ASSIGNED FORUM PROBLEM 3

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Chapter 8.5: *Integration of Rational Functions by Partial Fractions*

$$\textbf{41.} \quad \int \frac{cosy}{sin^2y + siny - 6}dy \tag{1}$$

How Can We Solve This Problem?

Since this problem is on Chapter 8.5: *Integration of Rational Functions by Partial Fractions*, I thought we'd have to use an integration technique involving partial fractions, which we will learn in *Lesson 14*. However, since we haven't learned about partial fractions yet - as of June 28, 2022 - I learned by trial and error that this problem can be solved by using u-substitution.

Solution

Let's start backwards. Here's the solution:

$$\textbf{41.} \quad \int \frac{cosy}{sin^2y + siny - 6}dy = \frac{2}{\sqrt{23}}tan^{-1}\left(\frac{2}{\sqrt{23}}(siny + \frac{1}{2})\right) + C$$

As you can see, the integral - or the solution - of this problem is in the form of $tan^{-1}x + C$. This is because it turns out that the problem can be rewritten in what we learned in *Lesson 5* as the integrand of an inverse trigonometric function. We learned in *Lesson 5* that if we differentiate an inverse trigonometric function like $tan^{-1}x$, we get $\frac{1}{1+x^2}$. In other words, $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$. Now, what this means is that $\tan^{-1} x$ should also be the integral of $\frac{1}{1+x^2}$.

This is the reason why the following equation, which I got from from *Lesson 5*, holds true:

$$\int \frac{1}{a^2 + u^2}du = \frac{1}{a}\tan^{-1}(\frac{u}{a}) + C$$

Now, let's go back to our problem. What we'll see is that **41.** $\int \frac{cosy}{sin^2y+siny-6}dy$ can be rewritten with u-substitution and with completing the square. It can eventually be rewritten in the form of $\int \frac{1}{1+u^2}du$; hence, the solution is in the form of $tan^{-1}u + C$.

Steps

Let's do u-substitution and complete the square:

41.
$$\int \frac{cosy}{sin^2y + siny - 6}dy = \int \frac{cosy}{u^2+u+6}dy \qquad (u = siny)$$

$$= \int \frac{cosy}{u^2+u+6} \cdot \frac{du}{cosy} \qquad (dy = \frac{du}{cosy})$$

$$= \int \frac{1}{u^2+u+6}du$$

$$= \int \frac{1}{(u+\frac{1}{2})^2 + \frac{23}{4}}du \qquad \text{(completing the square)}$$

$$= \int \frac{1}{w^2 + \frac{23}{4}}du \qquad (w = u + \frac{1}{2})$$

$$= \int \frac{1}{w^2 + \frac{23}{4}}dw \qquad (du = dw)$$

$$= \int \frac{1}{\frac{23}{4}(\frac{4}{23}w^2 + 1)}dw \qquad \text{(algebraically rewriting)}$$

$$= \int \frac{4}{23} \cdot \frac{1}{\frac{4}{23}w^2 + 1}dw$$

$$= \frac{4}{23} \int \frac{1}{\frac{4}{23}w^2 + 1}dw$$

$$= \frac{4}{23} \int \frac{1}{z^2+1}dw \qquad (z = \frac{2}{\sqrt{23}}w)$$

$$= \frac{4}{23} \int \frac{1}{z^2+1} \cdot \frac{\sqrt{23}}{2}dz \qquad (dw = \frac{\sqrt{23}}{2}dz)$$

$$= \frac{4}{23} \cdot \frac{\sqrt{23}}{2} \int \frac{1}{z^2+1}dz$$

$$= 2 \cdot \frac{23^{\frac{1}{2}}}{23^1} \int \frac{1}{z^2+1}dz$$

$$= 2 \cdot 23^{\frac{1}{2}-1} \int \frac{1}{z^2+1}dz$$

$$= 2 \cdot 23^{-\frac{1}{2}} \int \frac{1}{z^2+1}dz$$

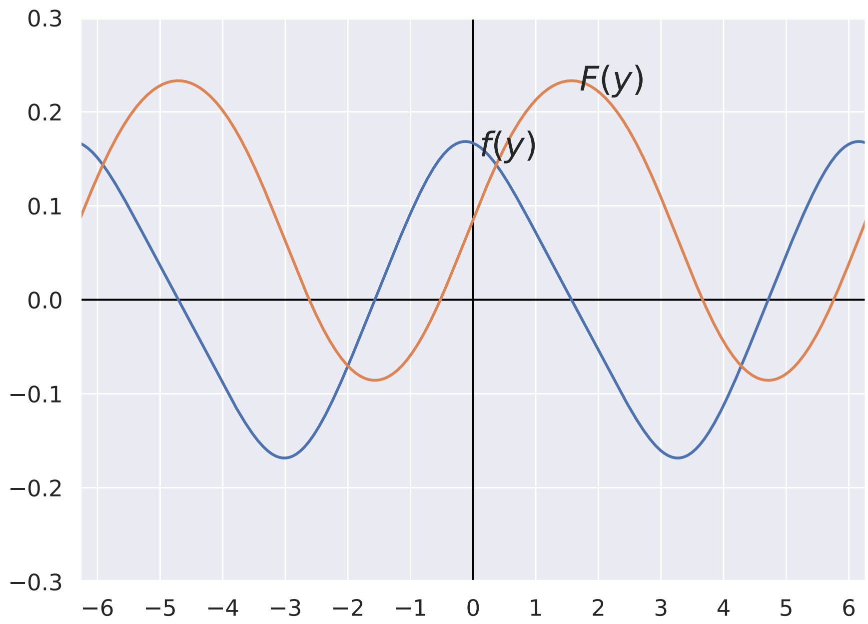
$$= \frac{2}{\sqrt{23}} \int \frac{1}{z^2+1}dz$$

$$= \frac{2}{\sqrt{23}}tan^{-1}z + C \qquad \left(\int \frac{1}{a^2+u^2}du = \frac{1}{a}\tan^{-1}(\frac{u}{a}) + C\right)$$

Finally, we can substitute back $u = siny$, $w = u + \frac{1}{2}$, and $z = \frac{2}{\sqrt{23}}$.

$$\textbf{41.} \quad \int \frac{cosy}{sin^2y + siny - 6}dy = \frac{2}{\sqrt{23}}tan^{-1}z + C$$
$$= \frac{2}{\sqrt{23}}tan^{-1}\left(\frac{2}{\sqrt{23}}w\right) + C$$
$$= \frac{2}{\sqrt{23}}tan^{-1}\left(\frac{2}{\sqrt{23}}(u + \frac{1}{2})\right) + C$$
$$= \frac{2}{\sqrt{23}}tan^{-1}\left(\frac{2}{\sqrt{23}}(siny + \frac{1}{2})\right) + C$$

Geometrically Verifying the Integral



Although I don't exactly know how to verify an integral geometrically, let's try to see if the integrand and the integral at least make any intuitive sense. One thing I've noticed from the graph above is that when $f(y)$ is at 0, $F(y)$ is either at its local minimum or maximum. This is one of the properties we learned in Calculus I: $f(y)$ is the derivative of $F(y)$.

Analytically Verifying the Integral

In fact, we know from the Fundamental Theorem of Calculus that $F\prime(x) = f(x)$.[1] So, we can analytically verify our integral by differentiating it. The derivative of the integral should be the same as the integrand of the original problem. Let's use *Maple* to differentiate our integral:

$$diff\left(\frac{2}{\sqrt{23}}\cdot\arctan\left(\frac{2}{\sqrt{23}}\cdot\left(\sin(y)+\frac{1}{2}\right)\right), y\right)$$

$$\frac{4\cos(y)}{23\left(\frac{4\left(\sin(y)+\frac{1}{2}\right)^2}{23}+1\right)} \tag{1}$$

$\overset{\text{simplify trig}}{=}$

$$-\frac{\cos(y)}{\cos(y)^2-\sin(y)-7} \tag{2}$$

Conclusion

The derivative of the integral indeed happens to be the integrand of our original problem. Therefore, we can finally conclude:

$$\textbf{41.}\quad \int \frac{cosy}{sin^2y+siny-6}dy = \frac{2}{\sqrt{23}}tan^{-1}\left(\frac{2}{\sqrt{23}}(siny+\frac{1}{2})\right)+C$$

Again, although we solved this problem using u-substitution and completing the square and using the integral of an $arctan(x)$ function, this problem comes from Chapter 8.5: *Integration of Rational Functions by Partial Fractions*, which means there must be another way of solving this problem by using the partial fractions technique.

See you later on AFP4. Thank you for your time!

[1]Hass et al., *Thomas' calculus* (14th ed.), Pearson, p. 280.