Applied Linear Algebra in Data Analysis

Linear Systems and Matrix Operations: Part 1

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Linear Equations

What is a linear equation? – *An algebraic equation in which the highest power of any variable is 1.*

$$5x_1 - 2x_2 + 11x_3 - 12x_4 = 23$$

x_1, x_2, x_3, x_4 are variables or unknowns in the above equation. Numbers infront of the variables are their coefficients.

The general form of a linear equation in n variables $x_1 \ldots x_n$ is,

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

where $a_1, a_2, \ldots a_n$ are known coefficients and b is a constant.

Geometry of Linear Equations

Let's now look at the geometry of linear equations. We will start with an equation in one variable.

$$ax_1 = b \implies x_1 = \frac{b}{a} \quad a \neq 0$$

There is one unknown variable, x_1 which is a real number, i.e. $x_1 \in \mathbb{R}$, and $a, b \in \mathbb{R}$.

There are geometric views we can associate with this equation. We will call these two views:

- **Row view** – Geometric view in the solution space, i.e. the space of x_1.
- **Column view** – Geometric view in the coefficient space, i.e. the space of a and b.

Geometry of Linear Equations

Row Space View

Consider the following equation

$$2x_1 = 4 \implies x_1 = \frac{4}{2} = 2$$

There is one unknown variable, x_1 which is a real number, i.e. $x_1 \in \mathbb{R}$.

The solution space in this case is the real line $\mathbb{R}$. The geometric view will be the locus of all points that satisfy the above equation.

$x_1 = 2$

-5   -4   -3   -2   -1   0   1   2   3   4   5

Geometry of Linear Equations

Column Space View

Consider the same equation

$$2x_1 = 4 \implies x_1 = \frac{4}{2} = 2$$

The coefficients associated with the unknowns can also be indicated on the real line.

In this space we will choose to represent the coefficients a and b as arrows pointing from the origin to their corresponding values.

We will refer to these arrows as *vectors* $\implies$ a, b are vectors in $\mathbb{R}$.

Geometry of Linear Equations

Column Space View

Vector Scaling – Multiplying a by a real number "scales" the vector in $\mathbb{R}$. This operation stretches or shrinks the vector a by a factor of $|\alpha|$. If α is negative, the vector also flips direction.

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Vector Scaling – Multiplying a by a real number "scales" the vector in $\mathbb{R}$. This operation stretches or shrinks the vector a by a factor of $|\alpha|$. If α is negative, the vector also flips direction.

$$a \cdot x_1 = b \iff \text{How much should we scale } a \text{ to obtain vector } b?$$

This gives x_1 a clear interpretation – it's the *scaling factor*.

Geometry of Linear Equations

Column Space View

The column view gives us another way to look at the simple linear equation.

$$a \cdot x_1 = b$$

where, a and b are fixed and x_1 is an unknown variable.

While, the right hand side (RHS) represents a fixed vector b in $\mathbb{R}$. The LHS represents a set of infinite number of vectors in $\mathbb{R}$, obtained by different amounts of scaling $x_1 \in \mathbb{R}$.

$$\mathcal{C}(a) = \{x_1 \cdot a \mid x_1 \in \mathbb{R}\}$$

This set $\mathcal{C}(a)$ is called the column space of the vector a.

Geometry of Linear Equations

Column Space View

The set $\mathcal{C}(a)$ looks like this:

$$\mathcal{C}(a) = \{\cdots, \;\; -1.2 \cdot a, \;\; 2.124 \cdot a, \;\; 5 \cdot a, \;\; 0 \cdot a, \;\; -a, \;\; \cdots\}$$

This is the set of all values that can be produced by scaling a by different amounts.

$$\begin{cases} \mathcal{C}(a) = \mathbb{R}, & a \neq 0 \\ \mathcal{C}(a) = \{0\}, & a = 0 \end{cases}$$

This way of looking at equation $a \cdot x_1 = b$ makes it clear when an equation is solvable, and when its not.

Geometry of Linear Equations

Example 1: $5 \cdot x_1 = 10$ $a = 5, \ b = 10$

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$\therefore$ The equation is solvable and the solution is $x_1 = 2$.

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Example 2: $0 \cdot x_1 = 10 \quad a = 0, \ b = 10$

$\mathcal{C}(0) = \{0\}$ since $a = 0$, $b = 10 \notin \mathcal{C}(0)$
$\therefore$ The equation is **not** solvable.

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$\mathcal{C}(0) = \{0\}$ since $a = 0, \; b = 0 \in \mathcal{C}(0)$
$\therefore$ The equation is solvable and has infinitely many solutions, $x_1 \in \mathbb{R}$.

Three Possible Solutions

$$a \cdot x = b$$

Unique Solution

- $a \neq 0, b \in \mathbb{R}$
- *Always solvable*
- $\mathcal{C}(a) = \mathbb{R}$
- $b \in \mathcal{C}(a)$
- $x = \dfrac{b}{a}$

No Solution

- $a = 0, b \in \mathbb{R}$
- *Not solvable*
- $\mathcal{C}(0) = \{0\}$
- $b \notin \mathcal{C}(0)$

Infinite Solutions

- $a = 0, b = 0$
- *Always solvable*
- $\mathcal{C}(0) = \{0\}$
- $b \in \mathcal{C}(0)$
- $x \in \mathbb{R}$

Remember: *The equation is solvable if and only if* $b \in \mathcal{C}(a)$.

Geometry of Linear Equations

Let's now look at a linear equation in two variables.
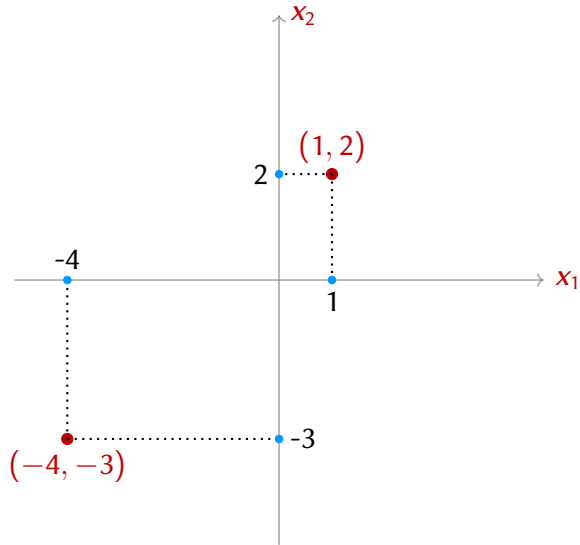
$$a_1 x_1 + a_2 x_2 = b$$

where, $a_1, a_2, b \in \mathbb{R}$ are fixed and x_1, x_2 are the unknown variables.

We once again have two geometric views, except things a bit more interesting now.

▶ The solution space is not the real line anymore! There are two real numbers $x_1, x_2 \in \mathbb{R}$. The set of two real numbers is the real plane $\mathbb{R}^2$.

The Real Plane – $\mathbb{R}^2$

Every point in the plane is represented by two real numbers (x_1, x_2).
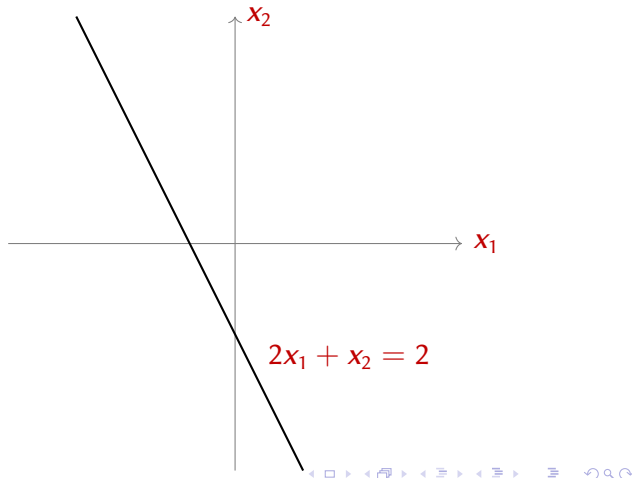
Geometry of Linear Equations

Row Space View

Consider the following equation.

$$2x_1 + x_2 = -2$$

The solution space is $\mathbb{R}^2$ and locus of all points that satisfy the above equation is shown as the black line.

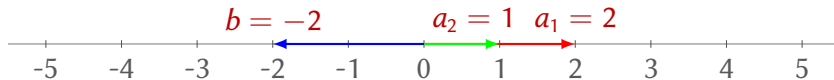We now have infinitely many solutions.

$2x_1 + x_2 = 2$

Geometry of Linear Equations

Column Space View

Consider the same equation

$$2x_1 + x_2 = -2$$

The coefficients associated with the unknowns can be seen as vectors on the real line.

The left hand side $2x_1 + x_2$ represents the two geometric operations – vector scaling and vector addition.

Geometry of Linear Equations
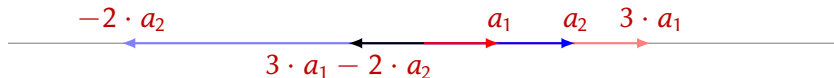
Column Space View

Vector Addition – The above geometric operation is the vector addition operation in $\mathbb{R}$.

Vector addition in $\mathbb{R}$ is simply the sum of the real numbers representing the vectors.

Geometry of Linear Equations

Column Space View

Linear Combination – An operation on vectors that involves both vector scaling and vector addition. The linear combination of vectors a_1 and a_2 with scalars $x_1, x_2 \in \mathbb{R}$ is given by,

$$x_1 \cdot a_1 + x_2 \cdot a_2$$

We will refer to the scaler or scaling factors x_1 and x_2 as the mixtures of the linear combination.

Column Space View

Now, we can provide an interpretation for x_1 and x_2 in the equation $a_1 x_1 + a_2 x_2 = b$.

Geometry of Linear Equations

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Now, we can provide an interpretation for x_1 and x_2 in the equation $a_1 x_1 + a_2 x_2 = b$.

$$a_1 \cdot x_1 + a_2 \cdot x_2 = b \quad \Longleftrightarrow \quad \text{What mixture of the linear combination of } a_1 \text{ and } a_2 \text{ produces } b?$$

Geometry of Linear Equations

Column Space View

$$a_1 \cdot x_1 + a_2 \cdot x_2 = b$$

As before, the LHS of the above equation can produce an infinite number of vectors in $\mathbb{R}$. We define this set as the column space of set $A = \{a_1, a_2\}$.

$$\mathcal{C}(A) = \{x_1 \cdot a_1 + x_2 \cdot a_2 \mid x_1, x_2 \in \mathbb{R}\}$$

There are two possibilities for the set of all possible values for the linear combinations of a_1 and a_2.

$$\begin{cases} \mathcal{C}(A) = \mathbb{R}, & a_1 \neq 0 \ \text{ OR } \ a_2 \neq 0 \\ \mathcal{C}(A) = \{0\}, & a_1 = 0 \ \text{ AND } \ a_2 = 0 \end{cases}$$

Geometry of Linear Equations

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$\mathcal{C}(A) = \mathbb{R}$ since $a_1, a_2 \neq 0, b = 2 \in \mathcal{C}(A)$

$\therefore$ The equation is solvable and had infinitely many solutions $(x_1, 2 - x_1), \ x_1 \in \mathbb{R}$.

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Example 2: $2 \cdot x_1 + 0 \cdot x_2 = 10 \quad a_1 = 5, a_2 = 0, \; b = 10$

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Example 3: $0 \cdot 0 \cdot x_1 + 0 \cdot x_2 = -3$ $a_1 = a_2 = 0, \ b = -3$

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$\mathcal{C}(A) = \{0\}$ since $a_1 = a_2 = 0, b = -3 \notin \mathcal{C}(0)$
$\therefore$ The equation is **not** solvable.

Two Possible Solutions

$$a_1 \cdot x_1 + a_2 \cdot x_2 = b$$

No Solution

- $a = 0, b \in \mathbb{R}$
- *Not solvable*
- $\mathcal{C}(0) = \{0\}$
- $b \notin \mathcal{C}(0)$

Infinite Solutions

- $a = 0, b = 0$
- *Always solvable*
- $\mathcal{C}(0) = \{0\}$
- $b \in \mathcal{C}(0)$
- $x \in \mathbb{R}$

Remember*: The equation is solvable with infinitely many solution, if and only if $b \in \mathcal{C}(a)$.*

What is the difference between the two equations?

The two linear equations in 1 and 2 unknown variables are different in the types of solutions.

$$\begin{cases} a \cdot x_1 = b & \longrightarrow \text{3 possible solutions} \\ a_1 \cdot x_1 + a_2 \cdot x_2 = b & \longrightarrow \text{2 possible solutions} \end{cases}$$

Why is there a difference?

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Why is there a difference?
More unknowns than equations $\implies$ more freedom to choose values.

But this cannot be the complete answer, because $0 \cdot x_1 = 0$ also has infinitely many solutions for $b = 0$, where the number of unknowns and equations are the same!

Linear Independence determines the number of solutions

Whether a solvable linear equation has unique or infinite solutions is determined by the property of linear independence of the set of vectors - formed by the coefficients associated with the unknown variables in the linear equations.

Linear independence is a property of a set of vectors.

Linear independence tells us if there is "more than what is needed" in the set.

Consider a set A of vectors (real numbers for now): $A = \{a_1, a_2, \cdots a_n\}$, $a_i \in \mathbb{R}$.

Definition

A set of vectors A is *linearly independent* if none of the vectors in the set can be expressed as a linear combination of the others.

Linear Independence

A test for linear linear independence. Consisder the linear combination of a set of vectors A which produces another vector b,

$$\alpha_1 \cdot a_1 + \alpha_2 \cdot a_2 + \cdots \alpha_n \cdot a_n = b, \;\; \alpha_i \in \mathbb{R}$$

What mixtures (values of α_i) of the above linear combination produce the zero vector $b = 0$?

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ is the only set of values that produces $b = 0$, then the set of vectors is linearly independent.

Even if any one of the $\alpha_i \neq 0$, then the set is linear dependent.

Linear Independence

If any one of the $\alpha_i \neq 0$, then the following is true (assuming $\alpha_j \neq 0$),

$$a_j = -\sum_{i=1, i \neq j}^{n} \frac{\alpha_i}{\alpha_j} \cdot a_i \implies \begin{cases} a_j \text{ can be expressed as a linear combination} \\ \text{of the other vectors in the set } A. \end{cases}$$

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ is the only set of values that produces $b = 0$, then no vector in the set can be expressed as a linear combination of the others.

What happens when all $\alpha_i = 0$ except $\alpha_j \neq 0$? Obviously , the set is linearly dependent. What more can we say about the set?

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$$a_j = 0$$

A set containing the zero vector is linearly dependent!

Linear Independence: Examples

Example 1: $A = \{1.5\}$

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The set is linearly dependent since $\alpha_1 \cdot 2 + \alpha_2 \cdot 1 = 0 \implies \alpha_2 = -2 \cdot \alpha_1$.

General conditions for solutions of a linear equation.

Now, we can state the general conditions for the solutions of a linear equation,

$$a_1 \cdot x_1 + a_2 \cdot x_2 + \cdots + a_n \cdot x_n = b$$

where, $a_i, b \in \mathbb{R}$ are fixed and x_i are the unknown variables.

▶ If the set $A = \{a_1, a_2, \cdots a_n\}$ is linearly independent and $b \in \mathcal{C}(A)$, then the equation has a unique solution.

▶ If the set $A = \{a_1, a_2, \cdots a_n\}$ is linearly dependent and $b \in \mathcal{C}(A)$, then the equation has infinitely many solutions.

▶ If $b \notin \mathcal{C}(A)$, then the equation has no solution.

Some additional concepts

We will now make some observations about some of the sets we have seen so far.

Let's first define an important concept from set theory.

Definition

A set S is said to *closed under an operation* if performing the operation on any elements of the set produces an element that is also in the set.

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Examples:

- The set of integers is closed under addition.
- The set of integers is closed under multiplication.
- The set of even integers is closed under addition.

- The set of odd integers is *NOT* closed under addition.
- The set of integers is *NOT* closed under division.
- The set of integers is *NOT* closed under square roots.

Some additional concepts: Vector Spaces & Subspace

Definition

A set *S* that is closed under linear combination of its elements is called a vector space. The elements of a vector space are called vectors.

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Is $\{0\}$ a vector space?

Some additional concepts: Vector Spaces & Subspace

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$\mathbb{R}$ is a vector space.

Is $\{0\}$ a vector space? Yes, it is! Linear combinations of 0 always produce 0.

But $\{0\} \subset \mathbb{R}$ and is a vector space itself. Such vector spaces are called subspaces.

$\{0\}$ is a subspace of $\mathbb{R}$. It is also called the *trivial subspace*.

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What about the closed interval $[0, 1]$?

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$\{0\}$ is a subspace of $\mathbb{R}$. It is also called the *trivial subspace*.

What about the closed interval $[0, 1]$? No, it is not a vector space since it is not closed under linear combination. For example, $0.5 + 0.7 = 1.2 \notin [0, 1]$.

Some additional concepts: Span of set of vector

For the linear equation, $x_1 \cdot a_1 + x_2 \cdot a_2 + \cdots + x_n \cdot a_n = b$, the column space of $A = \{a_1, a_2, \cdots a_n\}$ is a vector space! Why?

Some additional concepts: Span of set of vector

For the linear equation, $x_1 \cdot a_1 + x_2 \cdot a_2 + \cdots + x_n \cdot a_n = b$, the column space of $A = \{a_1, a_2, \cdots a_n\}$ is a vector space! Why?

$\mathcal{C}(A)$ is the set of all linear combinations of the vectors in A. This important concept has a special name – the span of a set of vectors A.

Definition

The *span* of a set of vectors $A = \{a_1, a_2, \ldots, a_n\}$ is the set of all possible linear combinations of the vectors in A.

The span of set A is denoted as $\text{span}(A)$.

$$\text{span}(A) = \{\alpha_1 \cdot a_1 + \alpha_2 \cdot a_2 + \cdots + \alpha_n \cdot a_n \mid \alpha_i \in \mathbb{R}\}$$

Important Vector Space Concepts

Vector Space

A set closed under linear combinations.

Subspace

A subset of a vector space that is itself a vector space.

Span

The set of all linear combinations of a set of vectors.