Eg: P.T. Kernel is Normal Subgrp.

~~Closure~~:

$$\phi(g x g^{-1}) = \bar{e}$$

we need to prove the above

$$\phi(x) = \bar{e} \quad [x \in K]$$

$$\phi(x) \cdot \phi(gg^{-1}) = \bar{e}$$

$$[\phi(e) = \bar{e}]$$

$$\phi(gxg^{-1}) = \bar{e}$$

<u>Mid 2- End</u>

Automorphism is a homomorphism of a group onto itself.

If G is a ~~is~~ homorphiasm to $\bar{G}$, then $\bar{G}$ is also a hoamomorphism to G.

$$G \simeq \bar{G}$$

$$\Rightarrow \bar{G} \simeq G.$$

$$\phi : G \to \bar{G}$$

then $\phi^{-1} : \bar{G} \to G$.

If $h_1, h_2 \in \bar{G}$

$$\phi^{-1}(h_1 h_2) = \phi^{-1}(\phi(h_1') \cdot \phi(h_2'))$$

$$\Rightarrow \phi^{-1}(\phi(h_1' h_2'))$$

$$\Rightarrow h_1' h_2'$$

$$\Rightarrow \phi^{-1}(h_1) \cdot \phi^{-1}(h_2).$$

so if G is homomorphic to $\bar{G}$, $\bar{G}$ is also homomorphic to G. That is the reason why we assumed that there exists some element in G which is satisfies

↳ Onto

$$\phi(g) = h'$$

$A(S) = S_3$ Symmetric Group.

$A(G) \Rightarrow$ group of Automorphisms of G onto itself

To prove: $A(G)$ is a subgroup of $A(S)$.

Consider 2 elements $T_1, T_2 \in A(G)$.

$$T_1(xy) = (T_1 x)(T_1 y)$$
$$T_2(xy) = (T_2 x)(T_2 y)$$

$$T_1 T_2 (xy) = T_1 (T_2 x)(T_2 y)$$
$$\Rightarrow (T_1 T_2 x)(T_1 T_2 y)$$

If $T_1, T_2 \in A(G)$

Closure property.

If $T \in A(G)$

$T^{-1} \in A(G)$?

T^{-1} exists as $T \in A(S) \supset A(G)$

T is 1-1 & onto.

$$T(T^{-1}x)(T^{-1}y) = (TT^{-1})x\,(TT^{-1})y$$
$$\Rightarrow xy$$

$$T^{-1}(x)\,T^{-1}(y) = T^{-1}(xy).$$

We have proved $A(G)$ is a subgrp of $A(S)$

Now does this contain only the identity element?

Let $x_0 \in G$

G is Abelian

$$x_0 = x_0^{-1}$$

Choose a mapping T: $T(x_0) = x_0^{-1}$

where $T \neq I$.

$$T(xy) = (xy)^{-1}$$
$$= y^{-1}x^{-1}$$
$$= x^{-1}y^{-1}$$
$$= T(x)\,T(y)$$

Non Abelian:

$g \in G$, $T_g : G \to G$

$T_g x = g^{-1} x g \quad \forall x \in G$.

$x, y \in G$

$T_g x = g^{-1} x g$

T_g is onto

given $y \in G$

Let $x = g y g^{-1}$

$T_g(x) = g^{-1} x g$

$\Rightarrow g^{-1}(g y g^{-1}) g$

$\Rightarrow y$

Every y has a pre-image x.

$$T_g(x) = T_g(y)$$

$$g^{-1}xg = g^{-1}yg$$

$$\Rightarrow x = y$$

Let $x, y \in G$

$$T_g(xy) = g^{-1}(xy)g$$

$$\Rightarrow (g^{-1}xg)(g^{-1}yg) = (T_g x)(T_g y)$$

The group of such mappings T_g : Inner Automorphism :

$$T_g, T_h \in I(G)$$

$$xT_gT_h = (g^{-1}xg)T_h$$

$$= h^{-1}(g^{-1}xg)h$$

$$\Rightarrow (gh)^{-1}x(gh)$$

$$\Rightarrow xT_{gh}$$

Let there be a mapping ψ from

$$G \longrightarrow A(G).$$

$$\psi(g) = T_g$$

ψ is homomorphism :

$$\psi(gh) = T_{gh} = T_gT_h = \psi(g)\psi(h)$$

$\Rightarrow$ Kernel of ψ

$$= \{g \in G \mid \psi(g) = I\}$$

$\psi(g) = T_g = I$.

$T_g(x) = g^{-1}xg = \hat{I}x = x$.

$xg = gx$

$\therefore$ Kernel is the Center of the grp.

$$\begin{array}{ccc} \psi(G) & \rightarrow & I(G) \\ | & \diagup & \\ G/Z & & \end{array}$$

$\frac{G}{Z} \simeq I(G)$.

Eg: Integers ; $Tx = -x$. Is this Automorphism

$\pm$ Reals ; $Tx = x^2$ Is this Automorphism

Permutation group

S: set of mappings of n elements onto themselves.

A(S) ⇒ { group of all 1-1 mappings of S onto S }

S_3 : $x_1 \to x_2$
$x_2 \to x_1$
$x_3 \to x_3$

$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_1 & x_3 \end{pmatrix} \to$ Permutation

$\theta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$: Permutation

Let $\theta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$ $\psi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$

$\theta\psi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$

$\psi\theta \Rightarrow \theta(\psi(1)) = \theta(1) = 3$
$\theta\psi = \psi(\theta(1)) = \psi(3) = 2$

$\theta^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$

$\theta^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

$\boxed{\theta^3 = I}$

n elements : after n iterations, we get back θ.
but here $(4 \to 4)$ is redundant so θ^3.

Let S be a set

$\theta \in A(S)$

given 2 elements $a, b \in S$

$a \equiv_\theta b$: if $b = a\theta^i$

[Read as a is congruent to b mod θ]

($i \Rightarrow$ +ve or -ve or 0 $\Rightarrow$ Integer)

We can reach b from a ~~the~~ by iterating through θ i times.

$$\psi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

$$1 \equiv_\psi 3$$

as $3 = (1)\psi^2$

$\psi(1) = 2 \quad \psi^2(1) = \psi(2) = 3$.

Theorm : The above relation is an equivalence relation.

Reflexive: a should be congruent to a.

$a \equiv_\theta a \quad$ if $a = a\theta^i$

$a = a\theta^0$

$\therefore$ Reflexive

$f(x) = x$: Identity if $f(x)$ is defined $\forall x$.

else: fixed pt. mapping

Symmetric: if $a \equiv_\theta b$,

$a = b\theta^i$

$\because \theta \in$ group of mappings, inverse exists.

$\Rightarrow b = a\underbrace{\theta^{-i}}$

Inverse mapping : i times

Transitive: $a \equiv_\theta b$, $b \equiv_\theta c$

$b = a\theta^i$ $\quad$ $c = b\theta^j$

$\Rightarrow (a\theta^i)\theta^j$

$\Rightarrow a\theta^{i+j}$

$\Rightarrow a \equiv_\theta c$

<u>Orbit</u>

$$\theta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 6 & 4 \end{pmatrix}$$

Orbit of 1 : $\underbrace{(1, 2)}_{\text{cycle}}$

Orbit of 3 : (3)

Orbit of 4 : $(4\,5\,6)$.

Eg: ~~O~~ given an Orbit; find the permutation

$(1\,2\,3)$

$$C_{1_\theta}: \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 1 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}$$

$C_2 : (5\ 6\ 4\ 1\ 8)$

$$C_{2\theta}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 2 & 3 & 1 & 6 & 4 & 7 & 5 & 9 \end{pmatrix}$$

Find the permutation which arises from C_1 followed by C_2.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 8 & 1 & 6 & 4 & 7 & 5 & 9 \end{pmatrix} [C_{1\theta} C_{2\theta}]$$

(Single cycle corresponding to both cycle).

Every permutation can be represented as a product of '2-cycles'.

An ordered pair is a 2 cycle.

2-cycles = Transpositions

$(1\,2\,3) = (1\,2)(1\,3)$

$1 \to 2 \to 3 \to 1$

$1 \to 2 \to 1 \to 3 \to 1$

2 is going to 3 via 3

m cycle : $(1, 2 \ldots m) = (1\,2)(1\,3)\ldots(1\,m)$

This is not unique:

$(1\,2\,3) = (1\,2)(1\,3)$

$= (3\,1)(3\,2)$

$3 \to 1 \to 3 \to 2 \to 3$

Eg: Find the Orbits and Cycles of following permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 1 & 6 & 7 & 9 & 8 \end{pmatrix}$$

Orbit (1) = (1 2 3 4 5)

Orbit (8) = (8 9).

Orbit (6, 7) $\rightarrow$ (6)(7)

Cycles: (1 2 3 4 5) (6) (7) (8 9)

$\Rightarrow$ Disjoint

Eg: Express the ~~above~~ below permutation as product of disjoint cycles:

Final Ans

$\theta = \underbrace{(1\,5)}_{T_1}\ \underbrace{(1\,6\,7\,8\,9)}_{T_2}\ \underbrace{(4\,5)}_{T_3}\ \underbrace{(1\,2\,3)}_{T_4}$

T =

$T(1) = T_1 T_2 T_3 T_4(1)$

$\Rightarrow T_1 T_2 T_3 (2) \Rightarrow T_1 T_2 (2) = 2$

$T(2) = 3 \quad T(3) = 6 \quad T(4) = 1 \quad T(5) = 4$

$T(6) = 7 \quad T(7) = 8 \quad T(8) = 9 \quad T(9) = 1$

$$T: \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 6 & 1 & 4 & 7 & 8 & 9 & 1 \end{pmatrix}$$

Eg: $\underbrace{(1\,2)}_{T_1}\ \underbrace{(1\,2\,3)}_{T_2}\ \underbrace{(1\,2)}_{T_3}$

T =

$T(1) = 3 \quad T(2) = 1 \quad T(3) = 2$

(1 3 2).

Eg: Find the cycle structure of all powers of T: (1 2 3 4 5 6 7 8).

$T^2 = T \cdot T$

$T^2(1) = 3$	$T^2(2) = 4$
$T^2(3) = 5$	$T^2(4) = 6$
$T^2(5) = 7$	$T^2(6) = 8$
$T^2(7) = 1$	$T^2(8) = 2$

$\Rightarrow (1\,3\,5\,7)\,(2\,4\,6\,8)$

$T^3 = T \cdot T \cdot T$

$(4\,7\,2)\,(5\,8\,3\,6\,1)$

- Even Permutations: Even nos. of $2's$ cycles in representation

To see the effect of Odd permutation:

Define P $(x_1 x_2 x_3 x_4 x_5) = \prod_{i<j} (x_i - x_j)$

Let $\Rightarrow \Theta (1\,3\,4)(2\,5)$

$\Theta = (1\,3)(1\,4)(2\,5)$

Apply Θ on P;

$P = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)$
$(x_2 - x_3)(x_2 - x_4)(x_2 - x_5)(x_3 - x_4)$
$(x_3 - x_5)(x_4 - x_5)$

If we permute over the indices,

$\theta = (1\ 3\ 4)(2\ 5)$

$\theta(P)$ can be written as:

$$(x_3 - x_5)(x_3 - x_4)(x_3 - x_1)(x_3 - x_2)$$
$$(x_5 - x_4)(x_5 - x_1)(x_5 - x_2)(x_4 - x_1)$$
$$(x_4 - x_2)(x_1 - x_2).$$

Odd permutation ~~can~~ changes the sign of the function.

Even perm. · Even perm ⇒ Even permutation

Even . Odd = Odd

Alternating group:

S_n is a group of all mappings of n elements onto themselves.

[Each element is an permutation].

A_n: Set of even permutation

A_n is a subgroup of S_n:

Closure: Even . Even = Even

Inverse: Inverse of an permutation exists and is also even.

No. of elements in A_n: $\frac{n!}{2}$

define ψ : $\psi(S_n) = W = \{1, -1\}$

$$\psi(s) = 1 \quad s \text{ is even}$$
$$\quad\;\; = -1 \quad s \text{ is odd}$$

Is ψ homomorphic ?

E-E : $\psi(\underbrace{s_1 s_2}_{\substack{\text{Even}\\ \downarrow \\ 1}}) = 1 = \underset{\substack{\downarrow\\1}}{\psi(s_1)} \cdot \underset{\substack{\downarrow\\1}}{\psi(s_2)}$

E-O : $\psi(\underset{\substack{\downarrow\\ \text{odd}\\ \downarrow\\ -1}}{s_1 s_2}) = -1 = \underset{1}{\psi(s_1)} \cdot \underset{-1}{\psi(s_2)}$

O-O : $\psi(\underbrace{s_1 s_2}_{\substack{\text{Even}\\ 1}}) = \underset{\substack{\downarrow\\-1}}{\psi(s_1)} \cdot \underset{\substack{\downarrow\\-1}}{\psi(s_2)}$

$$\begin{tikzcd}
S_n \arrow[r, "\psi"] \arrow[d] & W \\
\frac{S_n}{A_n} \arrow[ur, "\sim"] &
\end{tikzcd}$$

One-One
Onto

so $O(W) = O\left(\frac{S_n}{A_n}\right)$.

Kernel of S_n under ψ :

$$\{ s \mid \psi(s) = 1 \}$$

$\hookrightarrow$ Even permutations

$$\therefore O\left(\frac{S_n}{A_n}\right) = 2$$

$$\Rightarrow O(A_n) = \frac{n!}{2}$$

~~Eg: Compute~~

Inverse of a permutation : $s\begin{pmatrix} a_1 & a_2 \cdots & a_n \\ b_1 & b_2 \cdots & b_n \end{pmatrix}$

$In: \begin{pmatrix} b_1 & b_2 \cdots & b_n \\ a_1 & a_2 \cdots & a_n \end{pmatrix}$ $\quad s.I_n = \begin{pmatrix} a_1 & a_2 \cdots & a_n \\ a_1 & a_2 \cdots & a_n \end{pmatrix}$

$\rightarrow$ Identity

Eg: Find out $a^{-1} b a$

$a = (1\ 3\ 5)(1\ 2)$

$b = (1\ 5\ 7\ 9)$

$a = \cancel{(3\ 5\ 2)} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix}$

$\Rightarrow (1\ 3\ 5\ 2)$

$a^{-1} = (2\ 5\ 3\ 1)$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix}$

$b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 2 & 3 & 4 & 7 & 6 & 9 & 8 & 1 \end{pmatrix}$

$a^{-1} b a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 7 & 2 & 4 & 5 & 6 & 9 & 8 & 3 \end{pmatrix}$

$\Rightarrow (2\ 7\ 9\ 3)$

Eg: P.T. the smallest subgroup of S_n containing $(1, 2)$ and $(1, 2, \cdots n)$ is S_n.

Or in other words these cycles generate S_n.

We need to show, every transposition can be generated from the product of these cycles

$(n\ n{-}1\ \dots\ 2\ 1)(1\ 2)(1\ 2\ 3\ \dots\ n) = (2, 3)$

$(n\ n{-}1\ \dots\ 3\ 2\ 1)(2\ 3)(1\ 2\ 3\ 4\ \dots\ n) = (3, 4)$

$\vdots \qquad\qquad = (4, 5)$

$\vdots \qquad\qquad \vdots$

In this manner, I generated 2 cycles of the form $(i, i+1)$

So we got one type of 2-cycles

$(1\ 2)(2\ 3)(1\ 2)$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$= (1\ 3)$.

$(1\ 3)(3\ 4)(1\ 3)$

$\Rightarrow (1\ 4)$

$(1\ 4)(4\ 5)(1\ 4)$

$= (1\ 5)$

$(1\ n{-}1)(n{-}1, n)(1, n{-}1)$

$= (1, n)$.

$\Rightarrow (1, i)$ is done

$(1, i)(1, j)(i, 1)$

$\Rightarrow (i, j)$.

If $a, b \in G$, b is said to be conjugate to a if $\exists$ an element c such that

$$b = c^{-1} a c$$

$$b \sim a$$

Conjugacy class:
$C_a \Rightarrow \{c \mid a \sim c\}$

This is an equivalence relation.

Reflexive: $a = a^{-1} a a$

Symmetric: $b = x^{-1} a x$

$$\Rightarrow (x^{-1})^{-1} b = a x$$

$$\underbrace{(x^{-1})^{-1} b x^{-1}}_{\text{These belong to the group.}} = a$$

Symmetric

Transitive: $a \sim b$, $b \sim c$

$$a = x^{-1} b x$$

$$b = y^{-1} c y$$

$$a = x^{-1} y^{-1} c y x$$

$$\Rightarrow (yx)^{-1} c (yx)$$

Transitive.

If C_a is the no. of elements in Class of a;

Total Number of elements in G : $\sum_{a \in G} C_a$

dont Count Classes which have already come.

Normalizer of:

① An element: $N(a) = \{x \in G \mid xa = ax\}$

② A subgroup: $N(H) = \{x \in G \mid xh = hx \ \forall h \in H\}$

↳ Centralizer

Center $N(G) = \{x \in G \mid xg = gx \ \forall g \in G\}$

P.T. $N(a)$ is a subgroup.

Let $x, y \in N(a)$.

$ax = xa$, $ay = ya$

Closure: (xy) should belong to $N(a)$.

$axy = xya$.

$\Rightarrow (ax)y$

$\Rightarrow x(ay)$

$\Rightarrow xya$

Inverse: $x^{-1}a = ax^{-1}$

$xa = ax$

$a = x^{-1}ax$

$ax^{-1} = x^{-1}a$

⇒ The cosets formed by the Normalizer form the conjugate class of an element

If G is a finite group.

$$C_a = \frac{O(G)}{O(N(a))}$$

<u>Method</u> Pick 2 elements in the same right coset of $N(a)$ and show that they $\in C_a$.

Let $x, y \in$ same right coset of $N(a)$

$y = nx$ where $n \in N(a)$.

$na = an$.

$y^{-1} = (nx)^{-1} = x^{-1}n^{-1}$.

$y^{-1}ay = x^{-1}n^{-1}a\,nx$

$\Rightarrow x^{-1}n^{-1}\,nax$

$\Rightarrow x^{-1}ax$

$\Rightarrow x, y \in C(a)$

<u>Claim-2</u> On the other hand if x, y are in diff right cosets of $N(a)$.

$(x^{-1}ax) \neq (y^{-1}ay)$

Suppose not:

$x^{-1}ax = y^{-1}ay$

$yx^{-1}axy^{-1} = a$

$\Rightarrow yx^{-1} \in N(a)$.

$\Rightarrow$ contradiction.

Eg: S_3 : $(12), (23), (1,3),$
$(123), (132), (1)(2)(3).$

Find cong class of (12).

$C(1,2) = b\ \exists\ (x)^{-1}(12)(x)$
$= (12)$

Let $x = (12)$
$(12)^{-1}(12)(12) \Rightarrow (12).$