EXAM 2 OUT-OF-CLASS COMPONENT

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Chapter 2.5: *Rational Numbers and Form*

(a) Prove the following conjecture:
If x is rational and y is rational, then $x + y$ is rational.

Proof

In order to prove this generalization, the first thing I did was to translate it into more primitive terms by using the sentence-form definition of rational numbers.

"If x is rational and y is rational, then $x + y$ is rational" is equivalent to

$$\forall\, x, y \in \mathbb{R},\;\; \big(x \in Q \;\text{ and }\; y \in Q\big) \Rightarrow (x + y) \in Q \tag{1}$$

In order to prove this,

$$\text{Let } x, y \in Q \tag{2}$$

Using the sentence-form definition of rational numbers, we get

$$\text{Then, }\; \exists\, p, q \in \mathbb{Z} \;\text{ s.t. }\; x = \frac{p}{q} \text{ and } q \neq 0 \tag{3}$$

$$\exists\, m, n \in \mathbb{Z} \;\text{ s.t. }\; y = \frac{m}{n} \text{ and } n \neq 0 \tag{4}$$

What I'm trying to do here is to show that both the hypothesis and the conclusion are true for all $x, y \in \mathbb{R}$. Proving that this conditional sentence is always true proves that it is a true generalization.

$$\text{Then, }\; (x + y) = \frac{p}{q} + \frac{m}{n} \tag{5}$$

$$= \frac{pn + qm}{qn} \tag{6}$$

From here, how can I know if $(x + y) = \frac{pn+qm}{qn}$ is a rational number? Just to remind myself, here's what it means to be a rational number:

x is a rational number iff $\exists\, a, b \in \mathbb{Z}\;\; s.t.\;\; x = \frac{a}{b}$ and $b \neq 0$

So, in this case, what I had to do was to show that both the numerator $pn + qm$ and the denominator qn as in $(x + y) = \frac{pn+qm}{qn}$ are integers.

How can I know whether or not $pn + qm$ and qn are integers? I used Prior Results given by Dr. Perry. First, *Prior Result 0.2* states that integers are closed under addition. This means that an addition of integers results in another integer. So, $pn+qm$ results in an integer. Likewise, *Prior Result 0.4* states that integers are closed under multiplication. So, qn results in an integer, too. Going back to the original expression, we now know that $(x + y) = \frac{pn+qm}{qn}$ always results in an integer over integer.

One more thing. In order for $(x+y) = \frac{pn+qm}{qn}$ to be a rational number, both the numerator and denominator have to be integers, which we just proved. However, there's one more condition. The denominator $qn \neq 0$. This, however, is already true because we defined $q \neq 0$ in (3) and $n \neq 0$ in (4).

$$\text{Since } (x + y) = \frac{pn + qm}{qn} \text{ and } p, q \in \mathbb{Z} \text{ and } m, n \in \mathbb{Z} \text{ and } qn \neq 0, \tag{7}$$

$$\exists\ a = (pn + qm) \in \mathbb{Z},\ b = (qn) \in \mathbb{Z}\ \ s.t.\ (x + y) = \frac{a}{b} \text{ and } b \neq 0 \tag{8}$$

$$\therefore\ \forall\ x, y \in \mathbb{R},\ (x \in Q \text{ and } y \in Q) \Rightarrow (x + y) \in Q \tag{9}$$

(b) Select a false conjecture from the list and provide a counterexample.
D. If x is irrational and y is irrational, then $x + y$ is rational.

Counterexample

First, translate.

$$\forall\, x, y \in \mathbb{R},\ \ \big[(x \notin Q \ \text{ and } \ y \notin Q) \Rightarrow (x + y) \in Q\big] \tag{10}$$

Negation of a conditional sentence is an existment statement with the hypothesis being true and the conclusion being false.

$$\exists\, x, y \in \mathbb{R} \ \text{ s.t. } \ \big[(x \notin Q \ \text{ and } \ y \notin Q) \text{ and } (x + y) \notin Q\big] \tag{11}$$

Any value of $x, y \in \mathbb{R}$ that makes this negation true will suffice. I'll go with π.

$$\text{Let } x = \pi,\ y = \pi, \tag{12}$$
$$\text{Then, } x = \pi \notin Q \text{ and } y = \pi \notin Q, \text{ but} \tag{13}$$
$$(x + y) = \pi + \pi = 2\pi \notin Q \tag{14}$$
$$\therefore\ \exists\, x, y \in \mathbb{R} \ \text{ s.t. } \ \big[(x \notin Q \ \text{ and } \ y \notin Q) \text{ and } (x + y) \notin Q\big] \tag{15}$$

The question of "how can we prove if π is an irrational number?" still remains, but it seems like our class will be covering this topic in the *Chapter 3.4*.

(c) Select a true conjecture from the list and give a proof for why it is true.
E. If x is rational and $x + y$ is irrational, then y is irrational.

Proof

Again, translation first.

$$\forall\, x, y \in \mathbb{R},\ \ \big(x \in Q \ \text{ and } \ (x+y) \notin Q\big) \Rightarrow y \notin Q \tag{16}$$

At first glance, it's hard to know how to prove this, but we can use logical equivalence to make this easier to understand. Using *A Version of the Contrapositive*, we get

$$(A \text{ and } B) \Rightarrow C \tag{17}$$

$$\text{LE to } \ (A \text{ and not}(C)) \Rightarrow \text{not}(B) \tag{18}$$

$$\forall\, x, y \in \mathbb{R},\ \ \big(x \in Q \ \text{ and } \ (x+y) \notin Q\big) \Rightarrow y \notin Q \tag{19}$$

$$\text{LE to } \ \forall\, x, y \in \mathbb{R},\ \ \big(x \in Q \ \text{ and } \ y \in Q\big) \Rightarrow (x+y) \in Q \tag{20}$$

This generalization therefore happens to be the same as the generalization (1) from the first question. Beautiful!

$$\forall\, x, y \in \mathbb{R},\ \ \big(x \in Q \ \text{ and } \ y \in Q\big) \Rightarrow (x+y) \in Q$$

Thanks to the logical equivalence, we now know that this question is acutally the same as the first question. The same proof for the first question can be used for this question.

$$\text{Let } \ x \in Q \text{ and } y \in Q, \tag{21}$$

$$\text{Then, } \ \exists\, p, q \in \mathbb{Z} \ \text{ s.t. } \ x = \frac{p}{q} \text{ and } q \neq 0 \tag{22}$$

$$\exists\, m, n \in \mathbb{Z} \ \text{ s.t. } \ y = \frac{m}{n} \text{ and } n \neq 0 \tag{23}$$

$$(x+y) = \frac{pn+qm}{qn} \ \text{ and } \ p, q \in \mathbb{Z} \ \text{ and } \ m, n \in \mathbb{Z} \ \text{ and } \ qn \neq 0 \tag{24}$$

$$\exists\, a = (pn+qm) \in \mathbb{Z},\ b = (qn) \in \mathbb{Z} \ \ s.t. \ \ (x+y) = \frac{a}{b} \text{ and } b \neq 0 \tag{25}$$

$$\therefore\ \forall\, x, y \in \mathbb{R},\ (x \in Q \ \text{ and } \ y \in Q) \Rightarrow (x+y) \in Q \tag{26}$$