Permutations

STAT 330 - Iowa State University

Outline

In this lecture students will learn about permutations. The permutation number gives us the number of ways to select k objects from n when we have an ordered sample, without replacement. We will use the permutation number to find probabilities.

Ordered With Replacement

A box has n items numbered $1, \ldots, n$. Draw k items with replacement. (A number can be drawn twice).

$\boxed{1, 2, \ldots n}$ $\Downarrow$ k

Sample Space: $\Omega = \{\underline{(x_1, \ldots, x_k)} : x_i \in \{1, \ldots, n\}\}$

What is $|\Omega|$?

Break complex action into a series of k single draws.

$\underline{n}\ \underline{n}\ \underline{n}\ \ldots\ \underline{n}$

1. n possibilities for x_1
2. n possibilities for x_2

   $\vdots$

k. n possibilities for x_k

$\boxed{0, 1, 2, \ldots n}$ $n = 10$

$\Downarrow$

$k = 5$

Multiplication principle: $|\Omega| = n \cdot n \cdot n \cdots n = n^k$

10^5

Permutation

Ordered Without Replacement

A box has n items numbered $1, \dots, n$. Select k items without replacement. This means once a number is chosen, it can't be selected again.

$\cancel{(1\,2\,3\,1)}$

Sample Space: $\Omega = \{\underline{(x_1, \dots, x_k)} : x_i \in \{1, \dots, n\}, \underline{x_i \neq x_j}\}$

What is $|\Omega|$?

Break complex action into a series of k single draws.

$\underline{n} \quad \underline{n-1} \quad \underline{n-2} \quad \dots \quad \underline{n-(k-1)}$

1. n possibilities for x_1
2. $n-1$ possibilities for x_2
3. $n-2$ possibilities for x_2

   $\vdots$

k. $n-(k-1)$ possibilities for x_k

Multiplication principle: $\underline{|\Omega|} = n \cdot (n-1) \cdot (n-2) \cdots (n-(k-1))$

This is equivalent to $\frac{n!}{(n-k)!}$

Permutation

1, 2, 3 | 123, 132, ...

Definition

A *permutation* is an ordering of k distinct objects chosen from n objects. This is another name for the *ordered without replacement* scenario.

"In the box"

we choose

Theorem

$P(n, k)$, called the *permutation number*, is the number of permutations of k distinct objects out of n objects.

$$P(n,k) = \frac{n!}{(n-k)!}$$

Note (factorials): $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$

$0! = 1$

Ex. $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

Ex 4 books

How many ways to order those books?

Permutation Example

Example 5:

Out of a group of 10 students, I choose 3 distinct students to give prizes to. How many ways can I select 3 students?

$n = 10 \qquad k = 3$

$$
\begin{aligned}
P(n, k) &= \frac{n!}{(n-k)!} \\
P(10, 3) &= \frac{10!}{(10-3)!} \\
&= \frac{10!}{7!} \\
&= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}} \\
&= 10 \cdot 9 \cdot 8 = 720
\end{aligned}
$$

Permutation Example

Example 6:

University phone exchange starts with 641 – _ _ _ _

What is the probability that a randomly selected phone number contains 7 distinct digits?

Sample space: (All possibilities for 4 chosen numbers)

$|\Omega| =$ $\underline{10}\ \underline{10}\ \underline{10}\ \underline{10} \Rightarrow |\Omega| = 10^4$

Event: (4 chosen numbers are distinct - no repeats!)

$|A| =$ $\boxed{n=7} \Downarrow k=4$ $\quad P(7,4) = \frac{7!}{3!} = \frac{7\cdot 6\cdot 5\cdot 4\cdot \cancel{3!}}{\cancel{3!}} = 840$

$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{840}{10^4} = \boxed{.084}$

Recap

Students should be familiar with permutations and the permutation number. They should be able to apply it to probability questions under ordered without replacement samples.