ASSIGNED FORUM PROBLEM 3

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Chapter 8.5: *Integration of Rational Functions by Partial Fractions*

$$\textbf{41.} \quad \int \frac{cosy}{sin^2y + siny - 6}dy \tag{1}$$

How Can We Solve This Problem?

Since this problem is on Chapter 8.5: *Integration of Rational Functions by Partial Fractions*, I thought we'd have to use an integration technique involving partial fractions, which we will learn in *Lesson 14*. However, since we haven't learned about partial fractions yet - as of June 28, 2022 - I learned by trial and error that this problem can be solved by using u-substitution.

Solution

Let's start backwards. Here's the solution:

$$\textbf{41.} \quad \int \frac{cosy}{sin^2y + siny - 6}dy = \frac{2}{\sqrt{23}}tan^{-1}\left(\frac{2}{\sqrt{23}}(siny + \frac{1}{2})\right) + C$$

As you can see, the integral - or the solution - of this problem is in the form of $tan^{-1}x + C$. This is because it turns out that the problem can be rewritten in what we learned in *Lesson 5* as the integral of an inverse trigonometric function. We learned in *Lesson 5* that if we differentiate an inverse trigonometric function like $tan^{-1}x$, we get $\frac{1}{1+x^2}$. In other words, $\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$. Now, what this means is that $\tan^{-1}x + C = \int \frac{1}{1+x^2}$.

This, by the way, is the reason why the following equation from *Lesson 5* holds true:

$$\int \frac{1}{a^2 + u^2}du = \frac{1}{a}\tan^{-1}(\frac{u}{a}) + C$$

Now, let's go back to the start of our problem. What we'll see now is that the problem **41.** $\int \frac{cosy}{sin^2y+siny-6}dy$ can be rewritten with u-substitution, completing the square, and the integral of an inverse trigonometric function – it can eventually be rewritten in the form of $\int \frac{1}{1+u^2}du$; hence, the solution is in the form of $tan^{-1}u + C$.

Steps

Starting with u-substitution and completing the square:

41.

$$\int \frac{cosy}{sin^2y + siny - 6}dy = \int \frac{cosy}{u^2+u+6}dy \qquad (u = siny)$$

$$= \int \frac{cosy}{u^2+u+6} \cdot \frac{du}{cosy} \qquad (dy = \frac{du}{cosy})$$

$$= \int \frac{1}{u^2+u+6}du$$

$$= \int \frac{1}{(u+\frac{1}{2})^2 + \frac{23}{4}}du \qquad \text{(completing the square)}$$

$$= \int \frac{1}{w^2 + \frac{23}{4}}du \qquad (w = u + \frac{1}{2})$$

$$= \int \frac{1}{w^2 + \frac{23}{4}}dw \qquad (du = dw)$$

$$= \int \frac{1}{\frac{23}{4}(\frac{4}{23}w^2+1)}dw \qquad \text{(algebraically rewriting)}$$

$$= \int \frac{4}{23} \cdot \frac{1}{\frac{4}{23}w^2+1}dw$$

$$= \frac{4}{23}\int \frac{1}{\frac{4}{23}w^2+1}dw$$

$$= \frac{4}{23}\int \frac{1}{z^2+1}dw \qquad (z = \frac{2}{\sqrt{23}}w)$$

$$= \frac{4}{23}\int \frac{1}{z^2+1} \cdot \frac{\sqrt{23}}{2}dz \qquad (dw = \frac{\sqrt{23}}{2}dz)$$

$$= \frac{4}{23} \cdot \frac{\sqrt{23}}{2}\int \frac{1}{z^2+1}dz$$

$$= 2 \cdot \frac{23^{\frac{1}{2}}}{23^1}\int \frac{1}{z^2+1}dz$$

$$= 2 \cdot 23^{\frac{1}{2}-1}\int \frac{1}{z^2+1}dz$$

$$= 2 \cdot 23^{-\frac{1}{2}}\int \frac{1}{z^2+1}dz$$

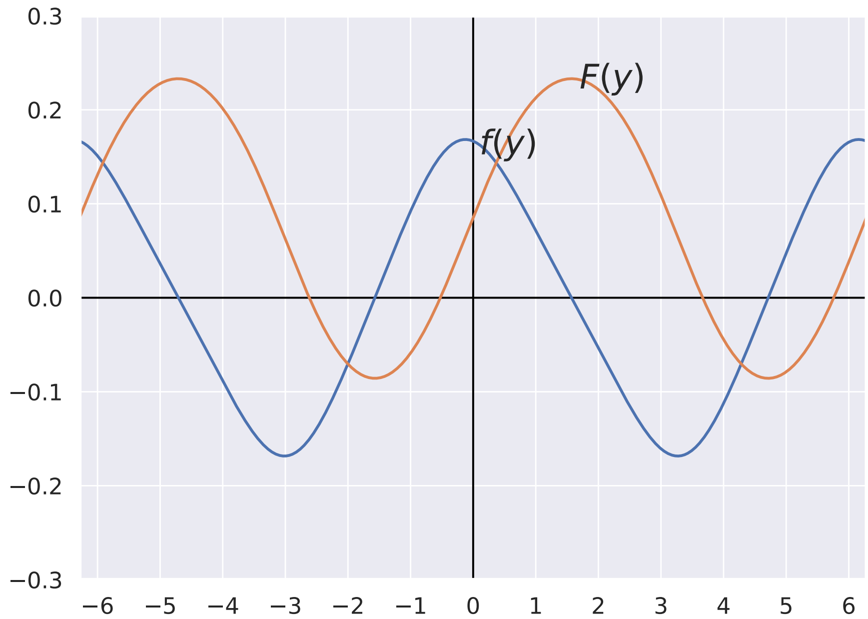
$$= \frac{2}{\sqrt{23}}\int \frac{1}{z^2+1}dz$$

$$= \frac{2}{\sqrt{23}}tan^{-1}z + C \qquad \left(\int \frac{1}{a^2+u^2}du = \frac{1}{a}\tan^{-1}(\frac{u}{a}) + C\right)$$

Finally, we can substitute back every u-substitution we made:

41. $$\begin{aligned}\int \frac{cosy}{sin^2y + siny - 6}dy &= \frac{2}{\sqrt{23}}tan^{-1}z + C \\ &= \frac{2}{\sqrt{23}}tan^{-1}\left(\frac{2}{\sqrt{23}}w\right) + C && (z = \frac{2}{\sqrt{23}}w) \\ &= \frac{2}{\sqrt{23}}tan^{-1}\left(\frac{2}{\sqrt{23}}(u + \frac{1}{2})\right) + C && (w = u + \frac{1}{2}) \\ &= \frac{2}{\sqrt{23}}tan^{-1}\left(\frac{2}{\sqrt{23}}(siny + \frac{1}{2})\right) + C && (u = siny)\end{aligned}$$

Geometrically Verifying the Integral



Let's see if we got the correct integral by graphing it. One thing I've noticed from this graph is that whenever $f(y)$ is at 0, $F(y)$ is either at its minimum or maximum. For instance, at $f(y) \approx -1.6$, $F(y)$ is at its minimum. Notice that this is one of the properties we learned in Calculus I.

The integrand of our original problem $f(y)$ seems to be the derivative of the integral we got, which is $F(y)$.

Analytically Verifying the Integral

In fact, we know from the Fundamental Theorem of Calculus that $F'(x) = f(x)$.[1] This means we can analytically verify our integral by differentiating it. The derivative of our integral should be the integrand of the original problem. Let's use *Maple* to differentiate our integral:

$$diff\left(\frac{2}{\sqrt{23}} \cdot \arctan\left(\frac{2}{\sqrt{23}} \cdot \left(\sin(y) + \frac{1}{2}\right)\right), y\right)$$

$$\frac{4\cos(y)}{23\left(\frac{4\left(\sin(y)+\frac{1}{2}\right)^2}{23}+1\right)} \tag{1}$$

simplify trig

$$-\frac{\cos(y)}{\cos(y)^2 - \sin(y) - 7} \tag{2}$$

Conclusion

The derivative of the integral indeed is equal to the integrand of the original problem. Therefore, we can finally conclude that the integral we got is the correct integral:

$$\textbf{41.} \quad \int \frac{cosy}{sin^2y + siny - 6}dy = \frac{2}{\sqrt{23}}tan^{-1}\left(\frac{2}{\sqrt{23}}(siny + \frac{1}{2})\right) + C$$

Plus, although we solved this problem using u-substitution, completing the square, and the integral of an $arctan(x)$ function, this problem comes from Chapter 8.5: *Integration of Rational Functions by Partial Fractions*, so keep in mind that there must be another way of solving this problem by using the partial fractions technique as well.

Thank you for your time, and I'll see you in two weeks for AFP4!

Update

June 30, 2022 - I've learned about partial fractions, so now I know an alternative way to solve this problem, too.

$$\begin{aligned}
\textbf{41.} \quad \int \frac{cosy}{sin^2y + siny - 6}dy &= \int \frac{1}{u^2 + u + 6}du && (u = siny) \\
&= \int \frac{\frac{1}{5}}{u-2} + \frac{-\frac{1}{5}}{u+3}du && \text{(partial fraction decomposition)} \\
&= \frac{1}{5}(ln|2 - siny| - ln|3 + siny|) + C
\end{aligned}$$

Now, we've got a form different from the first answer we got, which was in th form of $arctan(x)$, but plotting the first answer and this answer on *Maple* gives you the same output. In other words, although in a different form, they are the same, meaning that both answers are the correct integral.

[1] Hass et al., *Thomas' calculus* (14th ed.), Pearson, p. 280.