Topological properties:

E.g. · Cannot distinguish $\overset{X}{[0,1]}$ from $\overset{Z}{[0,2]}$ in terms of continuous functions.

· Can distinguish $[0,1] = X$ from $Y = [0,1] \cup [2,3]$. Namely

Property ☺ : X has ☺ if $\forall f : X \to \{0,1\}$ continuous map, f is constant.

Property ☆ : X has ☆ if $\forall x, y \in X$, $\exists f : [0,1] \to X$ continuous map s.t. $f(0) = x$ & $f(1) = y$

Dangers with sets:

$$X = \{ S \text{ a set} : S \notin S \} \quad \Big\{ \underline{\text{Not permitted}} \Big\} \qquad \%$$

Question: Does $X \in X$?

- If $X \notin X$, by defn $X \in X$.
- If $X \in X$, by defn $X \notin X$.

Solution:
- Carefully define 'well-formed' expressions (which give sets)
- Axioms that give existence for sets. (in terms of other sets)

Carefully define arbitrary collections.

· Define $\{X_\alpha\}_{\alpha\in A}$, A 'index set' (e.g. $\mathbb{N}$). $\begin{cases}(a,b) \leftrightarrow \{\{a\},\{a,b\}\} \\ (c,d)\end{cases}$

· Try: function on A – to what?

· What is a function anyway? $\{\{x\},\{x,f(x)\}\}$

· A function $f: X \to Y$ is identified with its graph; what is this?

$$\Gamma(f) = \{(x, f(x)) : x \in X\} \subset X \times Y$$

· But not all subsets are graphs; have properties that characterize graphs

· $\forall x \in X$, $\exists y \in Y$ s.t. $(x,y) \in \Gamma$

· $\forall x \in X$, $\forall y_1, y_2 \in Y$ if $(x,y_1) \in \Gamma$ & $(x,y_2) \in \Gamma$ then $y_1 = y_2$

For collections, no <u>codomain</u>, instead have 'graph-like sets'

$\underline{\{X_\alpha\}_{\alpha\in A}}$: 'Graph-like set' corresponding to $\underset{\sim}{\Gamma} := \{(\alpha, X_\alpha) : \alpha \in A\}$

all collection of sets, not a set

Formally, $\Gamma = \Gamma(X_\alpha)$ is a set

. we do not have $\Gamma \subset X \times Y$, instead $\Gamma \subset X \times \text{Set}$

- If $p \in \Gamma$, then $\exists$ S set s.t. $p = (\alpha, S)$, $\alpha \in A$.
- If $\alpha \in A$, $\exists$ S set s.t. $(\alpha, S) \in \Gamma$
- If $\alpha \in A$, S_1, S_2 sets s.t. $(\alpha, S_1) \in \Gamma$ & $(\alpha, S_2) \in \Gamma$ then $S_1 = S_2$

Then X_α is the unique set s.t. $(\alpha, X_\alpha) \in \Gamma$.

<u>Furstenberg's topology</u>: On $\mathbb{Z}$,

<u>Basis</u>: Arithmetic progressions $S(a,b) = \{a+nb : n \in \mathbb{Z}\} \subset \mathbb{Z}$

· This is a basis as finite intersections of arithmetic progressions are empty or arithmetic progressions (<u>Ex</u>)

· <u>Notice</u>: · Basic sets are closed.

· Finite non-empty sets are not open, i.e. complements of finite sets are not closed.

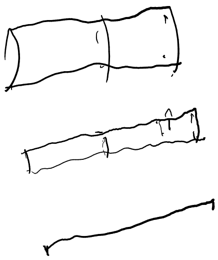
Suppose there were only finitely many primes then

$$\underbrace{\mathbb{Z} \setminus \{-1, 1\}}_{\substack{\text{contradiction} \\ \to \text{closed}}} = \underbrace{\bigcup_{p \text{ prime}}}_{\text{finite union}} \underbrace{S(0,p)}_{\text{closed}}$$

__Why ψ-metrics:__ Let $d_n : X \times X \to X$ be metrics s.t.

$d_n \to d_\infty$, i.e. $\forall x, y \in X$, $d_n(x,y) \to d_\infty(x,y)$

$[0,1]\times S^1$



__Then:__

- $d_\infty(x,y) \geq 0 \quad \forall x, y \in X$
- $d_\infty(x,y) = d_\infty(y,x) \quad \forall x, y \in X$
- $d_\infty(x,z) \leq d_\infty(x,y) + d_\infty(y,z)$ (Exercise)

__But:__ $d_\infty(x,y) = 0$ can hold for $x \neq y$.

__E.g.__ $X = \mathbb{R}^2$, $d_n((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + \frac{1}{n}|y_1 - y_2|$

Then $d_n \to d_\infty$, $d_\infty((x_1,y_1),(x_2,y_2)) = |x_1 - x_2|$

Characterizing interior : $S \subset X$, X topological space

- 'largest' w.r.t. partial order $A \subset B$
- may not in general exist

E.g. $a, b \in \mathbb{N}$, $\gcd(a,b)$ is the largest common divisor w.r.t the partial order $m|n$, i.e.

E.g. For polynomials with integer coefficients $\mathbb{Z}[x]$, we do not have a g.c.d. in general, e.g. $2, x$

Defn: $d = \gcd(a,b) \Leftrightarrow d|a \wedge d|b$ & $m|a$ & $m|b \Rightarrow m|d$ } i.e. the set $\{d \in \mathbb{N} : d|a \ \& \ d|b\}$ has a maximum w.r.t. the order $m|n$.

Theorem: If d, d' are both gcd's of a & b, then $d = d'$

int(S): (1) int$(S) \subset S$

(2) int(S) is open

(3) If $V \subset S$ is open, $V \subset$ int(S) (maximality)

X space

E.g. We say $F \subset X$ is the 'finite-core' of X if

(a) F is finite

(b) $F \subset X$

(c) If A is finite and $A \subset X$ then $A \subset F$

Propn: If F_1 & F_2 are both finite cores of X, then $F_1 = F_2$

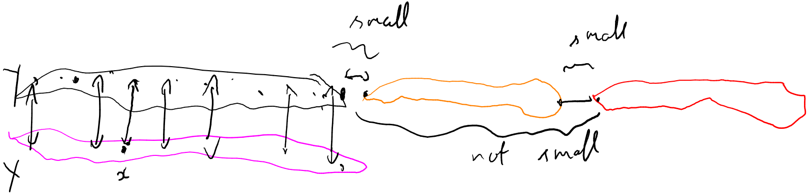
Pf: As F_1 is a finite core & F_2 satisfies (a) & (b), take $A = F_2$ in (c) to show $F_2 \subset F_1$

|||^{ly} $F_1 \subset F_2$,

Thus $F_1 = F_2$

□

Distances between sets:



Sets $X, Y \subset Z$ bounded

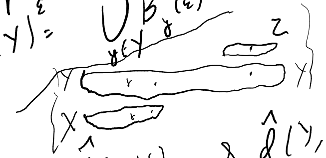
Candidates: $\cdot\ \inf\{d_Z(x,y) : x\in X,\ y\in Y\}$

for $d_Z(X,Y)$

$\cdot\ \sup\{d_Z(x,y) : x\in X,\ y\in Y\}$: $d_Z(X,X)\neq 0$ in general

What work: minimax:

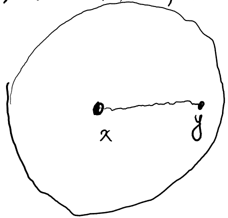
$$d^+(X,Y) = \sup_{x\in X}\Big(\inf_{y\in Y} d_Z(x,y)\Big) = \inf\{\varepsilon>0 : N_\varepsilon(Y)\supset X\}$$ where $N_\varepsilon(Y) = \bigcup_{y\in Y} B_y(\varepsilon)$



$$d(X,Y) = \max\{d^+(X,Y),\ d^+(Y,X)\}$$

$\hat d(X,Y)$ & $\hat d(Y,Z)$ are small, but not $\hat d(X,Z)$

Another candidate $\hat d = \min\{\dots\}$



Rk: $d(x,y) = \inf\{\varepsilon>0 : y\in B_\varepsilon(x)\}$

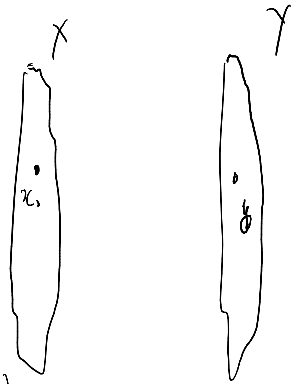
Propn: $d^+(X,Y) := \sup_{x\in X}\{\underbrace{\inf_{y\in Y} d_2(x,y)}\} = \underset{D}{\inf}\{\varepsilon>0 : X \subset N_\varepsilon(Y)\}$

$D-\delta \notin$, no $X_x \not\subset N_{D-\delta}(Y)$

Pf: **Lemma:** $\inf_{y\in Y} d_2(x_0,y) < \varepsilon$ for fixed $x_0 \in X \iff x_0 \in N_\varepsilon(Y)$

pf: $\inf_{y\in Y}\{d_2(x_0,y) < \varepsilon\} \implies \exists\, y_0\in Y$ s.t. $d_2(x_0,y_0)<\varepsilon$

$\implies x_0 \in B_\varepsilon(y_0) \subset N_\varepsilon(Y) = \bigcup_{y\in Y} B_\varepsilon(y)$



Conversely, if $x_0\in N_\varepsilon(Y)$, then $x_0 \in B_\varepsilon(y_0)$ for some $y_0 \in Y \implies d_2(x_0,y_0)<\varepsilon \implies \inf_{y\in Y}\{d_2(x_0,y)\}<\varepsilon$

Rest as exercise.

Alternative definition: $d(X,Y) = \inf\{\varepsilon>0 : X\subset N_\varepsilon(Y) \text{ and } Y \subset N_\varepsilon(X)\}$.

Then $d(\emptyset,\emptyset)=0$ but $d(\emptyset,X) := \infty$ (or D) if $X\neq\emptyset$.

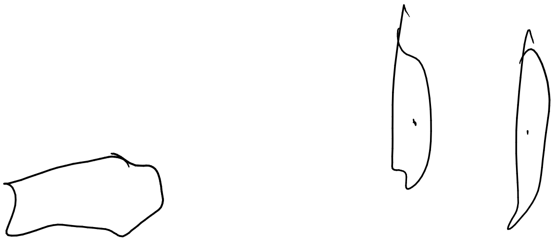
Modified definition if Z bounded, $\operatorname{diam}(Z) \leq D$

$$d_H^{(D)}(X, Y) = \inf\left(\{ \varepsilon > 0 \, ; \, X \subset N_\varepsilon(Y) \text{ and } Y \subset N_\varepsilon(X)\} \cup \{D\}\right)$$

- If $X \neq \emptyset \neq \emptyset$, $\quad d_H(X, Y) = d_H^{(D)}(X, Y)$

- If $X \neq \emptyset$, $\quad d_H(X, \emptyset) = D$

- $d_H(\emptyset, \emptyset) = 0$



Nowhere dense $\operatorname{int}(\bar{A}) = \emptyset$; $X \setminus \bar{A}$ is dense, i.e. $\overline{X \setminus \bar{A}} = X$

$$\operatorname{int}(\bar{A}) = \emptyset \iff \forall x \in X, \quad x \notin \operatorname{int}(\bar{A})$$

not
||

$$\iff \forall x \in X, \ \neg\left(\exists V \text{ open}, \ x \in V \text{ s.t. } V \subset \bar{A}\right)$$

$$\iff \forall x \in X, \ \forall V \text{ open s.t. } x \in V, \ V \not\subset \bar{A}$$

$$\iff \forall x \in X, \ \forall V \text{ open s.t. } x \in V, \ V \cap (X \setminus \bar{A}) \neq \emptyset$$

$$\iff \forall x \in X, \ x \in \overline{(X \setminus \bar{A})}$$

$$\iff X \setminus \bar{A} \text{ is dense.}$$

$X' =$ limit points of X

· $X^0 = X$; $X^{n+1} = (X^n)'$

Want: X s.t. $X^n \neq \emptyset$ and $X^{n+1} = \emptyset$

Soln 1: $X = \{1, 2, \dots, n\}$, open sets: $[i] = \{k : 1 \leq i \leq k\}$ for $i \in \{1, \dots, n\}$

· $\{1\}$ is open, so 1 is isolated.

· if $j \geq 1$, $U = [i]$ open s.t. $j \in U$, $1 \in U \cap \{X \setminus \{j\}\}$,

i.e. $\{1\} = X$

So $X' = \{2, \dots, n\}$ with topology $X' \cap [i] = [2, i]$, $i \in \{2, \dots, n\}$

· Proceed inductively

Soln 2:

Dense point construction: Given X space,

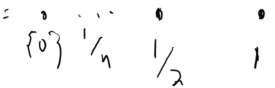
Let $\hat{X} = X \cup \{\infty\}$, with topology

$$\Omega_{\hat{X}} = \{ U \cup \{\infty\} : U \subset X \text{ open} \} \cup \{\emptyset\}$$

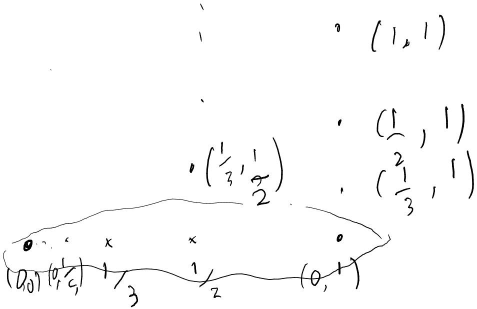
- Then $\{\infty\}$ is dense and isolated.
- So $(\hat{X})' = X$

Hence we inductively construct.

- X^0 = discrete, e.g. single point.
- $X^{n+1} = \widehat{(X^n)}$

Soln 3: $X =$  $= \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}, n > 0\}$

has $X' = \{0\}$, $X'' = X^{2} = \emptyset$



$X \subset \mathbb{R}^n = \{ (x_1, \dots, x_n) : x_i \in \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}\}$

Or product.

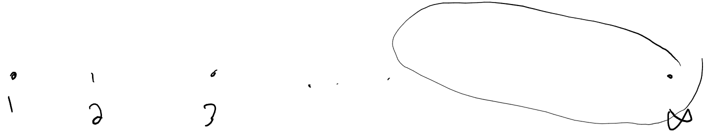
Soln 4: e.g. $\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ with topology

$$\{U \subset \hat{\mathbb{N}} : \text{if } \infty \in U,\ \hat{\mathbb{N}} \setminus U \text{ is finite}\}$$

Given X, consider $\hat{X} = X \amalg (X \times \mathbb{N})$ with topology

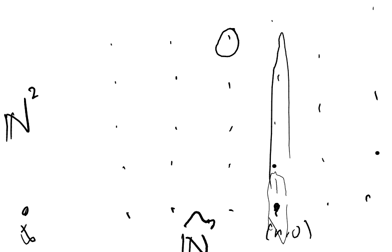
$\Omega = \{U \subset \hat{X} : x \in U$ for $x \in X$, then $(x,n) \in U$ for all but finitely many $n\}$

· if $x \in X$, U open, $x \in U$, then $(x,n) \in U \cap \hat{X} \setminus \{x\}$ for all but finitely many.



Here $\hat{\mathbb{N}}' = \{\infty\}$, $\hat{\mathbb{N}}^2 = \emptyset$

Hence $\hat{X}' = X$ as $\{(x,n)\}$ is open



X s.t. $X^2 \neq \emptyset$, $X^3 = \emptyset$,

$X = \mathbb{N}^2 \cup \hat{\mathbb{N}}$ with topology

$\{U \subset X :$ · $\infty \in U \Rightarrow X \setminus U$ is finite

· $n \in U \Rightarrow \#\{(n,m) \notin U\}$ is finite$\}$

Then $X' = \hat{\mathbb{N}}$

• d Euclidean metric on $\mathbb{R}^2$

$\{(x,y) \in \mathbb{R}^2 : d((x,y),(a,b)) + d((x,y),(c,d)) = d((a,b),(c,d))\}$

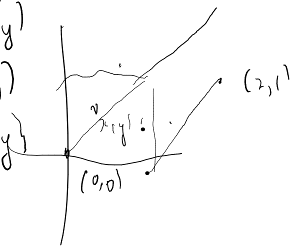
is the line segment from (a,b) to (c,d)

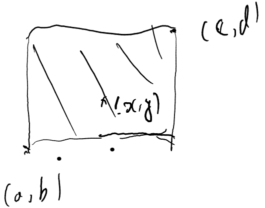
e.g. $d_1((0,0),(n,m)) = n+m$

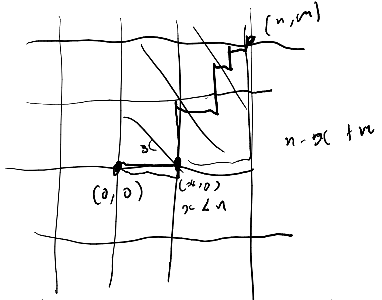
$d_\infty((x,y),(0,0)) = \max\{x, y\}$

$d_\infty((x,y),(2,1))$

$= \max\{2-x, 1-y\}$







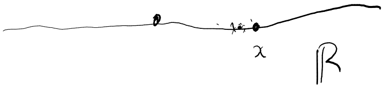
$d_1((x,y),(a,b)) = (x-a) + (y-b)$

$d_1((x,y),(c,d)) = (c-x) + (d-y)$

$d_1 \ldots + d \ldots = (c-a) + (d-b) = d((a,b),(c,d))$

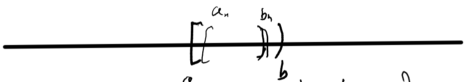
E.g. $\mathbb{R}^2 \supset \mathbb{N}^2 \longrightarrow \mathbb{R}$; $(p,q) \longmapsto p/q$, then limit points are $\mathbb{R}^+$

no limit points



Sorgenfrey line : $\mathbb{R}_l$

- Basis is half-open intervals $[a, b)$
- The set (a,b) is open as $(a,b) = \bigcup_{n=1}^{\infty} [a+\frac{1}{n}, b)$
- Finer than $\mathbb{R}$



- **Significance:** (1) In $\mathbb{R}$, $\{(a,b) : (a,b) \in \mathbb{Q}\}$ is a (countable) basis;

(2) In $\mathbb{R}_l$, $\{[a,b) : (a,b) \in \mathbb{Q}\}$ is <u>not</u> a basis

Why (1) : - If $a, b \in \mathbb{R}$, $a < b$, $\exists\, a_n \downarrow a$ & $b_n \uparrow b$ with $a_n, b_n \in \mathbb{Q}$ & $a_n < b_n$

- Then $\bigcup_{n \in \mathbb{N}} (a_n, b_n) = (a,b)$

Why not (2): - Above does not work as $\bigcup_{n \in \mathbb{N}} [a_n, b_n) \neq (a,b)$

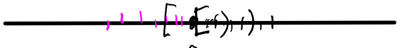
- In general, if $a \notin \mathbb{Q}$, $a < b$, $b \in \mathbb{R}$, $c, d \in \mathbb{Q}$ with $[c,d) \subset [a,b)$. Then $a \notin [c,d)$. Thus $a \notin \bigcup_{c,d \in \mathbb{Q}} [c,d)$

<u>Convergence</u>: $a_n \to a$ in X if $\forall\, U \subset X$ open, $x \in U$, $\exists\, N > 0$ s.t.

$$n > N \Rightarrow a_n \in U.$$

In $\mathbb{R}_\ell$, $a_n \to a$ means $a_n \to a$ (from the right).

i.e. $a_n \to a$ in $\mathbb{R}$ <u>and</u>



$\{n \in \mathbb{N} : a_n < a\}$ is finite.

• Cantor set : $C = \{a_n\}$, $n \in \{0,1\}$

Mapped to $[0,1]$ by $\varphi : \{a_n\} \longmapsto \sum_{n=0}^{\infty} \frac{2a_n}{3^n}$, image $\hat{C}$

Facts: • If a_n & b_n agree up to N, $d_E(\varphi(a_n), \varphi(b_n)) \leq \frac{1}{3^N}$

$|a+b| \leq |a|+|b|$

• If $d_E(\varphi(a_n), \varphi(b_n)) < \frac{1}{3^N}$, then $a_n = b_n \ \forall n < N$

$1/3$ $\quad$ $2/3$

$a_0 = 0$, $b_0 = 1$

$\sum_{n=1}^{\infty} \frac{2a_n}{3^n} \leq \frac{1}{3}$, $\sum_{n=1}^{\infty} b_n \geq 0$

$\therefore \sum_{n=0}^{\infty} \frac{2b_n}{3^n} \geq \frac{2}{3}$

Pf

$$\left| \sum_{n=0}^{\infty} \frac{2a_n}{3^n} - \sum_{n=0}^{\infty} \frac{2b_n}{3^n} \right|$$

$$= \left| \sum_{n=N+1}^{\infty} \frac{2a_n}{3^n} - \sum_{n=N+1}^{\infty} \frac{2b_n}{3^n} \right|, \quad a_n = b_n \text{ if } n \leq N$$

$$\leq \sum_{n=N+1}^{\infty} 2 \frac{|a_n - b_n|}{3^n}$$

$$\leq 2 \sum_{n=N}^{\infty} \frac{1}{3^N} \ldots$$

Defi: $\varphi_\infty(x) := \frac{x}{1+x}$, $\varphi_\infty : [0,\infty) \to [0,1)$

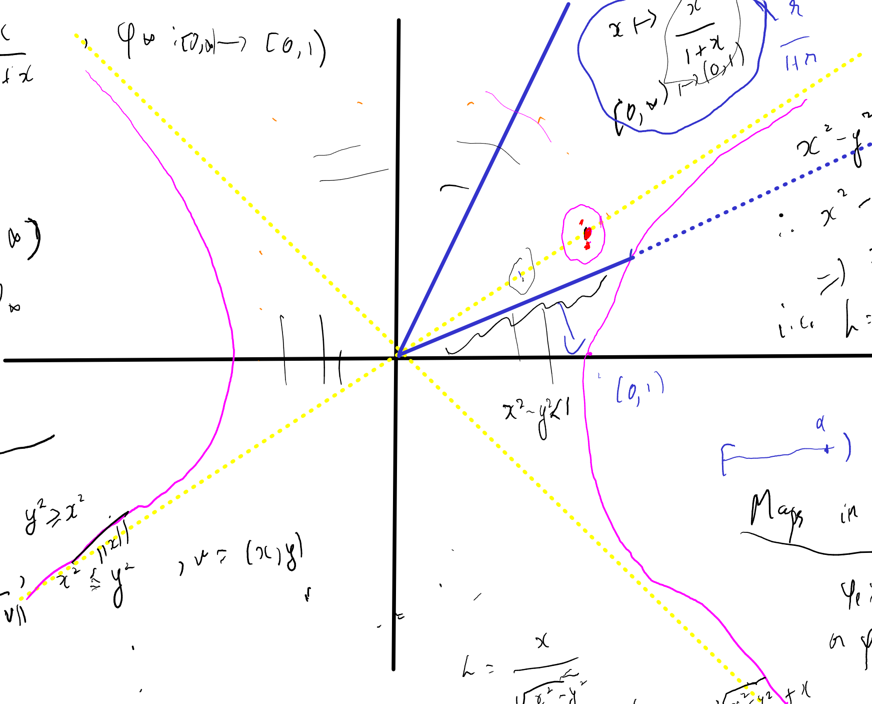
$\varphi_L(x) = \frac{1+L}{L}\,\varphi_\infty(x)$

$x \mapsto x$

$[0,L) \overset{incl}{\hookrightarrow} [0,\infty) \xrightarrow{\varphi_\infty} [0,1)$

image $[0, \frac{L}{1+L})$

$$\Phi(v) = \begin{cases} \frac{v}{1+\|v\|}, & y^2 \geq x^2 \\ \frac{\sqrt{x^2-y^2}+x}{x}\,\frac{v}{1+\|v\|}, & x^2 \geq y^2 \end{cases}, \quad v = (x,y)$$



$x \mapsto \frac{x}{1+x}$

$[0,\infty) \mapsto [0,1)$

$\frac{x}{1+x}$

$x^2 - y^2 = 1$, $y = mx$, $m<1$

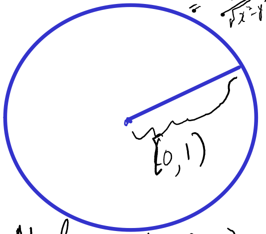
$\therefore x^2 - m^2x^2 = 1$

$\Rightarrow x = \frac{1}{\sqrt{1-m^2}} = \tan\theta = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}} = \frac{x}{\sqrt{x^2-y^2}}$

i.e. $L = \frac{1}{\sqrt{1-m^2}}$

$x^2 - y^2 < 1$

$[0,1)$



$[0,1)$

Maps in direction: Need $x \ll L$, L large $\Rightarrow$ $\varphi_L(x) \approx \varphi_\infty(x)$

$\varphi_L : [0,L) \to [0,1)$

or $\varphi_\infty : [0,\infty) \to [0,1)$

$x \mapsto \frac{x}{1+x}$

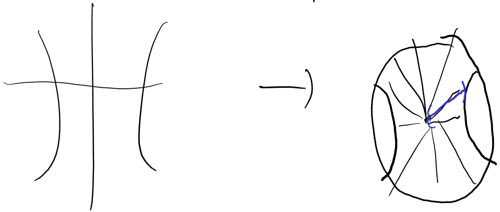
$L = \frac{x}{\sqrt{x^2-y^2}}$

So $\frac{1+L}{L} = \frac{\sqrt{x^2-y^2}+x}{x}$

$\mathbb{R}^2 \to \mathbb{D}^2$

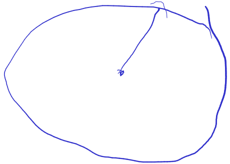
$(r, \theta) \mapsto \left(\frac{r}{1+r}, \theta\right)$

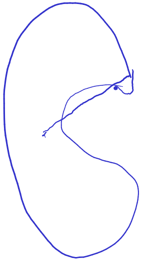
or $\quad x \mapsto \frac{x}{1+\|x\|}$



$\mathbb{R}^2 \to \mathbb{D}^2$

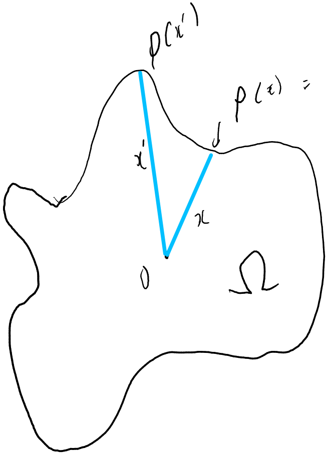
$x \mapsto \frac{x}{1+\|x\|}$

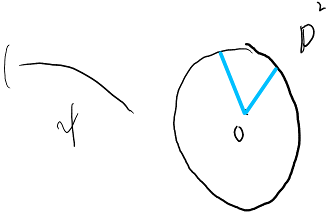




$\rho(x) = \sup\{\, \|y\| : y = tx \text{ for some } t > 0,\ y \in \Omega \,\}$

$\rho(x) = d(p_x, 0)$, p_x unique point on ray in ∂R





$\Psi : D^2 \to \Omega,$

$\Psi(x) = x \cdot \rho(x)$

$\Phi : \Omega \to D^2$

by $\Phi(x) = \dfrac{x}{\rho(x)}$

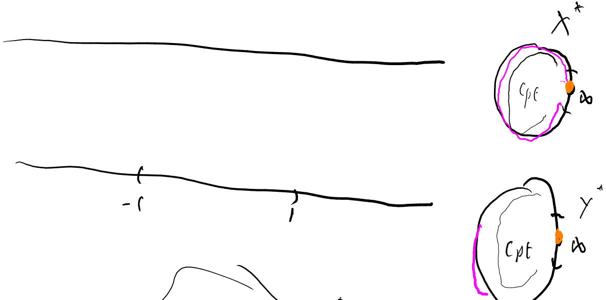
X, Y, $X \hookrightarrow X^*$, $Y \in Y^*$, assume X, Y locally compact, Hausdorff

$f : X \to Y$ extends to $f^* : X^* \to Y^*$

Theorem: f^* is continuous if f is proper.

E.g. $X = Y = \mathbb{R}$, $f(x) = \dfrac{x}{1+|x|}$

$f^*(\infty) = \infty$.



$X^* = Y^* = S^1 = \mathbb{R} \cup \{\infty\}$

Sketch:
- X^* is compact, Y^* is Hausdorff.
- f^* is continuous as f is proper
  - at points in X, f is cont. $\Rightarrow$ f^* is cont
  - proper $\Rightarrow$ cont. at ∞.

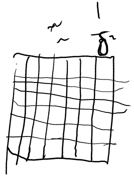
$\underline{\omega_1}$: $\mathbb{R}$ with well-ordering. $\ll$

$S = \{ \alpha \in \mathbb{R} : \{x \in \mathbb{R} : x \ll \alpha\}$, is uncountable $\}$

$\omega_1 = \min S$, exists by well-ordering

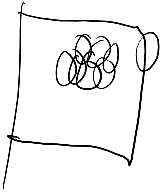
$X = [0,1] \times [0,1]$, $\quad d < 2$

Prop: $\nexists\, C_d$ s.t. $[0,1]\times[0,1]$ is covered by $C_d \left(\frac{1}{\varepsilon}\right)^d$ ε-balls



Pf: · Area of a ε-ball is $\pi \varepsilon^2$

· Area of $\underset{\overset{\shortparallel}{1}}{\text{region covered}}$ $\leq$ sum of areas of ε-balls



$$= C_d \cdot \frac{1}{\varepsilon^d} \cdot \pi \varepsilon^2$$

$$= \pi C_d \cdot \varepsilon^{2-d}$$

if $d < 2$, for ε small enough $\underbrace{\pi C_d \cdot \varepsilon^{2-d}}_{\substack{\downarrow \text{ as } \varepsilon \to 0 \\ 0}} < 1$, a contradiction.

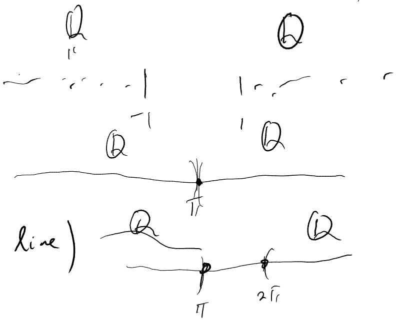
- $\mathbb{Q}$: metric space; ordered set (hence order topology)
- 2 completions:
  - as metric space (Cauchy)
  - as ordered set (Dedekind)

  } both $\mathbb{R}$

metric{ $\mathbb{Q}$ as ordered sets

$\parallel$

$\mathbb{Q} \cap (\mathbb{R} \setminus [-1,1])$



Generalizations:
- Metric very general, but
- Order topology: 'large sets' (long line)

E.g. $(0,1)$ & $\mathbb{R}$ are homeomorphic, isomorphic as ordered sets

$(0,1)$: not complete, completion $[0,1]$

$\mathbb{R}$: complete

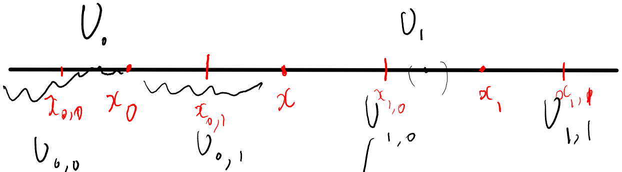
1. Let X be a non-empty, connected, Hausdorff topological space. Suppose that for all points $x \in X$, the space $X \setminus \{x\}$ has exactly two connected components. Prove or disprove that (the underlying set of) X must be uncountable.

Sketch of proof: . We construct points like Cantor diagonalization as intersections of sets $\subset X$. Show that these intersections are non-empty.

(1) Pick $x \in X$. Call components of $X \setminus \{x\}$ as U_0 and U_1

· U_0 & U_1 are open in X

· $x \in \bar{U}_i$, $i = 0, 1$ (otherwise U_i is closed in X)



(2) Split along $x_0 \in U_0$ (i.e., consider $X \setminus \{x_0\}$) to get $V_0 \wedge V_1$, with $x \in V_1$

Claim: $V_0 \subset U_0$. Let $U_{0,0} = V_0$, $U_{0,1} = V_1 \cap U_0$

V_0 is connected & $V_0 \subset X \setminus \{x\}$ ⟹ $V_0 \subset U_0$ or $V_0 \subset U_1$

$V_0 \subset U_0$ ∵ $x_0 \in \bar{V}_0$, $V_0 \subset U_0$ as

· Inductively define $U_{i_1, \dots, i_n}$ where $i_j \in \{0,1\}$ for all j.

Lemma: Let $\{i_n\}_{n\geq 1}$ be a sequence of 0's & 1's. (not all 0's or 1's)

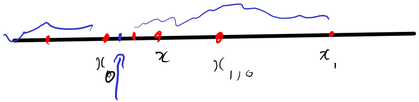
Then $\bigcap_{n=1}^{\infty} U_{i_1,\dots,i_n} \neq \emptyset$

· there are disjoint.

· there are uncountably many.

Pf: Otherwise we contradict openness of X,

· Namely, the sets $U_{j_1,\dots,j_p}$ are all open, so unions are open.



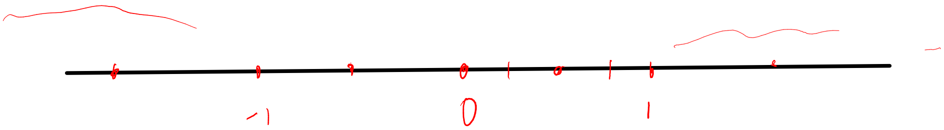
· We have an ordering on sequences of 0,1's.

· We have open sets in X

$$W_- = \bigcup \{U_{j_1,\dots,j_p} : (j_1,\dots,j_p) < (i_1,i_2,\dots,i_n,\dots)\}$$

$$W_+ = \bigcup \{U_{j_1,\dots,j_p} : (j_1,\dots,j_p) > (i_1,\dots,\dots)\} \cup \text{points } x_{j_1,\dots,j_p} \text{ s.t. } (j_1,\dots)$$

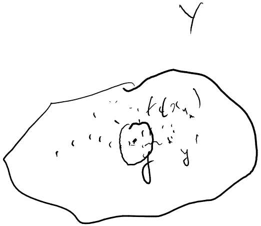
· If $\bigcap_{n=1}^{\infty} U_{i_1,\dots,i_n} = \emptyset$, then $X = W_- \cup W_+$, so X is disconnected.


-1
0
1

Let X and Y be compact metric spaces and let $f : X \to Y$ be a function. Assume the following:

if $\{x_n\}_{n\geq 1}$ is a sequence of points in X converging to a point $x \in X$ such that the sequence $\{f(x_n)\}_{n\geq 1}$ converges to a point $y \in Y$, then we must have $f(x) = y$.

Prove that f is continuous.



- We show $x_n \to x \Rightarrow f(x_n) \to f(x) = y$

  For $x \in X$

- Suppose not, then $\exists\, \varepsilon > 0$ s.t.

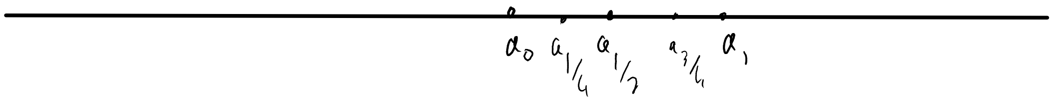
  $d(f(x_n), y) > \varepsilon$ for infinitely many x_n, i.e. a subsequence $\{x_{n_k}\}$.

- By passing to a further subsequence, can assume $f(x_{n_k}) \to y'$ for some y'.

- We conclude $y' \neq y$

  But $x_{n_k} \to x$ & $f(x_{n_k}) = y$, a contradiction.

- We label countably many points as a_α, $\alpha = p/2^n$ a dyadic rational in $[0,1]$, preserving order inductively on k
- To do this, given $a_{p/2^n}$, p odd, take a point between $a_{\frac{p-1}{2^n}}$ and $a_{\frac{p+1}{2^n}}$



- Given any $\alpha \in [0,1]$, define $a_\alpha = \sup\{a_\beta : \beta \text{ dyadic rational}, \beta < \alpha\}$