Applied Linear Algebra in Data Analysis

What is this course about?

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What is this course about?

This is an introductory course on high dimensional thinking.

The world is multidimensional

- Most things/systems/ideas of interest in science and engineering are multidimensional.
- Undergraduate engineering programs focus on mathematical skills for representing, analysing, and designing relatively simple systems $\longrightarrow$ Univariate **Calculus**!
  - Linear single-input-single-output (SISO) system theory (used in signal processing and control theory).
  - Fourier, Laplace, z-transforms for understanding 1D temporal or spatial phenomena.
  - Study of dynamics in mechanical engineering.
- Most applications or systems we deal with have multiple degree-of-freedom, various causes and effects, multiple inputs and outputs, multiple measurements, multiple parameters etc.
- Dealing with such multidimensional problems requires an additional set of mathematical tools and a different way of thinking.

The world is also uncertain

- Noise and uncertainty are inherent in the real world.
- All measurements are corrupted by unwanted sources and the outcomes of all our actions are also influenced by a variety of factors.
- There is often some structure to this uncertainty, which can be exploited.
- Dealing with uncertainity too requires an additional set of mathematical and statistical tools.

We are often faced with multiple options

- There are multiple options for solving a problem. For example,
  - Choosing the inputs to a system to get a desired output.
  - Estimating a parameter from a set of measurements.
  - Choose the design parameters for a given set of specifications.
- In such situations, we need to search through the *options space* to find the “best” one.
- Various mathematical tools exist for formulating and efficiently solving a wide range of problems.

Three major topics are covered in this course

Linear Algebra

- First step in high dimensional thinking.

Probability theory

- Principled approach to navigate uncertainity.

Optimization

- Approaches for searching spaces for the best solutions.

Linear Algebra

- Algebra (Arabic: *al-jabr* 'reunion of broken parts') is the study of variables and the rules for manipulating these variables in formulas (Source: Wikipedia).
- Linear algebra is the study of linear system of equations.

$$\left.\begin{array}{r} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = y_1 \\ \vdots \\ a_{m1}x_1 + a_{32}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = y_m \end{array}\right\} \longrightarrow \mathbf{Ax} = \mathbf{y}$$

These are useful in a surprisingly wide range of real world problems!

$$\begin{aligned} x_1, x_2, \ldots x_n &: \text{inputs/parameters of the problem} \\ y_1, y_2 \ldots y_m &: \text{outputs or measurements} \\ a_{ij} &: \text{physics of the problem} \end{aligned}$$

- Linear algebra provides tools for: understanding, manipulating, and efficiently solving such problems.

Probability Theory

- Branch of mathematics dealing with probability - a measure of uncertainty of events.
- Provides the tools for representing, analysing, and manipulating uncertainity of events.
- Some essential concepts: probability distributions/densities, random variables, expectation, variance, covariance, conditional probability, Bayes rule, etc.
- We will also look at the Gaussian or Normal distribution, arguably the most important distribution in statistics.

Mathematical Optimization

- Mathematical optimization/programming is the selection of a best element, with regard to some criterion, from a some set of available alternatives. (Source: Wikipedia)

- We will deal only with continuous optimization problems, where involves searching over the spaces of real numbers ($\mathbb{R}^n$).

- Criteria for choosing the best element is often expressed as a function ($f(\mathbf{x})$) of the elements from the search space $\mathbb{R}^n$.

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \qquad \text{subject to } g_i(\mathbf{x}) \leq 0, \; i = 1, \ldots, m$$

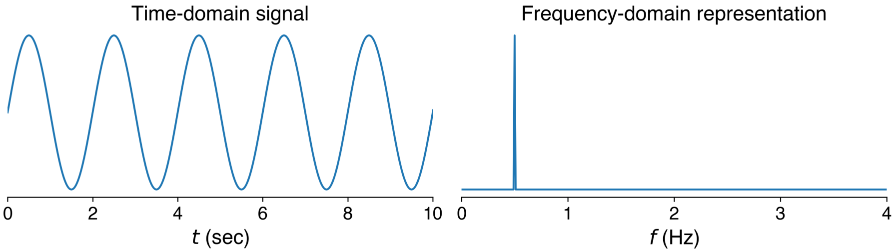
What kind of problems can we solve with these tools?

- Signal processing
- Control theory
- Robotics
- Statistics
- Machine learning
- Medicine
- Economics
- ...

Signal Processing

We often deal with temporal signal that can be visualized as a function of time. Is this the only way to represent a signal?

A sinusoidal signal's time domain representation is shown below. We don't need the signal's entire time record to covney its information. We only need three numbers - the amplitude, frequency, and phase of the sinusoid. These are sufficient to reconstruct the signal. They can be obtained by Fourier transforming the signal into the frequency domain, i.e. a different perspective of the signal.



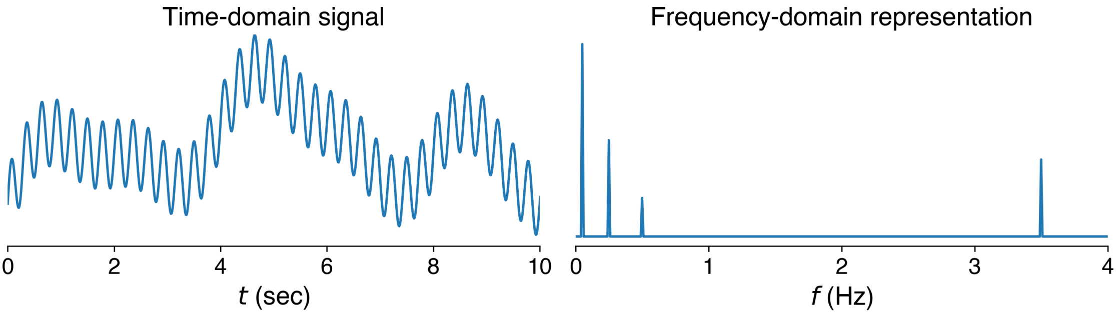
Signal Processing

- Expressing signals as a linear combinations of other signals (basis functions) is at the core of most of signal processing.

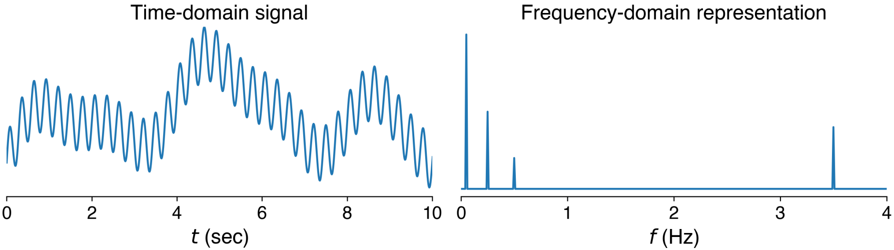
$$x(t) = c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t)$$

- $\{c_i\}_i$ is another representation of the signal $x(t)$ in terms of $\{\phi_i(t)\}_i$.
- The Fourier transform is one such representation of a signal.
- $\{\phi_i\}_i$ are called the basis functions. Different basis functions gives different types of transforms.
- Manipulating $\{c_i\}_i$ will allow us to extract or suppress different types of information from $x(t)$.

Signal Processing



Signal Processing



What is so special about the Fourier transform? Are there other transforms that might be better suited for certain types of signals? Can I design my own transform?

Thinking of signals and images as entities residing in a high-dimensional space turns out to be very useful way to think about signals and how to process them.

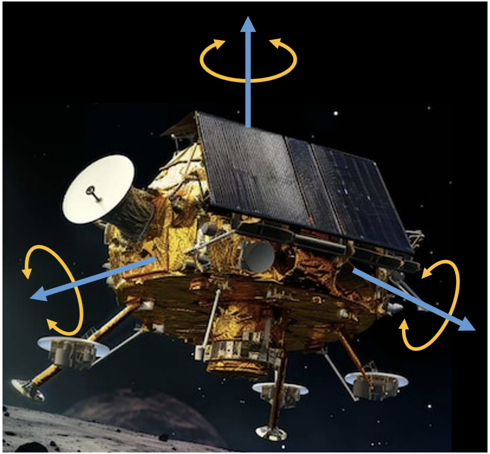
Control theory

A spacecraft has 6 DOF - 3 translational x, y, z and 3 rotational ϕ_1, ϕ_2, ϕ_3.
We need both position and velocity information to control the spacecraft.

The spacecraft has multiple sensors that measure position and velocity.

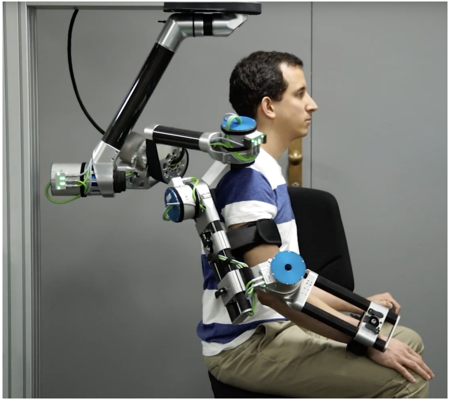
Multiple thrusters, reaction-wheels are used to control the spacecraft dynamics.

How do we get the best estimate of the spacecraft position and velocity? Which and by how much do we activate the different thrusters and reaction-wheels to control the spacecraft?



Source: http://surl.li/klulx

Robotics



Rehab Robot [Source: http://surl.li/klupd]

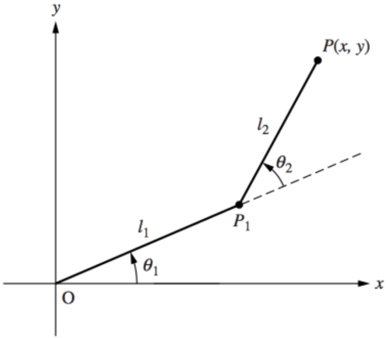
Surgical Robot [Source: http://surl.li/klurm]

Robotics

Kinematics and dynamics of serial robots are highly non-linear.

Kinematics of a 2-link planar robot

$$x = l_1 \cos(\theta_1) + l_1 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin(\theta_1) + l_1 \sin(\theta_1 + \theta_2)$$



The differential kinematics however has is linear relationship:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \mathbf{J}(\theta_1, \theta_2) \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$\mathbf{J}(\theta_1, \theta_2)$: Jacobian matrix

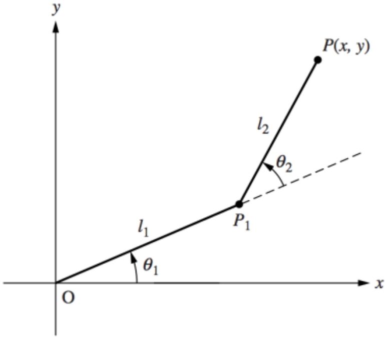
Robotics

Dynamics of serial robots.

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} + \mathbf{J}^\top \mathbf{f}$$

where,

- $\mathbf{q}$: Joint angles
- $\mathbf{M}(\mathbf{q})$: Inertia matrix
- $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$: Coriolis and centrifugal forces
- $\mathbf{g}(\mathbf{q})$: Gravity forces
- $\boldsymbol{\tau}$: Joint torques



Statistics

The data that we deal with is often heterogeneous, high-dimensional, and noisy.

Core of statistics:

- Representing such data
- Evaluating the relationships between different variables
- Building and estimating models for inference and decision making

Proability theory, Calculus, Linear algebra, and Optimiztion are employed for solving these problems efficiently.

Statistics

Health-related physical fitness (HRPF) can be good indicator for the developments of chronic diseases [1].
HRPF can be difficult to measure and has multiple components - HGS, STS, and VO_2 max.
Can we predict these HRPF component's from easily measurable variables – age, height, weight, BMI, and % body fat?

$$y_j = \beta_0 + \beta_1 x_{j1} + \beta_2 x_{j2} + \cdots + \beta_n x_{jn} + \epsilon_j \longrightarrow \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

$$\min_{\boldsymbol{\beta}} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 \implies \hat{\boldsymbol{\beta}} = \left(\mathbf{X}^\top \mathbf{X}\right)^{-1} \mathbf{X}^\top \mathbf{y}$$

[1] Kim, S.-W. et al., Front. Physiol. 12, 668055 (2021).

What if we want to have a minimalistic model – good performance with fewest number of variables?

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

Ridge Regression:

$$\min_{\boldsymbol{\beta}} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 + \lambda\|\boldsymbol{\beta}\|_2^2 \implies \hat{\boldsymbol{\beta}} = \left(\mathbf{X}^\top\mathbf{X} + \lambda\mathbf{I}\right)^{-1}\mathbf{X}^\top\mathbf{y}$$

LASSO Regression:

$$\min_{\boldsymbol{\beta}} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 + \lambda\|\boldsymbol{\beta}\|_1$$

Machine learning

In machine learning, we are interested in modelling complex models through the use of data.

Data is almost always high dimensional in such applications with complex structure.

Models we are interested in fitting a more complex than linear models.

- Linear/Logistic Regression
- Decision trees, random forests
- Support vector machines
- Neural networks
- ...

Why do this course?

This is will prepare you to comfortably work with such problems.