HOMEWORK 3.1

Soobin Rho

October 27, 2022

Chapter 3.1: *Inequalities*

B10. Prove Theorem 10, transitivity:

If $a < b$ and $b < c$, then $a < c$.

Proof

The first thought that came to my mind was, can I use Theorem 1.6.1 (Transitivity) "If $A \Rightarrow B$ and $B \Rightarrow C$, then $A \Rightarrow C$" to solve this problem? Maybe I could, but then I couldn't find a way to translate our original problem into this this theorem. So, I took a different approach.

Assume $a < b$ and $b < c$,	
Then, $a < b$	by assumption
$\Leftrightarrow a - a < b - a$	Theorem 4
$\Leftrightarrow 0 < b - a$	Axiom 1
Also, $b < c$	by assumption
$\Leftrightarrow b - b < c - b$	Theorem 4
$\Leftrightarrow 0 < c - b$	Axiom 1
Since the sum of positive numbers is positive,	by Axiom 1B
$(b - a) + (c - b) > 0$	
$\Leftrightarrow -a + c > 0$	
$\Leftrightarrow -a > -c$	
$\Leftrightarrow a < c \quad \square$	Theorem 7

B11. Reproduce and complete the given parts of the proof of Theorem 8F, and finish the proof:

If $a \neq 0$, then $a^2 > 0$.

Proof

If $a \neq 0$, then a is positive or $-\, a$ is positive.	Axiom 1

This is in the form of $H \Rightarrow (A \text{ or } B)$. So, I'll prove this by proving both cases – i.e. both the case in which a is positive and the other case in which $-a$ is positive.

If a is positive, then $a > 0$	Axiom 1A
Also, since $a^2 = a \cdot a$,	Prior Result 0
and since a product of two positive numbers is positive,	Prior Result 0
$a^2 > 0$	Theorem 7
If $-\, a$ is positive, then $-\, a > 0$	Axiom 1A
Also, since $a^2 = a \cdot a$,	Prior Result 0
and since a product of two negative numbers is positive,	Prior Result 0
$a^2 > 0 \quad \square$	Theorem 7

Reflection on myself: I feel guilty about just assuming that a product of (two positive numbers and two negative numbers) is positive becaues of Prior Result 0. Is there a better way of proving this?

Update: I just found out that Axiom 1C actually states that the product of positive numbers is positive!

B13. Prove Theorem 13B:

(Multiplying the Sides of an Inequality by a Number)
If $c < 0$ and $a < b$, then $ca > cb$.

Proof

Assume $c < 0$ and $a < b$,	
Then, $a < b$ implies $0 < b - a$	Theorem 4
Therefore, $b - a$ is positive.	Axiom 1A
Also since $-c$ is positive,	by assumption
$0 < (b - a)(-c)$	Axiom 1C
$\Leftrightarrow 0 < -cb + ca$	Prior Result 0
$\Leftrightarrow -ca < -cb$	Theorem 4
$\Leftrightarrow ca > cb \quad \square$	Theorem 7

B20. Resolve Conjecture 20:

If $c \geq 1$**, then** $cx \geq x$**.**

Counterexample

My intuition tells me that this conjecture is false. Thus, I'll come up with a counterexample, where the conjecture's negation is true.

$$\text{Let } c = 2, x = -20,$$
$$\text{Then } c = 2 \geq 1, \text{ but } cx = (2)(-20) = -40 < -20 = x$$

B24. Fix Conjecture 20 and provide a proof of it:

Conjecture 20: If $c \geq 1$, then $cx \geq x$.

Proof

Conjecture 20 is a false generalization, but it can become a true generalization if we put one more condition into the hypothesis:

$$\text{If } c \geq 1 \text{ and } x > 0, \text{ then } cx \geq x.$$

In other words, x needs to be a positive number. Here's proof:

Assume $c \geq 1$ and $x > 0$,	
Then, c and x are positive numbers.	Axiom 1A
Also, a product of positive numbers is positive.	Axiom 1C
Therefore, $c \cdot x$ is a poistive number.	
Then, $c \geq 1$,	By assumption
$c \cdot x \geq 1 \cdot x$	Theorem 14A
$\leftrightarrow cx \geq x \quad \square$	

B27. Resolve this conjecture:

If, for all $\epsilon > 0$, $x < c + \epsilon$, then $x < c$.

Counterexample

This is false. The following counterexample shows a case in which the hypothesis is true but the conclusion is false.

Let $x = 1, c = 1,$

Let $\epsilon > 0,$ By hypothesis

Then $x = 1 < 1 + \epsilon = c + \epsilon$, but $x = 1 \geq 1 = c$