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Lecture - 16

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⇒ Problems:

① If a vector space V has a basis with n vectors, then every basis for V has exactly n-vectors — Show?

② Find the dimension of the vector space W of symmetric 2×2 matrices / $W = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

③ In each case, determine whether S is a basis of V

× a) $V = P_2$; $S = \{1+x, 2-x+x^2, -1+3x+x^2, 3x-2x^2\}$

× b) $V = M_{2\times2}$, $S = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \right\}$

✓ c) $V = P_2$, $S = \{1+x, x+x^2, 1+x^2\}$

→ ② $W = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$S \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} = W$

→ $a_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow a_1 = a_2 = a_3 = 0$

Q $T: R^2 \to R^3$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x-y \\ 3x+4y \end{bmatrix}$$

Check whether T is a linear transform or not

$\rightarrow$ ① $T\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ 2(x_1+x_2)-(y_1+y_2) \\ 3(x_1+x_2)-4(y_1+y_2) \end{bmatrix}$

$$= \begin{bmatrix} x_1 + x_2 \\ (2x_1-y_1) + (2x_2-y_2) \\ (3x_1-4y_1) + (3x_2-4y_2) \end{bmatrix}$$

$$= T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = T(\bar{u})+T(\bar{v})$$

first method

$\rightarrow$ ① Let $\bar{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in R^2$, $\bar{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in R^2$

$T(\bar{u}+\bar{v}) = T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) \rightarrow$ ① same proof as above.

$\rightarrow$ $T(c\bar{u}) = cT(\bar{u})$

Property $\rightarrow$ Let A be a $m \times n$ matrix. Then show the matrix transformation $T_A: R^n \to R^m$ defined by

$$T_A(\bar{x}) = A\bar{x} \qquad (\text{for } \bar{x} \text{ in } R^n)$$

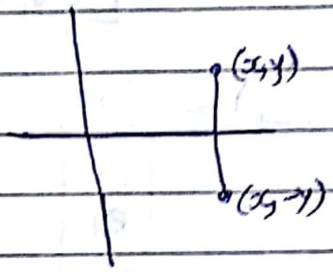
is a linear transformation.

Proof: Let $\bar{u}, \bar{v} \in R^n$

$\rightarrow$ $$T(\bar{u}+\bar{v}) = A(\bar{u}+\bar{v}) = A\bar{u} + A\bar{v} = T(\bar{u}) + T(\bar{v}).$$

$\rightarrow$ $$T(c\bar{u}) = A(c\bar{u}) = cA(\bar{u}) = cT(\bar{u}).$$

$\rightarrow$ Let $F: R^2 \rightarrow R^2$ be the transf. that each point is its reflection in the x-axis. Show that F is a linear transf.



$$F\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

$$\begin{bmatrix} x \\ -y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\bar{x}.$$

$\rightarrow$ Let $F: R^2 \rightarrow R^2$ be the transf. that rotates each point $90°$ counter-clockwise about the origin. Is it L.T.

$$F\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

$$\begin{bmatrix} -y \\ x \end{bmatrix} = y\begin{bmatrix} -1 \\ 0 \end{bmatrix} + x\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\bar{x}$$

→ Let $T: R^n \to R^m$ be a linear Transform. Then T is a matrix transformation. More specifically, $T = T_A$ where A is the $m \times n$ matrix $[T(\bar{e}_1), T(\bar{e}_2) \dots T(\bar{e}_n)]$

where $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n$ are the basis vectors in R^n.

$$T(\bar{e}_i) \to m \times 1$$

① $T: R^2 \to R^3$ ② $S: R^3 \to R^4$

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 - x_2 \\ 3x_1 + 4x_2 \end{bmatrix} \qquad S\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2y_1 + y_3 \\ 3y_2 + y_3 \\ y_1 + y_2 \\ y_1 + y_2 + y_3 \end{bmatrix}$$

→ ① $R^2 \to \bar{e}_1, \bar{e}_2$.

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(e_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$$A = [T(e_1); T(e_2)]$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}$$

② $T(e_1) = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$$T(e_2) = [0 \;\; 3 \;\; 1 \;\; 1]^T$$

$$T(e_3) = [1 \;\; 1 \;\; 0 \;\; 1]^T$$

$$A = [T(e_1); T(e_2); T(e_3)]$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

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⇒ System of Linear Equations.

→ $T: R^3 \to R^2$.

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2y \\ x+y-3z \end{bmatrix}$$

Let $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$, $C = \{\bar{e}_1, \bar{e}_2\}$ be the basis of R^3 and R^2 respectively. Find the matrix.

→ Let V and W be two finite dimensional vector spaces with basis B and C respectively.

$B = \{v_1, v_2, \dots, v_n\}$.
If $T: V \to W$ is a linear transform, then the $m \times n$ matrix is defined by

$$A = \left[(T(v_1))_C, (T(v_2))_C, \dots, (T(v_n))_C\right]$$

↳ $T(v_1)$ expressed in basis C.

→ $T(e_1) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$T(e_2) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $T(e_3) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

Case I: If $C = \{\bar{e}_1, \bar{e}_2\}$

$$\begin{bmatrix}1\\1\end{bmatrix} = \bar{e}_1 + \bar{e}_2 = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}$$

$$[T(\bar{e}_1)]_c = \begin{bmatrix}1\\1\end{bmatrix}, \qquad [T(\bar{e}_2)]_c = \begin{bmatrix}-2\\1\end{bmatrix},$$

$$[T(\bar{e}_3)]_c = \begin{bmatrix}0\\-3\end{bmatrix}$$

Case II: If $c = \{\bar{e}_2, \bar{e}_1\} = \left\{ \overset{\bar{e}_2}{\begin{bmatrix}0\\1\end{bmatrix}}, \overset{\bar{e}_1}{\begin{bmatrix}1\\0\end{bmatrix}} \right\}$

$$[T(\bar{e}_2)]_c = \begin{bmatrix}-2\\1\end{bmatrix} = \cancel{\bar{e}_2 - 2\bar{e}_1} \quad \bar{e}_2 - 2\bar{e}_1$$

$$= \begin{bmatrix}1\\-2\end{bmatrix}$$

$$T(\bar{e}_2)_c = \begin{bmatrix}-3\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & -3\\1 & -2 & 0\end{bmatrix}_{(\bar{e}_2, \bar{e}_1)}$$

$$\begin{bmatrix}1 & -2 & 0\\1 & 1 & -3\end{bmatrix}_{(\bar{e}_1, \bar{e}_2)}$$

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$L : V_F \to W_F$ } $\{L(V,W), F\}$ is also going to form a vector space
$\to +,$

→ The sum $f+g$ of two linear mappings $f, g \in L(V_F, W_F)$ is defined by this rule.

$(f+g)(x) = f(x) + g(x) \quad \forall x \in V.$

and the scalar multiplication by

$(cf)(x) = cf(x)$ where $c \in F, x \in V$.

(i) If $f, g \in L(V, W)$
then $f+g \in L(V, W)$

1st part $\{L(V,W), +\} \to$ commutative group
→ closure
→ associative
→ identity
→ inverse.
→ commutative.

Also need to show $f+g$ is L.T.

→ Since f, g are L.T.

$$\left.\begin{aligned} f(ax+by) &= af(x) + bf(y) \\ g(ax+by) &= ag(x) + bg(y) \end{aligned}\right\} \quad (a, b \in F,\ x, y \in V)$$

$$\begin{aligned}(f+g)(ax+by) &= f(ax+by) + g(ax+by) \\ &= af(x) + bf(y) + ag(x) + bg(y). \\ &= a(f(x)+g(x)) + b(f(y)+g(y)) \\ &= a(f+g)(x) + b(f+g)(y).\end{aligned}$$

(Closure Proved)

Associativity

$$[(f+g)+h](x) = (f+g)(x) + h(x)$$
$$= f(x) + g(x) + h(x)$$
$$= f(x) + (g+h)(x)$$

Existence of Identity

$Z: V \to W$ s.t. $Z \in L(V, W)$

$Z(x) = \bar{0}_w \quad \forall x \in V$

$Z(ax+by) = 0.$

$aZ(x) + bZ(y) = a.0 + b.0$

$= 0.$

$(f+Z)(x) = f(x) + Z(x)$

$= f(x) + \bar{0}_w = f(x) \quad \forall f \in L(V, W)$

$\forall x \in V.$

$f+Z = f = Z+f.$

Inverse.

$f \in L(V, W)$

$-f \in L(V, W)$

$f + (-f) = (-f) + f = Z$

So, $(f+g)(x) = f(x) + g(x)$

$= g(x) + f(x)$

$= (g+f)(x)$

$(\because W$ is a commutative group$)$

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6. $f \in L(V,W) \Rightarrow cf \in L(V,W)$
7. $c(f+g) = cf + cg$
8. $(a+b)f = af + bf$
9. $I \cdot f = f$.

$\rightarrow$ 6. $(cf)(ax+by) = c\{af(x) + bf(y)\}$ $(f \rightarrow L.T)$

$= (c.a)f(x) + (c.b)f(y)$ (dist. property)

$= (a.c)f(x) + (b.c)f(y)$ ($\because F$ is a field)

$= a(cf(x)) + b(cf(y))$

$\Rightarrow$ Linear Functional.

$L : V_F \rightarrow F$ Eg:- scalar product, vector product, norm, trace of M

$\hookrightarrow L(ax+by) = aL(x) + bL(y)$

Eg: Let $V(F)$ be a vector space of n-square matrices over F.
Let $\phi : V \rightarrow F$ be a trace map. defined by

$$\phi(A) = a_{11} + a_{22} + \cdots + a_{nn}$$
$$= tr(A). \quad (A = [a_{ij}]_{n \times n})$$

Show that ϕ is linear functional.

$\rightarrow$ $\phi(k_1A + k_2B) = k_1\phi(A) + k_2\phi(B)$ $\rightarrow$ To show.

$$\phi(k_1A + k_2B) = \phi\{k_1[a_{ij}] + k_2[b_{ij}]\}$$
$$= \phi[[k_1a_{ij} + k_2b_{ij}]]$$
$$= (k_1a_{11} + k_2b_{11}) + (k_1a_{22} + k_2b_{22})$$
$$+ \cdots + (k_1a_{nn} + k_2b_{nn})$$

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Prob

$= (k_1 a_{11} + k_1 a_{22} + \cdots + k_1 a_{nn}) + (k_2 b_{11} + k_2 b_{22} + \cdots + k_2 b_{nn})$

$= k_1 \phi(A) + k_2 \phi(B)$

~~The vector space~~

→ $V^* = L(V, F) \rightarrow \boxed{\text{dual space}}$ / conjugate space

also forms a vector space w.r.t vector addition and scalar multiplication.

① $(\phi + \delta)(x) = \phi(x) + \delta(x)$ $\quad \forall x \in V$

② $(k\phi)(x) = k\phi(x)$ $\quad \forall \phi(x), \delta(x) \in V^*$

dim $V^* = ?$ $\boxed{\text{DIY}}$.

→ dim $V = n$ $\quad$ dim $W = m$.

dim $\{L(V, W)\} \rightarrow nm$.

dim $V^* = \dim \{L(\underset{\downarrow\, n}{V}, \underset{\downarrow\, 1}{F})\}$

$= n$.

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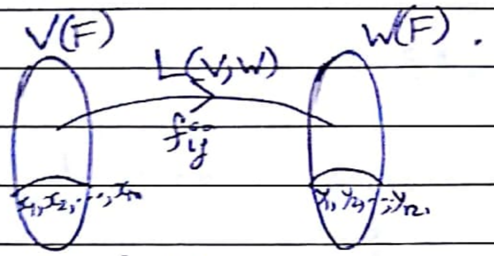
Prob. If $V(F)$ and $W(F)$ are vector spaces of dimension 'm' and 'n' respectively over F, then show that the space $L(V,W)(F)$ is of dimension $m \cdot n$ over F.

→ 

$i = 1, 2, \ldots, m$
$j = 1, 2, \ldots, n$.

$$\{f_{11}, f_{12}, \ldots, f_{mn}\}$$

$f_{ij} : V \to W$ $\qquad i = 1 \ldots m$.
$\qquad\qquad\qquad\qquad j = 1 \ldots n$

$$f_{ij}(x_k) = \begin{cases} y_j & i = k \\ 0 & i \neq k \end{cases}$$ Mapping defined.

Eg:- $f_{13}(x_1) = y_3$.

→ ① To prove: $f = \text{span}\{f_{ij}\}$
$\{f_{ij}\}$ is L.I

$f \in L(V,W)$
$f(x_i) \in W$

$f(x_i) = a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n$, $\quad a_{ij} \in F$
$\qquad 1 \leq i \leq m$.

$$= \sum_{j=1}^{n} a_{ij} y_j$$

$$f(x_k) = \sum_{j=1}^{n} a_{kj} y_j$$

$$= \sum_{j=1}^{n} a_{kj} f_{kj}(x_k)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} f_{ij}(x_k) \rightarrow \text{Since, } f_{ij}(x_i) = y_j \quad i=k; \quad = 0 \quad i \neq k.$$

$$\Rightarrow f = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} f_{ij}$$

Thus f can be expressed as a linear comb. of $\{f_{ij}\}$.

Hence span $\{f_{ij}\} = L(V,W)$.

→ 2 $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} f_{ij} = \bar{0}$ → added in $L(V,W)$ where $c_{ij} \in F$.

$$\Rightarrow \bar{0} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} f_{ij}(x_k)$$

$$= c_{kj} y_j$$

Since y_j is basis in W → L.I

$c_{kj} = 0$.

By mapping diff. k we get $c_{ij} = 0$.

→ $\dim\{L(V,V)\} = n^2$.

$\dim\{L(V,F)\} = n \times 1 = n$.

V^*

Prob → Let $\{x_1, x_2, \dots, x_n\}$ be a basis of $V(F)$ over F. Let $\phi_1, \phi_2, \dots, \phi_n \in V^*$ be the linear functionals defined by ~~$\phi_i(x_j) = \delta_{ij}$~~

$$\phi_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then show that $\{\phi_1, \phi_2, \dots, \phi_n\}$ is a basis for V^*

$\rightarrow$ $\phi_1(x_1) = 1,\ \phi_1(x_2) = 0, \dots, \phi_1(x_n) = 0$

$\phi_2(x_1) = 0,\ \phi_2(x_2) = 1, \dots, \phi_2(x_n) = 0.$

$\rightarrow$ $\{\phi_1, \phi_2, \dots, \phi_n\}$ spans V^* $\Big\} \rightarrow$ ϕ is L.C of $\{\phi_1, \phi_2, \dots, \phi_n\}$

Let $\phi \in V^*$

$\phi(x_1) = k_1$

Let $\phi(x_i) = k_i$, $k_i \rightarrow$ scalars.

Now, $k_1\phi_1 + k_2\phi_2 + \dots + k_n\phi_n = \sigma \in V^*$

$$\sigma(x_i) = (k_1\phi_1 + \dots + k_n\phi_n)(x_i)$$
$$= k_i = \phi(x_i) \qquad \sigma = \phi.$$

$\therefore$ ϕ can be expressed as L.C of $\{\phi_1, \phi_2, \dots, \phi_n\}$ vectors.

$\therefore$ span $\{\phi_1, \phi_2, \dots, \phi_n\} = L(V, F)$

$\rightarrow$ ② $a_1\phi_1 + \dots + a_n\phi_n = 0 \in V^*$

$$(a_1\phi_1 + \dots + a_n\phi_n)(x_i) = 0$$

$\Rightarrow$ $a_1\phi_1(x_i) + \dots + a_n\phi_n(x_i) = 0$

$a_i\phi_i(x_i) = 0 \Rightarrow a_i = 0.$

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$\Rightarrow \{\phi_1, \phi_2, \cdots, \phi_n\}$ are L.I

Problem Let $\{x_1, x_2, \cdots, x_n\}$ be a basis of V and let $\{\phi_1, \phi_2, \cdots, \phi_n\}$ be a dual basis of V^*, then any vector $x \in V$

$$x = \phi_1(x)x_1 + \phi_2(x)x_2 + \cdots + \phi_n(x)x_n$$

and any linear functional $\delta \in V^*$

$$\delta = \delta(x_1)\phi_1 + \delta(x_2)\phi_2 + \cdots + \delta(x_n)\phi_n$$

$\rightarrow$ (1) Let $x \in V$

$$x = a_1x_1 + a_2x_2 + \cdots + a_nx_n \quad \{a_n \in F\}$$

$$\begin{aligned}\phi_i(x) &= \phi_i(a_1x_1 + \cdots + a_nx_n)\\ &= \sum_j a_j \phi_i(x_j)\\ &= a_i \phi_i(x_i)\\ &= a_i\end{aligned}$$

$$\therefore x = \phi_1(x)x_1 + \phi_2(x)x_2 + \cdots + \phi_n(x)x_n$$

(2) Let $\delta \in V^*$

$$\delta = a_1\phi_1 + a_2\phi_2 + \cdots + a_n\phi_n \quad \{a_n \in F\}$$

$$\delta(x_k) = a_k$$

$$\therefore \delta = \sum_{i=1}^{n} \delta(x_i)\phi_i$$

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System of Linear Equations

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$ax + by = c$ — (1)

$ax + by + cz = d$ — (2)

→ A linear equation in the n variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

(where the coefficients $a_1, a_2, \cdots, a_n$ and b are constants).

① $3x - 4y = -1$

② $r - \frac{1}{2}s - \frac{15}{3}t = 9$

③ $x_1 + 5x_2 = 3 - x_3 + 2x_4$

④ $xy + 2z = 1$

⑤ $\sin xy - 3x_2 + 2^{x_3} = 0$

⑥ $\frac{x}{y} + z = 2$

A solⁿ of linear eqⁿ $a_1x_1 + \cdots + a_nx_n = b$ is a vector $[s_1, s_2, \ldots, s_n]$ whose components will satisfy the eqⁿ, when we substitute $x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$.

→ A system of linear eqⁿ is a finite set of linear eqⁿ each with same variables.

→ A solⁿ of a system of linear eqⁿ is a vector that is simultaneously a solⁿ of each eqⁿ ~~is~~ in the system.

① $x - y = 3$
$2x - 2y = 6$
(infinite solⁿ)

② $x + y = 4$
$x + y = 7$
(no solⁿ)

③ $x - y = 1$
$x + y = 3$
((2,1) is a solⁿ → unique solⁿ)

→ a) unique solⁿ → (consistent system)
b) infinite solⁿ → (consistent system)
c) no solⁿ → (inconsistent system)

→ Solve the system
Triangular form structure

$$\left.\begin{aligned} x - y - z &= 2 \\ y + 3z &= 5 \\ 5z &= 10 \end{aligned}\right\} \text{Back substitution.}$$

$z = 2, y = -1, x = 3$

→ Solve the system

$$x - y - z = 2$$
$$3x - 3y + 2z = 16.$$
$$2x - y + z = 9$$

$$\underbrace{\left(\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{array}\right)}_{\text{augmented matrix}} \qquad \underbrace{\begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & 2 \\ 2 & -1 & 1. \end{pmatrix}}_{\text{coeff. matrix}}$$

→ Subtract 3 times of 1st eqⁿ from 2nd eqⁿs.

$$R_2 - 3R_1$$

$$\begin{aligned} x - y - z &= 2 \\ 5z &= 10 \\ 2x - y + z &= 9 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 2 & -1 & 1 & 9 \end{array}\right)$$

→ Subtract 2 times the 1st eqⁿ from 3rd eqⁿ.

$$R_3 - 2R_1$$

$$\begin{aligned} x - y - z &= 2 \\ 5z &= 10 \\ y + 3z &= 5 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & 3 & 5 \end{array}\right)$$

→ Interchange eqⁿ (2) and (3).

$$R_{23}$$

$$\begin{aligned} x - y - z &= 2 \\ y + 3z &= 5 \\ 5z &= 10 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right)$$

→ $(3, -1, 2)$ solⁿ.

→ A matrix is in row echolon form if it satisfies the following properties.

1) Any row consisting of entirely of zero's are at the bottom.
2) In each non-zero row, the first non-zero entry (called the leading entities ~~entries~~) is in a column to the left ~~of~~ any leading entities below it.

→ Row operations:

1) Interchange two rows (R_{ij} or $R_i \leftrightarrow R_j$)

2) Add a multiple of a row to another row $(R_i + kR_j)$

3) Multiply a row by a non-zero constant (kR_1)

→ Reduce the following matrix to an echolen form

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}$$

① $R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow R_3 - 2R_1$

$R_4 \leftarrow R_4 + R_1$

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix}$$

② R_{23}

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix}$$

③ $R_4 \leftarrow R_4 + 3R_2$, $R_3 \leftarrow \frac{1}{8} R_3$

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 29 & 29 & -5 \end{bmatrix}$$

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④ $R_4 \leftarrow R_4 - 29R_3$.

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix}$$

→ Matrices A and B are <u>row equivalent</u> if there is a sequence of elementary row operations that converts A into B.

23/3/18 **Lecture - 20**

⇒ Gaussian Elimination

1. Write the augmented matrix of the system of linear equation.
2. Use elementary row operations to reduce the augmented matrix to row echolen form.
3. Using back substitution, solve the equivalent system that corresponds to the row reduced matrix.

Q.1 Solve the system

$$2x_2 + 3x_3 = 8$$
$$2x_1 + 3x_2 + x_3 = 5$$
$$x_1 - x_2 - 2x_3 = -5$$

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$$\begin{pmatrix} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & -2 & -5 \\ 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -2 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 2 & 3 & 8 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix} \Rightarrow x_3 = 2,\ x_2 = 1,\ x_1 = 0$$

$$w - x - y + 2z = 1$$
$$2w - 2x - y + 3z = 3$$
$$-w + x - y \quad = -3$$

$$\begin{pmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{pmatrix} \xRightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xRightarrow{R_3 \to R_3 - 2R_2} \begin{aligned} w - x - y + 2z &= 1 \\ y - z &= 1 \\ y &= 1 + z \end{aligned}$$

$\therefore\ w - x - 1 - z + 2z = 1$

$\therefore\ w - x + z = 2$

Let $x = s,\ z = t \Rightarrow w = \begin{pmatrix} 2 + s - t \\ s \\ t \\ 1 + t \end{pmatrix}$

Q.3 $x_1 - x_2 + 2x_3 = 3$

$x_1 + 2x_2 - x_3 = -3$

$2x_2 - 2x_3 = 1$

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & -2 & 1 \end{pmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 2 & -2 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2/3}$$

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & -2 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_2]{} \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

No solⁿ.

⇒ Gauss Jordan Elimination

→ A matrix is in reduced row echolen form if it satisfies the following properties.

1. It is in row echolen form.

* 2. The leading entry in each non-zero row is 1 (leading 1)

* 3. Each column containing a leading 1 has zero everywhere else.

Eg:-
$$\begin{pmatrix} 1 & 2 & 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1. Write the augmented matrix of the system of linear equation.

2. Use elementary row operations to reduce the augmented matrix to reduced row echolen form.
3. Solve.

Q.1
$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 9 \\ 2x_1 - x_2 + x_3 &= 0 \\ 4x_1 - x_2 + x_3 &= 4 \end{aligned} \Rightarrow \begin{pmatrix} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 0 & -9 & 13 & -32 \end{pmatrix} \Rightarrow$$

$R_3 \to R_3 - R_2$
$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 0 \\ 2 & 0 & 0 & 4 \end{array}\right) \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 2 & -1 & 1 & 0 \\ 1 & 2 & 3 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 2 & 3 & 7 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 0 & 5 & -1 \end{pmatrix}$$

$\cancel{x_3 = -1}$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1/5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 19/5 \\ 0 & 0 & 1 & -1/5 \end{pmatrix}$$

Check sol^n

~~Jacob~~

⇒ Iterative Methods for solving system of linear eq^n.

Jacobi

$x_1 \leftarrow a_1 x_1 + b_1 x_2 = c_1$
$x_2 \leftarrow a_2 x_1 + b_2 x_2 = c_2$

$$x_1 = \frac{c_1 - b_1 x_2}{a_1}$$

$$x_2 = \frac{c_2 - a_2 x_1}{b_2}$$

Q-1 $7x_1 - x_2 = 5$
$3x_1 - 5x_2 = -7$

$$x_1 = \frac{5 + x_2}{7} \qquad x_2 = \frac{+7 + 3x_1}{5}$$

$x_1 = 0,\ x_2 = 0 \rightarrow x_1 = 5/7,\ x_2 = 7/5$
$\quad = 0.714 \quad = 1.4$

$\downarrow$

$x_1 = 0.976, x_2 = 1.949 \leftarrow x_1 = 0.914,\ x_2 = 1.829$

$\downarrow$

$x_1 = 0.993, x_2 = 1.985 \rightarrow x_1 = 1,\ x_2 = 2$

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Lecture - 21

⇒ Gauss Seidal iterative method

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}\right\} \qquad x_1^{(1)} = \frac{b_1 - a_{12}x_2^{(0)}}{a_{11}} \qquad x_2^{(1)} = \frac{b_2 - a_{21}x_1^{(1)}}{a_{22}}$$

$$x_1^{(2)} = \frac{b_1 - a_{12}x_2^{(1)}}{a_{11}} \qquad x_2^{(2)} = \frac{b_2 - a_{21}x_1^{(2)}}{a_{22}}$$

Q.1 $7x_1 - x_2 = 5$ $\qquad x_1^{(0)} = 0$

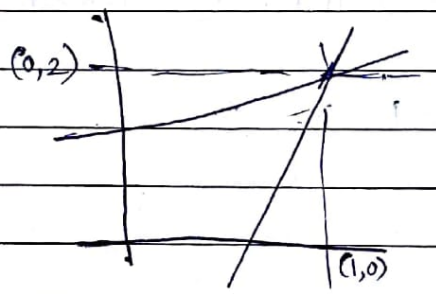
$3x_1 - 5x_2 = -7$ $\qquad x_2^{(0)} = 0$

→ $x_1^{(1)} = \dfrac{5 + x_2^{(0)}}{7}$ $\qquad x_2^{(1)} = \dfrac{7 + 3x_1^{(1)}}{5}$

$x_1^{(1)} = \dfrac{5}{7}$ $\qquad x_2^{(1)} = \dfrac{7 + \frac{15}{7}}{5} = \dfrac{64}{7 \times 5}$

$x_1^{(2)} = 0.975$ $\qquad x_1^{(3)} = 0.99$ $\qquad x_1^{(4)} = 1$

$x_2^{(2)} = 1.985$ $\qquad x_2^{(3)} = 1.9$ $\qquad x_2^{(4)} = 2$



⇐ How both iterative methods converge

→ General for n eq^n :

$$\begin{aligned} x_1 &\leftarrow a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ x_2 &\leftarrow a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\vdots \\ x_n &\leftarrow a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{aligned}$$

Q.2 $x_1 - x_2 = 1$ $\qquad x_1^{(0)} = 0$

$2x_1 + x_2 = 5$ $\qquad x_2^{(0)} = 0$

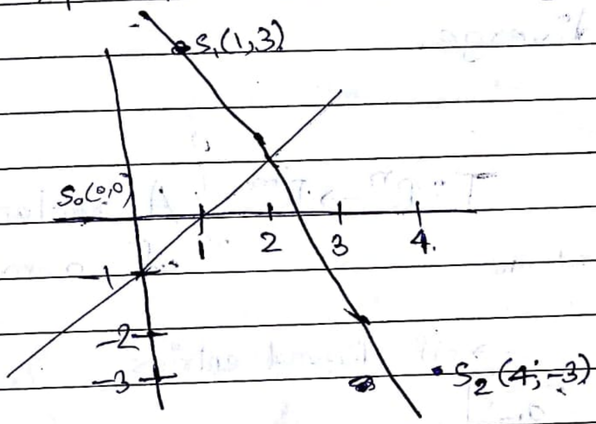
$x_1^{(1)} = 1 + x_2^{(0)}$ $\qquad x_2^{(1)} = 5 - 2x_1^{(1)}$

$x_1^{(1)} = 1$ $\qquad x_2^{(1)} = 3$

$x_1^{(2)} = 4$ $\qquad x_2^{(2)} = -3$

$x_1^{(3)} = -2$ $\qquad x_2^{(3)} = 9$

$x_1^{(4)} = 10$ $\qquad x_2^{(4)} = -15$



→ $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$

$\vdots$

$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

We say that A is strictly diagonally dominant if

$$\left.\begin{aligned} |a_{11}| &> |a_{12}| + |a_{13}| + \dots + |a_{1n}| \\ |a_{22}| &> |a_{21}| + \dots + |a_{2n}| \\ &\vdots \\ |a_{nn}| &> |a_{n1}| + \dots + |a_{nn}| \end{aligned}\right\} \quad |a_{ii}| > \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \quad \forall i$$

→ If this is satisfied then it will always converge.
If this is not satisfied then it may converge or diverge.

→ $\Big[\quad\Big]_{m \times n}$ $m \times n$ → columns, ↳ rows

$T : R^n \to R^m$ } A rectangle array of numbers

→ off diagonal entries

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix} \qquad A = [a_{ij}]_{m \times n}$$

↳ diagonal entries → diagonal matrix.

→ If $m = n$ → The matrix is called an square matrix.

→ A diagonal matrix where all diagonal entries are equal is called 'scalar matrix'.

→ $C = A + B$ (when sizes are same)
↳ matrix addition.

→ Scalar Multiplication.

$$cA = c\,[a_{ij}]$$

→ Matrix Multiplication.

$$C = AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nr} \end{bmatrix}_{n\times r}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

→ A is of size $m\times n$.
A^T is obtained by interchanging rows and column $A^T_{n\times m}$

$A^T = A \Rightarrow a_{ij} = a_{ji}$ → symmetric matrix.

$A^T = -A$ → skew symmetric

$a_{ij} = -a_{ji}$

$a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$.

→ $AA^{-1} = A^{-1}A = I$

$$A^{-1} = \text{adj}A \cdot \frac{1}{(\det A)}$$

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Lecture - 22

$$A = \begin{array}{c} (-1)^{1+1} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

Eg: $x + 2y = 3.$
$3x + 4y = -2.$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$Ax = b.$$
$$x = A^{-1}b$$

① If. A is invertible, then A^{-1} is also invertible. and

$$(A^{-1})^{-1} = A$$

② If A is an invertible matrix and c is a non-zero scalar, then cA is invertible and.

$$(cA)^{-1} = \frac{1}{c}A^{-1}$$

③ If A and B are invertible matrices of the same size then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

④ If A is an invertible matrix and n is a positive integer, then A^{-n} is defined by

$$A^{-n} = (A^{-1})^n = (A^n)^{-1}.$$

Def^n An elementary matrix is any matrix that can be ~~denoted~~ obtained by performing elementary row operation on an identity matrix.

① $R_i \longleftrightarrow R_j$
② kR_j
③ $R_i \pm kR_j$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

$$I_4 \xrightarrow{3R_2} E_1$$
$$I_4 \xrightarrow{R_1 \leftrightarrow R_3} E_2$$
$$I_4 \xrightarrow{R_4 - 2R_2} E_3$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & & & \\ a_{31} & & & \\ a_{41} & & & a_{44} \end{bmatrix} \xrightarrow{3R_2} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 3a_{21} & 3a_{22} & 3a_{23} & 3a_{24} \\ a_{31} & & & \\ a_{41} & & & \end{bmatrix}$$

$$'' \; E_1 A$$

→ Let E be an elementary matrix obtained by performing elementary row operation on I_n. If same elementary row operation is performed on an $n \times n$ matrix A, the result is the same elementary matrix EA.

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$I_3 \xrightarrow{R_2 \leftrightarrow R_3} E_1$ $I_3 \xrightarrow{4R_2} E_2$ $I_3 \xrightarrow{R_3 + 2R_1} E_3$

$I_3 \xleftarrow{R_2 \leftrightarrow R_3} E_1$ $I_3 \xleftarrow{\frac{1}{4}R_2} E_2$

$E_1^2 = E_1 \circ E_1 = I_3$ $E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$E_1 = E_1^{-1}$

→ Let A be a $n \times n$ matrix. The following statements are equivalent statements:

a) A is invertible

b) $A\bar{x} = \bar{b}$ has an unique sol^n for any $b \in R^n$.

c) $A\bar{x} = \bar{0}$ has a trivial sol^n.

d) The reduced row echolon form of A is I_n.

e) A is a product of elementary ~~matrix~~ matrices.

→ $d \Rightarrow e$

$A \xrightarrow{E_1} \xrightarrow{E_2} \cdots I_n$

$$A' = E_1 A$$
$$E_2 A' = E_2 E_1 A$$
$$(E_j \dots E_2 E_1) A = I_n$$
$$A = (E_j \dots E_2 E_1)^{-1} I_n$$
$$= E_1^{-1} E_2^{-1} \dots E_j^{-1}$$

→ Let A be a square matrix. If seq. of elementary row operation reduces A to I_n then same seq. of ~~ele~~ elementary row operation transform I to A^{-1}.

$$(E_1 E_2 \dots E_k) A = I$$

$$IA = A$$
$$((E_1 E_2 \dots E_k) I) A = I$$
$$(E_1 E_2 \dots E_k) I = A^{-1}$$

⇒ Gauss Jordan method for finding inverse.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 + 2R_3]{R_1 + R_3} \quad \xrightarrow{R_1 - 2R_2}$$

$$A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}.$$

→ $A = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$ $\qquad$ $A = \begin{bmatrix} 2 & 1 & -4 \\ -4 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix}$

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Lecture-23

Eigenvalues and Eigenvectors

→ If A is a $n \times n$ matrix. The eigenvalues of A is a scalar quantity 'λ' such that $\boxed{A\bar{x} = \lambda\bar{x}}$, where $\bar{x}$ is the eigenvector corresponding to the eigenvalues 'λ'.

→ $\underset{n\times n}{A}\bar{x} = \lambda\bar{x}$

↳ real/complex

$$(A - \lambda I_n)\bar{x} = 0$$

⇒ Caylay Hamiltan Theorem.

→ Every square matrix satisfies its own characteristic eq^n.

$$\det(A - \lambda I_n) = 0$$

Q. Find out the eigenvalues and eigenvectors of the matrix A.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

$\rightarrow$ $\det(A - \lambda I_n) = 0$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda^2(4-\lambda) + 2 - 5\lambda = 0$$

$$\therefore \quad 4\lambda^2 - \lambda^3 - 5\lambda + 2 = 0$$

$$\therefore \quad (\lambda-1)^2(\lambda-2) = 0$$

$\therefore \lambda = 1$ or 2. $\leftarrow$ Eigenvalues

<u>Case-I</u> $\lambda_1 = \lambda_2 = 1$

$\rightarrow$ $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{bmatrix} \bar{x} = 0 \qquad \bar{x} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\lambda=1}$

Sir Method $\rightarrow$ $(A - I \mid 0) = \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{array}\right]$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 - x_3 = 0$$
$$x_2 - x_3 = 0$$
$$\therefore x_1 = x_2 = x_3 = t$$

span $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ geometric multiplicity $= 1$

Case II $\lambda = 2$.

$$(A - 2I)\bar{x} = 0$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 2 & -5 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 - \frac{x_3}{4} = 0$$

$$x_2 - \frac{x_3}{2} = 0$$

$$x_3 = t, \quad x_1 = \frac{t}{4}, \quad x_2 = \frac{t}{2}$$

span $\left\{\begin{bmatrix}1/4\\1/2\\1\end{bmatrix}\right\}$ gem $= 1$

Q. Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

$$(A - \lambda I) = 0$$

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(-\lambda)(-1-\lambda) + \lambda = 0$$

$$\therefore (1+\lambda)^2(-\lambda) + \lambda = 0$$

$$\therefore (\lambda^2+2\lambda+1)(-\lambda) + \lambda = 0$$

$$-(\lambda^2+2\lambda)\lambda = 0$$

$$-\lambda^2(\lambda+2) = 0$$

$$\therefore \lambda = 0 \text{ or } -2.$$

Case I $\lambda = 0$

$$\begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix} \bar{x} = 0 \qquad \bar{x} = \begin{bmatrix} t \\ k \\ t \end{bmatrix}_{\lambda=0}$$

gem = 2.

Case II $\lambda = -2.$

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{bmatrix} \bar{x} = 0 \qquad \bar{x} = t\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}_{\lambda=-2.}$$

gem = 1.

(1) → A square matrix A is invertible iff 0 is not an eigenvalue of A.

→ Let A be $n \times n$ matrix with eigenvalue λ and corresponding eigenvector $\bar{x}$.

(2) a) For any positive integer 'n', λ^n is an eigen value of A^n with corr. eigenvector $\bar{x}$.

(3) b) If A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigen vector $\bar{x}$.

~~Q A square matrix A is invertible iff 0 is not an eigenvalue of A~~

(1)
$$\det A \neq 0$$
$$\Rightarrow \det(A - 0I) \neq 0$$
$\Rightarrow$ 0 is not an eigenvalue of A

(2) For $n=1$.
$$A\bar{x} = \lambda \bar{x}$$

Let us assume it is true for $n=k$

$$A^k \bar{x} = \lambda^k \bar{x}$$

$$A^{k+1}\bar{x} = A(A^k \bar{x})$$

$$A(\lambda^k \bar{x}) = \lambda^k (A\bar{x}) = \lambda^{k+1}\bar{x}$$

(3)
$$Ax = \lambda x$$
$$A^{-1}A x = A^{-1}\lambda x \qquad (\because A \text{ is invertible})$$
$$\therefore x = \lambda A^{-1} x$$

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$$\therefore A^{-1}x = \frac{1}{\lambda}x.$$

Q. Let A be $n\times n$ matrix and let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be distinct eigenvalues of A with corresponding eigenvalues $v_1, v_2, \cdots, v_m$. Then show that $v_1, v_2, \cdots, v_m$ are linearly independent.

→ Let us assume $v_1, \ldots, v_{k+1}, \ldots, v_m$ are L.D.

$$v_{k+1} = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

$(v_1, v_2, \cdots, v_k$ are LD$)$ —— ①

$$A v_{k+1} = c_1 (A v_1) + c_2 (A v_2) + \cdots + c_k (A v_k)$$

$$\lambda_{k+1} v_{k+1} = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_k \lambda_k v_k$$ —— ②

$$\lambda_{k+1} v_{k+1} = c_1 \lambda_{k+1} v_1 + c_2 \lambda_{k+1} v_2 + \cdots + c_k \lambda_{k+1} v_k$$

(Mult. λ_{k+1} in ① on both sides) —— ③

③ − ②

$$0 = c_1(\lambda_1 - \lambda_{k+1}) v_1 + c_2 (\lambda_2 - \lambda_{k+1}) v_2 + \cdots + c_k (\lambda_k - \lambda_{k+1}) v_k$$

$$c_i (\lambda_i - \lambda_{k+1}) = 0 \quad \forall i$$

$c_i \neq 0 \;\because$ LD, $\quad (\lambda_i - \lambda_{k+1}) \neq 0 \;\because$ eigenvalues are unique

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Lecture - 24

Q.1 Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be a complete set of eigenvalues (repetition included) of the $(n \times n)$ matrix A.

Prove that: $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

$tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. (5)

Q.2 Show that if sparse matrix A can be partitioned as

$$A = \left[\begin{array}{c|c} P & Q \\ \hline 0 & S \end{array}\right],$$ where P and S are sparse matrices.

then the characterstic polynomial of A is

$$C_A(\lambda) = C_P(\lambda)\, C_S(\lambda).$$ (5)

Q.3 Solve

$w - x - y + 2z = 1$

$2w - 2x - y + 3z = 3$

$-w + x - y \quad = -3$ (5)

Q.4 Show that a rotation about the origin through an angle θ defines a L.T from $\mathbb{R}^2 \to \mathbb{R}^2$ and find its standard matrix. (5)

Q.5 $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a L.T.

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ x + y - 3z \end{bmatrix}$$ Let, $B = \{e_1, e_2, e_3\}$, $C = \{e_2, e_1\}$

Find the matrix T. (5)

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① Show that $\{1, \cos t, \cos^2 t, \ldots, \cos^6 t\}$ is a linearly independent set of functions defined on $\mathbb{R}$? (5)

② Show that A and A^T have the same eigen-values? (5)

③ Apply the Gauss-seidal method to solve the system.

$$10x_1 + x_2 - x_3 = 18$$
$$x_1 + 15x_2 + x_3 = -12$$
$$-x_1 + x_2 + 20x_3 = 17$$

(5)

④ If A is an invertible matrix $(n \times n)$ and B is $(n \times p)$ matrix. Show that equation $AX = B$ has unique sol^n $A^{-1}B$ (5)

⑤ Let H be a non-zero subspace of V and let $T(H)$ be the set of images of vectors in H. Prove $\dim T(H) \leq \dim H$.