Chapter 9: Multiple and logistic regression

OpenIntro Statistics, 4th Edition

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Introduction to multiple regression

Multiple regression

- Simple linear regression: Bivariate - two variables: y and x

$$y = b_0 + bx$$

- Multiple linear regression: Multiple variables: y and $x_1, x_2, \cdots$

$$y = b_0 + b_1x_1 + b_2x_2 + \ldots + b_nx_n.$$

Note: x is the explanatory variable and y is the response variable.

Purpose: to determine the strength and character of the relationship between one dependent (response) variable and a series of other variables (known as independent or explanatory variables)

Response or Dependent variable is usually continuous

Poverty vs. region (east, west)

$$\widehat{poverty} = 11.17 + 0.38 \times west$$

- Explanatory variable: region, *reference level:* east
- *Intercept:* The estimated average poverty percentage in eastern states is 11.17%

Predictors with Several Variables

When fitting a regression model with a categorical variable that has k levels where k > 2, software will provide a coefficient for k-1 of those levels. For the last level that does not receive a coefficient, this is the **reference level** and the coefficients for all other levels are considered relative to this reference level.

Poverty vs. region (east, west)

$$\widehat{poverty} = 11.17 + 0.38 \times west$$

- Explanatory variable: region, *reference level:* east
- *Intercept:* The estimated average poverty percentage in eastern states is 11.17%
  - This is the value we get if we plug in *0* for the explanatory variable
- *Slope:* The estimated average poverty percentage in western states is 0.38% higher than eastern states.
  - Then, the estimated average poverty percentage in western states is 11.17 + 0.38 = 11.55%.
  - This is the value we get if we plug in *1* for the explanatory variable

Poverty vs. region (northeast, midwest, west, south)

Which region (northeast, midwest, west, or south) is the reference level?

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	9.50	0.87	10.94	0.00
region4midwest	0.03	1.15	0.02	0.98
region4west	1.79	1.13	1.59	0.12
region4south	4.16	1.07	3.87	0.00

(a) northeast

(b) midwest

(c) west

(d) south

(e) cannot tell

Poverty vs. region (northeast, midwest, west, south)

Which region (northeast, midwest, west, or south) has the lowest poverty percentage?

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	9.50	0.87	10.94	0.00
region4midwest	0.03	1.15	0.02	0.98
region4west	1.79	1.13	1.59	0.12
region4south	4.16	1.07	3.87	0.00

(a) northeast

(b) midwest

(c) west

(d) south

(e) cannot tell

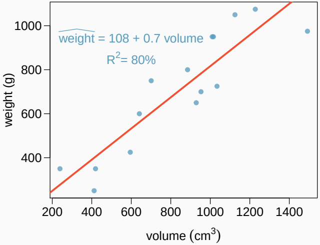
Weights of books

	weight (g)	volume (cm^3)	cover
1	800	885	hc
2	950	1016	hc
3	1050	1125	hc
4	350	239	hc
5	750	701	hc
6	600	641	hc
7	1075	1228	hc
8	250	412	pb
9	700	953	pb
10	650	929	pb
11	975	1492	pb
12	350	419	pb
13	950	1010	pb
14	425	595	pb
15	725	1034	pb

From: Maindonald, J.H. and Braun, W.J. (2nd ed., 2007) "Data Analysis and Graphics Using R"

Weights of books (cont.)

The scatterplot shows the relationship between weights and volumes of books as well as the regression output. Which of the below is correct?



(a) Weights of 80% of the books can be predicted accurately using this model.

(b) Books that are 10 cm^3 over average are expected to weigh 7 g over average.

(c) The correlation between weight and volume is $R = 0.80^2 = 0.64$.

(d) The model underestimates the weight of the book with the highest volume.

Modeling weights of books using volume

somewhat abbreviated output...

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 107.67931   88.37758   1.218    0.245
volume        0.70864    0.09746   7.271 6.26e-06


Residual standard error: 123.9 on 13 degrees of freedom
Multiple R-squared: 0.8026,Adjusted R-squared: 0.7875
F-statistic: 52.87 on 1 and 13 DF,  p-value: 6.262e-06
```

Determining the reference level

Based on the regression output below, which level of `cover` is the reference level? Note that `pb`: paperback.

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	197.9628	59.1927	3.34	0.0058
volume	0.7180	0.0615	11.67	0.0000
cover:pb	-184.0473	40.4942	-4.55	0.0007

(a) paperback

(b) *hardcover*

Determining the reference level

Which of the below correctly describes the roles of variables in this regression model?

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	197.9628	59.1927	3.34	0.0058
volume	0.7180	0.0615	11.67	0.0000
cover:pb	-184.0473	40.4942	-4.55	0.0007

(a) response: weight, explanatory: volume, paperback cover

(b) response: weight, explanatory: volume, hardcover cover

(c) response: volume, explanatory: weight, cover type

(d) response: weight, explanatory: volume, cover type

Linear model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	197.96	59.19	3.34	0.01
volume	0.72	0.06	11.67	0.00
cover:pb	-184.05	40.49	-4.55	0.00

$$\widehat{weight} = 197.96 + 0.72\ volume - 184.05\ cover : pb$$

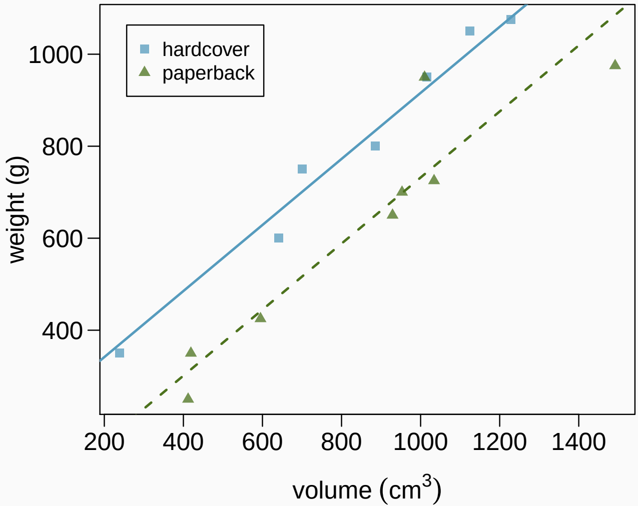
1. For *hardcover* books: plug in *0* for `cover`

$$\begin{aligned}\widehat{weight} &= 197.96 + 0.72\ volume - 184.05 \times 0 \\ &= 197.96 + 0.72\ volume\end{aligned}$$

2. For *paperback* books: plug in *1* for `cover`

$$\begin{aligned}\widehat{weight} &= 197.96 + 0.72\ volume - 184.05 \times 1 \\ &= 13.91 + 0.72\ volume\end{aligned}$$

Visualising the linear model



Interpretation of the regression coefficients

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	197.96	59.19	3.34	0.01
volume	0.72	0.06	11.67	0.00
cover:pb	-184.05	40.49	-4.55	0.00

- *Slope of volume:* All else held constant, books that are 1 more cubic centimeter in volume tend to weigh about 0.72 grams more.
- *Slope of cover:* All else held constant, the model predicts that paperback books weigh 184 grams lower than hardcover books.
- *Intercept:* Hardcover books with no volume are expected on average to weigh 198 grams.
    - Obviously, the intercept does not make sense in context. It only serves to adjust the height of the line.

Prediction

Which of the following is the correct calculation for the predicted weight of a paperback book that is 600 cm^3?

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	197.96	59.19	3.34	0.01
volume	0.72	0.06	11.67	0.00
cover:pb	-184.05	40.49	-4.55	0.00

(a) 197.96 + 0.72 * 600 - 184.05 * 1

(b) 184.05 + 0.72 * 600 - 197.96 * 1

(c) 197.96 + 0.72 * 600 - 184.05 * 0

(d) 197.96 + 0.72 * 1 - 184.05 * 600

Why bother?

Why bother with another approach for calculating R^2 when we had a perfectly good way to calculate it as the correlation coefficient squared?

- *For single-predictor linear regression, having three ways to calculate the same value may seem like overkill.*
- *However, in multiple linear regression, we can't calculate R^2 as the square of the correlation between x and y because we have multiple xs.*
- *And next we'll learn another measure of explained variability, adjusted R^2, that requires the use of the third approach, ratio of explained and unexplained variability.*

Collinearity between explanatory variables

poverty vs. % female head of household

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	3.31	1.90	1.74	0.09
female_house	*0.69*	0.16	4.32	0.00

poverty vs. % female head of household and % female hh

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-2.58	5.78	-0.45	0.66
female_house	*0.89*	0.24	3.67	0.00
white	0.04	0.04	1.08	0.29

Collinearity between explanatory variables (cont.)

- Two predictor variables are said to be collinear when they are correlated, and this *collinearity* complicates model estimation.

  Remember: Predictors are also called explanatory or independent variables. Ideally, they would be independent of each other.

- We don't like adding predictors that are associated with each other to the model, because often times the addition of such variable brings nothing to the table. Instead, we prefer the simplest best model, i.e. *parsimonious* model.
- While it's impossible to avoid collinearity from arising in observational data, experiments are usually designed to prevent correlation among predictors.

R^2 vs. adjusted R^2

	R^2	Adjusted R^2
Model 1 (Single-predictor)	0.28	0.26
Model 2 (Multiple)	0.29	0.26

- When <u>any</u> variable is added to the model R^2 increases.
- But if the added variable doesn't really provide any new information, or is completely unrelated, adjusted R^2 does not increase.

Adjusted R^2

Adjusted R^2

$$R^2_{adj} = 1 - \left(\frac{SS_{Error}}{SS_{Total}} \times \frac{n-1}{n-p-1} \right)$$

where n is the number of cases and p is the number of predictors (explanatory variables) in the model.

- Because p is never negative, R^2_{adj} will always be smaller than R^2.
- R^2_{adj} applies a penalty for the number of predictors included in the model.
- Therefore, we choose models with higher R^2_{adj} over others.

Model selection

Model selection strategies

Based on what we've learned so far, what are some ways you can think of that can be used to determine which variables to keep in the model and which to leave out?

Backward-elimination

1. Start with the full model
2. Drop one variable at a time and record R^2_{adj} of each smaller model
3. Pick the model with the highest increase in R^2_{adj}
4. Repeat until none of the models yield an increase in R^2_{adj}

`step` function in R

The `step` function in R does a similar backward elimination process, however it uses a different metric called AIC (Akaike Information Criterion) instead of adjusted R^2 to do the model selection.

```
Call:
lm(formula = profevaluation ~ beauty + gender + age + formal +
    native + tenure, data = d)

Coefficients:
       (Intercept)              beauty          gendermale
          4.628435            0.105546            0.208079
               age           formalyes   nativenon english
         -0.008844            0.132422           -0.243003
tenuretenure track       tenuretenured
         -0.206784           -0.175967
```

Best model: beauty + gender + age + formal + native + tenure

Forward selection

1. Start with regressions of response vs. each explanatory variable
2. Pick the model with the highest R^2_{adj}
3. Add the remaining variables one at a time to the existing model, and once again pick the model with the highest R^2_{adj}
4. Repeat until the addition of any of the remanning variables does not result in a higher R^2_{adj}

Backward-elimination: $p - value$ approach

Step	Variables included & p-value									
Full	beauty	gender male	age	formal yes	lower yes	native nonenglish	minority yes	students	tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.04	0.29	0.06	*0.35*	0.30	0.02	0.02
Step 1	beauty	gender male	age	formal yes	lower yes	native nonenglish		students	tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.04	*0.38*	0.03		0.34	0.02	0.01
Step 2	beauty	gender male	age	formal yes		native nonenglish		students	tenure tenure track	tenure tenured
	0.00	0.00	0.01	0.05		0.02		*0.44*	0.01	0.01
Step 3	beauty	gender male	age	formal yes		native nonenglish			tenure tenure track	tenure tenured
	0.00	0.00	0.01	*0.06*		0.02			0.01	0.01
Step 4	beauty	gender male	age			native nonenglish			tenure tenure track	tenure tenured
	0.00	0.00	0.01			*0.06*			0.01	0.01
Step 5	beauty	gender male	age						tenure tenure track	tenure tenured
	0.00	0.00	0.01						0.01	0.01

Best model: beauty + gender + age + tenure

Adjusted R^2 vs. p-value approaches

- The two approaches are similar, but they sometimes lead to different models, with the adjusted R^2 approach tending to include more predictors in the final model.
- When the sole goal is to improve prediction accuracy, use R^2 . This is commonly the case in machine learning applications.
- When we care about understanding which variables are statistically significant predictors of the response, or if there is interest in producing a simpler model at the potential cost of a little prediction accuracy, then the p-value approach is preferred.
- Regardless of the approach we use, our job is not done after variable selection – we must still verify the model conditions are reasonable.

Checking model conditions using graphs

Modeling conditions

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p$$

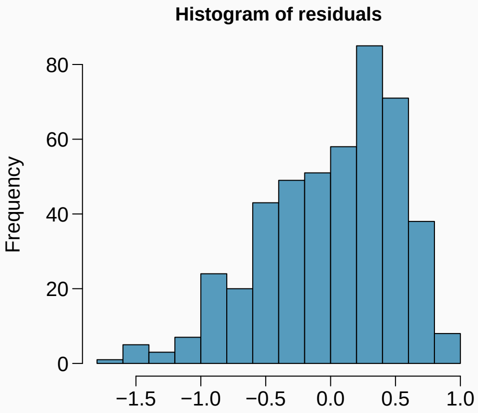
The model depends on the following conditions

1. residuals are nearly normal (less important for larger data sets)
2. residuals have constant variability
3. residuals are independent
4. each variable is linearly related to the outcome

We often use graphical methods to check the validity of these conditions.

(1) nearly normal residuals

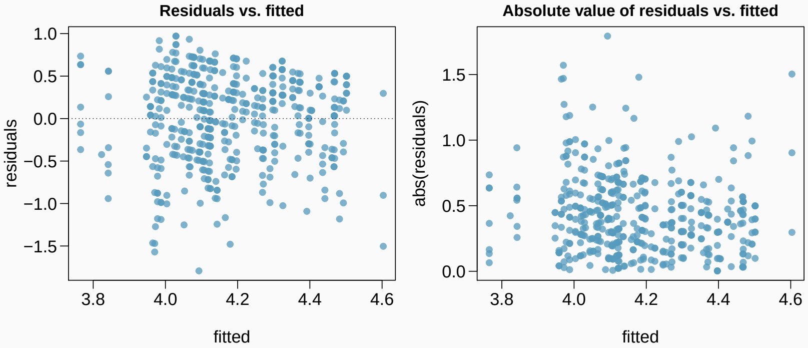
normal probability plot and/or histogram of residuals:



Does this condition appear to be satisfied?

(2) constant variability in residuals

scatterplot of residuals and/or absolute value of residuals vs. fitted (predicted):



Does this condition appear to be satisfied?

Checking constant variance - recap

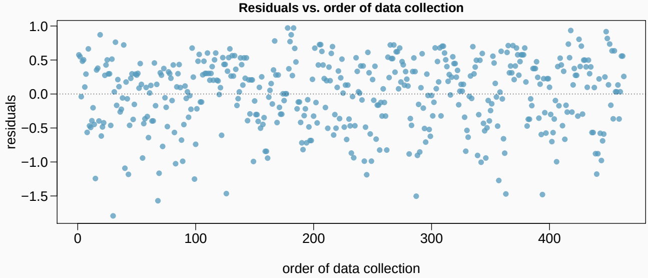
- When we did simple linear regression (one explanatory variable) we checked the constant variance condition using a plot of *residuals vs. x*.
- With multiple linear regression (2+ explanatory variables) we checked the constant variance condition using a plot of *residuals vs. fitted*.

Why are we using different plots?

In multiple linear regression there are many explanatory variables, so a plot of residuals vs. one of them wouldn't give us the complete picture.

(3) independent residuals

scatterplot of residuals vs. order of data collection:



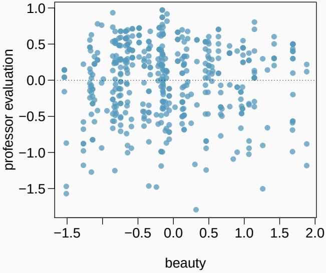
Does this condition appear to be satisfied?

More on the condition of independent residuals

- Checking for independent residuals allows us to indirectly check for independent observations.
- If observations and residuals are independent, we would not expect to see an increasing or decreasing trend in the scatterplot of residuals vs. order of data collection.
- This condition is often violated when we have time series data. Such data require more advanced time series regression techniques for proper analysis.

(4) linear relationships

scatterplot of residuals vs. each (numerical) explanatory variable:



Does this condition appear to be satisfied?

Note: We use residuals instead of the predictors on the y-axis so that we can still check for linearity without worrying about other possible violations like collinearity between the predictors.

Several options for improving a model

- Transforming variables
- Seeking out additional variables to fill model gaps
- Using more advanced methods that would account for challenges around inconsistent variability or nonlinear relationships between predictors and the outcome

Transformations

If the concern with the model is non-linear relationships between the explanatory variable(s) and the response variable, transforming the response variable can be helpful.

- Log transformation ($\log y$)
- Square root transformation ($\sqrt{y}$)
- Inverse transformation ($1/y$)
- Truncation (cap the max value possible)

It is also possible to apply transformations to the explanatory variable(s), however such transformations tend to make the model coefficients even harder to interpret.

Models can be wrong, but useful

All models are wrong, but some are useful. - George Box

- No model is perfect, but even imperfect models can be useful, as long as we are clear and report the model's shortcomings.
- If conditions are grossly violated, we should not report the model results, but instead consider a new model, even if it means learning more statistical methods or hiring someone who can help.

Logistic regression

Regression so far ...

At this point we have covered:

- Simple linear regression
  - Relationship between numerical response and a numerical or categorical predictor
- Multiple regression
  - Relationship between numerical response and multiple numerical and/or categorical predictors

What we haven't seen is what to do when the predictors are weird (nonlinear, complicated dependence structure, etc.) or when the response is weird (categorical, count data, etc.)

Odds

Odds are another way of quantifying the probability of an event, commonly used in gambling (and logistic regression).

Odds

For some event E,

$$\text{odds}(E) = \frac{P(E)}{P(E^c)} = \frac{P(E)}{1 - P(E)}$$

Similarly, if we are told the odds of E are x to y then

$$\text{odds}(E) = \frac{x}{y} = \frac{x/(x+y)}{y/(x+y)}$$

which implies

$$P(E) = x/(x+y), \quad P(E^c) = y/(x+y)$$

Example - Donner Party

In 1846 the Donner and Reed families left Springfield, Illinois, for California by covered wagon. In July, the Donner Party, as it became known, reached Fort Bridger, Wyoming. There its leaders decided to attempt a new and untested rote to the Sacramento Valley. Having reached its full size of 87 people and 20 wagons, the party was delayed by a difficult crossing of the Wasatch Range and again in the crossing of the desert west of the Great Salt Lake. The group became stranded in the eastern Sierra Nevada mountains when the region was hit by heavy snows in late October. By the time the last survivor was rescued on April 21, 1847, 40 of the 87 members had died from famine and exposure to extreme cold.

From *Ramsey, F.L. and Schafer, D.W. (2002). The Statistical Sleuth: A Course in Methods of Data Analysis (2nd ed)*

Example - Donner Party - Data

	Age	Sex	Status
1	23.00	Male	Died
2	40.00	Female	Survived
3	40.00	Male	Survived
4	30.00	Male	Died
5	28.00	Male	Died
⋮	⋮	⋮	⋮
43	23.00	Male	Survived
44	24.00	Male	Died
45	25.00	Female	Survived

Example - Donner Party - EDA

Status vs. Gender:

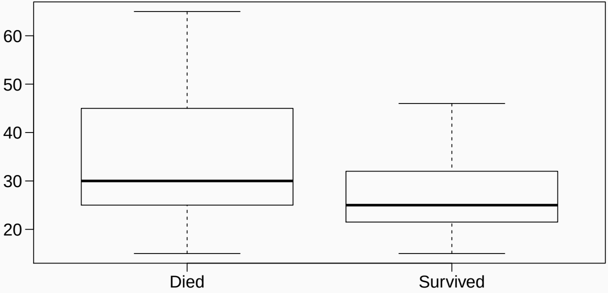
	Male	Female
Died	20	5
Survived	10	10

Example - Donner Party - EDA

Status vs. Gender:

	Male	Female
Died	20	5
Survived	10	10

Status vs. Age:



Example - Donner Party

It seems clear that both age and gender have an effect on someone's survival, how do we come up with a model that will let us explore this relationship?

Even if we set Died to 0 and Survived to 1, this isn't something we can transform our way out of - we need something more.

One way to think about the problem - we can treat Survived and Died as successes and failures arising from a binomial distribution where the probability of a success is given by a transformation of a linear model of the predictors.

Generalized linear models

It turns out that this is a very general way of addressing this type of problem in regression, and the resulting models are called generalized linear models (GLMs). Logistic regression is just one example of this type of model.

All generalized linear models have the following three characteristics:

1. A probability distribution describing the outcome variable
2. A linear model
   - $\eta = \beta_0 + \beta_1 X_1 + \cdots + \beta_n X_n$
3. A link function that relates the linear model to the parameter of the outcome distribution
   - $g(p) = \eta$ or $p = g^{-1}(\eta)$

Logistic Regression

Logistic regression is a GLM used to model a binary categorical variable using numerical and categorical predictors.

We assume a binomial distribution produced the outcome variable and we therefore want to model p the probability of success for a given set of predictors.

To finish specifying the Logistic model we just need to establish a reasonable link function that connects η to p. There are a variety of options but the most commonly used is the logit function.

Logit function

$$logit(p) = \log\left(\frac{p}{1-p}\right), \text{ for } 0 \le p \le 1$$

Properties of the Logit

The logit function takes a value between 0 and 1 and maps it to a value between $-\infty$ and ∞.

Inverse logit (logistic) function

$$g^{-1}(x) = \frac{\exp(x)}{1 + \exp(x)} = \frac{1}{1 + \exp(-x)}$$

The inverse logit function takes a value between $-\infty$ and ∞ and maps it to a value between 0 and 1.

This formulation also has some use when it comes to interpreting the model as logit can be interpreted as the log odds of a success, more on this later.

The logistic regression model

The three GLM criteria give us:

$$y_i \sim \text{Binom}(p_i)$$

$$\eta = \beta_0 + \beta_1 x_1 + \cdots + \beta_n x_n$$

$$\text{logit}(p) = \eta$$

From which we arrive at,

$$p_i = \frac{\exp(\beta_0 + \beta_1 x_{1,i} + \cdots + \beta_n x_{n,i})}{1 + \exp(\beta_0 + \beta_1 x_{1,i} + \cdots + \beta_n x_{n,i})}$$

LOGISTIC REGRESSION

Logistic regression is a generalized linear model where the outcome is a two-level categorical variable. The outcome, Yi, takes the value 1 (in our application, this represents a callback for the resume) with probability pi and the value 0 with probability 1 – pi. Because each observation has a slightly different context, e.g. different education level or a differentnumber of years of experience, the probability pi will differ for each observation. Ultimately, it is this probability that we model in relation to the predictor variables.

Good to use when response variable is binary or categorical

LOGISTIC REGRESSION REQUIREMENTS

There are two key conditions for fitting a logistic regression model:

1. Each outcome Yi is independent of the other outcomes.
2. Each predictor xi is linearly related to logit(pi) if all other predictors are held constant.

GLMs can be thought of as a two-stage modeling approach. We first model the response variable using a probability distribution, such as the binomial or Poisson distribution. Second, we model the parameter of the distribution using a collection of predictors and a special form of multiple regression. Ultimately, the application of a GLM will feel very similar to multiple regression, even if some of the details are different.

Good to use when response variable is binary or categorical

Example - Donner Party - Model

In R we fit a GLM in the same was as a linear model except using `glm` instead of `lm` and we must also specify the type of GLM to fit using the `family` argument.

```
summary(glm(Status ~ Age, data=donner, family=binomial))

## Call:
## glm(formula = Status ~ Age, family = binomial, data = donner)
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)  1.81852    0.99937   1.820   0.0688 .
## Age         -0.06647    0.03222  -2.063   0.0391 *
##
##     Null deviance: 61.827  on 44  degrees of freedom
## Residual deviance: 56.291  on 43  degrees of freedom
## AIC: 60.291
##
## Number of Fisher Scoring iterations: 4
```

Lower AIC is better

Example - Donner Party - Prediction

	Estimate	Std. Error	z value	Pr(>|z|)
(Intercept)	1.8185	0.9994	1.82	0.0688
Age	-0.0665	0.0322	-2.06	0.0391

Model:

$$\log\left(\frac{p}{1-p}\right) = 1.8185 - 0.0665 \times \text{Age}$$

Odds / Probability of survival for a newborn (Age=0):

$$\log\left(\frac{p}{1-p}\right) = 1.8185 - 0.0665 \times 0$$

$$\frac{p}{1-p} = \exp(1.8185) = 6.16$$

$$p = 6.16/7.16 = 0.86$$

Example - Donner Party - Prediction (cont.)

Model:

$$\log\left(\frac{p}{1-p}\right) = 1.8185 - 0.0665 \times \text{Age}$$

Odds / Probability of survival for a 25 year old:

$$\log\left(\frac{p}{1-p}\right) = 1.8185 - 0.0665 \times 25$$

$$\frac{p}{1-p} = \exp(0.156) = 1.17$$

$$p = 1.17/2.17 = 0.539$$

Odds / Probability of survival for a 50 year old:

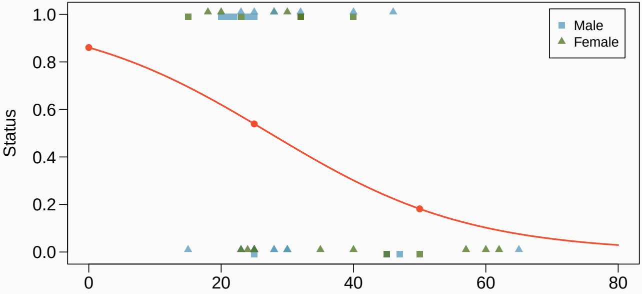
$$\log\left(\frac{p}{1-p}\right) = 1.8185 - 0.0665 \times 50$$

$$\frac{p}{1-p} = \exp(-1.5065) = 0.222$$

$$p = 0.222/1.222 = 0.181$$

Example - Donner Party - Prediction (cont.)

$$\log\left(\frac{p}{1-p}\right) = 1.8185 - 0.0665 \times \text{Age}$$



Example - Donner Party - Interpretation

	Estimate	Std. Error	z value	Pr(>|z|)
(Intercept)	1.8185	0.9994	1.82	0.0688
Age	-0.0665	0.0322	-2.06	0.0391

Simple interpretation is only possible in terms of log odds and log odds ratios for intercept and slope terms.

Intercept: The log odds of survival for a party member with an age of 0. From this we can calculate the odds or probability, but additional calculations are necessary.

Slope: For a unit increase in age (being 1 year older) how much will the log odds ratio change, not particularly intuitive. More often then not we care only about sign and relative magnitude.

Example - Donner Party - Age and Gender

```
summary(glm(Status ~ Age + Sex, data=donner, family=binomial))

## Call:
## glm(formula = Status ~ Age + Sex, family = binomial, data = donner)
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)  1.63312    1.11018   1.471   0.1413
## Age         -0.07820    0.03728  -2.097   0.0359 *
## SexFemale    1.59729    0.75547   2.114   0.0345 *
## ---
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 61.827  on 44  degrees of freedom
## Residual deviance: 51.256  on 42  degrees of freedom
## AIC: 57.256
##
## Number of Fisher Scoring iterations: 4
```

Gender slope: When the other predictors are held constant this is the log odds ratio between the given level (Female) and the reference level (Male).

Example - Donner Party - Gender Models

Just like MLR we can plug in gender to arrive at two status vs age models for men and women respectively.

General model:

$$\log\left(\frac{p_1}{1-p_1}\right) = 1.63312 + -0.07820 \times \text{Age} + 1.59729 \times \text{Sex}$$

Male model:

$$\begin{aligned}\log\left(\frac{p_1}{1-p_1}\right) &= 1.63312 + -0.07820 \times \text{Age} + 1.59729 \times 0 \\ &= 1.63312 + -0.07820 \times \text{Age}\end{aligned}$$

Female model:

$$\begin{aligned}\log\left(\frac{p_1}{1-p_1}\right) &= 1.63312 + -0.07820 \times \text{Age} + 1.59729 \times 1 \\ &= 3.23041 + -0.07820 \times \text{Age}\end{aligned}$$

Example - Donner Party - Gender Models (cont.)