HOMEWORK 3.4

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November 11, 2022

Proofs by Contradiction or Contrapositive

B3-B9. Do proofs by contrapositive or contradiction.

B3. $Conjecture: \ If \ for \ all \ \epsilon > 0, \ x < c + \epsilon, \ then \ x \le c.$

This can be proven with proof by contrapositive $\forall x, c \in \mathbb{R}, \ \ x > c \Rightarrow \exists\, \epsilon > 0 \ \ s.t. \ \ x \geq c + \epsilon$

$$\begin{aligned}
&Let \ x, c \in \mathbb{R}, \ where \ \ x > c \\
&Choose \ \ \epsilon = x - c \\
&Since \ \ x > c, \ x - c > 0 && \text{(Theorem 3.1.3)} \\
&Also, \ c + \epsilon = c + (x - c) = x \\
&Therefore, \ c + \epsilon \leq x && \square
\end{aligned}$$

B4. $Prove: \ There \ are \ no \ integers \ a \ and \ b \ such \ that \ b^2 = 4a + 2.$

Proof by contradiction:

$$\begin{aligned}
& Suppose \ \exists\, a, b \in \mathbb{Z} \ \ s.t. \ \ b^2 = 4a + 2 && \text{(Means of contradiction)} \\
& b^2 = 4a + 2 \\
\Leftrightarrow\ & b^2 = 2(2a + 1) \\
\Rightarrow\ & \exists\, k \in \mathbb{Z} \ \ s.t. \ \ b^2 = 2k, \ where \ \ k = 2a + 1 && \text{(Definition of even)} \\
\Rightarrow\ & b^2 \ \ is \ \ an \ \ even \ \ number. \\
& Since \ \ b^2 \ \ is \ \ even, \ b \ \ is \ \ even. && \text{(Theorem 2.2.15)} \\
\Rightarrow\ & \exists\, j \in \mathbb{Z} \ \ s.t. \ \ b = 2j \\
& However, \ b^2 = (b)^2 = (2j)^2 = 4j^2 = 2k = 4a + 2 \\
\Leftrightarrow\ & 4j^2 = 4a + 2 \\
\Leftrightarrow\ & j^2 = 2a + 1 && \text{(Definition of odd)} \\
\Rightarrow\ & Odd \ \ and \ \ even \ \ at \ \ the \ \ same \ \ time. && \square
\end{aligned}$$

B6. *Prove* : *If* n^3 *is even, so is* n.

Proof by contrapositive $\exists\, j \in \mathbb{Z}$ *s.t.* $n = 2j + 1 \Rightarrow \exists\, k \in \mathbb{Z}$ *s.t.* $n^3 = 2k + 1$

$$\begin{aligned}
& Let\ \ j \in \mathbb{Z}\ \ and\ \ choose\ \ n = 2j+1 && \text{(By hypothesis)}\\
& Then,\ n = 2j+1 && \\
\Leftrightarrow\ & n^3 = (2j+1)^3 && \\
\Leftrightarrow\ & n^3 = 2(2j^3 + 5j^2 + 3j) + 1 && \text{(Definition of odd)}\\
& Thus,\ \ n^3 = 2k+1 = 2(2j^3+5j^2+3j)+1 && \square
\end{aligned}$$

B7. *Prove* : *Let* a *and* b *be real numbers. If* $a > 0$ *and* $b > 0$, *then* $\frac{2}{a} + \frac{2}{b} \neq \frac{4}{a+b}$

Proof by contradiction:

$$\begin{aligned}
& Suppose\ \ a > 0,\ b > 0,\ \frac{2}{a} + \frac{2}{b} = \frac{4}{a+b} && \text{(Means of contradiction)}\\
& \frac{2}{a} + \frac{2}{b} = \frac{4}{a+b} && \\
\Leftrightarrow\ & \frac{2b+2a}{ab} = \frac{4}{a+b} && \\
\Leftrightarrow\ & (2b+2a)(a+b) = 4ab && \\
\Leftrightarrow\ & 2(a^2 + 2ab + b^2) = 4ab && \\
\Leftrightarrow\ & a^2 + 2ab + b^2 = 2ab && \\
\Leftrightarrow\ & a^2 + b^2 = 0 && \\
& However,\ since\ \ a > 0\ \ and\ \ b > 0,\ a^2 > 0\ \ and\ \ b^2 > 0 && \text{(Axiom 3.1.1C)}\\
& So,\ a^2 + b^2 > 0 && \text{(Axiom 3.1.1B)}\\
& Thus, a > 0\ \ and\ \ b > 0 \Rightarrow \frac{2}{a} + \frac{2}{b} \neq \frac{4}{a+b} && \square
\end{aligned}$$

B9. $Prove: \ \sqrt{3} \ is \ not \ a \ rational \ number.$

This can be proven with proof by contradiction – i.e. supposing that $x^2 = 3 \Rightarrow x \ \ is \ \ rational.$

$$
\begin{aligned}
& Suppose \ \ x^2 = 3 \ \ and \ \ x \ \ is \ \ rational && \text{(Means of contradiction)} \\
& Then, \ x = \frac{m}{n}, \ where \ \ m, n \in \mathbb{Z} && \text{(Definition of rational)} \\
& Also, \ m \ \ and \ \ n \ \ have \ \ no \ \ common \ \ factor. && \\
& (\frac{m}{n})^2 = 3 && \text{(By hypothesis)} \\
\Leftrightarrow \ & m^2 = 3n^2 && \text{(Definition of even)} \\
\Rightarrow \ & m^2 \ \ is \ \ even. && \\
\Rightarrow \ & m \ \ is \ \ even. && \text{(Theorem 2.2.15)} \\
& m^2 = 3n^2 && \\
& test && \square
\end{aligned}
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