Random Variables & Distributions

STAT 330 - Iowa State University

Outline

In this lecture, students will learn what a random variable is. We start with discrete random variables. Students are introduced to the probability distribution of a discrete random variable that can be described with:

1. A probability mass function (pmf)
2. A cumulative distribution function (cdf)

Random Variable

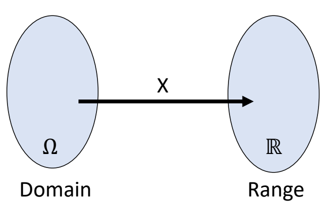
Random Variable

- Random variables (R.V.) connect a random experiment to data
- Denote random variables with capital letters (X, Y, Z, etc)
- The values of a R.V. are determined by the outcome of a random experiment.

Definition

A *random variable (R.V.)* is a function that maps the sample space (Ω) to real numbers ($\mathbb{R}$)

$$X : \Omega \to \mathbb{R}$$



Random Variable Cont.

Example 1: Suppose you toss 3 coins, and observe the face up for each flip. $\Omega = \{HHH, HHT, \dots, TTT\}$. $|\Omega| = 8$

We are interested in the number of heads we obtain in 3 coin tosses.

What is the random variable X?

$X = \#$ of heads in 3 coin tosses

Notation:

$X \equiv$ Random variable

$x \equiv$ Realized value

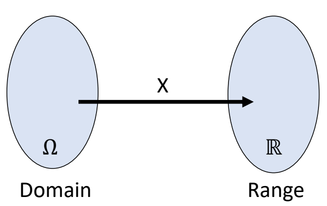
$X = x \Rightarrow$ "random variable X takes on the value x".

$\{X = x\}$ is just an event

Consider the event 1 or 2 heads. This is $\{X = 1\} \cup \{X = 2\}$

Types of Random Variables

Types of Random Variables



Two types of random variables:

Discrete Random Variable

Sample space (Ω) maps to finite or countably infinite set in $\mathbb{R}$

Ex: $\{1, 2, 3\}$, $\{1, 2, 3, 4, \ldots\}$

Continuous Random Variable

Sample space (Ω) maps to an uncountable set in $\mathbb{R}$.

Ex: $(0, \infty)$, $(10, 20)$

Image of a Random Variable

Definition

The *image* of a random variable is defined as the values the random variable can take on.

$$Im(X) = \{x : x = X(\omega) \text{ for some } \omega \in \Omega\}$$

Example 2:

1. Put a disk drive into service. Let $Y =$ time till the first major failure. $Im(Y) = (0, \infty)$.
   Image of Y is an interval (uncountable)
   $\rightarrow Y$ is a continuous random variable.
2. Flip a coin 3 times. Let $X = \#$ of heads obtained.
   $Im(X) = \{0, 1, 2, 3\}$. Image of X is a finite set
   $\rightarrow X$ is a discrete random variable.

Probability Mass Function

Probability Mass Function

Two things to know about a random variable X:

(1) What are the values X can take on? (what is its image?)

(2) What is the probability that X takes on each value?

(1) and (2) together gives the *probability distribution* of X.

Definition

Let X be a discrete random variable.
The *probability mass function (pmf)* of X is $p_X(x) = \mathbb{P}(X = x)$.

Properties of pmf:

1. $0 \leq p_X(x) \leq 1$
2. $\sum_{x \in Im(x)} p_X(x) = 1$

Probability Mass Function Cont.

Example 3: Which of the following are *valid* probability mass functions (pmfs)?

1.

x	-3	-1	0	5	7
$p_X(x)$	0.1	0.45	0.15	0.25	0.05

2.

y	-1	0	1.5	3	4.5
$p_Y(y)$	0.1	0.45	0.25	-0.05	0.25

3.

z	0	1	3	5	7
$p_Z(z)$	0.22	0.18	0.24	0.17	0.18

Probability Mass Function Cont.

Example 4: Suppose you toss 3 coins, and observe the face up for each flip. We are interested in the number of heads we obtain in 3 coin tosses.

1. Define the random variable X.

2. What is the image of X?

3. What is the pmf of X? (find $p_X(x)$ for all x)

Probability Mass Function Cont.

Cumulative Distribution Function

Cumulative Distribution Function

Definition

The *cumulative distribution function (cdf)* of X is

$$F_X(t) = \mathbb{P}(X \leq t)$$

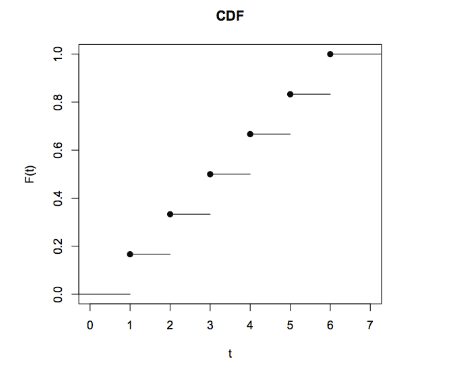
- The pmf is $p_X(x) = \mathbb{P}(X = x)$, the probability that R.V. X is equal to value x.
- The cdf is $F_X(t) = \mathbb{P}(X \leq t)$, the probability that R.V. X is less than or equal to t.

Relationship between pmf and cdf

- $F_X(t) = \mathbb{P}(X \leq t) = \sum_{x \leq t} p_X(x) = \sum_{x \leq t} \mathbb{P}(X = x)$

Properties of CDFs

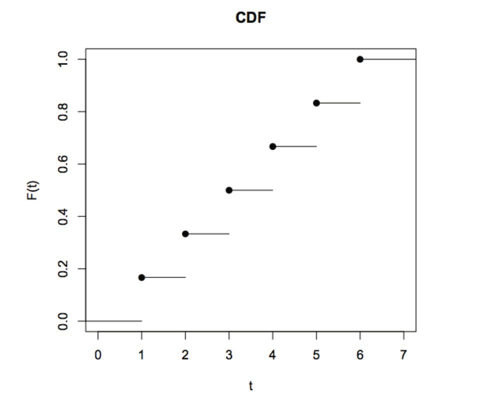
1. $0 \leq F_X(t) \leq 1$
2. F_X is non-decreasing (if $a \leq b$, then $F(a) \leq F(b)$.
3. $\lim_{t\to-\infty} F_X(t) = 0$ and $\lim_{t\to\infty} F_X(t) = 1$
4. F_X is right-continuous with respect to t



Cumulative Distribution Function Cont.

Example 6: Roll a fair die. Let X = the number of dots on face up

x	1	2	3	4	5	6
(pmf) $p_X(x)$	1/6	1/6	1/6	1/6	1/6	1/6
(cdf) $F_X(x)$	1/6	2/6	3/6	4/6	5/6	1



Cumulative Distribution Function Cont.

Example 7: Suppose you toss 3 coins, and observe the face up for each flip. We are interested in the number of heads we obtain in 3 coin tosses.

From example 4, the pmf is

x	0	1	2	3
(pmf) $p_X(x)$	1/8	3/8	3/8	1/8
(cdf) $F_X(x)$				

What is the cdf of X?

Recap

Students should now be familiar with the concept of a random variable. They should be able to present the probability distribution in table form. They should be able to use the pmf or cdf to find probabilities for a discrete random variable.