1. Mechanical Eigenvalues and Eigenvectors.

(a). We have $(A-\lambda I_2)\cdot\vec{x}=\vec{0}$, and $A-\lambda I_2=\begin{bmatrix}5-\lambda & 0\\ 0 & 2-\lambda\end{bmatrix}$

So we need $\det(A-\lambda I_2)=(5-\lambda)(2-\lambda)-0\cdot 0=0$. so eigenvalues $\lambda_1=2$, $\lambda_2=5$.
Each value will have its own corresponding eigenvector.

① $\lambda_1=2$, so $(A-2I_2)\vec{x}=\begin{bmatrix}3 & 0\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\vec{0}\Rightarrow\left[\begin{array}{cc|c}3 & 0 & 0\\ 0 & 0 & 0\end{array}\right]\Rightarrow\left[\begin{array}{cc|c}1 & 0 & 0\\ 0 & 0 & 0\end{array}\right]$

So the eigenvectors are $\begin{bmatrix}0\\ 1\end{bmatrix}\alpha_1$ with $\alpha_1\in\mathbb{R}$.

② $\lambda_2=5$. so $(A-5I_2)\vec{x}=\begin{bmatrix}0 & 0\\ 0 & -3\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\vec{0}\Rightarrow\left[\begin{array}{cc|c}0 & 0 & 0\\ 0 & -3 & 0\end{array}\right]\Rightarrow\left[\begin{array}{cc|c}0 & 0 & 0\\ 0 & 1 & 0\end{array}\right]$

So the associated eigenvector are $\begin{bmatrix}1\\ 0\end{bmatrix}\alpha_2$, $\alpha_2\in\mathbb{R}$.

Thus. The eigenvalues are $\boxed{\lambda_1=2,\ \lambda_2=5}$; the eigenspace is $\boxed{\mathrm{span}\left\{\begin{bmatrix}0\\ 1\end{bmatrix},\begin{bmatrix}1\\ 0\end{bmatrix}\right\}}$

(b). We have $(A-\lambda I_2)\cdot\vec{x}=\vec{0}$, and given $A=\begin{bmatrix}22 & 6\\ 6 & 13\end{bmatrix}$. So $A-\lambda I_2=\begin{bmatrix}22-\lambda & 6\\ 6 & 13-\lambda\end{bmatrix}$

So, we need $\det(A-\lambda I_2)=(22-\lambda)(13-\lambda)-6\cdot 6=0\Rightarrow\lambda^2-35\lambda+250=0$

so, $(\lambda-10)(\lambda-25)=0$, so we have eigenvalues $\lambda_1=10$, $\lambda_2=25$. and so:

① $\lambda_1=10$. so $(A-10I_2)\cdot\vec{x}=\begin{bmatrix}12 & 6\\ 6 & 3\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\vec{0}\Rightarrow\left[\begin{array}{cc|c}12 & 6 & 0\\ 6 & 3 & 0\end{array}\right]$ R_2: Subtract $\frac{1}{2}\cdot R_1$

So. $\left[\begin{array}{cc|c}12 & 6 & 0\\ 0 & 0 & 0\end{array}\right]$ which means that $12x_1+6x_2=0\Rightarrow x_2=-2x_1$.

So the associated eigenvectors are. $\begin{bmatrix}1\\ -2\end{bmatrix}\alpha_1$, with $\alpha_1\in\mathbb{R}$

② $\lambda_2=25$. so $(A-25I_2)\cdot\vec{x}=\begin{bmatrix}-3 & 6\\ 6 & -12\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\vec{0}\Rightarrow\left[\begin{array}{cc|c}-3 & 6 & 0\\ 6 & -12 & 0\end{array}\right]$. R_2: Add $2\cdot R_1$.

So, $\left[\begin{array}{cc|c}-3 & 6 & 0\\ 0 & 0 & 0\end{array}\right]$ which gives $-3x_1+6x_2=0\Rightarrow x_1=2x_2$.

So the associated eigenvectors are $\begin{bmatrix}2\\ 1\end{bmatrix}\alpha_2$. $\alpha_2\in\mathbb{R}$.

Thus. the eigenvalues are: $\boxed{\lambda_1=10.\ \lambda_2=25}$; the eigenspace is $\boxed{\mathrm{span}\left\{\begin{bmatrix}1\\ -2\end{bmatrix},\begin{bmatrix}2\\ 1\end{bmatrix}\right\}}$.

(c) Since $A=\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}=\begin{bmatrix}\cos\frac{\pi}{6} & -\sin\frac{\pi}{6}\\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6}\end{bmatrix}$, so this is a special matrix that would rotate any vector it applies to by $\frac{\pi}{6}$ $(=30^\circ)$ counterclockwise

Now, we have $(A-\lambda I_2)\cdot\vec{x}=\vec{0}$. so $\begin{bmatrix}\frac{\sqrt{3}}{2}-\lambda & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}-\lambda\end{bmatrix}=A-\lambda I_2$, so we have

$\det(A-\lambda I_2)=(\frac{\sqrt{3}}{2}-\lambda)\cdot(\frac{\sqrt{3}}{2}-\lambda)-\frac{1}{2}\cdot(-\frac{1}{2})=\lambda^2-\sqrt{3}\lambda+\frac{3}{4}+\frac{1}{4}=\lambda^2-\sqrt{3}\lambda+1=0$.

$\Rightarrow(\lambda-\frac{\sqrt{3}+i}{2})(\lambda-\frac{\sqrt{3}-i}{2})=0$. so eigenvalues $\lambda_1=\frac{\sqrt{3}+i}{2}$, $\lambda_2=\frac{\sqrt{3}-i}{2}$. So

① $\lambda_1=\frac{\sqrt{3}+i}{2}$. so $(A-\frac{\sqrt{3}+i}{2}I_2)\cdot\vec{x}=\begin{bmatrix}-\frac{i}{2} & -\frac{1}{2}\\ \frac{1}{2} & -\frac{i}{2}\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\vec{0}\Rightarrow\left[\begin{array}{cc|c}-\frac{i}{2} & -\frac{1}{2} & 0\\ \frac{1}{2} & -\frac{i}{2} & 0\end{array}\right]$

R_2: Subtract $i\cdot R_1\Rightarrow\left[\begin{array}{cc|c}-\frac{i}{2} & -\frac{1}{2} & 0\\ 0 & 0 & 0\end{array}\right]$ which gives $-\frac{i}{2}x_1-\frac{1}{2}x_2=0.\Rightarrow x_2=-ix_1$.

So the associated eigenvectors are $\begin{bmatrix}1\\ -i\end{bmatrix}\alpha_1$ where $\alpha_1\in\mathbb{R}$.

② $\lambda_2=\frac{\sqrt{3}-i}{2}$. so $(A-\frac{\sqrt{3}-i}{2}I_2)\cdot\vec{x}=\begin{bmatrix}\frac{i}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{i}{2}\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\vec{0}\Rightarrow\left[\begin{array}{cc|c}\frac{i}{2} & -\frac{1}{2} & 0\\ \frac{1}{2} & \frac{i}{2} & 0\end{array}\right]$

R_2. Add $iR_1\Rightarrow\left[\begin{array}{cc|c}\frac{i}{2} & -\frac{1}{2} & 0\\ 0 & 0 & 0\end{array}\right]$ which gives $\frac{i}{2}x_1-\frac{1}{2}x_2=0\Rightarrow x_2=ix_1$.

So the associated eigenvectors are $\begin{bmatrix}1\\ i\end{bmatrix}\alpha_2$ where $\alpha_2\in\mathbb{R}$.

Thus. the eigenvalues are $\boxed{\lambda_1=\frac{\sqrt{3}+i}{2}\quad\lambda_2=\frac{\sqrt{3}-i}{2}}$

and the eigenspace is $\boxed{\mathrm{span}\left\{\begin{bmatrix}1\\ -i\end{bmatrix},\begin{bmatrix}1\\ i\end{bmatrix}\right\}}$.

(e). We have $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and $(A - \lambda I_2) \cdot \vec{x} = 0$. with $A - \lambda I_2 = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}$

Since we need $\det(A - \lambda I_2) = (2-\lambda)(2-\lambda) - 0 \cdot 0 = 0$, so eigenvalues $\lambda_1 = \lambda_2 = 2$. which is a repeated eigenvalue of $\lambda = 2$.

Now, since $(A - 2I_2) \cdot \vec{x} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$, so the eigenspace is all of $\mathbb{R}^2$

This makes sense as $\forall \vec{v} \in \mathbb{R}^2$, with $\lambda = 2$, we have $A\vec{v} = \lambda \vec{v}$. A basis for $\mathbb{R}^2$ is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ by our definition (our regular view).

Thus, the eigenvalue is $\boxed{\lambda = 2}$; the eigenspace is $\boxed{\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}}$

2. Counting The Paths of a random Surfer.

(a). With $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, since $a_{21} = 1$. so there's $\boxed{1}$ one-hop path from webpage 1 to 2.

Then, $A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, since for A^2, $a_{21} = 0$, so there's $\boxed{\text{0 two-hop path}}$ from webpage 1 to webpage 2.

Then, since $A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

Since for A^3, $a_{21} = 1$ again, so there's $\boxed{\text{1 three-hop path}}$ from webpage 1 to webpage 2.

(b). By definition of transition matrix, so $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Consider when $\lambda = 1$.

So $(T - \lambda I_2) \cdot \vec{x} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$ R_2: Add R_1. so:

$\Rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ which gives $-x_1 + x_2 = 0$, so $x_1 = x_2$

so the eigenvectors this $\lambda = 1$ gives are: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \alpha$, $\alpha \in \mathbb{R}$.

To have the values of this eigenvector sum to 1, so $1 \cdot \alpha + 1 \cdot \alpha = 1 \Rightarrow \alpha = 0.5$, and so the specific eigenvector we're looking for is $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$.

Thus, the steady-state frequency for website 1 is $\boxed{0.5}$, for webpage 2 is $\boxed{0.5}$.

(c). From the graph, we can compute the adjacency matrix for graph B to be: $\boxed{B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}$

(d). Since $B^2 = B \cdot B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ So, with $a_{31} = 1$ in B^2, so there is $\boxed{\text{1 two-hop path}}$ from webpage 1 to webpage 3.

Then, $B^3 = B^2 \cdot B = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 2 & 0 & 3 & 2 \end{bmatrix}$ With $a_{21} = 1$ in B^3, so there is $\boxed{\text{1 three-hop path}}$ from webpage 1 to webpage 2.

(e). First, we calculate Graph B's transition matrix T, which by definition, gives: $T = \begin{bmatrix} 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} & 0 \end{bmatrix}$

Then using IPython, we discover that the normalized eigenvector (Sum to one) corresponding to eigenvalue $\lambda = 1$ is: $\begin{bmatrix} 0.333 \\ 0.167 \\ 0.125 \\ 0.375 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{8} \\ \frac{3}{8} \end{bmatrix}$

Thus, the steady-state frequency for webpage 1 is $\boxed{\frac{1}{3}}$, webpage 2 is $\boxed{\frac{1}{6}}$, webpage 3 is $\boxed{\frac{1}{8}}$, webpage 4 is $\boxed{\frac{3}{8}}$

(f). The adjacency matrix for graph C is:

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(g). There is $\boxed{\text{0 path}}$ from webpage 1 to webpage 3.

As we can see from Graph C, it is a disconnected graph, and so surfers from webpage 1 can only get to webpage 2 or back to webpage 1, which implies that there's no path between webpages 1 and 3, and thus, there's 0 path from webpage 1 to webpage 3.

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3. Noisy Images

(a). Since $\vec{y} = A\vec{x} + \vec{n}$, so $A\vec{x} = \vec{y} - \vec{n}$. Multiplying both sides by A^{-1}.

So: $\vec{x} = I_N\vec{x} = (A^{-1}A)\vec{x} = A^{-1}(A\vec{x}) = A^{-1}(\vec{y} - \vec{n}) \Rightarrow \boxed{\vec{x} = A^{-1}(\vec{y} - \vec{n})}$

(b). Since $\vec{\hat{w}} = \alpha_1\lambda_1\vec{b}_1 + \cdots + \alpha_N\lambda_N\vec{b}_N$ where $\vec{w} = \alpha_1\vec{b}_1 + \cdots + \alpha_N\vec{b}_N$, so this means that:
for eigenvectors with large eigenvalues, the noise signals along will be $\boxed{\text{amplified}}$
vice versa, for eigenvectors with small eigenvalues, the noise will be: $\boxed{\text{attenuated}}$.

(c). A_1 is an $\boxed{\text{identity matrix } (100 \times 100)}$

$\boxed{\text{Yes,}}$ there are differences between A_2 and A_3 by inspection.

(d). By decreasing the absolute value of the eigenvalues, the pictures get much more vague, which means that the noises are amplified with small eigenvalues.

(e). Proof. Suppose λ is an eigenvalue of a matrix A where A is invertible.
So, there exists a corresponding eigenvector $\vec{v} \neq \vec{0}$ such that $A\vec{v} = \lambda\vec{v}$, $\lambda \neq 0$
Since A has an inverse A^{-1}, multiply both sides by A^{-1}, and we get:

$$A^{-1}(A\vec{v}) = A^{-1}(\lambda\vec{v}). \Rightarrow (A^{-1}A)\vec{v} = A^{-1}\lambda\vec{v}.$$

By definition of Inverses, so $A^{-1}A = I_N$, so $(A^{-1}A)\vec{v} = I_N \cdot \vec{v} = \vec{v}$
which gives: $\vec{v} = A^{-1}\lambda\vec{v}$. Then, since λ is a constant, $\lambda \neq 0$.
Divide both sides by λ and we have $\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v} \Leftrightarrow A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$.
By definition, with $\vec{v} \neq \vec{0}$, so $\frac{1}{\lambda}$ is an eigenvalue of matrix A^{-1}.
Q.E.D.

4. The Dynamics of Romeo and Juliet's Love Affair.

(a). For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $a+b = c+d$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Consider $A\vec{v}_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix}$.

① Since $a+b = c+d$, so $\begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \end{bmatrix} = (a+b) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Since $\vec{v}_1 \neq 0$ and $(a+b)$ is a constant,

So, $\vec{v}_1$ by definition is an eigenvector of $\vec{A}$. Its corresponding eigenvalue $\boxed{\lambda_1 = a+b}$

② Similarly. for $\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}$, we have $A\vec{v}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} b \\ -c \end{bmatrix} = \begin{bmatrix} ab-bc \\ bc-cd \end{bmatrix}$

Since $a+b = c+d$, so $a-c = d-b$, and so $(a-c) \cdot \vec{v}_2 = \begin{bmatrix} (a-c)b \\ (a-c)(-c) \end{bmatrix} = \begin{bmatrix} (a-c)b \\ (b-d)(-c) \end{bmatrix} =$

$= \begin{bmatrix} ab-bc \\ cd-bc \end{bmatrix}$, which implies that $A\vec{v}_2 = (a-c)\vec{v}_2$. Since $\vec{v}_2 \neq 0$, $(a-c)$ is constant,

so again, $\vec{v}_2$ by definition is an eigenvector of $\vec{A}$. Its corresponding eigenvalue is $\boxed{\lambda_2 = a-c}$

Thus, the eigenspace is $\boxed{\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}b\\-c\end{bmatrix}\right\}}$ or equivalently, $\text{span}\left\{\begin{bmatrix}a\\a\end{bmatrix}, \begin{bmatrix}b\\-c\end{bmatrix}\right\}$

(b). Since $A = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$ is just a special case of the generalized part (a). with $a+b = c+d$. Specifically, $a = d = 0.75$, $b = c = 0.25$

So the first eigenpair is $\lambda_1 = a+b = 0.75+0.25 = 1$, and $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The second eigenpair is $\lambda_2 = a-c = 0.75-0.25 = 0.5$, and $\vec{v_2} = \begin{bmatrix} b \\ -c \end{bmatrix} = \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix}$

Thus. the eigenpairs are: $\boxed{\left(1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \text{ and } \left(0.5, \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix}\right)}$.

(c) To find the set of points $\{\vec{s_*} \mid A\vec{s_*} = \vec{s_*}\}$ is equivalent to finding the eigenvectors corresponding to $\lambda = 1$. so $(A - I_2) \cdot \vec{S_*} = \vec{0}$ $\Rightarrow \begin{bmatrix} -0.25 & 0.25 \\ 0.25 & -0.25 \end{bmatrix} \cdot \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \vec{0}$.

$\Rightarrow \left[\begin{array}{cc|c} -0.25 & 0.25 & 0 \\ 0.25 & -0.25 & 0 \end{array}\right]$. R_2: Add R_1. $\Rightarrow \left[\begin{array}{cc|c} -0.25 & 0.25 & 0 \\ 0 & 0 & 0 \end{array}\right]$ which gives $-0.25 s_1 + 0.25 s_2 = 0$ $\Rightarrow s_1 = s_2$

So the steady states are $\boxed{\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}}$.

(d). We've shown that A is a special case for our generalized situation in part (a), and so we proved that (with $b = c = 0.25$). $\begin{bmatrix} b \\ -c \end{bmatrix} = \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix}$ is an eigenvector; which means that $\forall \alpha \in \mathbb{R}$, $\begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix} \alpha$ is also an eigenvector. Take $\alpha = 4$, so $\begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix} \cdot 4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and

so $\text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix}\right\}$ are eigenvectors with an eigenvalue $\lambda_2 = a - c = 0.5$.

Thus. $\forall \vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$, $\vec{S}[1] = A\vec{S}[0] = \lambda_2 \vec{s}[0] = 0.5\,\vec{s}[0] \in \text{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Thus, $\vec{S}[n] = 0.5^n\, \vec{s}[0] = \begin{bmatrix} 0.5^n \\ -0.5^n \end{bmatrix}$ which means that as $\boxed{n \to \infty,\ \vec{s}[n] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$.

which implies that Romeo and Juliet both have a neutral stance towards each other.

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(e). Since here, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. with $a=b=c=d$, so A is just another special case of part (a), with $a+b=c+d=2$.

So, the first eigenpair is: $\lambda_1 = a+b = 2$ and $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and

the second eigenpair is: $\lambda_2 = a-c = 1-1 = 0$. and $\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Thus, the eigenpairs are: $\boxed{\left(2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \text{ and } \left(0, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)}$

f). Since $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector, so $\text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ are all eigenvectors.

which means that $\vec{S}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ is an eigenvector, so we have that $\vec{s}[1] = A\vec{s}[0] = \lambda_2 \vec{s}[0] = \vec{0}$, so $\forall n \geq 1$, $\vec{s}[n] = A\vec{s}[n-1] = \vec{0}$.

Thus, their relationship again falls into a $\boxed{\text{neutral stance towards each other}}$

Specifically, as $n \to \infty$, $\boxed{\vec{s}[n] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

(g). Since $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector, so $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is also an eigenvector.

So, $\vec{s}[1] = A\vec{S}[0] = \lambda_1 \vec{s}[0] = 2\vec{s}[0] = 2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

which we can then generalize that $\forall n \in \mathbb{N}$, $\vec{s}[n] \in \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$\Rightarrow$ they are all eigenvectors.

which implies that $\vec{s}[n] = A\vec{s}[n-1] = \lambda_1 \vec{s}[n-1] = 2^n \vec{s}[0] = \begin{bmatrix} 2^n \\ 2^n \end{bmatrix}$.

Thus, as $n \to \infty$, $2^n \to \infty$, so $\boxed{\vec{s}[n] = \infty \cdot S[0]}$ if $R[0] = J[0] > 0$.

In other words, Romeo and Juliet would each have stronger love/like towards each other. and thus, developing a stronger relationship over time. and for $R[0] < 0$, $\boxed{\text{vice versa. (stronger hatred)}}$

(h) Again. Since $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ with $a=d=1$. $b=c=-2$, this is yet another special case of part (a), with $a+b = c+d = -1$.

So ① the first eigenpair is $\lambda_1 = a+b = -1$. and $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

② the second eigenpair is $\lambda_2 = a-c = 1-(-2) = 3$, $\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

Thus, the eigenpairs are. $\boxed{\left(-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \text{ and } \left(3, \begin{bmatrix} -2 \\ 2 \end{bmatrix}\right)}$

(i). Since $\begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with $-2 \in \mathbb{R}$, so $\text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} -2 \\ 2 \end{bmatrix}\right\}$ are all eigenvectors.

which implies $\vec{s}[0]$ is an eigenvector; so $\vec{s}[1] = A\vec{S}[0] = \lambda_2 \vec{s}[0] = 3\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.

In other words, $\forall n \in \mathbb{N}$, $\vec{s}[n] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ is an eigenvector, with $\vec{s}[n] = 3^n \vec{s}[0]$.

→ Case 1: if $R[0] > 0$, $J[0] < 0$, then Romeo will have growing love/like for Juliet, while Juliet will have growing hate towards Romeo

In other words, as $n \to \infty$, $3^n \to \infty$, so $\vec{s}[n] = \begin{bmatrix} \infty \\ -\infty \end{bmatrix}$.

→ Case 2: if $R[0] < 0$, $J[0] > 0$, then vice versa, Romeo has growing hatred towards Juliet, and Juliet having growing love/like towards Romeo.

In other words, as $n \to \infty$, $3^n \to \infty$, so $\vec{s}[n] = \begin{bmatrix} -\infty \\ \infty \end{bmatrix}$.

(j) Similar to our argument in (g), so $\forall n \in \mathbb{N}$. $\vec{s}[n] \in \text{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and so $\vec{s}[n] = (-1)^n \vec{s}[0]$.

Thus, as $n \to$ $\boxed{\text{we can't decide } \vec{s}[n]}$ and similarly we can't decide Romeo and Juliet's exact relationship – we only know they either maintained initial states or swapped entirely.

6. Homework Process and Study Group

I worked alone without getting any help, except asking questions and reading posts (especially answers from the GSIs) on Piazza as well as reading the Notes of the course.

EE16A Homework 5

Question 2: Counting The Paths of a Random Surfer

In [47]:
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

T = np.array([
    [0, 1, 1/3, 1/3],
    [0, 0, 1/3, 1/3],
    [0, 0, 0, 1/3],
    [1, 0, 1/3, 0]
])

eig_val, normalized_eig = np.linalg.eig(T)

print(eig_val)
print()

eig_vec = normalized_eig[:,0]
print("Correponding normalized eig_vec for lambda = 1 is:")
print(eig_vec)
print()

print("Corresponding eig_vec that has values sum to 1 is:")
print(eig_vec / sum(eig_vec))
```

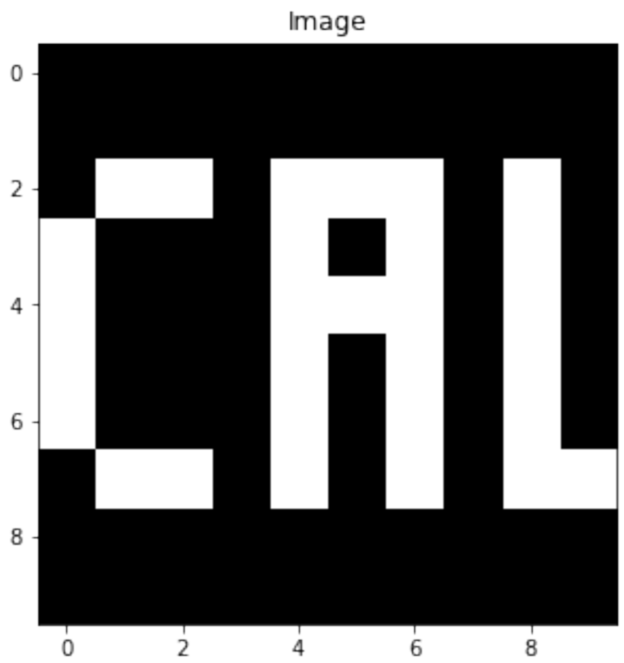
```
[ 1.        +0.j         -0.33333333+0.47140452j -0.33333333-0.47140
452j
 -0.33333333+0.j        ]

Correponding normalized eig_vec for lambda = 1 is:
[-0.61357199+0.j -0.306786   +0.j -0.2300895 +0.j -0.69026849+0.j]

Corresponding eig_vec that has values sum to 1 is:
[0.33333333-0.j 0.16666667-0.j 0.125     -0.j 0.375     -0.j]
```

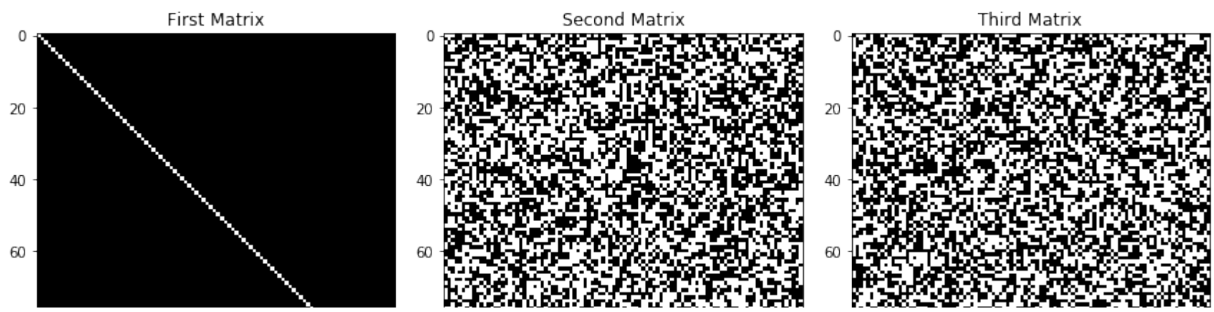
Question 3: Noisy Images

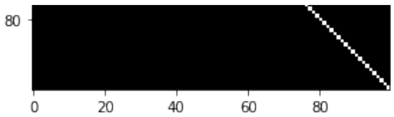
In [48]:
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

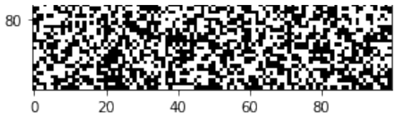
Question 3: Part c

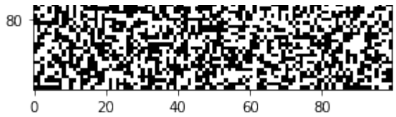
In [50]:
```
# Let's load some data to start off with.
A3 = np.loadtxt("cond_10e6.txt", delimiter=',').reshape(100,100)
A2 = np.loadtxt("cond_1e3.txt", delimiter=',').reshape(100,100)
A1 = np.eye(100)
img = np.loadtxt("image.txt", delimiter=',').reshape(10,10)

# The code below displays the image and the set of masks.
plt.figure(figsize=(5,5))
plt.imshow(img,cmap='gray')
plt.title('Image')
plt.figure(figsize=(12,5))
plt.subplot(131)
plt.imshow(A1,cmap='gray')
plt.title('First Matrix')
plt.subplot(132)
plt.imshow(A2,cmap='gray')
plt.title('Second Matrix')
plt.subplot(133)
plt.imshow(A3,cmap='gray')
plt.title('Third Matrix')
plt.tight_layout()
```











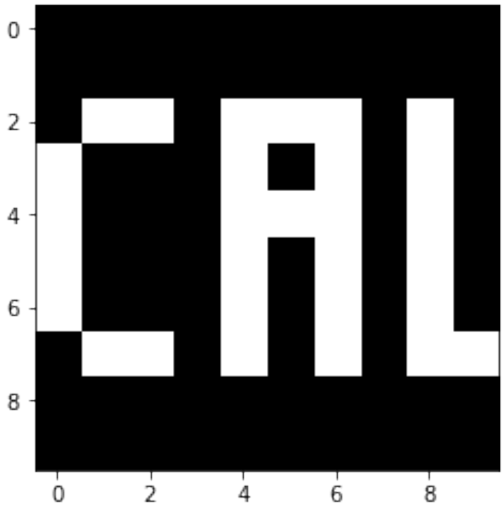
Question 3: Parts d

In [51]:
```
# We'll use numpy.random to make some noise.
noise = np.random.normal(0.5,0.1)

# Lets compute the b vector for each matrix and add some noise to the 
b1 = A1.dot(img.reshape(100)) + noise
b2 = A2.dot(img.reshape(100)) + noise
b3 = A3.dot(img.reshape(100)) + noise
```

In [52]:
```
# First, let's compute x1 after adding noise and find the minimum eige
x1 = np.linalg.inv(A1).dot(b1)
eigenvalues1 = np.linalg.eig(A1)[0]
print("Is the matrix invertible?", abs(np.linalg.det(A1)) > 0.5)
print("The smallest eigenvalue is:", min(np.absolute(eigenvalues1)))
print("Number of eigenvectors:", len(eigenvalues1))
plt.imshow(x1.reshape(10,10), cmap='gray')
```

```
Is the matrix invertible? True
The smallest eigenvalue is: 1.0
Number of eigenvectors: 100
```

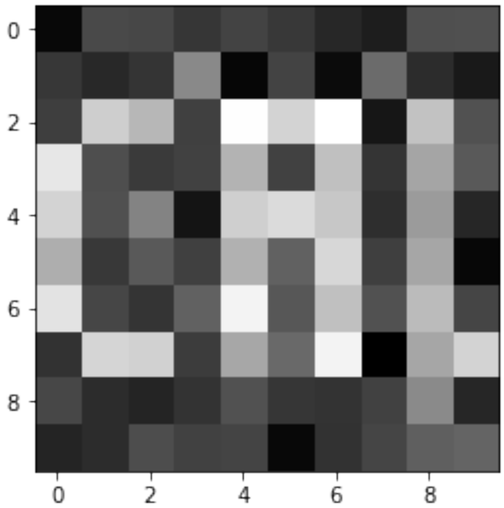
Out[52]: <matplotlib.image.AxesImage at 0x10fe5f5f8>



In [53]:
```
# Now let's compute x2 and find the minimum eigenvalue of A2.
x2 = np.linalg.inv(A2).dot(b2)
eigenvalues2 = np.linalg.eig(A2)[0]
print("Is the matrix invertible?", abs(np.linalg.det(A2)) > 0.5)
print("The smallest eigenvalue is:", min(np.absolute(eigenvalues2)))
print("Number of eigenvectors:", len(eigenvalues2))
plt.imshow(x2.reshape(10,10), cmap='gray')
```

```
Is the matrix invertible? True
The smallest eigenvalue is: 0.29516363308630184
Number of eigenvectors: 100
```

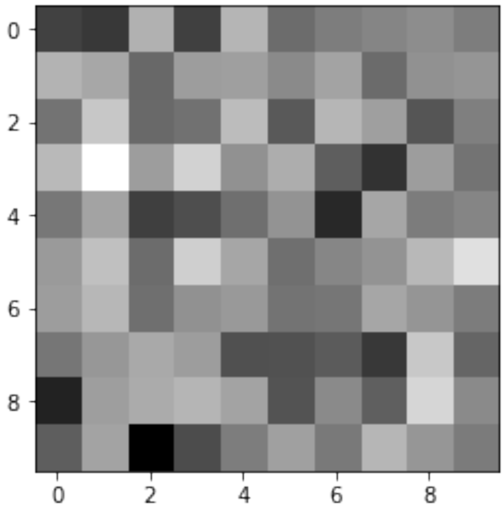
Out[53]:
```
<matplotlib.image.AxesImage at 0x119db5828>
```



In [54]:
```
# Now let's compute x3 and find the minimum eigenvalue of A3.
x3 = np.linalg.inv(A3).dot(b3)
eigenvalues3 = np.linalg.eig(A3)[0]
print("Is the matrix invertible?", abs(np.linalg.det(A3)) > 0.5)
print("The smallest eigenvalue is:", min(np.absolute(eigenvalues3)))
print("Number of eigenvectors:", len(eigenvalues3))
plt.imshow(x3.reshape(10,10), cmap='gray')
```

```
Is the matrix invertible? True
The smallest eigenvalue is: 1.2184217510026823e-05
Number of eigenvectors: 100
```

Out[54]: <matplotlib.image.AxesImage at 0x11a624cc0>



In [7]:

In [ ]:

1. (a). $\det(A-\lambda I_2) = \det\left(\begin{bmatrix} 5-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = (5-\lambda)(2-\lambda) - 0 = 0. \Rightarrow \lambda = 5, 2.$

① $\boxed{\lambda_1 = 5} \Rightarrow (A-5I_2)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -3 & 0 \end{array}\right] \Rightarrow y = 0 \Rightarrow$ eigenspace: $\boxed{\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}}$.

② $\boxed{\lambda_2 = 2.} \Rightarrow (A-2I_2)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 3 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x = 0 \Rightarrow$ eigenspace: $\boxed{\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}}$

(b). $\det(A-\lambda I_2) = \det\left(\begin{bmatrix} 22-\lambda & 6 \\ 6 & 13-\lambda \end{bmatrix}\right) = (22-\lambda)(13-\lambda) - 36 = 0. \Rightarrow \lambda = 25, 10.$

① $\boxed{\lambda_1 = 25} \Rightarrow (A-25I_2)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} -3 & 6 & 0 \\ 6 & -12 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -3 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x = 2y$

so eigenspace correspondingly is: $\boxed{\text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}}$

② $\boxed{\lambda_2 = 10} \Rightarrow (A-10I_2)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 12 & 6 & 0 \\ 6 & 3 & 0 \end{array}\right]$

$\Rightarrow \left[\begin{array}{cc|c} 12 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow y = -2x \Rightarrow$ corresponding eigenspace is: $\boxed{\text{span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}}$

(d). $\det(A-\lambda I_2) = \det\left(\begin{bmatrix} \sqrt{3}/2-\lambda & -1/2 \\ 1/2 & \sqrt{3}/2-\lambda \end{bmatrix}\right) = \left(\frac{\sqrt{3}}{2}-\lambda\right)\left(\frac{\sqrt{3}}{2}-\lambda\right) - \left(\frac{1}{2}\right)\cdot\left(-\frac{1}{2}\right) = 0$

$\Rightarrow \lambda^2 - \sqrt{3}\lambda + 1 = 0. \Rightarrow \lambda = \frac{\sqrt{3}+i}{2}, \frac{\sqrt{3}-i}{2}$

① $\boxed{\lambda_1 = \frac{\sqrt{3}+i}{2}}$. So $\left(A - \frac{\sqrt{3}+i}{2}I_2\right)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} -i/2 & -1/2 & 0 \\ 1/2 & -i/2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -i/2 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow y = -ix \Rightarrow$ so corresponding eigenspace is: $\boxed{\text{span}\left\{\begin{bmatrix} 1 \\ -i \end{bmatrix}\right\}}$

② $\boxed{\lambda_2 = \frac{\sqrt{3}-i}{2}}$. So $\left(A - \frac{\sqrt{3}-i}{2}I_2\right)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} i/2 & -1/2 & 0 \\ 1/2 & i/2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} i/2 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow y = ix \Rightarrow$ corresponding eigenvector: $\boxed{\text{span}\left\{\begin{bmatrix} 1 \\ i \end{bmatrix}\right\}}$

3. (c). See IPython Notebook. Matrix A_1 is the identity matrix.
And there are almost no visible differences between matrices A_2 and A_3.

(d). Notice that we're considering eigenvalue in absolute value.
As the absolute value of the smallest eigenvalue decreases, the noise $\boxed{\text{increases}}$.

4. (a). ① For $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $A\vec{v_1} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = (a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ since $a+b = c+d$.
Thus, $\lambda_1 = a+b$, its eigenspace is $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

② For $\vec{v_2} = \begin{bmatrix} b \\ -c \end{bmatrix}$, so $A\vec{v_2} = \begin{bmatrix} ab-bc \\ bc-cd \end{bmatrix} = (a-c)\begin{bmatrix} b \\ -c \end{bmatrix} = (d-b)\begin{bmatrix} b \\ -c \end{bmatrix}$ since $a+b = c+d$.
which gives $a-c = d-b$. so $\lambda_2 = a-c$, its eigenspace is $\text{span}\left\{\begin{bmatrix} b \\ -c \end{bmatrix}\right\}$.