ASSIGNED FORUM PROBLEM 4

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Chapter 8.8: *Improper Integrals*

$$\textbf{33.} \quad \int_{-1}^{\infty} \frac{d\theta}{\theta^2 + 5\theta + 6} \tag{1}$$

Solution

The first thing that comes to my mind when I look at this problem is that the denominator can be factored into $(\theta + 3)(\theta + 2)$ which means our problem becomes

$$\textbf{33.} \quad \int_{-1}^{\infty} \frac{d\theta}{\theta^2 + 5\theta + 6} = \int_{-1}^{\infty} \frac{d\theta}{(\theta + 3)(\theta + 2)} \tag{2}$$

After that, we can use partial fraction decomposition to make the integrand easier to integrate.

$$\int_{-1}^{\infty} \frac{d\theta}{(\theta + 3)(\theta + 2)} = \int_{-1}^{\infty} \frac{A}{\theta + 3} + \frac{B}{\theta + 2} d\theta \tag{3}$$

We can then multiply both sides of the equation with $(\theta+3)(\theta+2)$. For the sake of readability, let's skip the integral notation for now and look at only the coefficients now.

$$1 = A(\theta + 2) + B(\theta + 3) \tag{4}$$

$$(0)\theta + (1) = (A + B)\theta + (3A + 2B) \tag{5}$$

Solving for these two equations $A + B = 0$ and $3A + 2B = 1$ gives us

$$A = 1 \tag{6}$$

$$B = -1 \tag{7}$$

In other words, if we put everything back to the original equation together, we get something that's much more easier to integrate:

$$\int_{-1}^{\infty} \frac{d\theta}{(\theta + 3)(\theta + 2)} = \int_{-1}^{\infty} \frac{1}{\theta + 3} + \frac{-1}{\theta + 2} d\theta$$

Well, why is it easier to integrate? It's because $\int \frac{1}{x}$ is elegantly $ln|x| + C$.

$$\int_{-1}^{\infty} \frac{d\theta}{(\theta+3)(\theta+2)} = \Big[ln|\theta+2| - ln|\theta+3| \Big]_{-1}^{\infty} \tag{8}$$

$$= ln2 \tag{9}$$

Sorry! I skipped over an important part of how this works. If you look at the equation above, we have an integral with infinity. Specifically, the function we have here is what we call Type 1 improper integral. By the way, there are two types of improper integrals.

Type 1 improper integral refers to integrals like ours, where one or both of the bounds have infinity as in $\int_{-1}^{\infty}$

Type 2 improper integral, on the other hand, refer to integrals whose bounds have a discontinuity between them. Anyway, our function doesn't have any discontinuity, so let's see how to evaluate a Type 1 improper integral.

First of all, from Lesson 16 *Integrals with the infinite*, we learned that

$$\int_{-1}^{\infty} f(\theta) = \Big[\lim_{d\to\infty} f(\theta) \Big] - \Big[f(-1) \Big] \tag{10}$$

Therefore, our problem is equal to

$$= \Big[\lim_{d\to\infty} ln|d+2| - ln|d+3| \Big] - \Big[ln|(-1)+2| - ln|(-1)+3| \Big] \tag{11}$$

$$= \Big[\lim_{d\to\infty} ln\frac{|d+2|}{|d+3|} \Big] - \Big[ln|1| - ln|2| \Big] \tag{12}$$

$$= \Big[\lim_{d\to\infty} ln\frac{d+2}{d+3} \Big] - \Big[-ln2 \Big] \tag{13}$$

Now, we can use L'Hôpital's rule that we learned from Lesson 15 *Limits with the infinite.*

$$\Big[\lim_{d\to\infty} ln\frac{d+2}{d+3} \Big] = \Big[\lim_{d\to\infty} ln\frac{d+2}{d+3} \cdot \frac{\frac{d}{d}}{\frac{d}{d}} \Big] \tag{14}$$

$$= \Big[\lim_{d\to\infty} ln\frac{1+\frac{2}{d}}{1+\frac{3}{d}} \Big] \tag{15}$$

$$= ln\frac{\lim_{d\to\infty} \big[1+\frac{2}{d}\big]}{\lim_{d\to\infty} \big[1+\frac{3}{d}\big]} \tag{16}$$

$$= ln\frac{1}{1} \tag{17}$$

$$= 0 \tag{18}$$

Putting this back to the original problem, we finally get

$$\textbf{33.} \quad \int_{-1}^{\infty} \frac{d\theta}{\theta^2+5\theta+6} = \Big[0 \Big] - \Big[-ln2 \Big] = ln2 \tag{19}$$

Verifying the Integral

Let's verify our integral by differentiating the integrand and see if both are equal to each other.

On Maple, let's type the following, which basically means differentiate $ln|\theta+2| - ln|\theta+3|$ which is the integral we got: $\mathrm{diff}\Big(\ln(\mathrm{abs}(\theta+2)) - \ln(\mathrm{abs}(\theta+3))\Big)$ Then, the output is the same as the integrand of our original question, which is $\frac{1}{\theta^2+5\theta+6}$

Therefore, we can finally confirm that at least the indefinite version of our answer is a correct answer to our problem. To be honest, however, I think there must be a better way of verifying our answer. One thought I came across was to use Simpson's Rule - which we learned in June, 2022 - to approximate the definite integral of **33.** $\int_{-1}^{\infty} \frac{d\theta}{\theta^2+5\theta+6}$ However, it has infinity as a boundary.

So, I couldn't come up with another way of verifying our answer, but I'll ask this question on our Unit 4 Forum as a discussion :) Thank you for reading my AFP, and hope to see you again for AFP5!