Law of Total Probability

STAT 330 - Iowa State University

outlined

In this lecture students will be introduced to:

1. The idea of a cover/partition of a sample space
2. Setting up a tree diagram for a partitioned sample space
3. The Law of Total Probability

Tree Diagram

Tree Diagram

Example 1: Suppose you randomly select one of 3 boxes, and then randomly select a coin from inside the box. The contents of the boxes are . . .

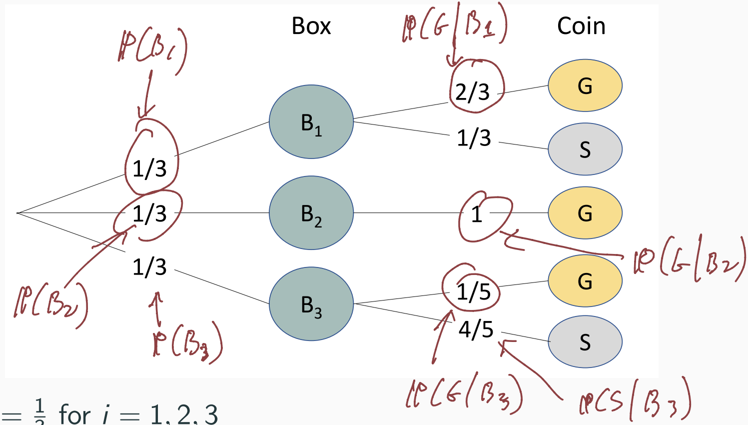
- Box 1: 2 gold coins, 1 silver coin
- Box 2: 3 gold coins
- Box 3: 1 gold coin, 4 silver coins

Let events $B_i = i^{th}$ box is selected for $i = 1, 2, 3$,
$G =$ gold coin selected, and $S =$ silver coin selected.

We can visualize this *step-wise procedure* with a *tree diagram*.

Using a Tree Diagram

A tree diagram shows all possible outcomes of step-wise procedures



$\mathbb{P}(B_i) = \frac{1}{3}$ for $i = 1, 2, 3$

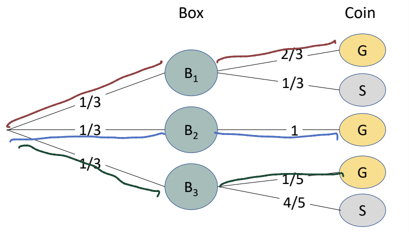
$\mathbb{P}(G|B_1) = \frac{2}{3}$, $\mathbb{P}(S|B_1) = \frac{1}{3}$

$\mathbb{P}(G|B_2) = 1$

$\mathbb{P}(G|B_3) = \frac{1}{5}, \mathbb{P}(S|B_3) = \frac{4}{5}$

Using a Tree Diagram Cont.

What is the probability of choosing a gold coin $\mathbb{P}(G)$?



- What are the *"total"* different paths to get to gold coin?

  $(B_1 \cap G)$ or $(B_2 \cap G)$ or $(B_3 \cap G)$

- These are disjoint events

$$\mathbb{P}(G) = \mathbb{P}(B_1 \cap G) + \mathbb{P}(B_2 \cap G) + \mathbb{P}(B_3 \cap G)$$

$$= \mathbb{P}(B_1)\mathbb{P}(G|B_1) + \mathbb{P}(B_2)\mathbb{P}(G|B_2) + \mathbb{P}(B_3)\mathbb{P}(G|B_3)$$

$$= (1/3)(2/3) + (1/3)(1) + (1/3)(1/5) = .622$$

This calculation is done using *Law of Total Probability.*

Law of Total Probability

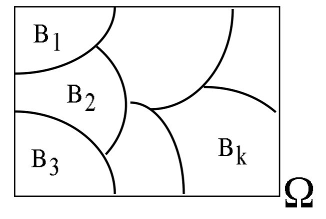
Cover/Partition

Definition:

A collection of events $B_1, \dots B_k$ is a *cover* or *partition* of Ω if

1. the events are pairwise disjoint ($B_i \cap B_j = \emptyset$ for $i \neq j$), and
2. the union of the events is Ω ($\bigcup_{i=1}^{k} B_i = \Omega$).

We can represent a cover using a Venn diagram:



Note: In a tree diagram, the branches of the tree form a cover.

Law of Total Probability

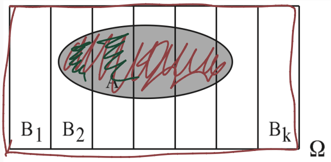
Theorem (Law of Total Probability)

If the collection of events $B_1, \ldots, B_k$ is a cover of Ω, and A is an event, then

$$\boxed{\mathbb{P}(A) = \sum_{i=1}^{k} \mathbb{P}(A|B_i)\mathbb{P}(B_i).}$$

Proof

- $A = (B_1 \cap A) \cup \ldots \cup (B_k \cap A)$
- $\mathbb{P}(A) = \mathbb{P}(B_1 \cap A) + \ldots + \mathbb{P}(B_k \cap A)$
  $= \mathbb{P}(A|B_1)\mathbb{P}(B_1) + \ldots + \mathbb{P}(A|B_k)\mathbb{P}(B_k)$



Recap

Students should now be familiar with:

1. The idea of a cover/partition of a sample space
2. Setting up a tree diagram for a partitioned sample space
3. The Law of Total Probability