ASSIGNED FORUM PROBLEM 5

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Chapter 10.10: *Applications of Taylor Series*

Use series to approximate the values with an error of magnitude less than 10^{-8}.

$$\textbf{19.} \quad \int_0^{0.1} \frac{sinx}{x}\, dx \tag{1}$$

How?

First thing I did when I looked at this problem was to list everything I can remember about this problem. What did we learn in Unit 5? In Lesson 21, we learned that power series "can be integrated term by term." In Lesson 22, we learned that $sinx$ is a Taylor series, which in turn tells us that $sinx$ is also a power series because a Taylor series is also a power series at the same time. Thus, here's what I thought: we're going to find how to represent this problem in the form of a Taylor series, and we'll integrate that term by term.

On the last page of our assigned reading, which was Chapter 10.10 *Applications of Taylor Series* page 640, we learned that $sinx$ is a frequently used Taylor series in the form of

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \tag{2}$$

Let's use this to find the Taylor series representation of our problem. We can do that just by multiplying the above Taylor series with $\frac{1}{x}$

$$\textbf{19.} \quad \int_0^{0.1} \frac{sinx}{x}\, dx = \int_0^{0.1} \frac{1}{x}\, sinx\, dx = \int_0^{0.1} \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\, dx \tag{3}$$

$$= \int_0^{0.1} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\, dx \tag{4}$$

Before diving into the process of evaluating the above integral, let me quickly describe what the overall process of solving this problem would look like. We'll first integrate term by term. We'll be able to approximate using the Alternating Series Estimation Theorem, and then we'll verify our answer by plotting $\frac{sinx}{x}$ on a graph and then looking at its area.

Steps

$$\textbf{19.} \quad \int_0^{0.1} \frac{sinx}{x}\,dx = \int_0^{0.1} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\,dx \tag{5}$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \right]_0^{0.1} \tag{6}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (0.1)^{2n+1}}{(2n+1)(2n+1)!} \tag{7}$$

$$= 0.1 - \frac{0.1^3}{3\cdot 3!} + \frac{0.1^5}{5\cdot 5!} - \frac{0.1^7}{7\cdot 7!} + \cdots \tag{8}$$

Approximating

Now, Alternating Series Estimation Theorem tells us that in an alternating series - just like what we have above - the greatest error value that its total sum can have is the first neglected term. So, that's what we're gonna do. We'll keep plugging in to the summation formula we have above, until we find an n^{th} term that has what our problem requires us to have – which is an error of magnitude less than 10^{-8}.

We could do this by using Maple too, just so you know. Go to Lesson 22. There's a maple example file for showing how to "approximate the value of using a power series with an error less than 0.001" I, however, found the n^{th} term just by plugging in the first few values.

Here's what I got. $n_3 = 0.00000001\overline{6}$ This is very close to the answer we need, but it has 7 leading zeros, so it's just a little bit short of what the problem asks us to have. Thus, going to the next term, we have $n_4 = 0.00000000000283\ldots$ Thus, we just found that the summation of the first three terms is what we're looking for.

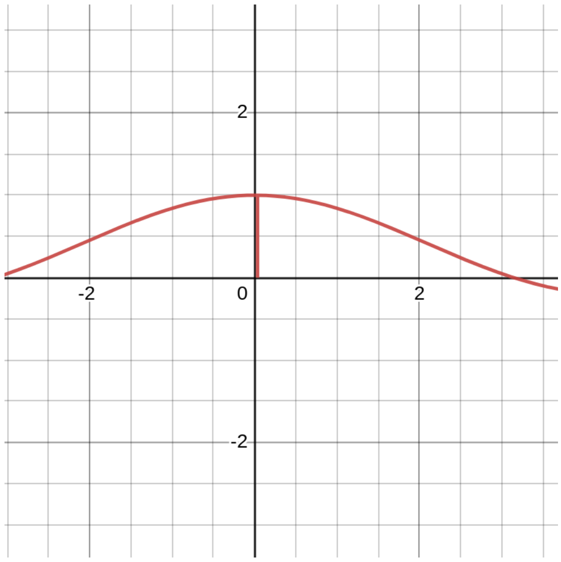
In conclusion, the answer to our problem is this:

$$\textbf{19.} \quad \int_0^{0.1} \frac{sinx}{x}\,dx = 0.1 - \frac{0.1^3}{3\cdot 3!} + \frac{0.1^5}{5\cdot 5!} \tag{9}$$

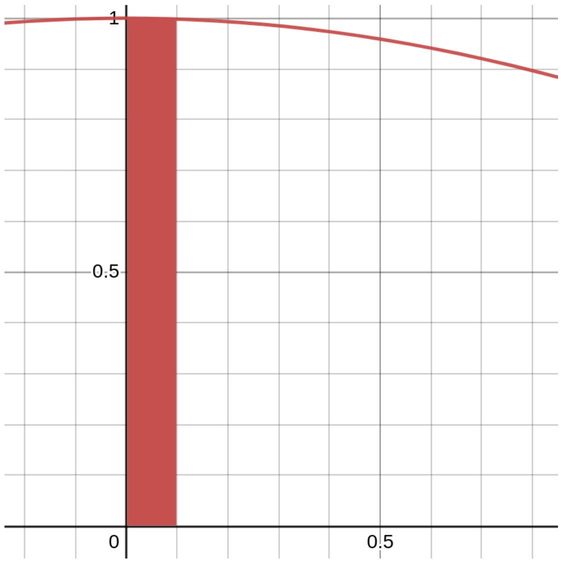
$$\approx 0.09994446\overline{1} \tag{10}$$

Verification

Here's how $\frac{sinx}{x}$ looks like.



And here's the same graph zoomed in to the area we need.



Approximately, the area under the curve - the red rectangle - has the width of 0.1 and the height of around 1, which means its area is $0.1 \cdot 1 \approx 0.1$ This indeed confirms - at least geometrically - that the answer we got is correct:

$$\textbf{19.} \quad \int_0^{0.1} \frac{sinx}{x}\, dx = 0.1 - \frac{0.1^3}{3 \cdot 3!} + \frac{0.1^5}{5 \cdot 5!} \tag{11}$$

$$\approx 0.09994446\overline{1} \tag{12}$$