5AG07
Nonlinear structural mechanics by finite element method.

Introduction to nonlinear elasticity

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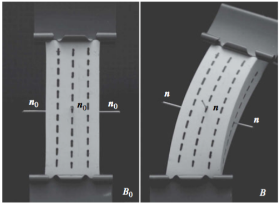
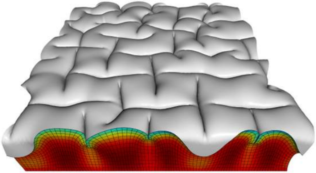
Introduction

Nonlinear elasticity





Hyperelastic blocs in large deformations

1. Quick review of tensor algebra.
2. Kinematics with large deformations
3. Statics, stress tensors.
4. Consitutive laws: Isotropic hyperelastic materials.
5. Variational formulation and numerical resolution strategy.

1-3 are review of a class in Continuum Mechanics.

References

- D.Bigoni, Nonlinear Solid Mechanics Bifurcation Theory and Material Instability. Cambridge University Press, 2012, ISBN:9781107025417
  - Chapter 3: 3.1, 3.2, 3.3 till Eq.(3.36), 3.3, 3.4, 3.5, 3.6 till Eq.(3.135)
  - Chapter 4: 4.1, 4.2.1, 4.2.2
  - Chapter 5: At least one of the proposed examples

- Class notes from Kerstin Weinberg available here:
  - Tensor calculus: `http://mech2.pi.tu-berlin.de/weinberg/Lehre/fem2/Appendix.pdf`
  - Kinematics: `http://mech2.pi.tu-berlin.de/weinberg/Lehre/fem2/Chapter2.pdf`
  - Statics: `http://mech2.pi.tu-berlin.de/weinberg/Lehre/fem2/Chapter3.pdf`
  - Constitutive laws: `http://mech2.pi.tu-berlin.de/weinberg/Lehre/fem2/Chapter4.pdf`

- Alternative references:
  - P. Chadwick, Continuum Mechanics: Concise Theory and Problems, Dover 1998
  - M.E. Gurtin, An Introduction to Continuum Mechanics. Academic Press, New York (1981).
  - P.Wriggers Nonlinear finite element methods: available from SU: `https://link.springer.com/content/pdf/10.1007%2F978-3-540-71001-1.pdf`

Linear Algebra

- $\underline{a}, \underline{b}, \underline{c} \in \mathbb{V}$: vectors in $\mathcal{R}^3$
- $b = \{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$: orthonormal basis in $\mathbb{V} \equiv \mathcal{R}^3$
- $A, B, C \in \mathrm{Lin}$: second order tensors from $\mathcal{R}^3$ to $\mathcal{R}^3$
- Component representation in b (repeated indices are summed):

$$\underline{a} = a_i \underline{e}_i, \qquad a_i = \underline{a} \cdot \underline{e}_i$$

$$A = A_{ij} \underline{e}_i \otimes \underline{e}_j, \qquad A_{ij} = A\underline{e}_i \cdot \underline{e}_j$$

Let be $(\underline{a}, \underline{b}, \underline{c})$ three vectors, being θ the angle between $\underline{a}$ and $\underline{b}$, ϕ the angle between $\underline{c}$ and the normal $\underline{n}$ to the plane defined by $(\underline{a}, \underline{b})$

Scalar product: $\boxed{\underline{a} \cdot \underline{b} = a_i b_i = \|\underline{a}\| \|\underline{b}\| \cos\theta}$

- Length of a vector: $\|\underline{a}\| = \sqrt{\underline{a} \cdot \underline{a}} = \sqrt{a_i\, a_i}$
- Angle between two vectors: $\cos\theta = \dfrac{\underline{a} \cdot \underline{b}}{\sqrt{(\underline{a} \cdot \underline{a})(\underline{b} \cdot \underline{b})}}$

Vector product: $\boxed{\underline{a} \times \underline{b} = \epsilon_{ijk} a_i b_j \underline{e}_k = \|\underline{a}\| \|\underline{b}\| \sin\theta\, \underline{n}, \quad \underline{n} \perp (\underline{a}, \underline{b})}$

- Surface of the parallelogram defined by $\underline{a}$ and $\underline{b}$: $\|\underline{a} \times \underline{b}\|$.

Triple product:

$$\boxed{\underline{c} \cdot (\underline{a} \times \underline{b}) = \epsilon_{ijk} a_i b_j c_k = \|\underline{a}\| \|\underline{b}\| \|\underline{c}\| \sin\theta \cos\phi = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.}$$

- Volume of the parallelepiped defined by $\underline{a}$, $\underline{b}$, $\underline{c}$: $\underline{c} \cdot (\underline{a} \times \underline{b})$.

Second order tensors: notation

- Lin: space of linear applications from $\mathcal{R}^3$ to $\mathcal{R}^3$.
- A transforms a vector into another vector: $\underline{c} = A\underline{b}$
- Tensor product between two vectors:

$$\boxed{(\underline{a} \otimes \underline{b})\underline{c} = (\underline{a} \cdot \underline{c})\underline{b}}$$

- Transpose:

$$\boxed{A\underline{a} \cdot \underline{b} = \underline{a} \cdot A^T\underline{b}, \quad A^T_{ij} = A_{ji}}$$

- Determinant
  - $\det(A) = \det(A_{ij})$
  - $\det(AB) = \det(A)\det(B), \quad \det(I + \underline{a} \otimes \underline{b}) = 1 + \underline{a} \cdot \underline{b}$
  - The determinant gives the change of volume of the parallelepiped defined by $(\underline{a}, \underline{b}, \underline{c})$ under the action of A :

$$\boxed{A\underline{c} \cdot (A\underline{a} \times A\underline{b}) = \det(A)\ \underline{c} \cdot (\underline{a} \times \underline{b})}$$

Second order tensors: subspaces of Lin

- sym $\equiv \{A \in \mathrm{Lin}, A = A^T\}$: $A\underline{a} \cdot \underline{b} = \underline{a} \cdot A\underline{b}$
- skw $\equiv \{A \in \mathrm{Lin}, A = -A^T\}$ $A\underline{a} \cdot \underline{b} = -\underline{a} \cdot A\underline{b}$
- orth $\equiv \{Q \in \mathrm{Lin},\, Q^T = Q^{-1}\}$: $Q\underline{a} \cdot Q\underline{b} = \underline{a} \cdot \underline{b}$
- orth$^+$ $\equiv \{Q \in \mathrm{Lin},\, Q^T = Q^{-1}, \det(Q) > 0\}$, rotations
- orth$^-$ $\equiv \{Q \in \mathrm{Lin},\, Q^T = Q^{-1}, \det(Q) < 0\}$, reflections

Second order tensors: two important theorems

Spectral decomposition of a symmetric tensor

For every $A \in \text{sym}$, exists $\underline{a}_i$ and $\alpha_i \in \mathcal{R}$ such that

$$A\underline{a}_i = \alpha_i \underline{a}_i, \qquad A = \alpha_1(\underline{a}_1 \otimes \underline{a}_1) + \alpha_2(\underline{a}_2 \otimes \underline{a}_2) + \alpha_3(\underline{a}_3 \otimes \underline{a}_3)$$

Definition: square root of a symmetric tensor (definition):
Considering the spectral decomposition of $A \in \text{sym}$,

$$\sqrt{A} = \sqrt{\alpha_1}(\underline{a}_1 \otimes \underline{a}_1) + \sqrt{\alpha}_2(\underline{a}_2 \otimes \underline{a}_2) + \sqrt{\alpha}_3(\underline{a}_3 \otimes \underline{a}_3)$$

Polar decomposition of a positive definite tensor

For every A such that $\det(A) > 0$, there exist unique $U, V \in$ sym and $R \in$ orth$^+$ such that:

$$A = RU = VR, \qquad U = \sqrt{A^T A}, \quad V = \sqrt{AA^T}.$$

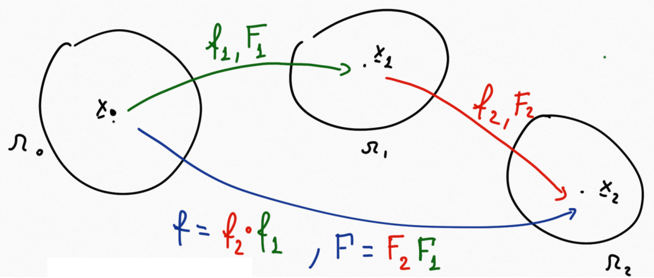
Kinematics

1. Transformations
   - Jacobian, homogeneous transformations
   - Transformation of line element, volume element, surface element
2. Deformations
   - Change of lengths (strecthing) and angles (shearing)
   - Cauchy and Green-Lagrange deformation tensors
   - Rigid tranformations
   - Multiplicative decomposition of a transformation
   - Polar decomposition in rigid transformation and pure deformation
   - Overview of the possible measures of deformations: C, E, U, $\log(U)$, ϵ
3. Movements
   - Lagragian description, velocity, acceleration
   - Eulerian velocity field and deformation rate.

Kinematics: cheatsheet

- $\underline{x} = \underline{f}(\underline{x}_0)$: transformation.
- $F = \nabla f$: gradient of the transformation(Lagrangian-Eulerian)
- $F = RU, \quad U = \sqrt{F^T F}$, polar decomposition in rigid rotation and pure deformation.
- Deformation measures:
  - $C = F^T F$, Cauchy stretch tensor (Lagrangian)
  - $E = (C - I)/2$, Green-Lagrange (Lagrangian)
  - $U = \sqrt{F^T F}$
  - $\log(U)$ logarithmics strain (Lagrangian)
  - $B = (FF^T - I)/2$, Cauchy - left (Eulerien).
- Rigid transformation: $\underline{x} = Q(\underline{x}_0 - \underline{c}) + \underline{v}, \quad Q \in \mathrm{orth}^+$
- $\underline{v}(\underline{x}, t)$ Eulerian velocity field.
- $L = \mathrm{grad}(v) = D + W$, strain rates.
- $\dot{F} = LF$, $\dot{E} = F^T DF$
- $\rho J = \rho_0$ (mass balance)

Composition of two transformations



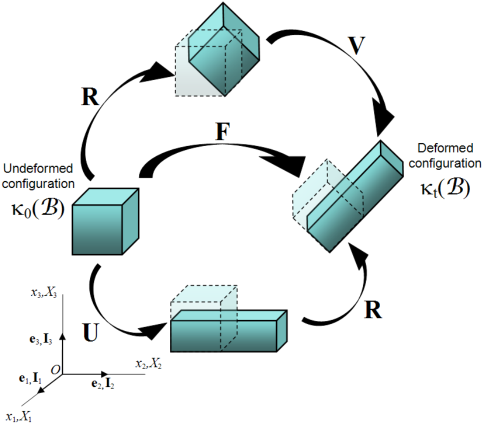
- First transformation f_1: $\underline{x}_1 = \underline{f}_1(\underline{x}_0)$, $F_1 = \dfrac{\partial \underline{x}_1}{\partial \underline{x}_0}$
- Second transformation f_2: $\underline{x}_2 = \underline{f}_2(\underline{x}_1)$, $F_2 = \dfrac{\partial \underline{x}_2}{\partial \underline{x}_1}$
- Composition of the two transformations $f = f_2 \odot f_1$:

$$\boxed{\underline{x}_2 = f(\underline{x}_0) = f_2(f_1(\underline{x}_0)), \quad F = \frac{\partial \underline{x}_2}{\partial \underline{x}_0} = \frac{\partial \underline{x}_2}{\partial \underline{x}_1}\frac{\partial \underline{x}_1}{\partial \underline{x}_0} = F_2 F_1}$$

Polar decomposition

For every A such that $\det(A) > 0$, there exist unique $U, V \in$ sym and $R \in$ orth$^+$ such that:

$$A = RU = VR, \qquad U = \sqrt{A^T A}, \quad V = \sqrt{AA^T}.$$



Statics

1. Equilibrium equations and stress tensors
   - Eulerian description, Cauchy stress tensor
   - Lagrangian description, first and second Piola-Kirchhoff stress tensors
2. Power balance
   - Power of external forces and energy balance
   - Duality (in the sense of power) between stress and strain measures

Equilibrium: Eulerian description

(We follow the notation in Bigoni for the stress tensors).

- Cauchy stress tensor (usually noted as σ)

$$T : \underbrace{\underline{n}}_{\text{normal in } \partial\Omega} \quad \to \quad \underbrace{\underline{t} = T\underline{n}}_{\text{force/surface in}\Omega}$$

- Equilibrium equations:

$$\begin{aligned} \underline{\operatorname{div} T} + \underline{b} = 0 \quad & \text{pn } \Omega \\ T\underline{n} = \underline{g} \quad & \text{on } \partial_g\Omega \end{aligned}$$

- The fundamental issue here is that the deformed configuration Ω is part of the unkwnows of the problems.

Equilibrium: Lagrangian description

- First Piola-Kirchhoff stress tensor (Lagrangian-Eulerien):

$$S: \underbrace{\underline{n_0}}_{\text{normal on } \Omega_0} \to \underbrace{\underline{t} = S\underline{n_0},}_{\text{force/surface on } \Omega} \qquad \boxed{S = JTF^{-T}}$$

- Equilibrium equations:

$$\begin{aligned} \underline{\mathrm{Div} S} + \underline{b_0} &= 0 \quad \text{on } \Omega_0 \\ S\underline{n_0} &= \underline{g}_0 \quad \text{on } \partial_g \Omega_0 \end{aligned}$$

- Second Piola-Kirchhoff stress tensor (purely Lagrangien).

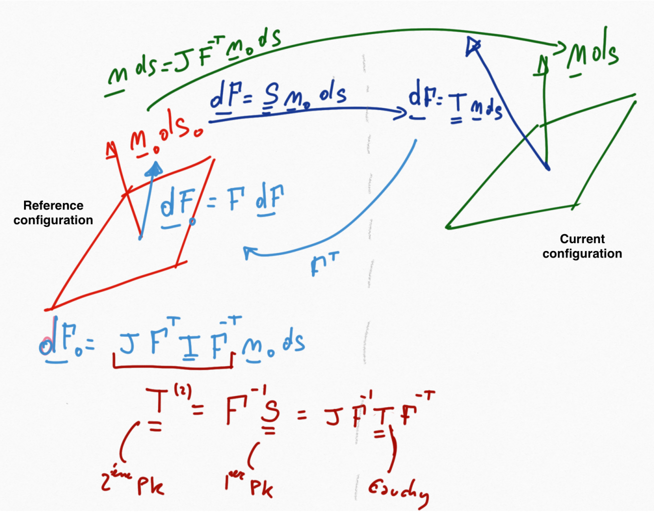
$$\boxed{T^{(2)} = F^{-1} S = F^{-1} TF^{-T}}$$

- $S \notin \text{sym}$, but $SF^T = FS^T$
- $T^{(2)} \in \text{sym}$

Statics: résumé

- Stress measures:
  - T Cauchy stress tensor (Eulerian)
  - $S = J\sigma F^{-T}$ First Piola-Kirchhoff stress tensor (Lagrangien-Eulerien).
  - $T^{(2)} = F^{-1}\, S$ Second Piola-Kirchhoff stress tensor (Lagrangien).
- $SF^T = FS^T$
- $\mathrm{Div}(S) + \underline{b}_0 = 0$ on Ω_0, $S\underline{n_0} = \underline{g}_0$ on $\partial_f\Omega_0$ (Lagrangian equilibrium)
- $\mathrm{div}(T) + \underline{b} = 0$ on Ω, $T\underline{n} = \underline{g}$ on $\partial_f\Omega$ (Eulerieean equilibrium)
- Stress power: $P_{int} = \int_\Omega T \cdot D\, dx = \int_{\Omega_0} S \cdot \dot{F}\, dx_0 = \int_{\Omega_0} T^{(2)} \cdot \dot{E}\, d\underline{x}_0$

Stress tensors



Stress power and power balance

- Stress power (or internal power or *puissance des efforts interieurs*)

$$P_{int} = \int_{\Omega} T \cdot D \, dx = \int_{\Omega_0} S \cdot \dot{F} \, dx_0 = \int_{\Omega_0} T^{(2)} \cdot \dot{E} \, d\underline{x}_0$$

- Power balance (Eulerian version)

$$\underbrace{\int_{\Omega} \underline{b} \cdot v \, dx + \int_{\partial\Omega} \underline{g} \cdot \underline{v} \, ds}_{\text{Puissance externe}} = \underbrace{\int_{\Omega} T \cdot D \, dx}_{\text{Puissance interne}} + \frac{d}{dt} \underbrace{\int_{\Omega} \frac{\rho}{2} \underline{v} \cdot \underline{v} \, dx}_{\text{Energie cinétique}}$$

- Power balance (Lagrangian version)

$$\underbrace{\int_{\Omega_0} \underline{b}_0 \cdot \dot{\underline{x}} \, dx_0 + \int_{\partial\Omega_0} \underline{g}_0 \cdot \dot{\underline{x}} \, ds_0}_{\text{Puissance externe}} = \underbrace{\int_{\Omega_0} S \cdot \dot{F} \, dx_0}_{\text{Puissance interne}} + \frac{d}{dt} \underbrace{\int_{\Omega_0} \frac{\rho_0}{2} \dot{\underline{x}} \cdot \dot{\underline{x}} \, dx}_{\text{Energie cinétique}}$$

The strain work per unit of volume in the reference configuration can be written in the following form:

$$\boxed{S \cdot \dot{F} = T^{(2)} \cdot \dot{E} = J \, T \cdot D}$$

Derivatives

Derivatives

The directional derivative $f'(u)(v)$ of a function $f: u \to f(u)$ is found using the following definition, that can be applied to scalar, vector, tensor valued function of scalar, vector, tensor fields:

$$Df(u)[v] = f'(u)(v) = \left. \frac{d}{dh} f(u + hv) \right|_{h=0}$$

Important derivatives are

- Determinant of a tensor field (see e.g. [1]):

$$det'(A)(B) = det(A)B^{-T}, \qquad J'(u)(v) = J(u)\mathrm{Div}(v)$$

-

$$E'(u)(v) = \mathrm{sym}(F^T \nabla v), \quad C'(u)(v) = 2E'(u)(v)$$

Notation:

- Lagrangian gradient: $\mathrm{Grad} u = \nabla u = \partial u_i / \partial x_{0j}$;
- Eulerian gradient: $\mathrm{grad} u = \partial u_i / \partial x_j$

[1] `https://en.wikipedia.org/wiki/Tensor_derivative_(continuum_mechanics)`

Invariants and their derivatives

The three invariants of a second order tensor A are

$$I_1(A) = \mathrm{tr} A$$
$$I_2(A) = \frac{1}{2}\left[(\mathrm{tr} A)^2 - \mathrm{tr} A^2\right]$$
$$I_3(A) = \det(A)$$

Their derivatives are

$$\frac{\partial I_1}{\partial A} = \boldsymbol{1}$$
$$\frac{\partial I_2}{\partial A} = I_1 \ \boldsymbol{1} - A^T$$
$$\frac{\partial I_3}{\partial A} = \det(A) \ [A^{-1}]^T = I_2 \ \boldsymbol{1} - A^T \ (I_1 \ \boldsymbol{1} - A^T) = (A^2 - I_1 \ A + I_2 \ \boldsymbol{1})^T$$

Cayley–Hamilton theorem says that the invariants verify:

$$A^3 - I_1 A^2 + I_2 A - I_3 \boldsymbol{1} = 0$$

where $\boldsymbol{1}$ is the second-order identity tensor.

Derivative of the determinant

Proof, see Gurtin

We have to compute $\det(A + hB)$ and take the derivative wrt h. Using

$$\det(A - \lambda I) = -\lambda^3 + I_1(A)\lambda^2 - I_2(A)\lambda + I_3(A)$$

with $\lambda = -1$ we get

$$\det(A + I) = 1 + \operatorname{tr}(A) + o(A)$$

$$\begin{aligned}\det(A + hB) &= \det(A(hA^{-1}B + I)) = \det(A)\det(hA^{-1}B + I)) \\ &= (\det(A) + h\det(A)\operatorname{tr}(A^{-1}B) + o(B)) \\ &= (\det(A) + h\det(A)\, A^{-T} \cdot B + o(B))\end{aligned}$$

Hence

$$\det(A')(B) = \left.\frac{\det(A + hB)}{dh}\right|_{h=0} = \det(A)\, A^{-T} \cdot B$$

Exercises (TD)

Let be

- $F = \nabla u,\ C = F^T F,\ E = (C - I)/2$
- $S = \partial W/\partial F, T^{(2)} = \partial W/\partial E$
- $W(u) = W_F(F(u))$ a scalar function (the energy density)

Show that

1. $E'(u)(v) = \mathrm{sym}(F^T \nabla v), \quad C'(u)(v) = 2E'(u)(v)$
2. $(F^{-1})'(u)(v) = F^{-1}(u)\, \nabla v\, F^{-1}(u) = F^{-1}(u)\, \mathrm{grad}(v)$
3. $W'(u)(v) = \dfrac{\partial W}{\partial F} \cdot \nabla v, \qquad W'(u)(v) = \dfrac{\partial W}{\partial E} \cdot F^T \nabla v, \qquad \underbrace{\dfrac{\partial W}{\partial F}}_{S} = F \underbrace{\dfrac{\partial W}{\partial E}}_{T^{(2)}}$
4. $W''(u)(v)(z) = \left(\dfrac{\partial^2 W}{\partial E^2} F^T \nabla z \right) \cdot F^T \nabla v + \underbrace{\dfrac{\partial W}{\partial E}}_{T^{(2)}} \cdot (\nabla z \cdot \nabla v)$

   In the following we will introduce the fourth order tensor

$$\mathbb{C} := \frac{\partial^2 W}{\partial E^2} = \frac{\partial T^{(2)}}{\partial E}$$

representing the linearised material stiffness

Constitutive laws

Hyperelastic materials

- Stress power per unit of volume in the reference configuration:

$$S \cdot \dot{F} = T^{(2)} \cdot \dot{E} = JT \cdot D$$

- Internal energy density (we consider isothermal process):

$$W(F) = \hat{W}(E)$$

- Hyperlastic materials do not dissipate energy, so for any admissible deformation rate $\dot{F}$ (or $\dot{E}$) stress power and variation of the internal energy must coincide:

$$S \cdot \dot{F} = \frac{\partial W}{\partial F} \cdot \dot{F} \Rightarrow \boxed{S = \frac{\partial W}{\partial F}} \text{ and similarly } \boxed{T^{(2)} = \frac{\partial W}{\partial E}} = F^{-1} S$$

- In hyperelastic materials the strain energy is a state function and the work required to pass from a deformation state F_1 to F_2 does not depend on the path.

Properties of the energy function

- Objectivity (the energy is invariant for rigid rotations):

$$W(F) = W(RU) = W(U)$$

  W can be defined in terms of F, E, C with suitable change of variables, e.g.:

$$W_F(F) = W_C(F^TF) = W_E\left(\frac{F^TF - I}{2}\right),$$

  When written in terms of C or E automatically verifies the objectivity.

- Note that

$$\frac{\partial W_F}{\partial F} = \frac{\partial W_E}{\partial E}\frac{\partial E}{\partial F} = F\frac{\partial W_E}{\partial E} = 2F\frac{\partial W_C}{\partial C}$$

- With an abuse of notation, we will often omit the subscript, e.g. write $W(E)$ instead of $W_E(E)$.

Properties of the energy function

- The well-posedness of the hyperelastic problem requires the energy to be quasi-convex:

$$\int_{\Omega_0} W(F+\nabla u)\, dx_0 \geq \int_{\Omega_0} W(F)\, dx_0, \qquad \forall u \quad \text{rég.}$$

- A sufficient condition for quasi-convexity is polyconvexity:

$$W(F) = \phi(F, \mathrm{cof}(F), \det(F)),$$

where ϕ is convex in each variable separately .

- A "good" energy should verify the growth conditions:

$$\det F \to (0^+, \infty) \Rightarrow W(F) \to \infty.$$

Examples:

- An energy that is polyconvex and satisfy the growth condition:

$$W_F(F) = \frac{\mu}{2}(F \cdot F - 3) - 2\mu \ln J + \frac{\lambda}{2} \ln J^2$$

- An energy function that is not polyconvex and do not verify the growth condition:

$$W_E(E) = \frac{\lambda}{2}(\mathrm{tr} E)^2 + \mu E \cdot E$$

Isotropic elasticity

For isotropic materials the strain energy density is an isotropic function of the deformation tensor. As such it can be written in terms of invariants.

- Isotropic function in terms of the invariants:

$$W(F) = W(I_1(C), I_2(C), I_3(C))$$

  where we denote by $I_i(C)$ the i-th invariant of C. This gives the second Piola-Kirchhoff stress

$$\begin{aligned} T^{(2)} = 2\frac{\partial W}{\partial C} &= 2\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial C} + 2\frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial C} + 2\frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial C} && (1) \\ &= 2\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\mathbf{1} - 2C\frac{\partial W}{\partial I_2} + 2\frac{\partial W}{\partial I_3} I_3\, C^{-1} && (2) \end{aligned}$$

- Isotropic function in terms of the eigenvalues:

$$W(C) = W(\lambda_1, \lambda_2, \lambda_3)$$

  where $(\lambda_1, \lambda_2, \lambda_3) = \mathrm{eig}(C)$.

Hyperlastic materials: example of compressible laws

- Kirchhoff-Saint Venant model:

$$\boxed{W_E(E) = \frac{\lambda}{2}(\mathrm{tr}E)^2 + \mu E \cdot E, \quad T^{(2)} = \lambda(\mathrm{tr}E)I + 2\mu E}$$

  - It is a direct extension of linear elastic behaviour
  - The energy is finite for $\det F \to 0^+, \infty$. The problem is not well-posed for large deformations.
  - Frequently used in the large-displacement small-deformations regime (e.g. slender structures such as beam and plates).

- Neo-Hookean model :

$$\boxed{W_F(F) = \frac{\mu}{2}(F \cdot F - 3) - 2\mu \ln J + \frac{\lambda}{2} \ln J^2}$$

  - The energy is polyconvex and compressible, with $W_F(F) \to \infty$ pour $\det F \to 0^+, \infty$.
  - Largely used for slightly compressible materials.

Decomposition is isochoric and volumetric energy

In many situations can be useful to decompose the energy in the contribution due to volume change (volumetric part) and those associated to pure shear (isochoric part).

- Decomposition of the transformation gradient (F) in isochoric ($\bar{F}$) and volumetric (J) parts

$$F = J^{1/3}\bar{F}, \qquad \det \bar{F} = 1$$

$$C = J^{2/3}\bar{C}, \qquad \det \bar{C} = 1$$

- Decomposition of the internal energy:

$$W(C) = W_{\mathrm{vol}}(J) + W_{\mathrm{iso}}(\bar{C})$$

- Example: volumetric growth can be modelled by prescribing a target J, say $J_0 \neq 0$ through an energy in the form:

$$W(C) = W_{\mathrm{vol}}(J/J_0) + W_{\mathrm{iso}}(\bar{C})$$

Incompressible hyperelastic materials

The volume change is null and the volumetric part of the deformation energy vanishes.

- Incompressibility constraint

$$\boxed{J = detF = 1} \Rightarrow \boxed{\frac{\partial J}{\partial F} = JF^{-T}}$$

- Relation between internal energy and stress with constraint:

$$\left(S - \frac{\partial W}{\partial F}\right) \cdot \dot{F} = 0, \quad \forall F : \frac{\partial J}{\partial F} \cdot \dot{F} = 0$$

Hence there is a constitutively undetermined term collinear to F^{-T}

$$\boxed{S = \frac{\partial W}{\partial F} - p\, F^{-T}, \quad T = -p\mathbf{1} + \frac{\partial W}{\partial F} F^{T}}$$

where p is a constant (pressure) undertermined from the constitutive laws. Its value is found via equilibrium and boundary conditions. The pressure is the Lagrange multiplier associated to the constraint.

Example of incompressible hyperelastic materials

- Neo-Hooke:

$$\boxed{W_F(F) = \frac{\mu}{2}(\mathrm{tr}(F^T F) - 3) = \frac{\mu}{2}(F \cdot F - 3)}$$

- Mooney–Rivlin:

$$\boxed{W_F(F) = \frac{\mu_1}{2}(F \cdot F - 3) + \frac{\mu_2}{2}(\mathrm{tr}(F^{-1} \cdot F^{-1}) - 3)}$$

- Ogden:

$$\boxed{W_F(F) = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3)}$$

Example: traction in plane-strain

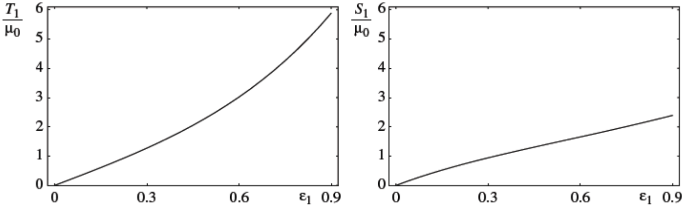


Figure 5.5. Uniaxial plane strain tension of a Mooney-Rivlin incompressible elastic material. The Cauchy and nominal stresses (normalised through division by the shear modulus at the unloaded state μ_0) are reported on the left and on the right, respectively, as functions of the logarithmic strain.

from Bigoni, Nonlinear solid mechanics

Example: traction in plane-strain

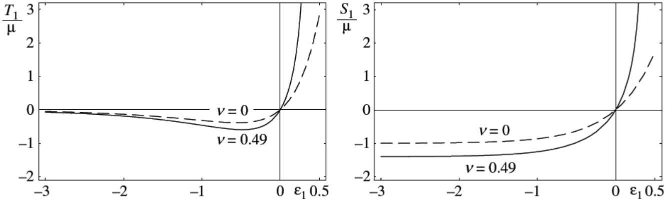


Figure 5.8. Uniaxial plane strain tension and compression of a Kirchhoff–Saint Venant material. Axial Cauchy T_1 and nominal S_1 stress (normalised through division by μ) versus the logarithmic strain ε_1. Two values of Poisson's ratio have been considered, namely, $\nu = 0$ and $\nu = 0.49$. In tension, the material becomes progressively stiff, and the stress becomes infinite when the transversal stretch tends to zero. In compression, the material exhibits softening.

from Bigoni, Nonlinear solid mechanics

Variational Formulation

5. Potential energy, variational formulation and numerical solution strategy

Content

1. Total potential energy and variational formulation of the equilibrium
2. Linearisation
3. Newton algorithm
4. Second derivative and stability
5. Eulerian buckling

Energy minimisation problem

We consider a solid with a reference configuration Ω_0, submitted to imposed displacements $\underline{u}_0$ on $\partial_u\Omega_0$, dead volume forces $\underline{b}_0$ and surface tractions $\underline{g}_0$ on $\partial_g\Omega_0$, expressed as density per unity of volume or surface in the reference configuration. The **total potential energy** is

$$\mathcal{E}(\underline{u}) = \underbrace{\int_{\Omega_0} W(I + \nabla\underline{u})\, dx_0}_{\text{elastic energy}} - \underbrace{\int_{\Omega_0} \underline{b}_0 \cdot \underline{u}\, dx_0 - \int_{\partial_g\Omega_0} \underline{g}_0 \cdot \underline{u}\, ds_0}_{\text{Potential of dead loads}}$$

The stable equilibria are the solutions of

$$\min_{\underline{u}\in\mathcal{U}} \mathcal{E}(\underline{u}), \qquad \mathcal{U} \equiv \{\underline{u} \text{ reg. } : \underline{u} = \underline{u}_0 \text{ on } \partial_u\Omega_0\}$$

Variational formulation of the equilibrium

A point u is a local minimum of $\mathcal{E}$ only if the local variation of the energy in any test direction $v \in \mathcal{U}_0 \equiv \{u \text{ reg. } : \underline{u} = \underline{0} \text{ on } \partial_u\Omega_0\}$ is not negative, i.e. only if for sufficiently small h

$$0 \leq \mathcal{E}(\underline{u} + h\,\underline{v}) - \mathcal{E}(\underline{u}) = \underbrace{\left.\frac{d\mathcal{E}(\underline{u} + h\,\underline{v})}{dh}\right|_{h=0}}_{\mathcal{E}'(\underline{u})(\underline{v})} + o(|h|)$$

Since for any $v \in \mathcal{U}_0$, $-v \in \mathcal{U}_0$, we obtain the following first order optimality conditions (stationarity) of the energy, giving the variational formulation of the equilibrium conditions:

$$\boxed{\text{Find } \underline{u} \in \mathcal{U} : \quad \mathcal{E}'(\underline{u})(\underline{v}) = 0, \quad \forall \underline{v} \in \mathcal{U}_0}$$

with

$$\begin{aligned}
\mathcal{E}'(\underline{u})(\underline{v}) &= \left.\frac{d\mathcal{E}(\underline{u} + h\,v)}{dh}\right|_{h=0} \\
&= \int_{\Omega_0} \underbrace{\frac{\partial W}{\partial F}}_{S} \cdot \nabla \underline{v}\, dx_0 - \int_{\Omega_0} \underline{b_0} \cdot \underline{v}\, dx_0 - \int_{\partial_g\Omega_0} \underline{g}_0 \cdot \underline{v}\, ds_0
\end{aligned}$$

Exercise: Show that the variational formulation above implies the Lagrangian version of the equilibrium equations.

Linearised problem (Newton algorithm)

Solving the variational equation of the previous slide is a nonlinear problem, that can be solved through successive linearizations using the Newton algorithm. Given a starting point $\underline{u}_0 \in \mathcal{U}$, we can expand as follow the equilibrium condition

$$0 = \mathcal{E}'(\underline{u}_0 + \underline{w})(\underline{v}) = \mathcal{E}'(\underline{u}_0)(\underline{v}) + \underbrace{\left.\frac{d\mathcal{E}'(\underline{u} + h\,\underline{w})(\underline{v})}{dh}\right|_{h=0}}_{\mathcal{E}''(\underline{u}_0)(\underline{v})(\underline{w})} + o(|h|)$$

and determine a tentative variation $w \in \mathcal{U}_0$ by solving the following linearized problem

$$\boxed{\text{Find } \underline{w} \in \mathcal{U}_0 : \mathcal{E}''(\underline{u}_0)(\underline{v})(\underline{w}) = -\mathcal{E}'(\underline{u}_0)(\underline{v}), \forall \underline{v} \in \mathcal{U}_0}$$

where we define the second derivative of the energy as the following symmetric bilinear form:

$$\boxed{\mathcal{E}''(\underline{u})(\underline{v})(\underline{w}) = \left.\frac{d\mathcal{E}'(\underline{u} + h\,\underline{w})(\underline{v})}{dh}\right|_{h=0}}$$

Newton algorithm

- Give an initial $\underline{u}_0$ and set $i = 0$
- While $\texttt{err} > \texttt{tol}$ and $\texttt{i} < \texttt{i}_{\texttt{max}}$:
  1. Solve the linearized problem

$$\boxed{\text{Find } \underline{w} \in \mathcal{U}_0 : \mathcal{E}''(\underline{u}_0)(\underline{v})(\underline{w}) = -\mathcal{E}'(\underline{u}_0)(\underline{v}), \forall \underline{v} \in \mathcal{U}_0}$$

  2. Update

$$\begin{array}{rcl} u_0 & \leftarrow & u_0 + w \\ i & \leftarrow & i + 1 \\ \texttt{err} & \leftarrow & \|\texttt{assemble}(\mathcal{E}'(\underline{u}_0)(\underline{v}))\| \end{array}$$

We need to calculate the first and second derivatives of the energy.

Derivatives of the strain energy density

- In terms of F and S, $W(F(u))$

$$W'(u)(v) = \frac{\partial W}{\partial F} \cdot F'(u)(v) = S \cdot \nabla v$$
$$W''(u)(v)(w) = \left(\frac{\partial S}{\partial F} \nabla w\right) \cdot \nabla v = \left(\frac{\partial S}{\partial F}\right) \cdot (\nabla v \otimes \nabla w)$$

- In terms of E and $T^{(2)}$, $W(E(u))$

$$W'(u)(v) = \frac{\partial W}{\partial E} \cdot E'(u)(v) = T^{(2)} \cdot \mathrm{sym}(F^T \nabla v) = T^{(2)} \cdot (F^T \nabla v)$$
$$W''(u)(v)(w) = \left(\frac{\partial T^{(2)}}{\partial E} (F^T \nabla w)\right) \cdot (F^T \nabla v) + T^{(2)} \cdot (\nabla w^T \nabla v)$$

where we introduce the elastic tangent stiffness (fourth-order) tensor

$$\mathbb{C}_E = \frac{\partial T^{(2)}}{\partial E}$$

Second derivative of the energy functional

Using the result of the previous slide we can write the first and second derivative of the strain energy

$$
\begin{aligned}
\mathcal{E}'(\underline{u})(\underline{v}) &= \int_{\Omega_0} T^{(2)} \cdot (F^T \nabla v)\, \mathrm{d}x_0 \\
\mathcal{E}''(\underline{u})(\underline{v})(\underline{w}) &= \underbrace{\int_{\Omega_0} \mathbb{C}_E (F^T \nabla w) \cdot (F^T \nabla v)\, \mathrm{d}x_0}_{\text{elastic stiffness}} + \int_{\Omega_0} \underbrace{T^{(2)} \cdot (\nabla w^T \nabla v)}_{\text{geometric stiffness}}\, \mathrm{d}x_0
\end{aligned}
$$

where $T^{(2)}$, F, and $\mathbb{C}_E$ depend on u.

The UFL components of FEniCS allows us to define the directional derivatives using symbolic automatic differentiation. This is the syntax:

```
u, v, w = Function(V), TestFunction(V), TrialFunction(V)
energy = function_that_you_have_to_write(u)
denergy_v = derivative(u,v)
ddenergy_v_w = derivative(denergy_v,w)
```

Stability

Given a state u_0 solution of the equilibrium problem $(\mathcal{E}'(u_0)(v) = 0)$, the stability of the equilibrium can be assessed by looking at the second order optimality condition:

$$0 \leq \mathcal{E}(\underline{u} + h\,\underline{v}) - \mathcal{E}(\underline{u}) = \underbrace{\left.\frac{d\mathcal{E}(\underline{u} + h\,\underline{v})}{dh}\right|_{h=0}}_{\mathcal{E}'(\underline{u})(\underline{v})=0} + \underbrace{\left.\frac{d^2\mathcal{E}(\underline{u} + h\,\underline{v})}{dh^2}\right|_{h=0}}_{\mathcal{E}''(\underline{u})(\underline{v})=\mathcal{E}''(\underline{u})(\underline{v})(\underline{v})} + o(h^2)$$

hence to the sign of the second derivative

$$H(\underline{u}) = \mathcal{E}''(\underline{u})(\underline{v}) = \mathcal{E}''(\underline{u})(\underline{v})(\underline{v})$$

After a finite element discretisation H is a matrix, and its sign can be assessed by looking at the sign of the smallest eigenvalue. If the smallest eigenvalue is positive, the equilibrium is stable; if it is negative the equilibrium is unstable.
Special packages (e.g. `SLEPc`) provide efficient parallel algorithms to solve eigenvalue problems and find the n smallest eigenvalues. However this is an expensive operation in term of computational resources.

Euler buckling revisited

Consider a stiff prismatic solid under uniaxial compression. The stress tensor is in the form

$$T^{(2)} = \lambda\, \underline{e}_1 \otimes \underline{e}_1$$

Let us assume that the solid is sufficiently stiff and neglect the displacement and strain along the fundamental solution: $u_0 \simeq 0$, $F(u_0) \simeq I$. Hence

$$\mathcal{E}''(\underline{u})(\underline{v})(\underline{w}) \simeq \underbrace{\int_{\Omega_0} \mathbb{C}_E(\nabla w)\cdot(\nabla v)\,\mathrm{d}x_0}_{\text{elastic stiffness } K(w,\,v)} + \lambda \underbrace{\int_{\Omega_0} ((\nabla w^T \nabla v)\underline{e}_1)\cdot \underline{e}_1 \mathrm{d}x_0}_{\text{geometric stiffness } G(w,\,v)}$$

Solving the following generalised eigenvalue problem

$$\boxed{\text{Find } \lambda < 0, \quad w \in \mathcal{U}_0 : \quad K(w,v) + \lambda\, G(w,v) = 0, \quad \forall v \in \mathcal{U}_0}$$

gives the critical buckling loads λi and the corresponding buckling modes w^i.

Exercice: How is the problem modified in the case of soft solids, where we cannot neglect the deformation before buckling?

Recent research on instability in hyperelastic solids

Examples of elastic instabilities due to growth

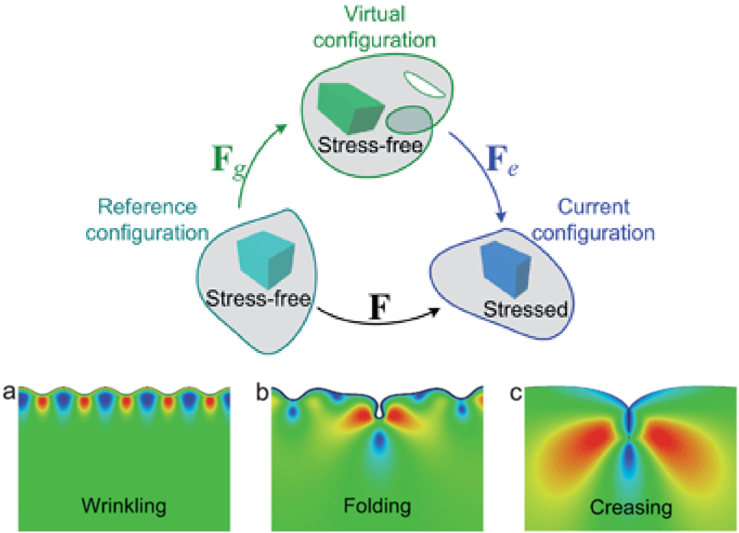
and possible subjects for the final project!

Recent review papers:

- B.Li, Y.P.Cao, X.-Q.Feng, H.Gao, Mechanics of morphological instabilities and surface wrinkling in soft materials: a review, Soft Matter, 2012, 8, 5728-5745, DOI: 10.1039/C2SM00011C
- Z.Liu, S.Swaddiwudhipong, W.Hong, Pattern formation in plants via instability theory of hydrogels, Soft Matter, 2013, 9, 577-587, 10.1039/C2SM26642C (Paper)

Examples of elastic instabilities due to growth

Typical phenomena:



Examples of elastic instabilities due to growth

B. Li, H. P. Zhao and X. Q. Feng, J. Mech. Phys. Solids, 2011, 59, 610–624

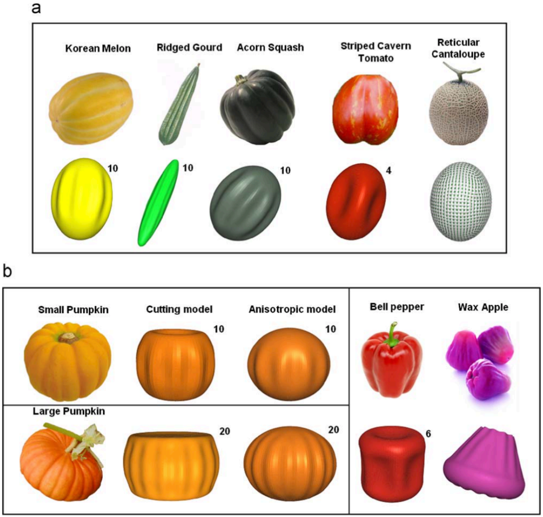
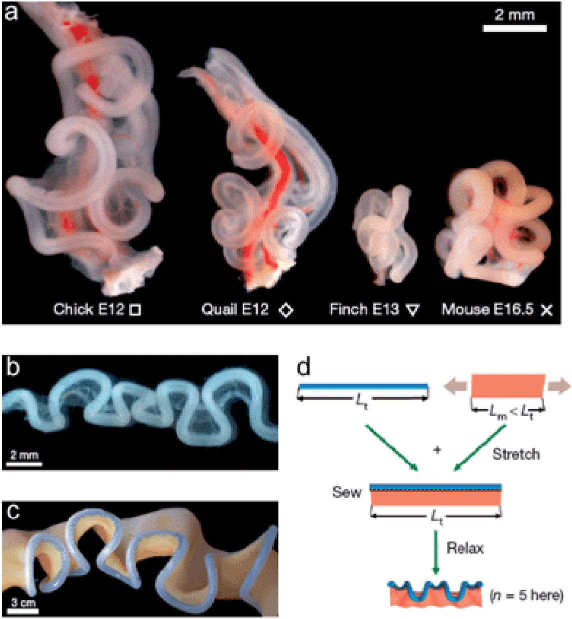


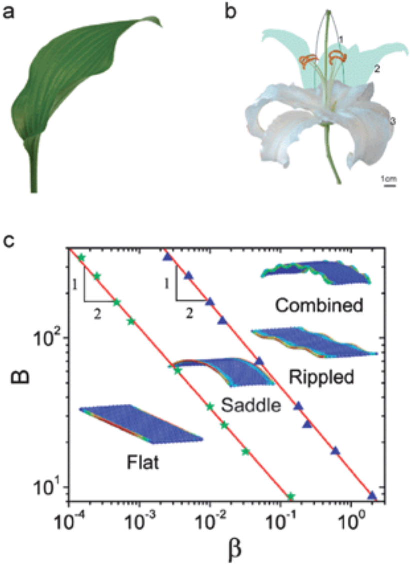
Fig. 9. (a) Implications for fruit morphogenesis: the morphologies of several fruits and vegetables are compared with the simulated buckle shapes of model spheroids. The effective geometry/material parameters used for simulation are: Korean melon ($R/t = 15$, $k = 1.3$, $\bar{E}_f/\bar{E}_s = 30$), ridged (silk) gourd ($R/t = 4$, $k = 5$, $\bar{E}_f/\bar{E}_s = 30$), acorn squash ($R/t = 17$, $k = 1.2$, $\bar{E}_f/\bar{E}_s = 30$), striped cavern tomato ($R/t = 5$, $k = 1.3$, $\bar{E}_f/\bar{E}_s = 100$), bell pepper ($R/t = 8$, $k = 1.3$, $\bar{E}_f/\bar{E}_s = 100$), reticular cantaloupe ($R/t = 75$, $k = 1.3$, $\bar{E}_f/\bar{E}_s = 5$). (b) Implications for fruit morphogenesis with derivational or anisotropic spheroids. Small pumpkin simulant by cutting geometrical model ($a/t = 20$, $k' = 0.8$, $\bar{E}_f/\bar{E}_s = 30$) and by anisotropic growth model ($R/t = 25$, $k = 0.8$, $\bar{E}_f/\bar{E}_s = 30$, grow along hoop direction). Large pumpkin simulant by cutting model ($a/t = 50$, $k' = 0.6$, $\bar{E}_f/\bar{E}_s = 30$) and by anisotropic growth model ($R/t = 65$, $k = 0.8$, $\bar{E}_f/\bar{E}_s = 30$, grow along hoop direction). Wax apple is approximated as a cone with $\bar{E}_f/\bar{E}_s = 20$, cone angle of 30°, and $q/t = 30$.

Examples of elastic instabilities due to growth

T. Savin, N. A. Kurpios, A. E. Shyer, P. Florescu, H. Liang, L. Mahadevan and C. J. Tabin, Nature, 2011, 476, 57–62



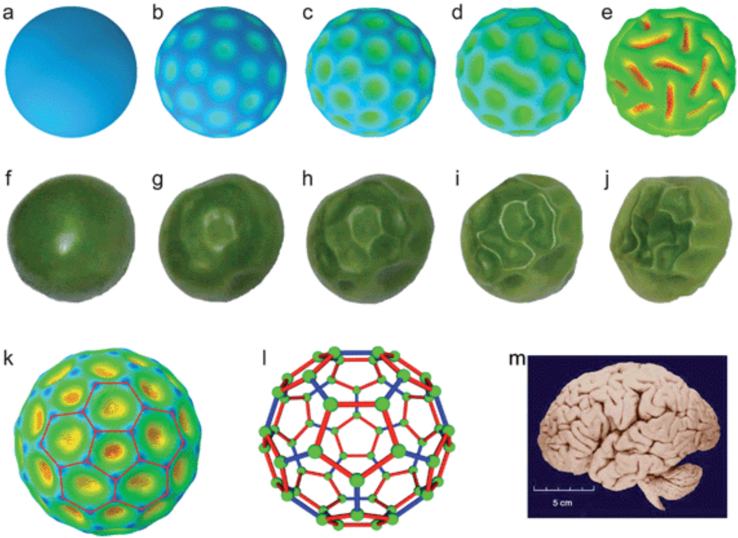
Examples of elastic instabilities due to growth



H. Liang and L. Mahadevan, Proc. Natl. Acad. Sci. U. S. A., 2009, 106, 22049–22054

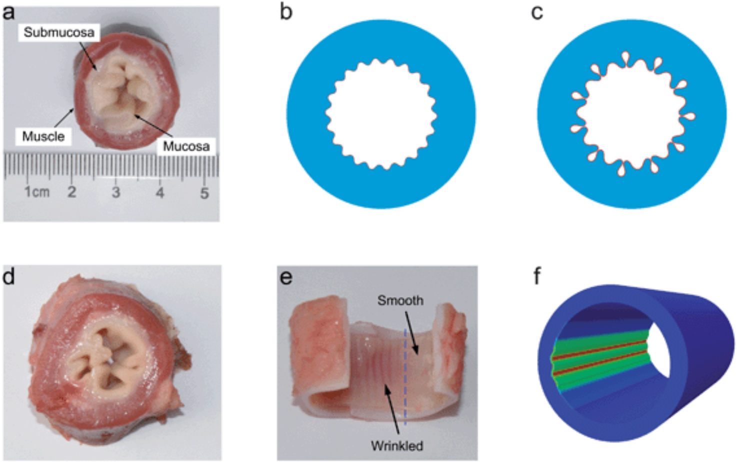
Examples of elastic instabilities due to growth

S. Hill and C. A. Walsh, Nature, 2005, 437, 64–67



Examples of elastic instabilities due to growth

B. Li, Y. P. Cao, X. Q. Feng and H. Gao, J. Mech. Phys. Solids, 2011, 59, 758–774



Some useful formulas

Exemples: Kirchhoff-Saint Venant

$$
\begin{aligned}
W_E(E) &= \frac{\lambda}{2}(\mathrm{tr}(E))^2 + \mu E \cdot E \\
T^{(2)} &= \lambda\,\mathrm{tr}(E) I + 2\mu E \\
\mathbb{C}[E] &= \lambda\,\mathrm{tr}(E) I + 2\mu E
\end{aligned}
$$

Exemples: Neo-Hooke incompressible

$$
\begin{aligned}
W_C(C) &= \frac{\mu}{2}(\mathrm{tr}(C) - 3) - p(J-1)\\
W_E(E) &= \mu\,\mathrm{tr}(E) - p(J-1)\\
W_F(F) &= \frac{\mu}{2}(F\cdot F - 3) - p(J-1)\\
T^{(2)} &= \mu\, I - pJC^{-1}\\
S &= \mu F - pJF^{-T}\\
\mathbb{C}_E[E] &= pJC^{-1}(\mathrm{D}\,C[E] - E)\,C^{-1}\\
\mathbb{C}_F[\hat{F}] &= \mu\hat{F} - p\,J\,F^{-T}(\hat{F} - \hat{F}^T)F^{-T}
\end{aligned}
$$

Relations utiles:

$$
\frac{\partial J}{\partial F} = JF^{-T}, \quad \frac{\partial J}{\partial C} = \frac{1}{2}JC^{-1}, \quad \frac{\partial J}{\partial E} = JC^{-1}, \qquad \mathrm{D}\,C^{-1}[E] = -C^{-1}\mathrm{D}\,C[E]\,C^{-1}
$$

Exemples: Neo-Hooke compressible (à verifier et completer)

$$
\begin{aligned}
W_C(C) &= \frac{\mu}{2}(\mathrm{tr}(C) - 3) - 2\mu \ln J + \frac{\lambda}{2} \ln J^2 \\
W_E(E) &= \mu\, \mathrm{tr}(E) - 2\mu \ln J + \frac{\lambda}{2} \ln J^2 \\
W_F(F) &= \frac{\mu}{2}(F \cdot F - 3) - 2\mu \ln J + \frac{\lambda}{2} \ln J^2 \\
T^{(2)} &= \dots \\
S &= \mu F + g(J)\frac{\partial J}{\partial F}, \quad g(J) = (-2\mu + \lambda \ln J) J^{-1} \\
\mathbb{C}_E[E] &= \dots \\
\mathbb{C}_F[\hat{F}] &= \mu \hat{F} + g'(J)\frac{\partial J}{\partial F}\hat{F}\frac{\partial J}{\partial F} + g(J)\frac{\partial^2 J}{\partial F^2}\hat{F}
\end{aligned}
$$

Relations utiles:

$$
\frac{\partial J}{\partial F} = JF^{-T}, \quad \frac{\partial J}{\partial C} = \frac{1}{2}JC^{-1}, \quad \frac{\partial J}{\partial E} = JC^{-1}, \qquad \mathrm{D}\, C^{-1}[E] = -C^{-1}\mathrm{D}\, C[E]C^{-1}
$$

$$
\frac{\partial^2 J}{\partial F^2}\hat{F} = \frac{\partial J}{\partial F}\hat{F}F^{-T} - JF^{-T}\hat{F}^T F^{-T},
$$