Dual space (Also known as conjugate space).

- Linear Operator: LT from V to V.
- " functional: LT from V to F.

Dual space involves the set of all ~~all these sets of~~ LFs.

$L(V,F) \rightarrow$ Dual space

$\rightarrow$ Forms a vector space.

Let $V(F)$ be a vector space over F. A linear transformation from $V(F)$ to F is called linear functional. That is $\phi : V \to F$, satisfying that $\phi(ax+by) = a\phi(x) + b\phi(y)$ for every $x, y \in V$, $a, b \in F$.

The vector space $L(V,F)(F)$ consisting of linear functionals is called a dual space of $V(F)$ and is denoted by $V^*(F)$.

$\therefore \phi_i \in V^*(F)$.

Q) $V(F) \rightarrow n$ square matrix.

$\phi : V \to F$.

$\phi(A) = tr[A]$. Show that ϕ is a linear functional.

$$\phi(A+B) = tr[A+B]$$

$$= \sum_{i=1}^{n} C_{ii} \qquad \text{where} \quad C = A+B$$

$$= \sum_{i=1}^{n} (A+B)_{ii}$$

$$= \sum_{i=1}^{n} (A_{ii} + B_{ii})$$

$$= \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii}$$

$$= \phi(A) + \phi(B).$$

$$\phi(cA) = tr[cA]$$

$$= \sum_{i=1}^{n} c a_{ii}$$

$$= c \sum_{i=1}^{n} a_{ii}$$

$$= c\, tr[A].$$

$$= c\,\phi(A).$$

9) Dim. of dual space?

$$m \times n = m \times 1 \quad [n = 1]$$

$$= m.$$

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Q) Let $\{x_1, x_2, \dots, x_n\}$ be a basis of $V(F)$ over F.
Let $\phi_1, \dots, \phi_n \in V^*$ be linear functionals defined by

$$\phi_i(x_j) = \delta_{ij} = 1 \text{ if } i = j$$
$$= 0 \text{ if } i \neq j.$$

Then show that $\{\phi_1, \dots, \phi_n\}$ is a basis of $V^*(F)$

Ans) Proof of ~~check if~~ L.I.

$$(a_1\phi_1 + a_2\phi_2 + \dots + a_n\phi_n)(x_i) = 0(x_i)$$

$$a_i = 0$$

$\Rightarrow$ Similarly, for all values of i, $a_i = 0$

$\therefore \{\phi_1, \dots, \phi_n\}$ is L.I.

Proof of ~~check if~~ spans $V^*(F)$

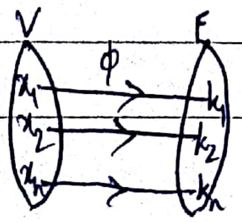
~~Let $f \in V^* F$, $f : V \to F$.~~

~~$f(x) = f(c_1x_1 + \dots + c_nx_n)$~~

~~or, $f(x) = c_1f(x_1) + \dots + c_nf(x_n)$.~~

~~$\therefore$ for $x = x_i$, $f(x) = c_i$~~

Let ϕ be any arbitrary element in V^* and let $\phi(x_1) = k_1, \dots, \phi(x_n) = k_n$.



$k_1\phi_1 + k_2\phi_2 + \dots + k_n\phi_n = \sigma \in V^*$.

$\therefore \sigma(x_i) = (k_1\phi_1 + \dots + k_n\phi_n)(x_i)$

$= k_1\phi_1(x_i) + \dots + k_i\phi_i(x_i) + \dots + k_n\phi_n(x_i)$

$= k_i$.

$k_i \equiv \phi(x_i) \equiv \sigma(x_i) \quad \forall\, x_i$.

$\Rightarrow \phi \equiv \sigma$.

$\therefore \phi = k_1\phi_1 + k_2\phi_2 + \dots + k_n\phi_n$.

$\therefore \{\phi_1, \dots, \phi_n\}$ spans $V^*(F)$.

$\therefore \{\phi_1, \dots, \phi_n\}$ is a basis of $V^*(F)$.

Q) $\mathbb{R}^2_B = \{x_1 = (2,1),\ x_2 = (3,1)\}$. ← Basis in $\mathbb{R}^2$.

Find the dual basis $\{\phi_1, \phi_2\}$.

$\phi_1(x_1) = 1, \quad \phi_1(x_2) = 0$

$\phi_2(x_1) = 0, \quad \phi_2(x_2) = 1$.

$\phi_1 = \begin{cases} 1 & \text{if } x_i = (2,1) \\ 0 & \text{~~otherwise~~ if } x_i = (3,1) \end{cases}$

$\phi_2 = \begin{cases} 1 & \text{if } x_i = (3,1) \\ 0 & \text{~~otherwise~~ if } x_i = (2,1) \end{cases}$

Let $\phi_1((x,y)) = ax + by$, $\phi_2((x,y)) = cx+dy$ [Linear functional]

$\phi_1((2,1))=1$, $\phi_1((3,1))= 0$.

$\Rightarrow$ $2a + b=1$
$3a+ b= 0$

$a = -1$, $b = 3$.

$\phi_2((2,1)) = 0$, $\phi_2((3,1)) = 1$.

$2c+d= 0$
$3c+d= 1$

$\Rightarrow$ $c=1$, $d=-2$.

$\therefore$ $\phi_1((x,y)) = -x+3y$
$\phi_2((x,y)) = x-2y$.

9) Let $\{x_1, x_2, \dots, x_n\}$ be a basis of V and let $\{\phi_1, \phi_2, \dots, \phi_n\}$ be a dual basis of V^*, then show that any vector $x \in V$ is written as

$$x = \phi_1(x)\, x_1 + \phi_2(x)\, x_2 + \dots + \phi_n(x)\, x_n \quad \text{—} (1)$$

and any linear functional $\sigma \in V^*$ is written as

$$\sigma = \sigma(x_1)\phi_1 + \sigma(x_2)\phi_2 + \dots + \sigma(x_n)\phi_n \quad \text{—} (2)$$

Ans). Since $\{x_1, \ldots, x_n\}$ is a basis in V.

$x \in V$, $x = a_1 x_1 + \ldots + a_n x_n$ $(a_i \in F)$.

$$\boxed{\phi_i(x) = a_i}$$

Since $\{\phi_1, \ldots, \phi_n\}$ is a basis in V^*.

$\sigma \in V^*$, $\sigma = c_1 \phi_1 + \ldots + c_n \phi_n$. $(c_i \in F)$.

Could have used the above ~~result~~ as well.

$\sigma(x_i) = c_i \phi_i(x_i)$ $\left[\text{Since } \phi_j(x_i) = 0,\ \phi_i(x_i) = 1\right]$.

$\therefore \boxed{\sigma(x_i) = c_i}$.

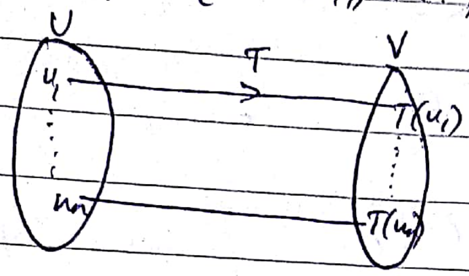
Alt.

$$\begin{aligned}
\sigma(x) &= \sigma(a_1 x_1 + \ldots + a_n x_n) \\
&= a_1 \sigma(x_1) + \ldots + a_n \sigma(x_n) \\
&= \phi_1(x) \sigma(x_1) + \ldots + \phi_n(x) \sigma(x_n). \\
&= \sigma(x_1) \phi_1(x) + \ldots + \sigma(x_n) \phi_n(x) \quad [F \text{ is commutative}] \\
&= \left(\sigma(x_1)\phi_1 + \ldots + \sigma(x_n)\phi_n\right)(x)
\end{aligned}$$

Date __/__/__

Matrices and Linear Transformations

Let $U(F)$ and $V(F)$ be 2 vector spaces over the field $(F, +, \cdot)$ and let $T: U \to V$ be a Linear Transformation. Again, let $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_m\}$ be the bases of $U(F)$ and $V(F)$ respectively. Since T maps U into V, $\{T(u_1), \dots, T(u_n)\}$ will belong to V.



$$T(u_1) = a_{11} v_1 + a_{12} v_2 + \dots + a_{1m} v_m$$

$$T(u_2) = a_{21} v_1 + a_{22} v_2 + \dots + a_{2m} v_m.$$

$$\vdots$$

$$T(u_n) = a_{n1} v_1 + a_{n2} v_2 + \dots + a_{nm} v_m.$$

$[T(u_1) \quad T(u_2) \quad \dots \quad T(u_n)]_{m \times n}$ where $T(u_i)$ is a column vector $m \times 1$.

$\Big[$

Continued after 'System of Linear Equations'

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System of linear Equations

$ax + by = c$

$ax + by + cz + d = 0$

For an eq^n to be linear, the no. of variables don't matter, just their degree should be 1.

A linear equation in the n-variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form
$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ (where $a_1, \ldots, a_n$ are coeff.)

Q) i) $3x - 4y = 1$ linear

ii) $r - \frac{1}{2}s - \frac{15}{3}t = 9$ linear.

(iii) $x_1 + 5x_2 = 3 - x_3 + 2x_4$ linear.

(iv) $\sqrt{2}\,x + \frac{\pi}{4}y - \left(\sin\frac{\pi}{5}\right)z = 1$ linear

(v) $xy + 2z = 1$ Non linear.

(vi) $\frac{x}{y} + z = 2$ Non linear.

(vii) $x_1^2 - x_2^3 = 3$ Non linear.

Solution of a linear eq^n.

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

The vector $[s_1, s_2, \ldots, s_n]$ which satisfies the eq^n is a solution.

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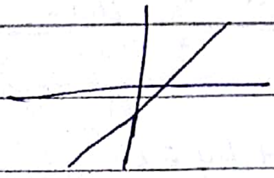
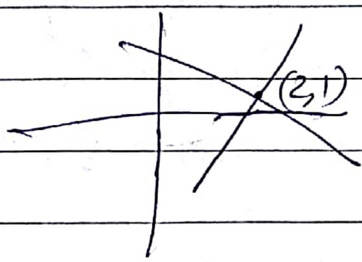
$3x - 4y = -1$.



$(1,1)$ $\quad$ $(0, \frac{1}{4})$ $\quad$ Basically, represents a straight line.

$\therefore$ Infinite solⁿ.

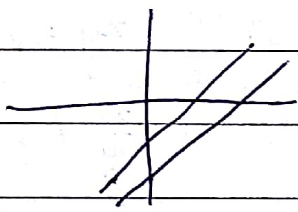
$$\left.\begin{aligned} 2x - y &= 3 \\ x + 3y &= 5 \end{aligned}\right\} \rightarrow (2,1)$$



Unique solⁿ.

__Examples__

$x - y = 1$ $x + y = 3$	$x - y = 2$ $2x - 2y = 4$	$x - y = 1$ $x - y = 3$
Unique solⁿ. Geometrically intersecting.	Infinite solⁿ. Both represent the same line.	Parallel lines $\Rightarrow$ 0 solⁿ



$\underbrace{\qquad\qquad\qquad\qquad}_{\text{There is solution.}}$

Called consistent system of equations. $\qquad$ Inconsistent system of equations.

If the constant part of the eqⁿs are 0, we can be assured that there's atleast 1 solⁿ $\rightarrow$ origin.

$\therefore$ Consistent sys. of eqⁿs.

__/__/__

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0 \end{aligned}\right\} \text{CONSISTENT.}$$

Case 1

$x - y = 1$

$x + y = 3$

Case 2

$x - y = 1$ — Substituting $y=1$ in ① called backward substitution.

$y = 1$

Same sol^n. (2,1) ∴ called Equivalent systems.

In case 2, we can find the sol^n by merely looking.

∴ If we have a mechanism by which case 1 can be converted to case 2 (equivalent) easily, ~~ease~~ solving of the system of eq^ns would be easy.

$$\begin{aligned} x - y - z &= 2 \\ y + 3z &= 5 \\ 5z &= 10 \end{aligned} \longrightarrow (3, -1, 2).$$

However,
$$\left.\begin{aligned} x - y - z &= 2 \\ 3x - 3y + 2z &= 16 \\ 2x - y + z &= 9. \end{aligned}\right\} \longrightarrow (3, -1, 2).$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

Coefficient Matrix

Augmented Matrix

We need to show how a sys. of eq^ns is equivalent to another sys. of eq^ns.

Lets convert this to a form suitable for backward substitution.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{array}\right] \qquad \begin{aligned} x-y-z&=2 \\ 3x-3y+2z&=16. \\ 2x-y+z&=9. \end{aligned}$$

Step 1. Subtract $3\times ①$ from ②, i.e. $R2' \leftarrow R2-3R1$.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 2 & -1 & 1 & 9 \end{array}\right] \qquad \begin{aligned} x-y-z&=2 \\ 5z&=10 \\ 2x-y+z&=9. \end{aligned}$$

Step 2. Subtract $2\times ①$ from ③, i.e. $R3' \leftarrow R3-2R1$.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & 3 & 5 \end{array}\right] \qquad \begin{aligned} x-y-z&=2 \\ 5z&=10 \\ y+3z&=5. \end{aligned}$$

Step 3 Interchange 2nd & 3rd eqn's. i.e. $R_2 \leftrightarrow R_3$.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right] \qquad \begin{aligned} x-y-z&=2 \\ y+3z&=5 \\ 5z&=10. \end{aligned}$$

Defn. A matrix is in row echelon form if it satisfies the following properties :-

1) Any row(s) consisting entirely of zeros is (are) at bottom.

2) In each nonzero row, the first nonzero entry (called the leading entry) is in a column to the left of any leading entries below it.

Q) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{bmatrix}$ Is this a Row-Echelon Matrix ?

YES.

Row Operations

1) Interchange ~~of~~ 2 rows.
2) Multiply a row by a non-zero constant.
3) Add a multiple of a row to another row.

Using these row operations,

$$\begin{bmatrix} \quad \end{bmatrix} \longrightarrow \begin{bmatrix} \quad \end{bmatrix}$$ Row Echelon Matrix.

called Gaussian Elimination. (Direct Method).

Steps of Gaussian Elimination :-

1) Write the augmented matrix.
2) Use row operations to reduce the augmented matrix to row echelon form.
3) Solve the sys. of eq^{n}s., using back substitution.

Q) Solve :-

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array}\right]$$

$R1 \leftrightarrow R2$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \\ 1 & -1 & -2 & -5 \end{array}\right]$$

$R3' \leftarrow R3 - \frac{1}{2} R_1$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \\ 0 & -2.5 & -2.5 & -7.5 \end{array}\right] \xrightarrow{R_3' \leftarrow 2R_3} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \\ 0 & -5 & -5 & -15 \end{array}\right]$$

$R_3' \leftarrow R_3 + \frac{5}{2} R_2$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \\ 0 & 0 & 2.5 & 5 \end{array}\right]$$

i.e. $2x + 3y + z = 5$

$2y + 3z = 8$

$2.5z = 5$ $\rightarrow$ $(0, 1, 2)$.

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9)

$w - x - y + 2z = 1$

$2w - 2x - y + 3z = 3$

$-w + x - y = -3.$

~~$② - ① \Rightarrow z = -w + x + 2.$~~

~~Now, $w - x - y + 2(-w + x + 2) = 1 \Rightarrow -w + x - y = -3.$~~

~~$2w - 2x - y + 3(-w + x + 2) = 3 \Rightarrow -w + x - y = -3.$~~

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right]$$

$$\Big\downarrow R_2' \leftarrow R_2 - 2R_1$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right] \xrightarrow{R_3' \leftarrow R_3 + R_1} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right]$$

$$\Big\downarrow R_3' \leftarrow R_3 + 2R_2$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore w - x - y + 2z = 1$

$y - z = 1 \Rightarrow y = z + 1$

Let $z = t$, $y = t + 1$.

$\Rightarrow w - x - t - 1 + 2t = 1 \Rightarrow w - x + t = 2$

$\Rightarrow w = x - t + 2$

Let $x = s$, $w = 2 + s - t$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 + s - t \\ s \\ 1 + t \\ t \end{bmatrix}$$

~~Row~~ Reduced Row Echelon Form

1) ~~Rf~~ It is in Row Echelon form.
2) The leading entry in each nonzero row is 1.
3) Each column containing a leading 1 has 0 everywhere else.

Hence, once converted to ~~Row~~ reduced row Echelon form, the backward substitution step isn't reqd. You directly get $x_i = b_i$, ~~etc~~ $\forall\, i$.

This is called Gauss-Jordan Elimination (Direct Method)

Steps :-

1) Write augmented matrix.
2) Use elementary row operations ~~to~~ to reduce the augmented matrix to reduced row echelon form.

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Q) Solve the system of eq^ns by Gauss Jordan elimination Method.

$$\sqrt{2}x + y + 2z = 1$$
$$\sqrt{2}y - 3z = -\sqrt{2}$$
$$-y + \sqrt{2}z = 1.$$

$$\left[\begin{array}{ccc|c} \sqrt{2} & 1 & 2 & 1 \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & -1 & \sqrt{2} & 1 \end{array}\right] \xrightarrow{R_1' \to \frac{1}{\sqrt{2}} R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{\sqrt{2}} & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & -1 & \sqrt{2} & 1 \end{array}\right]$$

$$R_2' \to \frac{R_2}{\sqrt{2}}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{\sqrt{2}} & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 1 & -\frac{3}{\sqrt{2}} & -1 \\ 0 & -1 & \sqrt{2} & 1 \end{array}\right] \xrightarrow{R_1' \to R_1 + \frac{1}{\sqrt{2}} R_3} \left[\begin{array}{ccc|c} 1 & 0 & (\sqrt{2}+1) & \sqrt{2} \\ 0 & 1 & -\frac{3}{\sqrt{2}} & -1 \\ 0 & -1 & \sqrt{2} & 1 \end{array}\right]$$

$$R_3' \to R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & (\sqrt{2}+1) & \sqrt{2} \\ 0 & 1 & -\frac{3}{\sqrt{2}} & -1 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{array}\right] \xrightarrow{R_1' \to R_1 + (2+\sqrt{2}) R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & -\frac{3}{\sqrt{2}} & -1 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{array}\right]$$

$$R_2' \to R_2 - 3R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{array}\right] \xrightarrow{R_3' \to -\sqrt{2} R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Ans : $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -1 \\ 0 \end{bmatrix}$.

Q) $a+b+c+d=4.$

$a+2b+3c+4d=10.$ Solve by Gaussian Elimination.

$a+3b+6c+10d=20.$

$a+4b+10c+20d=35.$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 4 & 10 \\ 1 & 3 & 6 & 10 & 20 \\ 1 & 4 & 10 & 20 & 35 \end{array}\right] \xrightarrow{R_2' \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 1 & 3 & 6 & 10 & 20 \\ 1 & 4 & 10 & 20 & 35 \end{array}\right]$$

$$\xrightarrow{R_3' \to R_3 - R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & 4 \\ 1 & 4 & 10 & 20 & 35 \end{array}\right] \xrightarrow{R_4' \to R_4 - R_1 - 3R_2 - 3R_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$d=1 \Rightarrow c=1 \Rightarrow b=1 \Rightarrow a=1.$

$$\therefore \text{Ans} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

CONTD after Compilers (last section).

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Indirect / Intuitive Methods.

$7x_1 - x_2 = 5$ Quite easy to guess: (1,2)

$3x_1 - 5x_2 = -7$

Jacobi's Method.

$$x_1 = \frac{5 + x_2}{7} \qquad \text{Initial Guess} \quad \left.\begin{array}{l} x_1^{(0)} = 0 \\ x_2^{(0)} = 0 \end{array}\right\}.$$

$$x_2 = \frac{7 + 3x_1}{5}$$

$$\left.\begin{array}{l} x_1^{(1)} = \dfrac{5 + x_2^{(0)}}{7} = \dfrac{5}{7} \\[2ex] x_2^{(1)} = \dfrac{7 + 3x_1^{(0)}}{5} = \dfrac{7}{5} \end{array}\right\} \qquad \left.\begin{array}{l} x_1^{(2)} = \dfrac{5 + 7/5}{7} = \dfrac{32}{35} \\[2ex] x_2^{(2)} = \dfrac{7 + 3 \cdot \frac{5}{7}}{5} = \dfrac{64}{35} \end{array}\right\}.$$

	(0)	(1)	(2)	(3)	(4)	(5)	(6)
x_1	0	5/7	32/35	239/245	1216/1225	0.998	0.999
x_2	0	7/5	64/35	341/175	2432/1225	1.996	1.999

$$\left.\begin{array}{l} x_1^{(3)} = \dfrac{5 + 64/35}{7} = \dfrac{239}{245} \\[2ex] x_2^{(3)} = \dfrac{7 + 3 \cdot \frac{32}{35}}{5} = \dfrac{341}{175} \end{array}\right\} \qquad \begin{array}{l} x_1^{(4)} = \dfrac{5 + 341/175}{7} = \dfrac{1216}{1225} \\[2ex] x_2^{(4)} = \dfrac{7 + 3 \cdot 239/245}{5} \\[2ex] \quad = \dfrac{2432}{1225}. \end{array}$$

Gauss-Seidal Method

$x_1 = \frac{5 + x_2}{7}$ $\quad x_1^{(0)} = 0$

$\quad\quad\quad\quad\quad x_2^{(0)} = 0.$

$x_2 = \frac{7 + 3x_1}{5}.$

$x_1^{(1)} = \frac{5 + x_2^{(0)}}{7} = 5/7$ $\quad x_1^{(2)} = \frac{5 + x_2^{(1)}}{7} = \frac{5 + \frac{64}{35}}{7} = \frac{239}{245}$

$x_2^{(1)} = \frac{7 + 3x_1^{(1)}}{5} = \frac{64}{35}$ $\quad x_2^{(2)} = \frac{7 + 3x_1^{(2)}}{5} = \frac{7 + 3.\frac{239}{245}}{5} = \frac{2432}{1225}$

	(0)	(1)	(2)	(3)	(4)	(5)
x_1	0	5/7	239/245	8557/8575		
x_2	0	64/35	2432/1225	85696/42875		

$x_1^{(3)} = \frac{5 + 2432/1225}{7} = \frac{8557}{8575}$

$x_2^{(3)} = \frac{7 + 3.\frac{8557}{8575}}{5} = \frac{85696}{42875}.$

9) $3x_1 - x_2 = 1.$

$-x_1 + 3x_2 - x_3 = 0.$ $\quad$ Solve by

$-x_2 + 3x_3 - x_4 = 1.$ $\quad$ (a) Jacobi

$-x_3 + 3x_4 = 1.$ $\quad$ (b) Gauss-Seidal.

Matrices (continued)

$V^m \to W^n$. $\qquad \begin{bmatrix} & \end{bmatrix}_{n \times m}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}_{n \times m}.$$

Diagonal Entries

Off diagonal entries

- Trace of a matrix, $tr[A] = \sum_i a_{ii}$.
- If $m = n$ and $A = [a_{ii}]$ it is called a diagonal matrix.

- $A = B$ if their orders are equal and $[a_{ij}] = [b_{ij}]$ entries.
- $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$. (have to be of same order).
- $A^T \qquad a_{ij} \to a^T_{ji}$
- Symmetric Matrix : $A = A^T$. i.e. $a_{ij} = a_{ji} \ \forall i, j$.
- Skew-Symmetric Matrix : $A = -A^T$ i.e. $a_{ij} = -a_{ji} \ \forall i, j$.
  $\therefore a_{ii} = 0 \ \forall i$. $[\because a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0]$.

- $AB \neq BA$

- $A_{m \times n}$ $\quad B_{n \times p}$ necessary condition for mult. to be possible.

$$\begin{bmatrix} \vdots & & \\ a_{i1} & a_{i2} \cdots & a_{in} \\ & & \end{bmatrix}_{m \times n} \times \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}_{n \times p} \longrightarrow [c_{ij}].$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

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Inverse of a Matrix

If A is a matrix of size $(n \times n)$, then the inverse $A'(n \times n)$ is a matrix such that:

$$AA' = A'A = I_n$$

$$\boxed{A^{-1} = \frac{adj(A)}{\det A}}.$$

Adjoint = $(\text{Cofactor matrix})^T$.

If $\det A \neq 0$, the matrix is ~~invert~~ invertible.

If A is invertible, then show that the inverse is unique.

Let A' and A'' be 2 inverses of A.

then, $AA' = A'A = I$ —— ①

$AA'' = A''A = I$ —— ②

$$\begin{aligned} A' &= A' I \\ &= A'(AA'') \\ &= (A'A)A'' \\ &= I A'' \\ &= A''. \end{aligned}$$

[Proved].

If A is invertible, $n \times n$ matrix, then the system of linear equations $Ax = b$ has the unique sol^n $x = A^{-1}b$ for any b in $\mathbb{R}^n$.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1,$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n.$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \equiv Ax = b.$$

If $x = A^{-1}b$, $A(A^{-1}b) = (AA^{-1})b = Ib = b$.

Lets show that it is unique.

Let y be another solⁿ to $Ax = b$.

$Ay = b \Rightarrow A^{-1}(Ay) = A^{-1}b$.

$\Rightarrow Iy = A^{-1}b$.

$\Rightarrow y = A^{-1}b$.

$\therefore y = x$. $\Rightarrow$ Unique.

[Proved]

Q) Use the inverse of the matrix to solve :-

$x + 2y = 3$
$3x + 4y = -2$.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad b = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

$\text{adj}(A) = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T$

$A^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}^T = \frac{1}{2}\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} b = \frac{1}{2} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -16 \\ 11 \end{bmatrix}.$$

<u>Properties of inverse of a matrix</u>.

1) $(A^{-1})^{-1} = A$.

2) $(cA)^{-1} = \frac{1}{c} A^{-1}$.

3) $(AB)^{-1} = B^{-1} A^{-1}$.

4) $(A^T)^{-1} = (A^{-1})^T$

5) $(A^n)^{-1} = (A^{-1})^n$.

If A is an invertible matrix, and n is a positive integer, then A^{-n} is defined by

$$A^{-n} = (A^{-1})^n = (A^n)^{-1}.$$

Elementary Matrices

An Elementary matrix is any matrix that is obtained by performing an elementary row operation~~(s)~~ on an identity matrix.

g) $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}$ $E_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}$ $E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}_{4\times4}$.

What operations have been applied to get E_i's ?

$\underline{\underline{E_1}}$: $R_2' \leftarrow 3R_2$

$\underline{E_2}$: $R_1' \longleftrightarrow R_3$.

E_3 : $R_4' \leftarrow -2R_2 + R_4$.

g) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}_{4\times3}$ Find E_1A, E_2A, E_3A.

$E_1A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$ $E_2A = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$

$$E_3 A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41}-2a_{21} & a_{42}-2a_{22} & a_{43}-2a_{23} \end{bmatrix}$$

$\therefore$ Equivalent to applying the corresponding row operations on the matrix A itself.

Let E be an elementary matrix obtained by performing row operation on I_n. If the same elementary row operation is performed on $n \times n$ matrix A, the result is same EA.

$$I \underset{\frac{1}{3}R_2}{\overset{3R_2}{\rightleftarrows}} E_1 . \qquad I \underset{R_{13}}{\overset{R_{13}}{\rightleftarrows}} E_2$$

$$I \underset{R_4+2R_2}{\overset{R_4-2R_2}{\rightleftarrows}} E_3 .$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{E_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{E_1^{-1}} = I .$$

(Also an Elementary Matrix)

$E_2^{-1} = E_2$.

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}.$$

The fundamental theorem of Invertible Matrices

Let A be $n \times n$ matrix. Then the following statements are equivalent :-

1) A is invertible
2) $Ax = b$ has a unique sol^n for every $b \in \mathbb{R}^n$.
3) $Ax = 0$ has only the trivial sol^n.
4) The reduced row echelon form of A is I_n.
5) A is a product of elementary matrix.

Assume (1) to be true.

(1) $\Rightarrow$ (2) (already proved).

Now, for $Ax = 0$, $x = A^{-1} 0 = 0$ [Since $X0 = 0$ $\forall X$.]

$\therefore$ (2) $\Rightarrow$ (3).

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$$a_{11}x_1 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{n1}x_1 + \cdots + a_{nn}x_n = 0$$

$$\Downarrow$$

$$x_1 = 0$$
$$x_2 = 0$$
$$\ddots$$
$$x_n = 0$$

On applying Gaussian Elimination on this, we'll get I_n

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \end{array}\right] \quad \therefore \quad (4).$$

Now since (4) is true,

$$A \xrightarrow{R_1} \xrightarrow{R_2} \cdots \xrightarrow{R_k} I_n.$$

$$(E_k \cdots E_2 E_1) A = I_n.$$

$$A = (E_k \cdots E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}.$$

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Q) If possible, express $A = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$ as a product of elementary matrices.

$$(E_k \cdots E_1)A = I \implies A = (E_k \cdots E_1)^{-1}.$$

$$\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} = (E_k \cdots E_1).$$

$$\therefore (E_k \cdots E_1)^{-1} \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore (E_k \cdots E_1)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 3 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 3 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} = E_k \cdots E_1$$

$$\underset{A}{\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}} \xrightarrow{R_1' \leftarrow R_1 - R_2} \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \xrightarrow{R_2' \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \xrightarrow{R_2' \leftarrow \frac{1}{3} R_2} \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}.$$

$$\therefore E_3 E_2 E_1 A = I$$

$$E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}.$$

$$E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \qquad A = (E_3 E_2 E_1)^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right)^{-1}$$

$\therefore A = E_1^{-1} E_2^{-1} E_3^{-1}$.

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}.$$

\# Let A be a square matrix. If a sequence of elementary row operations reduces A to I, then the same sequence will transform I to A^{-1}.

$$A \xrightarrow{R_1} \xrightarrow{R_2} \cdots \xrightarrow{R_k} I \qquad \underbrace{(E_k E_{k-1} \cdots E_1)}_{B} A = I.$$

$$I \xrightarrow{R_1} \xrightarrow{R_2} \cdots \xrightarrow{R_k} A^{-1}$$

$\therefore BA = I$.

$\Rightarrow B = A^{-1}$.

$\therefore (E_k E_{k-1} \cdots E_1) I = A^{-1}$.

$\Rightarrow BI = A^{-1}$.

Q) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right] \quad [R_2' \leftarrow R_2 - 2R_1]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right] \quad [R_3' \leftarrow R_3 - R_1 + \tfrac{1}{2} R_2].$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & -2 & 1/2 & 1 \end{array}\right] \quad [R_2' \leftarrow \tfrac{1}{-2} R_2].$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -3/2 & -5 \\ 0 & 1 & -3 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & -2 & 1/2 & 1 \end{array}\right] \quad (R_1' \leftarrow R_1 - 2R_2 - 5R_3)$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -3/2 & -5 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1/2 & 0 \end{array}\right] \quad [R_2' \leftarrow R_2 + 3R_3]$$

$\therefore \quad A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 0 \end{bmatrix}.$

Q) Find the inverse of $\begin{bmatrix} 2 & 1 & -4 \\ -4 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix}$. using Gauss Jordan method.

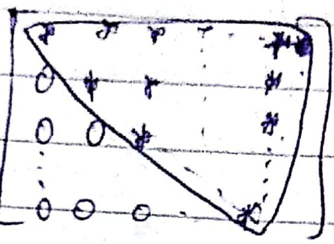
$$\left[\begin{array}{ccc|ccc} 2 & 1 & -4 & 1 & 0 & 0 \\ -4 & -1 & 6 & 0 & 1 & 0 \\ -2 & 2 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 2 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 3 & -6 & 1 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} R_2' \leftarrow R_2 + 2R_1 \\ R_3' \leftarrow R_3 + R_1 \end{array}$$

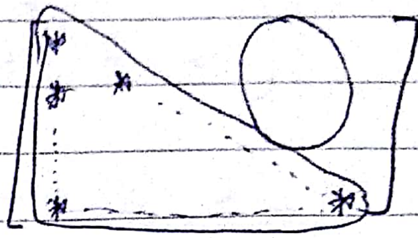
$$= \left[\begin{array}{ccc|ccc} 2 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 6 & -2 & 0 & 1 \end{array}\right] \qquad R_3' \leftarrow R_3 - 3R_1$$

$$= \left[\quad\right] \qquad R_1' \leftarrow R_1 -$$

Upper Triangular Matrix



Lower Triangular Matrix



Let A be a square matrix. A factorization of A as $A = LU$ where L is the unit lower triangular and U is the upper triangular, is called LU factorization.

Q) $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix}$ Express A as LU.

Step 1: Convert A to some U.

$$A \xrightarrow{R_1} \cdots \xrightarrow{R_k} U$$

$$U = (E_k \cdots E_1)\, A \Rightarrow A = (E_k \cdots E_1)^{-1}\, U$$

$$A = \underbrace{E_1^{-1} E_2^{-1} \cdots E_k^{-1}}_{\text{will be } L_u \text{ (unit lower } \Delta r \text{ matrix)}}\, U$$

$\therefore A = L_u U$.

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$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} \xrightarrow[\textcircled{2}\ R_3' \leftarrow R_3 + R_1]{\textcircled{1}\ R_2' \leftarrow R_2 - 2R_1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 6 & 8 \end{bmatrix} \xrightarrow{\textcircled{3}\ R_3' \leftarrow R_3 + 2R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$\therefore \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\therefore \quad A = \underbrace{(E_3 E_2 E_1)^{-1}}_{L} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$E_3 E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}.$$

$$(E_3 E_2 E_1)^{-1} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

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Eigen values and eigen vectors

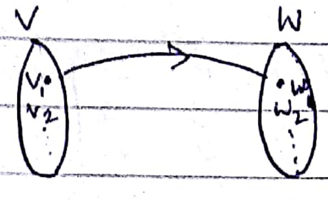
What is an eigen value of a matrix? (N×N matrix)

$$Ax = \lambda x$$

λ ↳ Eigen value.

What is the importance of finding eigen values of a matrix?

Consider two vector spaces V, W.



An L.T. from V to W can be represented as a matrix.

$\{v_1, v_2, \ldots, v_n\}$ $\{w_1, w_2, \ldots, w_m\}$

Construct a matrix corresponding to this L.T. :-

Let $T : R^3 \to R^2$ $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2y \\ x+y-3z \end{bmatrix}$

Let $B = \{e_1, e_2, e_3\}$ be the basis of R^3. Let $\{\bar{e}_2, \bar{e}_1\}$ be the basis of R^2. Find the matrix wrt. B & C.

For B; $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.

C: $e_1 = (1,0)$, $e_2 = (0,1)$.

Dim. of transformation matrix : 2×3.

$$\begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$ [Since order of basis is changed]

If $C : (\bar{e}_1, \bar{e}_2)$ basis then

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -3 \end{bmatrix}.$$

Similarly, if we change $\bar{e}_1, \bar{e}_2$ to :-

$C:$ $\bar{e}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\bar{e}_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Find matrix representation of the linear transformation.

$$T(e_1) = T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$[T(e_1)]_c = \bar{e}_2 + \bar{e}_1.$$

$$T(e_2) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad T(e_2) = e_2 - 2e_1.$$

$$T(e_3) = \begin{bmatrix} 0 \\ -3 \end{bmatrix} = -3e_2.$$

$$(T(e_1), T(e_2), T(e_3)) :- \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix}.$$

So changing order of basis vectors makes a difference. But something remains fundamental (eigen value remains same)

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New scenario :-

$B : \{e_1, e_2, e_3\}$; $C : \{e_1, e_2, e_3\}$.

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2y \\ x+y-3z \\ x+y \end{bmatrix}.$$

Consider $C' : \{e_3, e_2, e_1\}$.

Find T_{BC} and $T_{BC'}$.

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

$$T(e_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

Characteristic eq^n

Let $\underbrace{(A - \lambda I_n)}_{\text{characteristic polynomial.}} = 0$.

if you have diagonal matrix, the diagonal elements are itself eigen values.

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$$A - x I_3 = \begin{array}{c} \begin{bmatrix} \lambda_1 - x & 0 & 0 \\ 0 & \lambda_2 - x & 0 \\ 0 & 0 & \lambda_3 - x \end{bmatrix} \end{array} \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array}$$

$\Downarrow$

Let $A - xI = 0$, This is an eigen basis

There are 2 basis associated with this

8) Find the eigen values and eigen vectors of the following matrix.

A represents an L.T. from $\mathbb{R}^3 \to \mathbb{R}^3$ wrt. B & C

If B & C were eigen basis $\Rightarrow$ A_{BC} will be diagonal matrix.

$$A_{BC} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{bmatrix}$$

we get

$$(\lambda - 1)^2 (\lambda - 2) = 0 \Rightarrow \lambda = 1, 2$$

The no. of times an eigen value occurs is called the algebraic multiplicity.

Here, alg. multiplicity $(1) = 2$

$(2) = 1$.

For $\lambda = 1$, $Ax = \lambda x \Rightarrow Ax - x = 0 \Rightarrow (A-I)x = 0$

if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [A - I \mid 0]$

$$= \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{array}\right]$$

On solving, $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 - x_3 = 0$

$x_2 - x_3 = 0$

Free variable : x_3.

Let $x_3 = t$:- $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Eigen space ? dim ?

$\downarrow$ $\hookrightarrow = 1$

Corresponding to $\lambda = 1$

(Geometric Multiplicity).

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What are similar matrices?

Let A and B be $(n \times n)$ matrices. We say that A is similar to B if there is an invertible $(n\times n)$ matrix P such that $P^{-1}AP = B$. If A is similar to B then we write $A \sim B$.

If A is similar to B, $P^{-1}AP = B$

$$P^{-1}A = BP^{-1}$$

$$\Rightarrow \boxed{A = PBP^{-1}}.$$

$$\Rightarrow \boxed{AP = PB}$$

Q) If $A \sim B$, can we say $B \sim A$?

Yes: $P^{-1}AP = B \Rightarrow A = PBP^{-1} = (P^{-1})^{-1}B(P^{-1})$

$\therefore B \sim A$.

Q) Is $A \sim A$? $(I^{-1})AI = A$ Yes.

Q) If $A \sim B$ and $B \sim C$, then $A \sim C$?.

$P^{-1}AP = B$.

$\Rightarrow A = PBP^{-1}. = (P^{-1})^{-1}B(P^{-1})$.

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Q) If A, B are $(n \times n)$ matrices, such that $A \sim B$ show that $\det(A) = \det(B)$.

$$P^{-1}AP = B$$

$$\det(P^{-1}AP) = \det(B)$$

$$\det(P^{-1})\det(A)\det(P) = \det(B) \quad \text{--- (1)}$$

$$PP^{-1} = I$$

$$\det(PP^{-1}) = \det(I) = 1$$

$$\Rightarrow \det(P)\det(P^{-1}) = 1 \Rightarrow \det(P^{-1}) = \frac{1}{\det(P)}$$

$\therefore$ From (1),

$$\frac{1}{\cancel{\det(P)}} \times \det(A) \times \cancel{\det(P)} = \det(B)$$

$$\Rightarrow \boxed{\det(A) = \det(B)}$$

Necessary condition but not sufficient.

Q1) If A, B are $n \times n$ matrices, such that $A \sim B$ show that A and B have same characteristic polynomial.

~~$P^{-1}AP = B$~~

Q2) Find out whether they are similar matrices or not :-

i) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

Ans 2) i) ~~Let there exist P such that~~

det (A) = -3

det (B) = 3

Since det (A) != det (B) $\Rightarrow A \nsim B$.

ii) det (A) = -4

det (B) = -4

Since det (A) = det (B) $\Rightarrow$ they might be similar.

Check characteristic polynomial.

$$\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

i.e. $(1-\lambda)(2-\lambda) - 6 = 0$

$\Rightarrow 2 - 3\lambda + \lambda^2 - 6 = 0$

$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$ — (1)

Let there exist P s.t.

$P^{-1} A P = B$.

If $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$\frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}.$$

$$\frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a+3c & b+3d \\ 2a+2c & 2b+2d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}.$$

$\frac{1}{(ad-bc)}$ [

For B, $\begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-1-\lambda) - 3 = 0$.

$\Rightarrow -1 + \lambda^2 - 3 = 0$

$\Rightarrow \lambda^2 - 4 = 0$ — (2)

(1) $\neq$ (2) $\Rightarrow A \nsim B$.

Unitary matrix. ~~Rotation of basis~~

If $B = U A U^{-1}$

$\underbrace{B = U A}_{\text{Same Eigen values}}$

Eigen values do not change by rotating basis etc.
↓
fundamental.

Diagonalization

An $n \times n$ matrix A is diagonalizable if there is a diagonal matrix D such that A is similar to D i.e. there exists $n \times n$ invertible matrix P such that $P^{-1} A P = D$.

Q) Let A be an $(n \times n)$ matrix then A is diagonalizable if and only if A has n linearly independent eigen vectors.

1. Suppose $A \sim D$,

$\boxed{D = P^{-1} A P}$ (P is an invertible matrix)

$\Rightarrow AP = PD$.

Let the columns of the matrix P be $\{P_1, P_2, \ldots, P_n\}$

$$\Rightarrow A \, [P_1 \; P_2 \cdots P_n] = [P_1 \; P_2 \cdots P_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

$$\Rightarrow [AP_1 \;\; AP_2 \cdots AP_n] = [\lambda_1 P_1 \;\; \lambda_2 P_2 \cdots \lambda_n P_n].$$

$$AP_i = \lambda_i P_i .$$

$\therefore$ Entries of the diagonal matrix are the eigen values of the matrix A.

Since P is invertible, column vectors are LI.

(2) DIY.

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Q) If possible, find a matrix P that diagonalizes

i) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$ ii) $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$.

Calc. eigen vectors of A (should be L.I.).

i) $\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{bmatrix}$

$-\lambda\left[\lambda(\lambda-4)+5\right] + \left[-2\right] = 0$

$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$.

$(\lambda-1)\left[\lambda^2 - 3\lambda + 2\right] = 0$

$\Rightarrow (\lambda-1)^2(\lambda-2) = 0$.

$\Rightarrow \lambda = 1, 1, 2$.

Same Eigen values $\Rightarrow$ L.D. eigen vectors
$\Rightarrow$ Not diagonazable.

$\cancel{Ax = \lambda x}$

$\cancel{Ax = x}$ ~~and~~ $\cancel{Ax = 2x}$

$\cancel{(A-I)x = 0}$

(ii) $\begin{bmatrix} (-1-\lambda) & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & (-1-\lambda) \end{bmatrix}$

$(-1-\lambda)\left[-\lambda(-1-\lambda) \right] + [\lambda] = 0$.

$\Rightarrow -\lambda^3 - \lambda^2 - \lambda^2 - \cancel{\lambda} + \cancel{\lambda} = 0$.

$\Rightarrow -\lambda^3 - 2\lambda^2 = 0$.

$\Rightarrow \lambda^2(\lambda + 2) = 0$.

$\lambda = 0, 0, -2$.

Ans =

$P = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$