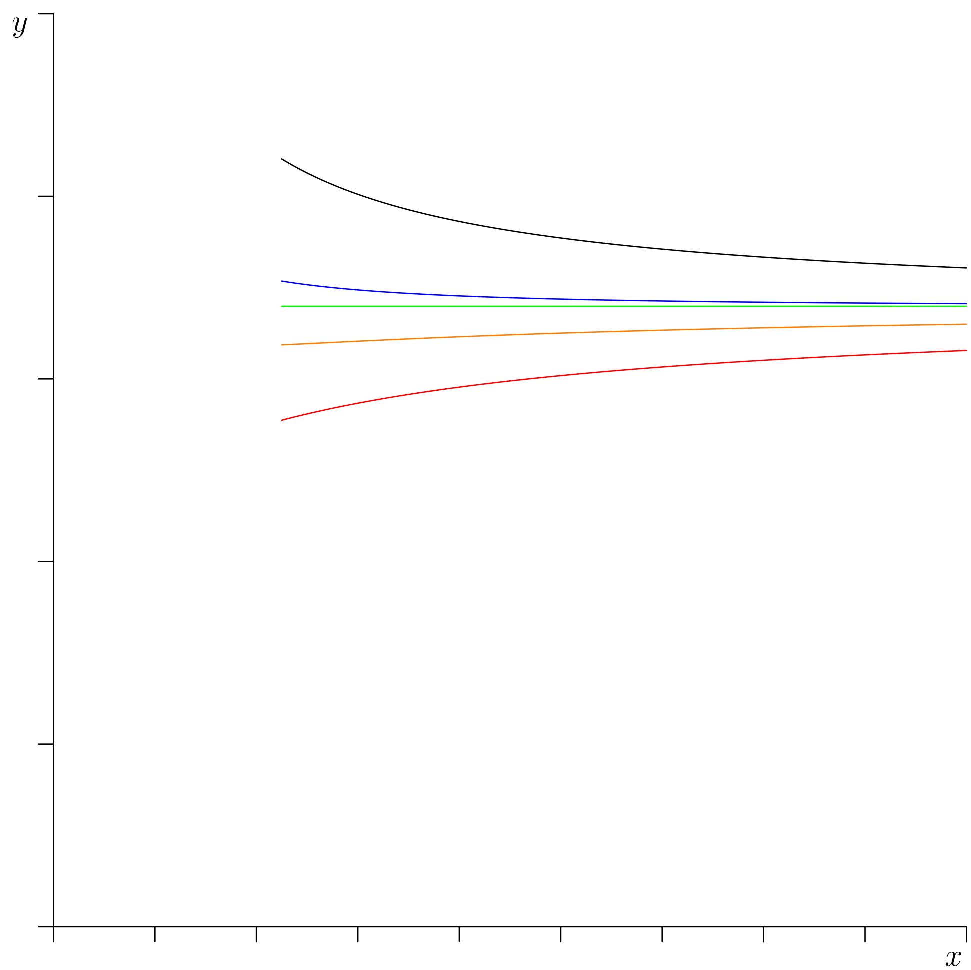

Figure 1: Graphs of $y = a_\alpha(x)$ for $\alpha = 0.15$, red, 0.35, orange, 0.5, blue, and 0.65, black. The green horizontal line is $y = \mathrm{e}$.

2. Taking the derivative, rearranging and expanding in series:

$$\begin{aligned} g'(x) &= \log(1+1/x) + \frac{x+\alpha}{1+1/x} \cdot \frac{-1}{x^2} \\ &= \left(\frac{1}{x^2+x}\right)\left((x^2+x)\left(\frac{1}{x} - \frac{1/2}{x^2} + \frac{1/3}{x^3} - \cdots\right) - (x+\alpha)\right) \\ &= \left(\frac{1}{x^2+x}\right)\left((\frac{1}{2} - \alpha) + \sum_{k\geq 2} \frac{(-1)^{k-1}}{x^{k-1}}\right) \\ &= \left(\frac{1}{x^2+x}\right)\psi_c(x) \end{aligned}$$

where $\psi_c(x) = \left(\frac{1}{2} - c\right) + \sum_{k\geq 2} \frac{(-1)^{k-1}}{x^{k-1}}$.

3. From the definitions

$$\frac{\psi_\alpha(x)}{x^2+x} = g'_\alpha(x) = \frac{a_{\alpha(x)}}{a_\alpha(x)}$$

and $x^2 + x > 0$ and $a_\alpha(x) > 0$, so $\psi_\alpha(x)$, $g'_\alpha(x)$ and $f_\alpha(x)$ all have the same sign. Therefore to prove the theorem it suffices to consider $\psi_\alpha(x)$ in terms of α.

4. The absolute ratio of consecutive terms in the alternating series in the definition of $\psi_\alpha(x)$ is

$$\frac{\frac{1}{(k+1)(k+2)}\frac{1}{x^k}}{\frac{1}{k(k+1)}\frac{1}{x^{k-1}}} = \frac{k}{k+2}\left(\frac{1}{x}\right) < 1$$

for $k \geq 2$ and $x > 1$.

5. The alternating series has terms that decrease in magnitude and so

$$\frac{1}{2} - \alpha - \frac{1}{6x} < \psi_\alpha(x) < \frac{1}{2} - \alpha$$

for $x > 1$ and $\alpha \geq 1/2$.

6. From the right inequality, $\psi_\alpha(x) < \frac{1}{2} - \alpha \leq 0$ for $\alpha \leq \frac{1}{2}$ and $x > 1$. This means $a'_\alpha(x) < 0$ for $\alpha \leq \frac{1}{2}$ and $x > 1$. So $a_\alpha(x)$ decreases for $\alpha \leq \frac{1}{2}$ and $x > 1$. This proves the first part of the lemma.

7. When $\alpha < \frac{1}{2}$ and $x > 1$ we have $\frac{1}{2} - \alpha - \frac{1}{6x}$ which is equivalent to $x(3 - 6\alpha)$ which in turn is equivalent to $x \geq \frac{1}{3-6c}$.

8. Then from the left hand inequality,

$$\psi_c(x) > \frac{1}{2} - \alpha - \frac{1}{6x} \geq 0$$

for $x > \max[1, \frac{1}{3-6\alpha}]$.

□

Corollary 1.

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e}.$$

Proof. 1. Write $g_\alpha(x)$ as a power series as done for $g'_\alpha(x)$:

$$\begin{aligned} g_\alpha(x) &= (x+\alpha)\log\left(1 + \frac{1}{x}\right) \\ &= (x+\alpha)\left(\frac{1}{x} - \frac{1/2}{x} + \frac{1/3}{x^3} - \cdots\right) \\ &= 1 + \sum_{k\geq 1}\left(\frac{k}{k+1} - c\right)\frac{(-1)^k}{kx^k} \end{aligned}$$

2. Now $\lim_{x\to\infty} g_\alpha(x) = 1$ for all α.

3. Then

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = \lim_{n\to\infty} \mathrm{e}^{g_0(n)} = \mathrm{e}.$$

□

More is true, namely

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^{x+\alpha} = \lim_{n\to\infty} \mathrm{e}^{g_\alpha(n)} = \mathrm{e}$$

and what is more, the lemma says exactly how members of the parametrized family approach e,

$$\left(1 + \frac{1}{n}\right)^{n+\alpha_1} < \mathrm{e} < \left(1 + \frac{1}{n}\right)^{n+\alpha_2}$$

for $\alpha_1 < 1/2$ and $\alpha_2 \geq 1/2$ and n sufficiently large.

Now separately derive some equivalent formulations of these inequalities. First for $\alpha_1 < 1/2$, the following are equivalent

$$\mathrm{e} > \left(1+\frac{1}{n}\right)^{n+\alpha_1}$$

$$\mathrm{e}\left(\frac{n}{n+1}\right)^{n+\alpha_1} > 1$$

$$\frac{\mathrm{e}^{n+1}}{(n+1)^n(n+1)_1^\alpha} > \frac{\mathrm{e}^n}{n^n n_1^\alpha}$$

$$\frac{(n+1)!}{(n+1)_1^\alpha}\left(\frac{\mathrm{e}}{n+1}\right)^{n+1} > \frac{n!}{n_1^\alpha}\left(\frac{\mathrm{e}}{n}\right)^n$$

Similarly

$$\frac{(n+1)!}{(n+1)_2^\alpha}\left(\frac{\mathrm{e}}{n+1}\right)^{n+1} < \frac{n!}{n_2^\alpha}\left(\frac{\mathrm{e}}{n}\right)^n.$$

Introducing the parametrized family of sequences

$$\frac{n}{n^\alpha}\left(\frac{\mathrm{e}}{n}\right)^n,$$

the sequence increases with n for $\alpha < 1/2$ and $n > \max[1, 1/(3-6\alpha)]$ and the sequence decreases with n for $\alpha \geq 1/2$ and $n > 1$. For fixed n, the sequence is a decreasing function of α.

By monotonicity, the sequences

$$\frac{n!}{n^\alpha}\left(\frac{\mathrm{e}}{n}\right)^n,$$

must approach 0, ∞, or some finite L as $n \to \infty$. If $\alpha < 1/2$, the sequence is increasing and so the limit must be either ∞ or L. If some $\alpha_0 < 1/2$ the limit is $L < \infty$, then for fixed $\beta \in (\alpha_0, 1/2)$,

$$\lim_{n\to\infty}\frac{n!}{n^\beta}\left(\frac{\mathrm{e}}{n}\right)^n = \lim_{n\to\infty}\frac{n^{\alpha_0}}{n^\beta}\frac{n!}{n_0^\alpha}\left(\frac{\mathrm{e}}{n}\right)^n =$$

$$\lim_{n\to\infty}\frac{n^{\alpha_0}}{n^\beta}\cdot\lim_{n\to\infty}\frac{n!}{n_0^\alpha}\left(\frac{\mathrm{e}}{n}\right)^n = 0\cdot L = 0$$

which is not possible, so the limit must be ∞ for all $\alpha < 1/2$.

Similarly, if $\alpha > 1/2$, the sequence is decreasing to a limit of 0, see the exercises.

This leaves just $\alpha = 1/2$ the only possible member of the family of sequences that could approach a finite limit as $n \to \infty$. The goal is now to show this finite limit exists.

Stirling's Formula from sequences and series

Rearranging slightly, let $a_n = \frac{n!}{n^{n+1/2}\mathrm{e}^{-n}}$. Then $\frac{a_n}{a_{n+1}} = \left(\frac{n+1}{n}\right)^{n+1/2}\mathrm{e}^{-1}$ and $\log\left(\frac{a_n}{a_{n+1}}\right) = (n+1/2)\log(1+1/n) - 1$.

Lemma 3. *For* $|x| < 1$,

$$\log\left(\frac{1+x}{1-x}\right) = 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)x^{2k+1}}.$$

Proof. Left as an exercise. □

Note then that

$$\log\left(1+\frac{1}{n}\right) = \log\left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}\right) = 2\sum_{k=0}^{\infty}\frac{1}{(2k+1)(2n+1)^{2k+1}}.$$

Then

$$\begin{aligned}\log\left(\frac{a_n}{a_{n+1}}\right) &= \left(\frac{2n+1}{2}\right)\left(2\sum_{k=0}^{\infty}\frac{1}{(2k+1)(2n+1)^{2k+1}}\right) - 1 \\ &= \sum_{k=1}^{\infty}\frac{1}{(2k+1)(2n+1)^{2k}} = \sum_{k=0}^{\infty}\frac{1}{(2k+3)(2n+1)^{2k+2}}.\end{aligned}$$

Now coarsely estimating the denominators

$$\log\left(\frac{a_n}{a_{n+1}}\right) \le \frac{1}{2(2n+1)^2}\sum_{k=0}^{\infty}\frac{1}{(2n+1)^{2k}}.$$

Lemma 4.

$$\log(a_{n+1}) < \log\left(a_n\right) < \frac{1}{12n} - \frac{1}{12(n+1)} + \log(a_{n+1}).$$

Proof. 1. Let $f(x) = (x + 1/2) \log(1 + 1/x) - 1$, so $\log\left(\frac{a_n}{a_{n+1}}\right) = f(n)$.

2. An exercise proves $f(x) \to 0$ as $x \to \infty$.
3. Also $f'(x) = \log(1 + 1/x) - \frac{2x+1}{2x^2+2x}$.
4. Because $f'(x) < 0$ for $x > 1$ (proof left as an exercise) $f(x)$ is decreasing from $(3/2)\log(2) - 1 \approx 0.0397$ to 0 as x increases.
5. Hence $0 < \log\left(\frac{a_n}{a_{n+1}}\right)$.
6. By Lemma 3

$$\log\left(\frac{a_n}{a_{n+1}}\right) \leq \frac{1}{3(2n+1)^2} \sum_{k=0}^{\infty} \frac{1}{(2n+1)^{2k}}.$$

7. The sum is a geometric sum, so

$$0 < \log\left(\frac{a_n}{a_{n+1}}\right) < \frac{1}{3(2n+1)^2} \frac{1}{1 - (1/(2n+1)^2)} = \frac{1}{12n(n+1)}.$$

8. Expand $1/(12n(n+1))$ in partial fractions and add $\log(a_{n+1})$ throughout to get

$$\log(a_{n+1}) < \log(a_n) < \frac{1}{12n} - \frac{1}{12(n+1)} + \log(a_{n+1}).$$

□

Define $x_n = \log(a_n) - \frac{1}{12n}$, so Lemma 4 shows that x_n is an increasing sequence

$$x_n = \log(a_n) - \frac{1}{12n} < \log(a_{n+1}) - \frac{1}{12(n+1)} = x_{n+1}. \tag{2}$$

Define $y_n = \log(a_n)$, and then the left-side inequality in Lemma 4 shows that y_n is a decreasing sequence that is, $y_{n+1} < y_n$. By the definition of x_n and y_n, $x_n < y_n$ and $|x_n - y_n| = 1/(12n)$, so $|x_n - y_n| \to 0$ as $n \to \infty$. Therefore $\sup x_n = \inf y_n$ and call the common value λ. By continuity, $\lim_{n\to\infty} a_n = e^\lambda$.

Using elementary properties of limits

$$\mathrm{e}^{\lambda} = \lim_{n\to\infty} a_n = \frac{(\lim_{n\to\infty} a_n)^2}{\lim_{n\to\infty} a_{2n}} = \lim_{n\to\infty} \frac{(a_n)^2}{a_{2n}}\,. \tag{3}$$

However,

$$\frac{(a_n)^2}{a_{2n}} = \sqrt{\left(\frac{2}{n}\right)} \frac{2\cdot 4\ldots(2n-2)\cdot 2n}{1\cdot 3\ldots(2n-3)\cdot(2n-1)}\,. \tag{4}$$

The demonstration is left as an exercise.

Lemma 5 (Wallis' Formula)**.**

$$\lim_{n\to\infty} \left(\frac{(2n)\cdot(2n)\ldots 2\cdot 2}{(2n+1)\cdot(2n-1)\cdot(2n-1)\ldots 3\cdot 3\cdot 1} \right) = \frac{\pi}{2}.$$

Proof. See the proofs in Wallis Formula. □

Using the continuity of the square root function

$$\lim_{n\to\infty} \left(\sqrt{\frac{1}{2n+1}} \right) \left(\frac{(2n)\cdot(2n-2)\ldots 4\cdot 2}{(2n-1)\cdot(2n-3)\ldots 5\cdot 3\cdot 1} \right) = \sqrt{\frac{\pi}{2}}\,.$$

Now multiplying both sides by 2 and rewriting the leading square root sequence, get

$$\lim_{n\to\infty} \left(\sqrt{\frac{2}{n}\,\frac{2n}{2n+1}} \right) \left(\frac{(2n)\cdot(2n-2)\ldots 4\cdot 2}{(2n-1)\cdot(2n-3)\ldots 5\cdot 3\cdot 1} \right) = \sqrt{2\pi}\,.$$

Then since

$$\lim_{n\to\infty} \sqrt{\frac{2n}{2n+1}} = 1,$$

equation 4 is

$$\mathrm{e}^{\lambda} = \lim_{n\to\infty} a_n = \sqrt{\left(\frac{2}{n}\right)} \frac{2\cdot 4\ldots(2n-2)\cdot 2n}{1\cdot 3\ldots(2n-3)\cdot(2n-1)} = \sqrt{2\pi}\,.$$

Equivalently, unwrapping the definition of $a_n = \frac{n!}{n^{n+1/2}\mathrm{e}^{-n}}$ this is exactly Stirling's Formula

$$n! \sim \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n}\,.$$

Using the definitions $x_n = \log(a_n) - \frac{1}{12n}$ and $y_n = \log(a_n)$, the inequality $x_n < y_n$, and the least upper bound and greatest lower bound limit in equation (3) we can express Stirling's Formula in inequality form

$$\sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n} < n! < \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n+1/(12n)} .$$

This is almost as good as the inequality (1). This also gives a proof of the simple and useful inequality about Stirling's Formula, for all n

$$\sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n} < n! .$$

Remark. This proof of Stirling's Formula and the inequality (1) is the easiest, the shortest and the most elementary of the Stirling's Formula proofs. These are all definite advantages. The main disadvantage of this proof is that it requires the form of Stirling's Formula. However, this form follows from investigation of a natural family of functions motivated by the definition of Euler's constant, the base of the natural logarithms.

Stirling's Formula from Wallis' Formula

Lemma 6. *For $\alpha \in \mathbb{R}$, the sequence*

$$a_\alpha(n) = \left(1 + \frac{1}{n}\right)^{n+\alpha}$$

is decreasing if $\alpha \in [\frac{1}{2}, \infty)$, and increasing for $n \geq N(\alpha)$ if $\alpha \in (-\infty, \frac{1}{2})$.

Remark. An illustration of representative sequences with $\alpha = 1$ (green) and $\alpha = 0$ (red) is in Figure 2.

Remark. This lemma is based on a remark due to I. Schur, see [5], problem 168, on page 38 with solution on page 215.

Proof. 1. The derivative of the function $f(x) = \left(1 + \frac{1}{x}\right)^{x+\alpha}$ (defined on $[1, \infty)$) is

$$f'(x) = \left(1 + \frac{1}{x}\right)^{x+\alpha} \left(\log\left(1 + \frac{1}{x}\right) - \frac{x+\alpha}{x(x+1)}\right).$$

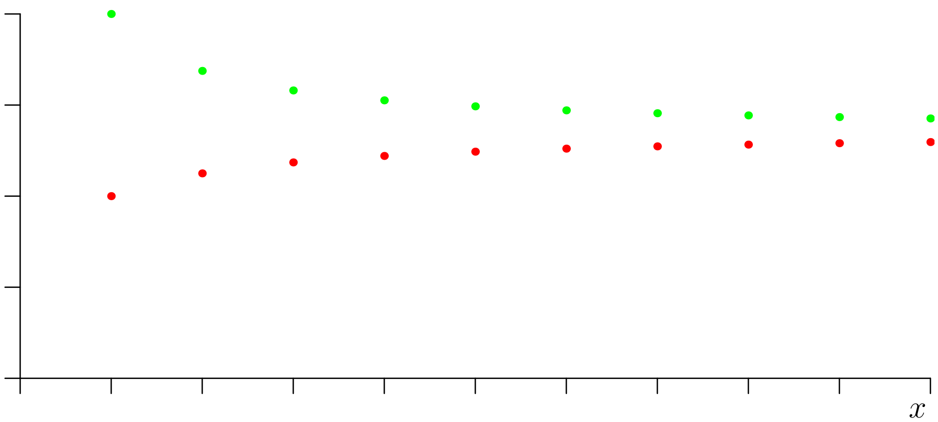

Figure 2: Example of the lemma with $\alpha = 1$ (green) and $\alpha = 0$ (red).

2. Let
$$g(x) = \left(\log\left(1+\frac{1}{x}\right) - \frac{x+\alpha}{x(x+1)}\right)$$
then
$$g'(x) = \frac{(2\alpha-1)x+\alpha}{x^2(x+1)^2}$$
and $\lim_{x\to\infty} g(x) = 0$.

3. It follows that $g(x) < 0$ and so $f'(x) < 0$ when $\alpha \geq 1/2$ and $x \geq 1$, and $f'(x) > 0$ when $\alpha < 1/2$ and $x \geq \max(1, \frac{\alpha}{1-2\alpha})$. The monotonicity of $a_\alpha(n)$ follows.

$\square$

From the lemma, for every $\alpha \in (0, 1/2)$ there is a positive integer $N(\alpha)$ such that
$$\left(1+\frac{1}{k}\right)^{k+\alpha} < \mathrm{e} < \left(1+\frac{1}{k}\right)^{k+1/2}$$
for all $k \geq N(\alpha)$. As a consequence, we get
$$\prod_{k=n}^{2n-1}\left(1+\frac{1}{k}\right)^{k+\alpha} < \mathrm{e}^n < \prod_{k=n}^{2n-1}\left(1+\frac{1}{k}\right)^{k+1/2}.$$

Rearrange the products with telescoping cancellations, using the upper bound on the right as an example.

$$
\begin{aligned}
&\prod_{k=n}^{2n-1} \left(1+\frac{1}{k}\right)^{k+1/2} \\
&\quad= \left(1+\frac{1}{n}\right)^{n+1/2} \left(1+\frac{1}{n+1}\right)^{n+1+1/2} \cdots \left(1+\frac{1}{2n-1}\right)^{2n-1+1/2} \\
&\quad= \left(\frac{n+1}{n}\right)^{n+1/2} \left(\frac{n+2}{n+1}\right)^{n+1+1/2} \cdots \left(\frac{2n}{2n-1}\right)^{2n-1+1/2} \\
&\quad= \left(\frac{n+1}{n}\right)^{1/2} \left(\frac{n+2}{n+1}\right)^{1/2} \cdots \left(\frac{2n}{2n-1}\right)^{1/2} \cdot \\
&\qquad\quad \left(\frac{n+1}{n}\right)^{n} \left(\frac{n+2}{n+1}\right)^{n+1} \cdots \left(\frac{2n}{2n-1}\right)^{2n-1} \\
&\quad= \left(\frac{2n}{n}\right)^{1/2} \cdot \left(\frac{(2n)^{2n-1}}{n^n \cdot (n+1) \cdots (2n-1)}\right) \\
&\quad= 2^{1/2} \cdot \frac{(2n)^{2n-1}}{n^n \cdot (n+1) \cdots (2n-1)} \\
&\quad= 2^{1/2} \cdot \frac{2^{2n-1} \cdot n^{n-1}}{(n+1) \cdots (2n-1)}.
\end{aligned}
$$

Multiply by the last fraction by $n!/n^n$ and Write more compactly,

$$
\begin{aligned}
&\frac{n!}{n^n} \cdot 2^{1/2} \cdot \left(\frac{2^{2n-1} \cdot n^{n-1}}{(n+1) \cdots (2n-1)}\right). \\
&\quad= 2^{1/2} \cdot \frac{2^{n-1} \cdot 2^n \cdot n!}{n \cdot (n+1) \cdots (2n-1)} \\
&\quad= 2^{1/2} \cdot \frac{(2n)!!}{(2n-1)!!}.
\end{aligned}
$$

This uses the **double factorial** notation $n!! = n \cdot (n-2) \cdots 4 \cdot 2$ if n is even, and $n!! = n \cdot (n-2) \cdots 3 \cdot 1$ if n is odd. The lower bound product on the left is similar.

After multiplying through by $n!/n^{n+1/2}$,

$$
\frac{2^\alpha}{\sqrt{n}} \cdot \frac{(2n)!!}{(2n-1)!!} < \frac{n!\mathrm{e}^n}{n^{n+1/2}} < \frac{2^{1/2}}{\sqrt{n}} \cdot \frac{(2n)!!}{(2n-1)!!}
$$

for all $n \geq N(\alpha)$. Using the Wallis Formula,

$$2^{\alpha}\sqrt{\pi} \leq \liminf_{n\to\infty} \frac{n!\mathrm{e}^{n}}{n^{n+1/2}} \leq \limsup_{n\to\infty} \frac{n!\mathrm{e}^{n}}{n^{n+1/2}} \leq \sqrt{2\pi}.$$

Stirling's formula follows by passing to the limit as $\alpha \to 1/2$.

Remark. This proof generalizes to an asymptotic formula for the Gamma function using log-convexity of the Gamma function. See [1].

Related Asymptotic Formulas for Binomial Coefficients

Theorem 7.

$$\left(\frac{n}{k}\right)^{k} \leq \binom{n}{k} \leq \left(\frac{\mathrm{e}n}{k}\right)^{k}$$

Proof. 1. Start with

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+2)(n-k+1)}{k(k-1)\cdots 2\cdot 1}.$$

2. Since

$$\frac{n}{k} \leq \frac{n-i}{k-i}$$

for all $i = 0, 1, \dots k-1$, the left inequality is immediate.

3. Since

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+2)(n-k+1)}{k(k-1)\cdots 2\cdot 1} \leq \frac{n^{k}}{k!}$$

and by a step in the proof of Theorem 1

$$\frac{1}{k!} \leq \left(\frac{e}{k}\right)^{k}$$

the right inequality follows.

□

Theorem 8.

$$\binom{n}{k} \sim \left(\frac{n^{k}}{k!}\right)$$

if $k = o(n^{1/2})$ as $n \to \infty$.

Proof. 1. The statement of the theorem is equivalent to showing

$$\lim_{n\to\infty} \frac{n(n-1)\cdots(n-k+1)}{n^k} = \lim_{n\to\infty} \prod_{j=1}^{k-1} (1 - j/n) = 1$$

if $k = o(n^{1/2})$.

2. In turn, this is equivalent to showing

$$\lim_{n\to\infty} \log\left(\prod_{j=1}^{k-1} (1 - j/n)\right) = \lim_{n\to\infty} \sum_{j=1}^{k-1} \log\left(1 - j/n\right) = 0.$$

3. Using $\log(1-x) = x + O(x^2)$, the sum is

$$\sum_{j=1}^{k-1} \log\left(1 - j/n\right) = \sum_{j=1}^{k-1} \left(j/n + O(j^2/n^2)\right) = \frac{k(k-1)}{n} + \frac{1}{n^2}O(k^3).$$

4. Now the hypothesis $k = o(n^{1/2})$ comes into play so $\lim_{n\to\infty} \frac{k(k-1)}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{n^2}O(k^3) = 0$ establishing the desired limit.

$\square$

Section Ending Answer

The geometric summation formula is

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

which converges uniformly for $|x| < 1$. Then

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$$

and

$$\log(1+x) = \int_0^x \frac{1}{1+t}\, \mathrm{d}t = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$$

where the term-by-term integration is justified by the uniform convergence for $|x| < 1$.

Sources

The weak form of Stirling's Formula is taken from [6]. The motivating derivation of the sequences from the family of functions is from [2]. The first sequence proof is adapted from the sketch in Kazarinoff [3] based on the note by Nanjundiah [4]. The second sequence proof using derivatives and monotonicity is adapted from [1]. The binomial coefficient limits are from lecture notes by Xavier Perez Gimenez.

Problems to Work for Understanding

1: Show $\sqrt[n]{2n+1} \to 1$ as $n \to \infty$.

2: Show for $|x| < 1$,

$$\log\left(\frac{1+x}{1-x}\right) = 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)x^{2k+1}}\,.$$

3: Show that if $\alpha > 1/2$

$$\lim_{n\to\infty} \frac{n!}{n^\alpha}\left(\frac{\mathrm{e}}{n}\right)^n = 0.$$

4: Let $f(x) = (x+1/2)\log(1+1/x) - 1$. Show $f(x)$ decreases to 0 as $x \to \infty$.

5: Show

$$\frac{(a_n)^2}{a_{2n}} = \sqrt{\left(\frac{2}{n}\right)} \frac{2 \cdot 4 \ldots (2n-2) \cdot 2n}{1 \cdot 3 \ldots (2n-3) \cdot (2n-1)}\,.$$

6: Show the derivative of the function $f(x) = \left(1+\frac{1}{x}\right)^{x+\alpha}$ (defined on $[1,\infty)$) is

$$f'(x) = \left(1+\frac{1}{x}\right)^{x+\alpha}\left(\log\left(1+\frac{1}{x}\right) - \frac{x+\alpha}{x(x+1)}\right).$$

Reading Suggestion:

References

[1] Dorin Ervin Dutkay, Constantin P. Niculescu, and Florin Popvici. Stirling's formula and its extension for the Gamma function. *American Mathematical Monthly*, 120:737–740, October 2013. available as http:dx.doi.org/10.4169/amer.math.monthly.120.08.737.

[2] Adam Hammett. Euler's Limit and Stirling's Estimate. *College Mathematics Journal*, 51(5):330–336, November 2020.

[3] Nicholaus D. Kazarinoff. *Analytic Inequalities*. Holt, Rinehart and Winston, 1961.

[4] T. S. Nanjundiah. Note on Stirling's formula. *American Mathematical Monthly*, 66:701–703, 1959.

[5] G. Pólya and G. Szegö. *Problems and Theorems in Analysis I: Series, Integrals, Theory of Functions*. Classics in Mathematics. Springer Verlag, 1998. Reprint of the 1978 version.

[6] S. Saks and A. Zygmund. *Analytic Functions*. Elsevier Publishing, 1971.

Outside Readings and Links:

Solutions

1: Take logarithms and consider $\log(2n+1)/n$ which approaches 0 as $n \to 0$. Then $\sqrt[n]{2n+1} \to \mathrm{e}^0 = 1$ as $n \to \infty$.

2:

$$\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} - \sum_{k=0}^{\infty} \frac{x^{k+1}}{(2k+1)}$$

$$= -2 \sum k = 1, 3, 5, \ldots \frac{x^{k+1}}{k}$$

$$= -2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)}.$$

3: If $\alpha > 1/2$, the sequence is positive and decreasing and so the limit must be either 0 or $L > 0$. If for some $\alpha_0 > 1/2$ the limit is $L < 0$, then for fixed $\beta \in (1/2, \alpha_0)$,

$$\lim_{n\to\infty} \frac{n!}{n^\beta}\left(\frac{\mathrm{e}}{n}\right)^n = \lim_{n\to\infty} \frac{n^{\alpha_0}}{n^\beta}\frac{n!}{n_0^\alpha}\left(\frac{\mathrm{e}}{n}\right)^n =$$

$$\lim_{n\to\infty} \frac{n^{\alpha_0}}{n^\beta} \cdot \lim_{n\to\infty} \frac{n!}{n_0^\alpha}\left(\frac{\mathrm{e}}{n}\right)^n = \infty \cdot L = \infty$$

which is not possible, so the limit must be 0 for all $\alpha > 1/2$.

4: This is essentially the content of steps 1 through 6 of Lemma 2. Using the notation in the proof in step 6

$$f'(x) = a'_{1/2}(x) = -\frac{1}{6x} < \psi_\alpha(x) < 0.$$

Using, for instance L'Hospital's rule, $\lim_{x\to\infty}(x+1/2)\log(1+1/x) = 1$, or expanding as a power series in terms of $1/x$

$$(x+1/2)\log(1+1/x) = 1 + \frac{1}{12y^2} - \frac{1}{12y^3} + \ldots$$

so $\lim_{x\to\infty}(x+1/2)\log(1+1/x) = 1$.

5: Recall $a_n = \frac{n!}{n^{n+1/2}\mathrm{e}^{-n}}$ so $(a_n)^2 = \frac{n!\cdot n!}{n^{2n+1}\mathrm{e}^{-2n}}$. Also $a_{2n} = \frac{(2n)!}{(2n)^{2n+1/2}\mathrm{e}^{-2n}}$. Then

$$\begin{aligned}\frac{(a_n)^2}{a_{2n}} &= \left(\frac{n!\cdot n!}{(2n)!}\right)\frac{2^{2n}2^{1/2}n^{2n}n^{1/2}}{n^{2n}n} \\ &= \sqrt{\frac{2}{n}}\left(\frac{n!\cdot n!}{(2n)!}\right)2^{2n} \\ &= \sqrt{\frac{2}{n}}\left(\frac{n!}{1\cdot 3\cdots(2n-3)\cdot(2n-1)}2^n\right) \\ &= \sqrt{\left(\frac{2}{n}\right)}\frac{2\cdot 4\cdots(2n-2)\cdot(2n)}{1\cdot 3\cdots(2n-3)\cdot(2n-1)}.\end{aligned}$$

6: Consider $\log(f(x)) = (x+\alpha)\log(1+1/x)$ so

$$\begin{aligned}[\log(f(x))]' &= \frac{f'(x)}{f(x)} \\ &= \log\left(1+\frac{1}{x}\right) + (x+\alpha)\cdot\frac{x}{x+1}\cdot\frac{-1}{x^2} \\ &\log\left(1+\frac{1}{x}\right)(x+\alpha)\cdot\frac{-1}{x(x+1)}.\end{aligned}$$

Then

$$f'(x) = \left(1+\frac{1}{x}\right)^{x+\alpha}\left(\log\left(1+\frac{1}{x}\right) - \frac{x+\alpha}{x(x+1)}\right).$$

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