Unit 4 Homework: Random Variables

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1. Best Game in the Casino

(a) How much do you get paid if the coin comes up heads 3 times?

$$X = \begin{cases} \$0 & \text{0 heads} \\ \$2 & \text{1 head} \\ \$4 & \text{2 heads} \\ \$\text{x} & \text{3 heads} \end{cases}$$

$$P(X) = \begin{cases} 1/8 & \text{0 heads} \\ 1/4 & \text{1 head} \\ 1/4 & \text{2 heads} \\ 1/8 & \text{3 heads} \end{cases}$$

$$E(X) = 0 * \frac{1}{8} + 2 * \frac{1}{4} + 4 * \frac{1}{4} + x * \frac{1}{8} = 6$$

$$\frac{x}{8} = 6 - \frac{2}{4} - \frac{4}{4} = \frac{18}{4}$$

$$x = 36$$

You would get paid \$36 if the coin comes up heads 3 times.

(b) Cumulative probability function for winning from the game

$$F(X) = P(X \leq x) = \begin{cases} 1/8 & \text{x} = 0 \\ 3/8 & \text{x} = 1 \\ 5/8 & \text{x} = 2 \\ 1 & \text{x} = 3 \end{cases}$$

2. Processing Pasta

(a) Cumulative probability function of L

$$F(L) = P(L \leq l) = \int_{-\infty}^{l} f(y)dy = \int_{0}^{l} \frac{y}{2}dy = \frac{y^2}{4} = \frac{l^2}{4}$$

(b) Expected length of the pasta E(L)

$$E(L) = \int_{-\infty}^{\infty} l * f(l)dl = \int_{0}^{2} l * \frac{l}{2} = \int_{0}^{2} \frac{l^2}{2} = \frac{l^3}{2 * 3} = \frac{l^3}{6}|_0^2 = \frac{2^3}{6} - 0 = \frac{4}{3}$$

3. The Warranty is Worth It

(a) Expected payout from the contract, E(X) = E(g(T)) using the expression for the expectation of a function of a random variable

$$f(x) = \begin{cases} 1 & 0 \leq \text{x} \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

$$E(X) = E(g(T)) = \int_{-\infty}^{\infty} g(t) * f(x)dx = \int_{0}^{1} 100(1-t)^{\frac{1}{2}} * 1dx = 100 \int_{0}^{1} (1-t)^{\frac{1}{2}} * dx$$

$$= -100 * \frac{2}{3}(1-t)^{\frac{3}{2}} = -\frac{200}{3}(1-t)^{\frac{3}{2}}|_0^1 = \frac{200}{3}$$

\$66.67

(b) E(X) another way, by first characterizing the random variable X.

i. Value for T that results in the payoff X = x

$$X = x$$

$$F(X) = P(X \leq x) = \begin{cases} 0 & \text{x<0} \\ \sqrt{100(1-t)} & 0 \leq \text{x} \leq 100 \end{cases}$$

$$100(1-t)^{\frac{1}{2}} = x$$
$$(1-t)^{\frac{1}{2}} = \frac{x}{100}$$
$$\sqrt{(1-t)} = \frac{x}{100}$$
$$1 - t = \frac{x^2}{10000}$$
$$t = 1 - \frac{x^2}{10000}$$

ii. Condition for T that is equivalent to X ≤ x.

From i), we know

$$t = 1 - \frac{x^2}{10000}$$

The T equivalent of X ≤ x would be

$$T \geq 1 - \frac{x^2}{10000}$$

iii. Probability that X ≤ x using the condition above (cumulative probability function of X)

$$F(Y) = P(X \leq x) = \int_{0}^{x} f(y)dy = \int_{0}^{x} 1 - \frac{y^2}{10000}dy = x - \frac{x^3}{30000}$$

iv. Take a derivative to compute the probability density function for X

$$f(x) = F'(X) = (x - \frac{x^3}{30000})dx = 1 - \frac{x^2}{10000}$$

v. Use the pdf of X to compute E(X)

$$E(X) = \int_{-\infty}^{\infty} x * f(x)dx = \int_{0}^{100} x * (1 - \frac{x^2}{10000})dx$$
$$= \frac{x^2}{2} - \frac{x^4}{40000}|_0^{100} = 10000 - \frac{1000000}{40000} = 2500$$

This is clearly wrong..

4. The Baseline for Measuring Deviations

(a) Expression for E(Y) and use properties of expectation

$$E(Y) = \int_{-\infty}^{\infty} (x-t)^2 f(x)dx = \frac{x^3 * f(x)}{3} - x^2 t * f(x) + t^2 * f(x)$$

(b) Taking a partial derivative with respect to t, compute the value of t that minimizes E(Y)

$$\frac{d}{dt}(\frac{x^3 * f(x)}{3} - x^2 t * f(x) + t^2 * f(x)) = t * f(x) - x^2 * f(x)$$
$$t * f(x) = x^2 * f(x)$$
$$t = x^2$$

(c) What is the value of E(Y) for this choice of t?

$$E(Y) = \frac{x^3 * f(x)}{3} - x^2 t * f(x) + t^2 * f(x) = \frac{x^3 * f(x)}{3} - x^4 * f(x) + x^4 * f(x)$$