Combinations

STAT 330 - Iowa State University

Outline

In this lecture students will learn about combinations. The combination number gives us the number of ways to choose k objects from n when we have an unordered sample, without replacement. We will use the combination number to find probabilities.

Combination

Unordered Without Replacement

Select k objects out of n objects with *no replacement* where *order does not matter*.

$$\Omega = \{(x_1, \ldots, x_k) : x_i \in \{1, \ldots, n\}, x_i \neq x_j\}$$

To derive $|\Omega|$ for this scenario, we can go back to how it was derived for permutations (where order mattered).

- Step 1: Select k objects from n (order doesn't matter)
- Step 2: Order the objects (there is $k!$ ways to order objects)

$P(n, k) = (\text{number of ways to select } k \text{ objects unordered}) \cdot k!$

Number of ways to select k objects unordered $= \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!k!}$

Combination

Definition

A *combination* is a subset of k objects from n objects. This is another name for *unordered without replacement* scenario.

Theorem

$C(n,k)$, called the *combination number*, is the number of combinations of k objects chosen from n.

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

- $C(n,k)$ or $\binom{n}{k}$ is read "n choose k"

Combination Example

Example 7: Lottery (pick-five)

The lottery picks 5 numbers from $\{1, \ldots, 49\}$ without replacement as the "winning numbers". You choose 5 numbers and win if you pick at least 3 of the winning numbers.

1. What is the probability you match all 5 winning numbers?
2. What is the probability you win?

Easiest way to do combination problems is to draw a picture of the problem by visualizing a box of items you are selecting from. Break the box into sections according to the problems.

Here, we break the box into "winning" and "non-winning" numbers.

Combination Example

1. What is the probability you match all 5 winning numbers?

Event: To match all 5 winning numbers – we need to choose 5 numbers from "winning" and group, and 0 numbers from the "non-winning" group. This is done in ...

$$|A| = \binom{5}{5} \cdot \binom{44}{0} = \frac{5!}{(5-5)!5!} \frac{44!}{(44-0)!0!} = \frac{5!}{0!5!} \frac{44!}{44!0!} = \frac{5!}{1 \cdot 5!} \frac{44!}{44! \cdot 1} = 1$$

Sample Space: How many total ways are there to choose 5 numbers from 49 numbers (all possibilities). This is done in ...

$$|\Omega| = \binom{49}{5} = \frac{49!}{(49-5)!5!} = \frac{49!}{44!5!} = 1,906,884$$

$$\mathbb{P}(\text{match all}) = \frac{\binom{5}{5} \cdot \binom{44}{0}}{\binom{49}{5}} = \frac{1}{1,906,884} = 0.000005$$

Combination Example

2. What is the probability you win? (Recall that you win if you match at least 3 “winning” numbers.)

$\mathbb{P}(\text{win}) = \mathbb{P}(\text{match at least 3}) =$
$\mathbb{P}(\text{match 3}) + P(\text{match 4}) + \mathbb{P}(\text{match 5})$

Combination Example

Counting Summary

Method	# of Possible Outcomes
Ordered with replacement	n^k
Ordered without replacement	$P(n,k) = \frac{n!}{(n-k)!}$
Unordered without replacement	$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$

Recap

Students should be familiar with combinations and the combination number. They should be able to apply it to probability questions under unordered without replacement samples.