Mid1 - ~~B~~ Mid2

Eg: Evaluate $\oint \tanh z \, dz$

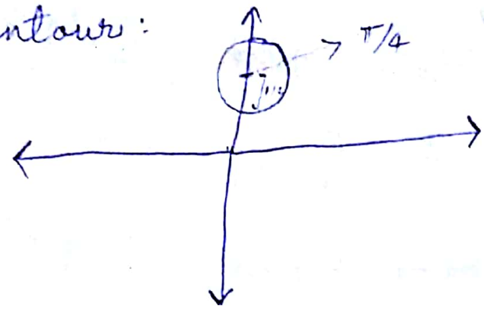
$c = \left| z - \frac{\pi i}{4} \right| = \frac{1}{2}$

Tanhz is singular at pts where $\cosh z = 0$

$\cosh z = \cos(iz)$

$\therefore 0$ at $z = \pm \frac{\pi i}{2}, \pm \frac{3\pi i}{2}, \ldots$

Contour:



$\therefore$ points are outside the contour.

$\therefore I = 0.$

Eg: $\oint \frac{7z - 6}{z^2 - 2z}$ Contour: Given that Contour has ~~Z~~ 2, 0.

$$\frac{7z-6}{z^2-2z} = \frac{az}{(z-2)} + \frac{bz - 2b}{(z)}$$

$a + b = 7.$

$b = 3.$

~~b~~ $a = 4$

$$\oint \left(\frac{4}{z-2} + \frac{3}{z} \right) dz$$

$\Rightarrow 14\pi i$

(Use the result

$\oint (z - z_0)^m \, dz = 2\pi i$

$m = -1.$

Q: ## Cauchy's Integral Formula

Let $f(z)$ be Analytic in a simply connected domain D then for any point z_0 in D, and any simple closed ~~point~~ path C in D, that encloses z_0;

$$\oint_C \frac{f(z)}{z-z_0}\, dz = f(z_0)\, 2\pi i$$

Proof: $f(z) = f(z_0) + [f(z) - f(z_0)]$

$$\Rightarrow I = \underbrace{\oint \frac{f(z_0)}{z - z_0}}_{f(z_0).\, 2\pi i} + \underbrace{\oint \frac{f(z) - f(z_0)}{z - z_0}}_{\text{Should prove that this integration}}$$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \frac{\epsilon}{e}$$

Length of Contour: $2\pi e$

$C \Rightarrow$ (Circle of radius e)

By ML theorm: $I' < \frac{\epsilon}{e} . 2\pi e$

$I' \to 0$

as $f(z)$ tends to $f(z_0)$

Eg: $\oint \frac{e^z}{z-2}$ $C \Rightarrow$ has 2 in it

$I = e^z \big|_{z=2} \times 2\pi i$

$\therefore I = 2\pi i e^2$

4 $\displaystyle\oint_C \frac{z^2+1}{z^2-1}$

① Contour: Circle centered at -1 and radius 1.

$$\oint \frac{\left(\frac{z^2+1}{z-1}\right)}{z+1} \Rightarrow -2\pi i$$

Note that $z=1$ is not present in the contour.

② Contour: circle with center $z=1$.

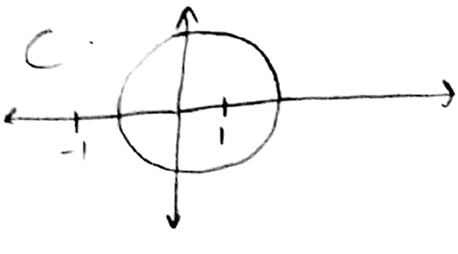
$I = 2\pi i$

③ $z=i$ as the center

No singularity.

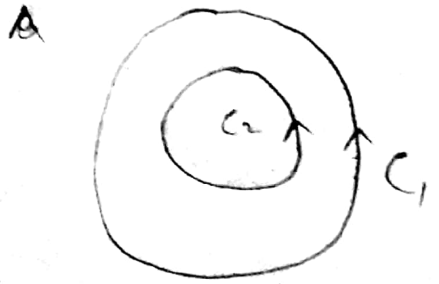
$\therefore I = 0$.

④ C.



same as second case

Note: If there are multiple contours,



$$\Rightarrow f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-z_0} - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-z_0}$$

Eg: $\int \frac{e^{3z}}{3z - i}$ $C : |z| = 1$

singularity : $i/3$.

$$\Rightarrow \frac{1}{3} \int \frac{e^{3z}}{z - i/3}$$

$$\Rightarrow \frac{1}{3} (2\pi i) \cdot e^{i/3 \cdot 3} \Rightarrow \frac{2\pi i}{3} e^{i}$$

Eg $\oint_C \frac{e^{z}}{z - 2i}$ $C : |z - 2i| = 4$

Lies inside the cont.

$$\Rightarrow 2\pi i \, e^{2i}$$

Result

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0} \, dz$$

$\cancel{f'(z_0) = \lim_{\Delta}}$ Goal to find $f'(z_0)$.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \Rightarrow ①$$

By Cauchy's Integral formula:

$$\oint \frac{f(z)\, dz}{z - (z_0 + \Delta z)} = 2\pi i \, f(z_0 + \Delta z) \longrightarrow ②$$

$$\oint \frac{f(z)}{z - z_0} = 2\pi i f(z_0) \longrightarrow ③$$

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{1}{\Delta z\, 2\pi i}\oint \frac{f(z)\,dz}{z-(z_0+\Delta z)} - \frac{1}{2\pi i}\oint \frac{f(z)\,dz}{z-z_0}$$

$$\Rightarrow \frac{1}{\Delta z\, 2\pi i}\oint\left[\left(\frac{f(z)}{z-(z_0+\Delta z)} - \frac{f(z)}{z-z_0}\right)\right] dz$$

$$\Rightarrow \frac{1}{\Delta z\, 2\pi i}\left(\frac{\Delta z \cdot f(z)\quad dz}{(z-(z_0+\Delta z))(z-z_0)}\right)$$

$$\Rightarrow \frac{1}{2\pi i}\int \frac{f(z)}{(z-z_0)^2} \qquad (\text{AS } \Delta z \to 0)$$

$$\Rightarrow \boxed{f'(z_0) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-z_0)^2}\, dz}$$

Similarly $\quad f''(z_0) = \dfrac{2!}{2\pi i}\oint \dfrac{f(z)}{(z-z_0)^3}\, dz$

Class

Taylor's Series :

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a$$

$+ \dots$

Although you are adding infinitely many terms, it results in a finite no. as all these terms are < 1
This is called convergence.

Now this result is valid only if $|z|<1$.
This is called radius of convergence.

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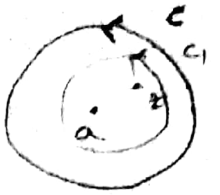
Proof: (Taylor series)



z is a pt in C

a is the center

Consider C_1 : It is another circle centered at a enclosing z.

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w-z}\, dw$$

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left(\frac{1}{1-\underbrace{\left(\frac{z-a}{w-a}\right)}}\right)$$

This quantity is less than 1.

$$\Rightarrow \frac{1}{w-a}\left(1+\left(\frac{z-a}{w-a}\right)^1+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^n\right)$$

$n \to \infty$

$$f(z) = \frac{1}{2\pi i}\int_{C_1} f(w) \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{C_1} \frac{f(w)\, dw}{(w-a)^{n+1}} (z-a)^n$$

$$\boxed{a_n = \frac{1}{2\pi i} \int \frac{f(w)}{(w-a)^{n+1}}\, dw}$$

This is $\frac{f^{n}(a)}{n!}$

$\Rightarrow$ Proved.

Eg: $\dfrac{1}{1-z}$ about $z_0 = 2i$

$$\Rightarrow \frac{1}{1-z+2i-2i} = \frac{1}{1-2i}\left(\frac{1}{1-\left(\frac{z-2i}{1-2i}\right)}\right)$$

$$\Rightarrow \frac{1}{1-2i}\left(\sum_{n=0}^{\infty}\left(\frac{z-2i}{1-2i}\right)^n\right)$$

$$\sum_{n=0}^{\infty}\int_c \cancel{\frac{1}{1-2i}} \cdot \left(\frac{z-2i}{1-2i}\right)^{n+1} dz$$

$$\Rightarrow \frac{1}{1-a} + \frac{1}{(1-a)^2}(z-a) + \cdots$$

Eg: $f(z) = e^z$ around $z=0$.

$$\Rightarrow \cancel{1 + f(z) +}$$

$$1 + z + \frac{z^2}{2!} + \cdots$$

Eg:

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Eg: $f(z) = \dfrac{e^z}{1-z}$ around $z=0$, ~~r~~ find radius of convergence.

$$e^z \Rightarrow 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$f(z) = \frac{1}{1-z} \Rightarrow 1 \cancel{+} - z + \frac{2z^2}{2!} - \frac{3z^3}{3!} + \frac{4z^4}{4!} \cancel{+} \cdots$$

$$\Rightarrow 1 + z + \frac{z^2}{1!} \cancel{+} + \frac{z^3}{\cancel{2!}} + \cdots$$

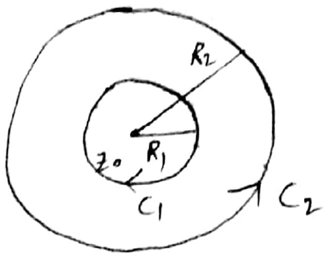
Multiply $\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)\left(1 - z + \frac{2!z^2}{2!} \cancel{+} - \frac{3!z^3}{3!} + \cdots\right)$

$\Rightarrow$

Laurent's Theorem

f(z) is analytic inside and on the boundary of the ring shaped region R bounded by 2 concentric circles C_1 and C_2 with center at a and respective radii r_1, r_2

Then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}$



$$f(z) = \underbrace{\frac{1}{2\pi i} \int_{C_2} \frac{f(\alpha)\,d\alpha}{\alpha - z}}_{I_1} + \underbrace{\frac{1}{2\pi i} \int_{C_1} \frac{f(\alpha)\,d\alpha}{\alpha - z}}_{I_2}$$

on C_2

$$\left| \frac{z - z_0}{\alpha - z_0} \right| < 1$$

on C_1

$$\left| \frac{\alpha - z_0}{z - z_0} \right| < 1.$$

$$I_2 \Rightarrow \frac{1}{\alpha - z} \Rightarrow \frac{1}{(\alpha - z_0) - (z - z_0)} = \frac{-1}{(z - z_0)\left(1 - \frac{(\alpha - z_0)}{z - z_0}\right)}$$

$$\Rightarrow \frac{-1}{z-z_0}\left[\sum_{n=0}^{\infty}\left(\frac{\alpha-z_0}{z-z_0}\right)^n\right]$$

$$-\sum_{n=0}^{\infty}\frac{(\alpha-z_0)^n}{(z-z_0)^{n+1}}$$

$$\Rightarrow -\sum_{m=1}^{\infty}\frac{(\alpha-z_0)^{m-1}}{(z-z_0)^m}$$

$$I_2 = \sum_{m=1}^{\infty}\left[\frac{-1}{2\pi i}\right]\int f(\alpha)\,\frac{(\alpha-z_0)^{m-1}}{(z-z_0)^m}\,d\alpha$$

Change direction

$$I_2 = \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n}$$

$$b_n = \frac{1}{2\pi i}\int \frac{f(\alpha)\,d\alpha}{(\alpha-z_0)^{-n+1}}$$

<u>Poles</u> If $f(z) = \dfrac{1}{z-a}$, $z=a$ is pole of order 1.

$\dfrac{1}{(z-a)^2}$,, $z=a$ is a pole of order 2.

and so on.

Removable singularity : If you can remove singularity by manipulating $f(z)$.

Essential singularity : When there are infinite no. of terms.

Eg: $\dfrac{e^{2z}}{(z-1)^3}$ find the expansion

$$z-1=u$$

$$z=u+1$$

$$\Rightarrow \frac{e^{2u}}{u^3}\cdot e^2$$

$$\Rightarrow \frac{e^2}{u^3}\left[1+2u+\frac{(2u)^2}{2!}+\frac{(2u)^3}{3!}+\cdots\right]$$

$$\Rightarrow e^2\left[\frac{1}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{2^2}{2!(z-1)}+\frac{2^3}{3!}+\cdots\right]$$

Triple pole.

q: $(z-3)\sin\left(\dfrac{1}{z+2}\right)$

$$z+2=u$$

$$(u-5)\sin\left(\frac{1}{u}\right)$$

$$1-x+\frac{x^3}{3!}-\frac{x^5}{5!}+\cdots$$

$$(u-5)\left(1-\frac{1}{u}+\frac{1}{(3!)u^3}-\frac{1}{(5!)u^5}+\cdots\right)$$

Essential Singularity.

Eg: $\dfrac{z}{(z+1)(z+2)} \Rightarrow \dfrac{-1}{z+1} + \dfrac{2}{z+2}$

$\Rightarrow \dfrac{-1}{(z+2)-1} = \dfrac{1}{1-(z+2)}$

$|(z+2)|<1$

$1+(z+2)+(z+2)^2+\dots + \dfrac{2}{z+2}$

simple pole at $z=-2$.

$0<|z+2|<1$

Only in this region, it is convergent.

Eg: $\dfrac{1}{z^2(z-3)^2}$; $\underline{z=3}$

$z-3=u \qquad \Rightarrow \dfrac{1}{(u+3)^2u^2} \Rightarrow \dfrac{1}{9u^2\left(1+\frac{u}{3}\right)^2}$

$\Rightarrow \dfrac{1}{9u^2}\left(1+(-2)\dfrac{u}{3}+\dfrac{(-2)(-3)}{2!}\left(\dfrac{u}{3}\right)^2+\dfrac{(-2)(-3)(-4)}{3!}\left(\dfrac{u}{3}\right)^3 \dots\right.$

$\Rightarrow \dfrac{1}{9u^2}-\dfrac{2}{27u}+\dfrac{1}{27}-\dfrac{4u}{243}+\dots$

Double pole

$0<|z-3|<3.$

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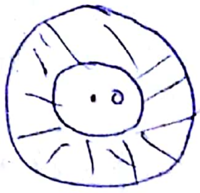
Eg: v $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent series valid for

Expand

(i) $1 < |z| < 3$ (b) $|z| > 3$ (c) $0 < |z+1| < 2$

(d) $|z| < 1$

(i)



$1 < |z| < 3$

$\Rightarrow \left|\dfrac{z}{3}\right| < 1 , \dfrac{1}{|z|} < 1$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$$

$$\frac{1}{2}\left(\frac{1}{z\left(1+\frac{1}{z}\right)}\right) = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \dots\right)$$

$$\frac{1}{2}\left(\frac{1}{z+3}\right) \Rightarrow \frac{1}{6}\left(\frac{1}{1+\left(\frac{z}{3}\right)}\right)$$

$$\Rightarrow \frac{1}{6}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \dots\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \dots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \dots\right)$$

Maths Class

Singularity : function is not analytic at z_0 but is analytic at other pts in the neighbourhoods.

Isolated : Singular at z_0, but is not singular in an ϵ neighbourhood.

$$0 < |z - z_0| < \epsilon$$

Eg: $\dfrac{(z+i)^3}{z^2 (z+i)^2}$

$z = 0, -i$: Isolated singularity

Eg: $\dfrac{1}{\sin(\pi/z)}$ Singular at $z = \frac{1}{n}$ $n = \pm 1, \pm 2, \ldots$

All points except $n = 0$, isolated singularity

For $z \to 0$ $\sin \pi/z$ flutuates very fast for even an ϵ change, the function is either $+1$ or -1. The function is not defined.

Similarly $\tan\left(\frac{\pi}{2z}\right)$.

Principle part : All -ve powers in Laurent series

If no. of singularities is infinite : essential Singularity

Removable Singularity : When there exists limit tending to z_0 for $f(z)$ but is not defined at z_0, we call it removable singularity.

In principal part of Laurent's series:

① No principal part: Removable singularity

② Finite number of nonzero terms · pole

③ Infinite no. of non zero terms: essential pole.

Zero of a function: $f(z) = 0$.

Zero of order n: $f(z_0) = f'(z_0) = \cdots f^{n-1}(z_0) = 0$

~~Residue of a function [Method to integrate]~~

Methods for finding ~~poles~~ Residue:

① I (Simple poles): So only one term in the laurent's series.

$$\therefore b = \lim_{z \to z_0} (z - z_0) f(z)$$

② When it is not simple, pole of order m:

$$b_1 = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z - z_0)^m f(z) \right\}$$

Maths Class

Cauchy Residue theorm

$$\oint f(z)\,dz = \sum_{k=0}^{n} 2\pi i\ (\mathrm{Res}(k))$$

when there are n singularities inside the contour.

Eg: $\oint \frac{4-3z}{z^2-z} \Rightarrow \oint \frac{4-3z}{z(z-1)}$

(a) 0, 1 are inside

$$\mathrm{Res}(0) = \frac{\left(\frac{4-3z}{z-1}\right)_{z=0}}{1} = -4$$

$$\mathrm{Res}(1) = 1$$

$$I = (-4+1)\,2\pi i = -6\pi i$$

(b) 0, inside, 1 outside : $-8\pi i$

(c) 1 inside, 0 outside : $2\pi i$

(d) both are outside : 0

Real Applications of Contour Integration:

1. $I = \int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$

Let's say it's not easily integrable

C : $z = e^{i\theta}$

$dz = i\, e^{i\theta}$

$$d\theta = \frac{-i\,dz}{z}$$

Now replace $\sin\theta \rightarrow \frac{z - z^{-1}}{2}$

$$\cos\theta \rightarrow \frac{z + z^{-1}}{2}$$

Eg: $I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}$

$$\cos\theta = \frac{z + \bar{z}}{2} \qquad d\theta = \frac{-i\,dz}{z}$$

$$\Rightarrow \oint \frac{-i\,dz}{z\left(1 + \frac{az + \frac{a}{z}}{2}\right)} \Rightarrow \oint \frac{-2i\,dz}{2z + az^2 + a}$$

$$\Rightarrow -i\frac{2}{a} \oint dz \frac{1}{\left[z^2 + \frac{2}{a}z + 1\right]}$$

$= 0$ at

$$z = \frac{\frac{-2}{a} \pm \sqrt{\frac{4}{a^2} - 4}}{2}$$

But only $\frac{\frac{-2}{a} + \sqrt{\frac{4}{a^2} - 4}}{2}$ is inside contour.

$$\Rightarrow \frac{1}{a}\left(-1 + \sqrt{1 - a^2}\right)$$

$$\Rightarrow 2\pi i\left(-i\frac{2}{a}\right)\left(\frac{1}{2\left(\frac{-1}{a} + \frac{\sqrt{1-a^2}}{a}\right) + \frac{2}{a}}\right)$$ [Apply the Residue formula

$$\Rightarrow \frac{4\pi}{a} \times \frac{1}{\frac{2\sqrt{1-a^2}}{a}}$$

$$\Rightarrow \frac{4\pi}{a\frac{2\sqrt{1-a^2}}{a}} \Rightarrow \frac{2\pi}{\cancel{a}\sqrt{1-a^2}}$$

Eg: $\int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{5 - 4\cos\theta}$

$$\Rightarrow -i \oint \frac{dz}{z} \frac{(z^2+z^{-2})/2}{5 - 4\left(\frac{z+z^{-1}}{2}\right)}$$

$$\Rightarrow -i \oint \frac{dz\left(z^2+\frac{1}{z^2}\right)}{z\left(10-4\left(z+\frac{1}{z}\right)\right)} \Rightarrow 4z^2-10z+4$$

$$10z - 4z^2 - 4 = 0$$

$$4z^2 - 10z + 4 = 0 \qquad \frac{10 \pm 6}{8}$$

$$2, \tfrac{1}{2}$$

$$z = 2, \frac{1}{2}, 0$$

outside

$$\Rightarrow \text{Residue at } 1/2: \quad \frac{\frac{1}{4}+4}{-6} = \frac{-17}{24}$$

Residue at $z = 0$: $\dfrac{\left[\dfrac{z^4+1}{4z^2-10z+4}\right]}{z^2} \Rightarrow f(z)$

$$\frac{d}{dz}\left(\frac{z^6+z^2}{4z^2-10z+4}\right)\Bigg|_{z=0.}$$

$$\frac{(4z^2-10z+4)\;(6z^5+2z) - (z^6+z^2)\,(8z-10)}{(4z^2-10z+4)^2}$$

$$\Rightarrow \boxed{I = 2\pi\left(-\frac{5}{8}+\frac{17}{24}\right)}$$

Eg: $\displaystyle\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$

$\cos\theta \Rightarrow \dfrac{z+z^{-1}}{2}$ $\quad \sin\theta = \dfrac{z-z^{-1}}{2}$

$$\Rightarrow \int_0^{2\pi} \frac{\cancel{d\theta}}{} \;\frac{-i\,dz}{z}\left(\frac{1}{3-z-z^{-1}+\frac{z-z^{-1}}{2}}\right.$$

$$\Rightarrow \int_0^{2\pi} \frac{-2i\,dz}{z}\left(\frac{1}{6-2z-\frac{2}{z}+z\ldots}\right.$$

$$\Rightarrow \int_0^{2\pi} \frac{-2i\,dz}{z}\left(\frac{1}{6-z-\frac{3}{z}}\right)$$

$$z = 2-i,\ \underbrace{\left(\frac{2}{5}-\frac{i}{5}\right)}_{\text{Lies inside}}$$

Eg: $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ $\quad a > |b|$

$$\int \frac{-i\,dz}{z} \left(\frac{2}{2a + bz + \frac{b}{z}} \right)$$

$$\Rightarrow \oint \frac{-2i\,dz}{2az + bz^2 + b}$$

$bz^2 + 2az + b = 0.$

$4(a^2-b^2)$

$D = 2\sqrt{a^2-b^2}$

$z = \dfrac{-a \pm \sqrt{a^2-b^2}}{b}$

$z = \dfrac{-a}{b} \pm \sqrt{\left(\frac{a}{b}\right)^2 - 1}$

Singularity

$$\Rightarrow (-2i) \int \frac{dz}{b\left(z - \left(\frac{-a}{b} - \sqrt{\left(\frac{a}{b}\right)^2-1}\right)\right)\left(z - \left(\frac{-a}{b} + \sqrt{\ }\right)\right)}$$

$$\Rightarrow \quad \text{Residue} = \frac{1}{\cancel{-\frac{a}{b}} + 2\sqrt{\left(\frac{a}{b}\right)^2-1} + \cancel{\frac{a}{b}}}$$

$$\Rightarrow \frac{b}{2\sqrt{a^2-b^2}}$$

$$I = \frac{-2i}{b}(2\pi i)\left(\frac{b}{2\sqrt{a^2-b^2}}\right) \Rightarrow \frac{2\pi}{\sqrt{a^2-b^2}}$$

Eg: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$

$\Rightarrow \displaystyle\lim_{R \to \infty} \int_{-R}^{R} f(z)$.

Find roots of x^4+1

$x^4 = -1$

$r^4 e^{i4\theta} = (-1)\,1$

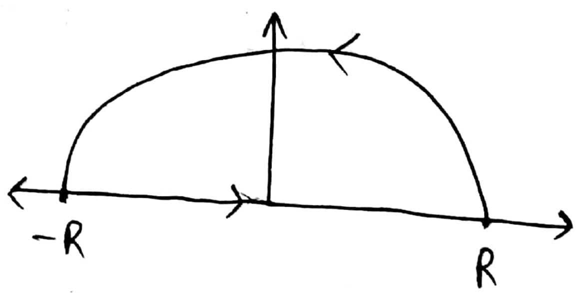
$r = 1$.

$e^{i4\theta} = e^{i(\pi \pm 2k\pi)}$

$\Rightarrow \theta = \frac{k\pi}{2} + \frac{\pi}{4}$

$\Rightarrow \underbrace{e^{\pi i/4}, e^{3\pi i/4}}_{\text{Only these pts lie inside}}, e^{5\pi i/4}, e^{7\pi i/4}$

Check for those singularities which lie in the full semi circle:



Eg: $\int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+5)^2}$

$x^2 - 2x + 5 = 0$

$x = \frac{2 \pm 4i}{2}$

$\Rightarrow 1 \pm 2i$

$$\left[\frac{1}{[z - 1 + 2i]^2} \right]'_{z=z_1} \Rightarrow \frac{-2}{(z_1 - z_2)^3} = \frac{1}{32i}$$

$$\therefore I = 2\pi i \left(\frac{1}{32i} \right) = \frac{\pi}{16}$$

Probability

A- priori probability — Probability is already known before the expt. It is measured as frequency probability.

For n mutually exclusive events:

$$P(A_1 \cup A_2 \cdots A_n) = P(A_1) + P(A_2) \ldots + P(A_n).$$

Thearms

1. If $\phi \Rightarrow$ empty set, P.T. $P(\phi) = 0$.

$$P(A) = P(A \cup \phi) = P(A) + P(\phi)$$

$$\Rightarrow P(\phi) = 0.$$

2. $P(A^c) = 1 - P(A)$ $\quad P(S) = P(A \cup A^c) = P(A) + P(A^c)$

$= 1.$

3. If $A \subset B$, $P(A) \leq P(B)$

$$A + \underbrace{(B/A)}_{B-A} = B$$

$$\Rightarrow P(B) = P(A) + \underbrace{P(B/A)}_{\geq 0}$$

$$\therefore P(A) \leq P(B)$$

4. $P(A/B) = P(A) - P(A \cap B)$

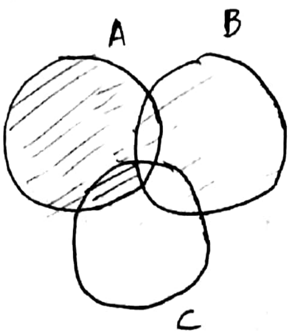
$$A = \underbrace{(A/B) \cup (A \cap B)}_{\text{disjoint}}$$

$$\Rightarrow P(A) = P(A/B) + P(A \cap B).$$

5. $P(A \cup B) = P(A/B) + P(B)$

$$\Rightarrow P(A) + P(B) - P(A \cap B)$$

6. $P(A \cup B \cup C) = \cancel{P(A)}\ P(A - B) + P(B) + P(C - A - B)$



$$\Rightarrow P(A) + P(B) - P(A \cap B)$$

$P(C) - [P(A \cap C) +$
$P(B \cap$
$+ P(A \cap B \cap C$

Eg: 3 marbles Randomly drawn 6 W, 5 B

Fan : 1 W, 2 Bl

$$\left({}^{3}C_1\right)\left(\frac{6}{11}\right)\left(\frac{{}^{5}C_2}{{}^{10}C_2}\right)$$

$$\Rightarrow \frac{3\times 6}{11}\times\frac{5\times 4}{2\times \frac{10\times 9}{2}} = \frac{4}{11}$$

Eg: 6 M, 9 W committee of 5 is to be selected

$$5 \swarrow\searrow 3M \quad 2W$$

$$\left({}^{5}C_3\right)\left(\frac{{}^{6}C_3}{{}^{15}C_3}\right)\left(\frac{{}^{9}C_2}{{}^{12}C_2}\right) + {}^{5}C_2\left(\frac{{}^{9}C_2}{{}^{15}C_2}\right)\left(\frac{{}^{6}C_3}{{}^{13}C_3}\right)$$

<u>Conditional probability</u>

If A, B are events in a sample space and $P(A) \neq 0$, $P(B) \neq 0$ then

$$P(A\cap B) = P(A)\cdot P(B|A) = P(B)\left(P(A|B)\right)$$

If $P(E/F) = P(E)$, E, F are independent.

If $P(A\cap B) = P(A)\cdot P(B)$ then A, B are independent

This can be extended to any no. of events.

PTO

Eg: A ⇒ event that the family has children with both genders.

B ⇒ Atmost one boy.

(i) A, B are independent if children = 3.

bbb, bbg, bgb, bgg, gbb, gbg, ggg, ggb.

Finish it off...

(ii) 2 children

bb, bg, gb, gg

$A = \{bg, gb\}$ $P(A) = 1/2$.

$B = \{bg, gb, gg\}$ $P(B) = 3/4$

dependent.

Eg: Given A, B are independent, P.T A^c, B^c are independent

$$P((A \cup B)^c) = 1 - P(A \cup B)$$

$$\Rightarrow (1 - P(A))(1 - P(B))$$

$$\Rightarrow P^c(A)\ P^c(B)$$

$$\Rightarrow P(A^c \cap B^c) = (P(A))^c\ (P(B))^c$$

Eg: 25% failed Maths

15% Chem

10% Both

(a) If he failed in chem, Prob. that he failed in maths?

$$P(M \cap C) = P(M / C) \cdot P(C)$$

$$\Rightarrow 10/15$$

(b) failed in Maths, P(failed in chem)?

$$\Rightarrow 10/25$$

~~$\not\!\!E$~~

Eg: Box 1 : 10 bulbs : 4 defective

2 : 6 1

3 : 8 3

A box is drawn at random and a bulb is drawn at random. P(defective)

$$P(\text{def} / A) \cdot P(A) + P(\text{def} / B) \cdot P(B) + P(\text{def} / C) \cdot P(C)$$

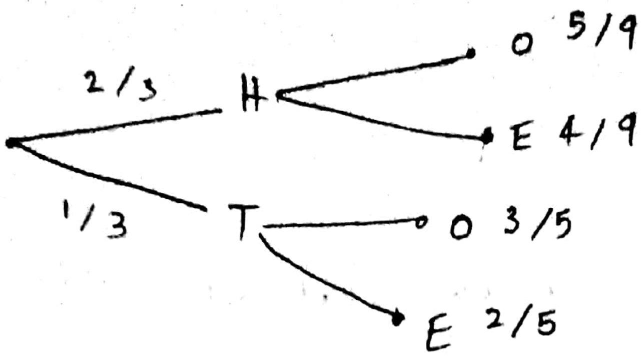
$$\Rightarrow \frac{1}{3}\left(\frac{4}{10} + \frac{1}{6} + \frac{3}{8}\right) = \frac{113}{360}$$

Eg: $P(H) = \frac{2}{3}$ $P(T) = 1/3$.

If (head) : select no. from 1 - 9

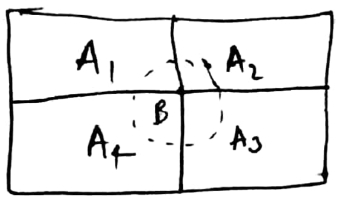
Else : select no. from 1 - 5.

P(Even) = ?



$$P(\text{Even}) = \frac{2}{3} \times \frac{4}{9} + \frac{1}{3} \times \frac{2}{5}$$

Partitioning an Event



~~R(A1 A A2~~

① Pairwise disjoint

② Exhaustine

③

Given some B, $P(B) = \sum_{i=1}^{4} B \cap A_i$

Theorm of total probability:

$$P(B) = \sum_{i=1}^{n} P(B \mid A_i)\, P(A_i)$$

$\boxed{\text{Baye's Theorm}}$

$$P(A_i \mid B) = \frac{P(B \mid A_i) \times P(A_i)}{P(B \mid A_1)\, P(A_1) + P(B \mid A_2)\, P(A_2) + \cdots + P(B \mid A_n) \cdots}$$

Eg: 4% of Men > 6 feet
1% Women > 6 feet

Women : $\frac{3}{5}$, Men : $\frac{2}{5}$

student selected among > 6 feet, P(woman) = ?

$P(M) = \frac{2}{5}$ $P(F) = \frac{3}{5}$

~~P(T|M) =~~

$$P(F|T) = \frac{P(T|F) \cdot P(F)}{P(T|F) \cdot P(F) + P(T|M) \cdot P(M)}$$

$$\Rightarrow \frac{\frac{1}{100} \times \frac{3}{5}}{\frac{1}{100} \times \frac{3}{5} + \frac{4}{100} \times \frac{2}{5}}$$

$$\Rightarrow \frac{3}{11}$$

Eg: Machine A : 25% B : 35% C : 40%

Defective (A) = 5% D(B) = 4% D(C) = 2%

Given defective, P(It came from A) :

$$P(A|D) = \frac{P(D|A) \times P(A)}{\Sigma P(D|x) \times P(x)}$$

Substitute and get the answer!!!