

# Unit 4 Homework: Random Variables

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## 1. Best Game in the Casino

(a) How much do you get paid if the coin comes up heads 3 times?

$$X = \begin{cases} \$0 & 0 \text{ heads} \\ \$2 & 1 \text{ head} \\ \$4 & 2 \text{ heads} \\ \$x & 3 \text{ heads} \end{cases}$$

$$P(X) = \begin{cases} 1/8 & 0 \text{ heads} \\ 3/8 & 1 \text{ head} \\ 3/8 & 2 \text{ heads} \\ 1/8 & 3 \text{ heads} \end{cases}$$

$$E(X) = 0 * \frac{1}{8} + 2 * \frac{3}{8} + 4 * \frac{3}{8} + x * \frac{1}{8} = 6$$

$$\frac{x}{8} = 6 - \frac{3}{4} - \frac{3}{2} = \frac{15}{4}$$

$$x = 30$$

You would get paid \$30 if the coin comes up heads 3 times.

(b) Cumulative probability function for winning from the game

$$F(X) = P(X \leq x) = \begin{cases} 1/8 & x = 0 \\ 4/8 & x = 1 \\ 7/8 & x = 2 \\ 1 & x = 3 \end{cases}$$

## 2. Processing Pasta

(a) Cumulative probability function of L

$$F(L) = P(L \leq l) = \int_{-\infty}^l f(y)dy = \int_0^l \frac{y}{2}dy = \frac{y^2}{4} = \frac{l^2}{4}$$

(b) Expected length of the pasta E(L)

$$E(L) = \int_{-\infty}^{\infty} l * f(l)dl = \int_0^2 l * \frac{l}{2} = \int_0^2 \frac{l^2}{2} = \frac{l^3}{2 * 3} = \frac{l^3}{6} \Big|_0^2 = \frac{2^3}{6} - 0 = \frac{4}{3}$$

### 3. The Warranty is Worth It

(a) Expected payout from the contract,  $E(X) = E(g(T))$  using the expression for the expectation of a function of a random variable

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

$$E(X) = E(g(T)) = \int_{-\infty}^{\infty} g(t) * f(x) dx = \int_0^1 100(1-t)^{\frac{1}{2}} * 1 dx = 100 \int_0^1 (1-t)^{\frac{1}{2}} * dx$$

$$= -100 * \frac{2}{3} (1-t)^{\frac{3}{2}} = -\frac{200}{3} (1-t)^{\frac{3}{2}} \Big|_0^1 = \frac{200}{3}$$

\$66.67

(b)  $E(X)$  another way, by first characterizing the random variable  $X$ .

i. Value for  $T$  that results in the payoff  $X = x$

$$X = x$$

$$F(X) = P(X \leq x) = \begin{cases} 0 & \text{Otherwise} \\ \sqrt{100(1-t)} & 0 \leq x \leq 100 \end{cases}$$

$$100(1-t)^{\frac{1}{2}} = x$$

$$(1-t)^{\frac{1}{2}} = \frac{x}{100}$$

$$\sqrt{1-t} = \frac{x}{100}$$

$$1-t = \frac{x^2}{10000}$$

$$t = 1 - \frac{x^2}{10000}$$

ii. Condition for  $T$  that is equivalent to  $X \leq x$ .

i) gives us value for  $T$  in the case of  $P(T \geq g^{-1}(x))$ . We are however looking for a condition equivalent to  $X \leq x$  for  $T$ . Therefore the condition we are looking is as below:

$$P(T \leq g^{-1}(x))$$

This tells us that the we are interested in the life span of the server less than or equal to  $g^{-1}(x)$ .

iii. Probability that  $X \leq x$  using the condition above (cumulative probability function of  $X$ )

$$t = 1 - \frac{x^2}{10000}$$

The above equation from i) gives us  $P(T \geq g^{-1}(x))$ . However as we are trying to look at this from the perspective of  $P(T \leq g^{-1}(x))$  as seen in ii), we will need to invert this in respect to 1 as below:

$$F(X) = \frac{x^2}{10000}$$

iv. Take a derivative to compute the probability density function for  $X$

$$f(x) = F'(X) = \left(\frac{x^2}{10000}\right)' = \frac{x}{5000}$$

v. Use the pdf of  $X$  to compute  $E(X)$

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x * f(x) dx = \int_0^{100} x * \left(\frac{x}{5000}\right) dx \\ &= \frac{x^3}{15000} \Big|_0^{100} = \frac{1000000}{15000} = \frac{200}{3} \end{aligned}$$

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This matches what we got for (a).

## 4. The Baseline for Measuring Deviations

(a) Expression for  $E(Y)$  and use properties of expectation

$$E(Y) = E[(X - t)^2] = E[X^2] - 2tE[X] + t^2$$

(b) Taking a partial derivative with respect to  $t$ , compute the value of  $t$  that minimizes  $E(Y)$

(b)

$$\begin{aligned} E'[(X - t)^2] &= -2E[X] + 2t = 0 \\ t &= E[X] \end{aligned}$$

$$\frac{d}{dt}(E[X^2] - 2tE[X] + t^2) = 2t - 2E[X]$$

Since we are trying to find the value of  $t$  that minimizes  $E(Y)$

$$\begin{aligned} 2t - 2E[X] &= 0 \\ t &= E[X] \end{aligned}$$

(c) What is the value of  $E(Y)$  for this choice of  $t$ ?

$$E(Y) = E[X^2] - 2tE[X] + t^2 = E[X^2] - 2E[X]E[X] + E[X]^2 = E[X^2] - E[X]^2$$

This is the variance of  $X$   $V(X)$