Applied Linear Algebra in Data Analysis

Linear Systems and Matrix Operations: Part 2

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Simultaneous Linear Equations

We now move on to a more general case of linear equations – simultaneous linear equations.

We have p equations and q unknown variables.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1q}x_q &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2q}x_q &= b_2 \\
&\vdots \\
a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pq}x_q &= b_p
\end{aligned}
$$

where, $a_{ij},\ b_i \in \mathbb{R},\ 1 \leq i \leq p,\ 1 \leq j \leq q$ are fixed and x_j are the unknown variables.

The goal is to find values for the unknown variables $x_j,\ 1 \leq j \leq q$ that satisfy all the equations simultaneously.

Why care about simultaneous linear equations?

Numerous problem commonly appear in varied fields: Engineering, Medicine, Signal Processing, Statistics, Economics, etc.

- **Circuit analysis.** Solving for voltages and currents in electrical circuits with multiple loops and nodes.
- **Structural analysis.** Determining forces and displacements in structures.
- **Chemical reaction networks.** Finding concentrations of species in complex reaction systems.
- **Musculoskeletal modeling.** Estimating muscle forces during movement.
- **Medicine.** Linear regression models to understand relationships between variables.
- **Radiology.** Reconstructing images from projection data in CT scans.
- **Radiotherapy.** Optimizing radiation dose distributions for cancer treatment.
- **Operations Research.** Optimizing resource allocation in supply chain management.

Row View

Consider the following equations,

$$\begin{aligned} 2x_1 + x_2 &= 2 \\ x_1 - x_2 &= -1 \end{aligned}$$

There are two unknown variables, $x_1, x_2 \in \mathbb{R}$.

$\mathbb{R}^2$ is the solution space. The locus of all points that satisfy both equations is the solution.

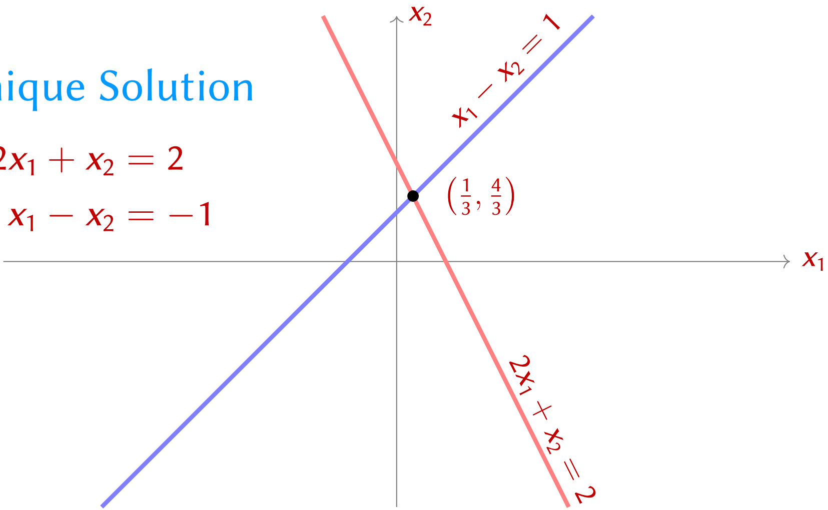
This is the standard view we are familiar with.

Each equation represents a geometric object in the solution space.

The solution to the system of equations is the intersection of these geometric objects.

Unique Solution

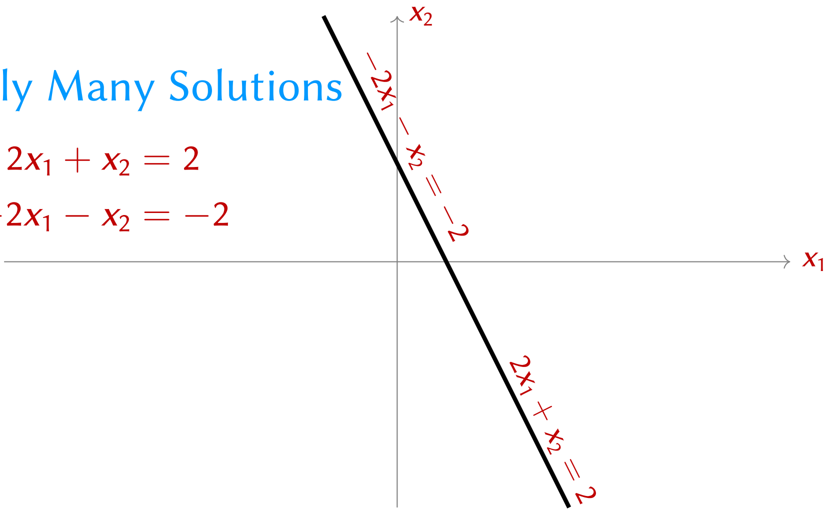
$$2x_1 + x_2 = 2$$
$$x_1 - x_2 = -1$$



Geometry of Linear Equations

Infinitely Many Solutions

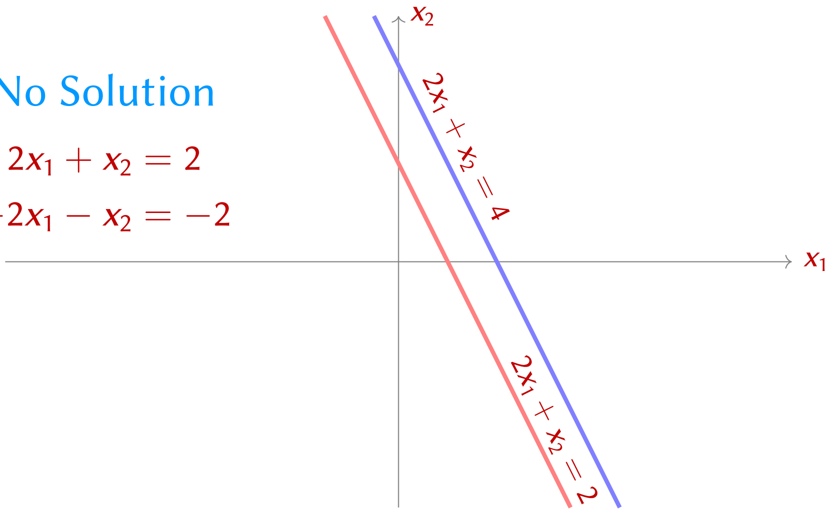
$$2x_1 + x_2 = 2$$
$$-2x_1 - x_2 = -2$$



Geometry of Linear Equations

No Solution

$$2x_1 + x_2 = 2$$
$$-2x_1 - x_2 = -2$$



Geometry of Linear Equations

Row View

$$\boxed{2x_1 + x_2 = +2}$$
$$\boxed{x_1 - x_2 = -1}$$

Column View

Deals with the equation's coefficient space. We group the coefficients correponding to the unknowns on the LHS and the constants on the RHS together.

$$\boxed{\begin{matrix} 2x_1 + x_2 \\ x_1 - x_2 \end{matrix}} \begin{matrix} = \\ = \end{matrix} \boxed{\begin{matrix} 2 \\ -1 \end{matrix}} \quad \longrightarrow \begin{bmatrix} 2x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

We have now introduce a mathematical object – the n-vector – to group multiple elements between square brackets.

n-vectors

We now define a new mathematical object – the *n-vector*.

Definition

An **n-vector** is an ordered collection of n numbers.

- n-vectors are represented by bold lowercase alphabets (e.g., $\mathbf{a}, \mathbf{x}, \ldots$).
- n-vectors are indexed and the order matters.
- The numbers in an n-vector are called its **components**.
- Components are represented using regular lowercase alphabets with subscripts.
- n-vectors are written using either column or row notation. But in this course, we will assume that all n-vectors are column vectors by default.

n-vectors

n-vectors are written as a column vector.

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \quad a_i \in \mathbb{R}$$

When we want to express it as a row vector, we will use the transpose notation, as follows.

$$\mathbf{a}^\top = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}^\top = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^n$$

The transpose operation changes columns to rows and vice versa, i.e. $\left(\mathbf{a}^\top\right)^\top = \mathbf{a}$.

n-vectors

We will refer to individual elements of an n-vector as scalars.

The transpose of scalar or 1-vector is itself. $x^\top = x, \quad x \in \mathbb{R}$.

When the individual elements of an n-vectors are complex numbers, then the vector belongs to $\mathbb{C}^n$.

The *dimension* or *size* of an n-vector is the number of components/elements of the vector.

Vectors $\mathbf{a}$ and $\mathbf{b}$ are *equal*, i.e., $\mathbf{a} = \mathbf{b}$, if and only if,

- both have the same size; and
- $a_i = b_i, i \in \{1, 2, 3, \ldots n\}$

Column View

$$\begin{aligned} 2x_1 + x_2 &= 2 \\ x_1 - x_2 &= -1 \end{aligned} \longrightarrow \begin{bmatrix} 2x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

The two equations involving only scalars is expressed as a single equation equating two 2-vectors, $\begin{bmatrix} 2x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \in \mathbb{R}^2$.

However, this vector equation hardly adds any new insight. Its simply the restatement of the original set of equations with fancy brackets!

The new insights become clear, when the LHS is further broken down in terms of two basic n-vector operations: (a) *scalar multiplication*, and (b) vector addition.

Two important operations on n-vectors

Vector Addition: Defined for two vectors of the same size, and the individual elements are added. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then we define vector addition as the following:

$$\mathbf{x} + \mathbf{y} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \triangleq \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \in \mathbb{R}^n$$

Thus, we can now rewrite the LHS of the column view as the following,

$$\begin{aligned} 2x_1 + x_2 &= 2 \\ x_1 - x_2 &= -1 \end{aligned} \longrightarrow \begin{bmatrix} 2x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ -x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Two important operations on n-vectors

Vector Scaling: Multiplication of a scalar and a vector. The individual elements of the vector are multiplied by the scalar. Let $\mathbf{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then we define vectoe scaling as the following:

$$\alpha \mathbf{x} \triangleq \alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \triangleq \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix} \in \mathbb{R}^n$$

Thus, we can now rewrite the LHS of the column view as the following,

$$\begin{aligned} 2x_1 + x_2 &= 2 \\ x_1 - x_2 &= -1 \end{aligned} \longrightarrow \begin{bmatrix} 2x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ -x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \longrightarrow x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Linear combination of n-vectors

We can now define a new operation on n-vectors, that is the combination of vector scaling and vector addition – the linear combination of n-vectors.

Definition

A **linear combination** of n-vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k \in \mathbb{R}^n$ is an expression of the form,

$$\alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \cdots + \alpha_k \mathbf{a}_k, \quad \alpha_i \in \mathbb{R}, \; 1 \leq i \leq k$$

where, α_i are scalars and are referred to as the **weights** or the **mixture** of the linear combination.

Thus, the LHS of linear equations under the column view can be expressed as a linear combination of p-vectors.

Geometry of Linear Equations

Column view In the following equation,

$$\begin{aligned} 2x_1 + x_2 &= 2 \\ x_1 - x_2 &= -1 \end{aligned} \longrightarrow \begin{bmatrix} 2x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Let, $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

We now have a nice interpreation that is consistent with when we had just a single equation,

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$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \quad \Longleftrightarrow$$

What linear combination of $\mathbf{a}_1$ & $\mathbf{a}_2$ produces the vector $\mathbf{b}$?

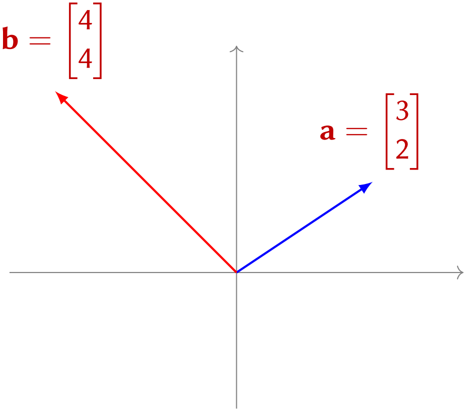
Geometry of n-vectors and their operations

We had visualized real numbers as vectors in $\mathbb{R}$. They were represented as a arrows from the origin to the location of the number in the real line.

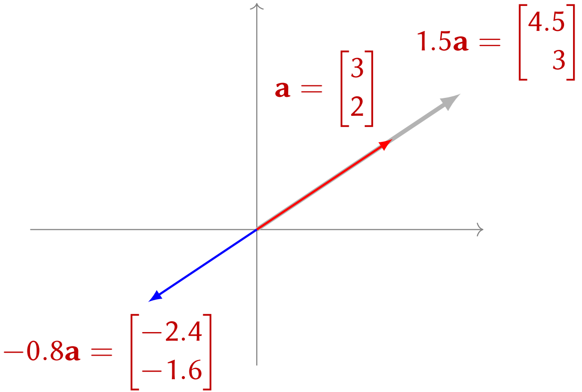
We can extend this concept to $\mathbb{R}^n$.

n-vectors are arrows pointing from the origin to a particiular spatial location represented by the components of the n-vector in $\mathbb{R}^n$.

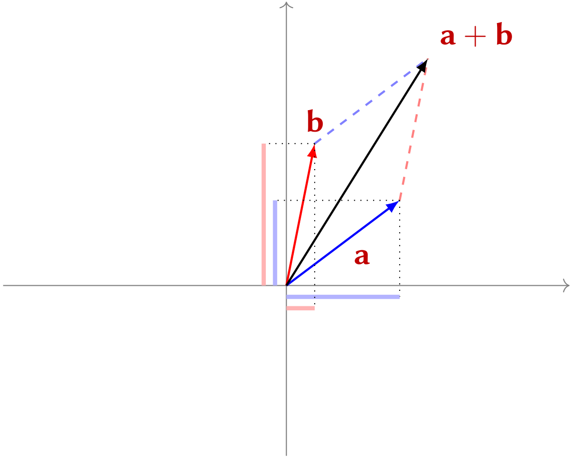
Visualizing 2-vectors in $\mathbb{R}^2$



Vector scaling in $\mathbb{R}^2$



Vector addition in $\mathbb{R}^2$



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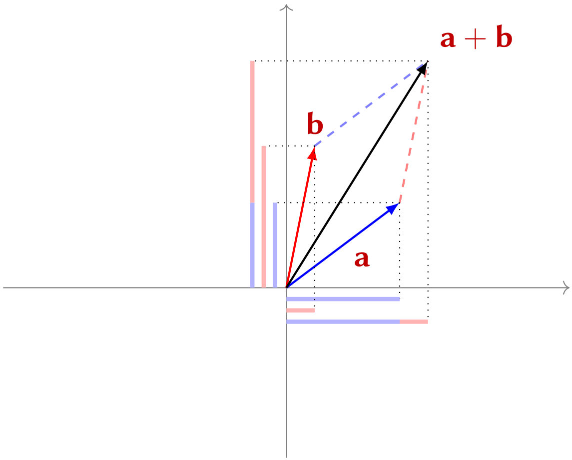
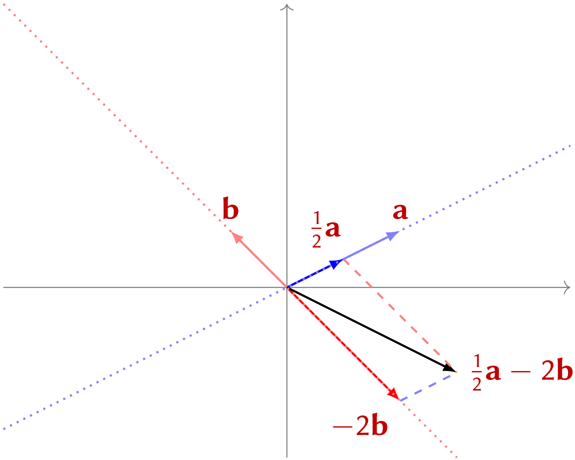


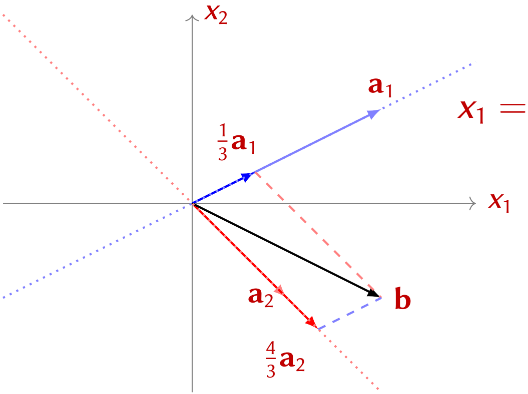
Linear combination of vectors in $\mathbb{R}^2$



Geometry of linear equations

Let's view the following problem in this column view's new geometric lens,

$$\begin{aligned} 2x_1 + x_2 &= 2 \\ x_1 - x_2 &= -1 \end{aligned}, \quad \mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$



$$x_1 = \frac{1}{3}, \quad x_2 = \frac{4}{3}$$

Some useful vectors in $\mathbb{R}^n$

The following some useful, special vectors that we will encounter:

Zero vector: The vector with all its components equal to zero.

$$\mathbf{0}^\top = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^n$$

One vector: The vector with all its components equal to one.

$$\mathbf{1}^\top = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^n$$

Unit vector: A vector with all zero components except for one component which is equal to one. The unit vector along the i^{th} direction is denoted as $\mathbf{e}_i$.

$$\mathbf{e}_i^\top = \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^n$$

How many unit vectors are there in $\mathbb{R}^n$?

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How many unit vectors are there in $\mathbb{R}^n$? n unit vectors.

Vector space ideas in $\mathbb{R}^n$

Is $\mathbb{R}^2$ or $\mathbb{R}^n$ a vector space?

Vector space ideas in $\mathbb{R}^n$

Is $\mathbb{R}^2$ or $\mathbb{R}^n$ a vector space? Yes. Its closed under linear combinations of its elements.

Consider $\mathbf{a} \in \mathbb{R}^n$ What is the span of $\mathbf{a}$? i.e., $\text{span}\left(\{\mathbf{a}\}\right) = ?$

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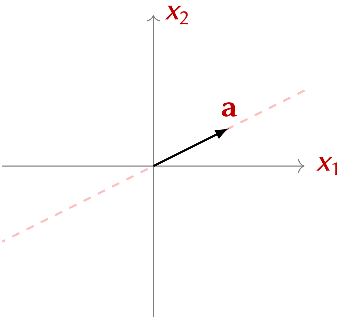
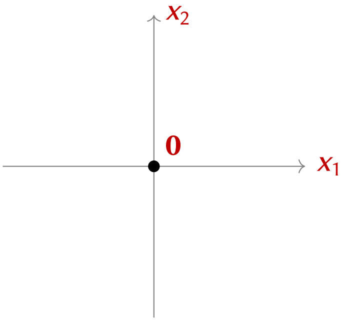
$$\text{span}\left(\{\mathbf{a}\}\right) = \{\alpha\mathbf{a} \mid \alpha \in \mathbb{R}\}$$

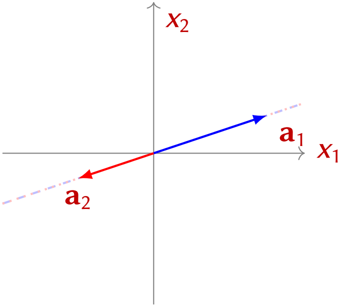




Vector space ideas in $\mathbb{R}^n$

Let's now consider the case of more than one vector from $\mathbb{R}^2$, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^2$.

$$\text{span}\left(\{\mathbf{a}_1, \mathbf{a}_2\}\right) = ?$$



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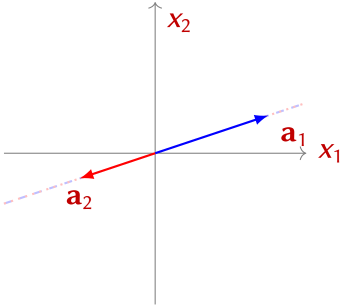

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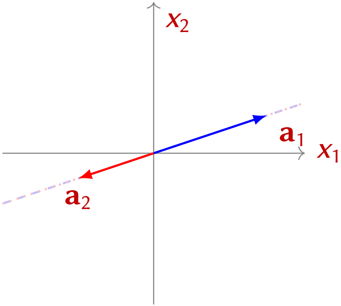

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Is the span a subspace of $\mathbb{R}^2$?

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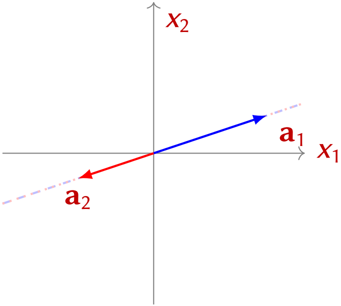

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Yes. But why?

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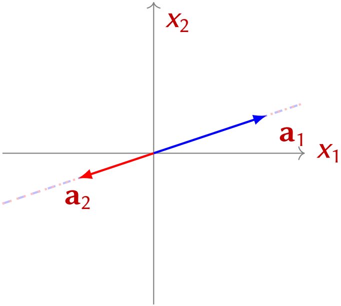

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Is the span a subspace of $\mathbb{R}^2$?

Yes. But why?

Is the set linearly indepedent?

No. There is a non-zero $\alpha \in \mathbb{R}$ such that $\mathbf{a}_1 = \alpha \cdot \mathbf{a}_2$.

Vector space ideas in $\mathbb{R}^n$

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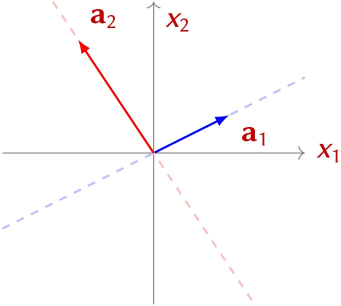

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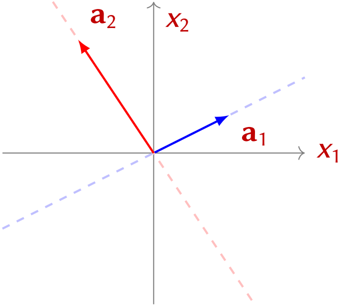

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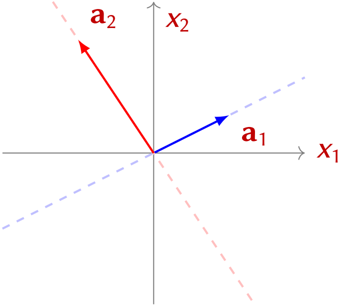

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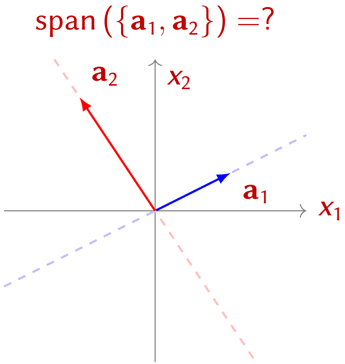

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Why?

We can generate any point in $\mathbb{R}^2$ through the linear combinations of $\mathbf{a}_1$ and $\mathbf{a}_2$.

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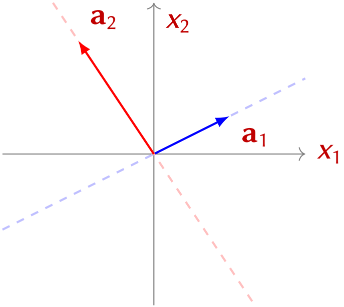

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Why?
We can generate any point in $\mathbb{R}^2$ through the linear combinations of $\mathbf{a}_1$ and $\mathbf{a}_2$.

Is the set linearly indepedent?
Yes. The only way we can generate $\mathbf{0}$ is $0 \cdot \mathbf{a}_1 + 0 \cdot \mathbf{a}_2$.

Vector space ideas in $\mathbb{R}^n$

Let's now consider the case of more than one vector from $\mathbb{R}^2$, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^2$.

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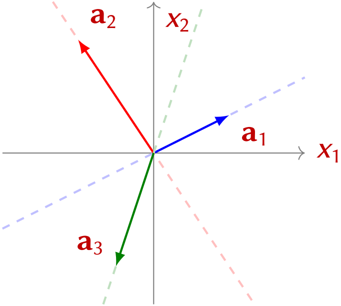

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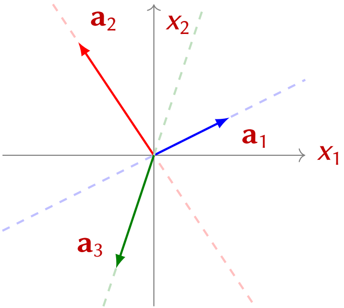
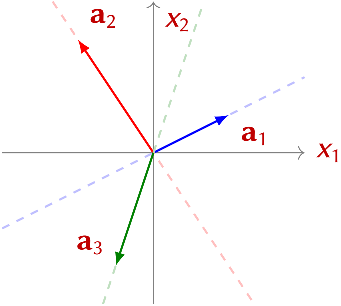


What is $\text{span}\left(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}\right)$?
$\mathbb{R}^2$.

Vector space ideas in $\mathbb{R}^n$

Let's now consider the case of more than one vector from $\mathbb{R}^2$, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^2$.

$\text{span}\left(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}\right) = ?$



What is $\text{span}\left(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}\right)$?
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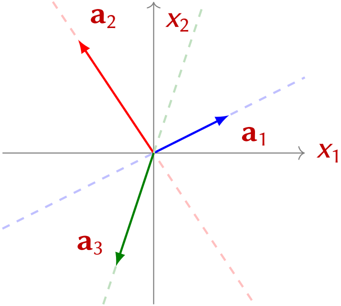
Why?

Vector space ideas in $\mathbb{R}^n$

Let's now consider the case of more than one vector from $\mathbb{R}^2$, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^2$.

$\text{span}\left(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}\right) = ?$

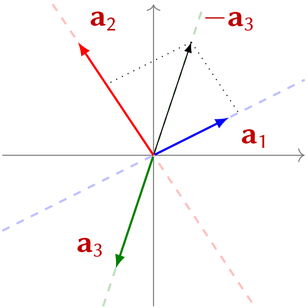


What is $\text{span}\left(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}\right)$?
$\mathbb{R}^2$.

Why?
We can generate any point in $\mathbb{R}^2$ through the linear combinations of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$.

Let's now consider the case of more than one vector from $\mathbb{R}^2$, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^2$.

$$\text{span}\left(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}\right) = ?$$



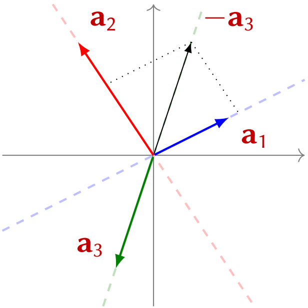
Is the set linearly indepedent?

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix} \quad \mathbf{a}_3 = \begin{bmatrix} -0.5 \\ -1.5 \end{bmatrix}$$

Vector space ideas in $\mathbb{R}^n$

Let's now consider the case of more than one vector from $\mathbb{R}^2$, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^2$.

$\text{span}\left(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}\right) = ?$



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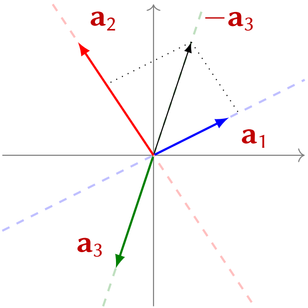
No. $\frac{9}{8}\mathbf{a}_1 + \frac{5}{8}\mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$.

Are there other combinations of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ that produce $\mathbf{0}$?

Vector space ideas in $\mathbb{R}^n$

Let's now consider the case of more than one vector from $\mathbb{R}^2$, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^2$.

$\text{span}\left(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}\right) = ?$



Is the set linearly indepedent?

$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix} \quad \mathbf{a}_3 = \begin{bmatrix} -0.5 \\ -1.5 \end{bmatrix}$

No. $\frac{9}{8}\mathbf{a}_1 + \frac{5}{8}\mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$.

Are there other combinations of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ that produce $\mathbf{0}$?

Yes. Infinitely many.

Solutions of Simultaneous Linear Equations

We will start with two equations, with one, two, and three unknows to understand the nature of the solutions.

Three possibilities: (a) No solution, (b) Unique solution, and (c) Infinitely many solutions.

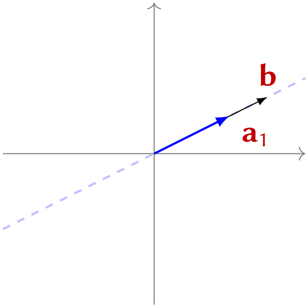
Two equations $p = 2$, one unknown $q = 1$:

$$\begin{aligned} a_{11}x_1 &= b_1 \\ a_{21}x_1 &= b_2 \end{aligned} \longrightarrow x_1 \cdot \mathbf{a}_1 = \mathbf{b}$$

where, $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Let $A = \{\mathbf{a}_1\}$ and $\mathcal{C}(A) = \text{span}(A)$.

What can we say about the solutions of this set of equations?

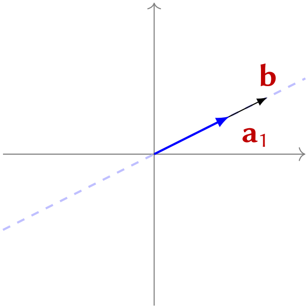
Solutions of Simultaneous Linear Equations

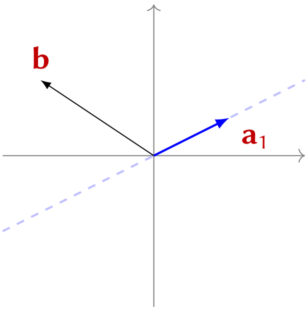


Unique solution: When $\mathbf{b} \in \mathcal{C}(A)$.

$$\begin{aligned} x_1 &= 1.5 \\ 0.5x_1 &= 0.75 \end{aligned} \longrightarrow x_1 \cdot \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.75 \end{bmatrix}$$

Solutions of Simultaneous Linear Equations





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No solution: When $\mathbf{b} \notin \mathcal{C}(A)$.

$$\begin{aligned} x_1 &= -1.5 \\ 0.5x_1 &= 1.0 \end{aligned} \longrightarrow x_1 \cdot \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -1.5 \\ 1.0 \end{bmatrix}$$

Solutions of Simultaneous Linear Equations

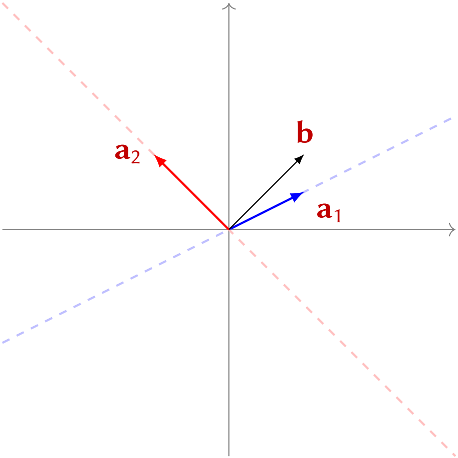
Two equations $p = 2$, Two unknowns $q = 2$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \longrightarrow x_1 \cdot \mathbf{a}_1 + x_2 \cdot \mathbf{a}_2 = \mathbf{b}$$

where, $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Let $A = \{\mathbf{a}_1, \mathbf{a}_2\}$ and $\mathcal{C}(A) = \text{span}(A)$.

What can we say about the solutions of this set of equations?

Solutions of Simultaneous Linear Equations



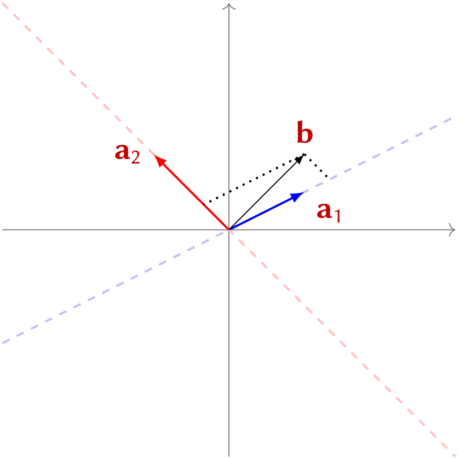
Unique solution: When $\mathbf{b} \in \mathcal{C}(A)$.

$$x_1 - x_2 = 1$$
$$0.5x_1 + x_2 = 1$$

$$x_1 \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_1 = \frac{4}{3}, \quad x_2 = \frac{1}{3}$$

Solutions of Simultaneous Linear Equations



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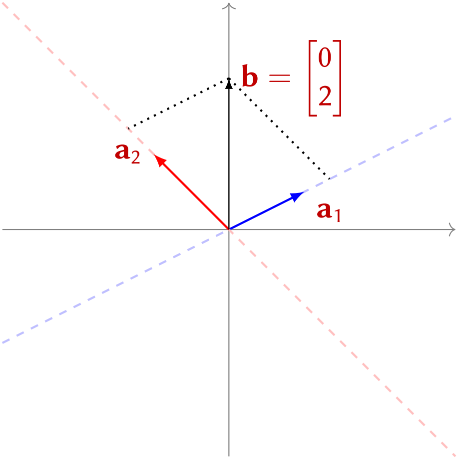
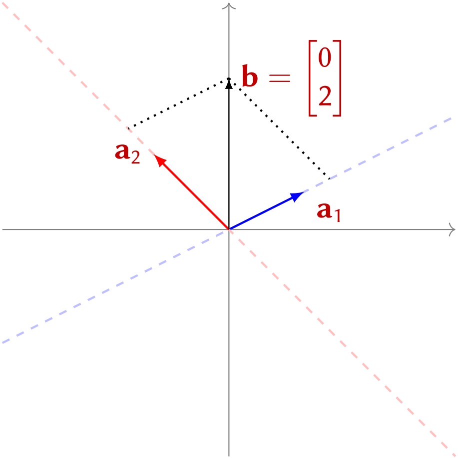
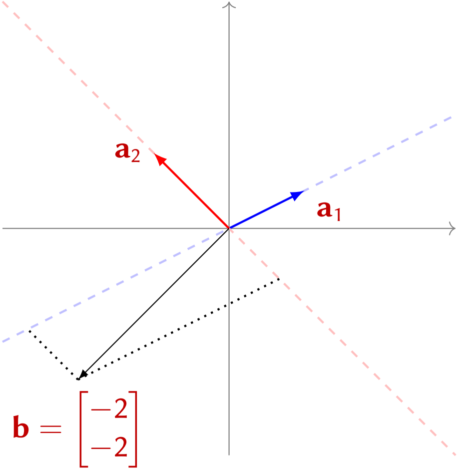
Solutions of Simultaneous Linear Equations



Solutions of Simultaneous Linear Equations





Solutions of Simultaneous Linear Equations

Two equations $p = 2$, Two unknowns $q = 2$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \longrightarrow x_1 \cdot \mathbf{a}_1 + x_2 \cdot \mathbf{a}_2 = \mathbf{b}$$

Can we ever have a situation where there are either no solutions or infinitely many solutions?

Solutions of Simultaneous Linear Equations

Two equations $p = 2$, **Two unknowns** $q = 2$:

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Can we ever have a situation where there are either no solutions or infinitely many solutions?

Yes. When $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly dependent, and

Solutions of Simultaneous Linear Equations

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Can we ever have a situation where there are either no solutions or infinitely many solutions?

Yes. When $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly dependent, and

$$\begin{aligned} \mathbf{b} \in \mathcal{C}(A) &\implies \text{Infinitely many solutions} \\ \mathbf{b} \notin \mathcal{C}(A) &\implies \text{No solution} \end{aligned}$$

Solutions of Simultaneous Linear Equations

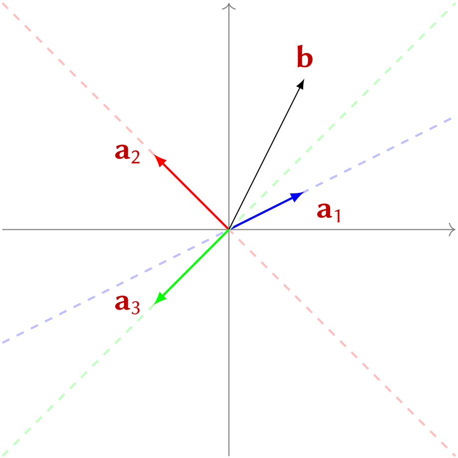
Two equations $p = 2$, Three unknowns $q = 3$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \end{aligned} \longrightarrow x_1 \cdot \mathbf{a}_1 + x_2 \cdot \mathbf{a}_2 + x_3 \cdot \mathbf{a}_3 = \mathbf{b}$$

where, $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Let $A = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{C}(A) = \text{span}(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\})$.

What can we say about the solutions of this set of equations?

Solutions of Simultaneous Linear Equations



Infinitely Many solution: When $\mathbf{b} \in \mathcal{C}(A)$.

$$
\begin{aligned}
x_1 - x_2 - x_3 &= 1 \\
0.5x_1 + x_2 - x_3 &= 2
\end{aligned}
$$

$$
x_1 \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}
$$

$$
x_1 = 2 + \frac{4}{3}\alpha, \quad x_2 = 1 + \frac{1}{3}\alpha, \quad x_3 = \alpha
$$

Solutions of Simultaneous Linear Equations

Two equations $p = 2$, Three unknowns $q = 3$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \end{aligned} \longrightarrow x_1 \cdot \mathbf{a}_1 + x_2 \cdot \mathbf{a}_2 + x_3 \cdot \mathbf{a}_3 = \mathbf{b}$$

Can we ever have a unique solution?

Solutions of Simultaneous Linear Equations

Two equations $p = 2$**, Three unknowns** $q = 3$**:**

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \end{aligned} \longrightarrow x_1 \cdot \mathbf{a}_1 + x_2 \cdot \mathbf{a}_2 + x_3 \cdot \mathbf{a}_3 = \mathbf{b}$$

Can we ever have a unique solution?

No. Why?

Solutions of Simultaneous Linear Equations

Two equations $p = 2$, Three unknowns $q = 3$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \end{aligned} \longrightarrow x_1 \cdot \mathbf{a}_1 + x_2 \cdot \mathbf{a}_2 + x_3 \cdot \mathbf{a}_3 = \mathbf{b}$$

Can we ever have a unique solution?
No. Why?

A set of more than two 2-vectors will always be linear dependent! A linearly dependent set cannot produce a unique solution.

Its as if there are too many 2-vectors in this case!

Solutions of Simultaneous Linear Equations

Two equations $p = 2$, Three unknowns $q = 3$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \end{aligned} \longrightarrow x_1 \cdot \mathbf{a}_1 + x_2 \cdot \mathbf{a}_2 + x_3 \cdot \mathbf{a}_3 = \mathbf{b}$$

where, $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Can we ever have no solution?

Solutions of Simultaneous Linear Equations

Two equations $p = 2$, **Three unknowns** $q = 3$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \end{aligned} \longrightarrow x_1 \cdot \mathbf{a}_1 + x_2 \cdot \mathbf{a}_2 + x_3 \cdot \mathbf{a}_3 = \mathbf{b}$$

where, $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Can we ever have no solution?

Yes. When $\mathbf{b} \notin \mathcal{C}(A)$.

When would this happen?

Solutions of Simultaneous Linear Equations

Two equations $p = 2$, Three unknowns $q = 3$:

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Can we ever have no solution?

Yes. When $\mathbf{b} \notin \mathcal{C}(A)$.

When would this happen?

When two or more of the vectors are collinear.

Solutions of Simultaneous Linear Equations

The nature of the solutions for $p = 2$ equations and q unknowns is a bit more complicated than the case of $p = 1$.

Let's summarize for what we have seen so far, for the case where $p = 2$ and $q = 1, 2, 3$.

$$\mathbf{b} \in \mathcal{C}(A) \rightarrow \begin{cases} \text{Unique solution,} & A \text{ is linearly independent.} \\ \text{Infinitely many solutions,} & A \text{ is linearly dependent.} \end{cases}$$

$$\mathbf{b} \notin \mathcal{C}(A) \rightarrow \text{No solution.}$$

This summary applies to any arbitrary p and q.