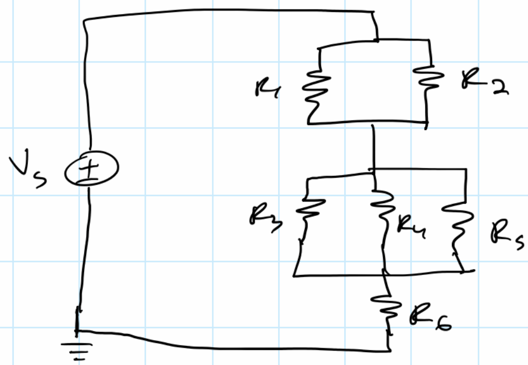
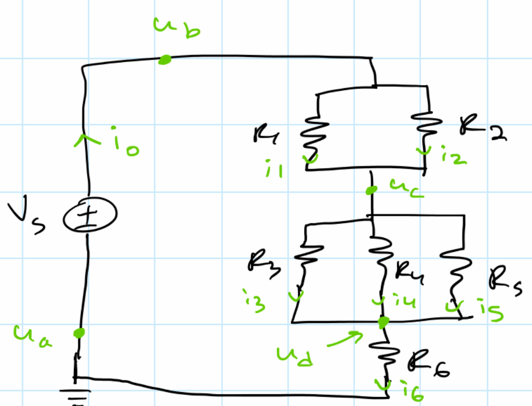
Fast Nodal Analysis Guide

Wednesday, October 24, 2018 7:14 PM



Question: Find the potentials/voltage/currents at every point.

Problem: Using our standard method of labelling currents & potentials, we run into a whole lot of variables!



Variables:

u_a, u_b, u_c, u_d

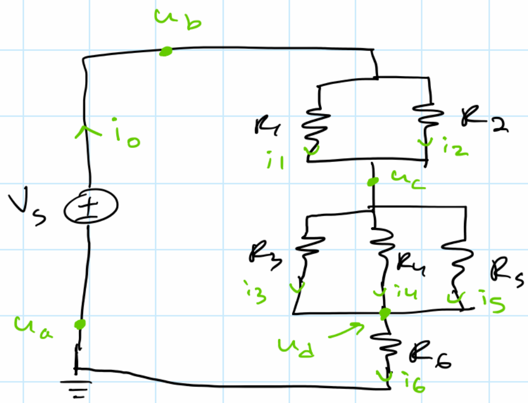
$i_0, i_1, i_2, i_3, i_4, i_5, i_6$

★ Idea: Do we really need to solve for all the variables or can we simplify what we are solving for?

Hint: We only need to solve for one type of variable (i.e. solve for only potentials or currents).

Which one is "better" to solve??

(Try to answer before seeing solution below?)



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☆ Idea: Do we really need to solve for all the variables or can we simplify what we are solving for?

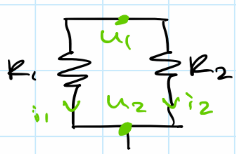
Hint: We only need to solve for one type of variable (i.e. solve for only potentials or currents).

Which one is "better" to solve??

(Try to answer before seeing solution before)

Solution: Solving for potentials is better! That way if we ever want to solve for some current through a resistor, we can use $V = IR$, and directly substitute u's to get $i_R = \frac{u^+ - u^-}{R}$.

Ex:



If we have solved for the potentials, we know the u's.

$$V = IR \Rightarrow I = \frac{V}{R}$$

$$\therefore \; i_1 = \frac{u_1 - u_2}{R_1}, \quad i_2 = \frac{u_1 - u_2}{R_2}$$

Note $\frac{u_1 - u_2}{R_1}$ & NOT $\frac{u_2 - u_1}{R_1}$ due to passive sign convention

Okay, back to the original problem.

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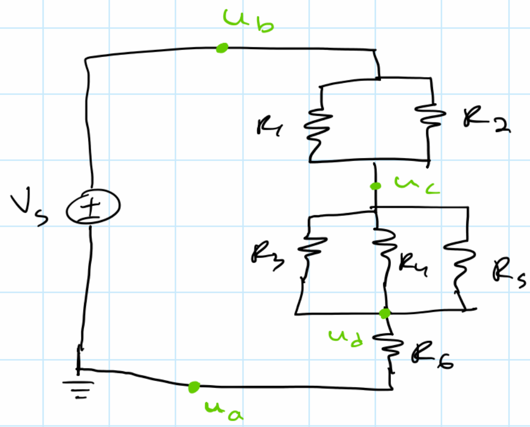


We want to only have potential variables.

Variables:

u_a, u_b, u_c, u_d

Note how u_a is on the wire connected with ground, so $u_a = 0$.

Can we solve for anything else?

Note that u_a and u_b are separated by a voltage source, so $u_b = u_a + V_s$ (or $u_b - u_a = V_s$).

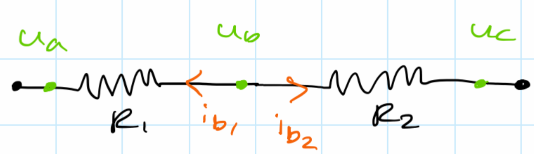
We now have $u_a = 0$, $u_b = V_s$, how can we solve for u_2 & u_3?

☆ This is really nice! Before we had to solve for 11 variables, now we only have 2!! ☆

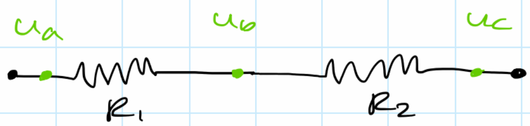
Solution: Use KCL and substitute $I = \frac{V}{R}$

Imagine this small example:

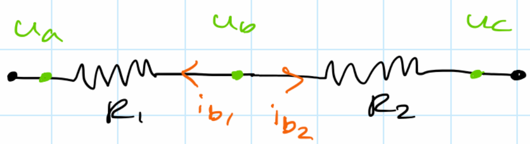
KCL at node ub.



Imagine this small example:



KCL at node ub.



$I_{out} = I_{in}$, so $i_{b1} + i_{b2} = 0$

We do not want any current variables, so we can use our buddy Ohm to help us out.

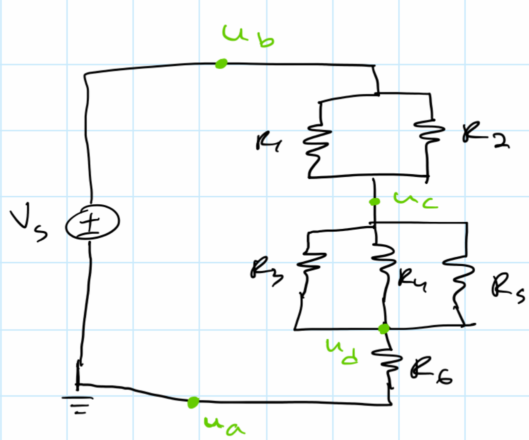
$V = IR \Rightarrow I = \frac{V}{R}$

$i_{b1} = \frac{u_b - u_a}{R_1}$, $i_{b2} = \frac{u_b - u_c}{R_2}$

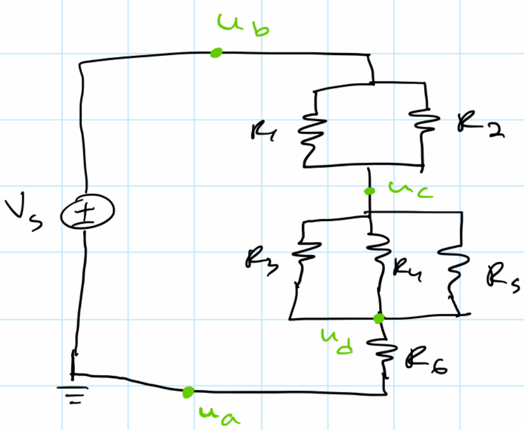
$i_{b1} + i_{b2} = 0 \Rightarrow \frac{u_b - u_a}{R_1} + \frac{u_b - u_c}{R_2} = 0$

We got rid of the current variables and only have equations with potentials now!

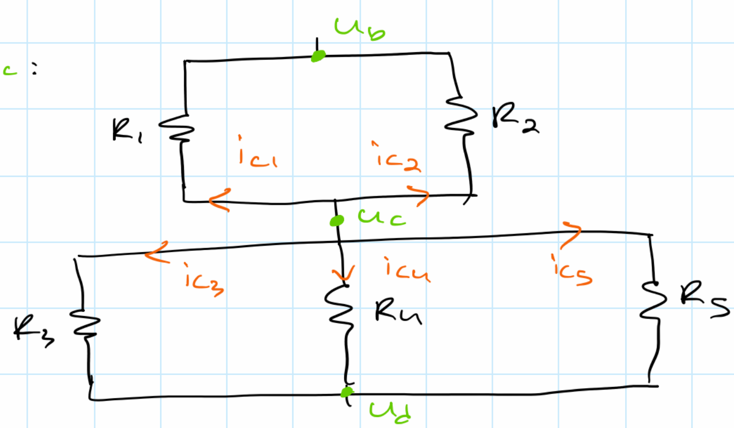
Back one last time to the original question:



Back one last time to the original question:



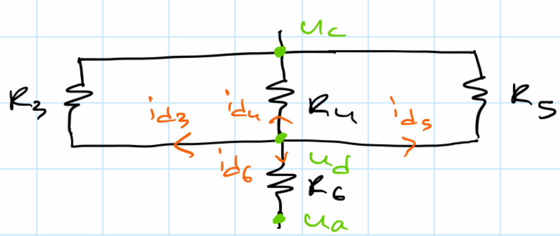
@ Node u_c:



$$i_{c1}+i_{c2}+i_{c3}+i_{c4}+i_{c5}=0$$

$$\frac{u_c-u_b}{R_1}+\frac{u_c-u_b}{R_2}+\frac{u_c-u_d}{R_3}+\frac{u_c-u_d}{R_4}+\frac{u_c-u_d}{R_5}=0$$

@ Node u_d:



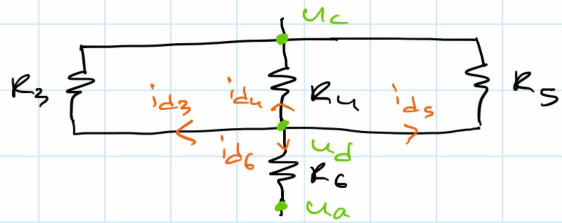
$$i_{d3}+i_{d4}+i_{d5}+i_{d6}=0$$

$$\frac{u_d-u_c}{R_3}+\frac{u_d-u_c}{R_4}+\frac{u_d-u_c}{R_5}+\frac{u_d-u_a}{R_6}=0$$

In total, we have the following equations:

$u_a=0$ $\quad$ $u_b=u_a+V_s$

@Node u_d:



$$i_{d3} + i_{d4} + i_{d5} + i_{d6} = 0$$

$$\frac{u_d - u_c}{R_3} + \frac{u_d - u_c}{R_4} + \frac{u_d - u_c}{R_5} + \frac{u_d - u_a}{R_6} = 0$$

In total, we have the following equations:

$u_a = 0$, $u_b = u_a + V_s$

$$u_c\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5}\right) + u_b\left(-\frac{1}{R_1} - \frac{1}{R_2}\right) + u_d\left(-\frac{1}{R_3} - \frac{1}{R_4} - \frac{1}{R_5}\right) = 0$$

$$u_d\left(\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5} + \frac{1}{R_6}\right) + u_c\left(-\frac{1}{R_3} - \frac{1}{R_4} - \frac{1}{R_5}\right) + u_a\left(-\frac{1}{R_6}\right) = 0$$

4 unknowns & 4 variables so we can solve!

Note that 2 of the variables are "trivial" ($u_a = 0$, $u_b = V_s$), so we have more like 2 unknowns & 2 equations!

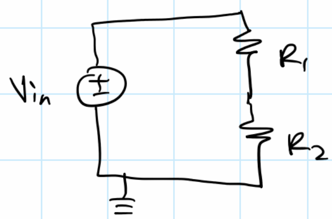
We can then solve using our matrix methods or by direct substitution/elimination (the latter is usually easier).

Codie, so what were the steps we did?

1. Label nodes
2. Get relations for all nodes connected to voltage sources
3. "Modified KCL" on all nodes not in step 2
4. Solve

We will now go through some examples. I encourage you to try to set up the equations for each circuit before going through the solutions.

Example 1:



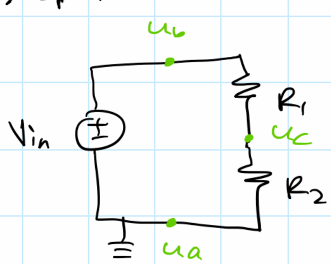
Step 1: Label nodes

Step 2: Get relations for all nodes connected to voltage sources

Step 3: "Modified" KCL on all nodes not in step 2.

Step 4: Solve!

Step 1:



Step 2:

$u_a = 0$, $\quad u_b - u_a = V_{in}$

Step 3:

$$\frac{u_c - u_b}{R_1} + \frac{u_c - u_a}{R_2} = 0$$

Note we are skipping writing $i_{c1} + i_{c2} = 0$ but that is implicitly what we are doing.

Step 4: $u_a = 0$

$$u_b - u_a = V_{in} \Rightarrow u_b = u_a + V_{in} = 0 + V_{in} = V_{in}$$

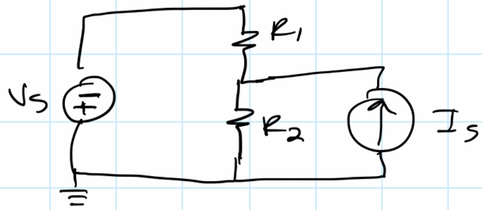
$$\frac{u_c - u_b}{R_1} + \frac{u_c - u_a}{R_2} = 0 \Rightarrow \frac{u_c - V_{in}}{R_1} + \frac{u_c - 0}{R_2} = 0$$

$$\Rightarrow u_c\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{V_{in}}{R_1}$$

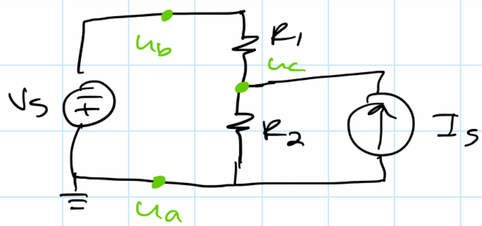
$$\Rightarrow u_c \cdot \left(\frac{R_1 + R_2}{\cancel{R_1} \cdot R_2}\right) = \frac{V_{in}}{\cancel{R_1}} \Rightarrow u_c = \frac{R_2}{R_1 + R_2} \cdot V_{in}$$

We got our regular voltage divider equation!

Example 2: Find the potential at each node.



Solution:



Note the negative as $\ominus$ instead of usual $\oplus$!!!

$u_a = 0$

$\underline{u_a - u_b} = V_s \Rightarrow 0 - u_b = V_s \Rightarrow u_b = -V_s$

positive side of voltage souce − negative side = voltage source

@ Node u_c: $\dfrac{u_c - u_b}{R_1} + \dfrac{u_c - u_a}{R_2} = I_s$ $(I_{out} = I_{in})$

← Don't be scared by the current souce, just think about how it affects the KCL equation!

Solving gets:

$u_a = 0, \quad u_b = -V_s$

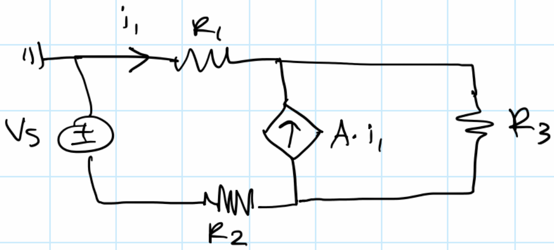
$u_c\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = I_s + \frac{u_b}{R_1} + \frac{u_a}{R_2} \Rightarrow$

$u_c\left(\frac{R_1 + R_2}{R_1 R_2}\right) = I_s - \frac{V_s}{R_1} \Rightarrow$

$u_c = \left(I_s - \frac{V_s}{R_1}\right) \cdot \frac{R_1 R_2}{R_1 + R_2}$

If we then wanted to solve for the current through R_1, we can use Ohms law and get $i_1 = \frac{u_a - u_c}{R_1}$ (or $\frac{u_c - u_a}{R_1}$ depending on direction of i_1) and substitute for u_a & u_c from above equations.

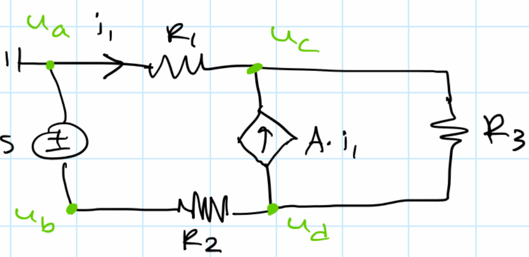
Example 3: Solve for potentials at each node.



$V_s = 5$

$R_1 = 1,\ R_2 = 2,\ R_3 = 3,\ A = 4$

Solution:



$u_a = 0$, $\quad u_a - u_b = V_s$, $\quad i_1 = \frac{u_a - u_c}{R_1}$

$\frac{u_c - u_a}{R_1} + \frac{u_c - u_d}{R_3} = A \cdot i_1$, $\quad \frac{u_d - u_b}{R_2} + A \cdot i_1 + \frac{u_d - u_c}{R_3} = 0$

We have 5 equations and 5 unknowns (i_1 counts as an unknown), so we can subsequently solve to get a unique solution.

① $\boxed{u_a = 0}$

② $u_a - u_b = 5 \Rightarrow 0 - u_b = 5 \Rightarrow \boxed{u_b = -5}$

③ $i_1 = \frac{0 - u_c}{1} = -u_c$

④ $\frac{u_c - 0}{1} + \frac{u_c - u_d}{3} = 4 \cdot (-u_c) \Rightarrow u_c + \frac{1}{3}u_c - \frac{1}{3}u_d = -4u_c \Rightarrow \frac{16}{3}u_c = \frac{1}{3}u_d$

$\Rightarrow u_c = \frac{1}{16}u_d$ or $16u_c = u_d$

⑤ $\frac{u_d - (-5)}{2} + 4 \cdot (-u_c) + \frac{u_d - u_c}{3} = 0 \Rightarrow \frac{u_d}{2} + \frac{5}{2} - 4u_c + \frac{u_d}{3} - \frac{u_c}{3} = 0$

$\Rightarrow -\frac{13}{3}u_c + \frac{5}{2} + \frac{5}{6}u_d = 0$

Substituting $u_d = 16u_c \Rightarrow -\frac{13}{3}u_c + \frac{5}{2} + \frac{5}{6} \cdot 16u_c = 0$

$\Rightarrow -\frac{13}{3}u_c + \frac{5 \cdot 8}{3}u_c = -\frac{5}{2} \Rightarrow \frac{27}{3}u_c = -\frac{5}{2} \Rightarrow \boxed{u_c = -\frac{5}{18}}$

$u_d = 16u_c = 16(-\frac{5}{18}) \Rightarrow \boxed{u_d = \frac{-40}{9}}$, $\quad i_1 = -u_c \Rightarrow \boxed{i_1 = \frac{5}{18}}$

Main takeaways

- Try to only solve for potentials
- 4 steps:
  - Label nodes
  - Relations with voltage sources
  - ☆ Modified KCL ☆
  - Solve!

(Food for thought, how can we handle circuits with capacitors using this technique?)

- Try to get a systematic approach in attacking circuit analysis problems!