UNIT 1 PROBABILITY, RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

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PART-A
SHORT QUESTIONS WITH SOLUTIONS

Q1. There are 5 items defective in a sample of 30 items. Find the probability that an item chosen at random from the sample is,

(i) Defective

(ii) Non-defective.

Answer :

Model Paper-I, Q1(b)

Let E be the event of selecting a defective item.

Given that,

$n(S) = 30$

$n(E) = 5$

$n(\bar{E}) = 30 - 5 = 25$

(i) **Defective**

The probability that an item chosen is defective is,

$$P(E) = \frac{n(E)}{n(S)} = \frac{5}{30} = \frac{1}{6}$$

(ii) **Non-defective**

The probability that an item chosen is non-defective is,

$$P(\bar{E}) = \frac{n(\bar{E})}{n(S)} = \frac{25}{30} = \frac{5}{6}$$

Q2. An integer is chosen from 20 to 30. Find the probability that it is a prime number.

Answer :

The integers from 20 to 30 are,

$S = \{20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\}$

$\therefore\ n(S) = 11$

Let E be the event of choosing prime number.

$E = \{23, 29\}$

$n(E) = 2$

$\therefore$ The required probability is,

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{11}$$

Q3. If E is any event, then prove that,

$$0 \le P(E) \le 1$$

Answer :

We know that,

$$\phi \le E$$

$$\Rightarrow \quad P\phi \le P(E)$$

$$0 \le P(E) \quad \text{... (i)} \qquad [\because P(\phi) = 0]$$

Again $E \le S$

$$P(E) \le P(S)$$

$$\therefore \quad P(E) \le 1 \quad \text{... (ii)} \qquad [\because P(S) = 1]$$

From equation (i) and (ii) we have,

$$0 \le P(E) \le 1$$

Hence proved.

Q4. Write short notes on conditional probability.

Answer : Model Paper-I, Q1(a)

Conditional probability a defined as the probability of occurrence of a particular value of one random variable when the other random variable has already occured. Let A and B be any two events. The probability of the happening of the event B given that A has already happened is denoted by $P(B/A)$ and is defined as,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \text{ if } P(A) \neq 0$$

(or)

$$P(A \cap B) = P(A) \mid P(B|A) \text{ if } P(A) \neq 0$$

Q5. If A and B are mutually independent events then prove that A and $\bar{B}$ are also independent events.

Answer :

Given that A and B are mutually independent, therefore we can write,

$$P(A \cap B) = P(A). P(B) \quad \text{... (1)}$$

To prove that A and $\bar{B}$ are also independent, we need to prove that,

$$P(A \cap \bar{B}) = P(A) .P(\bar{B}) \quad \text{... (2)}$$

Consider L.H.S of equation (2), we get,

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$= P(A) - P(A) .P(B) \text{ (From equation (1))}$$

$$= P(A). [1 - P(B)]$$

$$= P(A).P(\bar{B})$$

$$= \text{R.H.S}$$

Hence proved.

Q6. A problem in statistic is given to two students A and B. The probability that A solves the problem is 1/2 and that of B's to solve it is 2/3. Find the probability that the problem is solved.

Answer :

Let E be the event that the problem is solved.

Given that,

$$P(A) = \frac{1}{2} \quad ; \quad P(B) = \frac{2}{3}$$

$$\Rightarrow \quad P(\bar{A}) = 1 - P(A) \quad ; \quad P(\bar{B}) = 1 - P(B)$$

$$P(\bar{A}) = 1 - \frac{1}{2} \quad ; \quad P(\bar{B}) = 1 - \frac{2}{3}$$

$$P(\bar{A}) = \frac{1}{2} \quad ; \quad P(\bar{B}) = \frac{1}{3}$$

$\therefore$ The probability that the problem is solved,

$$P(E) = 1 - P(\bar{A}).P(\bar{B})$$

$$= 1 - \frac{1}{2} \times \frac{1}{3}$$

$$= 1 - \frac{1}{6} = \frac{6-1}{6}$$

$$= \frac{5}{6}$$

Q7. A problem is given to three students S_1, S_2 and S_3. The chances of solving the problems by them are 1/3, 3/4, and 2/5 respective. Find the probability that the problem will be solved if all of them try indepedently.

Answer : Model Paper-II, Q1(b)

The problem can be solved only if atleast one of the student can solve it.

$\therefore$ The probability of occurrence of atleast one of the three events is,

$$P(S_1 \cup S_2 \cup S_3) = 1 - P(\bar{S}_1).P(\bar{S}_2).P(\bar{S}_3)$$

$$= 1 - \left(1 - \frac{1}{3}\right).\left(1 - \frac{3}{4}\right).\left(1 - \frac{2}{5}\right)$$

$$= 1 - \left(\frac{2}{3}\right).\left(\frac{1}{4}\right).\left(\frac{3}{5}\right)$$

$$= 1 - \frac{6}{60}$$

$$= \frac{54}{60}$$

$$\therefore \quad P(S_1 \cup S_2 \cup S_3) = 0.9$$

Q8. A sample space S (for an experiment S) contains 25 equally likely outcomes. If an event A (for this experiment S) is such that Pr(A) = 0.24, how many outcomes are there in A?

Answer :

Given that,

For an event A,

$$n(S) = 25$$

$$Pr(A) = 0.24$$

Number of outcomes in A are,

$$n(A) = n(S)\ Pr(A)$$

Substituting the corresponding values above equation.

$$n(A) = 25 \times 0.24$$

$$\therefore \quad n(A) = 6$$

Q9. It two integers are selected, at random and without replacement, from {1, 2, 4, ... 99, 100}, what is the probability that integers are consecutive?

Answer :

Given that,

Sample space, $S = \{1, 2,3, 4, ...99, 100\}$

Two integers are selected at random in $100C_2$ ways without replacement

$n(S) = 4950$

Let, A be an event of selecting two consecutive integers.

$A = \{(1, 2)\ (2, 3)\ (3, 4)\ ....\ (99, 100)\}$

$n(A) = 99$

Probability of selecting two consecutive integers is,

$$Pr(A) = \frac{n(A)}{n(S)}$$

Substituting the corresponding values in above equation,

$$Pr(A) = \frac{99}{4950} = \frac{1}{50}$$

$$\therefore\ Pr(A) = \frac{1}{50}$$

Q10. Write short notes on Baye's theorem.

Answer : Model Paper-II, Q1(a)

An event Q occurs only if one of the set of mutually disjoint and exhaustive events $R_1, R_2, R_3,....,R_k$ occurs. If the $P(R_i) \neq 0$ for each k then for any arbitrary event Q which is a subset of $\bigcup_{k=1}^{n} R_k$ with $P(Q) > 0$,

$$P(R_k | Q) = \frac{P(R_k)P(Q | R_k)}{\sum_{i=1}^{k} P(R_k)P(Q | R_k)}$$

The above equation is known as 'Baye's theorem'.

Q11. Define the term 'Random Variable'.

Answer : Model Paper-III, Q1(a)

A random variable is a variable which takes particular value (numerical value) and the value is determined by the result of the random experiment.

Always random variables are denoted by capital letters and the corresponding values by small letters.

For example, if a fair dice is rolled and if 'X' denotes the number obtained then 'X' is called a random variable.

Thus, 'X' can take any one of the particular values as 1, 2, 3, 4, 5 or 6, each with a probability $\frac{1}{6}$.

These values can be tabulated as follows,

X	1	2	3	4	5	6
P(X)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

All the possible outcomes of the random experiment is called as a 'sample space' and is denoted by 'S'.

The sum of all the probabilities of sample space is always 1.

Q12. Define discrete and continuous random variables with an example each.

Answer :

Discrete Random Variable

If a random variable 'X' takes a finite number of values or countably infinite of values, then it is known as '*Discrete random variable*'.

Example

Number of head in tossing three coins.

Number of tosses to get a tail etc.

Continuous Random Variable

If a random variable 'X' takes any of the possible values (fractional or integral values) at any point of time is called as continuous random variable.

Examples

Height, Weight etc.

Q13. Write in brief about discrete probability distribution.

Answer :

Discrete probability distribution is the probability distribution of a discrete random variable 'X' which takes only a finite number of variables. Random variable 'X' as function $f(x)$ satisfies the following conditions,

(i) $f(x) \geq 0$

(ii) $\Sigma f(x) = 1$

(iii) $P(X = x) = f(x)$

The examples of discrete probability distribution are binomial and Poisson distribution.

Q14. If $f(x) = e^{-x}$; $0 < x < \infty$, then find,

(i) $P(1 < X < 2)$

(ii) $F(1)$.

Answer :

Given that,

$$f(x) = e^{-x} \text{ for } 0 < x < \infty$$

(i) $$P(1 < X < 2) = \int_1^2 e^{-x} dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_1^2$$

$$= \left(\frac{e^{-2}}{-1}\right) - \left(\frac{e^{-1}}{-1}\right)$$

$$= -e^{-2} + e^{-1}$$

$$= e^{-1} - e^{-2}$$

$$= \frac{1}{e} - \frac{1}{e^2}$$

$$= \frac{e-1}{e^2}$$

$$\therefore \quad P(1 < X < 2) = \frac{e-1}{e^2}$$

(ii) $$F(1) = P(X < 1)$$

$$= P(0 < X < 1)$$

$$= \int_0^1 e^{-x} dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^1$$

$$= \frac{e^{-1}}{-1} - \frac{e^{-0}}{-1}$$

$$= -e^{-1} + e^{-0}$$

$$= -e^{-1} - 1$$

$$= 1 - e^{-1}$$

$$= 1 - \frac{1}{e}$$

$$= \frac{e-1}{e}$$

$$F(1) = \frac{e-1}{e}.$$

Q15. An urn contains 6 red and 4 white balls. Three balls are drawn at random. Obtain the probability distribution of the number of white balls drawn.

Answer : Model Paper-III, Q1(b)

Let X denotes the number of white balls drawn.

Then X is a random variable which can take values 0, 1, 2 or 3.

There are $6R + 4W = 10$ balls

Exhaustive cases of drawing 3 balls

$$P(x = 0) = P(\text{No white balls}) = \frac{{}^6C_3}{{}^{10}C_3} = \frac{5}{30}$$

$$P(x = 1) = P(\text{One white ball}) = \frac{{}^6C_2\,{}^4C_1}{{}^{10}C_3} = \frac{15}{30}$$

$$P(x = 2) = P(\text{Two white balls}) = \frac{{}^6C_1\,{}^4C_2}{{}^{10}C_3} = \frac{9}{30}$$

$$P(x = 3) = P(\text{Three white balls}) = \frac{{}^4C_3}{{}^{10}C_3} = \frac{1}{30}$$

Hence, the probability distribution of x is given by,

x	0	1	2	3
P(x)	$\frac{5}{30}$	$\frac{15}{30}$	$\frac{9}{30}$	$\frac{1}{30}$

PART-B

ESSAY QUESTIONS WITH SOLUTIONS

1.1 PROBABILITY

1.1.1 Sample Space, Events

Q16. What is sample space? List the different type of sample space.

Answer :

Model Paper-I, Q2(a)

Sample Space

The set of all possible outcomes of a random experiment is known as sample space and is denoted by S. Each element in the sample space is called sample point.

Examples

1. In tossing a coin, the possible outcomes are either a Head (H) or a Tail (T).

   Sample space, S = {H, T}

2. In tossing two coins or a coin tossed twice, the possible outcomes are HH, HT, TH, TT.

   Sample space, S = {HH, HT, TH, TT}

3. If a die is thrown, the possible outcomes are 1 or 2 or 3 or 4 or 5 or 6.

   Sample space, S = {1, 2, 3, 4, 5, 6}

4. If a die is thrown two times, the number of outcomes are 36. The required sample space is,

   Sample space S ={(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

   If a die is thrown for n times or n dice are thrown, the sample space consists of 6 n elements.

Types of Sample Space

(i) **Discrete Sample Space**

Discrete sample space is a sample space involving finite set of events.

Examples

(a) In tossing a coin, Sample space, S = {H, T}

(b) In rolling a die, S = {1, 2, 3, 4, 5, 6}

(ii) **Continuous Sample Space**

Continuous sample space is a sample space which contains infinite number of events.

Example

In an experiment of measuring room temperature from t_1 to t_2 (sec), sample space can vary between t_1 and t_2. i.e., $S = \{t_1 < S < t_2\}$.

Q17. What is an experiment? List and explain the different types of experiment.

Answer :

Experiment

An experiment is a physical action or process whose results are observed and noted. It may result in two or more outcomes.

Examples

(i) Tossing a fair coin is an experiment. (A coin is a circular metal disc, the two faces of which are somehow distinguishable and are called 'head' and 'tail'). Whether the coin will fall head up or tail up is unpredictable.

(ii) Rolling an unbiased die is an experiment. (A die is a solid cube, the six faces of which are marked by 1, 2, 3, 4, 5 and 6 dots, How many dots it will actually throw up is unpredictable and is subject to chance.

Types of Experiments

Experiments are classified into two types. They are,

1. Deterministic or predictable experiments
2. Undeterministic or random experiments.

1. Deterministic or Predictable Experiments

An experiment whose result can be predicted with certainty prior to the experiment is known as deterministic experiment.

Examples

(i) In adding of two numbers like 2 + 2, one is sure that the result will be 4.

(ii) While throwing a stone upward, one is sure that it will go upto certain height and then it will fall onto the ground.

2. Undeterministic or Random Experiment

An experiment when repeated under same condition, such that its result cannot be predicted with certainty but all possible results can be determined prior to the performance of experiment is known as random experiment.

Examples

(i) In tossing a coin, one is not sure whether a head or a tail will occur.

(ii) In throwing a die, one is not sure whether 1, 2, 3, 4, 5 or 6 will be obtained.

(iii) Lifetime of a human being.

(iv) Lifetime of a computer system.

Q18. What is an event? List and explain the different types and operations of event.

Answer : Model Paper-II, Q2(a)

Event

An event is a possible result (or) outcome of an experiment or a result of trial. It is a subset of sample spaces.

Example

When a coin is tossed, the probability of getting a trail is said to be an event.

Types of Event

Basically there are two types of events.

(i) Simple event

(ii) Compound event.

(i) Simple Event

The probability of happening or non-happening of a single event is considered as a simple event.

Example

When we are selecting two black coins from a box containing 10 white and 5 black coins.

(ii) Compound Event

If the joint occurrence of two or more event is considered then it is known as 'compound event' or 'composite event'.

Example

A box containing 5 white balls, 3 black balls and 8 red balls, when we draw 2 white balls in first draw, 2 black balls in second draw and 5 red balls in third draw.

Operations of Event

If A and B are any two events in 's', then the various operations of set theories on the subsets of the set 's' can be defined with respect to events of 'S' corresponding to a random experiment are 'E' as follows,

(i) Union

$A \cup B$ (Union of A and B) is the event that can occur if either the event A or the event B or both occurs.

Example

Let A = {1, 3, 5}, B = {2, 4, 6}. Then,

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

(ii) Intersection

$A \cap B$ (Intersection of A and B) is the event that can occur if both the events A and B occurs.

Example

Let A = {c, g, h, e}, B = {f, g, h, i}. Then,

$A \cap B = \{g, h\}$

(iii) Complementation

$\bar{A}$ (A complement or negation of an event A) is the event that can occur if the event A does not occur or simply it specifies the outcomes that are present in S but not in A.

Example

Let S = {1, 2, 3, 4, 5, 6} and A = {1, 3, 5}. Then,

$A' = \{2, 4, 6\}$

(iv) Mutually Exclusive

Two events are said to be mutually exclusive, if the occurrence of one event stops the occurrence of the other event in the same trial of an experiment. Symbolically, for any two mutually exclusive events say A and B, $A \cap B = \phi$. This implies that $P(A \cap B) = 0$.

Examples

(a) In the experiment of tossing a coin, the occurrence of the event head prevents the occurrence of the event tail.

(b) In a die, all the faces are mutually exclusive events i.e., occurrence of face 3 prevents occurrence of 1, 2, 4, 5 and 6.

PROBLEMS

Q19. Consider an experiment of flipping a coin wherein tail occurs on first flip and head occurs on second flip. On occurrence of tail the die is tossed once. Determine the sample space by constructing a tree diagram.

Solution :

In flipping a coin, the possible outcomes are either a Head (H) or a Tail (T).

The possibility of occurrence of head on tossing a coin twice is {HH, HT}. Similarly, the possibility of occurrence of tail on tossing a coin 'n' number of time is {T1, T2, T3, T4, T5}.

The tree diagram for flipping a coin will be as follows,

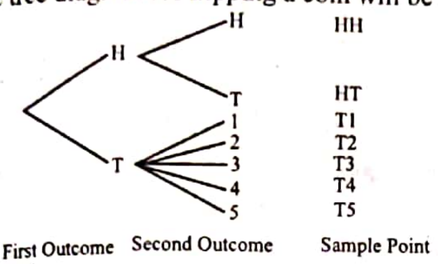


Figure: Tree Diagram for Flipping a Coin

∴ Sample space, S = {HH, HT, T1, T2, T3, T4, T5}.

Q20. The sample space for an experiment is S = {a, b, c, d, e, F, g, h}, where each outcome is equally likely. If event A = {a, b, c} and event B = {a, c, e, g}. Determine (i) union of events A and B (ii) Intersection of events A and B (iii) Complement of events A and B.

Solution :

Given that,

Sample space, S = {a, b, c, d, e, F, g, h}

Event, A = {a, b, c}

Event, B = {a, c, e, g}

(i) **Union of Events A and B**

Union of two events A and B is represented as $A \cup B$.

∴ $A \cup B$ = {a, b, c, e, g}

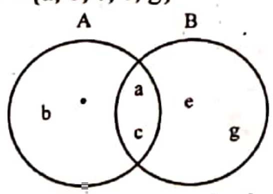


Figure: Venn Diagram for $A \cup B$

(ii) **Intersection of Events A and B**

Intersection of two events A and B is represented as $A \cap B$.

∴ $A \cap B$ = {a, c}

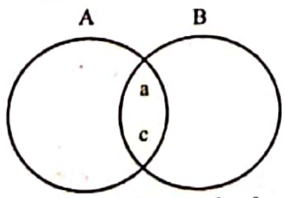


Figure: Venn Diagram for $A \cap B$

(iii) **Complement of Events A and B**

Complement of event A is represented as A'

∴ A' = {e, g}

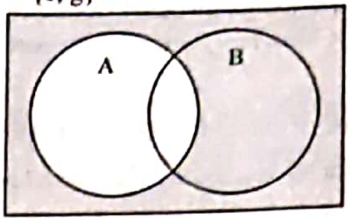


Figure: Venn Diagram for A'

Complement of event B is represented as B'

∴ B' = {b}

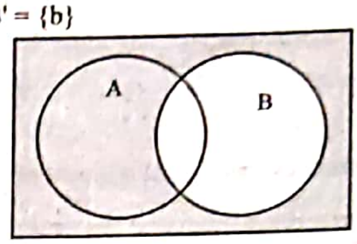


Figure: Venn Diagram for B'

1.1.2 Counting Sample Points

Q21. Give a brief introduction on rules of counting.

Answer : Model Paper-III, Q2(a)

Rules of Counting

Counting assumes importance in the study of probability theory because, the chance of occurrence of an event has to be assessed by counting the outcomes favourable to the happening of the event from among all possible outcomes. This process of counting with certain conditions is not as simple as it appears. The fundamental principle of counting includes the following rules.

Rule 1

If an operation is performed in any one of m ways and when it has been performed another operation can be performed in any one of n ways, then the number of ways in which both the operations can be performed together is $m \times n$.

Example

Find the number of exhaustive cases when a pair of dice are thrown once.

Solution

The number of ways the first die can land = 6

The number of ways the second die can land = 6

∴ The pair of dice can land together

$= 6 \times 6$

$= 36$

Rule 2

If an operation is performed in total of m_1 ways and for each of these ways another operation is performed in m_2 ways. Then, for each of first and second operation a third operation can be performed in m_3 ways. So, the sequence of n operations is performed in $m_1 m_2 \ldots\ldots m_n$ ways.

(Example) Problem

Sam is going to assemble a PC by himself. He has the choice of chips from two brands, a hard drive from four, memory from three and an accessory bundle from five local stores. How many different ways can Sam order the parts?

Solution

Number of choices for selecting chips, $n_1 = 2$

Number of choices for selecting a hard drive, $n_2 = 4$

Number of choices for selecting a memory, $n_3 = 3$

Number of choices for selecting an accessory bundle, $n_4 = 5$

Number of ways in which sam can order the parts is,

$$= n_1 \times n_2 \times n_3 \times n_4$$
$$= 2 \times 4 \times 3 \times 5$$
$$= 120$$

Therefore, the number of ways in which Sam can order parts is 120.

Q22. Write short notes on permutation.

Answer :

Permutation

Permutation refers to different arrangement of objects in a set where all the elements are different and distinguishable. These arrangements are without repetition of any individual object. If some objects are similar then, the permutations will be affected. In short, the permutations refer to separate arrangement of different objects contained in a set of elements.

Theorem 1 (Linear Permutation)

The total number of ways of arranging 'n' distinct objects among themselves in a linear fashion is $n!$.

(Example) Problem

A factory owner has received three new machines A, B and C. Find the number of ways in which these machines can be arranged.

Solution

Number of machines, $n = 3$

$\therefore$ Total number of ways in which machines can be arranged is $n!$

$$\Rightarrow \quad n! = 3!$$
$$= 3 \times 2 \times 1$$
$$= 6 \text{ ways.}$$

Theorem 2

A permutation of n objects taken r at a time is an ordered selection of r objects which is denoted by p(n, r) or npr.

Where,

$$nP_r = n(n-1)(n-2) \ldots\ldots (n-r+1)$$

(or)

$$= \frac{n!}{(n-r)!}$$

(Example) Problem

In a year, three awards (research, teaching, service) will be given to a class of 25 graduate students in statistics department. If each student can receive atmost one award, how many possible selections are there?

Solution

Number of students, $n = 25$

Number of awards given in each area, $r = 3$

$\therefore$ Total number of selections if each student receives atmost one award is,

$$25P_3 = \frac{25!}{(25-3)!}$$
$$= \frac{25!}{22!}$$
$$= 13{,}800.$$

Theorem 3

The number of permutations of 'n' number of objects that are arranged in a circle is $(n-1)!$.

Theorem 4 (Permutation with Repetition)

If there are 'n' objects in which n_1 is of type 1, n_2 is of type 2, . . . n_k of type k, then the number of ways in which they can be arranged is $\frac{n!}{n_1!n_2!\ldots n_k!}$

(Example) Problem

In a college football training session, the defensive coordinator needs to have 10 players standing in a row. Among these 10 players, there are 1 freshman, 2 sophomores, 4 juniors and 3 seniors. How many different ways can they be arranged in a row if only their class level will be distinguished?

Solution

Number of players, $n = 10$

Number of freshers in football training session, $n_1 = 1$

Number of sophomores in football training session, $n_2 = 2$

Number of juniors in football training session, $n_3 = 4$

Number of seniors in football training session, $n_4 = 3$

$\therefore$ Total number of different arrangements that can be made in a row can be given by,

$$\frac{n!}{n_1!n_2!\ldots.n_k!} = \frac{10!}{1!2!4!3!}$$
$$= 12{,}600 \text{ ways.}$$

Theorem 5

The total number of ways of dividing a group of n objects into r cells containing n_1 elements in the first cell, n_2 elements in the second cell and so on is given by,

$$\begin{bmatrix} n \\ n_1, n_2, n_3, \ldots\ldots, n_r \end{bmatrix} = \frac{n!}{n_1!n_2!n_3!,\ldots\ldots n_r!}$$

Where,

$$n_1 + n_2 + n_3 + \ldots\ldots n_r = n$$

(Example) Problem

In how many ways can 7 graduate students be assigned to 1 triple and 2 double hotel rooms during a conference?

Solution

Number of graduate students = 7

Number of triple rooms (3 beds) = 1

Number of double rooms (2 beds) = 2

∴ Total number of ways in which 7 graduate students can be assigned to one triple and two double rooms is,

$$\begin{bmatrix} 3 \\ 3, 2, 2 \end{bmatrix} = \frac{7!}{3!\,2!\,2!}$$

$$= 210 \text{ ways.}$$

Theorem 6 (Combination)

The number of ways of selecting 'r' objects from 'n' different objects irrespective of their arrangement is called as combinations.

It is denoted by ^nC_r.

$$^nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

(or)

$$= \frac{n!}{r!(n-r)!}$$

Combination of 'n' distinct objects by taking any number at a time is given by,

$$^nC_1 + {^nC_2} + ... {^nC_n} = 2^n - 1$$

A combination is a selection of objects irrespective of their arrangement.

The number of combinations of objects in different ways is entirely different from the number of their permutations.

The total number of possible combinations of a set of objects is always taken as 1.

For example, the possible arrangements from the set of objects 'a' and 'b' are 'b' are ab and ba. Irrespective of their order ab is same as ba, there is only one possible combination for this set.

(Example) Problem

A young boy asks his mother to get 5 Game-Boy cartridges from his collection of 10 arcade and 5 sports games. How many ways are there that his mother can get 3 arcade and 2 sports games?

Solution

Number of arcade collection, $n = 10$

Number of sports games collections, $r = 5$

∴ The number of ways of selecting 3 cartridges from 10 is,

$$\begin{bmatrix} 10 \\ 3 \end{bmatrix} = \frac{10!}{3!(10-3)!}$$

$$= 120$$

Similarly, the number of ways in which 2 cartridges can be selected from 5 is,

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \frac{5!}{2!\,3!}$$

$$= 10.$$

1.1.3 Probability of an Event

Q23. Give an overview of probability.

Answer :

The term 'probability' was coined by an Italian mathematician, 'Galileo'. He was the first man who measured probability quantitatively while dealing with the problems associated with rolling of dice in gambling.

The word 'probability' or 'chance' is the most common word used in our day-to-day life.

For example, in our daily life we use certain statements like 'Probably he may win the elections", "It is likely that India may win the match", "She may score above 90% in the coming examination" etc.

A probability is a quantitative measure of uncertain events. It helps in determining the chances of the occurrence of an event.

The systematic and extensive study of probability theory was made by 'B.Pascal', 'Pierre de Fermat', jacques Bernoulli' in mid-seventeeth century.

Probability is a part of our daily lives, as life is uncertain. In personal and managerial decisions, the use of probability theory helps the individual to make educated guess in matters where the outcome is uncertain. Probability helps in making decisions under uncertain conditions and predicts the outcome of an event.

The probability of an event E is defined as the sum of the weights of all sample points in E.

∴ $O \leq P(E) \leq 1$ such that, $P(\phi) + P(E_1) + ...$

When the result of an experiment can result in any of the N distinct equally likely outcomes and if exactly n of these outcomes corresponds to event E, then the probability of event E is,

$$P(E) = \frac{n}{N}$$

PROMBLES

Q24. A die is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the die, find P(E). Let A be the event that an even number turns up and let B be the event that a number divisible by 3 occurs. Find P(A∪B) and P(A∩B).

Solution :

On loading a die we get sample size as,

$$S = \{1, 2, 3, 4, 5, 6\}$$

Given that,

$P(\text{even number}) = 2 \times P(\text{odd number})$

Let $P(1) = x$, $P(3) = x$, $P(5) = x$

Then, $P(2) = 2x$, $P(4) = 2x$, $P(6) = 2x$

Total probability = 1 $\quad [\because P(s) = 1]$

$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$

$\Rightarrow \quad x + 2x + x + 2x + x + 2x = 1$

$\Rightarrow \quad 9x = 1$

$\therefore \quad x = \frac{1}{9}$

(i) Probability that a number less than 4 occurs on a single toss of a die is,

$$P(E) = P(1) + P(2) + P(3)$$
$$= x + 2x + x$$
$$= \frac{1}{9} + 2\left[\frac{1}{9}\right] + \frac{1}{9}$$
$$= \frac{4}{9}$$

From the sample space S, we have event $A = \{2, 4, 6\}$ and event $B = \{3, 6\}$

We get,

Now, $A \cup B = \{2, 3, 4, 6\}$ and $A \cap B = \{6\}$

[$\because$ Given that, event A contains even number and event B contains an even number divisible by 3.]

On substituting the probability of $\frac{1}{9}$ to every odd number and probability of $\frac{2}{9}$ to every even number we get,

$$P(A \cup B) = \frac{2}{9} + \frac{1}{9} + \frac{2}{9} + \frac{2}{9}$$

$$\therefore \quad P(A \cup B) = \frac{7}{9}$$

$$P(A \cap B) = \frac{2}{9} \qquad [\because A \cap B = \{6\} \text{ and 6 is an even number}]$$

Q25. A statistics class for engineers consists of 25 industrial, 10 mechanical, 10 electrical and 8 civil engineering students. If a person is randomly selected by the instructor to answer a question find the probability that the student choose.

(i) An industrial engineering major

(ii) A civil engineering or an electrical engineering major.

Solution :

Let 'I' denotes the number of students majoring in industrial engineering = 25

Let 'M' denotes the number of students majoring in mechanical engineering = 10

Let 'E' denotes the number of students majoring in electrical engineering = 10

Let 'C' denotes the number of students majoring in civil engineering = 8

$\therefore$ Total number of students in the class is given by,

$$N = I + M + E + C$$
$$= 25 + 10 + 10 + 8$$
$$= 53$$

(a) Probability that a Student Choose Industrial Engineering Major

Now, the number of students majoring in industrial engineering out of 53 students = 25(x)

$\therefore$ The probability of students selecting industrial engineering major at random is,

$$P(I) = \frac{n}{N}$$

$$= \frac{25}{53}$$

(b) Probability that a Student Choose Either a Civil Engineering or an Electrical Engineering Major

The number of students selecting civil engineering = 8

The number of students selecting electrical engineering = 10

The total number of students selecting either civil or electrical engineering majors is,

$$= [C \cup E]$$

$$= 8 + 10$$

$$= 18$$

$\therefore$ The probability of students selecting either civil or electrical engineering is,

$$P(C \cup E) = \frac{18}{53}$$

Q26. Four coins are tossed simultaneously what is the probability of getting.

(i) 2 heads

(ii) 3 heads

(iii) Atleast 3 heads.

Model Paper-I, Q2(b)

Solution :

If four coins are tossed simultaneously then the total number of possible outcomes are,

S = {(H H H H), (H H H T), (H H T H), (H H T T), (H T H H), (H T H T), (H T T H), (H T T T), (T H H H), (T H H T), (T H T H), (T H T T), (T T H H), (T T H T), (T T T H), (T T T T)}

$\therefore$ Total no. of outcomes = $n(S) = 16$

(i) Probability of Getting 2 Heads

From the sample space 'S' it can be noted that there are 6 events of getting 2 heads i.e.,

No. of favourable outcome = n(2 heads) = 6

$$\therefore \quad P(2 \text{ heads}) = \frac{n(2 \text{ heads})}{n(S)}$$

$$= \frac{6}{16}$$

$$= \frac{3}{8}$$

(ii) Probability of Getting 3 Heads

No. of favourable outcomes = n(3 heads) = 4

$$\therefore \quad P(3 \text{ heads}) = \frac{n(3 \text{ heads})}{n(S)}$$

$$= \frac{4}{16}$$

$$= \frac{1}{4}$$

(iii) Probability of Getting Atleast 3 Heads

No. of favourable outcomes = n(Atleast 3 heads) = 5

$$P(\text{Atleast 3 heads}) = \frac{n(\text{Atleast 3 heads})}{n(S)}$$

$$= \frac{5}{16}$$

Q27. Find the probability that a leap year selected at random will contain 53 sundays.

Solution :

Let the event of randomly choosing a leap year containing 53 sundays = E

The total no. of days in a leap year = 366 days = 52 weeks + 2 days

The remaining 2 days of the leap year will have the following combinations.

S = {(Monday, Tuesday),
(Tuesday, Wednesday),
(Wednesday, Thursday),
(Thursday, Friday),
(Friday, Saturday),
(Saturday, Sunday),
(Sunday, Monday)}.

From the above combinations it can be noted that out of 7 possible outcomes there are two outcomes that contains sunday.

$\therefore$ The no. of favourable outcomes for the event $E = 2$

$\therefore$ The required probability is,

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{7}$$

Q28. Two dice are thrown. What is the probability of getting,

(i) The sum is 10

(ii) Atleast 10.

Solution :

Given that two dice are thrown

$\therefore$ The total no. of possible out comes are,

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

$\therefore\ n(S) = 36$

(i) Probability of Getting the Sum as 10

Let A be the event of getting sum as 10

$\Rightarrow$ Favourable events A in S = {(4, 6), (5, 5), (6, 4)}

$n(A) = 3$

$$\therefore\ P(A) = \frac{n(A)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

(ii) Probability of Getting Atleast 10

Let B be the event of getting atleast 10

$\Rightarrow$ Favourable events B in S = {(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)}

$n(B) = 6$

$$\therefore\ P(B) = \frac{n(B)}{n(S)} = \frac{6}{36}$$

$$P(B) = \frac{1}{6}$$

Q29. A ware house contains 10 motors, three of which are defective (D). The other seven are in good (G) working condition. What is the probability that,

(i) One motor is defective

(ii) Atleast one motor is defective

(iii) Both motors are defective

(iv) Both motors are in good working condition.

(a) Without replacement

(b) With replacement when the inspection is done by two persons.

Solution :

Given that,

Number of motors = 10

Number of defective motors = 3

Number of non-defective motors = 7

Let, S_1 = {Defective, Good} $\Rightarrow$ {D, G} be the sample space for first experiment.

Let, S_2 = {Defective, Good} $\Rightarrow$ {D, G} be the sample space for second experiment.

Sample space, $S = S_1 \times S_2$ = {D, G,} {D, G}
= {DD, DG, GD, GG}

Let, X denotes the event of getting defective motors.

The probability of obtaining defective motors is,

$$Pr(X) = \frac{3}{10}$$

Let, Y denotes the event of getting motors in good working condition.

The probability of obtaining motors in good working condition is,

$$Pr(Y) = \frac{7}{10}$$

(a)

(i) Let, A_1 denotes the event of obtaining one motor is defective.

Sample space = $\{DG, GD\}$

Figure (a) represents the tree diagram of the motor selecting without replacement.

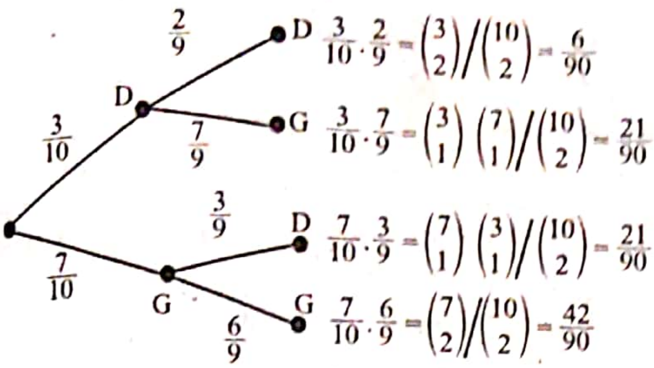


Figure (a)

From figure,

$$\left\{\frac{21}{90},\frac{21}{90}\right\}$$

The probability that one motor is defective is,

$$P(A_1)=\frac{21}{90}+\frac{21}{90}$$
$$=\frac{42}{90}$$
$$=\frac{7}{15}$$
$$\therefore \quad P(A_1)=\frac{7}{15}$$

(ii) Let, B_1 denotes the event of getting atleast one defective motor.

$$S=\{DG, GD, DD\}$$
$$=\left\{\frac{21}{90},\frac{21}{90},\frac{6}{90}\right\} \quad [\because \text{From figure (a)}]$$

The probability that atleast one motor is defective is,

$$P(B_1)=\frac{21}{90}+\frac{21}{90}+\frac{6}{90}$$
$$=\frac{48}{90}$$
$$=\frac{8}{15}$$
$$\therefore \quad P(B_1)=\frac{8}{15}$$

(iii) Let, C_1 be an event of obtaining both defective motors.

$$S=\{DD\}$$
$$=\left\{\frac{6}{90}\right\}$$

The probability that both motors are defective is,

$$P(C_1)=\frac{6}{90}=\frac{1}{15}$$
$$\therefore \quad P(C_1)=\frac{1}{15}$$

(iv) Let, E_1 be an event of obtaining motors in good working condition.

$$S=\{GG\}$$
$$=\frac{42}{90}$$

The probability that both motors are in good working condition is,

$$P(E_1)=\frac{42}{90}$$
$$=\frac{7}{15}$$
$$P(E_1)=\frac{7}{15}$$

(b)

(i) Let, A_2 be an event of obtaining exactly one defective motor,

$$S=\{DG, GD\}$$

Figure (b) represents the tree diagram of motor selection with replacement.

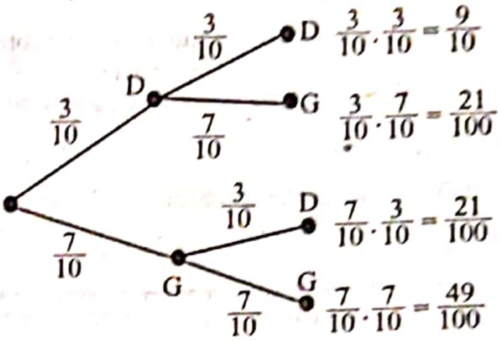


Figure (b)

$$S=\left\{\frac{21}{100},\frac{21}{100}\right\}$$

The probability that atleast one motor is defective is;

$$P(A_2)=\frac{21}{100}+\frac{21}{100}$$
$$=\frac{42}{100}$$
$$=\frac{21}{50}$$
$$\therefore \quad P(A_2)=\frac{21}{50}$$

(ii) Let, B be an event of obtaining atleast one defective motor.

$$S=\{DG, GD, DD\}$$
$$=\left\{\frac{21}{100},\frac{21}{100},\frac{9}{100}\right\} \quad [\because \text{From figure(b)}]$$

The probability that atleast one motor is defective is,

$$P(B_2)=\frac{21}{100}+\frac{21}{100}+\frac{9}{100}$$
$$=\frac{51}{100}$$
$$\therefore \quad P(B_2)=\frac{51}{100}$$

(iii) Let, C_2 be an event of obtaining both motors that are defective,

$$S = \{DD\}$$
$$= \frac{9}{100}$$

The probability that both motors are defective is,

$$P(C_2) = \frac{9}{100}$$
$$\therefore\ P(C_2) = \frac{9}{100}$$

(iv) Let, E_2 be an event of obtaining both motors in good working condition.

$$S = \{GG\}$$
$$= \frac{49}{100}$$

The probability that both motors are in good working condition is,

$$P(E_2) = \frac{49}{100}$$
$$\therefore\ P(E_2) = \frac{49}{100}$$

1.1.4 Additive Rules

Q30. What are the three axioms of probability?

Answer :

$P(E)$ is known as the probability of an event E, if it satisfies the following major axioms of probability.

Axiom 1

The probability of occurrence of an event (E) in a random experiment may be zero or any positive number. The probability should not be negative.

$$P(E) \geq 0$$

Axiom 2

The sample space S, is an event in itself consisting of all possible results. Therefore, it possesses the largest possible probability which is equal to one.

$$P(S) = 1$$

Axiom 3

The probability for the union of any number of mutually exclusive events such as Q and R, is equal to the sum of the individual event probabilities.

$$P(Q \cup R) = P(Q) + P(R)$$

Q31. State and prove addition theorem of probability for any two events.

Answer :

Statement

If A and B are any two events which are subsets of sample space S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

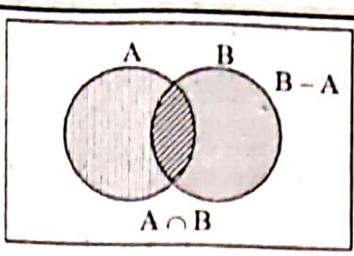
Proof

$A \cup B$ can be written as the union of two mutually exclusive events,

i.e., $A \cup B = A \cup (B - A)$

$$P(A \cup B) = P[A \cup (B - A)]$$

$$P(A \cup B) = P(A) + P(B - A) \quad [\because A \text{ and } (B - A) \text{ are disjoint}] \quad \ldots (1)$$



Again B can be written as the union of two mutually exclusive events,

$\therefore \quad B = (A \cap B) \cup (B - A)$

$P(B) = P[(A \cap B) \cup (B - A)]$

$P(B) = P(A \cap B) + P(B - A)$ $\quad$ [$\because (A \cap B)$ and $(B - A)$ are disjoint]

$P(B) - P(A \cap B) = P(B - A)$ ... (2)

From equations (1) and (2),

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Hence proved.

Note

If A and B are mutually exclusive, then

$A \cap B = \phi$

$\therefore \ P(A \cap B) = P(\phi) = 0$

$\therefore$ Addition theorem reduces to,

$P(A \cup B) = P(A) + P(B)$

Q32. Prove that Pr(A ∪ B ∪ C) = Pr(A) + Pr(B) + Pr(C) – Pr(A ∩ B) – Pr(B ∩ C) – Pr(A ∩ C) + Pr(A ∩ B ∩ C).

Answer :

Let S be the sample for an experiment. For events $A, B, C \subseteq S$

$Pr(A \cup B \cup C) = Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr(A \cap C) - Pr(B \cap C) + Pr(A \cup B \cap C)$.

Consider,

$$Pr(A \cup B \cup C) = Pr((A \cup B) \cup C) = Pr(A \cup B) + Pr(C) - Pr((A \cup B) \cap C)$$

$$= Pr(A) + Pr(B) - Pr(A \cap B) + Pr(C) - Pr((A \cap C) \cup (B \cap C))$$

$$= Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - [Pr(A \cap C) + Pr(B \cap C) - Pr((A \cap C) \cap (B \cap C))]$$

$$= Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr(A \cap C) - Pr(B \cap C) + Pr(A \cap B \cap C).$$

$$\therefore Pr(A \cup B \cup C) = Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr(A \cap C) - Pr(B \cap C) + Pr(A \cap B \cap C).$$

Q33. Prove the following,

(i) If ϕ is the impossible event then, P(ϕ) = 0

(ii) If $\bar{A}$ is the complementary event of A, then P($\bar{A}$) = 1 – P(A)

(iii) If B ⊂ A, then P(B) ≤ P(A).

Model Paper-II, Q2(b)

Answer :

(i) From set theory we know that $S \cup \phi = S$ and $S \cap \phi = \phi$

Consider,

$P(S \cup \phi) = P(S)$

$P(S) + P(\phi) = P(S)$

$1 + P(\phi) = 1 \quad [\because P(S) = 1]$

$\therefore \quad P(\phi) = 0$

(ii) Let A and $\bar{A}$ be two mutually exclusive events,

$$\therefore \quad A \cup \bar{A} = S$$

$$P(A \cup \bar{A}) = P(A) + P(\bar{A}) - P(A \cap \bar{A})$$

Two mutually exclusive events cannot be independent.

$$P(A \cap \bar{A}) = 0$$

$$\therefore \quad P(A \cup \bar{A}) = P(A) + P(\bar{A})$$

$$= P(S)$$

$$= 1$$

So, $P(A) + P(\bar{A}) = 1$

$$P(\bar{A}) = 1 - P(A)$$

$\therefore$ The probability of the complementary event $\bar{A}$ of event A is given by $P(\bar{A}) = 1 - P(A)$.

(iii) Let A and B are two events of a sample space S such that $B \subseteq A$,

$$\therefore \quad B \subseteq A \Rightarrow P(A - B) \geq 0$$

$$P(A) - P(B) \geq 0$$

$$P(A) \geq P(B)$$

$$P(B) \leq P(A).$$

Hence Proved.

PROBLEMS

Q34. An airline in a small city has five departures each day. It is known that any given flight has a probability of 0.3 of departing late. For any given day find the probabilities that,

(a) No flights depart late

(b) All flights depart late and

(c) Three or more depart on time.

Solution :

Given that,

For an airline in a small city,

Number of departures on each day, $n = 5$

Probability of a flight being departed late, $P(l) = 0.3$

The probability that a flight being departed on time is obtained as,

$$P(nl) = 1 - P(l)$$

$$= 1 - 0.3 = 0.7$$

(a) The probability that no flights depart late on a given day is obtained as,

$$P_N = P(nl) . P(nl) . P(nl) . P(nl) . P(nl)$$

$$= [P(nl)]^5$$

$$= [0.7]^5 = 0.168$$

$$\therefore \quad P_N = 0.168$$

(b) The probability that all flights depart late on a given day is obtained as,

$$P_A = P(l)\, P(l)\, P(l)\, P(l)\, P(l)$$

$$= [P(l)]^5$$

$$= (0.3)^5 = 0.00243$$

$$\therefore \quad P_A = 0.00243$$

(c) The probability that only three or more flights depart on time is obtained as,

$$P(n \geq 3) = P(n = 3) + P(n = 4) + P(n = 5)$$

$$P(n \geq 3) = P(nl) . P(nl) . P(nl) . P(l) . P(l) + P(nl) . P(nl) . P(nl) . P(nl) . P(l) + P(nl) . P(nl) . P(nl) . P(nl) . P(nl)$$

$$= [P(nl)]^3 [P(l)]^2 + [P(nl)]^4 \times P(l) + [P(nl)]^5$$

$$= [0.7]^3 [0.3]^2 + [0.7]^4 [0.3] + [0.7]^5$$

$$= 0.03087 + 0.07203 + 0.168 = 0.2709$$

$$\therefore \quad P(n \geq 3) = 0.2709.$$

Q35. Yosi selects a card from a well-shuffled standard deck. What is the probability his card is a club or a card whose face value is between 3 and 7 inclusive?

A : The card drawn is a club

B : The face value of the card drawn is between 3 and 7 inclusive.

Solution :

Let A be an event that the card drawn is a club.

A pack of cards has 52 cards

Sample space = {13 clubs, 13 spade, 13 aces, 13 hearts}

$$n(S) = 52$$

The probability that the card drawn is a club.

$$P(A) = \frac{13}{52}$$

Let, B denotes an events of obtaining a card whose face value is between 3 and 7 inclusive,

$$B = \{3, 4, 5, 6, 7\}$$

[$\because$ clubs, spades, hearts, aces]

$$n(B) = 4\{3, 4, 5, 6, 7\}$$

$$= 4(5)$$

$$= 20$$

The probability that the face value of card drawn is between 3 and 7 inclusive is,

$$P(B) = \frac{n(B)}{n(s)}$$

$$= \frac{20}{52}$$

The probability that the card drawn is a club and whose face value is between 3 and 7 inclusive is,

$$P(A \cap B) = \frac{5}{52}$$

$$\left[\begin{array}{l} \because A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\} \\ B = \{3, 4, 5, 6, 7,\} \\ (A \cap B) = \{3, 4, 5, 6, 7\} \Rightarrow n(A \cap B) = 5 \end{array}\right]$$

The probability that the card drawn is a club or a card whose face value is between 3 and 7 includive is given as,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Substituting the corresponding values in above equation,

$$P(A \cup B) = \frac{13}{52} + \frac{20}{52} - \frac{5}{52}$$

$$= \frac{28}{52}$$

$$= \frac{7}{13}$$

$$\therefore \quad P(A \cup B) = \frac{7}{13}$$

Q36. John is going to graduate from an industrial engineering department in a university by the end of the semester. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from company A is 0.8 and his probability of getting an offer from company is 0.6. If he believes that the probability that he will get offer from atleast one company is 0.5, what is the probability that he will get offer from both the companies?

Solution :

Given that,

The probability of getting an offer from company, A = 0.8

The probability of getting an offer from company, B = 0.6

The probability that john may get offer from atleast one company i.e., $P(A \cup B) = 0.5$

From the additive rule, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad 0.5 = 0.8 + 0.6 - P(A \cap B)$$

$$\Rightarrow \quad P(A \cap B) = 0.8 + 0.6 - 0.5$$

$$= 0.9$$

$\therefore$ The probability that john will get offer from atleast one company is 0.9.

Q37. Let A, B and C denote respectively, the events that a book is favorably reviewed by three critics x, y and z. If $P(A) = \frac{2}{5}$, $P(B) = \frac{5}{7}$ and $P(C) = \frac{3}{5}$, then what is the probability that,

(i) All the three reviews will be favorable and

(ii) Majority reviews will be favorable.

Solution :

In the given problem it can be noted that A, B and C are mutually independent events.

Given that,

$$P(A) = \frac{2}{5} \Rightarrow P(\bar{A}) = 1 - P(A)$$

$$= 1 - \frac{2}{5} = \frac{3}{5}$$

$$P(B) = \frac{5}{7} \Rightarrow P(\bar{B}) = 1 - P(B)$$

$$= 1 - \frac{5}{7} = \frac{2}{7}$$

$$P(C) = \frac{3}{5} \Rightarrow P(\bar{C}) = 1 - P(C)$$

$$= 1 - \frac{3}{5} = \frac{2}{5}$$

(i) The probability that all the three reviews will be favourable is,

$$P(A \cap B \cap C) = P(A).\,P(B).\,P(C)$$

$$= \frac{2}{5} \times \frac{5}{7} \times \frac{3}{5}$$

$$= \frac{6}{35} \text{ (or) } 0.171$$

(ii) The probability that majority reviews will be favourable is same to the probability that atleast two of them will be favourable.

$\therefore$ The required probability is,

$$= P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C) + P(A \cap B \cap C)$$

$$= P(A).P(B).P(\bar{C}) + P(A).P(\bar{B}).P(C) + P(\bar{A}).P(B).P(C) + P(A).P(B).P(C)$$

$$= \left(\frac{2}{5} \times \frac{5}{7} \times \frac{2}{5}\right) + \left(\frac{2}{5} \times \frac{2}{7} \times \frac{3}{5}\right) + \left(\frac{3}{5} \times \frac{5}{7} \times \frac{3}{5}\right) + \left(\frac{2}{5} \times \frac{5}{7} \times \frac{3}{5}\right)$$

$$= \frac{4}{35} + \frac{12}{175} + \frac{9}{35} + \frac{6}{35}$$

$$= \frac{20 + 12 + 45 + 30}{175}$$

$$= \frac{107}{175} \text{ (or) } 0.611$$

1.1.5 Conditional Probability, Independence and the Product Rule

Q38. What is conditional probability?

Answer :

Conditional probability is defined as the probability of occurrence of a particular value of one random variable when the other random variable has already occured. Let A and B be any two events. The probability of the happening of the event B given that A has already happened is denoted by $P(B|A)$ and is defined as,

$$P(B|A) = \frac{P(A \cap B)}{P(A)},\ P(A) > 0$$

(or)

$$P(A \cap B) = P(A)\,P(B|A),\ P(A) > 0$$

Similarly, the probability of the happening of the event A given that B has already happened is denoted by $P(A \mid B)$ and is defined as,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}\ ; P(B) > 0$$

(or)

$$P(A \cap B) = P(B).P(A|B);\ P(B) > 0$$

Note

For $B|A$, A is the sample space and for $A|B$, B is the sample space.

Example

From a pack of cards, a king is drawn

$$\therefore \quad P(K_1) = \frac{4}{52}$$

$$= \frac{1}{13}\ K_1 \rightarrow \text{King is first drawn}$$

Let the card is not replaced

Now, there are 51 cards in the pack and there are 3 kings, the probability of drawing another king K_2 is,

$$P(K_2|K_1) = \frac{3}{51} = \frac{1}{17}$$

We now verify that,

$$P(K_2|K_1) = \frac{P(K_1 \cap K_2)}{P(K_1)}$$

$P(K_1 \cap K_2)$ is the event of drawing 2 kings simultaneously.

$$\therefore \quad P(K_1 \cap K_2) = \frac{{}^4C_2}{{}^{52}C_2} = \frac{6}{1326} = \frac{1}{221}$$

$$P(K_2|K_1) = \frac{1}{221} \times \frac{1}{\frac{1}{13}}$$

$$P(K_2|K_1) = \frac{1}{221} \times \frac{13}{1}$$

$$= \frac{1}{17} \text{ (verified)}$$

Q39. What is an independent event?

Answer :

Independent Event

Two or more events are said to be independent, if the happening or non happening of one event has no influence on the happening or non happening of the other event.

Let A and B be two events such that the happening of any one event does not affect the happening of the other event. In such case,

$$P(B/A) = P(B)$$

(or)

$$P(A/B) = P(A)$$

Then A and B are said to be independent events.

Examples

(i) The events of drawing two kings in succession with replacement are independent events.

(ii) If we draw a ball from a bag and replace it before drawing the second ball, the result of the second draw is independent of the first draw. In case, if the first ball drawn is not replaced then the second draw is dependent on the first draw.

Pair-wise Independent Events

The events $A_1, A_2, A_3, \ldots A_n$ are said to be pairwise independent events if and only if $P(A_i \cap A_j) = P(A_i)\, P(A_j)\ \forall\, i \neq j$, where $i, j = 1, 2, \ldots n$

Mutually Independent Events

The events $A_1, A_2, \ldots A_n$ are known as mutually independent events if the probability of their simultaneous occurrence is equivalent to their corresponding probabilities i.e.,

$$P(A_i \cap A_j) = P(A_i)\, P(A_j)$$

$$P(A_i \cap A_j \cap A_k) = P(A_i)\, P(A_j)\, P(A_k)$$

$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1)\, P(A_2) \ldots P(A_n)$ is true.

Q40. State and prove multiplication theorem of conditional probability.

Answer :

Statement

The probability of the simultaneous occurrences of any two events is equal to unconditional probability of any one event multiplied by the conditional probability of the other, i.e., if A and B are any two events, then,

$$P(A \cap B) = P(A)\, P(B/A),\ P(A) > 0$$

(or)

$$P(A \cap B) = P(B)\, P(A/B),\ P(B) > 0$$

Proof

Suppose the sample space has N occurences of which N_1 belong to the event A and N_2 belong to event B and N_3 occurences belong to compound event $A \cap B$.

Hence, conditional probabilities are,

$$P(A) = \frac{N_1}{N}$$

$$P(B) = \frac{N_2}{N}$$

$$\text{and } P(A \cap B) = \frac{N_3}{N}$$

Now, the conditional probability $P(A/B)$ refers to the sample space of N_2 occurences out of which N_3 occurences pertain to occurrence of A i.e., B has already happened.

$$P(A/B) = \frac{N_3}{N_2}$$

$$\text{And,} \quad P(B/A) = \frac{N_3}{N_1}$$

$$\text{Now} \quad P(A \cap B) = \frac{N_3}{N} = \frac{N_3}{N_1} \cdot \frac{N_1}{N}$$

$$P(A \cap B) = P(B/A).P(A) \quad \ldots (1)$$

$$\text{And,} \quad P(A \cap B) = \frac{N_3}{N} = \frac{N_3}{N_2} \cdot \frac{N_2}{N}$$

$$P(A \cap B) = P(A/B).P(B) \quad \ldots (2)$$

From equations (1) and (2) we have,

$$P(B/A) = \frac{P(A \cap B)}{P(A)},\ P(A) \neq 0$$

$$\text{And,} \quad P(A/B) = \frac{P(A \cap B)}{P(B)},\ P(B) \neq 0$$

$$\therefore \quad P(A \cap B) = P(A)\, P(B/A) \text{ or } P(B)\, P(A/B)$$

Q41. State and prove multiplication theorem of probability for 'n' events.

OR

State and prove multiplication law of probability for 'n' events

II, Q2(b)

Answer :

Statement

If $A_1, A_2, A_3, ...A_n$ are 'n' events, then

$P(A_1 \cap A_2 \cap ..... \cap A_n) = P(A_1) \cdot P(A_2 / A_1) \cdot P(A_3 / A_1 \cap A_2) ..... P(A_n / A_1 \cap A_2 \cap ... \cap A_{n-1})$.

Where, $P(A_a / A_b \cap A_c ... A_m)$ represents the conditional probability of the event A_a given that the events $A_b, A_c ... A_m$ have already happened.

Proof

This can be proved by mathematical induction.

Let $n = 2$, for two events A_1 and A_2

$P(A_1 \cap A_2) = P(A_1).P(A_2 / A_1)$

Hence the result is true for $n = 2$.

Let $n = 3$, for three events A_1, A_2 and A_3.

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P[A_1 \cap (A_2 \cap A_3)] \\ &= P(A_1).P(A_2 \cap A_3 / A_1) \\ &= P(A_1).P(A_2 / A_1) \cdot P(A_3 / A_1 \cap A_2). \end{aligned}$$

$\therefore$ The result is true for $n = 3$.

Let us assume that, the result is true for $n = z$.

$P[(A_1 \cap A_2 \cap A_3 \cap ..... \cap A_z)] = P(A_1) \cdot P(A_2 / A_1)\, P(A_3 / A_1 \cap A_2) ... P(A_z / A_1 \cap A_2 \cap ..... \cap A_{z-1})$

Now, we need to prove that the result is also true for $n = z + 1$

$$\begin{aligned} \therefore \quad P[(A_1 \cap A_2 \cap ... \cap A_z) \cap A_{z+1}] &= P(A_1 \cap A_2 \cap ... \cap A_z)\, P(A_{z+1} / A_1 \cap A_2 \cap ... \cap A_z) \\ &= P(A_1) \cdot P(A_2 / A_1) ... P(A_z / A_1 \cap A_2 \cap .... \cap A_{z-1}) \cdot P(A_{z+1} / A_1 \cap A_2 \cap ... \cap A_z). \end{aligned}$$

Therefore, the result is true for $n = z + 1$ events. Thus, by the principle of mathematical induction it can be inferred that the result is true for all positive integral values of n i.e.,

$$\boxed{P(A_1 \cap A_2 \cap ... \cap A_z) \cap A_{z+1} = P(A_1)P(A_2 / A_1)P(A_3 / A_1 \cap A_2) ... P(A_n / A_1 \cap A_2 \cap ... \cap A_{n-1})}$$

Q42. If A and B are independent events then show that $\bar{A}$ and $\bar{B}$ are also independent.

Answer :

Given that, A and B are independent, therefore we can write,

$$P(A \cap B) = P(A).P(B) \quad ... (1)$$

To prove that $\bar{A}$ and $\bar{B}$ are also independent, we need to prove that,

$$P(\bar{A} \cap \bar{B}) = P(\bar{A}).P(\bar{B}) \quad ... (2)$$

Consider L.H.S of equation (2), we get,

$$\begin{aligned} P(\bar{A} \cap \bar{B}) &= P(\overline{A \cup B}) && [\because \text{From Demorgan's Law}] \\ &= 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - [P(A) + P(B) - P(A).P(B)] && [\because \text{From equation (1)}] \\ &= 1 - P(A) - P(B) + P(A).P(B) \\ &= [1 - P(A)] - P(B)\,[1 - P(A)] \\ &= [1 - P(A)]\,.[1 - P(B)] \\ &= P(\bar{A})\,.P(\bar{B}) \\ &= \text{R.H.S} \end{aligned}$$

Hence proved.

Q43. If A, B and C are pairwise independent and A is independent of $B \cup C$ then A, B and C are mutually independent.

Answer :

Given that, A, B and C are pairwise independent,

$\therefore \quad P(A \cap B) = P(A).P(B)$

$P(B \cap C) = P(B).P(C)$

$P(A \cap C) = P(A).P(C)$

Also, given that A is independent of $B \cup C$.

$\therefore \quad P[A \cap (B \cup C)] = P(A).P(B \cup C)$ ... (1)

Consider L.H.S of equation (1), we get,

$P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$ [$\because$ From Distributive law]

$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$

$= P(A).P(B) + P(A).P(C) - P(A \cap B \cap C)$ ... (2)

Now, consider R.H.S of equation (1), we get,

$P(A).P(B \cup C) = P(A).[P(B) + P(C) - P(B \cap C)]$

$= P(A)[P(B) + P(C) - P(B).P(C)]$

$= P(A).P(B) + P(A).P(C) - P(A).P(B).P(C)$ ... (3)

On substituting equation (2) and (3) in equation (1), we have,

$P(A \cap B \cap C) = P(A).P(B).P(C)$

$\therefore \quad A$, B and C are mutually independent.

Q44. (a) Prove that the probability of the simultaneous occurrence of two independent events is equal to the product of their individual prpbabilities i.e, if A and B are two independent events, then $P(A \cap B) = P(A)\,P(B)$.

(b) If A and B are any two independent events then prove that, $P(A \cup B) = 1 - P(\bar{A})\,P(\bar{B})$.

Answer :

(a) We know from conditional probability,

$P(A \cap B) = P(A)\,P(B/A)$ ... (1)

$\because \quad A$ and B are independent

$P(B/A) = P(B)$ ... (2)

From equations (1) and (2), we have,

$P(A \cap B) = P(A)\,P(B)$

Note

If A and B are independent, then $(\bar{A}$ and $B)$, $(A$ and $\bar{B})$ and $(\bar{A}$ and $\bar{B})$ are also independent.

(b) Since, it is given that the two events A and B are independent.

$\therefore \quad$ We have,

$P(A \cap B) = P(A).\,P(B)$ ... (3)

Consider R.H.S,

$1 - P(\bar{A}).P(\bar{B}) = 1 - [\{1 - P(A)\}.\,\{1 - P(B)\}]$

$= 1 - [1 - P(B) - P(A) + P(A).P(B)]$

$= 1 - 1 + P(A) + P(B) - P(A).P(B)$

$= P(A) + P(B) - P(A \cap B)$ [$\because$ From equation (3)]

$= P(A \cup B)$

$=$ R.H.S

Hence proved.

Note

For 'n' independent events such as $A_1, A_2, \ldots A_n$ the probability of happening atleast one of the events is defined as,

$P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - P(\bar{A_1}).P(\bar{A_2}) \ldots (\bar{A_n})$

PROBLEMS

Q45. From a city population, the probability of selecting:

(i) A male or a smoker is 7/10.

(ii) A male smoker is 2/5 and

(iii) A male, if a smoker is already selected is 2/3.

Find the probability of selecting a smoker, if a male is first selected.

Solution :

Let, M be the event of seleting a male and S be the event of selecting a smoker.

Given that,

(i) $P(M \cup S) = \frac{7}{10}$

(ii) $P(M \cap S) = \frac{2}{5}$

(iii) $P(M/S) = \frac{2}{3}$

∴ The probability of selecting a smoker if a male is first selected is,

$$P(S/M) = \frac{P(S \cap M)}{P(M)} (or) \frac{P(M \cap S)}{P(M)}$$

Consider,

$$P(M/S) = \frac{2}{3}$$

$$\frac{P(M \cap S)}{P(S)} = \frac{2}{3}$$

$$\frac{\frac{2}{5}}{P(S)} = \frac{2}{3}$$

$$P(S) = \frac{2}{5} \times \frac{3}{2}$$

$$P(S) = \frac{3}{5}$$

We know that,

$$P(M \cup S) = P(M) + P(S) - P(M \cap S)$$

$$\frac{7}{10} = P(M) + \frac{3}{5} - \frac{2}{5}$$

$$P(M) = \frac{7}{10} - \frac{3}{5} + \frac{2}{5}$$

$$P(M) = \frac{7-6+4}{10} = \frac{5}{10}$$

$$P(M) = \frac{1}{2}$$

∴ The required probability is,

$$P(S/M) = \frac{P(M \cap S)}{P(M)}$$

$$P(S/M) = \frac{\frac{2}{5}}{\frac{1}{2}}$$

$$\boxed{P(S/M) = \frac{4}{5}}$$

Q46. The probabilities of x, y and z becoming managers are 4/9, 2/9 and 1/3 respectively. The probabilities that the bonus scheme will be introduced if x, y and z becomes managers are 3/10, 1/2 and 4/5 respectively.

(i) What is the probability that bonus scheme will be introduced and

(ii) If the bonus scheme has been introduced, what is the probability that the manager appointed was x?

Solution :

Let the probabilities of x, y and z becoming manager as $P(x)$, $P(y)$ and $P(z)$ respectively.

Let B be the event that the bonus scheme is introduced.

∴ The probabilities that bonus scheme will be introduced when x, y and z becomes manager are $P(B/x)$, $P(B/y)$ and $P(B/z)$ respectively.

Given that,

$$P(x) = \frac{4}{9}, P(y) = \frac{2}{9}, P(z) = \frac{1}{3}$$

$$P(B/x) = \frac{3}{10}, P(B/y) = \frac{1}{2}, P(B/z) = \frac{4}{5}$$

(i) The probability that bonus scheme will be introduced,

$$P(B) = P(x \cap B) + P(y \cap B) + P(z \cap B)$$

$$= P(x) \times P(B/x) + P(y) \times P(B/y) + P(z) \times P(B/z)$$

$$= \frac{4}{9} \times \frac{3}{10} + \frac{2}{9} \times \frac{1}{2} + \frac{1}{3} \times \frac{4}{5}$$

$$= \frac{2}{15} + \frac{1}{9} + \frac{4}{15}$$

$$= \frac{6+5+12}{45}$$

$$= \frac{23}{45}$$

(ii) If bonus scheme is introduced then the probability that the manager appointed was x,

$$P(x/B) = \frac{P(x).P(B/x)}{P(B)}$$

$$= \frac{\frac{4}{9} \times \frac{3}{10}}{\frac{23}{45}}$$

$$= \frac{12}{90} \times \frac{45}{23}$$

$$= \frac{6}{23}$$

Q47. If $P(A \cup B) = \frac{5}{6}$, $P(A \cap B) = \frac{1}{3}$ and $P(\bar{B}) = \frac{1}{2}$. Prove that the events A and B are independent.

Solution :

Given that,

$$P(A \cup B) = \frac{5}{6}; P(A \cap B) = \frac{1}{3}; P(\bar{B}) = \frac{1}{2}$$

Consider,

$$P(\bar{B}) = \frac{1}{2}$$

$$1 - P(B) = \frac{1}{2}$$

$$P(B) = 1 - \frac{1}{2}$$

$$P(B) = \frac{1}{2}$$

From addition theorem of probability we know that,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{5}{6} = P(A) + \frac{1}{2} - \frac{1}{3}$$

$$P(A) = \frac{5}{6} - \frac{1}{2} + \frac{1}{3}$$

$$P(A) = \frac{15-9+6}{18} = \frac{12}{18}$$

$$P(A) = \frac{2}{3}$$

To prove that the events A and B are independent, we have to show that $P(A \cap B) = P(A).\ P(B)$.

Consider R.H.S,

$$P(A).\ P(B) = \frac{2}{3} \times \frac{1}{2}$$

$$= \frac{1}{3}$$

$$= P(A \cap B) \quad [\because \text{Given } P(A \cap B) = \frac{1}{3}]$$

$$= \text{L.H.S}$$

$\because$ $P(A \cap B) = P(A).\ P(B)$

$\therefore$ A and B are independent.

Q48. Define pair-wise independent and mutually independent event. Let A, B and C denote respectively, the events that a book is favourably reviewed by three critics x, y and z. If $P(A) = \frac{2}{7}$, $P(B) = \frac{5}{8}$ and $P(C) = \frac{3}{7}$, then what is the probability that,

(i) All the three reviews will be favourable and

(ii) Majority will reviews will be favourable.

Solution :

Pair-wise Independent Event

For answer refer Unit-I, Q39, Topic: Pair-wise Independent Events.

Mutually Independent Event

For answer refer Unit-I, Q39, Topic: Mutually Independent Events.

Problem

In the given problem it can be noted that A, B and C are mutually independent events.

Given that,

$$P(A) = \frac{2}{7} \Rightarrow P(\bar{A}) = 1 - P(A)$$

$$= 1 - \frac{2}{7}$$

$$= \frac{5}{7}$$

$$P(B) = \frac{5}{8} \Rightarrow P(\bar{B}) = 1 - P(B)$$

$$= 1 - \frac{5}{8}$$

$$= \frac{3}{8}$$

$$P(C) = \frac{3}{7} \Rightarrow P(\bar{C}) = 1 - P(C)$$

$$= 1 - \frac{3}{7}$$

$$= \frac{4}{7}$$

(i) The probability that all the three reviews will be favourable is,

$$P(A \cap B \cap C) = P(A).\ P(B).\ P(C)$$

$$= \frac{2}{7} \times \frac{5}{8} \times \frac{3}{7}$$

$$= \frac{30}{392} \text{ (or) } 0.0765$$

(ii) The probability that majority reviews will be favourable is same to the probability that atleast two of them will be favourable.

$\therefore$ The required probability is,

$$= P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C)$$

$$+ P(\bar{A} \cap B \cap C) + P(A \cap B \cap C)$$

$$= P(A).P(B).P(\bar{C}) + P(A).P(\bar{B}).P(C)$$

$$+ P(\bar{A}).P(B).P(C) + P(A).P(B).P(C)$$

$$= \left(\frac{2}{7} \times \frac{5}{8} \times \frac{4}{7}\right) + \left(\frac{2}{7} \times \frac{3}{8} \times \frac{3}{7}\right) + \left(\frac{5}{7} \times \frac{5}{8} \times \frac{3}{7}\right)$$

$$+ \left(\frac{2}{7} \times \frac{5}{8} \times \frac{3}{7}\right)$$

$$= \frac{40}{392} + \frac{18}{392} + \frac{75}{392} + \frac{30}{392}$$

$$= \frac{40+18+75+30}{392}$$

$$= \frac{163}{392} \text{ (or) } 0.415$$

Q49. The odds against student X qualifying in an examination are 8 : 6 and odds in favour for the student Y for the same are 14 : 16. What are the chances that neither of them will qualify the examination?

Solution :

Let, E_1 and E_2 be the events of student X and Y qualifying in the examination.

Given that,

$P(E_1) = \frac{6}{14}$; $P(E_2) = \frac{14}{30}$

$\therefore P(\bar{E}_1) = 1 - P(E_1)$; $P(\bar{E}_2) = 1 - P(E_2)$

$= 1 - \frac{6}{14}$; $= 1 - \frac{14}{30}$

$= \frac{8}{14}$; $= \frac{16}{30}$

$\therefore$ The chances that neither of them will qualify the examination is,

$$P(\bar{E}_1 \cap \bar{E}_2) = P(\bar{E}_1).P(\bar{E}_2)$$

[$\because$ Both the events are independent]

$$P(\bar{E}_1 \cap \bar{E}_2) = \frac{8}{14} \times \frac{16}{30}$$

$$P(\bar{E}_1 \cap \bar{E}_2) = \frac{32}{105}.$$

Q50. The probability that a regularly scheduled flight departs on time is P(D) = 0.83. The probability that it arrives on time is P(A) = 0.82 and the probability that it departs and arrives on time is P(D ∩ A) = 0.78. Find the probability that a plane.

(a) Arrives on time, given that it departed on time

(b) Departed on time, given that it has arrived on time.

Solution :

Given that,

Probability that a regularly scheduled flight departs on time, $P(D) = 0.83$

Probability that a regularly scheduled flight arrives on time, $P(A) = 0.82$

Probability that flight arrives and departs on time, $P(D \cap A) = 0.78$

From the definition of conditional probability we have,

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$\therefore$ The probability that plane arrives on time when it is departed on time is,

$$P(A/D) = \frac{P(D \cap A)}{P(D)}$$

$$= \frac{0.78}{0.83}$$

$$= 0.94$$

Similarly, the probability that plane departed on time when it has arrived on time is,

$$P(D/A) = \frac{P(D \cap A)}{P(A)}$$

$$= \frac{0.78}{0.82}$$

$$= 0.95$$

Q51. Suppose that we have a fuse box containing 20 fuses of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Solution :

Let A denote the event that first fuse is defective.

Then, $(A \cap B)$ denotes the event that both fuses are defective.

I

D	N
5	15

First Succession

II

D	N
4	15

Second Succession

Probability that first fuse is defective is $P(A) = \frac{5}{20} = \frac{1}{4}$

Probability that second fuse is defective without replacing the first fuse is $P\left(\frac{B}{A}\right) = \frac{4}{19}$

$\therefore$ The probability that both fuses are defective is,

$$P(A \cap B) = P(A)\,P\left(\frac{B}{A}\right)$$

$$\Rightarrow P(A \cap B) = \left(\frac{1}{4}\right)\left(\frac{4}{19}\right)$$

$$= \frac{1}{19}$$

Q52. One bag contains 4 white balls and 3 black balls and a second bag contains 3 white balls and 5 black bag balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Solution :

Let B_1 denote the event that a black ball is drawn from bag1.

Let B_2 denote the event that a black ball is drawn from bag2.

Let W_1 denote the event that a white ball is drawn from bag1.

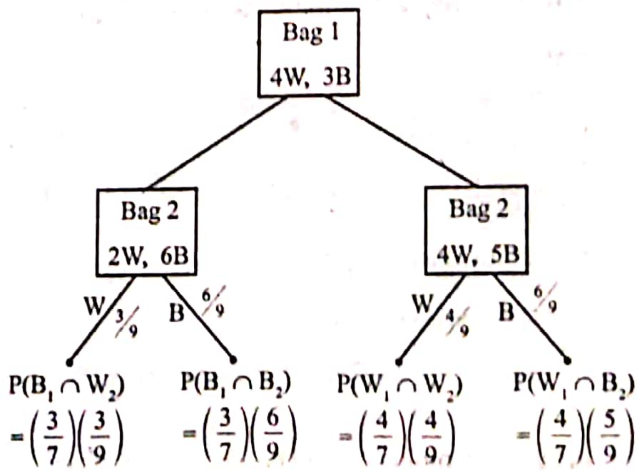


Figure: Tree Diagram

(i) Probability of 1st white ball and 2nd black ball

$$= P(W_1 \cap B_2)$$

$$= P(W_1)\, P\left(\frac{B_2}{W_1}\right)$$

$$= \left(\frac{4}{7}\right)\left(\frac{5}{9}\right)$$

$$= \frac{20}{63}$$

(ii) Probability of 1st black ball and 2nd black ball [since the ball drawn from first bag is black]

$$= P(B_1 \cap B_2)$$

$$= P(B_1)\, P\left(\frac{B_2}{B_1}\right)$$

$$= \left(\frac{3}{7}\right)\left(\frac{6}{9}\right)$$

$$= \frac{18}{63}$$

∴ The probability that a ball drawn from the second bag black is,

$$P[B_1 \cap B_2] + P[W_1 \cap B_2] = \left(\frac{18}{63}\right) + \left(\frac{20}{63}\right)$$

$$= \frac{38}{63}$$

Q53. A small town has one fire engine and one ambulance available from emergencies. The probability that the fire engine is available when needed is 0.98 and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burring building find the probability that both the ambulance and fire engine will be available assuming they operate independently.

Solution :

Let A denote the event that fire engine is available.

Let B deonte the event that amulance is available.

Given that,

Probability that fire engine is available when required is $P(A) = 0.98$

Probability that ambulance is available when called is $P(B) = 0.92$

From the definition of independent events we have,

$$P(A \cap B) = P(A)P(B)$$

∴ Probability that both fire engine and ambulance are available is,

$$P(A \cap B) = (0.98)\,(0.92)$$

$$= 0.9016$$

Q54. Three cards are drawn in succession, without replacement from an ordinary deck of playing cards. Find the probability that the event A1 ∩ A2 ∩ A3 occurs where A1 is the event that the first card is a red ace, A2 is the event that the second card is a 10 or a jack, and A3 is the event that the third card is greater than 3 but less than 7.

Solution :

- Total cards in a standard pack of cards = 52
  - Red Cards = 26
    - Diamonds = 13
    - Hearts = 13
  - Black Cards = 26
    - Spades = 13
    - Clubs = 13

Each group of 13 cards consists of,

(i) 2 to 10 numbered cards

13 = 9 + 4 → letter cards

↓

Number cards

(ii) 4 → An ace, a king, a Queen and a Jack.

(iii) A kind, a queen and a jack → also called as face cards or honour cards.

When three cards are drawn without replacement then,

Number of red ace = 2

Number of ten and jack = 2 * 4 = 8

Number of cards which are greater than 3 but less than 7 = 3 * 4 ⇒ 12

∴ Probability that first card is a red ace $P(A1) = \frac{2}{52}$

The set of remaining cards = 51

∴ Probability that second card is 10 or jack and first card is red ace is,

$$P\left(\frac{A2}{A1}\right) = \frac{8}{51}$$

The set of remaining cards = 50

∴ Probability that third card is greater than 3 but less than 7 when first card is a red ace and second is a 10 or a jack is,

$$P\left(\frac{A3}{A1 \cap A2}\right) = \frac{12}{50}$$

∴ Probability of occurence of event $A1 \cap A2 \cap A3$ is,

$$P(A1 \cap A2 \cap A3) = P(A1) * P\left(\frac{A2}{A1}\right) * \left(\frac{A3}{A1 \cap A2}\right)$$

$$= \left(\frac{2}{52}\right)\left(\frac{8}{51}\right)\left(\frac{12}{50}\right)$$

$$= \frac{8}{5525}$$

1.1.6 Bayes Rule

Q55. State and prove the theorem of total probability.

Answer :

Statement

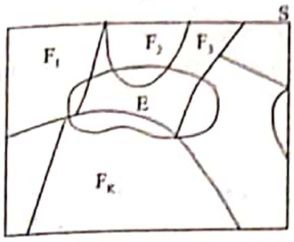
This theorem is called as rule of elimination.

If, $F_1, F_2 \ldots F_K$ are the partitions of sample spaces S with $P(F_i) \neq 0$ for $i = 1, 2 \ldots K$. Then for any event E of S, we have,

$$P(E) = \sum_{i=1}^{K} P(F_i \cap E) = \sum_{i=1}^{K} P(F_i)P(E/F_i)$$

Proof

Consider, the following figure,



Figure

Since, $F_i \ldots F_K$ are the partitions, therefore, we can write,

$$S = \bigcup_{i=1}^{k} F_i \qquad \ldots (1)$$

And $F_i \cap F_j = 0$ for any i and j, which means S and F_i are mutually disjoint sets,

From the figure above,

$$E = E \cap S = E \cap \left(\bigcup_{i=1}^{k} F_i\right) \quad [\because \text{From equation (1)}]$$

$$= E \cap (F_1 \cup F_2 \cup \ldots \cup F_K)$$

$$= (E \cap F_1) \cup (E \cap F_2) \cup \ldots \cup (E \cap F_K)$$

These are all mutually disjoint sets, therefore by using addition rule for these events, we can write,

$$P(E) = P[(E \cap F_1) \cup (E \cap F_2) \cup \ldots \cup (E \cap F_K)]$$

$$= P(F_1 \cap E) + P(F_2 \cap E) + \ldots + P(F_K \cap E) \qquad \ldots (2)$$

By applying multiplication rule to equation (2), we get,

$$P(E) = \sum_{i=1}^{K} P(F_i \cap E) = \sum_{i=1}^{K} P(F_i)P(E/F_i)$$

Q56. State and prove Baye's theorem.

Answer : Model Paper-I, Q3(a)

An event Q occurs only if one of the set of mutually disjoint and exhaustive events $R_1, R_2, R_3, \ldots, R_k$ occurs. If the $P(R_k) \neq 0$ for each k then for any arbitrary event Q which is a subset of $\bigcup_{k=1}^{n} R_k$ with $P(Q) > 0$, we have,

$$P(R_k|Q) = \frac{P(R_k)P(Q|R_k)}{\sum_{i=1}^{k} P(R_k)P(Q|R_k)}$$

The above equation is known as 'Baye's theorem'.

Statement

Let, a sample space S be partitioned into $E_1, E_2, E_3, \ldots E_n$ mutually exclusive events with $P(E_i) > 0$ $(1 \leq i \leq n)$ then for any event A of S, such that $P(A) > 0$, we have,

$$P\left(\frac{E_i}{A}\right) = \frac{P(E_i)P\left(\frac{A}{E_i}\right)}{\sum_{i=1}^{n} P(E_i)P\left(\frac{A}{E_i}\right)}$$

For $i = 1, 2, 3, \ldots n$

Proof

We know that,

$$A \subseteq \left(\bigcup_{i=1}^{n} E_i\right)$$

$$\therefore A \cap \left(\bigcup_{i=1}^{n} E_i\right) = A \begin{bmatrix} A \subseteq B \text{ then} \\ A \cap B = A \end{bmatrix}$$

$$A = \bigcup_{i=1}^{n} (A \cap E_i) \text{ [By distributive law]} \qquad \ldots (1)$$

i.e., $A = (A \cap E_1) \cup (A \cap E_2)(A \cup E_3) \cup \ldots \cup (A_n \cap E_n)$

Now, $E_1, E_2, E_3 \ldots E_n$ are all mutually exclusive events.

$\therefore (A \cap E_1) \cap (A \cap E_2) = \phi$ for $i \neq j$

By applying addition theorem of probability for mutually exclusive events, we have,

$$P(A) = P[(A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \ldots \cup (A \cap E_n)]$$

$$\Rightarrow P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) + \ldots + P(A \cap E_n)$$

$$\Rightarrow P(A) = P\left(\sum_{i=1}^{n} \cup (A \cap E_i)\right).$$

$$P(A) = \sum_{i=1}^{n} P(A \cap E_i) \qquad \ldots (2)$$

From multiplication rule, we have,

$$P(A) = \sum_{i=1}^{n} P(E_i).P\left(\frac{A}{E_i}\right) \quad [P(E_i) \neq 0] \qquad \ldots (A)$$

$$\text{i.e. } P(A \cap E_i) = P(E_i) \cdot P\left(\frac{A}{E_i}\right) \qquad \ldots (3)$$

and also $P(A \cap E_i) = P(A) \cdot P\left(\frac{E_i}{A}\right)$, which gives us,

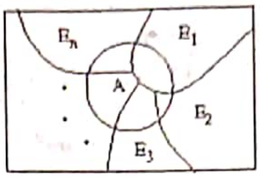
$$P\left(\frac{E_i}{A}\right) = \frac{P(A \cap E_i)}{P(A)} \qquad \ldots (4)$$

Substituting equation (3) in equation (4), we get,

$$P\left(\frac{E_i}{A}\right) = \frac{P(E_i) \cdot P\left(\frac{A}{E_i}\right)}{P(A)} \qquad \ldots (5)$$

Now, substituting the value of $P(A)$ from equation (A) into equation (5), we get,

$$P\left(\frac{E_i}{A}\right) = \frac{P(E_i) \cdot P\left(\frac{A}{E_i}\right)}{\sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{A}{E_i}\right)}$$



Figure

PROBLEMS

Q57. Suppose 5 men out of 100 and 25 women out 1,000 are colour blind. A colour blind person is chosen at random. What is the probability of his being male? (Assume that a person being male or female is equally probable).

Solution :

Let, M and F be the events of selecting a male and a female person respectively. Let 'C' be the event of selecting a colour blind person.

Given that,

$P(M) = P(F) = \frac{1}{2}$ [$\because$ Assuming that a person being male or female is equally probable]

$P(C/M) = \frac{5}{100} = \frac{1}{20}$

$P(C/F) = \frac{25}{1000} = \frac{1}{40}$

$\therefore$ The probability that the colour blind person chosen is male,

$$P(M/C) = \frac{P(M).P(C/M)}{P(M).P(C/M) + P(F).P(C/F)}$$

$$= \frac{\frac{1}{2} \times \frac{1}{20}}{\frac{1}{2} \times \frac{1}{20} + \frac{1}{2} \times \frac{1}{40}} = \frac{\frac{1}{40}}{\frac{1}{40} + \frac{1}{80}}$$

$$= \frac{\frac{1}{40}}{\frac{10+5}{400}}$$

$$= \frac{1}{40} \times \frac{400}{15}$$

$$= \frac{2}{3} \text{ (or) } 0.666$$

Q58. There are ten urns of which three contains 1 white and 9 black balls each; other three contain 9 white and 1 black balls each; and the remaining four contains 5 white and 5 black balls each. One of the urns is selected at random and a ball is taken from it. It turns out to be white. What is the probability that an urn containing 5 white and 5 black balls was selected?

Solution :

Let,

E_1 = Event of urns containing 1 white and 9 black balls

E_2 = Event of urns containing 9 white and 1 black balls

E_3 = Event of urns containing 5 white and 5 black balls

W = Event of drawing a white ball from the selected urn.

Given that,

$P(E_1) = \frac{3}{10}$ [∵ Out of 10 urns 3 contain 1 white and 9 black balls]

$P(E_2) = \frac{3}{10}$ [∵ Out of 10 urns 3 contain 9 white and 1 black balls]

$P(E_3) = \frac{4}{10}$ [∵ Out of 10 urns 4 contain 5 white and 5 black balls]

$P(W/E_1) = \frac{1}{10}$ [∵ Each E_1 contains 1 white ball out of 10 balls]

$P(W/E_2) = \frac{9}{10}$ [∵ Each E_2 contains 9 white balls out of 10 balls]

$P(W/E_3) = \frac{5}{10}$ [∵ Each E_3 contains 5 white balls out of 10 balls]

∴ The probability of drawing a white ball from the urn containing 5 white and 5 black balls is,

$$P(E_3/W) = \frac{P(E_3).P(W/E_3)}{P(E_1).P(W/E_1) + P(E_2).P(W/E_2) + P(E_3).P(W/E_3)}$$

$$= \frac{\frac{4}{10} \times \frac{5}{10}}{\frac{3}{10} \times \frac{1}{10} + \frac{3}{10} \times \frac{9}{10} + \frac{4}{10} \times \frac{5}{10}}$$

$$= \frac{\frac{20}{100}}{\frac{3}{100} + \frac{27}{100} + \frac{20}{100}}$$

$$= \frac{\frac{20}{100}}{\frac{50}{100}}$$

$$= \frac{20}{50}$$

$$= \frac{2}{5} \text{ or } 0.4$$

Q59. The chances of x, y, z becoming managers of a certain company are 4 : 2 : 3. The probabilities that bonus schemes will be introduced if x, y, z become managers are 0.3, 0.5 and 0.8 respectively. If the bonus scheme has been introduced, what is the probability that x introduces it.

Solution :

Let E_1, E_2 and E_3 be the events of becoming manager for x, y and z respectively.

And also, let 'B' be the event of introducing a bonus scheme.

Given that,

The probabilityes (chances) of x, y and z becoming the managers of a company are 4 : 2 : 3 respectively.

$\Rightarrow$ $P(E_1) = \frac{4}{9}$

$P(E_2) = \frac{2}{9}$

$P(E_3) = \frac{3}{9}$

Also, given that,

The probabilities that the bonus scheme will be introduced if x, y and z becomes manager are 0.3, 0.5 and 0.8 respectively.

$\Rightarrow$ $P(B/E_1) = 0.3 = \frac{3}{10}$

$P(B/E_2) = 0.5 = \frac{5}{10}$

$P(B/E_3) = 0.8 = \frac{8}{10}$

If the bonus scheme has been introduced, then the probability that x introduces it is,

$$P(E_1/B) = \frac{P(E_1).P(B/E_1)}{P(E_1)P(B/E_1) + P(E_2)P(B/E_2) + P(E_3)P(B/E_3)}$$

$$P(E/B) = \frac{\frac{4}{9} \times \frac{3}{10}}{\frac{4}{9} \times \frac{3}{10} + \frac{2}{9} \times \frac{5}{10} + \frac{3}{9} \times \frac{8}{10}}$$

$$= \frac{\frac{12}{90}}{\frac{12}{90} + \frac{10}{90} + \frac{24}{90}}$$

$$= \frac{\frac{12}{90}}{\frac{46}{90}}$$

$$= \frac{12}{46}$$

$$= \frac{6}{23} \text{ (or) } 0.261$$

Q60. There are two indentical bags containing 6 red and 4 white balls and 3 red and 5 white balls respectively. A fair die is tossed for the selection of bag. If the die shows 1 or 2, the first bag is selected otherwise second is selected. From the selected bag, a ball is drawn and is found to be red. What is the probability that the second bag was selected?

Solution : Model Paper-III, Q3(b)

Let E_1 and E_2 be the events of selecting 1st and 2nd bags.

Let 'E' be the event of drawing a red ball.

Given that,

No. of balls in 1st bag = 6 red + 4 white = 10

No. of balls in 2nd bag = 3 red + 5 white = 8

1st bag is selected if the die shows 1 or 2

∴ No. of favourable outcome for selecting 1st bag = 2

2nd bag is selected if the die show other than 1 or 2 (i.e., 3 or 4 or 5 or 6)

∴ No. of favourable outcome for selecting 2nd bag = 4

And, total no. of outcomes when a die is tossed = 6

$$\Rightarrow P(E_1) = \frac{2}{6} = \frac{1}{3}$$

$$P(E_2) = \frac{4}{6} = \frac{2}{3}$$

$$P(E/E_1) = \frac{6}{10} = \frac{3}{5} \quad [\because 1^{st} \text{ bag contain 6 red balls out of 10}]$$

$$P(E/E_2) = \frac{3}{8} \quad [\because 2^{nd} \text{ bag contains 3 red balls out of 8}]$$

∴ The probability that the 2nd bag was selected is,

$$P(E_2/E) = \frac{P(E_2).P(E/E_2)}{P(E_1).P(E/E_1) + P(E_2)P(E/E_2)}$$

$$= \frac{\frac{2}{3} \times \frac{3}{8}}{\frac{1}{3} \times \frac{3}{5} + \frac{2}{3} \times \frac{3}{8}}$$

$$= \frac{\frac{1}{4}}{\frac{1}{5} + \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{4+5}{20}}$$

$$= \frac{1}{4} \times \frac{20}{9}$$

$$= \frac{5}{9}$$

Q61. In a factory producing bolts, machines A, B and C manufacture respectively 25%, 30% , and 40% of total output of the total of their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective, what is probability that is was manufactured by machine A?

Solution :

Let, E_1, E_2, and E_3 be the event that a bolt can be selected at random is manufactured by the machines A, B and C respectively.

Also, let 'E' be the event of drawing a defective bolt.

Given that,

$$P(E_1) = 25\% = \frac{25}{100} = 0.25$$

$$P(E_2) = 30\% = \frac{30}{100} = 0.3$$

$$P(E_3) = 40\% = \frac{40}{100} = 0.4$$

The probability of drawing a defective bolt manufactured by machine A is,

$$P(E/E_1) = 5\% = \frac{5}{100} = 0.05$$

Similarly,

$$P(E/E_2) = 4\% = \frac{4}{100} = 0.04$$

$$P(E/E_3) = 2\% = \frac{2}{100} = 0.02$$

$\therefore$ The probablity that a defective bolt which is selected at random is manufactured by machine A is,

$$P(E_1/E) = \frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)+P(E_3)P(E/E_3)}$$

$$P(E_1/E) = \frac{0.25\times0.05}{0.25\times0.05+0.3\times0.04+0.4\times0.02}$$

$$= \frac{0.0125}{0.0125+0.012+0.008}$$

$$= \frac{0.0125}{0.0325}$$

$$= 0.38$$

Q62. There are three identical boxes containing respectively 1 white and 3 red balls, 2 white and 1 red balls, 4 white and 3 red balls. One box is chosen at random and 2 balls are drawn. They happen to be white and red. Find the probability that the balls were chosen from the second box.

Solution :

Let the event of drawing two balls (one white and one red) as 'E'

Let the events of choosing 1st, 2nd and 3rd boxes as E_1, E_2 and E_3 respectively.

Given that,

Total No. of boxes = 3

Total No. of balls in 1st box = 1white + 3 red = 4

Total No. of balls in 2nd box = 2 white + 1 red = 3

Total No. of balls in 3rd box = 4 white + 3 red = 7

$\Rightarrow$ The probability of choosing 1st *box* = $P(E_1) = \frac{1}{3}$

Similarly, $P(E_2) = P(E_3) = \frac{1}{3}$

The probability of drawing balls from 1st box = $P(E/E_1) = \frac{1}{4}\times\frac{3}{3}+\frac{3}{4}\times\frac{1}{3} = \frac{1}{2}$

$\because$ Probability of drawing first ball as white and second ball as red from 1st box containing total 4 balls is $\left(\frac{1}{4}\times\frac{3}{3}\right)$ and that of first ball as red and second ball as white from the same box is $\left(\frac{3}{4}\times\frac{1}{3}\right)$.

Similarly, $P(E/E_2) = \frac{2}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{2}{2}$

$= \frac{2}{3}$

$P(E/E_3) = \frac{4}{7}\times\frac{3}{6}+\frac{3}{7}\times\frac{4}{6}$

$= \frac{2}{7}+\frac{2}{7} = \frac{4}{7}$

$\therefore$ The probability that the balls are from the 2nd box is,

$$P(E_2/E) = \frac{P(E_2).P(E/E_2)}{P(E_1).P(E/E_1)+P(E_2).P(E/E_2)+P(E_3).P(E/E_3)}$$ [$\because$ From Baye's theorem]

$$= \frac{\frac{1}{3}\times\frac{2}{3}}{\frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{2}{3}+\frac{1}{3}\times\frac{4}{7}}$$

$$= \frac{\frac{2}{9}}{\frac{1}{6}+\frac{2}{9}+\frac{4}{21}}$$

$$= \frac{\frac{2}{9}}{\frac{21+28+24}{126}}$$

$$= \frac{2}{9}\times\frac{126}{73}$$

$$= \frac{252}{657}$$

$= 0.383$

Q63. In a certain assembly plant, three machines B1, B2 and B3 make 30%, 45% and 25% respectively of the products. It is known from past experience that 2%, 3% and 2% of the products made by each machine, respectively are defective. Now suppose that a finished product is randomly selected. What is the probability that it is defective?

Solution :

Let A denotes the event that a product is defective.

Let $B1$ denotes the event that a product is made by machine $B1$.

Let $B2$ denotes the event that a product is made by machine $B3$.

Given that,

$$P(B1) = 30\% = \frac{30}{100} = 0.3$$

$$P(B2) = 45\% = \frac{45}{100} = 0.45$$

$$P(B3) = 25\% = \frac{25}{100} = 0.25$$

$$P\left(\frac{A}{B1}\right) = 2\% = \frac{2}{100} = 0.02$$

$$P\left(\frac{A}{B2}\right) = 3\% = \frac{3}{100} = 0.03$$

$$P\left(\frac{A}{B3}\right) = 2\% = \frac{2}{100} = 0.02$$

From rule of elimination we have,

$$P(A) = P(B1)\,P\left(\frac{A}{B1}\right) + P(B2)\,P\left(\frac{A}{B2}\right) + P(B3)\,P\left(\frac{A}{B3}\right)$$

On substituting the probabilities of three machines in the above formula we get,

$P(A) = (0.3)(0.2) + (0.45)(0.03) + (0.25)(0.02)$

$= 0.006 + 0.0135 + 0.005$

$= 0.0245$

$\therefore$ The probability that product is defective is 0.0245.

Q64. In a certain assembly plant, three machines B1, B2 and B3 make 30%, 45% and 25% respectively of the products. It is known from past experience that 2%, 3% and 2% of the products made by each machine respectively are defective. Now, suppose that a finished product is randomly selected and found to be defective what is the probability that it was made by machine B3?

Solution :

Let A denote the event that product is defective.

Let $B1$ denote the event that product is made by machine $B1$.

Let $B2$ denote the event that product is made by machine $B3$.

Given that,

$$P(B1) = 30\% = \frac{30}{100} = 0.3$$

$$P(B2) = 45\% = \frac{45}{100} = 0.45$$

$$P(B3) = 25\% = \frac{25}{100} = 0.25$$

$$P\left(\frac{A}{B1}\right) = 2\% = \frac{2}{100} = 0.02$$

$$P\left(\frac{A}{B2}\right) = 3\% = \frac{3}{100} = 0.03$$

$$P\left(\frac{A}{B3}\right) = 2\% = \frac{2}{100} = 0.02$$

From Bayes rule we get,

$$P\left(\frac{B3}{A}\right) = \frac{P(B3)P\left(\frac{A}{B3}\right)}{P(B1)P\left(\frac{A}{B1}\right) + P(B2)P\left(\frac{A}{B2}\right) + P(B3)P\left(\frac{A}{B3}\right)}$$

On substituting the probabilities of machiens we get,

$$P\left(\frac{B3}{A}\right) = \frac{0.005}{0.006 + 0.0135 + 0.005}$$

$$= \frac{0.005}{0.0245}$$

$$= \frac{10}{49}$$

1.2 RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

1.2.1 Concept of a Random Variable, Discrete Probability Distributions, Continuous Probability Distributions

Q65. What is random variable? Explain with example.

Answer :

Random Variable

A random variable X on a sample space S is a function from S to the set of real numbers T, which assigns a real number $X(s)$ to each sample point S of S.

The function is given as,

$$X : S \rightarrow T$$

Notation

If x is a real number the set of all sample points in sample space S such that $X(s) = x$ is denoted briefly by writing $X = x$

Thus,

$$P(X = x) = P\{s : X(s) = x\}$$

$$P(X \leq a) = P\{s : X(s \in [-\infty, a]\}$$

$$P(a < X \leq b) = P(s : X(s \in (a, b])$$

Example

1. If a coin is tossed, then,

$S = \{s_1, s_2\}$ where $s_1 = H, s_2 = T$

$$X = \begin{cases} 1, \text{if } S = H \\ 0, \text{if } S = T \end{cases}$$

2. If a random experiment consists of rolling a die and reading the number of points on the upturned face.

The random variable X to consider is,

$X(s) = S;$

$S = 1, 2, \ldots\ldots, 6.$

If we are interested in whether the number of points is even or odd, we consider a random variable Y defined as follows,

$$Y(s) = \begin{cases} 0, \text{if S is even} \\ 1, \text{if S is odd} \end{cases}.$$

Q66. Write in brief about discrete probability distribution.

Answer : Model Paper-II, Q3(a)

Discrete probability distribution is the probability distribution of a discrete random variable 'X' which takes only a finite number of variables. Random variable 'X' as function $f(x)$ satisfies the following conditions,

(i) $f(x) \geq 0$

(ii) $\Sigma f(x) = 1$

(iii) $P(X = x) = f(x)$

The examples of discrete probability distribution are binomial and Poisson distribution.

Probability Function or Probability Mass Function

Let X be discrete random variable. The discrete probability function $f(x)$ for X is given by $f(x) = p(X = x)$ for all real x.

Since, probabilities cannot be negative, a probability function $f(x)$ cannot assume negative values.

Note that the probability associated with a sample space is 1. Thus, if we add the values of $f(x)$ over all possible values of X, the total should be 1.

Infact the following two properties completely characterize the probability function of a discrete random variable.

Properties

1. $f(x) \geq 0$ for all real numbers x

2. $\sum_{all\ x} f(x) = 1$

We can state the discrete probability distribution, probability function or probability mass function of a discrete random variable X as the function $f(x)$ satisfying the following condition,

(i) $f(x) \geq 0$

(ii) $\sum_{all\ x} f(x) = 1$

(iii) $P(X = x) = f(x)$.

Q67. Explain cumulative distribution function or distribution function.

Answer :

Cumulative Distribution Function (or) Distribution Function

There are many problems in which we are interested not only in the probability $f(x)$ for the value of the random variable 'x' but also in the probability $F(x)$ that the value of a random variable is less than or equal to x. We refer to the function $F(x)$ as the cumulative distribution functions.

Definition

Let X be a random variable then the function,

$F(x) = P(X \leq x)\ ; -\infty < x < \infty$ is called the distribution function of X.

If X is discrete,

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

If X is continuous,

$$F(x) = P(X \leq x) = P(-\infty < X \leq x)$$
$$= f(x)dx$$

$$F(x) = \begin{cases} 0 & -\infty < x < x_1 \\ f(x_1) & x_1 \leq x < x_2 \\ f(x_1) + f(x_2) & x_2 \leq x < x_3 \\ \vdots & \vdots \\ f(x_1) + \ldots\ldots + f(x_n) & x_n \leq x < \infty \end{cases}$$

Properties of Cumulative Distribution Function

1. $P(a < X \leq b) = F(b) - F(a)$

2. $0 \leq F(x) \leq 1,\ \forall\ x \in R$

3. F is non-decreasing i.e., if $x \leq y$, then $F(x) \leq F(y)$

4. $F(-\infty) = 0, F(+\infty) = 1$

5. F is continuous from the right at each point,

i.e., $F(a+) = F(a)$

$F(a+) - F(a-) = P(X = a)$

6. If X is a continuous random variable then $\frac{d}{dx}F(x) = f(x)$ at all points where $F(x)$ is differentiable.

Note

Properties (3), (4) and (5) are necessary and sufficient for F to be cumulative distribution function on R.

Q68. Write about continuous probability distribution.

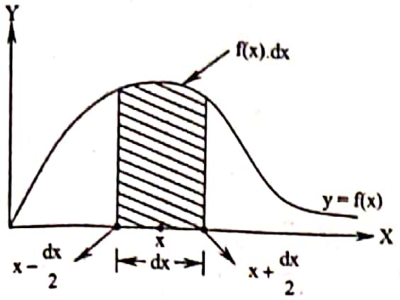
Answer :

Continuous probability distribution is a probability distribution of a continuous random variable. A continuous random variable is a variable which assumes infinite and uncountable set of values. Example of continuous probability distribution is 'Normal Distribution'.

Probability Density Function (p.d.f)

Consider the small interval $(x, x+dx)$ of length dx around the point x. Let $f(x)$ be any continuous function of x so that $f(x)dx$ represents the probability that X falls in the interval $(x, x + dx)$. Symbolically,

$$P(x \leq X \leq x + dx) = f(x)dx$$



In the figure, $f(x)$dx represents the area bounded by the curve $y = f(x)$, x-axis and the ordinates at the points x and $x + dx$.

The function $f(x)$ so defined is known as probability density function or simply density function of random variable X.

Note

1. The expression, $f(x)dx$ is known as the probability differential.

2. The curve $y = f(x)$ is known as the probability density curve or simply probability curve.

Definition

Probability density function of the random variable X is defined as,

$$f_x(x) = \underset{\delta x \to 0}{Lt} \frac{P(x \leq X \leq x + \delta x)}{\delta x}$$

(or)

$$f_x(x) = \frac{d}{dx} F_x(x)$$

i.e., derivative of cumulative distribution function gives the probability density function.

The plot of p.d.f of a discrete random variable is an impulse train.

The probability for a variate value to lie in the interval dx is $f(x)dx$ and hence, the probability for a variate value to fall in the finite interval $[a, b]$ is,

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

Which represents the area between the curve $y = f(x)$, x-axis and the ordinates at $x = a$ and $x = b$.

Since, the total probability is unity $\int_a^b f(x)dx = 1$.

Properties of Probability Density Function

(i) $f(x) \geq 0, -\infty < x < \infty$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(iii) The probability $P(E)$ given by,

$$P(E) = \int_E f(x)dx$$

(iv) $P(a < X < b) = \int_a^b f(x)dx$ = Area under $f(x)$ between ordinates $x = a$ and $x = b$.

Note

1. In case of discrete random variable, the probability at a point, i.e., $P(X = C)$ is not zero for some fixed C. However, in case of continuous random variable, the probability at a point is always zero.

   i.e., $P(X = C) = 0$, for all possible values of C.

2. $$P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b)$$

   i.e., inclusion or non-inclusion of endpoints, does not change the probability.

   Relation between probability density function and cumulative density function,

   (a) $F(x) = \int_{-\infty}^{x} f(x)dx \quad \forall x \in R$

   (b) $\frac{d}{dx}F(x) = f(x) \quad \forall x \in R$

PROBLEMS

Q69. Let X denote the minimum of the two numbers that appear when a pair of fair dice is thrown once. Determine the,

(i) Discrete probability distribution

(ii) Expectation

(iii) Variance.

Answer :

$P(X = 1) = P$ [(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (4, 1), (1, 4), (5, 1), (1, 5), (6, 1), (1, 6)]

$$= \frac{11}{36}$$

$P(X = 2) = P$ [(2, 2), (2, 3), (3, 2), (2, 4), (4, 2), (2, 5), (5, 2), (2, 6), (6, 2)]

$$= \frac{9}{36}$$

$P(X = 3) = P$[(3, 3), (3, 4), (4, 3), (3, 5), (5, 3), (3, 6), (6, 3)]

$$= \frac{7}{36}$$

$$P(X=4) = P[(4, 4), (4, 5), (5, 4), (4, 6), (6, 4)] = \frac{5}{36}$$

$$P(X=5) = P[(5, 5), (5, 6), (6, 5)] = \frac{3}{36}$$

$$P(X=6) = P[(6, 6)] = \frac{1}{36}$$

(i) Discrete Probability Distribution

X	1	2	3	4	5	6
P(X)	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

(ii) Expectation

Mean, $\mu = \Sigma X.P(X)$

$$= \frac{1\times11}{36} + \frac{2\times9}{36} + \frac{3\times7}{36} + \frac{4\times5}{36} + \frac{5\times3}{36} + \frac{6\times1}{36}$$

$$= \frac{11}{36} + \frac{18}{36} + \frac{21}{36} + \frac{20}{36} + \frac{15}{36} + \frac{6}{36}$$

$$= \frac{91}{36}$$

(iii) Variance

$$\text{Var}(X) = E(X^2) - \mu^2$$

$$E(X^2) = \Sigma X^2.P(X)$$

$$= \frac{1^2\times11}{36} + \frac{2^2\times9}{36} + \frac{3^2\times7}{36} + \frac{4^2\times5}{36} + \frac{5^2\times3}{36} + \frac{6^2\times1}{36}$$

$$= \frac{11}{36} + \frac{36}{36} + \frac{63}{36} + \frac{80}{36} + \frac{75}{36} + \frac{36}{36}$$

$$= \frac{301}{36}$$

Variance,

$$X = \frac{301}{36} - \left(\frac{91}{36}\right)^2$$

$$= 1.97$$

Q70. For the discrete probability distribution.

x	0	1	2	3	4	5	6	7
F	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

Determine,

(i) k

(ii) Mean

(iii) Variance

(iv) Smallest value of x such that $P(X \leq x) > \frac{1}{2}$.

Solution :

(i) k

We have $\Sigma P(x) = 1$ i.e., $\sum_{i=0}^{7} F_i$

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$10k^2 + 9k - 1 = 0$$

$$10k^2 + 10k - k - 1 = 0$$

$$10k(k+1) - 1(k+1) = 0$$

$$k = -1, \frac{1}{10}$$

k cannot be negative,

$$k = \frac{1}{10} \quad (\because P(X) \geq 0)$$

(ii) Mean

Mean, $\mu = \sum X\,P(X)$

$$= 0(0) + 1(k) + 2(2k) + 3(2k) + 4(3k) + 5(k^2) + 6(2k^2) + 7(7k^2+k)$$

$$= k + 4k + 6k + 12k + 5k^2 + 12k^2 + 49k^2 + 7k$$

$$= 66k^2 + 30k$$

$$= 66 \times \left(\frac{1}{10}\right)^2 + 30 \times \left(\frac{1}{10}\right) = \frac{66}{100} + 3 = 3.66$$

(iii) Variance

Variance $= E(X)^2 - [E(X)]^2$ [since $E(X) = xp(x)$]

$$= \sum X^2 P(X) - (3.66)^2$$

$$= 0^2(0) + 1^2(k) + 2^2(2k) + 3^2(2k) + 4^2(3k) + 5^2(k^2) + 6^2(2k^2) + 7^2(7k^2 + k)$$

$$= k + 8k + 18k + 48k + 25k^2 + 72k^2 + 343k^2 + 49k$$

$$= 440k^2 + 124k$$

$$= 440\left(\frac{1}{10}\right)^2 + 124\left(\frac{1}{10}\right)$$

$$= 4.4 + 12.4 = 16.8.$$

(iv) Smallest Value of X such that $P(X \leq x) > \frac{1}{2}$

The smallest value of x is determined by trial method.

$$P(X \leq 0) = 0$$

$$P(X \leq 1) = P(X=0) + P(X=1) = 0 + \frac{1}{10} = \frac{1}{10}$$

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= 0 + \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$$

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = \frac{5}{10}$$

$$P(X \leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} = \frac{8}{10} > \frac{1}{2}$$

Therefore, the smallest value of x for which,

$$P(X \leq x) > \frac{1}{2} \text{ is } x = 4.$$

Q71. A discrete random variable X has the mean 6 and variance 2. If it is assumed that the distribution is binomial, find the probability that $5 \le x \le 7$.

Solution :

Given that,

Mean, $np = 6$ ... (1)

Variance, $npq = 2$ ... (2)

Dividing equation (2) with equation (1),

$$\frac{npq}{np} = \frac{2}{6}$$

$$\Rightarrow q = \frac{1}{3}$$

$$\Rightarrow p = 1 - q$$

$$\Rightarrow = 1 - \frac{1}{3}$$

$$\Rightarrow p = \frac{2}{3}$$

Substituting the value of 'p' in equation (1),

$$np = 6$$

$$\Rightarrow n(2/3) = 6$$

$$\Rightarrow n = 6 \times \frac{3}{2}$$

$$\Rightarrow n = 9$$

$$\therefore n = 9, p = \frac{2}{3}, q = \frac{1}{3}$$

$$P(5 \le x \le 7) = P(x = 5) + P(x = 6) + P(x = 7 \quad ... (3)$$

$$B(x, n, p) = {}^nC_x p^x q^{n-x}$$

$$P(x = 5) = {}^9C_5 (2/3)^5 (1/3)^{9-5}$$

$$= {}^9C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^4$$

$$= {}^9C_5 \left(\frac{1}{3}\right)^9 .2^5 \quad ... (4)$$

$$P(x = 6) = {}^9C_6 (2/3)^6 (1/3)^{9-6}$$

$$= {}^9C_6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^3$$

$$= {}^9C_6 \left(\frac{1}{3}\right)^9 .2^6 \quad ... (5)$$

$$P(x = 7) = {}^9C_7 (2/3)^7 (1/3)^{9-7}$$

$$= {}^9C_7 \left(\frac{2}{3}\right)^7 \left(\frac{1}{3}\right)^2$$

$$= {}^9C_7 \left(\frac{1}{3}\right)^9 .2^7 \quad ... (6)$$

Substituting equations (4), (5) and (6) in equation (3),

$$P(5 \le x \le 7) = {}^9C_5 \left(\frac{1}{3}\right)^9 .2^5 + {}^9C_6 \left(\frac{1}{3}\right)^9 .2^6 + {}^9C_7 \left(\frac{1}{3}\right)^9 .2^7$$

$$= \left(\frac{1}{3}\right)^9 .2^5 [{}^9C_5 + {}^9C_6 .2^1 + {}^9C_7 .2^2]$$

$$= \frac{2^5}{3^9} [{}^9C_5 + 2.{}^9C_6 + 4.{}^9C_7]$$

$$= \frac{2^5}{3^9} [126 + 168 + 144] = \frac{2^5}{3^9} [438] = \frac{14016}{19683}$$

$$\therefore P(5 \le x \le 7) = 0.712.$$

Q72. A player tosses 3 fair coins. He wins ₹ 500 if 3 heads appear, ₹ 300 if 2 heads appear, ₹ 100 if 1 head occurs. On the other hand, he loses ₹ 1500 if 3 tails occur. Find the expected gain of the player.

Solution :

Let X be the expected gain of the player.

$\therefore$ The range of X is $\{-1500, 100, 300, 500\}$

Let s be the sample space of tossing.

3 fair coins appearing either with 3 heads, 2 heads, 1 head or no head.

$$\therefore n(s) = 2^3$$

$$= 8$$

$$= \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Let, E_1 be the event of getting all 3 heads.

$$n(E_1) = {}^3C_3$$

$$\therefore n(E_1) = 1$$

$\therefore$ The probability of winning ₹ 500 if 3 heads appear,

$$P(X = 3) = \frac{n(E_1)}{n(s)}$$

$$P(X = 3) = \frac{1}{8}$$

Let, E_2 be the event of getting 2 heads.

$$n(E_2) = {}^3C_2$$

$$n(E_2) = 3$$

$\therefore$ The probability of winning ₹ 300 if 2 heads appear

$$= P(X = 2)$$

$$= \frac{n(E_2)}{n(s)}$$

$$P(X = 2) = \frac{3}{8}$$

Let, E_3 be the event of getting 1 head.

$$n(E_3) = {}^3C_1$$

$$n(E_3) = 3$$

$\therefore$ The probability of winning ₹ 100 if 1 head appears

$$= P(X=1)$$

$$= \frac{n(E_3)}{n(s)}$$

$$P(X=1) = \frac{3}{8}$$

Let, E_4 be the event of getting 3 tails (i.e., no head)

$$n(E_4) = {}^3C_0$$

$$n(E_4) = 1$$

$\therefore$ The probability of losing ₹ 1500 if no head appears,

$$= P(X=0)$$

$$= \frac{n(E_4)}{n(s)}$$

$$P(X=0) = \frac{1}{8}$$

The probability distribution is,

X	– 1500	100	300	500
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$\therefore$ The expected gain of the player.

$$E(X) = \Sigma P(X)$$

$$= (-1500)\left(\frac{1}{8}\right) + 100\left(\frac{3}{8}\right) + 300\left(\frac{3}{8}\right) + 500\left(\frac{1}{8}\right)$$

$$= -\frac{1500}{8} + \frac{300}{8} + \frac{900}{8} + \frac{500}{8}$$

$$= \frac{-1500+300+900+500}{8} = \frac{200}{8}$$

$$= 25$$

$\therefore$ The expected gain = ₹ 25.

Q73. A bag contains 4 white and 6 black balls. A man draws 3 items from the box. Find the expected number of defective items he has drawn.

Solution :

Assume that there are 3 defective items in a bag containing 4 white and 6 black balls.

$\therefore$ Total number of items = 4 + 6 = 10

Number of defective items = 3

Number of good items = (10 – 3) = 7

Number of items drawn randomly = 3

$\therefore$ The random variable X can take the value 0, 1, 2.

Probability that, there are 0 defective items.

$$P(X=0) = \frac{{}^7C_3}{{}^{10}C_3}$$

$$P(X=0) = \frac{35}{120}$$

Probability that, there are 1 defective and 2 good items.

$$P(X=1) = \frac{{}^3C_1 \times {}^7C_2}{{}^{10}C_3} = \frac{3\times 21}{120}$$

$$P(X=1) = \frac{21}{40}$$

Probability that, there are 2 defective and 1 good item.

$$P(X=2) = \frac{{}^3C_2 \times {}^7C_1}{{}^{10}C_3} = \frac{3\times 7}{120}$$

$$P(X=2) = \frac{7}{40}$$

$\therefore$ The probability distribution of variable X is as follows,

X	0	1	2
P(X)	$\frac{35}{120}$	$\frac{21}{40}$	$\frac{7}{40}$

Q74. A random variable X has density function.

$$f(x) = \begin{cases} -ce^{-3x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

Find,

(i) The constant 'c'

(ii) P(1 < X < 2)

(iii) P(X ≥ 3)

(iv) P(X < 1).

Solution : **Model Paper-I, Q3(b)**

Given that,

$$f(x) = -ce^{-3x}$$

(i) Constant c

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow 0 + \int_{0}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{\infty} -ce^{-3x}\,dx = 1$$

$$\Rightarrow -c\int_{0}^{\infty} e^{-3x}\,dx = 1$$

$$\Rightarrow -c\left[\frac{e^{-3x}}{-3}\right]_0^{\infty} = 1$$

$$\Rightarrow \frac{c}{3}[e^{-3x}]_0^{\infty} = 1$$

$$\Rightarrow \frac{c}{3}[0-1] = 1$$

$$\therefore \quad c = -3$$

(ii) P(1 < X < 2)

$$P(1<X<2) = \int_1^2 f(x)dx$$

$$= \int_1^2 -ce^{-3x}dx$$

$$= -c\int_1^2 e^{-3x}dx$$

$$= -(-3)\left[\frac{e^{-3x}}{-3}\right]_1^2$$

$$= -1[e^{-3x}]_1^2$$

$$= -(e^{-6} - e^{-3})$$

$$= -(0.0024 - 0.049)$$

$$= 0.049 - 0.0024$$

$$P(1<X<2) = 0.0466$$

(iii) P(X ≥ 3)

$$P(X\geq 3) = \int_3^{\infty} f(x)\,dx$$

$$= \int_3^{\infty} -ce^{-3x}dx$$

$$= c\int_3^{\infty} -e^{-3x}dx$$

$$= -(-3)\left[\frac{e^{-3x}}{-3}\right]_3^{\infty}$$

$$= -[e^{-3x}]_3^{\infty}$$

$$= -[e^{-\infty} - e^{-9}] \quad [\because e^{-9} = 0.00012]$$

$$\therefore \; P(X\geq 3) = 0.00012$$

(iv) P(X < 1)

$$P(X<1) = \int_{-\infty}^{1} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^1 f(x)\,dx$$

$$= 0 + \int_0^1 f(x)dx$$

$$= \int_0^1 -ce^{-3x}dx = -c\int_0^1 e^{-3x}dx$$

$$= -(-3)\left[\frac{e^{-3x}}{-3}\right]_0^1$$

$$= -1[e^{-3x}]_0^1$$

$$= -1[e^{-3} - e^0]$$

$$= 1 - e^{-3}$$

$$= 1 - 0.049 \quad [\because e^{-3} = 0.049]$$

$$\therefore \; P(X<1) = 0.951.$$

Q75. Find the constant K such that, $f(x) = Kx^2$; if $0 < x < 3 = 0$; otherwise is a probability function.

(i) Find the distribution function F(x)

(ii) $P(1 < x \leq 2)$.

Solution :

According to the given data,

$$\int_0^3 f(x)dx = 1$$

$$\Rightarrow \int_0^3 Kx^2dx = 1 \qquad [\because f(x) = Kx^2]$$

$$\Rightarrow K\int_0^3 x^2dx = 1$$

$$\Rightarrow K\left[\frac{x^3}{3}\right]_0^3 = 1$$

$$\Rightarrow \frac{K}{3}[3^3 - 0^3] = 1$$

$$\Rightarrow \frac{K}{3}\times 3^3 = 1$$

$$\Rightarrow 9K = 1$$

$$\therefore K = \frac{1}{9}$$

(i) Distribution Function F(x)

$$F(x) = \int_0^x f(x)dx$$

$$= \int_0^x Kx^2dx = K\int_0^x x^2dx$$

$$= \frac{1}{9}\left[\frac{x^3}{3}\right]_0^x \qquad \left(\because K = \frac{1}{9}\right)$$

$$F(x) = \frac{x^3}{27}, \text{ for } 0 < x < 3$$

$$= 0, \quad \text{otherwise}$$

(ii) P(1 < x ≤ 2)

$$P(a<x\leq b) = \int_a^b f(x)dx$$

$$P(1<x\leq 2) = \int_1^2 Kx^2dx$$

$$= \int_1^2 \frac{1}{9}x^2dx \qquad \left(\because K = \frac{1}{9}\right)$$

$$= \frac{1}{9}\left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{9}\times\frac{1}{3}[2^3 - 1^3] = \frac{1}{27}[8-1]$$

$$\therefore \; P(1<x\leq 2) = \frac{7}{27}.$$

Q76. For the continuous probability function $f(x) = kx^2e^{-x}$ when $x \geq 0$, find,

(i) k

(ii) Mean

(iii) Variance.

Solution :

(i) k

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_{0}^{\infty} kx^2e^{-x}\,dx = 1$$

$$\Rightarrow k\left[x^2\int e^{-x}\,dx - \int\left(\left(\frac{d}{dx}x^2\right)\int e^{-x}\,dx\right)dx\right]_0^{\infty} = 1$$

$$\Rightarrow k\left[-x^2e^{-x} + \int(2x)e^{-x}\,dx\right]_0^{\infty} = 1$$

$$\Rightarrow k\left[-x^2e^{-x} + 2 - xe^{-x} + \int e^{-x}\,dx\right]_0^{\infty} = 1$$

$$\Rightarrow k\left[-x^2e^{-x} + 2\{-xe^{-x} - e^{-x}\}\right]_0^{\infty} = 1$$

$$\Rightarrow k\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^{\infty} = 1$$

$$\Rightarrow k\left[(-0-0-0) - (0-0-2e^{-0})\right] = 1$$

$$\therefore k = \frac{1}{2}$$

(ii) **Mean**

If x is a continuous random variable then mean $= E[x]$.

$$= \int_{-\infty}^{\infty} x f(x)dx$$

$$= \int_{0}^{\infty} x kx^2e^{-x}\,dx$$

$$= k\int_{0}^{\infty} x^3e^{-x}\,dx$$

$$= k\left[-x^3e^{-x} + \int 3x^2e^{-x}\,dx\right]_0^{\infty}$$

$$= k\left[-x^3e^{-x} + 3\left[-x^2e^{-x} + \int 2xe^{-x}\,dx\right]\right]_0^{\infty}$$

$$= k\left[-x^3e^{-x} + 3\left[-x^2e^{-x} + 2\left[-xe^{-x} + \int e^{-x}\,dx\right]\right]\right]_0^{\infty}$$

$$= k\left[-x^3e^{-x} + 3\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]\right]_0^{\infty}$$

$$= k\left[-x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x}\right]_0^{\infty}$$

$$= k\left[(-0-0-0-0-0) - (0-0-0-6)\right] = \frac{1}{2}[6] = 3$$

$\therefore$ Mean = 3

(iii) **Variance**

$$\sigma_x^2 = \text{Var}(x) = E[x^2] - [E(x)]^2$$

Now consider,

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{0}^{\infty} x^2(kx^2)e^{-x}\,dx$$

$$= k\int_{0}^{\infty} x^4e^{-x}\,dx$$

$$= k\left[-x^4e^{-x} + \int 4x^3e^{-x}\,dx\right]$$

$$= k\left[-x^4e^{-x} + 4\left[-x^3e^{-x} + \int 3x^2e^{-x}\,dx\right]\right]$$

$$= k\left[-x^4e^{-x} - 4x^3e^{-x} + 12\left[-x^2e^{-x} + \int 2xe^{-x}\,dx\right]\right]$$

$$= k\left[-x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} + 24\left[-xe^{-x} + \int e^{-x}\,dx\right]\right]$$

$$= k\left[-x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} - 24e^{-x}\right]_0^{\infty}$$

$$= \frac{1}{2}[0 - 4(0) - 12(0) - 24(0) - 24(0) - [0 - 4(0) - 12(0) - 24(0) - 24(1)]]$$

$$= \frac{1}{2}[0 - 0 - 0 - 0 - 0 - [0 - 0 - 0 - 0 - 24]]$$

$$= \frac{1}{2}[0 - (-24)] = \frac{1}{2}[24] = 12$$

$E[x^2] = 12$

Variance $= E[x^2] - [E[x]]^2 = 12 - 3^2 = 12 - 9$

$\therefore$ Variance = 3.

Q77. The probability density function is,

$y = k(3x^2 - 1)$, if $-1 \leq x \leq 2$

$= 0$, elsewhere

Find the value of k and find $P(-1 \leq x \leq 0)$.

Solution :

Given that,

$$y = f(x) = \begin{cases} k(3x^2 - 1), & -1 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

From the total probability,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{-1} f(x)\,dx + \int_{-1}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow 0 + \int_{-1}^{2} f(x)\,dx + 0 = 1$$

$$\Rightarrow \int_{-1}^{2} k(3x^2 - 1)\,dx = 1$$

$\Rightarrow \quad k\int_{-1}^{2}(3x^2-1)\,dx = 1$

$\Rightarrow \quad k\left[\frac{3x^3}{3}-x\right]_{-1}^{2} = 1$

$\Rightarrow \quad k[(2^3-2)-((-1)^3-(-1))] = 1$

$\Rightarrow \quad k[(8-2)-(-1+1)] = 1$

$\Rightarrow \quad k[6-0] = 1$

$\Rightarrow \quad 6k = 1$

$\therefore k = \frac{1}{6}$

Thus,

$$y = f(x) = \begin{cases} \frac{1}{6}(3x^2-1), & -1 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

$\therefore \quad P(-1 \le x \le 0) = \int_{-1}^{0} f(x)\,dx$

$= \int_{-1}^{0} \frac{1}{6}(3x^2-1)\,dx = \frac{1}{6}\int_{-1}^{0} 3x^2 - 1\,dx$

$= \frac{1}{6}\left[\frac{3x^3}{3}-x\right]_{-1}^{0}$

$= \frac{1}{6}[(0-0)-((-1)^3-(-1))]$

$= \frac{1}{6}[0-[(-1)+1]]$

$= \frac{1}{6}[0-0] = 0.$

Q78. If X is a continuous random variable with distribution.

$$f(x) = \begin{cases} \frac{1}{6}x+k, & \text{if } 0 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

Determine,

(i) The value of k

(ii) The mean

(iii) $P(1 \le x \le 2)$.

Solution :

(i) The Value of k

Given, $f(x) = \frac{1}{6}x + k$, if $0 \le x \le 3$

$\therefore \quad \int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \quad \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$

$\Rightarrow \quad \int_{-\infty}^{0} 0 + \int_{0}^{3}\left(\frac{1}{6}x+k\right)dx + \int_{3}^{\infty} 0 = 1$

i.e., $\int_{0}^{3}\left(\frac{x}{6}+k\right)dx = 1$

$\Rightarrow \quad \frac{1}{6}\int_{0}^{3} x\,dx + k\int_{0}^{3} 1\,dx = 1$

$\Rightarrow \quad \frac{1}{6}\left[\frac{x^2}{2}\right]_{0}^{3} + k\,[x]_{0}^{3} = 1$

$\Rightarrow \quad \frac{1}{6}\left[\frac{9}{2}\right] + k\,[3] = 1$

$\Rightarrow \quad \frac{3}{4} + 3k = 1$

$\Rightarrow \quad 12k = 4 - 3$

$\therefore \quad k = \frac{1}{12}$

(ii) Mean

$\int_{0}^{3} x \cdot f(x)\,dx$

$= \int_{0}^{3} x\left(\frac{x}{6}+k\right)dx$

$= \int_{0}^{3}\left(\frac{x^2}{6}+kx\right)dx$

$= \frac{1}{6}\int_{0}^{3} x^2\,dx + k\int_{0}^{3} x\,dx$

$= \frac{1}{6}\left[\frac{x^3}{3}\right]_{0}^{3} + k\left[\frac{x^2}{2}\right]_{0}^{3}$

$= \frac{1}{6}\left[\frac{27}{3}\right] + k\left[\frac{9}{2}\right]$

$= \frac{3}{2} + \frac{9}{2}k$

$= \frac{3}{2} + \frac{9}{2} \times \frac{1}{12}$

$= \frac{3}{2} + \frac{3}{8}$

$= \frac{12+3}{8}$

$= \frac{15}{8} = 1.875$

$\therefore$ Mean $= 1.875.$

(iii) $P(1 \le x \le 2)$

$P(1 \le x \le 2) = \int_{1}^{2} f(x)\,dx$

$= \int_{1}^{2}\left(\frac{x}{6}+k\right)dx$

$= \frac{1}{6}\int_{1}^{2} x\,dx + k\int_{1}^{2} 1\,dx$

$$= \frac{1}{6}\left[\frac{x^2}{2}\right]_1^2 + k\left[\frac{x}{1}\right]_1^2$$

$$= \frac{1}{6}\left[\frac{4}{2}\right] - \left[\frac{1}{2}\right] + k\,[2-1]$$

$$= \frac{1}{6}\left[\frac{3}{2}\right] + k \quad \left[\because k = \frac{1}{12}\right]$$

$$= \frac{1}{4} + \frac{1}{12}$$

$$= \frac{3+1}{12} = \frac{4}{12}$$

$$= \frac{1}{3}.$$

Q79. Probability density function of a random variable X is $f(x) = \begin{cases} \frac{1}{2}\sin x, \text{ for } 0 \le x \le \pi \\ 0, \text{ else where} \end{cases}$.

Find the mean and mode of the distribution.

Solution :

Given that,

$$f(x) = \begin{cases} \frac{1}{2}\sin x, & \text{for } 0 \le x \le \pi \\ 0, & \text{elsewhere} \end{cases}$$

(I) Mean of the Distribution

Mean of the distribution $= \int_{-\infty}^{\infty} xf(x)dx$

$$= \int_{-\infty}^{0} xf(x)dx + \int_{0}^{\pi} xf(x)dx + \int_{\pi}^{\infty} xf(x)dx$$

$$= \int_{-\infty}^{0} x(0)dx + \int_{0}^{\pi} x.\frac{1}{2}\sin x.dx + \int_{\pi}^{\infty} x(0)dx$$

$$= 0 + \frac{1}{2}\int_0^{\pi} x\sin x.dx + 0$$

$$= \frac{1}{2}\int_0^{\pi} x\sin x.dx$$

$$= \frac{1}{2}\left[-x\cos x - \int 1.(-\cos x)\right]_0^{\pi}$$

$$= \frac{1}{2}[-x\cos x + \sin x]_0^{\pi}$$

$$= \frac{1}{2}\left[\frac{-\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} - (0+0)\right]$$

$$= \frac{1}{2}\left[\frac{-\pi}{2}(0) + 1\right]$$

$$= \frac{1}{2}[0+1]$$

$$\therefore \text{Mean} = \frac{1}{2}$$

(II) Mode of the Distribution

Mode refers to the value of x for which $f(x)$ must be maximum.

For $f(x)$ to be maximum,

(a) $f'(x)$ must be equal to zero i.e., $f'(x) = 0$ and

(b) $f''(x)$ must be less than zero i.e., $f''(x) < 0$.

(a) $f'(x) = \frac{d}{dx}\left(\frac{1}{2}\sin x\right) = 0$

$$\Rightarrow \frac{1}{2}\cos x = 0$$

$$\Rightarrow \cos x = 0$$

$$\Rightarrow \cos x = \cos\frac{\pi}{2}$$

$$\therefore x = \frac{\pi}{2}$$

(b) $f''(x) = \frac{d}{dx}\left(\frac{1}{2}\cos x\right)$

$$= \frac{1}{2}(-\sin x)$$

$$f''(x) = \frac{-1}{2}\sin x$$

When, $x = \frac{\pi}{2}$

$$f''\left(\frac{\pi}{2}\right) = \frac{-1}{2}\sin\frac{\pi}{2}$$

$$= \frac{-1}{2} < 0$$

Since, the two conditions are satisfied, $f(x)$ is said to be maximum at $x = \frac{\pi}{2}$.

$\therefore$ Mode of the distribution, $x = \frac{\pi}{2}$.

Q80. A stockroom clerk returns three safety helmets at random to three steel mill employees who had previously checked them. If smith, Jones and Brown in that order ,receive one of the three hats, list the sample points for the possible orders of returning the helmets and find the value m of the random variable M that represents the number of correct matches.

Solution :

Let S, J and B denote Smith's Jones's and Brown's helmets.

The sample space for the possible orders or returning the helmets is,

S = {(SJB), (SBJ), BJS), (JSB), (JBS), BSJ)}

Let m be value of random variable 'M' that specifies the number of correct matches.

Sample Space	m
S J B	3
S B J	1
B J S	1
J S B	1
J B S	0
B S J	0

$\therefore \quad S = \{M/m = 0, 1, 2, 3\}$

Q81. A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Solution :

Assume that, there are x defective computers purchased by the school.

Given that,

Total number of laptop computers = 20

Number of defective items = 3

Number of good items = (20 – 3) = 17

Number of computers that are purchased randomly = 2

$\therefore$ The random variable X can take the value 0, 1, 2

Probability that there are 0 defective items.

$$P(X=0) = \frac{{}^3C_0 \times {}^{17}C_2}{{}^{20}C_2} = \frac{68}{95}$$

Probability that there is 1 defective computer is,

$$F(1) = P(X=1) = \frac{{}^3C_1 \times {}^{17}C_1}{{}^{20}C_2} = \frac{51}{190}$$

Probability that there are 2 defective computers is,

$$F(2) = P(X=2) = \frac{{}^3C_2 \times {}^{17}C_0}{{}^{20}C_2} = \frac{3}{190}$$

$\therefore$ The probability distribution of variable x is as follows,

x	0	1	2
f(x)	$\frac{68}{95}$	$\frac{51}{190}$	$\frac{3}{190}$

Q82. If a car agency sells 50% of its inventory of a certain foreign car equipped with side airbags, then,

(i) Find a formula for the probability distribution of the number of cars with side airbags among the next 4 cars sold by the agency.

(ii) Find the cumulative distribution function of the random variable x. Using f(x), verify that F(2) = $\frac{3}{8}$.

Solution :

Given that,

Probability of selling an automobile = 50%

$= 0.5$

$= \frac{1}{2}$

Number of cars sold by the agency = 4

$\therefore$ Number of sample points in the sample space that are equally likely to occur $\Rightarrow 2^4 = 16$

Inorder to get the number of ways of selling 3 cars with side airbags one can consider the number of ways in which two cells are partitioned into 4 outcomes. This partitioning of cells must be done in such a way that 3 cars with side airbags are assigned to one cell and model without side airbags are assigned to the other cell. i.e., ${}^4C_3$ ways.

(i) $f(x) = \frac{{}^4C_x}{16}$ [$F(x)$ at x = 0, 1, 2, 3, 4] is the formula for probability distribution of the number of cars with side airbags among the next 4 cars sold by the agency.

Since, $F(x) = \frac{{}^4C_x}{16}$,

$$f(0) = \frac{{}^4C_0}{16} = \frac{1}{16}$$

$$f(1) = \frac{{}^4C_1}{16} = \frac{4}{16} = \frac{1}{4}$$

$$f(2) = \frac{{}^4C_2}{16} = \frac{6}{16} = \frac{3}{8}$$

$$f(3) = \frac{{}^4C_3}{16} = \frac{4}{16} = \frac{1}{4}$$

$$f(4) = \frac{{}^4C_4}{16} = \frac{1}{16}$$

(ii) $F(0) = F(0) = \frac{1}{16}$

$$F(1) = F(0) + F(1) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

$$F(2) = F(0) + F(1) + F(2) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} = \frac{11}{16}$$

$$F(3) = F(0) + F(1) + F(2) + F(3) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} = \frac{15}{16}$$

$$F(4) = F(0) + F(1) + F(2) + F(3) + F(4)$$

$$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16}$$

$$= \frac{16}{16}$$

$$= 1$$

$\therefore$ $f(2) = F(2) - F(1)$ $[\because F(2) = F(1) + F(2)]$

$$= \frac{11}{16} - \frac{5}{16}$$

$$= \frac{3}{8}$$

Hence proved

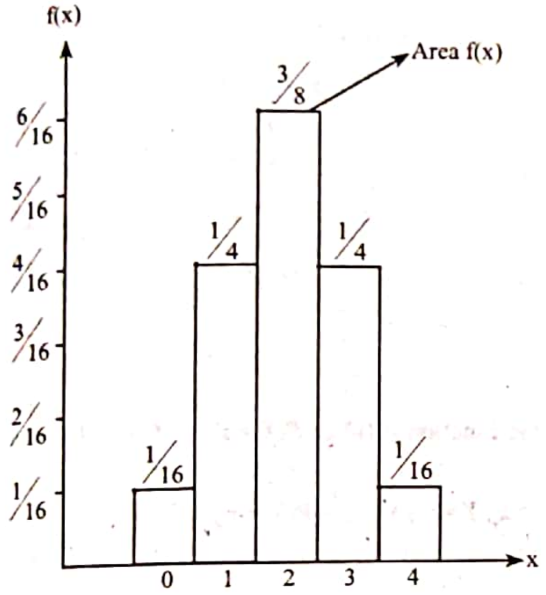


Figure: Probability Histogram

Q83. The department of energy (DOE) puts projects out on bid and generally estimates what a reasonable bid should be call the estimate b. The DOE has determined that the density function of the winning (low) bid is f(y) =

$$\begin{cases} \frac{5}{8}b, & \frac{2}{5}b \le y \le 2b \\ 0, & \text{elsewhere} \end{cases}$$

Find F(y) use it to determine the probability that the winning bid is less than the DOE's Preliminary estimate b.

Solution :

Given that,

$$F(y) = \int_{2b/5}^{y} \frac{5}{8b} dy \quad ; \text{for } \frac{2b}{5} < y < 2b$$

$$= \left(\frac{5t}{8b}\right)_{2b/5}^{y}$$

$$= \frac{5y}{8b} - \frac{1}{4}$$

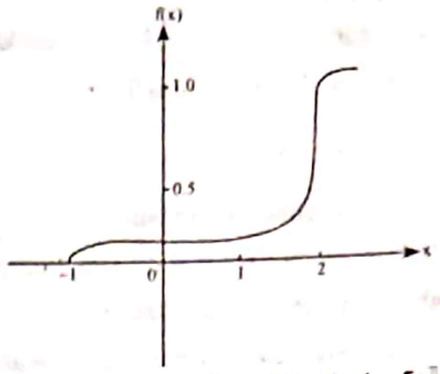


Figure: Continuous Cumulative Distribution Function

$$\therefore \quad F(y) = \begin{cases} 0, & y < \frac{2}{5}b \\ \frac{5y}{8b} - \frac{1}{4}, & \frac{2}{5}b \le y < 2b \\ 1, & y \ge 2b \end{cases}$$

Thus, the probability that the winning bid is less than the preliminary bid estimate b is,

$$P(y \le b) = F(b)$$

$$= \frac{5}{8} - \frac{1}{4}$$

$$= \frac{3}{8}$$

1.2.2 Statistical Independence

Q84. Discuss about bivariate distribution for discrete random variables (r.vs).

Answer :

The bivariate distribution of discrete r.v includes the following concepts.

1. Joint Probability Mass Function

Consider X and Y as two discrete r.v's defined on a sample space S with their values $x : x_1, x_2, \ldots x_n$ and $y : y_1, y_2 \ldots y_m$ respectively. If the product of both the variables are taken then the ordered pairs can be obtained are (x_i, y_j) where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Thus, the function defined on $X = x_i$ and $Y = y_j$ is known as joint probability mass function of (X, Y). It is represented as,

$P(X = x_i, Y = y_j)$ or $P(X = x_i \cap Y = y_j)$ or $P(X_i, Y_j)$ or simply P_{ij}.

The following table represents the joint probability function values

x/y	y_1	y_2	y_3	...	y_j	...	y_m	Total
x_1	P_{11}	P_{12}	P_{13}	...	P_{1j}	...	P_{1m}	P_1
x_2	P_{21}	P_{22}	P_{23}	...	P_{2j}	...	P_{2m}	P_2
x_3	P_{31}	P_{32}	P_{33}	...	P_{3j}	...	P_{3m}	P_3
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
x_n	P_{n1}	P_{n2}	P_{n3}	...	P_{nj}	...	P_{nm}	P_n
Total	p.1	p.2	p.3		p.j		p.m	1

Properties

The probability mass function $P(x_i, y_j)$ should meet the following conditions.

(i) $P(x_i, y_j) \geq 0$ for all values of i and j

(ii) $\sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) = 1$

2. Marginal Probability Functions

It is divided into two types as discussed below,

(a) Marginal Probability Function of X

If (X, Y) is a bivariate r.v for $i = 1, 2 .... n$ and $j = 1, 2, ...$ m then the function $P_x(x)$ or $P(X = x_i)$ or P_i is called marginal probability function of X which is defined as follows,

$$P_x(x_i) = P(X = x_i) = P(X = x_i, Y = y_1) + P(X = x_i, Y = y_2) + ... + P(X = x_i, Y = y_j) + ... + P(X = x_i, Y = y_m)$$

$$= P_{i1} + P_{i2} + ... + P_{ij} + ... + P_{im}$$

$$= \sum_{j=1}^{m} P_{ij}$$

(or)

$$= \sum_{j=1}^{m} P(x_i, y_j) \text{ for } i = 1, 2, ...; n.$$

Properties

(i) $P(X = x_i) \geq 0$ for all values of i

(ii) $\sum_{i=1}^{n} P(X = x_i) = 1$

(b) Marginal Probability Function of Y

If (X, Y) is a bivariate r.v for $i = 1, 2, ...n$ and $j = 1, 2, ... m$ then the function $P_y(y)$ or $P(Y = y_i)$ or P_j is called marginal probability function of Y which is defined as follows,

$$P_y(y_j) = P(Y = y_j) = P(X = x_1, Y = y_j) + P(X = x_2, Y = y_j) + ... + P(X = x_i, Y = y_j) + ... + P(X = x_n, Y = y_j)$$

$$= P_{1j} + P_{2j} + ... + P_{nj}$$

$$= \sum_{i=1}^{n} P_{ij}$$

(or)

$$= \sum_{i=1}^{n} P(x_i, y_j) \text{ for } j = 1, 2, ... m.$$

Properties

(i) $P(Y = y_j) \geq 0$ for all values of j

(ii) $\sum_{j=1}^{m} P(Y = y_j) = 1$

3. Conditional Probability Functions

It is divided into two types as discussed below,

(a) Conditional Probability Function of Y When X is Given

If (x, y) is a discrete bivariate r.v then the function $P(Y = y_j \mid X = x_i)$ or $P(y|x)$ is called conditional probability function of Y when X is given and is defined as,

$$P(Y = y_j \mid X = x_i) = \frac{P(X = x_i, Y = y_j)}{P(X = x_i)} \text{ for } P(x_i) > 0$$

Properties

(i) $P(Y = y_j \mid X = x_i) \geq 0$ for all values of j

(ii) $\sum_{j=1}^{m} P(Y = y_j \mid X = x_i) = 1$

(b) **Conditional Probability Function of X When Y is Given**

If (X, Y) is a discrete bivariate r.v then the function $P(X = x_i | Y = y_j)$ or $P(x | y)$ is called conditional probability function of X when Y is given and is defined as,

$$P(X = x_i | Y = y_j) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)} \text{ for } P(y_j) > 0$$

Properties

(i) $P(X = x_i | Y = y_j) \geq 0$ for all values of i

(ii) $\sum_{i=1}^{n} P(X = x_i | Y = y_j) = 1$

Q85. Discuss about bivariate distribution for continuous random variables.

Answer : Model Paper-III, Q3(a)

The bivariate distribution of continuous r.v includes the following concepts.

1. **Joint Probability Density Function**

The joint density function of (X, Y) denoted as $f(x, y)$ is defined as,

$$f(x, y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx dy$$

Properties

(i) $F(x,y) \geq 0$ for all values of x and y

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1$

(iii) The probability of an event that lie in the interval of x in (x_1, x_2) and y in (y_1, y_2) is,

$$P(x_1 \leq X \leq x_2, y \leq Y \leq y_2) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)\,dxdy$$

The first and second properties are used as tests to determine whether a function is a valid density function or not.

The individual density of 'x' and 'y' are obtained from $f(x, y)$ as,

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Where $f(x)$ and $f(y)$ are probability density function of 'x' and 'y' respectively.

2. **Marginal Probability Density Functions**

Marginal density functions are the density functions of individual random variables X and Y. They are given as,

$$f_x(x) = \frac{dF_X(x)}{dx}$$

$$f_y(y) = \frac{dF_Y(y)}{dy}$$

Where,

$f_x(x)$ and $f_y(y)$ are the marginal density function of x and y respectively. It can also be expressed in terms of joint density function as,

$$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy$$

$$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx$$

Properties

(i) $f(x)$ and $f(y) \geq 0$ for all values of x and y respectively

(ii) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(y)\,dy = 1$$

3. Conditional Probability Density Functions

It is divided into two types as discussed below,

(a) Conditional Probability Density Function of Y When X is Given

If two random variables say X and Y are jointly continuously distributed, then the function $f_{y/x}(y/x)$ or $f(y/x)$ is called conditional probability density function of Y when X is given and is defined as,

$$f(y|x) = \frac{f(x,y)}{f(x)} \text{ for } f(x) > 0$$

Properties

(i) $f(y|x) \geq 0$ for all values of y

(ii) $$\int_{-\infty}^{\infty} f(y \mid x)\,dy = 1$$

(b) Conditional Probability Density Function of X When Y is Given

If X and Y are two r.vs which are jointly continuously distributed, then the function $f_{x/y}(x \mid y)$ or $f(x \mid y)$ is called conditional probability density function of X when Y is given and is defined as,

$$f(x|y) = \frac{f(x,y)}{f(y)}, \text{ for } f(y) > 0$$

Properties

(i) $f(x \mid y) \geq 0$ for all values of x

(ii) $$\int_{-\infty}^{\infty} f(x \mid y)\,dx = 1$$

Q86. Give the distribution functions of bivariate random variable along with its properties.

Answer :

Distribution Function of Bivariate r.v

1. Joint Distribution Function

If (X, Y) is a bivariate random variable then the function $F_{xy}(x, y)$ or $F(x, y)$ is called joint distribution function or simply the distribution function which is a real valued function 'F' for all values of x and y. Mathematically it is defined as,

$$F_{xy}(x, y) = P(X \leq x, Y \leq y)$$

For discrete it is given by,

$$F(x, y) = \sum_{-\infty}^{x} \sum_{-\infty}^{y} P(X = x, Y = y)$$

For continuous it is given by,

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y)\,dx\,dy$$

Properties

The properties of joint distribution function of two random variables 'X' and 'Y' are,

(i) (a) $F_{XY}(-\infty, \infty) = 0$

(b) $F_{XY}(x, -\infty) = 0$

(c) $F_{XY}(-\infty, y) = 0$

(ii) $F_{XY}(\infty, \infty) = 1$

(iii) $0 \le F_{XY}(x, y) \le 1$

(iv) $F_{XY}(x, y)$ is a monotonic non-decreasing function.

(v) The joint distribution function of joint event

$$\{x_1 < X \le x_2, y_1 < Y \le y_2\} = F_{XY}(x_2, y_2) + F_{XY}(x_1, y_1) + F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1)$$

(vi) $F_X(x) = F_{XY}(x, \infty)$ and $F_Y(y) = F_{XY}(\infty, y)$

(vii) If $f(x, y)$ is the joint p.d.f of (x, y) then $f(x, y) = \dfrac{\partial^2 F(x,y)}{\partial x \partial y}$

2. Marginal Distribution Functions

(a) The marginal distribution function of X is given by,

$$\begin{aligned} F_x(x) = F(x) &= P(X \le x) \\ &= P(X \le x, Y < \infty) \\ &= \Sigma_y P(X \le x, Y = y) \\ &= F(x, \infty) \end{aligned}$$

Properties

(i) $P(k_1 < X \le k_2) = F(k_1) - F(k_2)$

(ii) $0 \le F(x) \le 1$

(iii) $F(-\infty) = 0, F(\infty) = 1$

(b) The marginal distribution function of Y is given by,

$$\begin{aligned} F_y(y) = F(y) &= P(Y \le y) \\ &= P(X < \infty, Y \le y) \\ &= \Sigma P(X = x, Y \le y) \\ &= F(\infty, y) \end{aligned}$$

Properties

(i) $P(k_3 < Y \le k_4) = F(k_4) - F(k_3)$

(ii) $0 \le F(y) \le 1$

(iii) $F(-\infty) = 0, F(\infty) = 1.$

3. Conditional Distribution Functions

(a) The conditional distribution function of Y when X is known is given by,

$$F_{y/x}(y|x) = F(y|x) = P(Y \le y|X = x)$$

Properties

(i) $P(k_3 < Y \le k_4|x) = F(k_4|x) - F(k_3|x)$

(ii) $0 \le F(y|x) \le 1$

(iii) $F(-\infty|x) = 0, F(\infty|x) = 1$

(b) The conditional distribution function of X when Y is known is given by,

$$F_{xy}(x/y) = F(x|y) = P(X \le x|Y = y)$$

Properties

(i) $P(k_1 < X \le k_2|y) = F(k_2|y) - F(k_1|y)$

(ii) $0 \le F(x|y) \le 1$

(iii) $F(-\infty|y) = 0, F(\infty|y) = 1$

Q87. State and explain statistical independence.

Answer :

Assume that X and Y are two random variables containing join probability distribution $f(x, y)$ and marginal distributions $g(x)$ and $h(y)$ respectively. Then, X and Y are statistically independent if and only if.

$\boxed{f(x, y) = g(x)\, h(y)\, \forall\, (x, y) \text{ within their range}}$

Assume that $X_1, X_2, \ldots, X_n$ are n random variables containing join probability distribution $f(x_1, x_2, \ldots, x_n)$ and marginal distribution $f_1(x_1), f_2(x_2), \ldots f_n(x_n)$ respectively. Then, $X_1, X_2, \ldots, X_n$ are said to be mutually statistical independent if and only if,

$\boxed{f(x_1, x_2, \ldots, x_n) = f_1(x_n), f_2(x_2), \ldots f_n(x_n)\, \forall\, (x_1, x_2, \ldots, x_n) \text{ within their range}}$

Proof

When $f(x/y)$ is not dependent on y, then $F(x/y) = g(x)$ and $f(x, y) = g(x)\, h(y)$

Since, $f(x/y) = g(x)$ we get,

$f(x, y) = f(x/y)\, h(y)$ ... (1)

Substituting equation (1) into the marginal distribution of X we get,

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$= \int_{-\infty}^{\infty} f(x/y)\, h(y)\, dy$$

Suppose if $f(x/y)$ does not depends on y, then we have,

$$g(x) = f(x/y) \int_{-\infty}^{\infty} h(y)\, dy$$

$$g(x) = f(x/y)\,(1) \qquad \left[\because \int_{-\infty}^{\infty} h(y)\, dy = 1\right]$$

Therefore, as $h(y)$ is the probability density function of Y we get,

$g(x) = f(x/y)$ and $f(x, y) = g(x)\, h(y)$.

PROBLEMS

Q88. The bivariate probability distribution of x and y is given below. Find,

(i) Marginal distribution of x and y

(ii) Conditional distribution of x given y = 1.

y \ x	−1	0	1
0	1/15	2/15	1/15
1	3/15	2/15	1/15
2	2/15	1/15	2/15

Solution :

The joint probability distribution is given as,

y \ x	− 1	0	1	P(y)
0	1/15	2/15	1/15	4/15
1	3/15	2/15	1/15	6/15
2	2/15	1/15	2/15	5/15
P(x)	6/15	5/15	4/15	1

Marginal Distribution of x and y

(i) The marginal distribution of x obtained from above table is,

x	– 1	0	1
P(x)	6/15	5/15	4/15

This can also be calculated as,

At $x = -1$

$$\Rightarrow \quad P(x=-1) = P(x=-1, y=0) + P(x=-1, y=1) + P(x=-1, y=2)$$

$$= \frac{1}{15} + \frac{3}{15} + \frac{2}{15} = \frac{6}{15}$$

At $x = 0$

$$\Rightarrow \quad P(x=0) = P(x=0, y=0) + P(x=0, y=1) + P(x=0, y=2)$$

$$= \frac{2}{15} + \frac{2}{15} + \frac{1}{15} = \frac{5}{15}$$

At $x = 1$

$$\Rightarrow \quad P(x=1) = P(x=1, y=0) + P(x=0, y=1) + P(x=1, y=2)$$

$$= \frac{1}{15} + \frac{1}{15} + \frac{2}{15} = \frac{4}{15}$$

The marginal distribution of y obtained from above table is,

y	0	1	2
P(y)	4/15	6/15	5/15

This can also be calculated as,

At $y = 0$

$$\Rightarrow \quad P(y=0) = P(x=-1, y=0) + P(x=0, y=0) + P(x=1, y=0)$$

$$= 1/15 + 2/15 + 1/15 = 4/15$$

At $y = 1$

$$\Rightarrow \quad P(y=1) = P(x=-1, y=1) + P(x=0, y=1) + P(x=1, y=1)$$

$$= \frac{3}{15} + \frac{2}{15} + \frac{1}{15} = \frac{6}{15}$$

At $y = 2$

$$\Rightarrow \quad P(y=2) = P(x=-1, y=2) + P(x=0, y=2) + P(x=1, y=2)$$

$$= \frac{2}{15} + \frac{1}{15} + \frac{2}{15} = \frac{5}{15}$$

(ii) **Conditional Distribution of x given y = 1**

(a) Conditional function of $x = -1$ when $y = 1$

$$P(x=-1 | y=1) = \frac{(x=-1, y=1)}{P(y=1)} = \frac{3/15}{6/15} = \frac{1}{2}$$

(b) Conditional function of $x = 0$ when $y = 1$

$$P(x=0 | y=1) = \frac{P(x=0, y=1)}{P(y=1)} = \frac{2/15}{6/15} = \frac{1}{3}$$

(c) Conditional function of $x = 1$ when $y = 1$

$$P(x=1 | y=1) = \frac{P(x=1, y=1)}{P(y=1)} = \frac{1/15}{6/15} = \frac{1}{6}$$

Q89. A two dimensional r.v(x,y) have a bivariate distribution given by,

$P(x = x; y = y) = \frac{x^2+y}{32}$ for x = 0, 1, 2 and y = 0, 1. Find the marginal distributions of x and y.

Solution :

Given that,

$$P(X=x, Y=y) = \frac{x^2+y}{32} \text{ for } x=0, 1, 2, \text{ and } y=0, 1$$

∴ The joint probability distribution and marginal distribution can be represented as,

x/y	0	1	$\sum_y P(x,y)$
0	0	1/32	1/32
1	1/32	2/32	3/32
2	4/32	5/32	9/32
$\sum_x P(x,y)$	5/32	8/32	

The marginal probability distribution of x is,

x	0	1	2
P(x = x)	1/32	3/32	9/32

The marginal probability distribution of y is,

y	0	1
P(y = y)	5/32	8/32

Q90. From the following bivariate probability of x and y find,

(i) $P(x \le 1, y = 2)$

(ii) $P(x \le 1)$

(iii) $P(y = 3)$

(iv) $P(y \le 3)$

(v) $P(x < 3, y \le 4)$

x/y	1	2	3	4	5	6	$\sum_y P(x,y)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
$\sum_x P(x,y)$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	1

Solution :

(i) $P(x \le 1, y=2) = P(x=0, y=2) + P(x=1, y=2) = 0 + \frac{1}{16} = \frac{1}{16}$

(ii) $P(x \le 1) = P(x+0) + P(x=1) = \frac{8}{32} + \frac{10}{16} = \frac{7}{8}$

(iii) $P(y=3) = \frac{11}{64}$

(iv) $P(y \le 3) = P(y=1) + P(y=2) + P(y=3) = \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$

(v) $P(x<3, y \le 4) = P(x=0, y=4) + P(x=1, y \le 4) + P(x=2, y \le 4)$

$$= \left(\frac{1}{32}+\frac{2}{32}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{32}+\frac{1}{32}+\frac{1}{64}+\frac{1}{64}\right)$$

$$= \frac{6}{64}+\frac{24}{64}+\frac{6}{64} = \frac{36}{64} = \frac{9}{16}$$

Q91. The joint pdf of two-dimensional random variable (x, y) is given by,

$f(x, y) = kx^2y \; ; 0 < x < 1 \; ; 0 < y < 1$

$= 0 \quad ;$ otherwise

(i) Find the value of 'k'

(ii) Find the marginal densities of x and y

(iii) Find the mean of x.

Solution :

(i) We know that,

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy = 1$$

$$\therefore \int_0^1\int_0^1 kx^2y \;\; dx\,dy = 1$$

$$\therefore k\int_0^1 x^2\left[\int_0^1 y\,dy\right]dx = 1$$

$$k\int_0^1 x^2\left[\frac{y^2}{2}\right]_0^1 dx = 1$$

$$k\int_0^1 x^2\left[\frac{1}{2}-0\right]dx = 1$$

$$\frac{k}{2}\int_0^1 x^2\,dx = 1$$

$$\frac{k}{2}\left[\frac{x^3}{3}\right]_0^1 = 1$$

$$\frac{k}{2}\left[\frac{1}{3}-0\right] = 1$$

$$\frac{k}{6} = 1$$

$$\boxed{\therefore k = 6}$$

(ii) The marginal density of x is given by,

$$f(x) = \int_0^1 f(x,y)\,dy$$

$$= \int_0^1 kx^2y\,dy$$

$$= \int_0^1 6x^2y\,dy$$

$$= 6x^2\left[\frac{y^2}{2}\right]_0^1 = 6x^2\left[\frac{1}{2}-0\right]$$

$$\boxed{\therefore f(x) = 3x^2;\; for\; 0 < x < 1}$$

The marginal density of y is given by,

$$f(y) = \int_0^1 f(x,y)\,dx$$

$$= \int_0^1 kx^2y\,dx$$

$$= \int_0^1 6x^2y\,dx$$

$$= 6y\left[\frac{x^3}{3}\right]_0^1 = 6y\left[\frac{1}{3}-0\right]$$

$$\boxed{\therefore f(y) = 2y \;;\; for\; 0 < y < 1}$$

(iii) The mean of x is given by,

$$E(x) = \int_0^1 xf(x)\,dx$$

$$= \int_0^1 x.3x^2\,dx$$

$$= \int_0^1 3x^3\,dx$$

$$= 3\left[\frac{x^4}{4}\right]_0^1$$

$$= 3\left[\frac{1}{4}-0\right]$$

$$\therefore \boxed{E(x) = \frac{3}{4}}$$

Q92. The random variable x and y have the joint density function,

$f(x, y) = 2 \; ; \; 0 < x < 1 \; ; 0 < y < x$

$= 0 \; ;$ otherwise

(a) Find marginal density functions of X and Y

(b) Find conditional density function of y given X = x, and conditional density function of x given Y = y

(c) Check for independence of X and Y.

Solution :

(a) The marginal density function of X is,

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

Limits of y is : $0 < y < x$

$$\therefore \quad f(x) = \int_0^x 2\,dy = 2y\Big|_0^x = 2x,\; 0 < x < 1$$

The marginal density function of y is,

$$f(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

Limits of x is : $0 < \boxed{y < x < 1}$

$$\therefore \quad f(y) = \int_{y}^{1} 2dx = 2x\Big|_{y}^{1} = 2(1-y),\ 0 < y < 1$$

(b) The conditional density function of y given x is,

$$f(y|x) = \frac{f(x, y)}{f(y)} = \frac{2}{2x}$$

$$= \frac{1}{x}, 0 < x < 1$$

The conditional density function of x given y is,

$$f(x|y) = \frac{f(x,y)}{f(y)}$$

$$= \frac{2}{2(1-y)} = \frac{1}{(1-y)}, 0 < y < 1$$

(c) Since $f(x)\,f(y) = 2x.2(1-y)$

$$= 4x(1-y) \neq f(x, y) \quad [\because f(x, y) = 2]$$

Hence, X and Y are not independent.

Q93. If X and Y are two random variables having joint density function.

$$f(x, y) = \frac{1}{8}(6-x-y) \ ; \quad 0 < x < 2, 2 < y < 4$$

$$= 0 \quad ; \quad \text{otherwise}$$

Find,

(i) $P(x < 1 \cap y < 3)$

(ii) $P(x + y < 3)$

(iii) $P(x < 1/y < 3)$.

Solution : Model Paper-II, Q3

(i) $P(x < 1 \cap y < 3)$

$$P(x < 1 \cap y < 3) = \int_{x=0}^{1} \int_{y=2}^{3} f(x, y)dydx$$

$$= \int_{0}^{1} \int_{2}^{3} \frac{1}{8}(6 - x - y)\, dydx$$

$$= \frac{1}{8}\int_{0}^{1}\left[6y - xy - \frac{y^2}{2}\right]_2^3 dx$$

$$= \frac{1}{8}\int_{0}^{1}\left[6y - xy - \frac{y^2}{2}\right]_2^3 dx$$

$$= \frac{1}{8}\int_0^1 [6(3-2) - x(3-2) - \frac{1}{2}(9-4)]dx = \frac{1}{8}\int_0^1 \left(6 - x - \frac{5}{2}\right)dx$$

$$= \frac{1}{8}\int_0^1 \left(\frac{7}{2} - x\right)dx = \frac{1}{8}\left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{8}\left[\frac{7}{2}(1-0) - \frac{1}{2}(1-0)\right]$$

$$= \frac{1}{8}\left[\frac{7}{2} - \frac{1}{2}\right] = \frac{1}{8}\cdot\frac{6}{2} = \frac{3}{8}$$

(ii) **P(x + y < 3)**

$$P(x+y<3) = \int_0^1\int_2^{3-x} f(x,y)dydx$$

$$= \int_0^1\int_2^{3-x} \frac{1}{8}(6-x-y)dy\ dx = \frac{1}{8}\int_0^1 \left[6y - xy - \frac{y^2}{2}\right]_2^{3-x} dx$$

$$= \frac{1}{8}\int_0^1 \left[6(3-x-2) - x(3-x-2) - \frac{1}{2}\{(3-x)^2 - 4\}\right]dx$$

$$= \frac{1}{8}\int_0^1 \left[6(1-x) - x(1-x) - \frac{1}{2}(9+x^2-6x-4)\right]dx$$

$$= \frac{1}{8}\int_0^1 \left[6 - 6x - x + x^2 - \frac{9}{2} - \frac{x^2}{2} + \frac{6}{2}x + \frac{4}{2}\right]dx$$

$$= \frac{1}{8}\int_0^1 \left(\frac{x^2}{2} - 4x + \frac{7}{2}\right)dx = \frac{1}{8}\left[\frac{x^3}{6} - \frac{4x^2}{2} + \frac{7x}{2}\right]_0^1$$

$$= \frac{1}{8}\left(\frac{1}{6}(1-0) - 2(1-0) + \frac{7}{2}(1-0)\right) = \frac{1}{8}\left[\frac{1}{6} - 2 + \frac{7}{2}\right]$$

$$= \frac{1}{8}\left(\frac{1-12+21}{6}\right) = \frac{1}{8}\times\frac{10}{6} = \frac{5}{24}$$

(iii) **P(x < 1/y < 3)**

$$P(x<1/y<3) = \frac{P(x<1 \cap y<3)}{p(y<3)}$$

Consider $P(y<3) = \int_0^2\int_2^3 f(x,y)dy\ dx = \int_0^2\int_2^3 \frac{1}{8}(6-x-y)dy\ dx$

$$= \int_0^2 \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^3 dx = \int_0^2 \frac{1}{8}\left[6(3-2) - x(3-2) - \frac{1}{2}(9-4)\right]dx$$

$$= \int_0^2 \frac{1}{8}\left[6 - x - \frac{1}{2}.5\right]dx = \frac{1}{8}\int_0^2\left(\frac{7}{2} - x\right)dx$$

$$= \frac{1}{8}\left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^2 = \frac{1}{8}\left[\frac{7}{2}(2-0) - \frac{1}{2}(4-0)\right] = \frac{1}{8}[7-2]$$

$$P(y<3) = \frac{5}{8}$$

$$P(x<1/y<3) = \frac{3/8}{5/8} = \frac{3}{5}$$

Q94. Suppose that the shelf life in years of a certain perishable food product packaged in cardboard containers is a random variable whose probability density function is given by,

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Let, X_1, X_2 and X_3 represent the shelf lives for three of these containers selected independently and find $P(X_1 < 2, 1 < X_2 < 3, X_3 > 2)$.

Solution :

Given that, the containers were selected randomly so let us suppose that the random variables X_1, X_2 and X_3 are statistically independent with joint probability density.

$$f(x_1, x_2, x_3) = f(x_1)f(x_2)f(x_3)$$

$$= e^{-x_1}e^{-x_2}e^{-x_3} \qquad [\because f(x) = e^{-x} \ \forall\ x < 0]$$

$$= e^{-x_1-x_2-x_3}, \text{ For } x_1 > 0, x_2 > 0, x_3 > 0 \text{ and } f(x_1, x_2, x_3) = 0$$

Therefore,

$$P(X_1 < 2, 1 < X_2 < 3, X_3 > 2) = \int_2^{\infty}\int_1^{3}\int_0^{2} e^{-x_1-x_2-x_3}\, dx_1 dx_2 dx_3$$

$$= (1 - e^{-2})(e^{-1} - e^{-3})e^{-2}$$

$$= (0.86(0.32)(0.14)$$

$$= 0.04$$