Applied Linear Algebra in Data Analysis

Tutorial & Assignments

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Contents

CHAPTER 1

Linear Systems and Matrix Operations - 02

1.1 Tutorial Problems

1. Which of the following sets forms a vector space?

   (a) $\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \ x_1, x_2 \in \mathbb{R} \text{ and } a_1x_1 + a_2x_2 = 0 \right\}$, where $a_1, a_2 \in \mathbb{R}$ are fixed constants.

   (b) $\{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{x}^\top \mathbf{x} = 1\}$.

   (c) $\left\{ \begin{bmatrix} x_1 \\ x_1^2 \end{bmatrix} \middle| \ x_1, x_2 \in \mathbb{R} \right\}$.

2. Which of the following is a subspace of $\mathbb{R}^3$?

   (a) $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \ x_1, x_2, x_3 \in \mathbb{R} \text{ and } x_1 + x_2 + x_3 = 0 \right\}$

   (b) $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \ x_1, x_2, x_3 \in \mathbb{R} \text{ and } x_1 + x_2 + 1 = 0 \right\}$

   (c) $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \ x_1, x_2, x_3 \in \mathbb{R} \text{ and } x_1^2 + x_2^2 + x_3^2 = 1 \right\}$

3. What geometrical object is represented by the following sets?

   (a) span $(\{\mathbf{a}_1\})$, where $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

   (b) span $(\{\mathbf{a}_1, \mathbf{a}_2\})$, where $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

   (c) span $(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\})$, where $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

4. A medical supply distributor must ship oxygen cylinders from two central plants (S_1 and S_2) to four hospitals (H_1, H_2, H_3, H_4) in one city. Plant S_1 can supply 40 cylinders; S_2 can provide 30 cylinders. The hospitals require the following numbers: (a) H_1: 25 cylinders; (b) H_2: 20 cylinders; (c) H_3: 15 cylinders; (d) H_4: 20 cylinders. Write down the set of linear equations corresponding to this problem. It should be clear that this problem cannot be solved. Why? Show that the system cannot be solved using the reduced row echelon form.

5. A hospital has 10 wards: A_1 to A_{10}. The hospital employs 55 attendants who are be allocated to these wards on a daily basis. You are given the task of allocating the attendants to the wards based on the following requirements:

(a) No attendant must be left unassigned.

(b) No attendant can be assigned to more than one ward.

(c) Even numbered wards have a high patient load than the odd numbered wards. Thus, the even numbers wards together will require 10 more attendants than the odd numbered wards.

(d) Ward A_1 requires 2 more attendants than Ward A_3.

(e) Ward A_2 requires twice as many attendants as Ward A_4.

Formulate the set of linear equations.

1.2 Assignment

1. A hospital has four special wards - A, B, C, and D, which require beds, patient monitors, ventilators, and nursing staff to operate effectively. The allocation is planned on a monthly basis, depending on the expected number of patients in these wards on that month. For the upcoming month, the hospital has the following resources and requirements:

   (a) There are a total of 100 beds, 60 patient monitors, 30 ventilators, and 40 nurse staff available for allocation.

   (b) Wards A and B together will required 10 more beds than wards C and D together.

   (c) Ward A has twice the number of beds as C.

   (d) All wards require an equal number of patient monitors.

   (e) Wards B and D will not require any ventilators.

   (f) Ward A and C together requires thrice the number of nurse staff as wards B and D together.

   You are given the job of allocating the resources to the four wards. Formulate this problem as a set of linear equations.

2. A meal planner must determine the servings of four food items — chicken, rice, daal, and milk — so the total diet exactly meets the following nutritional targets each day:

   - Exactly 1,400 calories
   - Exactly 90 grams of protein
   - Exactly 40 grams of fat
   - Exactly 150 grams of carbohydrates
   - Exactly 150 mg of vitamin C

   The nutritional content per serving for each of the four food items available to the meanl planner is as follows:

Food	**Calories**	**Protein (g)**	**Fat (g)**	**Carbs (g)**	**Vit C (mg)**
Chicken	220	43	5	0	0
Rice	200	4	0.5	45	0
Daal	50	4	0.5	10	75
Milk	100	9	2	11	2

   Formulate the system.

3. Which of the following sets forms a vector space?

   (a) $\{\mathbf{x} \mid x_1, x_2 \in \mathbb{R} \text{ and } a_1x_1 + a_2x_2 = 0\}$, where $a_1, a_2 \in \mathbb{R}$ are fixed constants.

   (b) $\left\{\mathbf{x} \;\middle|\; \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{a}^\top\mathbf{x} = b\right\}$, where $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ are fixed constants.

   (c) $\left\{\mathbf{x} \;\middle|\; \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{x}^\top\mathbf{x} = 1\right\}$.

   (d) $\{(x[0], x[1], x[2], \ldots x[N-1]) \mid x[i] \in \mathbb{R}, 0 \le i < N\}$.
   (The set of all real-valued time-domain signals of length N. $x[i]$ is the value of the signal at time instant i.)

4. A clinician is interested in estimating a patient's core body temperature T_{core} using multiple body surface measurements. Temperature sensors are placed at different body locations, each providing a reading that has a known proportional relationship to the core temperature. The measurements are:

   - Oral temperature: $T_{\text{oral}} = 0.95 T_{\text{core}}$
   - Axillary (armpit) temperature: $T_{\text{axillary}} = 0.90 T_{\text{core}}$
   - Tympanic (ear) temperature: $T_{\text{tympanic}} = 0.98 T_{\text{core}}$
   - Forehead temperature: $T_{\text{forehead}} = 0.92 T_{\text{core}}$

   During a clinical assessment, the following readings (in °C) are recorded: $T_{\text{oral}} = 36.5$, $T_{\text{axillary}} = 35.8$, $T_{\text{tympanic}} = 37.2$, $T_{\text{forehead}} = 36.0$. Formulate this as a system of linear equations to estimate T_{core}.

5. There is reason for you to believe that the proportionality constants relating the core temperature to body surface temperatures in the above problem are incorrect. So, you decide to do an experiment to estimate these proportionality constants by recruiting a sample of 100 healthy individuals on whom you make the following measurements:

   - Core body temperature using an invasive rectal thermometer (gold standard): $T_{\text{core}}^{(i)}$
   - Oral temperature: $T_{\text{oral}}^{(i)}$
   - Axillary temperature: $T_{\text{axillary}}^{(i)}$
   - Tympanic temperature: $T_{\text{tympanic}}^{(i)}$
   - Forehead temperature: $T_{\text{forehead}}^{(i)}$

   where $i = 1, 2, \ldots, 100$ denotes the individual subjects in your study. Assuming the linear model holds for each measurement location:

$$\begin{aligned} T_{\text{oral}}^{(i)} &= k_{\text{oral}}\, T_{\text{core}}^{(i)} \\ T_{\text{axillary}}^{(i)} &= k_{\text{axillary}}\, T_{\text{core}}^{(i)} \\ T_{\text{tympanic}}^{(i)} &= k_{\text{tympanic}}\, T_{\text{core}}^{(i)} \\ T_{\text{forehead}}^{(i)} &= k_{\text{forehead}}\, T_{\text{core}}^{(i)} \end{aligned}$$

   Formulate a system of linear equations to relating the known measurements from the 100 subjects to the four unknown proportionality constants k_{oral}, k_{axillary}, k_{tympanic}, and k_{forehead}.

6. Which of the following are subspace of $\mathbb{R}^n$?

   (a) $\left\{ \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top \,\middle|\, \sum_{i=1}^n x_i = 0,\ x_i \in \mathbb{R} \right\}$.

   (b) $\left\{ \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top \,\middle|\, x_4 = 1 \right\}$.

   (c) $\left\{ \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top \,\middle|\, \sum_{i=1}^n w_i x_i = 5,\ w_i, x_i \in \mathbb{R} \right\}$.

   (d) $\left\{ \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top \,\middle|\, \sum_{i=1}^n w_i x_i = 0,\ w_i, x_i \in \mathbb{R}, w_i > 0 \right\}$.

7. Consider the vector space of polynomials of order n or less.

$$\mathcal{P} = \left\{ \sum_{k=0}^{n} a_k x^k \,\middle|\, a_k \in \mathbb{R} \right\}, \text{ where, } x \in [0, 1]$$

   Show that polynomails of order strictly lower than n form subspaces of $\mathcal{P}$.

8. **Different types of combinations.** Consider the following vectors in $\mathbb{R}^2$.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} \quad \mathbf{a}_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

   Draw the locus of all points generated by the following combinations of these vectors. Note that all $c_i \in \mathbb{B}$.

(a) Linear combination: $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3$

(b) Linear combination: $c_2\mathbf{a}_2$

(c) Affine combination: $c_1\mathbf{a}_1 + c_2\mathbf{a}_2$, $c_1 + c_2 = 1$

(d) Affine combination: $c_2\mathbf{a}_2 + c_3\mathbf{a}_3$, $c_2 + c_3 = 1$

(e) Conic combination: $c_1\mathbf{a}_1 + c_2\mathbf{a}_2$, $c_i \geq 0$

(f) Conic combination: $c_1\mathbf{a}_1 + c_3\mathbf{a}_3$, $c_i \geq 0$

(g) Convex combination: $c_1\mathbf{a}_1 + c_2\mathbf{a}_2$, $c_i \geq 0$, $c_1 + c_2 = 1$

(h) Convex combination: $c_1\mathbf{a}_1 + c_3\mathbf{a}_3$, $c_i \geq 0$, $c_1 + c_3 = 1$

(i) Convex combination: $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3$, $c_i \geq 0$, $c_1 + c_2 + c_3 = 1$

CHAPTER 2

Linear Systems and Matrix Operations - 03

2.1 Tutorial Problems

1. Conisder the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & 1 \\ -3 & 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

   (a) Find the product of the two matrices $\mathbf{C} = \mathbf{AB}$ using the four views of matrix muliplication.

   (b) If we change $b_{23} = 0$. Can you compute the new matrix $\mathbf{C}$ without performing the entire matrix muliplication again?

   (c) If we increase the value of the elements of the 3^{rd} column of $\mathbf{A}$ by 1, how can we compute the new $\mathbf{C}$ without performing the entire matrix multiplication again?

   (d) If we insert a new row $\mathbf{1}^\top$ in $\mathbf{A}$ after the 2^{nd} row of $\mathbf{A}$, how can we compute the new $\mathbf{C}$ without performing the entire matrix multiplication again?

2. **Computational cost of different operations.** What is the computational cost of the following matrix operations? Computational cost refers to the number of arithmetic operations required to carry out a particular matrix operation. Computational cost is a measure of the efficiency of an algorithm. For example, consider the operation of vector addition, $\mathbf{a} + \mathbf{b}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. This requires n addition/subtraction operations and zero multiplication/division operations.

   (a) Matrix multiplication: $\mathbf{AB}$, where $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$

   (b) Inner product: $\mathbf{u}^T\mathbf{v}$

   Report the counts for the addition/subtraction and multiplication/division operations separately.

3. Prove $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

4. Prove that a matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$ can always be written as a sum of a symmetric matrix $\mathbf{S}$ and a skew-symmetric matrix $\mathbf{A}$.

$$\mathbf{M} = \mathbf{S} + \mathbf{A}, \;\; \mathbf{S}^T = \mathbf{S} \;\text{ and }\; \mathbf{A}^T = -\mathbf{A}$$

   Does this property also hold for a complex matrix $\mathbf{M} \in \mathbb{C}^{n \times n}$?

5. The trace of a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is defined as, $trace\,(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$. Prove the following,

   (a) $trace\,(\mathbf{A})$ is a linear function of $\mathbf{A}$.

   (b) $trace\,(\mathbf{AB}) = trace\,(\mathbf{BA})$

   (c) $trace\left(\mathbf{A}^T\mathbf{A}\right) = 0 \implies \mathbf{A} = 0$

6. Express the following set of linear equations as a matrix equation, $\mathbf{Ax} = \mathbf{b}$. Write down the exact dimensions of $\mathbf{A}, \mathbf{x}$ and $\mathbf{b}$, along with their elements. Write down the augmented matrix for each of these systems of equations.

(a)

$$\begin{aligned} 3x_1 + 4x_2 + 5x_3 &= 1 \\ 2x_1 + 3x_3 &= 2 \\ 4x_1 + 2x_2 - x_3 &= 3 \\ 5x_1 - 3x_2 + 4x_3 &= 4 \end{aligned}$$

(b)

$$\begin{aligned} y &= -23 \\ 3y &= 2 \\ \frac{1}{4}y &= 3 \end{aligned}$$

(c)

7. Find the complete solution for the following set of linear equations by reducing the augmented matrix to the reduced row echelon form. Find the rank of the matrix $\mathbf{A}$ and the augmented matrix, and list the number of basic and non-basic columns.

   (a)

$$\begin{aligned} a + b + c + d &= 10 \\ 2a - b + d &= 5 \\ a + 3c - d &= 4 \\ b + c &= 3 \end{aligned}$$

   (b)

$$\begin{aligned} 2x_1 + 3x_2 + x_3 + 4x_4 &= 25 \\ x_1 + 2x_2 + 3x_3 + x_4 &= 18 \\ 3x_1 + x_2 + 2x_3 + 5x_4 &= 32 \\ 4x_1 + 3x_2 + 2x_3 + x_4 &= 27 \\ 2x_1 + x_2 + 4x_3 + 3x_4 &= 29 \end{aligned}$$

8. A hospital has 10 wards: A_1 to A_{10}. The hospital employs 55 attendants who are be allocated to these wards on a daily basis. You are given the task of allocating the attendants to the wards based on the following requirements:

   (a) No attendant must be left unassigned.

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   Is this problem solvable (do not worry about having fractional and negative solutions)? What is the rank of the coefficient matrix of this system of equations?

   Solve this set of linear equations to find the number of attendants allocated to each ward. Note that the number of attendants must be a non-negative integer. You might have to make approximations to get integer solutions.

2.2 Assignment

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(a)

$$\begin{aligned}3z_5 + z_4 - 2z_6 + 2z_2 - z_3 + z_1 &= 5\\ z_6 + 4z_3 - z_4 + 2z_1 - z_2 + z_5 + 3 &= 0\\ 2z_4 + z_2 - z_5 + 3z_1 + 2z_3 + 4z_6 - 5 &= 2\\ -2z_4 - z_6 + 2z_5 - z_1 + z_3 + 3z_2 &= 4\end{aligned}$$

(b)

$$\begin{aligned}5f - 2i &= 3\\ 2b - e &= -1\\ c + 4g &= 8\\ a + h &= 7\\ -3d + j &= 2\\ a + 3d - j &= 6\\ 2b + e - g &= -4\end{aligned}$$

2. Consider a matrix $\mathbf{A} \in \mathbb{R}^{10^6 \times 5}$, and we are interested in computing the product $\mathbf{A}^\top \mathbf{A} \mathbf{A}^\top$. Should you compute the product as $\left(\mathbf{A}^\top \mathbf{A}\right) \mathbf{A}^\top$ or $\mathbf{A}^\top \left(\mathbf{A}\mathbf{A}^\top\right)$? Why?

3. A hospital has four special wards - A, B, C, and D, which require beds, patient monitors, ventilators, and nursing staff to operate effectively. The allocation is planned on a monthly basis, depending on the expected number of patients in these wards on that month. For the upcoming month, the hospital has the following resources and requirements:

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   You are given the job of allocating the resources to the four wards.

   Is this problem solvable (do not worry about having fractional and negative solutions)? What is the rank of the coefficient matrix of this system of equations?

   Solve this set of linear equations to find the number of attendants allocated to each ward. Note that the number of attendants must be a non-negative integer. You might have to make approximations to get integer solutions.

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6. There is reason for you to believe that the proportionality constants relating the core temperature to body surface temperatures in the above problem are incorrect. So, you decide to do an experiment to estimate these proportionality constants by recruiting a sample of 100 healthy individuals on whom you make the following measurements:

   - Core body temperature using an invasive rectal thermometer (gold standard): $T^{(i)}_{\text{core}}$
   - Oral temperature: $T^{(i)}_{\text{oral}}$
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   where $i = 1, 2, \dots, 100$ denotes the individual subjects in your study. Assuming the linear model holds for each measurement location:

$$
\begin{aligned}
T^{(i)}_{\text{oral}} &= k_{\text{oral}}\, T^{(i)}_{\text{core}} \\
T^{(i)}_{\text{axillary}} &= k_{\text{axillary}}\, T^{(i)}_{\text{core}} \\
T^{(i)}_{\text{tympanic}} &= k_{\text{tympanic}}\, T^{(i)}_{\text{core}} \\
T^{(i)}_{\text{forehead}} &= k_{\text{forehead}}\, T^{(i)}_{\text{core}}
\end{aligned}
$$

   Formulate a system of linear equations to relating the known measurements from the 100 subjects to the four unknown proportionality constants k_{oral}, k_{axillary}, k_{tympanic}, and k_{forehead}.

   Is this problem solvable (do not worry about having fractional and negative solutions)?

7. You are performing an experiment to characterize a force sensor, which is known to have non-linear response characteristics. You apply 3 different known forces to the sensor and record the sensor output voltage for the 3 forces. You hypothesize that the relationship between the applied force

Force (N)	**Sensor voltage** (V)
1	0.5
2	0.75
3	1.2

and the sensor voltage can be modeled as a polynomial function of order 2:

$$v = \beta_0 + \beta_1 f + \beta_2 f^2$$

We wish to use the experimental data to estimate the parameters β_0, β_1, and β_2 of this model. Can this problem be formulated as a set of linear equations? Is this problem solvable?

Later one you speak to a colleague who suggests that a cubic polynomial model might be more appropriate. Can the problem of fitting a cubic polynomial still be formulated as a set of linear equations now? Is this problem solvable with our current data?

You need to interface this sensor to a microcontroller and you find that this microcontroller can only handle polynomial functions or order 1. So you decide to fit a order 1 polynomial model for your sensor. Can this problem also be formulated as a set of linear equations? Is this problem solvable with our current data?

2.3 Additional Problems

1. Consider the following 5×4 matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 2 & 2 \\ 1 & -2 & 1 & 3 & 9 \\ -2 & 0 & 2 & -1 & -2 \\ 3 & 1 & 1 & -5 & 0 \end{bmatrix}$$

Compute the following:

(a) $\mathbf{a}_3$
(b) $\mathbf{a}_1^\top$
(c) $\tilde{\mathbf{a}}_2^\top$
(d) $\tilde{\mathbf{a}}_4$
(e) $\mathbf{a}_1\mathbf{a}_2^\top$
(f) $\tilde{\mathbf{a}}_3\mathbf{a}_2^\top$
(g) $\tilde{\mathbf{a}}_1\tilde{\mathbf{a}}_2^\top$
(h) $\tilde{\mathbf{a}}_1^\top\tilde{\mathbf{a}}_2$
(i) $\tilde{\mathbf{a}}_1\tilde{\mathbf{a}}_1^\top + \tilde{\mathbf{a}}_2\tilde{\mathbf{a}}_2^\top$
(j) $\mathbf{a}_3^\top\mathbf{a}_1 + \mathbf{a}_2^\top\mathbf{a}_4$

2. Consider the matrix $\mathbf{P} = \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_3 & \mathbf{e}_2\end{bmatrix}$. What does this P matrix do to a matrix $\mathbf{A} \in \mathbb{R}^{3\times 3}$ in the following operations? Try to compute these without performing the matrix multiplication and by using you understaning of the row and column views of matrix multiplication.

(a) $\mathbf{PA}$
(b) $\mathbf{AP}$
(c) $\mathbf{P}^2\mathbf{A}$
(d) $\mathbf{AP}^2$
(e) $\mathbf{PAP}$