Stable Matching

Concepts & Definitions**:**

- Matching: A set of edges without common vertices;
- Stable Matching: A matching where, given an order of preference for each element, there does not exist a pair (A, B) such that (A, B) would be better together than with their current matches;
- Perfect Matching: A matching such that every element is part of exactly 1 pair;
- Reworded: A perfect matching is such that every vertex v ∈ V(G) is incident to exactly one (u, v) ∈ E(G)

Theorems, Lemmas, Claims, [...] (w/o proofs)**:**

- *Boy Proposal Algorithm* always terminates in a stable matching;
- In BPA, no girl ever rejects a valid partner;
- In BPA, every boy is matched to the best possible valid choice;
- In BPA, every girl is matched to the worst possible valid choice;

Basics of Graph Theory

Concepts & Definitions:

- K_n is a complete graph on n vertices, $|V(K_n)| = n,\ |E(K_n)| = \binom{n}{2}$;
- P_n is a path on n vertices, $|V(P)| = n,\ |E(P)| = n - 1$
- C_n is a cycle on n vertices;
- A graph H is a subgraph of G if $V(H) \subseteq V(G),\ E(H) \subseteq E(G)$;
- A bipartition (A, B) of the vertex set of a graph G is a partition such that every edge has one end in A and one end in B;

Theorems, Lemmas, Claims, [...] (w/o proofs):

- Handshake lemma: $\sum\limits_{v\ in\ V(G)} deg_G(v) = 2|E(G)|$;
- If G is a forest, then $|V(G)| - |E(G)| = comp(G)$;
- If G is a connected graph and $deg_G(v) \leq 2\ \forall v \varepsilon V$, then G is a cycle or path;
- A graph G is bipartite if and only if it contains no odd cycles;

Matchings & Vertex Covers

Concepts & Definition):

- A set $M \subseteq E(G)$ is a matching of a graph G if every vertex in G is incident to at most one edge in M
- The matching number of a graph G, $\vartheta(G)$, is the maximum number of edges in a matching in G;
- A set $X \subseteq V(G)$ is a vertex cover of a graph G if every edge in G has at least one end in X;
- The minimum size of a vertex cover on the graph G, $\tau(G)$, is the minimum number of vertices in a vertex cover of G;
- A graph G is d-regular if all vertices in G have exactly d incident edges;

Theorems, Lemmas, Claims, [...] (w/o proofs):

- $\vartheta(G) \leq \tau(G)$;
- $\tau(G) \leq 2\vartheta(G)$;
- **König**: If G is bipartite, the $\vartheta(G) = \tau(G)$;
- If G is d-regular and bipartite, then it has a perfect matching;

Edge Colouring & Systems of Distinct Representatives:

Concepts & Definitions:

- $\chi'(G)$ is the minimum number of colours needed to appropriately colour the edges of G;
- $\Delta(G)$ is the maximum degree of a vertex in G;
- A system of distinct representatives of a collection of finite sets $S_1, S_2, ..., S_k$ is a sequence $(x_1, x_2, ..., x_k)\ such\ that\ x_i \varepsilon S_i\ \forall i$ and $x_i \neq x_j\ \forall i \neq j$;

Theorems, Lemmas, Claims, [...] (w/o proofs):

- $\chi'(G) \geq \Delta(G)$;
- If G is d-regular and bipartite, then $\chi'(G) = d$;
- **Hall**: If G is a graph with a bipartition (A, B), then there exists a matching M which uses all vertices in A if and only if $|N(S)| \geq |S|\ \forall\ S \subseteq A$;
- A collection of sets $S_1, S_2, ..., S_k$ has a system of distinct representatives if and only if $|\bigcup_{i \varepsilon I} S_I| \geq |I|$;

Graph Colouring:

Concepts & Definitions:

- A vertex colouring of a graph G is a map which assigns to every vertex of G a colour $C(v)$ such that, if vertices u and v are adjacent, $C(u) \neq C(v)$;
- A k-colouring is a colouring requiring k colours;
- $\chi(G)$, the chromatic number of G, is the minimum number k of colours needed for a k-colouring;
- A graph G is k-degenerate if for every subgraph H of G contains a vertex v such that $deg_H(v) \leq k$;

Theorems, Lemmas, Claims, [...] (w/o proofs):

- **Erdös**: For all $k,\ l\ > 0$, there exists a graph $G = G(k, l)$ such that $\chi(G) \geq k$ and every cycle in G has length $\geq l$;
- $\chi(G) \leq \Delta(G) + 1$;
- If G is a k-degenerate graph, then $\chi(G) \leq k + 1$;

Planar Graphs:

Concepts & Definitions:

- A graph G is said to be planar if it can be drawn such that its edges are disjoint except for their common endpoints;

- $Reg(G)$ denotes the number of regions on a planar graph;
- A graph H is a subdivision of a graph G if H can be obtained from G by replacing some edges by paths with the same endpoints, which otherwise don't share vertices;
- Contracting an edge with ends u, v in a graph G is done by deleting u, v, adding a new vertex w adjacent to all vertices in $G \setminus u \setminus v$ which were adjacent to either u or v in G. The result is denoted as G/e;
- A graph H is a minor of G if H can be obtained from G by repeatedly contracting edges, deleting edges, or deleting vertices;
- If H is a minor of G, we denote $H \leq_m G$;

<u>Theorems, Lemmas, Claims, [...] (w/o proofs)</u>:
- **Euler**: If G is a connected planar graph, then $|V(G)| - |E(G)| + Reg(G) = 2$;
- Let G be a planar graph with $|V(G)| \geq 3$, then $|E(G)| \leq 3|V(G)| - 6$;
- $\chi(G) \leq 4$ for every planar graph G;
- **Kuratowski**: A graph G is planar if and only if it does not contain a subgraph which is a subdivision of K_5 or $K_{3,3}$;
- Minor Variant: A graph G is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor;