# Applied Linear Algebra in Data Analysis

## Linear Systems and Matrix Operations: Part 1

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# Linear Equations

What is a linear equation? – *An algebraic equation in which the highest power of any variable is 1.*

$$5x_1 - 2x_2 + 11x_3 - 12x_4 = 23$$

$x_1, x_2, x_3, x_4$ are variables or unknowns in the above equation. Numbers infront of the variables are their coefficients.

The general form of a linear equation in $n$ variables $x_1 \dots x_n$ is,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \dots a_n$ are known coefficients and $b$ is a constant.

# Geometry of Linear Equations

Let's now look at the geometry of linear equations. We will start with an equation in one variable.

$$ax_1 = b \implies x_1 = \frac{b}{a} \quad a \neq 0$$

There is one unknown variable, $x_1$ which is a real number, i.e. $x_1 \in \mathbb{R}$, and $a, b \in \mathbb{R}$.

There are geometric views we can associate with this equation. We will call these two views:

- **Row view** – Geometric view in the solution space, i.e. the space of $x_1$.
- **Column view** – Geometric view in the coefficient space, i.e. the space of $a$ and $b$.

# Geometry of Linear Equations

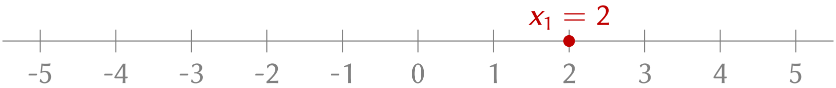
## Row View

Consider the following equation

$$2x_1 = 4 \implies x_1 = \frac{4}{2} = 2$$

There is one unknown variable, $x_1$ which is a real number, i.e. $x_1 \in \mathbb{R}$.

The solution space in this case is the real line $\mathbb{R}$. The geometric view will be the locus of all points that satisfy the above equation.



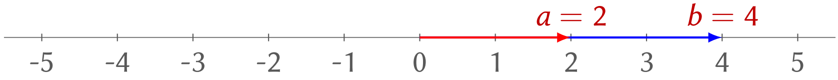
## Column View

Consider the same equation

$$2x_1 = 4 \implies x_1 = \frac{4}{2} = 2$$

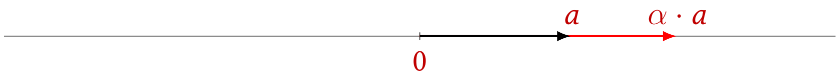
The coefficients associated with the unknowns can also be indicated on the real line.

In this space we will choose to represent the coefficients $a$ and $b$ as arrows pointing from the origin to their corresponding values.



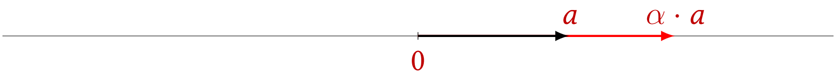

We will refer to these arrows as *vectors* $\implies$ $a, b$ are vectors in $\mathbb{R}$.

## Column View




Vector Scaling – Multiplying $a$ by a real number "scales" the vector in $\mathbb{R}$. This operation stretches or shrinks the vector $a$ by a factor of $|\alpha|$. If $\alpha$ is negative, the vector also flips direction.

## Column View

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$$a \cdot x_1 = b \iff$$ How much should we scale vector $a$ to obtain vector $b$?

This gives $x_1$ a clear interpretation – it's the *scaling factor*.

## Column View

The column view gives us another way to look at the simple linear equation.

$$a \cdot x_1 = b$$

where, $a$ and $b$ are fixed and $x_1$ is an unknown variable.

While, the right hand side (RHS) represents a fixed vector $b$ in $\mathbb{R}$. The LHS represents a set of infinite number of vectors in $\mathbb{R}$, obtained by different amounts of scaling $x_1 \in \mathbb{R}$.

$$\mathcal{C}\left(a\right) = \left\{x_1 \cdot a \mid x_1 \in \mathbb{R}\right\}$$

This set $\mathcal{C}\left(a\right)$ is called the column space of the vector $a$.

## Column View

The set $\mathcal{C}(a)$ looks like this:

$$\mathcal{C}(a) = \{\cdots, \quad -1.2 \cdot a, \quad 2.124 \cdot a, \quad 5 \cdot a, \quad 0 \cdot a, \quad -a, \quad \cdots\}$$

This is the set of all values that can be produced by scaling $a$ by different amounts.

$$\begin{cases} \mathcal{C}(a) = \mathbb{R}, & a \neq 0 \\ \mathcal{C}(a) = \{0\}, & a = 0 \end{cases}$$

This way of looking at equation $a \cdot x_1 = b$ makes it clear when an equation is solvable, and when its not.

# Geometry of Linear Equations

**Example 1:** $5 \cdot x_1 = 10 \quad a = 5,\ b = 10$

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$\therefore$ The equation is solvable and the solution is $x_1 = 2$.

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**Example 2:** $0 \cdot x_1 = 10 \quad a = 0, \; b = 10$

$\mathcal{C}(0) = \{0\}$ since $a = 0$, $b = 10 \notin \mathcal{C}(0)$
$\therefore$ The equation is **not** solvable.

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**Example 3:** $0 \cdot x_1 = 0 \quad a = 0,\ b = 0$

$\mathcal{C}(0) = \{0\}$ since $a = 0$, $b = 0 \in \mathcal{C}(0)$
$\therefore$ The equation is solvable and has infinitely many solutions,$x_1 \in \mathbb{R}$.

# Three Possible Solutions

$$a \cdot x = b$$

**Unique Solution**

- $a \neq 0, b \in \mathbb{R}$
- *Always solvable*
- $\mathcal{C}(a) = \mathbb{R}$
- $b \in \mathcal{C}(a)$
- $x = \frac{b}{a}$

**No Solution**

- $a = 0, b \in \mathbb{R}$
- *Not solvable*
- $\mathcal{C}(0) = \{0\}$
- $b \notin \mathcal{C}(0)$

**Infinite Solutions**

- $a = 0, b = 0$
- *Always solvable*
- $\mathcal{C}(0) = \{0\}$
- $b \in \mathcal{C}(0)$
- $x \in \mathbb{R}$

***Remember**: The equation is solvable if and only if* $b \in \mathcal{C}(a)$.

# Geometry of Linear Equations

Let's now look at a linear equation in two variables.

$$a_1x_1 + a_2x_2 = b$$

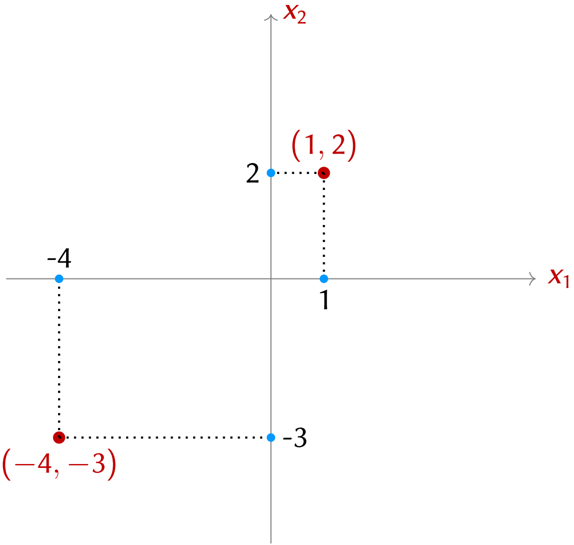
where, $a_1, a_2, b \in \mathbb{R}$ are fixed and $x_1, x_2$ are the unknown variables.

We once again have two geometric views, except things a bit more interesting now.

- The solution space is not the real line anymore! There are two real numbers $x_1, x_2 \in \mathbb{R}$. The set of two real numbers is the real plane $\mathbb{R}^2$.

# The Real Plane – $\mathbb{R}^2$

Every point in the plane is represented by two real numbers $(x_1, x_2)$.



# Geometry of Linear Equations

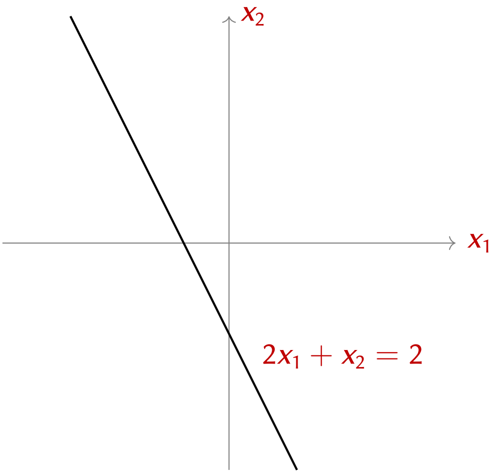
## Row View

Consider the following equation.

$$2x_1 + x_2 = -2$$

The solution space is $\mathbb{R}^2$ and locus of all points that satisfy the above equation is shown as the black line.

We now have infinitely many solutions.

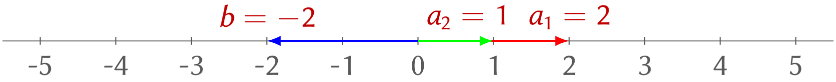


## Column View

Consider the same equation

$$2x_1 + x_2 = -2$$

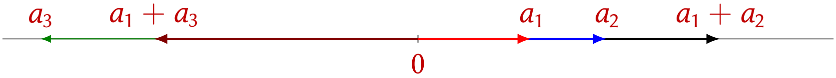
The coefficients associated with the unknowns can be seen as vectors on the real line.




The left hand side $2x_1 + x_2$ represents the two geometric operations – vector scaling and vector addition.

# Geometry of Linear Equations

## Column View

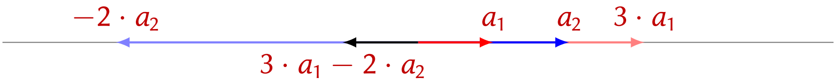



Vector Addition – The above geometric operation is the vector addition operation in $\mathbb{R}$.

Vector addition in $\mathbb{R}$ is simply the sum of the real numbers representing the vectors.

# Geometry of Linear Equations

## Column View

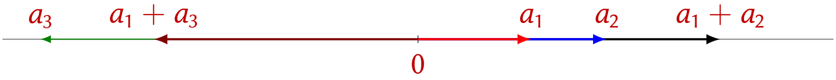



Linear Combination – An operation on a set of vectors, involving both vector scaling and vector addition. The linear combination of vectors $a_1$ and $a_2$ with scalars $x_1, x_2 \in \mathbb{R}$ is given by,

$$x_1 \cdot a_1 + x_2 \cdot a_2$$

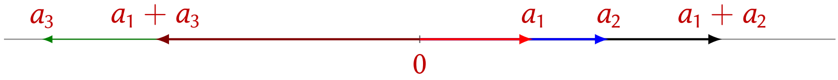
We will refer to the scaler or scaling factors $x_1$ and $x_2$ as the mixtures of the linear combination.

### Column View




Now, we can provide an interpretation for $x_1$ and $x_2$ in the equation $a_1x_1 + a_2x_2 = b$.

## Column View




Now, we can provide an interpretation for $x_1$ and $x_2$ in the equation $a_1x_1 + a_2x_2 = b$.

$$a_1 \cdot x_1 + a_2 \cdot x_2 = b \iff$$

What mixture of the linear combination of $a_1$ and $a_2$ produces $b$?

### Column View

$$a_1 \cdot x_1 + a_2 \cdot x_2 = b$$

As before, the LHS of the above equation can produce an infinite number of vectors in $\mathbb{R}$. We define this set as the column space of set $A = \{a_1, a_2\}$.

$$\mathcal{C}(A) = \{x_1 \cdot a_1 + x_2 \cdot a_2 \mid x_1, x_2 \in \mathbb{R}\}$$

There are two possibilities for the set of all possible values for the linear combinations of $a_1$ and $a_2$.

$$\begin{cases} \mathcal{C}(A) = \mathbb{R}, & a_1 \neq 0 \text{ OR } a_2 \neq 0 \\ \mathcal{C}(A) = \{0\}, & a_1 = 0 \text{ AND } a_2 = 0 \end{cases}$$

# Geometry of Linear Equations

**Example 1:** $x_1 + x_2 = 2 \quad a_1 = 1, a_2 = 1, \; b = 2$

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$\mathcal{C}(A) = \mathbb{R}$ since $a_1, a_2 \neq 0$, $b = 2 \in \mathcal{C}(A)$
$\therefore$ The equation is solvable and had infinitely many solutions $(x_1, 2 - x_1), \; x_1 \in \mathbb{R}$.

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$\mathcal{C}(A) = \{0\}$ since $a_1 = a_2 = 0$, $b = -3 \notin \mathcal{C}(0)$
$\therefore$ The equation is **not** solvable.

# Two Possible Solutions

$$a_1 \cdot x_1 + a_2 \cdot x_2 = b$$

**No Solution**

- $a = 0, b \in \mathbb{R}$
- *Not solvable*
- $\mathcal{C}(0) = \{0\}$
- $b \notin \mathcal{C}(0)$

**Infinite Solutions**

- $a = 0, b = 0$
- *Always solvable*
- $\mathcal{C}(0) = \{0\}$
- $b \in \mathcal{C}(0)$
- $x \in \mathbb{R}$

***Remember***: *The equation is solvable with infinitely many solution, if and only if* $b \in \mathcal{C}(a)$.

# What is the difference between the two equations?

The two linear equations in 1 and 2 unknown variables are different in the types of solutions.

$$\begin{cases} a \cdot x_1 = b & \longrightarrow 3 \text{ possible solutions} \\ a_1 \cdot x_1 + a_2 \cdot x_2 = b & \longrightarrow 2 \text{ possible solutions} \end{cases}$$

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Why is there a difference?
More unknowns than equations $\implies$ more freedom to choose values.

But this cannot be the complete answer, because $0 \cdot x_1 = 0$ also has infinitely many solutions for $b = 0$, where the number of unknowns and equations are the same!

# Linear Independence determines the number of solutions

Whether a solvable linear equation has unique or infinite solutions is determined by the property of linear independence of the set of vectors - formed by the coefficients associated with the unknown variables in the linear equations.

Linear independence is a property of a set of vectors.

Linear independence tells us if there is ”more than what is needed” in the set.

Consider a set $A$ of vectors (real numbers for now): $A = \{a_1, a_2, \cdots a_n\}$ , $a_i \in \mathbb{R}$.

**Definition**

(INCOMPLETE) A set of vectors $A$ is *linearly independent* if none of the vectors in the set can be expressed as a linear combination of the others.

# Linear Independence

**A test for linear independence**. Consisder the linear combination of a set of vectors $A$ which produces another vector $b$,

$$\alpha_1 \cdot a_1 + \alpha_2 \cdot a_2 + \cdots \alpha_n \cdot a_n = b, \quad \alpha_i \in \mathbb{R}$$

What mixtures (values of $\alpha_i$) of the above linear combination produce the zero vector $b = 0$?

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ is the only set of values that produces $b = 0$, then the set of vectors is linearly independent.

Even if any one of the $\alpha_i \neq 0$ for $b = 0$, then the set is linear dependent.

# Linear Independence

If any one of the $\alpha_i \neq 0$, then the following is true (assuming $\alpha_j \neq 0$),

$$a_j = -\sum_{i=1, i\neq j}^{n} \frac{\alpha_i}{\alpha_j} \cdot a_i \implies \begin{cases} a_j \text{ can be expressed as a linear combination} \\ \text{of the other vectors in the set } A. \end{cases}$$

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ is the only set of values that produces $b = 0$, then no vector in the set can be expressed as a linear combination of the others.

What happens when all $\alpha_i = 0$ except $\alpha_j \neq 0$? Obviously , the set is linearly dependent. What more can we say about the set?

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$$a_j = 0$$

A set containing the zero vector is linearly dependent!

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**Example 4:** $A = \{2, 1\}$

The set is linearly dependent since $\alpha_1 \cdot 2 + \alpha_2 \cdot 1 = 0 \implies \alpha_2 = -2 \cdot \alpha_1$.

# General conditions for solutions of a linear equation.

Now, we can state the general conditions for the solutions of a linear equation,

$$a_1 \cdot x_1 + a_2 \cdot x_2 + \cdots + a_n \cdot x_n = b$$

where, $a_i, b \in \mathbb{R}$ are fixed and $x_i$ are the unknown variables.

- If the set $A = \{a_1, a_2, \cdots a_n\}$ is linearly independent and $b \in \mathcal{C}(A)$, then the equation has a unique solution.
- If the set $A = \{a_1, a_2, \cdots a_n\}$ is linearly dependent and $b \in \mathcal{C}(A)$, then the equation has infinitely many solutions.
- If $b \notin \mathcal{C}(A)$, then the equation has no solution.

# Some additional concepts

We will now make some observations about some of the sets we have seen so far.

Let's first define an important concept from set theory.

## Definition

A set $S$ is said to *closed under an operation* if performing the operation on any elements of the set $S$ produces an element that is also in the set $S$.

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**Examples:**

- The set of integers is closed under addition.
- The set of integers is closed under multiplication.
- The set of even integers is closed under addition.
- The set of odd integers is *NOT* closed under addition.
- The set of integers is *NOT* closed under division.
- The set of integers is *NOT* closed under square roots.

# Some additional concepts: Vector Spaces & Subspace

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Is $\{0\}$ a vector space?

# Some additional concepts: Vector Spaces & Subspace

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A set $S$ that is closed under linear combination of its elements is called a vector space. The elements of a vector space are called vectors.

$\mathbb{R}$ is a vector space.

Is $\{0\}$ a vector space? Yes, it is! Linear combinations of $0$ always produce $0$.

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But $\{0\} \subset \mathbb{R}$ and is a vector space itself. Such vector spaces are called subspaces of a parent vector space.

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What about the closed interval $[0, 1]$? No, it is not a vector space since it is not closed under linear combination. For example, $0.5 + 0.7 = 1.2 \notin [0, 1]$.

# Some additional concepts: Span of set of vector

For the linear equation, $x_1 \cdot a_1 + x_2 \cdot a_2 + \cdots + x_n \cdot a_n = b$, the column space of $A = \{a_1, a_2, \cdots a_n\}$ is a vector space! Why?

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$\mathcal{C}(A)$ is the set of all linear combinations of the vectors in $A$. This important concept has a special name – the span of a set of vectors $A$.

## Definition

The *span* of a set of vectors $A = \{a_1, a_2, \ldots, a_n\}$ is the set of all possible linear combinations of the vectors in $A$.

The span of set $A$ is denoted as $\mathrm{span}(A)$.

$$\mathrm{span}(A) = \{\alpha_1 \cdot a_1 + \alpha_2 \cdot a_2 + \cdots + \alpha_n \cdot a_n \mid \alpha_i \in \mathbb{R}\}$$

# Summary

## Important Vector Space Concepts

### Vector Space

A set closed under linear combinations.

### Subspace

A subset of a vector space that is itself a vector space.

### Span

The set of all linear combinations of a set of vectors.