Expected Value and Variance

STAT 330 - Iowa State University

Outline

In this lecture, students will learn about two important properties of random variables:

1. Expectation
2. Variance

These will be defined and students will learn how to calculate each for a discrete random variable.

Expected Value

Expected Value

Example 1: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

What number of heads do we "expect" to get?

0 obtained $\frac{1}{8}$ of the time

1 obtained $\frac{3}{8}$ of the time

2 obtained $\frac{3}{8}$ of the time

3 obtained $\frac{1}{8}$ of the time

Intuitively, we can think about taking $0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$ as the "expected" number of heads

Expected Value

Definition

Let X be a discrete random variable. The *expected value* or *expectation* of $h(X)$ is

$$\mathbb{E}\left[h(X)\right] = \sum_x h(x)p_X(x) = \sum_x h(x)\mathbb{P}(X = x)$$

- The **MOST IMPORTANT** version of this is when h(x) = x

$$\mathbb{E}(X) = \sum_x xp_X(x) = \sum_x x\mathbb{P}(X = x)$$

- $\mathbb{E}(X)$ is usually denoted by μ
- $\mathbb{E}(X)$ is the weighted average of the x's, where the weights are the probabilities of the x's.

Expected Value Cont.

Example 2: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

Calculate the expected value of X.

$$\begin{aligned}
\mathbb{E}(X) &= \sum_x x p_X(x) \\
&= 0\mathbb{P}(X = 0) + 1\mathbb{P}(X = 1) + 2\mathbb{P}(X = 2) + 3\mathbb{P}(X = 3) \\
&= 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) \\
&= 1.5
\end{aligned}$$

Variance

Variance & Standard Deviation

Definition

The *variance* of a random variable X is

$$Var(X) = \mathbb{E}\big[(X - \mathbb{E}(X))^2\big] = \sum_x (x - \mathbb{E}(X))^2 \cdot p_X(x)$$

The *standard deviation* of a random variable X is

$$\sigma = \sqrt{Var(X)}$$

- $Var(X)$ is usually denoted by σ^2
- Units for variance is squared units of X.
- Units for the standard deviation is same as units for X.

SHORT CUT (use this formula to find variance)

$$\begin{aligned} Var(X) &= \mathbb{E}(X^2) - \big[\mathbb{E}(X)\big]^2 \\ &= \sum_x x^2 P(X = x) - \Big[\sum_x xP(X = x)\Big]^2 \end{aligned}$$

Variance Cont.

Example 3: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

Calculate the variance and standard deviation of X.

- $\mathbb{E}(X) = \sum_x x p_X(x) = 1.5$

- $\mathbb{E}(X^2) = \sum_x x^2 p_X(x) = 0^2\left(\frac{1}{8}\right) + 1^2\left(\frac{3}{8}\right) + 2^2\left(\frac{3}{8}\right) + 3^2\left(\frac{1}{8}\right) = 3$

Variance Cont.

- $Var(X) = \mathbb{E}(X^2) - \left[\mathbb{E}(X)\right]^2 = 3 - 1.5^2 = .75$

- $\sigma = \sqrt{Var(X)} = \sqrt{(}.75) = .866$

Properties of $\mathbb{E}(X)$ & $\mathrm{Var}(X)$

Operations with $\mathbb{E}(X)$ and Var(X)

X, Y are random variables; a, b, c are constants.

Operations with $\mathbb{E}(\cdot)$

1. $\mathbb{E}(aX) = a\mathbb{E}(X)$
2. $\mathbb{E}(aX + bY + c) = a\mathbb{E}(X) + b\mathbb{E}(Y) + c$

Operations with $Var(\cdot)$

3. $Var(aX) = a^2 Var(X)$
4. $Var(aX + b) = a^2 Var(X)$
5. $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y)$

   (when X,Y are independent, Cov(X,Y) $= 0$. We'll discuss more about independence and define covariance in later)

Chebyshev's Inequality

Chebyshev's Inequality

For any positive real number k, and random variable X with variance σ^2:

$$\mathbb{P}(|X - \mathbb{E}(X)| \leq k\sigma) \geq 1 - \frac{1}{k^2}$$

- Bounds the probability that X lies within a certain number of standard deviations from $\mathbb{E}(X)$

Recap

Students should now be familiar with the expected value and variance of a discrete random variable. They should be able to calculate both from a probability mass function.