8) If 2 vectors are linearly dependent, then show that one is scalar multiple of the other

$\bar{\alpha}, \bar{\beta} \in V(F) \Rightarrow L.D \Rightarrow a\bar{\alpha} + b\bar{\beta} = \bar{\phi}$ $(a, b \in F)$

Atleast one of them $\neq 0$.

$\therefore \quad a\bar{\alpha} = -b\bar{\beta}$

If $a \neq 0$, $\bar{\alpha} = -(a^{-1}b)\,\bar{\beta}$.

Multiplicative inverse.

$\therefore \quad \bar{\alpha} = c_1 \bar{\beta}$

Similarly if $\bar{\beta} \neq 0$, $\bar{\beta} = -(b^{-1}a)\,\bar{\alpha}$.

11.9.18.

9) Show that any superset of a linearly dependent set of vectors is also linearly dependent.

Let $V(F)$ be a vector space over the field F.

Let $\bar{\alpha}_i \in V(F)$ and the set $X = \{\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \dots, \bar{\alpha}_k\}$ are linearly dependent.

$\Rightarrow \exists$ scalars $c_1, c_2, \dots, c_k \in F$ such that $\sum_{i=1}^{k} c_i \bar{\alpha}_i = \phi$, — (1)

where not all c_i's are zero.

Now, consider a superset Y of $X = \{\bar{\alpha}_1, \dots, \bar{\alpha}_k, \bar{\alpha}_{k+1}, \dots, \bar{\alpha}_n\}$.

Now, $\underbrace{c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \dots + c_k\bar{\alpha}_k}_{\phi \text{ (from (1))}} + \underbrace{0\bar{\alpha}_{k+1} + \dots + 0\bar{\alpha}_n}_{} = \phi$

$[0\bar{\alpha} = \phi, \phi + \phi = \phi]$.

~~Abhijith~~ is made to do servant work. / /

$\therefore$ $\gamma = \{\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_m\}$ are L.D. [Hence, proved].

Q) The set of non-zero vectors $\{\bar{\alpha}_1, \bar{\alpha}_2, \dots \alpha_k\}$ is
★★ L.D iff one of them, say $\bar{\alpha}_r$, $2 \leq r \leq k$ is the linear combination of the preceding vector.

~~Let $\{\alpha_1, \alpha_2, \dots \alpha_k\}$ be L.D.~~

Let the set of vectors be L.I.

$\Rightarrow$ $\exists$ scalars $c_i \in F$, not all zero, such that

$$\sum_{i=1}^{k} c_i \alpha_i = \phi \quad \text{—(1)}$$

If c_r be the last non-zero coefficient.

If $r = 1 \Rightarrow c_1\alpha_1 = \phi$

$\boxed{\alpha_1 = \phi}$ CONTRADICTION.

[$\because$ α_i's are non-zero set of vectors].

Hence, $r > 1$.

Now, consider $c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \dots + c_r\bar{\alpha}_r = \phi$.

$\because c_r \neq 0$, c_r^{-1} exists.

$\Rightarrow c_r\alpha_r = -c_1\bar{\alpha}_1 - c_2\bar{\alpha}_2 - \dots - c_{r-1}\bar{\alpha}_{r-1}$.

$$\alpha_r = -(c_1c_r^{-1})\alpha_1 - (c_2c_r^{-1})\alpha_2 - \dots - (c_{r-1}c_r^{-1})\alpha_{r-1}.$$

$\{c_1c_r^{-1}, c_2c_r^{-1}, \dots, c_{r-1}c_r^{-1}\} \in F$.

$$\bar{\alpha}_r = c_1\bar{\alpha}_1 + \cdots + c_{r-1}\bar{\alpha}_{r-1} \quad (2 \le r \le k).$$

$\Rightarrow$ $c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \cdots + c_{r-1}\bar{\alpha}_{r-1} + (-1)\bar{\alpha}_r = \phi$

$\Rightarrow$ $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_r\} \to$ L.D.

and $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_k\}$ is a superset of this.

$\Rightarrow$ $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n\}$ is also linearly independent.

<u>Theorem</u> If the vector space $V(F)$ is spanned by a L.D. set $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_k\}$ then V can be generated by a proper subset of $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_k\}$.

Let $X = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_k\}$ is linearly dependent
$\Rightarrow$ one of the vectors is a L.C. of the others.

Let $\bar{\alpha}_r = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \cdots + c_{r-1}\bar{\alpha}_{r-1} + c_{r+1}\bar{\alpha}_{r+1} + \cdots + c_k\bar{\alpha}_k$, $c_1, c_2 \ldots c_k \in F$. —— (1)

Let $\beta \in V$.
$\because$ V is spanned by $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_k\}$.
$\Rightarrow \beta = b_1\bar{\alpha}_1 + \cdots + b_k\bar{\alpha}_k$, $b_1, \ldots, b_k \in F$

Now replace α_r by ①, we get,

$$\beta = b_1\bar{\alpha}_1 + b_2\bar{\alpha}_2 + \cdots + b_{r-1}\bar{\alpha}_{r-1} + b_r(c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \cdots + c_{r-1}\bar{\alpha}_{r-1} + c_{r+1}\bar{\alpha}_{r+1} + \cdots + c_k\bar{\alpha}_k) + b_{r+1}\bar{\alpha}_{r+1} + \cdots + b_k\bar{\alpha}_k .$$

$$= (b_1 + b_r c_1)\bar{\alpha}_1 + (b_2 + b_r c_2)\bar{\alpha}_2 + \cdots + (b_{r-1} + b_r c_{r-1})\bar{\alpha}_{r-1} + (b_{r+1} + c_{r+1} b_r)\bar{\alpha}_{r+1} + \cdots + (b_k + c_k b_r)\bar{\alpha}_k ,$$

Coeff. of the vectors $\in F$.

$\Rightarrow$ β is spanned by $Y = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{r-1}, \bar{\alpha}_{r+1}, \ldots, \bar{\alpha}_k\}$

$\Rightarrow$ $Y \subset X$.

Hence, V can be spanned by Y (a proper subset of X).

We can keep reducing the set as long as it is LD.

Defn

Basis

A subset S of a vector space $V(F)$ is said to be basis if :-

(i) ~~…~~ S is LI.

(ii) Any vector in V other than that in S can be expressed as a L.C. of vectors of S.

Defⁿ :- Dimension (rank) : The number of elements in the basis of the vector space $V(F)$ is called dimension and denoted as $\dim(V)$.

Finite dimensional vector space

$V_n(F) \Rightarrow$ Vector space over F, n is no. of elements in basis of $V(F)$.

Q) $(\mathbb{R}^3, \oplus) \to (\mathbb{R}, +, \cdot)$
where $(a,b,c) \oplus (d,e,f) = (a+d, b+e, c+f)$.

Show that $\alpha = \{(1,0,0), (0,1,0), (0,0,1)\}$ forms basis of $\mathbb{R}^3$

(i) → show it is LI.

$c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$.

$$\Rightarrow \left.\begin{aligned} c_1 + c_2(0) + c_3(0) &= 0 \\ c_1(0) + c_2 + c_3(0) &= 0 \\ c_1(0) + c_2(0) + c_3 &= 0 \end{aligned}\right\} \Rightarrow c_1 = c_2 = c_3 = 0.$$

$\Rightarrow$ Clearly X is L.I.

(ii) → take an arbitrary vector $\alpha \in V_3(F)$, show α can be expressed as a linear combination of $\{(1,0,0), (0,1,0), (0,0,1)\}$.

Basis of a Vector Space isn't unique.
Dimension of basis is unique. __/__/____

Let $\alpha = (a,b,c) \in V_3(F)$.
clearly, $a(1,0,0) + b(0,1,0) + c(0,0,1) = (a(1), b(1), c(1))$
$= (a,b,c)$.
[Hence (ii) Proved]

Q) Find another basis for R^3.

$(2,0,0), (0,2,0), (0,0,2)$ [Parallel to those].

Q) In R^4, does $(1,1,1,1), (0,1,1,1), (0,0,1,1), (0,0,0,1)$ form a basis?

(i) LI

$c_1(1,1,1,1) + c_2(0,1,1,1) + c_3(0,0,1,1) + c_4(0,0,0,1) = (0,0,0,0)$.

$\Rightarrow c_1(1) + c_2(0) + c_3(0) + c_4(0) = 0$
$\Rightarrow c_1 = 0$ — (1)

Similarly, $c_1 = c_2 = c_3 = c_4 = 0$. $\Rightarrow$ LI.

(ii) Spans.
Let $(a,b,c,d) \in \mathbb{R}^4$.
$\therefore$ $a(1,1,1,1) + (b-a)(0,1,1,1) + (c-b)(0,0,1,1) + (d-c)(0,0,0,1)$.
$= (a,b,c,d)$
$\therefore$ Basis [Proved]

Theorem Given a basis, vector, there is only 1 way of expressing the vector in terms of the basis.
$\Rightarrow$ In a vector space $V_n(F)$ with the basis set $B = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$, every vector $\bar{\alpha} \in V$ is uniquely expressed as the LC of vectors in B.

Proof:- $\sum c_i\alpha_i = \sum d_i\alpha_i \Rightarrow (c_i - d_i)\alpha_i = 0 \Rightarrow c_i - d_i = 0 \Rightarrow c_i = d_i$

Theore

Proof

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Theorem:- If $\{\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_n\}$ is a basis of $V_n(F)$ and $\bar{\beta}$ is a non-zero vector belonging to V, then $\bar{\beta} = \sum_{i=1}^{n} c_i \bar{\alpha}_i$ (where $c_i \in F$ and all c_i's are not zero).

If $c_j \neq 0$, then show that $\{\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_{j-1}, \bar{\beta}, \bar{\alpha}_{j+1}, \dots, \bar{\alpha}_n\}$ is a basis of $V_n(F)$.

Proof:-

$$\bar{\beta} = \sum_{i=1}^{n} c_i \bar{\alpha}_i$$

Let $c_j \neq 0$ for some j.

$$\therefore \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i \bar{\alpha}_i + c_j \bar{\alpha}_j - \bar{\beta} = \bar{\phi}$$

or,
$$\sum_{\substack{i=1 \\ i \neq j}}^{n} c_i \bar{\alpha}_i - \bar{\beta} = -c_j \bar{\alpha}_j$$

or,
$$-\sum_{\substack{i=1 \\ i \neq j}}^{n} c_j^{-1} c_i \bar{\alpha}_i + c_j^{-1} \bar{\beta} = \bar{\alpha}_j \quad \text{--- (1)} \quad [c_j^{-1} \text{ exists because } c_j \neq 0]$$

To prove:- $S = \{\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_{j-1}, \bar{\beta}, \alpha_{j+1}, \dots, \bar{\alpha}_n\}$ is a basis of $V_n(F)$.

↳ S is L.I

↳ S spans $V_n(F)$.

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① Let $\sum_{\substack{i=1 \\ i\neq j}}^{n} \delta_i \bar{\alpha_i} + \delta_j \bar{\beta} = \bar{\phi}$ $[\delta_i, \delta_j \in F]$.

$\Rightarrow \sum_{\substack{i=1 \\ i\neq j}}^{n} \delta_i \bar{\alpha_i} + \delta_j \left(\sum_{i=1}^{n} c_i \bar{\alpha_i} \right) = \bar{\phi}$.

$\Rightarrow \sum_{\substack{i=1 \\ i\neq j}}^{n} (\delta_i + c_i \delta_j) \bar{\alpha_i} + \delta_j c_j \bar{\alpha_j} = \phi$

$\Rightarrow \delta_i + c_i \delta_j = 0 \quad \forall i$ [Since $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n\}$ is L.I.]

$\delta_j c_j = 0$

Since $c_j \neq 0 \Rightarrow \delta_j = 0 \Rightarrow \delta_i = 0$.

$\therefore$ S is L.I.

② Let $\bar{v} \in V$ be any arbitrary vector in V.

$\therefore \bar{v} = \sum_{i=1}^{n} d_i \bar{\alpha_i}$ [because $\{\alpha_1, \ldots, \alpha_n\}$ is a basis of V]

Now, from ① $\bar{\alpha_j} = c_j^{-1} \bar{\beta} - \sum_{\substack{i=1 \\ i\neq j}}^{n} c_j^{-1} c_i \bar{\alpha_i}$

$\therefore \bar{v} = \sum_{\substack{i=1 \\ i\neq j}}^{n} d_i \bar{\alpha_i} + d_j \left(c_j^{-1} \bar{\beta} - \sum_{\substack{i=1 \\ i\neq j}}^{n} c_j^{-1} c_i \bar{\alpha_i} \right)$

or, $\bar{V} = \sum_{\substack{i=1\\ i\neq j}}^{n} d_i\bar{\alpha}_i + d_j c_j^{-1}\bar{\beta} - \sum_{\substack{i=1\\ i\neq j}}^{n} d_j c_j^{-1} c_i \bar{\alpha}_i$.

$$\bar{V} = \sum_{\substack{i=1\\ i\neq j}}^{n} \left(d_i - d_j c_j^{-1} c_i\right)\bar{\alpha}_i + d_j c_j^{-1}\bar{\beta}.$$

Since it can be represented as an LC of S.

$\therefore$ S spans $V_n(F)$.

Q) Show that the set $\{\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3\}$ is L.D. if the set $\{\bar{\alpha}_1 + a\bar{\alpha}_2 + b\bar{\alpha}_3, \bar{\alpha}_2, \bar{\alpha}_3\}$ is L.D. where $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3 \in V_n(F)$ and $a, b \in F$.

Given:- If $c_1(\bar{\alpha}_1 + a\bar{\alpha}_2 + b\bar{\alpha}_3) + c_2\bar{\alpha}_2 + c_3\bar{\alpha}_3 = \bar{\phi}$ — (1)

Then some $c_i \neq 0$.

(1) $\Rightarrow c_1\bar{\alpha}_1 + (c_1 a + c_2)\bar{\alpha}_2 + (c_1 b + c_3)\bar{\alpha}_3 = \bar{\phi}$.

~~Since $c_i \neq 0$ for some i $\Rightarrow$ $c_1 \neq 0$ or $c_1 a + c_2 \neq 0$ or $c_1 b + c_3 \neq 0$~~

~~$\therefore$ $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3$ is L.D.~~

If $c_1 \neq 0 \Rightarrow$ Proved.

Else if $c_1 = 0 \Rightarrow c_2\bar{\alpha}_2 + c_3\bar{\alpha}_3 = 0$ [Atleast one of c_2, c_3 has to be non-zero]

If $c_2 \neq 0 \Rightarrow c_1 a + c_2 \neq 0$

If $c_3 \neq 0 \Rightarrow c_1 b + c_3 \neq 0$

$\therefore$ Proved.

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Linear Sum of two subspaces

If W_1 and W_2 are two subspaces of $V(F)$, then the linear sum of two subspaces, denoted by W_1+W_2, is the set of sums ~~such that~~ $\bar{\alpha}_1+\bar{\alpha}_2$ such that $\bar{\alpha}_1 \in W_1$, $\bar{\alpha}_2 \in W_2$.

$$W_1+W_2 = \{\bar{\alpha}_1+\bar{\alpha}_2,\ \bar{\alpha}_1 \in W_1,\ \bar{\alpha}_2 \in W_2\}.$$

Q) If W_1 and W_2 are subspaces of $V(F)$, then show that W_1+W_2 is a subspace of V.

$W_1+W_2 \subset V$ trivially.

Now, let $\gamma_1, \gamma_2 \in W_1+W_2$

$\therefore \gamma_1 = \bar{\alpha}_1+\bar{\alpha}_2$ for some $\bar{\alpha}_1 \in W_1$, $\bar{\alpha}_2 \in W_2$

$\gamma_2 = \bar{\beta}_1+\bar{\beta}_2$. — - - $\bar{\beta}_1$. - . $\bar{\beta}_2$ - -

Now, $a\gamma_1+b\gamma_2 = a(\bar{\alpha}_1+\bar{\alpha}_2)+b(\bar{\beta}_1+\bar{\beta}_2)$

$= (a\bar{\alpha}_1+b\bar{\beta}_1)+(a\bar{\alpha}_2+b\bar{\beta}_2)$

$= \delta_1+\delta_2$ where $\delta_1 \in W_1$, $\delta_2 \in W_2$.

$[\because W_1$ is a subspace
$\Rightarrow a\bar{\alpha}_1+b\bar{\beta}_1 \in W_1]$

$\therefore a\gamma_1+b\gamma_2 \in W_1+W_2$

$\therefore$ It is a subspace.

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Def^n. Direct Sum

Let W_1 and W_2 be 2 subspaces of the vector space $V(F)$ such that $V = W_1 + W_2$. This linear sum is called the direct sum, denoted by $V = W_1 \oplus W_2$ if every vector $\bar{\gamma} \in V$ can be written in one and only one way as $\bar{\gamma} = \bar{\alpha} + \bar{\beta}$, where $\bar{\alpha} \in W_1$, $\bar{\beta} \in W_2$.

Theorem The necessary and sufficient condition that a vector space $V(F)$ is a direct sum of two subspaces W_1 and W_2 are $V = W_1 + W_2$ and $W_1 \cap W_2 = \{\phi\}$.

Proof:- ① ~~V is a direct sum~~

$V = W_1 + W_2$, necessary by definition.

~~Now, let $\bar{\gamma} \in V$~~

~~Let there exist 2 ways of decomposing $\bar{\gamma}$ into sum of 2 vectors.~~

~~$\therefore \bar{\gamma} = \bar{w}_1 + \bar{w}_2$ and $\bar{\gamma} = \bar{v}_1 + \bar{v}_2$, $\bar{w}_1 \neq \bar{v}_1$ and $\bar{w}_2 \neq \bar{v}_2$~~

Let $W_1 \cap W_2 = \{\phi\}$ and $V = W_1 + W_2$ $\Big\}$ $\xrightarrow{\text{To Prove}}$ $V = W_1 \oplus W_2$

~~Let $W_1 \cap W_2 = \bar{\alpha}$ where $\alpha \neq \phi$.~~

~~$\therefore \bar{\alpha} \in W_1, W_2, V$.~~

Lets assume V is not a direct sum of W_1 and W_2.

$\therefore$ $\bar{\alpha} \in V$, $\bar{\alpha} = \bar{\alpha}_1 + \bar{\alpha}_2 = \bar{\beta}_1 + \bar{\beta}_2$ where $\bar{\alpha}_1, \bar{\beta}_1 \in W_1$, $\bar{\alpha}_2, \bar{\beta}_2 \in W_2$.

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$\therefore \bar{\alpha}_1 + \beta . \bar{\alpha}_2 = \bar{\beta}_1 + \bar{\beta}_2$.

$\Rightarrow \bar{\alpha}_1 - \bar{\beta}_1 = \bar{\beta}_2 - \bar{\alpha}_2$. $\left(\bar{\alpha}_1 - \bar{\beta}_1 \in W_1, \; \bar{\beta}_2 - \bar{\alpha}_2 \in W_2 \right)$.

~~sufficient~~

$= \bar{X}$ (say).

Now, $\bar{X} \in W_1$ and $\bar{X} \in W_2$

$\Rightarrow \bar{X} \neq \phi$ and $\bar{X} \in W_1 \cap W_2$.

CONTRADICTION.

Hence V is a direct sum.

(2).

V is a direct sum $\xRightarrow{\text{To Prove}}$ $W_1 \cap W_2 = \{\phi\}$ and $V = W_1 + W_2$

Let $V = W_1 \oplus W_2$. $\Rightarrow V = W_1 + W_2$. (trivial).

Lets assume $\exists \alpha$, $\alpha \neq \phi$ and $\alpha \in W_1 \cap W_2$.

Now, $\bar{\alpha} = \underbrace{\bar{\alpha}}_{\in W_1} + \underbrace{\bar{\phi}}_{\in W_2}$

Also, $\bar{\alpha} = \underbrace{\bar{\phi}}_{\in W_1} + \underbrace{\bar{\alpha}}_{\in W_2}$. where $\bar{\alpha} \in V$.

Hence, contradiction

$\therefore W_1 \cap W_2 = \{\phi\}$.

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Linear Transformations

$(V,+) \xrightarrow{f} (W,+)$

$\vec{x} \rightarrow f(\bar{x})$

$\bar{y} \rightarrow f(\bar{y})$

$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ means

1^{st} basis vector $\hat{i}$ moves to $(1,2)$

& 2^{nd} . . . . $\hat{j}$ - - - $(3,1)$

For any transformation to be a linear transformation,

1) All lines should remain lines in W (even diagonal ones or any)

2) Origin should not move.

$L : V \rightarrow W$.

$L : V \rightarrow V$ — Linear Operator

$L : V \rightarrow F$. — Linear functional (One such example → Dot Product)

By 1^{st} Property (in def below) :-

$f(\phi + \phi) = f(\phi) + f(\phi) \Rightarrow f(\phi) + \phi = f(\phi) + f(\phi)$

$\Rightarrow f(\phi) = \phi$ [Origin doesn't move].

Def^n Let $U(F)$ and $V(F)$ be two vector spaces over the same field $(F,+,\cdot)$. A function from U to V is said to be linear transformation if (i) $f(\bar{x}+\bar{y}) = f(\bar{x}) + f(\bar{y})$ (ii) $f(c\bar{x}) = cf(\bar{x})$, $\forall\ x, y \in V,\ c \in F$.

$f(a\bar{x} + b\bar{y}) = af(\bar{x}) + bf(\bar{y})$.

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Example Let $M_{m\times n}(F)$ be a vector space over the field F, and let $[a_{ij}]_{m\times n}$ be a fixed matrix over F. Define a function $f: M_{m\times n} \to M_{m\times n}$ by
$f[b_{ij}] = [a_{ij}]\cdot[b_{ij}]$ $\forall$ $[b_{ij}] \in M_{m\times n}$.

Show that this is a linear transformation.

Let $c, d \in F$, $x, y \in M_{m\times n}$.
To prove:- $f(cx+dy) = cf(x) + df(y)$

$\therefore$ LHS $= f\left(c[x_{ij}] + d[y_{ij}]\right) = f\left([cx_{ij}] + [dy_{ij}]\right)$

$\cancel{= [cx_{ij}]\cdot[dy_{ij}]}$

$= [a_{ij}] \cdot \left(c[x_{ij}] + d[y_{ij}]\right)$

$= c[a_{ij}]\cdot[x_{ij}] + d[a_{ij}]\cdot[y_{ij}]$

$= c f([x_{ij}]) + d f([y_{ij}])$

$=$ RHS [Proved]

Q) Let $f: U \to V$ be a linear Transformation. If $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}$ are linearly Dependent in U, then show that $\{f(\bar{x}_1), f(\bar{x}_2), \dots, f(\bar{x}_n)\}$ are L.D. in V.

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Given :-

$a_1\bar{x}_1 + a_2\bar{x}_2 + \cdots + a_n\bar{x}_n = \phi$, for some ~~non-zero~~ scalars a_i's not all zero.

$\therefore f(a_1\bar{x}_1 + a_2\bar{x}_2 + \cdots + a_n\bar{x}_n) = f(\phi)$
[$f(\phi) = \phi$].

or, $a_1 f(\bar{x}_1) + a_2 f(\bar{x}_2) + \cdots + a_n f(\bar{x}_n) = \phi$, for some scalars a_i's not all zero.

$\therefore \{f(\bar{x}_1), f(\bar{x}_2), \ldots, f(\bar{x}_n)\}$ are L.D.

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Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for a finite dimensional vector space $V(F)$ and $\{y_1, y_2, \ldots, y_n\}$ be an arbitrary set of n vectors from $W(F)$. Then show that there exists a unique linear mapping $f: V \to W$ such that $f(x_i) = y_i$ for $i = 1$ to n.

Proof:- Let $x \in V$.

Any vector in $V(F)$ can be represented as $\sum_{i=1}^{n} a_i x_i$ for some a_i's $\in F$.

$x = a_1x_1 + a_2x_2 + \cdots + a_nx_n$. — (1) $(a_i \in F)$

Let us define a mapping $f: V \to W$.
such that $f(x) = a_1 y_1 + a_2 y_2 + \dots + a_n y_n$.

When $x = x_i$

$x_i = a_1 x_1 + \dots + a_i x_i + \dots + a_n x_n$.

$\Rightarrow a_1 x_1 + \dots + (a_i - 1) x_i + \dots + a_n x_n = 0$.

since basis $\Rightarrow$ L.I. $\Rightarrow$ All coefficients 0.

$\therefore$ ~~a~~. $a_1 = 0, \dots, a_i = 1, \dots, a_n = 0$.

Thus such a mapping exists ~~& is unique~~.
It is remaining to show that it is linear ~~& unique~~ & unique.

Now, let $x, y \in V$.

$x = a_1 x_1 + \dots + a_n x_n$

$y = b_1 x_1 + \dots + b_n x_n$.

$$f(x+y) = f\{(a_1 x_1 + \dots + a_n x_n) + (b_1 x_1 + \dots + b_n x_n)\}$$

$$= f\{(a_1+b_1) x_1 + \dots + (a_n+b_n) x_n\}.$$

$$= (a_1+b_1) y_1 + \dots + (a_n+b_n) y_n.$$

$$= (a_1 y_1 + \dots + a_n y_n) + (b_1 y_1 + \dots + b_n y_n).$$

$\therefore f(x+y) = f(x) + f(y)$.

Similarly, $f(cx) = f\{c(a_1x_1 + \cdots + a_nx_n)\}$

$$= f(ca_1x_1 + \cdots + ca_nx_n)$$

$$= ca_1y_1 + \cdots + ca_ny_n$$

$$= c(a_1y_1 + \cdots + a_ny_n)$$

$\therefore f(cx) = cf(x) \quad \forall \; c \in F.$

Finally, we show that <u>f is unique</u>.

Let us assume that f is not unique.

$\exists \; g : V \to W$ s.t. $g(x_i) = y_i$ for $i = 1 \ldots n$.

$g(x) = g(a_1x_1 + \cdots + a_nx_n)$

$g(x) = a_1 g(x_1) + \cdots + a_n g(x_n)$ [since g is linear].

$$= a_1y_1 + \cdots + a_ny_n.$$

$$= f(x) \quad \forall \; x.$$

$\therefore f$ is unique.

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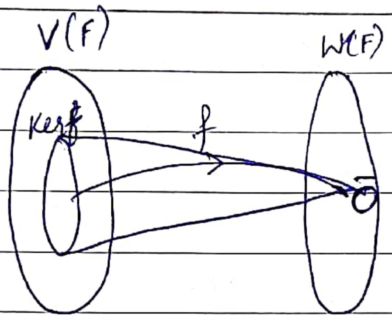
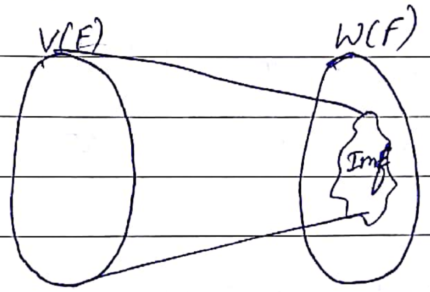


$f(x+y) = f(x) + f(y)$.

$f(cx) = cf(x)$.

~~Kerf : Kernel~~



Imf : Image (f) or Range set.

Let $f : V \to W$ be a linear mapping from a vector space $V(F)$ into a vector space $W(F)$. Then Ker(f) is the set of all $x \in V$, which are mapped to $\bar{0}$, the additive identity of W by f.

$$\text{Ker}(f) = \{x \in V \mid f(x) = \bar{0} \in W\}.$$

Let $f : V \to W$ be a L.T.. Then the set of all images of elements $x \in V$ is called the range set of image set.

$$R_f(V) = f(V) = \{y \in W \mid y = f(x) \text{ for some } x \in V\}$$

Theorem:- Let $f : V \to W$ be a Linear Transformation over the same field F. Then, ① $Ker(f)(F)$ is a subspace of $V(F)$.

② $R_f(V)(F)$ is a subspace of $W(F)$.

Proof:- ① Let $\bar{v}_1, \bar{v}_2 \in Ker(f)(F)$.

$\therefore f(\bar{v}_1) = \bar{0}, \quad f(\bar{v}_2) = \bar{0}$.

Now, $f(a\bar{v}_1 + b\bar{v}_2) = f(av_1) + f(bv_2)$

$= af(v_1) + bf(v_2) \quad \forall\ a, b \in F$.

$= a.\bar{0} + b.\bar{0}$

$= \bar{0}$.

$\therefore a\bar{v}_1 + b\bar{v}_2 \in Ker(f)(F) \cdot \forall\ \bar{v}_1, \bar{v}_2 \in Ker(f)(F)$.

$\therefore Ker(f)(F)$ is a subspace of $V(F)$.

② Let $\bar{w}_1, \bar{w}_2 \in R_f(V)(F)$.

$\therefore \exists\ \bar{v}_1, \bar{v}_2 \in V(F)$ s.t. $f(\bar{v}_1) = \bar{w}_1$, $f(\bar{v}_2) = \bar{w}_2$.

$\therefore af(\bar{v}_1) + bf(\bar{v}_2) = a\bar{w}_1 + b\bar{w}_2$

or, $f(a\bar{v}_1 + b\bar{v}_2) = a\bar{w}_1 + b\bar{w}_2$.

Since $a\bar{v}_1 + b\bar{v}_2 \in V \Rightarrow a\bar{w}_1 + b\bar{w}_2 \in R_f(V)(F) \ \forall\ \bar{w}_1, \bar{w}_2 \in R_f(V)(F)$

$\therefore R_f(V)(F)$ is a subspace of $W(F)$.

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Ker (f) is known as the null-space, and dim (Ker f) is called the nullity of linear transform.

Dim f(V) is called as the rank of the linear Transformation

↳ Rank of the matrix
= Dimension of the basis vectors.

Rank-Nullity Theorem

Let $f: V \to W$ be a linear Transformation from the vector space $V(F)$ to $W(F)$. Then, ~~dim (Ker f~~

$$\dim(V) = \dim(\operatorname{Ker} f) + \dim(f(V))$$

This is also called Sylvester law.

Proof :- Let $\dim(V) = n$.
Let $\operatorname{Ker} f \neq \{0\}$, say $\dim(\operatorname{Ker} f) = r \leq n$.

Let $\{x_1, x_2, \ldots, x_r\}$ be a basis of Ker(f).
Then we can extend $\{x_1, \ldots, x_r\}$ to a basis $\{x_1, x_2, \ldots, x_r, y_1, \ldots, y_{n-r}\}$ of V.

Now, for any $y \in f(V)$, $\exists x \in V$ s.t. $y = f(x)$

$x = a_1 x_1 + \cdots + a_r x_r + b_1 y_1 + \cdots + b_{n-r} y_{n-r}$.

$y = f(x) = a_1 f(x_1) + \cdots + a_r f(x_r) + b_1 f(y_1) + \cdots$
$\cdots + b_{n-r} f(y_{n-r})$.

[$\because f$ is linear]

$\therefore y = b_1 f(y_1) + \cdots + b_{n-r} f(y_{n-r})$ [$\because f(x_i) = 0$ since $f(x_i) \in \operatorname{Ker} f$]

$\therefore$ Any element $y \in f(V)$ can be expressed as a linear combination of $\{f(y_1), f(y_2), \ldots, f(y_{n-r})\}$.

$B = \{f(y_1), f(y_2), \ldots, f(y_{n-r})\}$

Now, lets prove that these are L.I.

$\therefore$ Lets assume $a_1 f(y_1) + \cdots + a_{n-r} f(y_{n-r}) = 0$.

$\Rightarrow f(a_1 y_1 + \cdots + a_{n-r} y_{n-r}) = 0$.

$\therefore (a_1 y_1 + \cdots + a_{n-r} y_{n-r}) \in \operatorname{Ker} f$.

Let $\hookrightarrow = m$.

$\therefore m = d_1 x_1 + \cdots + d_r x_r$ [$\because \{x_1, \ldots, x_r\}$ are basis of Ker f]

$\therefore d_1 x_1 + \cdots + d_r x_r = a_1 y_1 + \cdots + a_{n-r} y_{n-r}$

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$\Rightarrow$ $d_1x_1 + d_2x_2 + \cdots + d_r x_r - a_1 y_1 - \cdots - a_{n-r}y_{n-r} = 0$.

But $\{x_1, x_2, \ldots, x_r, y_1, \ldots, y_{n-r}\} \in$ basis of V.

$\therefore$ Each $d_i = 0$ & Each $a_i = 0$.

$\therefore \{f(y_1), f(y_2), \ldots, f(y_{n-r})\}$ is L.I. & spans $f(V)$.

$\therefore$ $\dim V = n = r + (n-r)$

$= \dim(\text{Ker} f) + \dim(f(V))$.

28.9.18.

Q) $f: V \to W$ is Linear Transformation.
If $\text{Ker}(f) = \{0\}$, then what can you say about $\dim(f(V))$?

$\therefore \dim(\text{Ker} f) = 0$

$\therefore \dim(f(V)) = \dim(V)$.

Q) Let $V(F)$ and $W(F)$ be finite dimensional vector spaces with $\dim V = \dim W$ and let $f: V \to W$ be L.T. Then it is one-to-one mapping iff f maps V onto W.

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$\dim V = \dim(\mathrm{Ker} f) + \dim(f(V))$ — (1)

~~$\mathrm{im} f(V) = W$~~

(1) lets assume f maps V onto W.

$\therefore f(V) = W$

$\therefore \dim(\mathrm{Ker} f) = 0$ (from (1)).

$\Rightarrow \mathrm{Ker}(f) = \{0\}$ ← Prove yourself

$\Rightarrow f$ is one-to-one.

(2) If f is one-to-one

$\mathrm{Ker}(f) = \{0\} \Rightarrow \dim(\mathrm{Ker}(f)) = 0$.

$\therefore \dim V = \dim(f(V))$ [by (1)]

or $\dim W = \dim(f(V))$ ← Prove yourself

$W = f(V)$

\# f is one-one iff $\mathrm{Ker}(f) = \{0\}$.

(1) way is trivial.

(2) lets assume $\mathrm{Ker}(f) = \{0\}$ but f is not one-one.

$f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

$\therefore f(x_1 - x_2) = f(x_1) - f(x_2)$

$f(x_1 - x_2) = 0$

$\therefore x_1 - x_2 \in \mathrm{Ker} f$ & $x_1 - x_2 \neq 0$.

$\therefore$ Contradiction

$\therefore f$ is one-one

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Non-Singular Transformation

A linear transformation f from $V(F)$ to $W(F)$ is called non-singular iff $\exists$ mapping f^* from $f(V)$ onto $V(F)$, s.t. $f^* \circ f = I$, where I is the Identity mapping on $V(F)$.

Q) We know that f is linear transformation. What about f^*?

$$f(a\bar{v}_1 + b\bar{v}_2) = af(\bar{v}_1) + bf(\bar{v}_2).$$

Now, $f^*(f(a\bar{v}_1 + b\bar{v}_2)) = a\bar{v}_1 + b\bar{v}_2$.

$$\begin{aligned} f^*(af(\bar{v}_1) + bf(\bar{v}_2)) &= f^*(f(a\bar{v}_1 + b\bar{v}_2)) \\ &= a\bar{v}_1 + b\bar{v}_2 \\ &= af^*(f(\bar{v}_1)) + bf^*(f(\bar{v}_2)) \end{aligned}$$

$\therefore$ f^* is linear.

Let $f: V \to W$ be a L.T., then show that the following statements are equivalent:-

a) f is non-singular

b) $\forall\, x, y \in V$, if $f(x) = f(y)$, then $x = y$.

c) $\mathrm{Ker}(f) = \{0\}$.

Lets assume (a) is true.

$\therefore$ f is non-singular.

$\therefore$ $\exists\ f^* : f(V) \to V(F)$ s.t. $f^* \circ f = I$.

Let $f(x) = f(y)$ for $x, y \in V$.

$\therefore$ $f^*(f(x)) = f^*(f(y))$

$\Rightarrow$ $x = y$

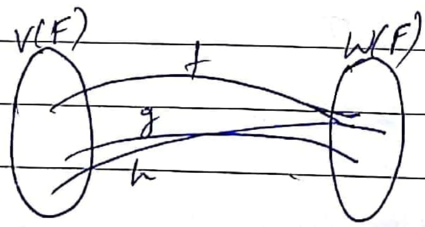
$\Rightarrow$ (b) is true.

Now, since f is one-one (from (b)).

$\therefore$ $Ker(f) = \{0\}$.

$\Rightarrow$ (c) is true.

#



$\{f, g, h, \ldots\}$.

$L(V, W)(F)$ is also a vector space.

Operations ~~vector addition~~ are defined by :-

① $(f+g)(x) = f(x) + g(x)$ $x \in V$.
↳ vector add^n.

② $(cf)(x) = cf(x)$ $x \in V$
↳ scalar mult.

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To Prove ① If $f, g \in L(V, W)$, then $f+g \in L(V, W)$.

② If $f \in L(V, W)$, then $cf \in L(V, W)$.

All Properties. Show the rest 8.

Proof :- ① Let $x, y \in V$, $a, b \in F$.

Since f, g are L.T.

$f(ax+by) = af(x) + bf(y)$.

$g(ax+by) = ag(x) + bg(y)$.

$$(f+g)(ax+by) = f(ax+by) + g(ax+by).$$
$$= af(x) + bf(y) + ag(x) + bg(y)$$
$$= a(f+g)(x) + b(f+g)(y).$$

$\therefore$ $f+g$ is a L.T.

$\therefore$ $f+g \in L(V, W)$.

② $$cf(ax+by) = c[f(ax+by)]$$
$$= c[af(x) + bf(y)]$$
$$= (c.a) f(x) + (c.b) f(y)$$
$$= (a.c) f(x) + (b.c) f(y)$$ [$\because c, a \in F$ & F is a field $\therefore c.a = a.c$]
$$= a(cf(x)) + b(cf(y)).$$

$cf \in L(V, W)$ is a L.T.

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(3) Let $f, g, h \in L(V,W)$, $x \in V$.

$\therefore ((f+g)+h)(x) = (f+g)(x) + h(x)$

$= (f(x) + g(x)) + h(x)$

$= f(x) + (g(x) + h(x))$

$= (f + (g+h))(x)$.

(4) Let $f \in L(V,W)$, $x \in V$.

$f(x) = f(x) + \theta_W$ where θ_W is the identity of W.

$\therefore$ Let ~~Z~~ $Z : V \to W$ be a mapping from V ~~to~~ W s.t. $Z(x) = \theta_W \ \forall \ x \in V$.

$Z(ax+by) = 0_W = 0_W + 0_W = a \cdot 0_W + b \cdot 0_W = a Z(x) + b Z(y)$.

$\therefore Z \in L(V,W)$ $\therefore (f + Z)(x) = f(x) = (Z + f)(x)$.

$\therefore Z$ is the identity of $L(V,W)$.

$\therefore 0 = Z$

(5) Let $f \in L(V,W)$, $x \in V$.

Let $(-f) : V \to W$ s.t. $(-f)(x) = -f(x)$ = additive inverse of $f(x)$

$(-f)(ax+by) = -[f(ax+by)] = -[af(x) + bf(y)] = a(-f(x)) + b(-f(y))$

$\therefore -f \in L(V,W)$ $\therefore$ ~~f~~ $(f + (-f))(x) = f(x) + (-f(x)) = 0$

$= ((-f) + f)(x) = Z$.

Prove others yourself.

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Q) $\dim(V) = m$, $\dim(W) = n$.

$\dim\{L(V, W)\} = m.n$.

<u>Theorem</u>: If $V(F)$ and $W(F)$ are vector spaces of dimension m and n, respectively, over F, then the space $L(V, W)(F)$ is of dimension $m.n$ over F.

<u>Proof</u>:- Let $\{x_1, x_2, \ldots, x_m\}$ be basis of V.

Let $\{y_1, y_2, \ldots, y_n\}$ be basis of W.

$\exists$ a linear mapping $f_{ij} : V \to W$ $(1 \leq i \leq m, 1 \leq j \leq n)$.

$f_{ij}(x_k) = 0$ if $i \neq k$ $\quad \boxed{f_{ij} \in L(V, W)}$

$= y_j$ if $i = k$.

$B = \{f_{ij}\}$ has $m.n$ elements.

<u>To show</u>: 1) $f_{ij} \in L(V, W)$.

2) B is the basis.

$f \in L(V, W)$

$\therefore f(x_i) \in W$.

$f(x_i) = a_{i1} y_1 + \cdots + a_{in} y_n$, $a_{ij} \in F$, $1 \leq i \leq m$.

$$= \sum_{j=1}^{n} a_{ij} y_j.$$

$$f(x_k) = \sum_{j=1}^{n} a_{kj} y_j .$$

$$f(x_k) = \sum_{j=1}^{n} a_{kj} f_{kj}(x_k) .$$

$$= \sum_{j=1}^{n} a_{kj} f_{kj}(x_k) + 0$$

$$(i = k) \qquad (i \neq k)$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} (c_{ij} f_{ij}(x_k)) \right) .$$

$$\therefore f = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} f_{ij}$$

$\therefore$ Any $f \in L(V,W)$ can be expressed as LC of $\{f_{ij}\}$

$\therefore$ $\{f_{ij}\}$ spans $L(V,W)$.

Now, let $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} f_{ij} = \bar{O}_W$. (To Prove:- LI).

Or,

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Dual space

- Linear operator: LT from V to V.
- " " functional: LT from V to F.

Dual space involves ~~all these sets of~~ the set of all LFs.

$L(V,F)$ → Dual space
→ Forms a vector space.

Let $V(F)$ be a vector space over F. A linear transformation from $V(F)$ to F is called linear functional. That is $\phi : V \to F$, satisfying that $\phi(ax+by) = a\phi(x) + b\phi(y)$ for every $x, y \in V$, $a, b \in F$.

The vector space $L(V,F)(F)$ consisting of linear functionals is called a dual space of $V(F)$ and is denoted by $V^*(F)$.

$\therefore \phi_i \in V^*(F)$.

Q) $V(F) \to n$ square matrix.
$\phi : V \to F$.
$\phi(A) = tr[A]$. Show that ϕ is a linear functional.

_/_/_

$$\phi(A+B) = tr[A+B]$$

$$= \sum_{i=1}^{n} C_{ii} \qquad \text{where} \quad C = A+B$$

$$= \sum_{i=1}^{n} (A+B)_{ii}$$

$$= \sum_{i=1}^{n} (A_{ii} + B_{ii})$$

$$= \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii}$$