Normal Distribution

STAT 330 - Iowa State University

Outline

In this lecture, students will learn about the Normal Distribution. They will be introduced to the Standard Normal Table (z-table) that will be used to find probabilities for a Normal random variable

Normal Distribution

Normal Distribution

Setup: The normal distribution is commonly used to model a wide variety of variables (weight, height, temperature. voltage, etc) due to its *"bell-shaped"* and *symmetric* shape.

If a random variable X follows a *normal distribution*,

$$X \sim N(\mu, \sigma^2)$$

where μ is the mean, and $\sigma^2 > 0$ is the variance

- Probability Density Function (pdf)

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \text{for } -\infty < x < \infty$$

- Expected Value: $\mathbb{E}(X) = \mu$
- Variance: $Var(X) = \sigma^2$

Normal PDF

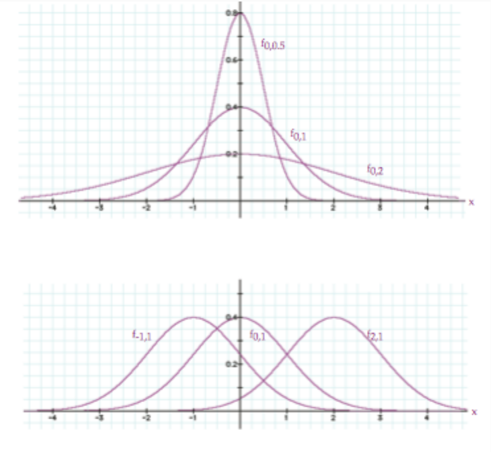


Figure 1: PDFs for normal distribution with various μ and σ^2

μ determines the location of the peak in the $x-$axis,

σ^2 determines the "width" of the bell shape.

Normal CDF

- Cumulative Distribution Function (cdf)

$$F_X(t) = \int_{-\infty}^{t} f(x)dx \qquad \text{(no closed form)}$$

→ The normal cdf does not have a closed form expression.

→ Use cdf table ($z-$table) of *standard normal distribution* $N(\mu = 0, \sigma^2 = 1)$ to obtain probabilities.

→ We need to *standardize* any normal random variable, X, into standard normal random variable, Z.

Standardization of Normal Distribution

Let $X \sim N(\mu, \sigma^2)$. Then,

1. $Z = \frac{X-\mu}{\sigma}$ is a standard normal random variable
2. $Z \sim N(0, 1)$ (normal distribution with $\mu = 0$, $\sigma^2 = 1$)

Standardization

Example 1: Suppose $X \sim N(20, 100)$. What is the probability that X is less than 23.5?

To find this probability, we usually ...

- Integrate the PDF (too difficult)
- Plug into CDF (no closed form for CDF of X)

Instead we standardize X, and obtain probabilities using the standard normal cdf table ($z-$table)

- The standardized R.V is $Z = \frac{X-\mu}{\sigma} = \frac{X-20}{\sqrt{100}} \sim N(0, 1)$
- The standardized observation is $z = \frac{x-\mu}{\sigma} = \frac{23.5-20}{\sqrt{100}} = 0.35$
- $\mathbb{P}(X < 23.5) = \mathbb{P}(Z < 0.35)$ (obtain this from $z-$table)

Standard Normal Distribution

Standard Normal Distribution

Suppose a random variable, X, follows a $N(\mu, \sigma^2)$ distribution. Then, $Z = \frac{X-\mu}{\sigma}$ follows a *standard normal distribution*

$$Z \sim N(0, 1)$$

- Probability Density Function (pdf)

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \qquad \text{for } -\infty < z < \infty$$

- Expected Value:

$$\mathbb{E}(Z) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{\mathbb{E}(X)-\mu}{\sigma} = \frac{\mu-\mu}{\sigma} = 0$$

- Variance:

$$Var(Z) = Var\left(\frac{X-\mu}{\sigma}\right) = Var\left(\frac{X}{\sigma}\right) = \frac{1}{\sigma^2} Var(X) = \frac{\sigma^2}{\sigma^2} = 1$$

Standard Normal CDF

- Cumulative Distribution Function (cdf)

$$F_Z(t) = \int_{-\infty}^{t} f(z)dz = \Phi(t) \qquad \text{(no closed form)}$$

→ Just like the normal cdf, the standard normal cdf does not have a closed form expression.

→ The cdf of N(0,1) random variable is denoted by $\Phi(t)$ (or more commonly $\Phi(z)$)

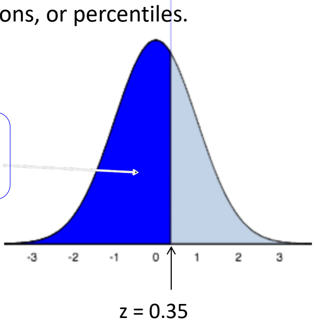
→ The values of the cdf, $\Phi(z)$, are found in the standard normal table (*z-table*)

Z-table (Standard Normal Table)

Z-table

- Z-Table gives proportion of normal curve less than a particular z score
  - Gives left-hand area (dark blue shaded region)
  - This is same as the *percentile* value for z
  - Can be referred to as areas, proportions, or percentiles.
  - Denoted P(Z< z)

Proportion of area less than z=0.35
Denoted as "P(Z<0.35)"



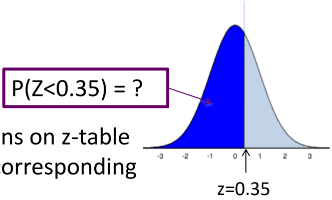
How to read the Z-table

- z values range from –3.99 to 3.99 on the z-table
- Row – ones and tenths place for z
- Column – hundredths place for z
- P(Z < z) found *inside* z-table

z	Second decimal place in z 0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	[illegible]	[illegible]	0.5239
0.1	0.5398	0.5438	0.5478	[illegible]	[illegible]	[illegible]	0.5636
0.2	0.5793	0.5832	[illegible]	[illegible]	[illegible]	[illegible]	0.6026
0.3	0.6179	[illegible]	[illegible]	[illegible]	0.6331	0.6368	0.6406
0.4	0.6554	[illegible]	[illegible]	0.6664	0.6700	0.6736	0.6772

Left-hand area/probability
P(Z < z)

How to read the Z-table



- Look up z = 0.35 in the margins on z-table
- Percentile/left-hand area is corresponding value inside z-table

z	Second decimal place in z						
	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772

P(Z < 0.35)

Examples

Z-table Practice

Suppose $Z \sim N(0, 1)$

1. $\mathbb{P}(Z < 1)$

2. $\mathbb{P}(Z > -2.31)$

Z-table Practice

Suppose $Z \sim N(0,1)$

3. $\mathbb{P}(0 < Z < 1)$

4. $\mathbb{P}(|Z| > 2)$

Normal Distribution Example

Suppose $X \sim N(1, 2)$, and we want to find $\mathbb{P}(1 < X < 2)$.

Normal Distribution Example

Recap

Students should now be familiar with the Normal distribution. They should be able to use the z-table to find probabilities for a Normal random variable.