Gamma Distribution

STAT 330 - Iowa State University

Outline

In this lecture, students will be introduced to the Gamma distribution.

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Setup: The gamma distribution is commonly used to model the total time for a procedure composed of α independent occurrences, where the time between each occurrence follows $Exp(\lambda)$

If a random variable follows a *Gamma distribution*,

$$X \sim Gamma(\alpha, \lambda)$$

where $\lambda > 0$ is there rate parameter, and $\alpha > 0$ is the shape parameter

- Probability Density Function (pdf)
  - $Im(X) = (0, \infty)$
  - $f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$

  where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ is called the "gamma function".

Gamma PDF

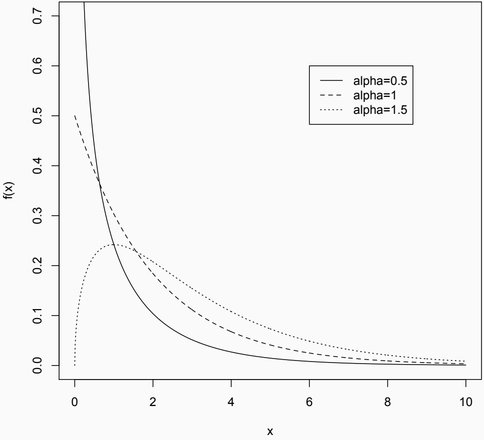

Figure 1: PDFs for gamma distribution with fixed λ and $\alpha = 0.5, 1, 1.5$

Gamma Distribution Summary

- Cumulative distribution function (cdf)

$$F_X(t) = \int_0^t f(x)dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^t x^{\alpha-1} e^{-\lambda x} dx$$

- Expected Value: $\mathbb{E}(X) = \frac{\alpha}{\lambda}$
- Variance: $Var(X) = \frac{\alpha}{\lambda^2}$

Examples

Gamma Distribution Example

Example 1: Compilation of a computer program consists of 3 blocks that are processed sequentially, one after the other. Each block is independent of the other blocks, and takes Exponential time with mean of 5 minutes. We are interested in the total compilation time.

- Total compilation time modeled using Gamma distribution.

  Define the R.V: T = total compilation time

  Distribution of T: $T \sim Gamma(\alpha, \lambda) \equiv Gamma(?, ?)$

- What value should we use for α and λ?
  - α is the number of independent occurrences (blocks) in the full procedure: $\alpha = 3$
  - Time for each occurrence (call this "T_i") is exponential with mean 5 min. $\mathbb{E}(T_i) = \frac{1}{\lambda} = 5 \rightarrow \lambda = \frac{1}{5}$

Gamma Distribution Example

$$T = \text{total compilation time}$$

$$T \sim \mathit{Gamma}\left(3, \frac{1}{5}\right)$$

1. What is the expected value of total compilation time?

2. What is the variance of total compilation time?

3. What is the probability for the entire program to be compiled in less than 12 minutes.

Gamma-Poisson Formula

Gamma Distribution Example

- Could answer the previous question by using the Gamma CDF directly (requires repeated integration by parts)
- Instead, simplify Gamma probabilities by turning it into a Poisson problem!
- Turn a Gamma random variable into Poisson random variable using the Gamma-Poisson formula.

Gamma-Poisson Formula

For $T \sim \text{Gamma}(\alpha, \lambda)$ and $X \sim \text{Pois}(\lambda t)$,

$$\mathbb{P}(T > t) = \mathbb{P}(X < \alpha)$$

and

$$\mathbb{P}(T \leq t) = \mathbb{P}(X \geq \alpha)$$

Gamma Distribution Example

3. What is the probability for the entire program to be compiled in less than 12 minutes.

- Step 1: Define our Gamma random variable:
  $T \sim Gamma(\alpha, \lambda) \equiv Gamma(3, \frac{1}{5})$
  We want to know $\mathbb{P}(T < t) = \mathbb{P}(T < 12) =?$
- Step 2: Convert the Gamma R.V (T) into a Poisson R.V (X):
  $X \sim Pois(\lambda t) \equiv Pois(\frac{1}{5} \cdot 12) \equiv Pois(2.4)$
- Step 3: Use Gamma-Poisson formula: $\mathbb{P}(T \leq t) = \mathbb{P}(X \geq \alpha)$

$$\begin{aligned}
\mathbb{P}(T < 12) &= \mathbb{P}(T \leq 12) \\
&= \mathbb{P}(X \geq 3) \\
&= 1 - F_X(2) \\
&= 1 - 0.5697 \\
&= 0.4303
\end{aligned}$$

Gamma Distribution Example

4. What is the probability that it takes at least 5 minutes to compile the entire program?

Recap

Students should be familiar with the Gamma distribution. They should be able to use the gamma-poisson formula to find probabilities for a Gamma random variable.