Lab 2: Probability Theory

W203: Statistics for Data Science

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1. Meanwhile, at the Unfair Coin Factory...

You are given a bucket that contains 100 coins. 99 of these are fair coins, but one of them is a trick coin that always comes up heads. You select one coin from this bucket at random. Let T be the event that you select the trick coin. This means that $P(T) = 0.01$.

a. To see if the coin you have is the trick coin, you flip it k times. Let H_k be the event that the coin comes up heads all k times. If you see this occur, what is the conditional probability that you have the trick coin? In other words, what is $P(T|H_k)$.

$$P(T|H_k) = \frac{0.01}{(0.99 * 0.5^k + 0.01 * 1^k)}$$

b. How many heads in a row would you need to observe in order for the conditional probability that you have the trick coin to be higher than 99%?

$$P(T|H_k) = 0.99 = \frac{0.01 * 1^k}{(0.99 * 0.5^k + 0.01 * 1^k)}$$

$$0.99(0.99 * 0.5^k + 0.01 * 1^k) = 0.01 * 1^k$$

$$\text{Since } 0.01 * 1^k = 0.01\text{, we can say}$$

$$0.99(0.99 * 0.5^k + 0.01) = 0.01$$

$$0.99 * 0.99 * 0.5^k = 0.01 - 0.99 * .01 = 0.0001$$

$$0.5^k = \frac{0.0001}{0.99 * 0.99}$$

$$log(0.5^k) = log(\frac{0.0001}{0.99 * 0.99})$$

$$log(0.5^k) = log(0.0001020304)$$

$$k * log(0.5) = log(0.0001020304)$$

$$k = log(0.0001020304)/log(0.5) = 13.25871$$

```
(q1b = log(0.0001020304)/log(0.5))
```

```
## [1] 13.25871
```

14 times

2. Wise Investments

You invest in two startup companies focused on data science. Thanks to your growing expertise in this area, each company will reach unicorn status (valued at $1 billion) with probability 3/4, independent of the other company. Let random variable X be the total number of companies that reach unicorn status. X can take on the values 0, 1, and 2. Note: X is what we call a binomial random variable with parameters $n = 2$ and $p = 3/4$.

a. Give a complete expression for the probability mass function of X.

$$P(X) = b(x; 2, 0.75) = \binom{2}{x}(.75)^x(.25)^{2-x}$$

b. Give a complete expression for the cumulative probability function of X.

$$F(x) = P(X \leq x) = b(x; 2, 0.75) = \sum_0^x b(x; 2, 0.75) = \sum_0^x \binom{2}{x}(.75)^x(.25)^{2-x}$$

c. Compute $E(X)$.

```r
n <- 2
p <- 3/4
q2c <- n*p
```

$$E(X) = np = 2 * 0.75 = 1.5$$

d. Compute $var(X)$.

```r
n <- 2
p <- 3/4
q2d <- n*p*(1-p)
```

$$V(X) = np(1-p) = 2 * 0.75(1 - 0.75) = 1.5 * 0.25 = 0.375$$

3. Relating Min and Max

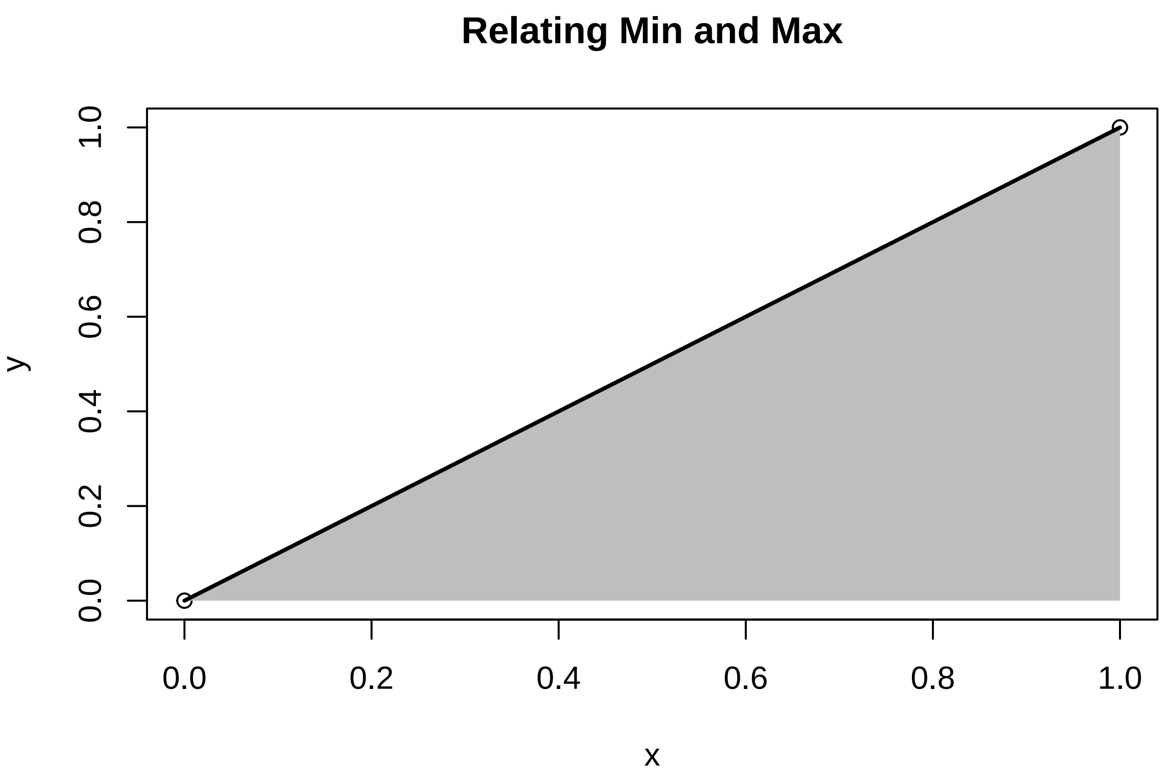
Continuous random variables X and Y have a joint distribution with probability density function,

$$f(x, y) = \begin{cases} 2, & 0 < y < x < 1 \\ 0, & otherwise. \end{cases}$$

You may wonder where you would find such a distribution. In fact, if A_1 and A_2 are independent random variables uniformly distributed on $[0, 1]$, and you define $X = max(A_1, A_2)$, $Y = min(A_1, A_2)$, then X and Y will have exactly the joint distribution defined above.

a. Draw a graph of the region for which X and Y have positive probability density.

```r
plot(c(0,1), c(0,1), xlab = "x", ylab = "y", main = "Relating Min and Max")
x <- c(0, 1, 1)
y <- c(0, 1, 0)
polygon(x, y, col = "grey", lty = 1, border = 0)
polygon(c(0, 1), c(0, 1), col = "black", lty = 1, lwd = 2)
```



b. Derive the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 2dy = 2y|_0^1 = 2$$

c. Derive the unconditional expectation of X.

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x2dx = x^2|_0^1 = 1$$

d. Derive the conditional probability density function of Y, conditional on X, $f_{Y|X}(y|x)$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2}{2} = 1$$

e. Derive the conditional expectation of Y, conditional on X, $E(Y|X)$.

$$E[Y|X] = \int_{-\infty}^{\infty} y * f_{Y|X}(y|x)dy = \int_0^1 ydy = \frac{y^2}{2}|_0^1 = \frac{1}{2}$$

f. Derive $E(XY)$. Hint: if you take an expectation conditional on X, X is just a constant inside the expectation. This means that $E(XY|X) = XE(Y|X)$.

$$E(XY) = E(XY|X) = XE(Y|X) = X * \frac{1}{2} = \frac{1}{2}$$

g. Using the previous parts, derive $cov(X,Y)$

$$Cov(X,Y) = E(XY) - \mu_X * \mu_Y = \frac{1}{2} - 1 * \frac{1}{2} = 0$$

4. Circles, Random Samples, and the Central Limit Theorem

Let $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ be independent random samples from a uniform distribution on $[-1, 1]$. Let D_i be a random variable that indicates if (X_i, Y_i) falls within the unit circle centered at the origin. We can define D_i as follows:

$$D_i = \begin{cases} 1, & X_i^2 + Y_i^2 < 1 \\ 0, & otherwise \end{cases}$$

Each D_i is a Bernoulli variable. Furthermore, all D_i are independent and identically distributed.

a. Compute the expectation of each indicator variable, $E(D_i)$. Hint: your answer should involve a Greek letter.

$$E(D_i) = \frac{\text{Area of Unit Circle}}{\text{Area of Square}} = \frac{1^2\pi}{2^2} = \frac{\pi}{4}$$

b. Compute the standard deviation of each D_i.

Since D_i is a Bernoulli variable, we can compute $V(D_i)$ as below:

$$V(D_i) = np(1-p) = 1 * \frac{\pi}{4}(1 - \frac{\pi}{4}) = \frac{\pi}{4}\frac{3\pi}{4} = \frac{3\pi^2}{16}$$

Then we take a square root of $V(D_i)$ to compute the standard deviation of D_i:

$$\sigma(D_i) = \sqrt{V(D_i)} = \sqrt{\frac{3\pi^2}{16}} = \frac{\pi}{4}\sqrt{3}$$

c. Let $\bar{D}$ be the sample average of the D_i. Compute the standard error of $\bar{D}$. This should be a function of sample size n.

$$\sigma(\bar{D}) = \sqrt{\frac{V(\bar{D})}{n^2}} = \sqrt{\frac{npq}{n^2}} = \sqrt{\frac{3\pi^2}{n16}} = \frac{\pi}{4}\sqrt{\frac{3}{n}}$$

d. Now let n=100. Using the Central Limit Theorem, compute the probability that $\bar{D}$ is larger than 3/4. Make sure you explain how the Central Limit Theorem helps you get your answer.

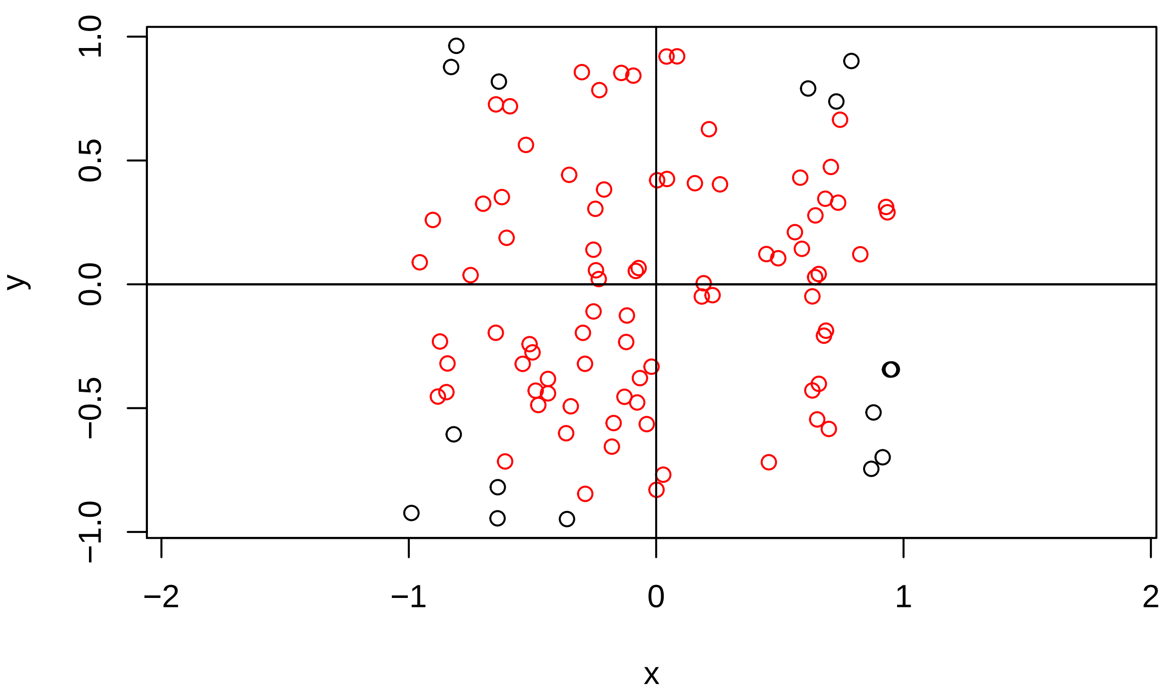
$$P(\bar{D} > \frac{3}{4}) = P\left(Z > \frac{\frac{3}{4} - \frac{\pi}{4}}{\frac{\pi}{4}\sqrt{\frac{3}{100}}}\right)$$

$$P\left(Z > \frac{\frac{1}{4}(3-\pi)}{\frac{\pi}{4} * \frac{1}{10}\sqrt{3}} = \frac{10(3-\pi)}{\pi\sqrt{3}}\right) = P(Z > -0.260) = 1 - \phi(-0.26) = 1 - .3974 = .6026$$

e. Now let $n = 100$. Use R to simulate a draw for $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$. Calculate the resulting values for $D_1, D_2, ...D_n$. Create a plot to visualize your draws, with X on one axis and Y on the other. We suggest using a command like the following to assign a different color to each point, based on whether it falls inside the unit circle or outside it. Note that we pass $d + 1$ instead of d into the color argument because 0 corresponds to the color white.

```
# Set seed
set.seed(898)

# Genereate random samples in the range of [-1, 1]
```

```r
x <- runif(100, min = -1, max = 1)
y <- runif(100, min = -1, max = 1)

# Function to compute Di given xi and yi
compD <- function(x, y) {
  value = 0

  if(x^2 + y^2 < 1) {
    value = 1
  }

  value
}

# Initialize list d
d = c()

# Compute D
for(i in 1:100)
  d[i] = compD(x[i], y[i])

# Plot the reults
plot(x, y, col=d+1, asp = 1, xlab = "x", ylab = "y")
abline(a = 0, b = 0, h = 0)
abline(a = 0, b = 0, v = 0)
```



f. What value do you get for the sample average, $\bar{D}$? How does it compare to your answer for part a?

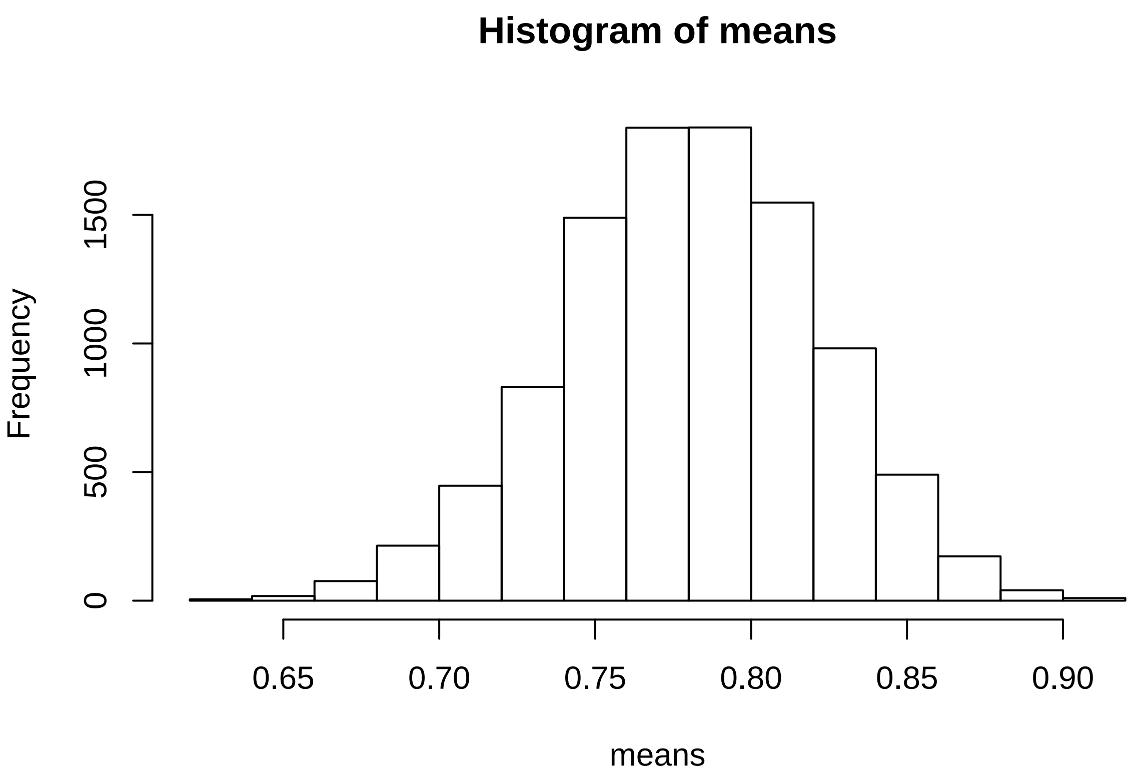
```r
# Compute sample average
(sample_average = length(d[d == 1])/length(d))
```

```
## [1] 0.84
```

$0.84 > \frac{\pi}{4}$ however it is relatively close.

g. Now use R to replicate the previous experiment 10,000 times, generating a sample average of the D_i each time. Plot a histogram of the sample averages.

```
# Function for generating random samples and computing average for the sample
execute_study <- function(seed) {
  # Generate random samples
  set.seed(seed)
  x <- runif(100, min = -1, max = 1)
  y <- runif(100, min = -1, max = 1)

  d = c()

  # Compute D
  for(i in 1:100)
    d[i] = compD(x[i], y[i])

  # Return sample average
  sample_average = length(d[d == 1])/length(d)
}

# Parameters
n = 10000
current_seed = 899

# Initialize list means
means = c()

for(i in 1:n) {
  means[i] = execute_study(current_seed)
  current_seed = current_seed + 1
}

hist(means)
```




```r
# Compute the mean of the sample means
mean(means)
```

```
## [1] 0.785265
```

h. Compute the standard deviation of your sample averages to see if it's close to the value you expect from part c.

```r
sd(means)
```

```
## [1] 0.04137077
```

From c, we would expect the standard of error in this case to be $\frac{\pi}{4}\sqrt{\frac{3}{10000}} = 0.0136$. Therefore the value I got from the experiment is considerably larger.