Mid-1 to Mid-2

No. of Cosets [Right or Left]

$O(G)/O(H)$.

Let us consider S_3:

$\phi = x_1 \to x_2$ $\quad\quad$ $\psi : x_1 \to x_2$
$\quad x_2 \to x_1$ $\quad\quad$ $x_2 \to x_3$
$\quad x_3 \to x_3$ $\quad\quad$ $x_3 \to x_1$.

Let $H = \{e, \phi\}$

Right Cosets	Left Cosets
$He = \{e, \phi\}$	$eH = \{e, \phi\}$
$H\psi = \{\psi, \phi\psi\}$	$\psi H = \{\psi, \psi\phi = \phi\psi^2\}$
$H\psi^2 = \{\psi^2, \phi\psi^2\}$	$\psi^2 H = \{\psi^2, \psi^2\phi = \phi\psi\}$

The right cosets need not be the same as left cosets.

If $N = \{e, \psi, \psi^2\}$

Right Cosets	Left Cosets
$Ne = \{e, \psi, \psi^2\}$	$eN = \{e, \psi, \psi^2\}$
$N\phi = \{\phi, \psi\phi, \psi^2\phi\}$	$\phi N = \{\phi, \phi\psi, \phi\psi^2\}$

Here, right cosets are same as left cosets, although not element by element.

Such groups (H) are called Normal Subgroups.

Definition (Normal subgroups)

A subgroup N of G is a normal subgroup of G if $\forall\, g \in G,\ n \in N$

$gng^{-1} \in N$

or equivalently $gNg^{-1} \subset N$
is contained in

$\Rightarrow \boxed{gNg^{-1} = N}$

result which is true for normal subgroups.

Result N is a normal subgroup of G iff

$gNg^{-1} = N \ \forall\, g \in G.$

① Assume $gNg^{-1} = N$

$\Rightarrow gNg^{-1} \subset N \quad \therefore N$ is normal.

②. Assume $gNg^{-1} \subset N$, P.T $gNg^{-1} = N$ → ①

True $\forall\, g$.

$\therefore g^{-1} N (g^{-1})^{-1} \in N$

$\Rightarrow g^{-1} N g \subset N$

$\Rightarrow g(\underbrace{g^{-1} N g}_{(x)}) g^{-1} \subset gNg^{-1}$

since x is contained in N.

But x is nothing but N.

$\therefore \boxed{N \subset gNg^{-1}}$ → ②

from ①, ② $gNg^{-1} = N$.

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~~Ex~~ Result: P.T. if N is a normal subgroup, every right coset is a left coset and vice versa.

$$\boxed{\begin{aligned} \text{Property: } (gng^{-1})^m &= gn^mg^{-1} \\ (gng^{-1})^2 &= gng^{-1} \cdot gng^{-1} \\ &\Rightarrow gn^2g^{-1} \end{aligned}}$$

Proof (Result): ① $gNg^{-1} = N$ (N is normal)

$$gNg^{-1} \cdot g = Ng$$

$$\Rightarrow gN = Ng$$

so if N is normal left coset = Right coset.

② Converse:

Let $g \in G$, gN - Left coset

$g \in gN$

$\therefore$ g is an element present in a left Coset gN.

No other ~~subgroup~~ Left coset of N has g.

Similarly g is also in Ng

Ng is unique.

$$\boxed{Ng = gN}$$

$$\Rightarrow \cancel{g}g^{-1} \cdot gNg^{-1} = Ngg^{-1}$$

$$= N$$

$\Rightarrow$ it is ~~a~~ Normal

If we consider a group of Cosets of G. normal sub-
(Quotient group) G/N.

Closure: $(Na . Nb)$ shd give back a right coset.

$N(a . N)b$

$N(Na)b$

$\Rightarrow N\underbrace{ab}_{\in G}$, $(N \times N = N)$

$\Rightarrow Nc$

which is a right coset.

Associative: $Na(Nb . Nc) = (Na . Nb) . Nc$

$\Rightarrow Na . N(bc)$ $\qquad Nabc$

$\Rightarrow Nabc$

Inverse: Ne is the Identity.

$\rightarrow Na^{-1}$

Problem If G is a group and H is a subgroup of index 2 in G, P.T. H is a normal subgroup.

2 cosets: ① He, ② Hx $\rightarrow x \in G$; xH.
eH

All Cosets togethers partition the group.
Left or right

$\therefore G = H \cup Hx$

$= H \cup xH$

$\therefore Hx = xH$

$H = xHx^{-1}$ $\therefore H$ is normal.

Eg: If N is a normal subgroup of G and H is any subgroup of G, P.T. NH is a subgroup of G.

Consider 2 elements n_1h_1, $\cancel{n_2h_2}$ ~~in NH~~

We need to P.T. $\cancel{(n_1h_1)(n_2h_2)^{-1}} \in NH$

T.P. $(n_1h_1)(n_2h_2)^{-1} \in NH \Rightarrow \cancel{n_1 h_1 h_2^{-1} n_2 \in NH}$

$\Rightarrow \cancel{n_1 h_x n_2 \in NH}$

$\Rightarrow n_1 h_1 h_2^{-1} n_2^{-1}$

$\Rightarrow n_1 \underbrace{h' n_2^{-1}}$

$\downarrow$

$h'N$

$= Nh'$ [Normal]

$n''h''$

$\Rightarrow n_1 n'' h''$

$\Rightarrow nh$

Eg: Suppose H is only subgroup of $O(H) = m$ of a finite group. Then ~~grp~~ P.T. H is a normal subgrp of G.

Let $H = \{h_1, h_2 \dots h_m\}$

let $x \in G$.

P.T. this is a subgrp

$xHx^{-1} = (\underbrace{xh_1x^{-1} \dots xh_mx^{-1}}_{\text{all these elements are distinct.}})$.

because if $xh_rx^{-1} = xh_sx^{-1}$

$\Rightarrow h_r = h_s$ [False].

That is we have generated 2 subgrps of $O(H)$.

$\therefore xHx^{-1} = H$

$\downarrow$

Normal.

Eg: Let N, M be normal subgroups. P.T. $N * M$ is a normal subgroup in G.

$gNg^{-1} = N$

$gMg^{-1} = M$

Also prove that $N * M$ is a subgroup.

$gNg^{-1} \cdot gMg^{-1} = N * M$

$\Rightarrow g(NM)g^{-1} = NM$.

Eg: ~~Ree~~. Let N, M be Normal Subgroups $N \cap M = e$.

P.T. $nm = mn$.

(n commutes with every element in M)

$nm \cdot (mn)^{-1} = e$

$\Rightarrow nmn^{-1}m^{-1} = e$.

To show: $nmn^{-1}m^{-1} \subset N \cap M$

① $n \underbrace{(mn^{-1}m^{-1})}_{\in N}$ $\cancel{= e}$.

$n n' \in N$

② $(\underbrace{nmn^{-1}}_{\in M}) m^{-1}$

$\Rightarrow m' M^{-1} \in M$.

$\therefore nmn^{-1}m^{-1} \in N, M$.

$\therefore nmn^{-1}m^{-1} \subset N \cap M$

$\Rightarrow nmn^{-1}m^{-1} = e$

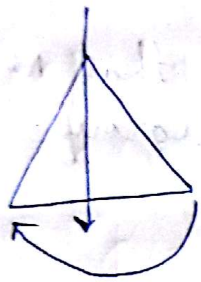
$\therefore mn = nm$

Dihedral Group

Let G be a group defined as:

$\boxed{x^i y^j}$ $i = (0, 1) \longrightarrow$ flipping

$j = (0, 1, \ldots n-1) \longrightarrow$ rotations ($2\pi/n$) diff rotations.



① $x^2 = e$ $y^n = e$

② $xy = y^{-1}x$

1. $(xy) \cdot (y^{-1}x) = x^2 = e$

2. $(xy) \cdot y = y^{-1}xy = y^{-1}y^{-1}x$
   $(xy^2) \Rightarrow y^{-2}x$

3. Similarly $xy^3 = y^{-3}x$.

What is the general form of:

$(x^i y^j)(x^k y^l) = ?$

$i = \{0, 1\}$

Let $k = 1$.

$\Rightarrow x^i y^j y^{-l} x$

$\Rightarrow x^i y^{j-l} x$

$\Leftrightarrow x^{i+1} y^{l-j}$

general form: $\boxed{x^{i+k} y^{(-1)^k \cdot j + l}}$

Homomorphism

Let ϕ be a mapping from G to G' where G, G' are 2 groups such that for (elements in G/G' may be of diff type)

$$\phi(x * y) = \phi(x) \times \phi(y)$$

Eg

Eg G is the group.

Eg: G is be the grp of non zero real numbers under multiplication

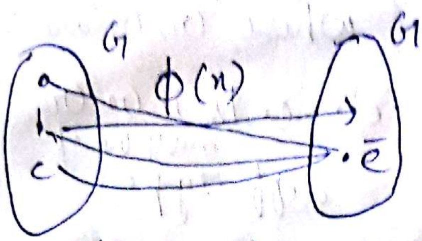
$$\phi(x) = x^2$$

Is $\phi(x)$ homomorphous.

Eg: $\phi(x) = 2^x$ Is this homomorphic (No)

Eg: $\phi(x) = (x+1)$. (No)

The Kernel K of a homomorphism :



(a, b, c) form the kernel.

Property of Kernel:

1. $\phi(e) = \bar{e}$

$$\phi(x)\cdot\bar{e} = \phi(x)$$
$$= \phi(xe)$$
$$\Rightarrow \phi(x)\cdot\phi(e)$$
$$\Rightarrow \bar{e} = \phi(e).$$

2. $\phi(x^{-1}) = (\phi(x))^{-1}$

$$\phi(e) = \bar{e}$$
$$\phi(x\cdot x^{-1}) = \bar{e}$$
$$\phi(x)\cdot\phi(x^{-1}) = \bar{e}$$
$$\phi(x) = (\phi(x^{-1}))^{-1}$$
$$\phi(x^{-1}\cdot x) = \bar{e}$$
$$\phi(x^{-1}) = (\phi(x))^{-1}$$

3. Kernel of a homomorphism is a normal subgroup.

if $x, y \in K$ $\quad \phi(x) = e \quad \phi(y) = e$.

To Prove: $x\cdot y \in K$; $\phi(xy) = \phi(x)\phi(y)$
$$= e\cdot e = e.$$

$$\therefore xy \in K$$

Inverse: $\phi(x) = e$

$(\phi(x))^{-1} = e^{-1} = e$.

$\Rightarrow \phi(x^{-1}) = e$

$\therefore x^{-1} \in K$.

it is a subgrp.

Normal: Consider gKg^{-1}.

$\phi(gKg^{-1})$

$\Rightarrow \phi(g) \cdot \underbrace{\phi(K)}_{e} \cdot \phi(g^{-1})$

$\phi(g \cdot g^{-1})$

$\Rightarrow \phi(e)$

$\Rightarrow e$.

Theorm

If ϕ is the homomorphism of G onto $\bar{G}$ with Kernel K then the set of all inverse images of $\bar{g} \in \bar{G}$ under ϕ in G is given by Kx where x is any particular inverse image of $\bar{g}$ in G.

if $(\bar{g} = \bar{e})$ we are done;

else if The inverse images of $\bar{g} \neq \bar{e}$.

Suppose $x \in G$ is one inverse of $\bar{g}$.

Let $k \in K$. $y = kx$

$\phi(y) = \phi(kx) = \phi(k) \cdot \phi(x) = \bar{e}\,\phi(x) = \phi(x)$
$= \bar{g}$

$\bar{g}$ is the image of Kx.

If Kx is not all of the inverse images of $\bar{g}$: Let $\phi(z) = \bar{g} = \phi(x)$

$$\Rightarrow \phi(z)\cdot(\phi(x))^{-1} = \bar{e}$$

$$\phi(\underbrace{z x^{-1}}) = \bar{e}$$

$$z x^{-1} \in K$$

$$z \in Kx$$

This mapping is one - one if $K = \{e\}$.

A Homo-morphism is isomorphism if it is Injective. It is isomorphic if it is bijective.
(One-One)

<u>Theorm</u>: Let ϕ be a homomorphism of G on $\bar{G}$ with kernel K.
Then $G/K \cong \bar{G}$
(Isomorphic)

$$\begin{array}{ccc} G & \xrightarrow{\phi} & \bar{G} \\ K \downarrow & \nearrow \psi & \\ G/K & & \end{array}$$

$g \in G \to \phi(g)$

$g \in G \to \frac{G}{K} (Kg)$

$\phi(g) = \phi(Kg') = \phi(K)\cdot\phi(g') = \bar{e}\,\phi(g')$
$= \phi(g')$

Mapping ψ is well defined.

$X, Y \in G/K$

$X = Kg \quad Y = Kf \quad XY = Kgf$

$$\psi(\phi XY) = \psi(Kgf)$$

$$\Rightarrow \phi(gf).$$

$$\Rightarrow \phi(g)\cdot\phi(f).$$

$$\Rightarrow \underset{X}{\psi(Kg)}\cdot\underset{Y}{\psi(Kf)}$$

For One-one:

If Kernel of ψ = Ident of K

$= \{X \in$

Eg: If H is a subgrp of G, then the

centralizer $C(H) = \{x = G \mid xh = hx \; \forall h \in H\}$

P.T. C(H) is a subgrp.

Closure: $xh = hx \; \forall h \in H$

for $x, y \in H$

$(xy)h = h(xy)$

$\hookrightarrow xhy$

$\Rightarrow hxy$

$\therefore$ Closure.

Inverse: $xh = hx$

$xhx^{-1} = h$

$hx^{-1} = x^{-1}h$

Eg: The center of G:

$\{z \in G \mid xz = zx \; \forall x \in G\}$

P.T. ~~C(H) is a~~ it is a subgroup.

Eg: P.T. center of G is a normal subgrp.

$\boxed{x\, C\, \cancel{(G)}\, x^{-1} = C}$

$\Rightarrow Cx \cdot x^{-1}$

$= C.$

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① $(n, o(G)) = 1$ $x, y \in \mathbb{Z}$

$\Rightarrow nx + o(G)y = 1$

$g^1 = g^{nx + o(G)y}$

$= (g^n)^x \cdot (g^{o(G)})^y$

$= e \times e = e.$

Qn → By the centralizer $C(H)$ of H

$= \{ x \in G \mid xh = hx \ \forall \ h \in H \}.$

P.T $C(H)$ is a subgroup of G.

Sol:

$x, y \in C(H)$

if $(xy)h = h(xy)$

$(xy)h = x(yh) = xhy$

$= hxy.$

Closure Property

$\Rightarrow xy \in C(H).$

Identity

If $a \in C(H)$ $ah = ha$ for some $h \in H$

S.T $a^{-1} \in C(H).$

$(ah)a^{-1} = (ha)a^{-1} = h.$

Inverse

$a^{-1}h = a^{-1}(aha^{-1})$

$= ha^{-1}.$

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Defⁿ The center of a group (Z)

$$Z = \{ z \in G \mid zx = xz \ \forall\ x \in G \}$$

P.T Z is a subgroup of G.

Solⁿ Same proof as above.

Replace h with some element of G.

$\boxed{\text{P.T It is Normal}}$

obvious

$g n g^{-1}$ $g h g^{-1}$

$= n g g^{-1}$

$\in N$ $gx = xg$