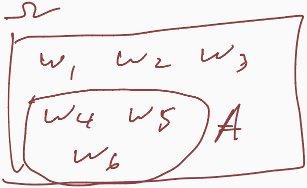
Introduction to Combinatorics

STAT 330 - Iowa State University

Outline

In this lecture students will be introduced to combinatorics. We will look at:

1. The Classical Definition of Probability
2. The Multiplication Rule
3. Ways to select objects



$$\mathbb{P}(A) = \sum \mathbb{P}(\omega_i)$$

Equally likely outcomes

n outcomes in Ω

$$\mathbb{P}(\omega_i) = \frac{1}{n}$$

Equally Likely Outcomes

Example 1: There are 4 chips in a box; 1 chip is defective. Randomly draw a chip from the box. What is the probability of selecting the defective chip?

- Common sense: $\mathbb{P}(\text{draw defective chip}) = \frac{1}{4}$ or 25%
- Using probability theory...

  Sample space:

  $$\Omega = \{g_1, g_2, g_3, d\}$$
  $$|\Omega| = 4$$

  Event:

  $$A = \text{"draw defective chip"} = \{d\}$$
  $$|A| = 1$$

  Probability of event: $\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{1}{4}$

Equally Likely Outcomes Cont.

$\boxed{\text{Classical Def of probability}}$

Theorem

If events in sample space are equally likely (i.e. $\mathbb{P}(\{\omega\})$ is same for all $\omega \in \Omega$), then the probability of an event A is given by:

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|},$$

where $|A|$ is the number of elements in set A (cardinality of A).

Equally Likely Outcomes Cont.

Example 2: There are 4 chips in a box; 1 chip is defective. Randomly draw 2 chips from the box. What is the probability that defective chip is among the 2 chosen?

Sample space: (All possibilities for drawing 2 chips)

$$\Omega = \big\{\{g_1, g_2\}, \{g_1, g_3\}, \{g_1, d\}, \{g_2, g_3\}, \{g_2, d\}, \{g_3, d\}\big\}$$

$$|\Omega| = 6$$

Event:

$$A = \text{"defective chip is among the 2 chips drawn"}$$

$$= \{\{g_1, d\}, \{g_2, d\}, \{g_3, d\}\}$$

$$|A| = 3$$

Probability of event: $\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{3}{6} = 1/2$

Multiplication Principle

fund. Theorem of
counting

Multiplication Principle

Multiplication Principle

If a complex action can be broken down into a series of k component actions, performed one after the other, where ...

- first action can be performed in n_1 ways
- second action can be performed in n_2 ways

$\vdots$

- last action can be performed in n_k ways

Then, the complex action can be performed in $n_1 n_2 \cdots n_k$ ways.

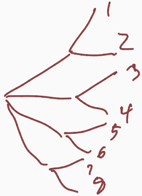
Multiplication Principle Cont.

Example 3: Your friend owns 4 shirts (red, blue, green, white), and 2 pants (blue, black). What are all the ways he can create an outfit by choosing a shirt and pants to wear?

$n_1 = 4$

$n_2 = 2$

$4 \times 2 = 8$



Example 4: Suppose licence plates are created as a sequence of 3 letters followed by 3 numbers. What is $|\Omega|$? (ie. how many license plates are in the sample space?)

$\underline{26}\ \underline{26}\ \underline{26}\quad \underline{10}\ \underline{10}\ \underline{10}$

$26^3 \cdot 10^3$

Sample selection

Sample Selection

Imagine picking k objects from a box containing n objects.

$\boxed{1, 2, 3, \ldots, n} \Rightarrow K$

Definitions

with replacement: After each selection, the object is put back in the box. It is possible to select the same object multiple times in the k selections.

without replacement: After each selection, the object is removed from the box. Cannot select the same object again.

ordered sample: Order of selected objects matters.

Example: Passwords . . . abc1 $\neq$ c1ba

unordered sample: Order of selected objects doesn't matter.

Example: Selecting people for a study.
(Mary, John, Susan) = (John, Mary, Susan)

3 Main Scenarios

There are *3 main scenarios* we will deal with ...

Suppose a box contains the letters "a", "b", "c"

$\boxed{a, b, c}$

$\Downarrow$

2

1. Ordered with replacement
   - Ex: Select 2 letters where repeat letters are allowed.

     $\Omega = \{(a, a), (b, b), (c, c), (a, b), (b, a),$
     $(a, c), (c, a), (b, c), (c, b)\}$

     $|\Omega| = 9$
2. Ordered without replacement
   - Ex: Select 2 letters where repeat letters aren't allowed.

     $\Omega = \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b)\}$

     $|\Omega| = 6$
3. Unordered without replacement
   - Ex: Consider "a", "b", "c" to be people, and you select 2 of them to be in your study. (Repeat letters aren't allowed)

     $\Omega = \{\{a, b\}, \{a, c\}, \{b, c\}\}$

     $|\Omega| = 3$
   - Here $\{a, b\}$ is same as $\{b, a\}$, so we only write one of them in the sample space.

Ultimately, we want to count up $|\Omega|$ for these scenarios.

Recap

Students should now be familiar with:

1. The Classical Definition of Probability
2. The Multiplication Rule
3. Ways to select objects

These ideas will be used in upcoming lectures for counting methods to assign probabilities under equally likely outcomes.