Class 4

Shikhar Saxena

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Contents

Stochastic Process $\{X(t), t \in T\}$

- $X(t) : \Omega \to \mathcal{S}$
- T and $\mathcal{S}$ are parameter and state space respectively

DTMC

- Obeys Markov Property

Consider Transition Probability Matrix

$$P = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}$$

Now if we consider $S_n = \sum X_i$ and $\hat{\mu}_n = S_n/n$ then $\hat{\mu}_n$ will depend on P and might not go as expected from Strong Law of Large Numbers (SLLN).

- $\overline{\mu}$: Initial Distribution over $\mathcal{S}$

Chapman Kolmogorov Equation for DTMC

- This follows only for homogeneous DTMCs
  - ⋆ For non-homogeneous, P depends on n (the timestep).

$$P^{(n+l)} = P^{(n)}P^{(l)}$$

Classification of States

Accessible State

- j is accessible from i if $p_{ij}^{n} > 0$ for some n
- Denoted as $i \to j$
- $i \to j$ and $j \to i$ then we say that i and j communicate. This is denoted by $i \leftrightarrow j$.

Communication is an equivalence relation.

$i \leftrightarrow i$ achieved in $P^0 = I$.

States which communicate belong to the same ***equivalence class***

Irreducible

Irreducible when $i \leftrightarrow j,\ \forall i, j \in \mathcal{S}$ i.e., only one communicating class.

Recurrent and Transient States

$$F_{ii} = P(\text{ever returning to } i \text{ having started in } i)$$

- State i is recurrent if $F_{ii} = 1$
- If $F_{ii} < 1$ then State i is transient
  - ⋆ Each communicating class share recurrence or transience

Limiting Probabilities

The dependence on initial distribution starts to vanish after long time.

$$\pi_j = \lim_{n\to\infty} p_{ij}^{(n)} = [\lim_{n\to\infty} P^n]_{ij}$$

Stationary Distribution

$$\pi = \pi P$$

Proof. Stochastic Matrices always have an eigenvalue 1, so stationary distribution will always exist. □

- πP is essentially the pmf of X_1 having picked X_0 according to π

- $\pi = \pi P$ says that the distribution of X_1 will also be π (if initial distribution was π)

Remark. *If Limiting Distribution exists, then it is same as stationary distribution.*