4-2-17 Lec-6

Even Abstract objects can be manipulated as vectors even if we may not conceive them to be. They should satisify 8 funda. properties. Symbol: $|a\rangle$

1. Addition b/w 2 objects ~~shd~~ should exist
Notion of
$|a\rangle + |b\rangle$ shd yield a vector

2. $\alpha |a\rangle \Rightarrow$ Multiplication with complex no. ~~shd~~ should yield a vector.

3. $\exists$ a null element $|0\rangle$ such that $|a\rangle + |0\rangle = |a\rangle$

4. $\forall |a\rangle \in S$, $\exists$ an inverse vector $|a'\rangle$, such that [reverse] $|a\rangle + |a'\rangle = |0\rangle$

5. $|a\rangle + |b\rangle = |b\rangle + |a\rangle$

6. $1\,|a\rangle = |a\rangle$

7. $\alpha(\beta|a\rangle) = (\alpha\beta)|a\rangle$

8. $(\alpha+\beta)|a\rangle = \alpha|a\rangle + \beta|a\rangle$
& $\alpha(|a\rangle + |b\rangle) = \alpha|a\rangle + \alpha|b\rangle$

If a set S satisifies the above properties it forms a vector field.
If $\alpha, \beta \in R$: Real vector field.
else: complex vector field.

Eg: Set of n-dimensional coloumn complex matrices: They form a vector field.
Set of nxn complex matrices: Yes.

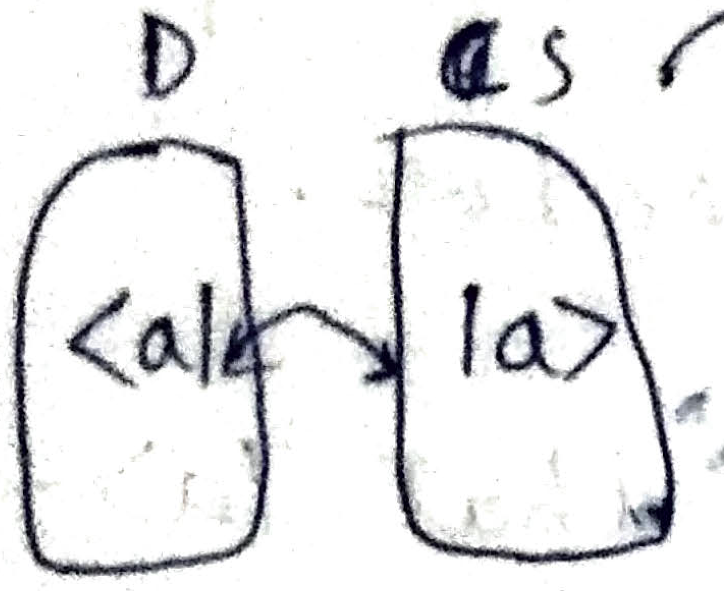
Note: $z|a\rangle = re^{i\theta}|a\rangle$

Basically, in the prev. example they form a set of ~~Matrices~~ Matrix space in the vector space.

Polynomials: form a vector space.

Set of all functions form a vector space.

Ket, Bra Sets



→ Duals of each other

One - One mapping.

for every element in D, there is ~~an image~~ a copy in S.

For scalar product:

1. $\langle b|a\rangle = \langle a|b\rangle^*$
2. $\langle c|(\alpha|a\rangle + \beta|b\rangle) = \alpha\langle c|a\rangle + \beta\langle c|b\rangle$
3. $\langle a|a\rangle \geq 0$, equality appears only if $|a\rangle = 0$.

Any ~~operation~~ vector spaces D, S which satisfies the above conditions ~~is can~~ are eligible for ~~come under~~ scalar product.

Eg: for coloumn Matrix, (Real Matrix) Generalise dot product.

D → Coloumn matrices S: Transpose of D.

Take product.

Eg: For complex coloumn matrix:

S: ————————

D: Take ~~conjug~~ Transpose of $|S$ and then take conjugate.

if $|z\rangle_S \Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \xrightarrow{X} |z\rangle_D \Rightarrow (a_1^*, a_2^*\ a_3^*) \xrightarrow{X^+}$

If: $\langle y|x\rangle = \alpha\langle y|a\rangle + \beta\langle y|b\rangle$

then $\langle x|y\rangle = \alpha^*\langle a|y\rangle + \beta^*\langle b|y\rangle$

<u>Note</u>: $\vec{A} \Rightarrow a_1, a_2, a_3$ $\vec{B} = b_1, b_2, b_3$

$\vec{A}\cdot\vec{B}$ need not only be

$a_1 b_1 + a_2 b_2 + a_3 b_3$.

<u>Linear Combination</u>

Linear Independency:

$$\sum_{i=1}^{n} \alpha_i |a_i\rangle = 0$$

$$\Rightarrow \alpha_i = 0 \ \forall i$$

Given 2 vectors, one can't be expressed as a sum of other multiplied by a constant.

<u>Lec-7</u>

In the complex vector space formed by complex nos; $\langle z_1|z_2\rangle = z_1^* z_2$ → conjugate

Cauchy Schwartz theorem

$|A| \cdot |B| \geq |A \cdot B|$

$$\text{Let } |c\rangle = |a\rangle - \underbrace{x}_{x \in R}\langle b|a\rangle |b\rangle$$

$$\langle c| = \langle a| - x \langle a|b\rangle \langle b|$$

$$\langle c|c\rangle \geq 0$$

$$\Rightarrow x^2 \langle b|a\rangle\langle a|b\rangle\langle b|b\rangle$$

$$- 2x\langle b|a\rangle\langle a|b\rangle + \langle a|a\rangle \geq 0.$$

$$(1 - x\langle b|b\rangle)^2 + \left(\frac{\langle a|a\rangle\langle b|b\rangle}{\langle a|b\rangle\langle b|a\rangle} - 1\right) \geq 0$$

If $x = \dfrac{1}{\langle b|b\rangle}$,

$$\langle a|a\rangle\langle b|b\rangle \geq \langle a|b\rangle\langle b|a\rangle$$

Generally when we speak about vectors we generally talk about distance. Is there a notion of dist in this abstract notation?

Yes; metric.

1) $p(a,b) = p(b,a)$
2) $p(a,b) = 0 \Rightarrow a, b$ are same vectors, $|a\rangle = |b\rangle$
3. $p(a,b) + p(b,c) \geq p(a,c)$ Triangle Inequality.

Good Standard: $\Delta^2_{AB} = |\vec{A} - \vec{B}|^2$

$$e(a,b)^2 = (\langle a| - \langle b|)(|a\rangle - |b\rangle)$$

In n-dimensional vector space, there are only n vectors which are independent.
You cant find one more vector which is independent of all the others.

The n-vectors form a basis;

given a basis and another vector, we can express it in terms of all others with unique constants:

<u>Orthonormal basis:</u> Vectors such that

$$\langle e_i | e_j \rangle = \delta_{ij}$$

example: $\hat{i}, \hat{j}, \hat{k}$

Given n-dimensions there will always be an orthonormal basis;

Procedure for finding the orthonormal basis:
(Gram Schmidt)

① Given $\{ |a_1\rangle, |a_2\rangle, \ldots |a_n\rangle \}$

$$|\hat{e}_1\rangle = \frac{|a_1\rangle}{\sqrt{\langle a_1 | a_1 \rangle}}$$

② Take 2nd vector; if it is $\perp$ to e_1, return e_2.

else $$|e_2\rangle = \frac{|a_2\rangle - \langle e_1 | a_2 \rangle |e_1\rangle}{N}$$

N → magnitude of ~~remainder~~ numerator

③ $$|e_3\rangle = \left(|a_3\rangle - \langle e_1 | a_3 \rangle |e_1\rangle - \langle e_2 | a_3 \rangle |e_2\rangle \right)$$

Questions

1. 3D position space.

$$\vec{A} = \hat{i} + 2\hat{j}$$
$$\vec{B} = -\hat{j} + \hat{k}$$
$$\vec{C} = \hat{i} - \hat{j} + \hat{k}$$

$$a.\hat{i} + 2a\hat{j} - bj + bk + ai - bj + ck = 0$$

$$a = 0 \quad b = 0 \quad c = 0.$$

Independent. $D = 2(\hat{j} - \hat{k})$

$$2ai + (2a - 2b)j + (b + c)k = 2\hat{j} - 2k$$

$$a = 0 \quad b = -1 \quad c = -1$$

2. Gram Schmidt of A, B, C

$$\hat{a}_1 = \frac{1}{\sqrt{5}}(\hat{i} + 2\hat{j})$$

$\cancel{(-j + k)}$ $\cancel{\hat{a}_2}$

$$-\hat{j} + k = \left(\frac{-2}{\sqrt{5}}\right)\left[\frac{1}{\sqrt{5}}(\hat{i} + 2\hat{j})\right] + \hat{a}_2$$

$$\frac{-2}{5}\hat{i} - \frac{4}{5}j + \hat{a}_2$$

$$\hat{a}_2 = \frac{2}{5}\hat{i} - \frac{1}{5}\hat{j} + \underline{\hat{k}}$$

$$\hat{a}_2 \Rightarrow \frac{1}{\cancel{\sqrt{6}}\ \sqrt{30}}(2\hat{i} - j + 5k)$$

$$\hat{a}_2 = \frac{\vec{B} - (a \cdot b)\vec{a}}{|\quad|} \quad \frac{2}{5}\hat{i} - \frac{1}{5}\hat{j} + k$$

$$(-j + k) + \left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right)(i + 2j)$$

$$\hat{e}_3 = (\hat{i} - j + k) - \frac{1}{\sqrt{5}}(-1)\left(\frac{1}{\sqrt{5}}(\hat{i} + 2\hat{j})\right)$$

$$- \frac{1}{\sqrt{30}} \cdot \frac{4}{15} \frac{(2i - j + 5k)}{\sqrt{30}}$$

$$\hat{i} - j + k + \frac{i}{5} + \frac{2}{5}j - \frac{8}{15}\hat{i} + \frac{4}{15}j - \frac{4}{3}k$$

$$\Rightarrow \frac{\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + -\frac{1}{3}\hat{k}}{\frac{1}{3}\sqrt{6}} = \frac{1}{\sqrt{6}}(2i - j - k)$$

3. Show that the foll. 4 Matrices are linearly ind:

$$M_1 = \overset{a}{\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}} \quad M_2 = \overset{b}{\begin{bmatrix} 0 & 1-5i \\ 1+5i & 0 \end{bmatrix}} \quad M_3 = \overset{c}{\begin{bmatrix} 6 & 2-4i \\ 2+4i & -6 \end{bmatrix}}$$

$$M_4 = \overset{d}{\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}}$$

$$6c + 7d = 0$$

$$4a + b - 5bi + 2c - 4ci = 0.$$

$$4a + b + 5bi + 2c + 4ci = 0.$$

$$-6c + 7d = 0.$$

$$c = 0, \; d = 0.$$

$$4a + b = 0 \quad b = 0 \quad a = 0.$$

Scalar product of a Matrix:

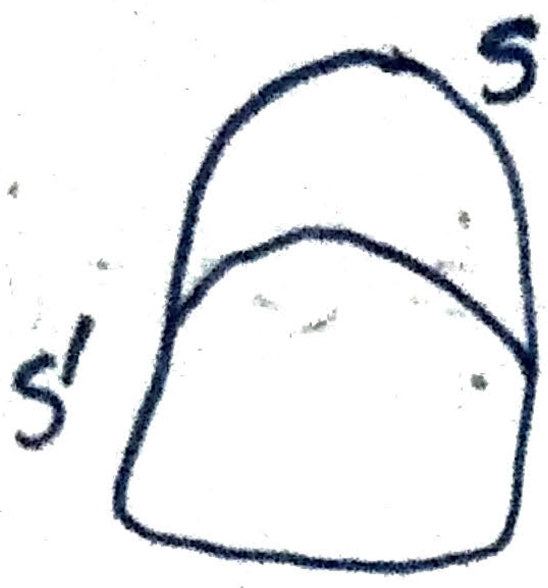
$$\langle M_i | M_j \rangle = \frac{1}{2} \text{Trace}(M_i^{+} M_j).$$

$$M_1^{+} = M_1 \; \langle M_1 | M_1 \rangle = \frac{1}{2} \text{Trace}($$

$$\hat{e}_1 = \frac{M_1}{|M_1|} = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \frac{1}{\sqrt{16}}$$

Subset of a Vector space: [Subspace]

It should be a subset of the original space and this set by itself should ~~be a~~ form a vector space.



(1) $|x\rangle, |y\rangle \in S'$, $|x\rangle + |y\rangle \in S'$

(2) $|x\rangle \in S'$, $\alpha |x\rangle \in S'$.

If S is the vector spaces and $S_1, S_2 \dots S_n$ are the subspaces of S.

if: ① $S_i \cap S_j = |0\rangle$

② $|x\rangle = \underbrace{|x_1\rangle + |x_2\rangle \dots |x_n\rangle}$

can be expressed unique way

They form the direct sum:

$$S = S_1 \oplus S_2 \oplus S_3 \oplus \dots \oplus S_n$$

$$\text{dimension}(S) = \sum_{i=1}^{n} \dim(S_i)$$

If S_1, S_2 form subspaces does $S_1 \cup S_2$ form a subspace? No.

eg: $S_1: \vec{x}$ $S_2: \vec{y}$

$S_1 \cup S_2$: Vectors along $\vec{x}, \vec{y}$ not $\vec{x} + \vec{y}$; Closure is not satisified

If at all they form a subspace, one of the foll must be true:

$S_1 \subseteq S_2$ or $S_2 \subseteq S_1$.

→ polynomials of degree $\leq n$ form a Vector space.

Even polynomials: subspace

All the Odd degree polynomials form an orthogonal space to even polynomials.

Linear Operators

If A is an operator;

$A|x\rangle = |y\rangle$ where $|x\rangle, |y\rangle$ are vectors in S.

example: $S_3 \Rightarrow P_2(x)$ $A = \frac{d}{dx}$

$\downarrow$

Polynomials with degree ≤ 2.

Properties:

1. $A = B$ iff $A|x\rangle = B|x\rangle \; \forall \; |x\rangle$
2. $(A+B)|x\rangle = A|x\rangle + B|x\rangle$
3. $(A.B)|x\rangle = A(B|x\rangle)$ | generally $AB \neq BA$
4. $A^m|x\rangle = \underbrace{A \cdot A \cdots}_{m \text{ times}} |x\rangle$

ex $P_2(x)$ $A = \frac{d}{dx}$ $B = x\frac{d^2}{dx^2}$

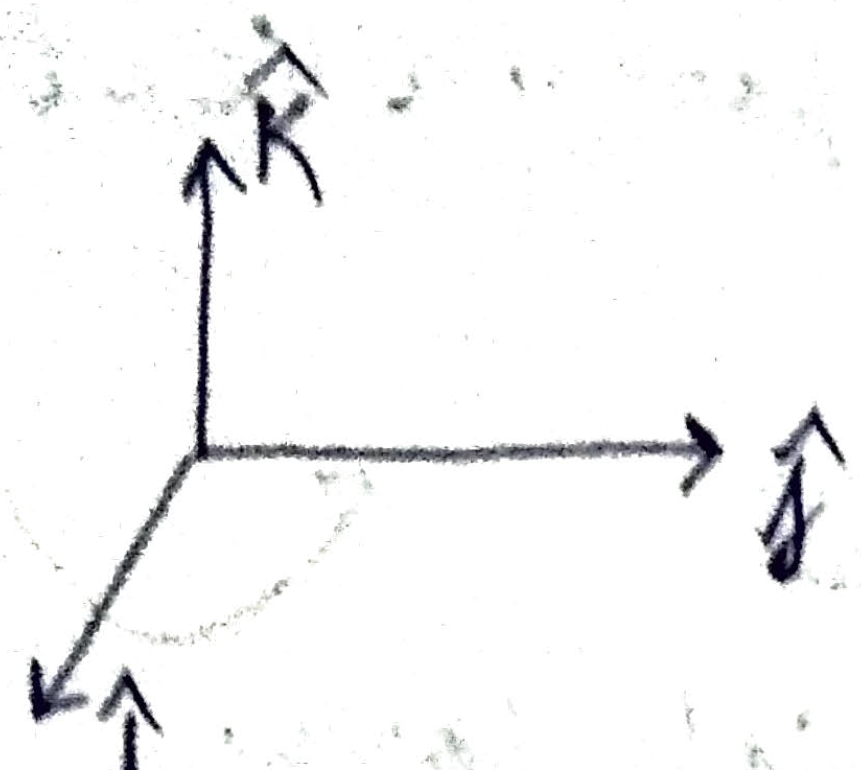
$AB = \frac{d^2}{dx^2} + x\frac{d^3}{dx^3} P(x)$

$BA = x\frac{d^3}{dx^3} P(x)$.

$$A^3 P_2(x) = 0.$$

Note: This doesn't mean ~~A^3 is null.~~

$P_2(x)$ is null.



R_1 = Rotation by $\pi/2$ about z-axis

R_2 = rotation by $\pi/2$ by x axis.

Clockwise : +ve

$$R_1 \hat{i} = \hat{j} \qquad \boxed{R_2 \hat{i} = \hat{i}}$$
$$R_1 \hat{j} = -\hat{i} \qquad R_2 \hat{j} = \hat{k}$$
$$\boxed{R_1 \hat{k} = \hat{k}} \qquad R \hat{k} = -\hat{j}$$

Invariant

$R_1 \cdot R_2 \Rightarrow$ $\hat{i} \to \hat{j}$, $\hat{j} \to \hat{k}$, $\hat{k} \to -\hat{i}$

$R_2 \cdot R_1 \Rightarrow$ $\hat{i} \to \hat{k}$, $\hat{j} \to -\hat{i}$, $\hat{k} \to -\hat{j}$ ~~Reverse order~~

$$\boxed{\text{Invariant: } \hat{i} + \hat{j} + \hat{k}}$$

Inverse of an operator:

(Right Inverse, Left Inverse).

$$A A_r^{-1} |x\rangle = \mathbb{I} |x\rangle = |x\rangle$$
$$A_l^{-1} A |x\rangle = \mathbb{I} |x\rangle = |x\rangle$$

$$A \cdot A_r^{-1} = I$$
$$A_l^{-1} A = I.$$

The left inverse cant exist in the foll ~~property~~ case:

$$\begin{cases} A|e_n\rangle = |e_{n-1}\rangle \\ A|e_1\rangle = 0 \end{cases}$$

Proof Let there be A_l^{-1} (left ~~right~~ inverse).

$$A_l^{-1} A|e_n\rangle = |e_n\rangle$$

Put $n=1$.

$A_l^{-1}(0)$ which is not equal to e_1.

In the foll case, right inverse doesn't exist:

$$A_r^{-1} |e_n\rangle = |e_{n+1}\rangle$$

$$A \cdot A_r^{-1} |e_n\rangle = |e_n\rangle.$$

If both left, right inverse exists, they are same and unique.

If left inverse is unique and it exists, right inverse exists and is same as right inverse.

In the dual space:

if in normal space $A|x\rangle = |y\rangle$

dual space: $\langle x| A^{\dagger} = \langle y|$.

We know $\langle z|x\rangle = \langle x|z\rangle^*$

then $\langle z|A^{\dagger}|x\rangle = \langle x|A|z\rangle^*$

Any Operator can be expressed as

$$A = |y\rangle\langle x|$$

suppose $A|w\rangle = |z\rangle$

$$|y\rangle\underbrace{\langle x|w\rangle}_{\text{complex Number}} = |z\rangle$$

vector along $|y\rangle$ times the complex no.

then $A^{\dagger} = |x\rangle\langle y|$

Theorm $\quad \langle b|A^{\dagger}|a\rangle = \langle a|A|b\rangle^{*}$

$$(\langle a|y\rangle^{*}\langle x|b\rangle)^{*}$$

$$\langle a|y\rangle^{*}\langle x|b\rangle^{*}$$

$$\langle y|a\rangle\langle b|x\rangle$$

reverse the order

$$\langle a|A^{\dagger}|b\rangle.$$

Unitary Matrices : $U^{\dagger}U = 1 = UU^{\dagger}$

Harmitian Matrix: $H^{\dagger} = H$

↳ Symmetric matrices in Real space.

If $|y\rangle = U|x\rangle$ then $\langle y| = \langle x|U^{\dagger}$

$$\langle y|y\rangle = \langle x|U^{\dagger}U|x\rangle$$

$$= \langle x|x\rangle$$

Unitary Operators do not change the length of a vector.

If $\{ |e_1\rangle, |e_2\rangle, \dots |e_n\rangle \}$
is an orthonormal basis,

The projection operator is:

$$P_i = |e_i\rangle\langle e_i|$$

Let $|x\rangle = x_1 |e_1\rangle + \cdots + x_n |e_n\rangle$.

$$P_1 |x\rangle = |e_1\rangle\langle e_1| (x_1 |e_1\rangle + \cdots)$$

$$\Rightarrow x_1 |e_1\rangle.$$

Identity operators: $\sum_{i=1}^{N} |e_i\rangle\langle e_i| = \mathbb{I}$

$$\boxed{P_i P_j = \delta_{ij} P_j}$$

$$\boxed{P_i^2 = P_i} \quad \boxed{P_i^3 = P_i}$$

Because of this property, Inverse of a projection doesn't exist

10th March

$$|x\rangle = \sum |e_i\rangle\langle e_i | x\rangle = \sum_1^n \langle x_i | e_i\rangle$$

$$\downarrow$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad A \to (a_{ji})$$

$$a_{ji} = \langle e_j | A | e_i\rangle$$

Eg: $f(x) = a_0 + a_1 x + a_2 x^2$

$A = \frac{d}{dx}$ $\quad |e_0\rangle = x^0 \quad |e_2\rangle$

$|e_1\rangle = x^1$

Now after finding a_{ji} foreach $i, j = (0, 1, 2)$

Operation A is
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Now check:

$$\frac{d}{dx}(5 + 5x + 3x^2) \quad [\text{Ans} : 5 + 6x]$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix}$$

Even the rotation problem, linear equations etc can be done by considering the operation as a matrix.

To check whether $AB|e_i\rangle = C|e_i\rangle$.
(product of $A \cdot B$)

Now we need to prove matrix(A) · matrix(B) = matrix(C)

$$C|e_i\rangle = \sum_{j=1}^{n} \underbrace{|e_j\rangle\langle e_j|}_{\text{identity}} \underbrace{C|e_i\rangle}_{c_{ji}}$$

$$AB|e_i\rangle = \sum_{j,k=1}^{n} |e_j\rangle \underbrace{\langle e_j|A|e_k\rangle}_{a_{jk}} \underbrace{\langle e_k|B|e_i\rangle}_{b_{ki}}$$

so $c_{ji} = \sum_k a_{jk} \cdot b_{ki}$

Eigen Vector

$A|x\rangle = \lambda|x\rangle$

operating A on $|x\rangle$ will give back

a vector in the same direction

scaled to a factor.

$|x\rangle$ is the Eigen vector of A.

$\lambda \to$ Eigen value.

$\frac{d}{dx}(e^{ikx}) = ik\, e^{ikx}$

Spectrum: $A \Rightarrow (\lambda_1, \lambda_2 \ldots \lambda_{n-1})$.

$\Rightarrow$ If $A|x_1\rangle = \lambda_1|x_1\rangle$

$A|x_2\rangle = \lambda_2|x_2\rangle$

prove that if $\lambda_1 \neq \lambda_2$ they are l·independent

$\to x_1, x_2$

if $x_1 = t x_2$

$A|x_1\rangle = \cancel{\lambda|x_1\rangle}\; A|t x_2\rangle$

$\cancel{A|t x_2\rangle = \lambda t}$

$\Rightarrow \lambda_1|x_1\rangle \Rightarrow \lambda_2 (t|x_2\rangle)$

$\Rightarrow \boxed{\lambda_1 = \lambda_2}$

$\Rightarrow$ if $A\{\lambda_1, \lambda_2 \ldots \lambda_n\}$

$A^m\{\lambda_1^m, \lambda_2^m \ldots \lambda_n^m\}$

$A|x\rangle = \lambda|x\rangle$

$A^2|x\rangle = A(A|x\rangle)$

$\Rightarrow A(\lambda|x\rangle)$

$\Rightarrow \lambda^2|x\rangle$.

and so on.

$\Rightarrow$ Eigen values of harmitian operator : Real

If $\lambda_1 \neq \lambda_2$; x_1, x_2 are $\perp$.

$H|x_1\rangle = \lambda_1 |x_1\rangle$

$\langle x_1| H^{\dagger} = \lambda_1^* \langle x_1|$

$\Rightarrow \langle x_1|H = \lambda_1^* \langle x_1|$

$(\langle x_1|H)(H|x_1\rangle) \Rightarrow \cancel{\lambda_1^* \langle x_1|x_1\rangle}$

$\langle x_1|H|x_1\rangle = \langle x_1|H^{\dagger}|x_1\rangle$

$\hookrightarrow A = A^T$

$\Rightarrow$ real

$\cancel{\langle \lambda_1 x_1}$

$\langle x_1|\underbrace{H^{\dagger}|H}|x_2\rangle = \langle \underbrace{\lambda_1^* x_1}|\underbrace{\lambda_2 x_2}\rangle$

independent

0