Continuous Random Variables

STAT 330 - Iowa State University

Outline

In this lecture, students will be introduced to continuous random variables. We look at how to use the various functions associated with continuous R.Vs to answer questions.

Continuous Random Variables

Discrete vs. Continuous R.Vs

<u>Discrete Random Variable</u>

Sample space (Ω) maps to finite or countably infinite set in $\mathbb{R}$

Ex: $\{1, 2, 3\}$, $\{1, 2, 3, 4, \ldots\}$

<u>Continuous Random Variable</u>

Sample space (Ω) maps to an uncountable set in $\mathbb{R}$.

Ex: $(0, \infty)$, $(10, 20)$

- We have already learned about discrete R.Vs (Lectures 5-9)
- All properties of discrete R.Vs have direct counterparts for continuous R.Vs
- Summations (Σ) used for discrete R.V's are replaced by integrals ($\int$) for continuous R.V's.

CDF of Continuous Random Variables

Definition

Let X be a continuous random variable. The *cumulative distribution function (cdf)* of X is

$$F_X(t) = \mathbb{P}(X \leq t)$$

- All cdf properties discussed earlier still hold
  1. $0 \leq F_X(t) \leq 1$
  2. F_X is non-decreasing (if $a \leq b$, then $F_X(a) \leq F_X(b)$).
  3. $\lim_{t\to-\infty} F_X(t) = 0$ and $\lim_{t\to\infty} F_X(t) = 1$
  4. F_X is right-continuous with respect to t
- The cdf for continuous R.V is also continuous (not a step function like in discrete case)

PDF $\longleftrightarrow$ CDF

Definition

For a continuous variable X with cdf F_X, the *probability density function (pdf)* of X is defined as:

$$f_X(x) = F'_X(x) = \frac{d}{dx}F_X(x)$$

Properties of pdf:

1. $f_X(x) \geq 0$ for all x,
2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

Additionally, for continuous R.V X,

- $F_X(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^{t} f_X(x)dx$ for any $t \in \mathbb{R}$
- $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx$ for any $a, b \in \mathbb{R}$
- $\mathbb{P}(X = a) = \mathbb{P}(a \leq X \leq a) = \int_a^a f_X(x)dx = 0$ for any $a \in \mathbb{R}$

Examples

Continuous R.V Example

Example 1: Let Y be the time (in yrs) until the first major failure of a new disk drive. Suppose the probability density function (pdf) of Y is given by

$$f_Y(y) = \begin{cases} 0 & y \leq 0 \\ e^{-y} & y > 0 \end{cases}$$

1. Check whether $f_Y(y)$ is a *valid* density function.

   We need to check the 2 properties of pdfs.

   (1) $f_Y(y)$ is non-negative function on $\mathbb{R}$

   (2) $\int_{-\infty}^{\infty} f_Y(y)dy = 1$

$$\int_{-\infty}^{\infty} f_Y(y)dy =$$

Continuous R.V Example Cont.

2. What is the probability that the 1^{st} major disk drive failure occurs within the first year?

   $\mathbb{P}(Y \leq 1) =$

Continuous R.V Example Cont.

3. What is the probability that the 1^{st} major disk drive failure occurs before the first year?

   $\mathbb{P}(Y < 1) =$

Continuous R.V. Example Cont.

4. What is the probability that the 1^{st} major disk drive failure occurs after the first year?

Continuous R.V. Example Cont.

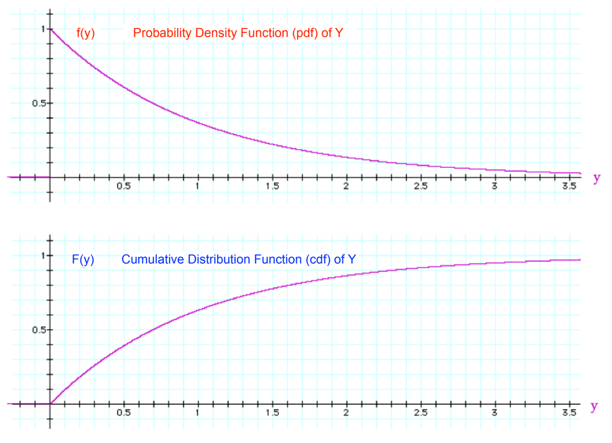
5. What is the probability that the 1^{st} major disk drive failure occurs after first year but before second year?

Continuous R.V. Example Cont.

6. What is the cumulative distribution function (cdf) of Y?

Continuous R.V Example Cont.

For Example 1, the pdf and cdf of Y are shown below.



Continuous R.V. Example Cont.

SHORT CUT: Use the cdf to calculate desired probabilities instead of integrating the pdf for each problem.

- Only need to integrate the pdf once to obtain the cdf
- Write any probability in terms of the cdf and plug in to solve

Back to Example 1:

- $\mathbb{P}(Y \leq 1) =$
- $\mathbb{P}(Y > 1) =$
- $\mathbb{P}(1 < Y < 2) =$

Additional Example

Example 2: Let X be a random variable with the following probability density function (pmf):

$$f_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } 0 \leq x \leq \frac{1}{2} \\ 2x & \text{for } \frac{1}{2} < x < 1 \\ 0 & \text{for } x \geq 1 \end{cases}$$

Additional Example Cont.

1. Give the cumulative distribution function (cdf) of X

Additional Example Cont.

Additional Example Cont.

2. What is the probability that X is less that 0.75?

Summary of Discrete & Continuous R.V.

Discrete R.V.

- Im(X) finite or countable infinite
- CDF: $F_X(t) = \mathbb{P}(X \leq t)$
  $= \sum_{x \leq t} p_X(x)$
- PMF: $p_X(x) = \mathbb{P}(X = x)$
- $\mathbb{E}(h(X)) = \sum_x h(x) p_X(x)$
- $\mathbb{E}(X) = \sum_x x p_X(x)$
- $Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$

Continuous R.V.

- Im(X) uncountable
- CDF: $F_X(t) = \mathbb{P}(X \leq t)$
  $= \int_{-\infty}^{t} f_X(x) dx$
- PDF: $f_X(x) = \frac{d}{dx} F_X(x)$
- $\mathbb{E}(h(X)) = \int_x h(x) f_X(x) dx$
- $\mathbb{P}(X) = \int_{-\infty}^{\infty} x f_X(x) dx$
- $Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$

Recap

Students should now be familiar with continuous random variables. They should know how to obtain a cdf and use the pdf or cdf to calculate probabilities for a continuous random variable. They should be aware of how to calculate expectation and variance in the continuous case.