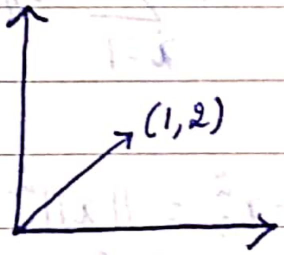
Orthogonality of Vectors

Norm: Let $x = (x_1, x_2, \ldots\ldots, x_n)$ be a vector in vector space $\mathbb{R}^n$, then the +ve length or size of vector x is called as norm of x and it is given by

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

Eg:



$$\|x\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Inner product: Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two vectors in $\mathbb{R}^n$, then

$$\langle x, y \rangle = x^T y = x_1 y_1 + x_2 y_2 + x_3 y_3 + \ldots + x_n y_n$$

$$= \sum_{i,j=1}^{n} x_i y_j$$ (is called inner product of x and y)

Note:

* $x^T y = \vec{x} \cdot \vec{y} = 0$ (orthogonal)
* Angle b/w two vectors is defined using inner product.

Orthogonality

Two vectors x and y are said to be orthogonal if the inner product $\langle x, y\rangle = 0$ i.e., $x^T y = 0$

$$x^T y = [x_1, x_2, \ldots, x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

$$= \sum_{i=1}^{n} x_i y_i = 0$$

Note:

* $\langle x, x\rangle = x_1^2 + x_2^2 + \ldots + x_n^2 = \|x\|^2$
* The zero vector is orthogonal to any/every vector in $\mathbb{R}^n$ $\because 0^T y = 0$
* The only vector orthogonal to itself is the zero vector $\because 0^T 0 = 0$
* If non-zero vectors $v_1, v_2, \ldots, v_n$ are mutually orthogonal then they are linearly independent

Eg: Show that $S = \{u_1, u_2, u_3\}$ is an orthogonal set where $u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$

Soln. $\langle u_1, u_2\rangle = u_1^T u_2 = [3\ 1\ 1] \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -3+2+1 = 0$

$\langle u_2, u_3\rangle = u_2^T u_3 = [-1\ 2\ 1] \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} = 1/2 - 4 + 7/2 = 0$

$\langle u_1, u_3\rangle = u_1^T u_3 = [3\ 1\ 1] \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} = -3/2 - 2 + 7/2 = 0$

$\Rightarrow$ given set is an orthogonal set.

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* If $v_1, v_2, \ldots, v_k$ are mutually orthogonal then there also linearly independent i.e., any vector cannot be written as the combination of other vectors.

Orthogonal subspaces

Two subspaces V and W are orthogonal subspaces of the vector space $\mathbb{R}^n$ if every vector in V is orthogonal to every vector in W and vice versa

Eg: (i) x axis and y axis in $\mathbb{R}^2$

(ii) x-axis and yz-plane in $\mathbb{R}^3$

(iii) x-axis and y-axis in $\mathbb{R}^3$

(iv) xz plane and yt plane in $\mathbb{R}^4$

Eg $(x,0,z,0)^T(0,y,0,t)=0$

(v) x axis and yt-plane in $\mathbb{R}^4$

(vi) y axis and xzt-plane in $\mathbb{R}^4$

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Fundamental theorem of Orthogonality

If A is any matrix of order $m \times n$ then

(i) $C(A)$ and $N(A^T)$ are orthogonal subspaces of $\mathbb{R}^m$

(ii) $C(A^T)$ and $N(A)$ are orthogonal subspaces of $\mathbb{R}^n$

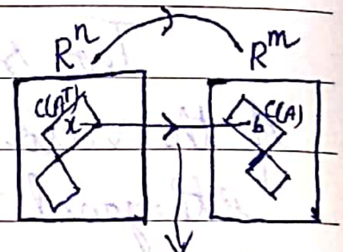
Proof: (i) Let $x \in C(A)$ and $y \in N(A^T)$

$\Rightarrow Au = x$ and $A^T y = 0$

Now, $x^T y = (Au)^T y = u^T (A^T y) = 0$

$\Rightarrow C(A) \perp N(A^T)$

$A_{m\times n}$



$Ax = b$

Here we have taken $x \in C(A)$

$\Rightarrow x$ is in $C(A)$

(ii) Let $x \in C(A^T)$ and $y \in N(A)$

$\Rightarrow A^T u = x$ and $Ay = 0$

Now, $x^T y = (A^T u)^T y = u^T (Ay) = 0$

$\Rightarrow C(A^T) \perp N(A)$

$y \in N(A^T)$

$\Rightarrow A^T y = 0$

A^T is x'd by all the special sol^n(s) then it must give zero then $y \in N(A^T)$

Note: (i) y is orthogonal to every combination of columns of A

$A^T y = 0$

$\Rightarrow y^T A = 0 \Rightarrow [y_1, y_2, \ldots, y_n][\text{col}1, \text{col}2, \ldots \text{col}n] = 0$

(ii) Consider a rank 1 matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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$$C(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \middle/ c_1 \in R \right\}$$ → line in $\mathbb{R}^3$

$$C(A^T) = \left\{ c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \middle/ c_1 \in R \right\}$$ → line in $\mathbb{R}^2$

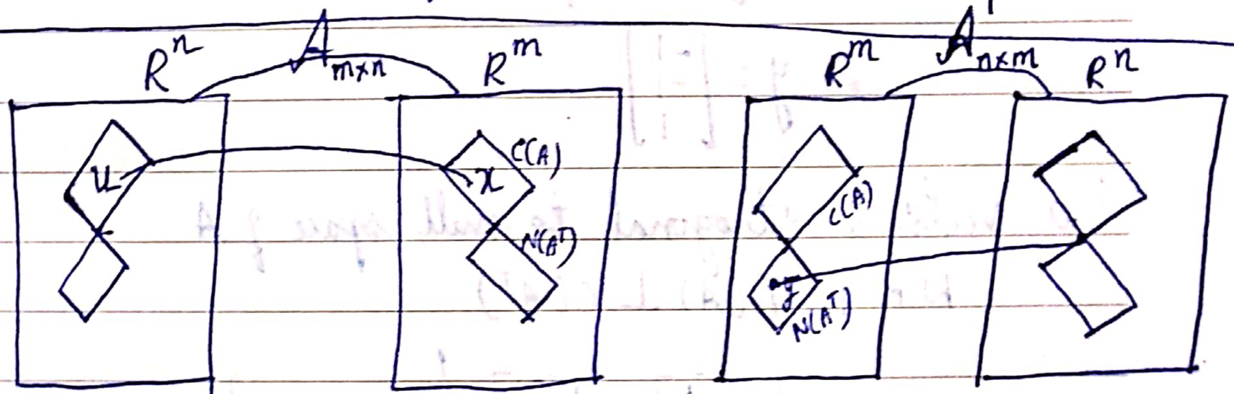
$$N(A) = \left\{ c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} \middle/ c_1 \in R \right\}$$

$$N(A^T) = \left\{ c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \middle/ c_1, c_2 \in R \right\}$$

$C(A)$ is a line passing through $(1,2,3)$
Since $C(A) \perp N(A^T)$, therefore, $N(A^T)$ must be $\perp^r$ to plane, $x+2y+3z=0$.

$N(A)$ contains the vector $x = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ which is orthogonal to all the rows of A

$N(A^T)$



$Au = x$ & $x \in C(A)$

$y \in N(A^T)$

$A^T y = 0$

i.e, A^T takes y to zero

why zero vector lies at this point ∵ there is no Subspace without origin. Hence zero vector is common point

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Eg: Find a vector x orthogonal to the row space of A, a vector y orthogonal to the column space of A and a vector z orthogonal to null space of A, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$$

Soln. // $N(A) = \left\{ c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \middle/ c_1 \in R \right\}$

Vector x orthogonal to row space of A

$C(A^T) \perp N(A)$

$\Rightarrow$ If we find N(A) then that is enough

$$\therefore x = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$ ✓

// a vector y orthogonal to column space of A

W.K.T $C(A) \perp N(A^T)$

So we have to find $N(A^T)$

$$N(A^T) = \left\{ c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \middle/ c_1 \in R \right\}$$

$$\therefore y = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

// a vector z orthogonal to null space of A

W.K.T $N(A) \perp C(A^T)$

$$C(A^T) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \middle/ c_1, c_2 \in R \right\}$$

$$z = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

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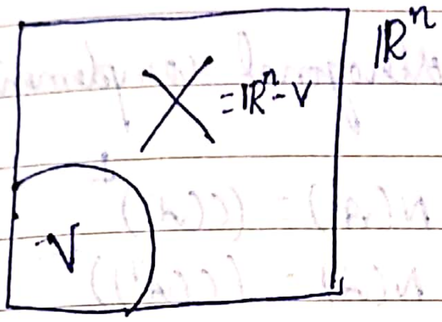
Orthogonal complements

Let V be a subspace of $\mathbb{R}^n$, then the orthogonal complement of V in $\mathbb{R}^n$ is the set of all vectors X such that X is orthogonal to all vectors in V



dim of V + dim of X
$= \dim$ of $\mathbb{R}^n = n$

It is denoted by $V^{\perp}$, read as V perp

Eg: (i) x-axis and y-axis in $\mathbb{R}^2$ are orthogonal complements
[dim [x-axis] = 1, dim [y-axis] = 1, dim [$\mathbb{R}^2$] = 2]

(ii) x-axis and y-axis in $\mathbb{R}^3$ are not orthogonal complements [dim(x) = 1, dim(y) = 1, dim$(\mathbb{R}^3)$ = 3)

(iii) x-axis and yz-plane are orthogonal complement in $\mathbb{R}^3$.

(iv) xz-plane and yt-plane are orthogonal compl. in $\mathbb{R}^4$

(v) x-axis and yt-plane are not orthogonal compl. in $\mathbb{R}^4$

(vi) y-axis and xzt-plane are orthogonal complement in $\mathbb{R}^4$.

Note: (i) Two subspaces V_1 and V_2 are orthogonal complements in a vector space V if
(a) V_1 and V_2 are orthogonal subspace
(b) $\dim(V_1) + \dim(V_2) = \dim(V)$

(ii) If $V^{\perp} = W$ then $W^{\perp} = V$

(iii) $V^{\perp\perp} = V$

Fundamental theorem of Linear algebra : Part 2

If A is any matrix of order $m \times n$ then

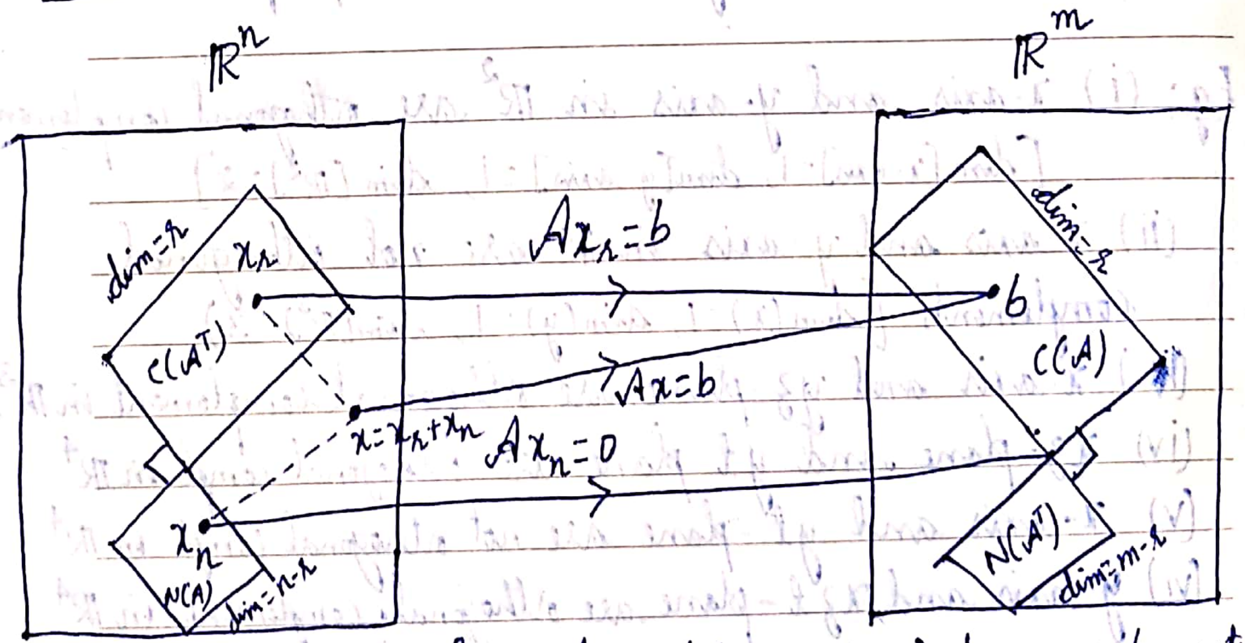
(i) $C(A)$ and $N(A^T)$ are orthogonal complements in $\mathbb{R}^m$

(ii) $C(A^T)$ and $N(A)$ are orthogonal complements in $\mathbb{R}^n$

i.e., $C(A) = (N(A^T))^\perp$ & $N(A^T) = (C(A))^\perp$

$C(A^T) = (N(A))^\perp$ & $N(A) = (C(A^T))^\perp$

Note



The true action $Ax = A(x_{row} + x_{null})$ of any m by n matrix

Let $Ax_r = b \rightarrow (1)$

$Ax_n = 0 \rightarrow (2)$

(1) + (2) $\Rightarrow A(x_r + x_n) = b$

$\Rightarrow Ax = b$

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Every vector $\overset{(b)}{Ax}$ is in the $C(A)$

$N(A)$ is carried to zero vector

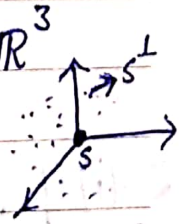
Real action is b/w $C(A^T)$ and $C(A)$

Every matrix transforms its $C(A^T)$ onto its $C(A)$

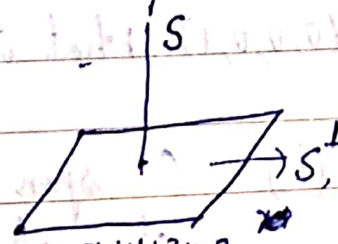
Problems

1. If S is the subspace of $\mathbb{R}^3$ containing only zero vector then what is $S^\perp$? If S is spanned by $(1,1,1)$, what is $S^\perp$ and basis for $S^\perp$? If S is spanned by $(1,1,1)$ and $(1,1,-1)$, what is the basis for $S^\perp$?

Solⁿ: If S contains only zero vector then $S^\perp$ is $\mathbb{R}^3$



If S is spanned by $(1,1,1)$ then $S^\perp$ is a plane in $\mathbb{R}^3$



$$Ax = 0$$

$$[1, 1, 1]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x+y+z=0$$

$$x=-y-z$$

The basis for the plane is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

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If S is spanned by $(1,1,1)$ and $(1,1,-1)$ then $S^{\perp}$ is a line

$$Ax = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$z = 0, \; x = -y$$

The basis for $S^{\perp}$ is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

2. (i) If S is the subspace of $\mathbb{R}^4$ containg only zero vectors. What is $S^{\perp}$? (ii) If S is spanned by the vector $(0,0,0,1)$. What is $S^{\perp}$? (iii) What is $(S^{\perp})^{\perp}$?

Sol^n (i) $S = \text{span}\{\text{zero vector}\}$

$S^{\perp}$ in $\mathbb{R}^4$

(ii) $S = \text{span}\{(0,0,0,1)\}$

$S^{\perp}$ is a plane in $\mathbb{R}^4$

(iii) $(S^{\perp})^{\perp} = S$

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3. Find all the vectors, which are $\perp^r$ to $(1,4,4,1)$ and $(2,9,8,2)$

Let $u(x,y,z,t)$ be any vector orthogonal to $a=(1,4,4,1)$ and $b=(2,9,8,2)$

$\Rightarrow u^T a = 0$ & $u^T b = 0$

$\Rightarrow x+4y+4z+t=0$ and $2x+9y+8z+2t=0$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 4 & 1 \\ 2 & 9 & 8 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{cccc:c} 1 & 4 & 4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 2$

$x+4y+4z+t=0$

$y=0$

choose $\left.\begin{array}{l} z=1 \quad t=0 \\ t=0 \quad z=1 \end{array}\right\}$ → free variables $\Rightarrow x=-4 \quad y=0$

$\Rightarrow x=-1 \quad y=0$

Thus the vectors which are $\perp^r$ to $(1,4,4,1)$ and $(2,9,8,2)$ are linear combination of $(-4,0,1,0)$ and $(-1,0,0,1)$

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4. Let S be the subspace of $\mathbb{R}^4$ containing all vectors with $x_1+x_2+x_3+x_4=0$. Find a basis for $S^\perp$. What is its dimension? Also find the basis for S

$x_1+x_2+x_3+x_4=0$ represents a 3D plane in $\mathbb{R}^4$ whose dimension is 3

$x+y=0$
1d line in $\mathbb{R}^2$

$x+y+z=0$ 2d line in $\mathbb{R}^3$

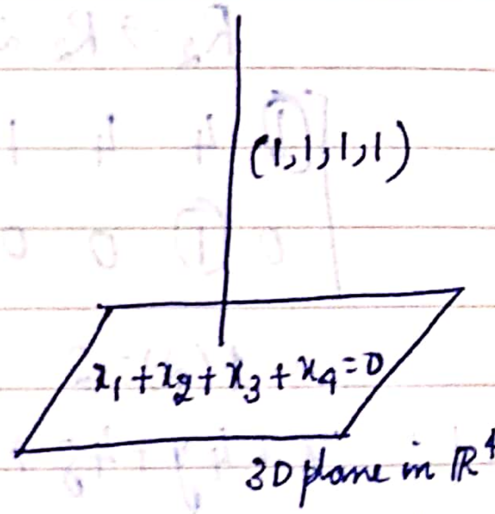
$\therefore S^\perp$ is of dimension 1 so that
$\dim(S) + \dim(S^\perp) = 4 = \dim(\mathbb{R}^4)$

$$\Rightarrow \underbrace{[1\ 1\ 1\ 1]}_{u^T} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{v} = 0$$

$\Rightarrow$ u and v are orthogonal

But $v \in S$

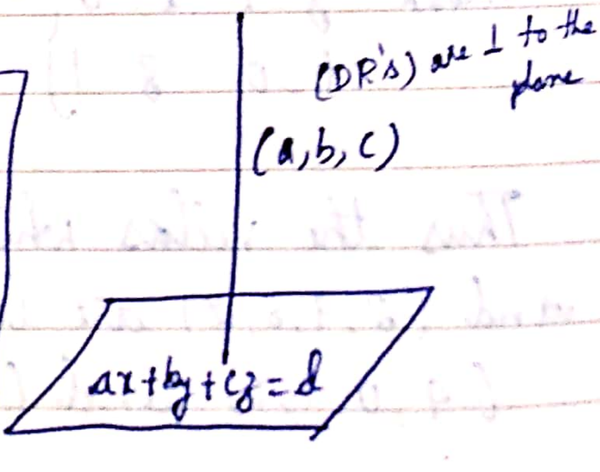
$\Rightarrow u \in S^T$



$\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $S^\perp$.

$x_1 = \cancel{gez} -x_2-x_3-x_4$

$$\text{Basis for } S = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$



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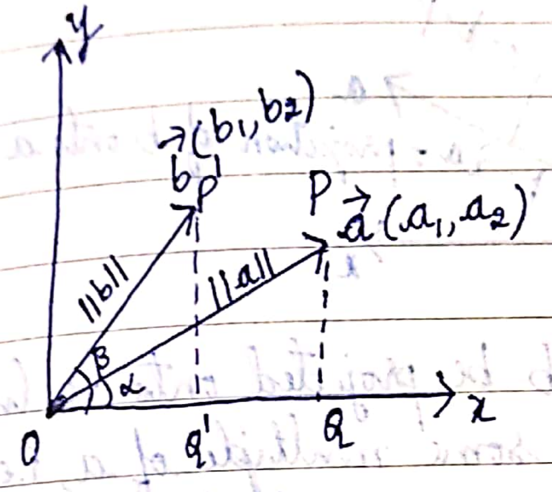
Relation b/w inner product and cosines



From the Δ^{le} OPQ, $\sin\alpha = \frac{a_2}{||a||}$, $\cos\alpha = \frac{a_1}{||a||}$

From the Δ^{le} OP'Q', $\sin\beta = \frac{b_2}{||b||}$, $\cos\beta = \frac{b_1}{||b||}$

Now consider $\theta = \beta - \alpha$

$$\Rightarrow \cos\theta = \cos(\beta-\alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$$

$$\Rightarrow \cos\theta = \frac{a_1 b_1}{||a||\,||b||} + \frac{a_2 b_2}{||a||\,||b||}$$

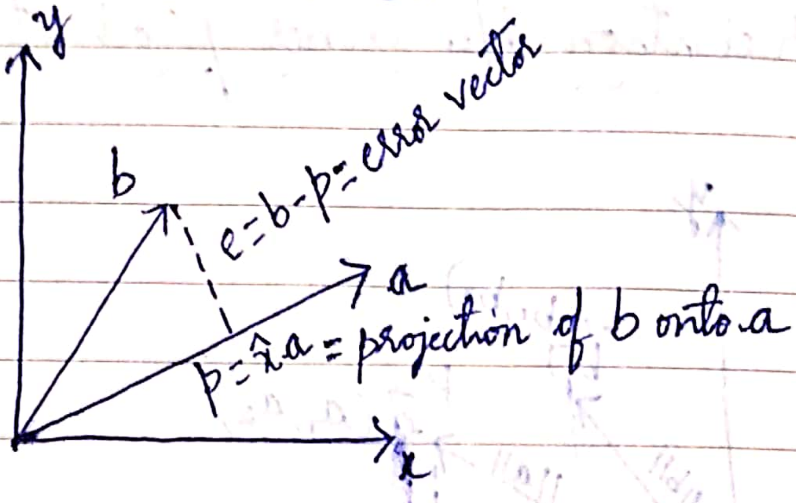
$$\Rightarrow \cos\theta = \frac{a_1 b_1 + a_2 b_2}{||a||.||b||}$$

$$\Rightarrow \cos\theta = \frac{\langle a, b\rangle}{||a||.||b||}$$

$$\therefore \quad \theta = \cos^{-1}\left[\frac{a^T b}{||a||\,||b||}\right]$$

This is the cosine of angle b/w two non-zero vectors a and b.

Projection onto a line



Let the vector b be projected onto a line. The point p must be some multiple of a, i.e., $p = \hat{x} a$ where $\hat{x}$ is scalar component

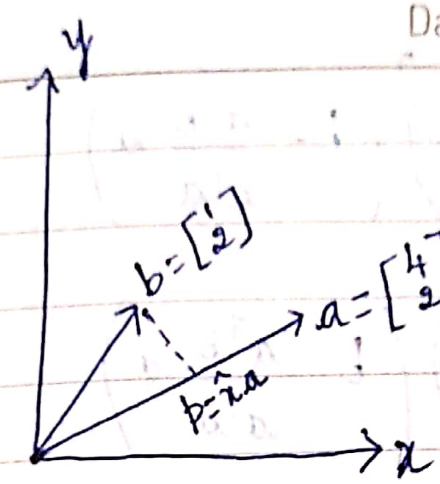
From the figure, $e \perp a$

$$\Rightarrow (b-p) \perp a$$
$$\Rightarrow (b-\hat{x}a) \perp a$$
$$\Rightarrow a^T(b-\hat{x}a) = 0$$
$$\Rightarrow a^Tb - \hat{x}a^Ta = 0$$
$$\Rightarrow \hat{x} = \frac{a^Tb}{a^Ta}$$

$$\therefore \quad p = \hat{x}a = \left[\frac{a^Tb}{a^Ta}\right] a$$

p is the projection point on the line 'a' which is closest to vector b.

Eg:



$$\hat{x} = \frac{\begin{bmatrix}4 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}}{\begin{bmatrix}4 & 2\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}} = \frac{8}{20} = \frac{2}{5}$$

$$p = \hat{x}a = \frac{2}{5}\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}8/5\\4/5\end{bmatrix}$$

Schwarz inequality

$$|\cos\theta| \leq 1 \text{ in } \mathbb{R}^n$$

(or)

$$\left|\frac{a^T b}{\|a\|\;\|b\|}\right| \leq 1$$

(or)

$$|a^T b| \leq \|a\|\,\|b\|$$

W.K.T. $\|e\|^2 = \|b-p\|^2 = \|b-\hat{x}a\|^2 =$

$$= \left\| b - \frac{a^T b}{a^T a} a \right\|^2$$

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$$\Rightarrow \left(b - \frac{a^T b}{a^T a} a\right)^T \left(b - \frac{a^T b}{a^T a} a\right)$$

$$\Rightarrow \left(b^T - \frac{a^T b^T}{a a^T} a\right)\left(b - \frac{a^T b}{a^T a} a\right)$$

$$\Rightarrow b^T b - \frac{a^T b^T b a}{a^T a} - \frac{a^T b^T b a}{a a^T} + \frac{a^T a^T b b^T a a}{a^2 a^T a^T}$$

$$\Rightarrow b^T b - \frac{b^T a^T b a}{a^T a} - \frac{a^T b^T a b}{a a^T} + \frac{a^T b^T a}{a a^T} \frac{a^T b a}{a^T a}$$

$$\Rightarrow b^T b - \frac{2(a^T b)^2}{a^T a} + \frac{(a^T b)^2}{a^T a}$$

but $\|e\|^2 \geq 0$

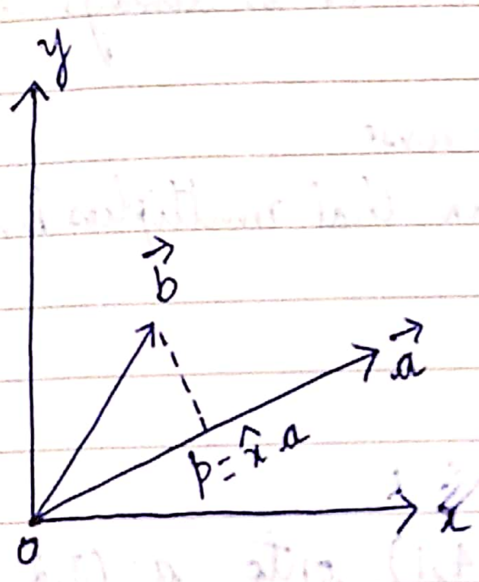
$$\Rightarrow b^T b - \frac{(a^T b)^2}{a^T a} \geq 0$$

$$\Rightarrow (b^T b)(a^T a) \geq (a^T b)^2$$

$$\Rightarrow |a^T b| \leq \|a\| \, \|b\|$$

* $\|b - a\|^2 = b^T b - 2a^T b + a^T a$

Projection matrix of Rank 1

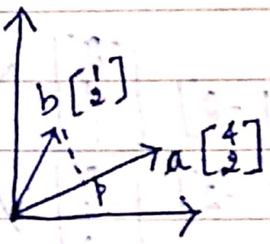


We have $p = \hat{x} a = a \hat{x} = a \dfrac{a^T b}{a^T a}$

$$p = \left(\frac{a a^T}{a^T a} \right) b$$

$\Rightarrow p = Pb$, where $P = \dfrac{a a^T}{a^T a}$ is a projection matrix

eg:



$$P = \frac{a a^T}{a^T a} = \frac{\begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} 4 & 2 \end{bmatrix}}{\begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}} = \frac{\begin{bmatrix} 16 & 8 \\ 8 & 4 \end{bmatrix}}{20} = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix}$$

$$Pb = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8/5 \\ 4/5 \end{bmatrix} = p$$

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Note: * Projection matrix is always symmetric
* $p^2 = p$
* p has no inverse
* p is a matrix that multiplies b and produces p

Problems

1. project $b:(2,4,1)$ onto $a:(3,2,5)$ and $c:(3,-1,-2)$

$$p = \hat{x}a = \left(\frac{a^Tb}{a^Ta}\right)a = \left(\frac{\begin{bmatrix}3 & 2 & 5\end{bmatrix}\begin{bmatrix}2\\4\\1\end{bmatrix}}{\begin{bmatrix}3 & 2 & 5\end{bmatrix}\begin{bmatrix}3\\2\\5\end{bmatrix}}\right)\begin{bmatrix}3\\2\\5\end{bmatrix}$$

$$= \begin{bmatrix}3/2\\1\\5/2\end{bmatrix}$$

$$p = \hat{x}c = \left(\frac{c^Tb}{c^Tc}\right)c = \left(\frac{\begin{bmatrix}3 & -1 & -2\end{bmatrix}\begin{bmatrix}2\\4\\1\end{bmatrix}}{\begin{bmatrix}3 & -1 & -2\end{bmatrix}\begin{bmatrix}3\\-1\\-2\end{bmatrix}}\right)\begin{bmatrix}3\\-1\\-2\end{bmatrix}$$

$$= \begin{bmatrix}0\\0\\0\end{bmatrix}$$

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2. What multiple of a is closed to the point b: (1,2,2) where a: (1,1,1). Find also the point closest to 'a' on the line passing through b.

Projection of 'b' onto a line 'a'

$$p = \hat{x}a = \left(\frac{a^T b}{a^T a}\right)a = \left(\frac{\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\2\\2\end{bmatrix}}{\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}}\right)\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}5\\5\\5\end{bmatrix}$$

Projection of 'a' onto a line 'b'

$$p = \hat{x}b = \left(\frac{b^T a}{b^T b}\right)b = \left(\frac{\begin{bmatrix}1 & 2 & 2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1 & 2 & 2\end{bmatrix}\begin{bmatrix}1\\2\\2\end{bmatrix}}\right)\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}5/9\\10/9\\10/9\end{bmatrix}$$

3. Project the vector b: (1,2,1) onto the line through a: (2,-3,1). Also check that the vector 'e' is orthogonal to 'a'. Discuss the reason

Projection of b on line through a

$$p = \hat{x}a = \left(\frac{a^T b}{a^T a}\right)a = \left(\frac{\begin{bmatrix}2 & -3 & 1\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}}{\begin{bmatrix}2 & -3 & 1\end{bmatrix}\begin{bmatrix}2\\-3\\1\end{bmatrix}}\right)\begin{bmatrix}2\\-3\\1\end{bmatrix}$$

$$= \begin{bmatrix}-3/7\\9/14\\-3/14\end{bmatrix}$$

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Now, $e = b - p = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -3/7 \\ 9/14 \\ -3/14 \end{bmatrix}$

$$e = \begin{bmatrix} 10/7 \\ 19/14 \\ 17/14 \end{bmatrix}$$

$$\langle e, a \rangle = e^T a = \begin{bmatrix} 10/7 & 19/14 & 17/14 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = 0$$

$\Rightarrow \quad e \perp a$

We know $e \cdot a = 0$

$\|e\| \, \|a\| \cos(\text{angle b/w them}) = 0$, θ

$\Rightarrow \|e\| \; \& \; \|a\| \neq 0$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = 90^\circ$.

4. Find the matrix P that projects every point in $\mathbb{R}^3$ onto the line of intersection of the plane $x+y+z=0$ and $x-z=0$

First, we need to find a vector on the line of intersection of the plane, which is the sol^n of matrix eq^n $Ax = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

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$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x + y + z = 0$$

$$-y - 2z = 0 \Rightarrow y = -2z$$

$$\Rightarrow x = z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} +z \\ -2z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

(OR)

$$x - z = 0 \Rightarrow x = z$$

$$x + y + z = 0 \Rightarrow y = -2z$$

$$\therefore (z, -2z, z) \text{ (or) } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The line passes through the point which are multiple of the special sol^n $(1,-2,1)$

Thus, we need to project all points onto the line passing through a: $(1,-2,1)$

Thus the projection matrix is

$$P = \frac{aa^T}{a^Ta} = \frac{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}} = \frac{\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}}{1+4+1} = \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$$

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5. Construct the projection matrices P_1 and P_2 onto the lines $a_1 : (1, 0)$ and $a_2 : (1, -1)$. Is it true that $(P_1 + P_2)^2 = P_1 + P_2$? Why or why not?

$$P_1 = \frac{a_1 a_1^T}{a_1^T a_1} = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P_2 = \frac{a_2 a_2^T}{a_2^T a_2} = \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix}}{\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$P_1 + P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$$(P_1 + P_2)^2 = (P_1 + P_2) \cdot (P_1 + P_2) = \begin{bmatrix} 5/2 & -1 \\ -1 & 1/2 \end{bmatrix}$$

$$P_1 + P_2 \neq (P_1 + P_2)^2$$

$\therefore$ $P_1 + P_2$ is not a projection matrix

<u>Note</u>: If $P_1 P_2 \neq 0$ then $P_1 + P_2$ is not a projection matrix.

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6. Let S be the two dimensional subspace of $\mathbb{R}^3$ Spanned by the vectors $v_1: (1,2,1)$ and $v_2: (1,-1,1)$. Write the vector $v: (-2,2,2)$ as the sum of a vector in S and a vector orthogonal to S.

The basis of $S^\perp$ is given by the matrix equation $Ax = 0$

where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

The plane is $C(A^T)$ of A and so the line is $N(A)$ of A

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

(2) Simply assume $A^T y = 0$ (orthogonal)

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$y = 0$; $x = -z$ (z is free variable)

$$x = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The line is spanned by the vector $(-1,0,1)$

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Now, by data

$$V = a v_1 + b v_2 + c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} a+b-c \\ 2a-b \\ a+b+c \end{bmatrix}$$

$$\Rightarrow a = 2/3, \quad b = -2/3, \quad c = 2$$

Thus, the vector in S is given as

$$u = \frac{2}{3} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \left(-\frac{2}{3}\right) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

and vector in $S^{\perp}$ is

$$v = 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

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Least squares for one variable

Consider $a_1 x = b_1$
$a_2 x = b_2$
$a_3 x = b_3$
i.e, $Ax = b$

This system is solvable, if & when b_1, b_2, b_3 are in the ratio $a_1 : a_2 : a_3$
[The point b is in the same line as column A]

Suppose the system is inconsistent, then choose x such that x minimizes the avg error E in 'm' eq^{n}s.

The most convenient average comes from the sum of the squares of the errors
i.e, $E^2 = e_1^2 + e_2^2 + e_3^2$
$= (a_1 x - b_1)^2 + (a_2 x - b_2)^2 + (a_3 x - b_3)^2$

If there is a exact sol^n, then $E = 0$
If there is no solution, then the graph of E^2 will be a parobola.

The minimum error is at the lowest point where the derivative is zero (I^{st} principle method)

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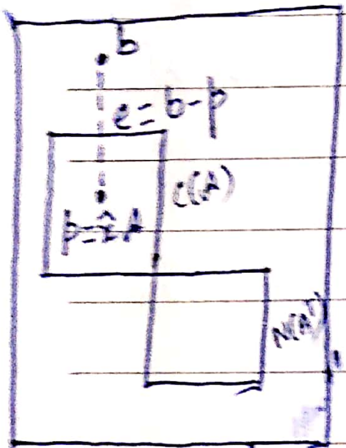
$$\frac{dE^2}{dx} = 2(a_1x-b_1)a_1 + 2(a_2x-b_2)a_2 + 2(a_3x-b_3)a_3 = 0$$

$$\Rightarrow a_1^2x + a_2^2x + a_3^2x - (a_1b_1 + a_2b_2 + a_3b_3) = 0$$

$$\therefore \hat{x} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{a^Ta} = \frac{a^Tb}{a^Ta}$$ is the least square solⁿ

Least squares for several variables



* Here $\hat{x}$ is a vector

Consider $A_{m\times n} X_{n\times 1} = b_{m\times 1}$

Suppose $Ax=b$ has no solution, then we look for best possible approximate solution i.e, we look for the vector p in $C(A)$ which is closest to b.

The system $Ax=b$ is reduced to $A\hat{x}=p$

From the fig: $e \perp C(A) \Rightarrow (b-p) \perp C(A)$

$$\Rightarrow (b - A\hat{x}) \perp C(A)$$

$$\Rightarrow b - A\hat{x} \in N(A^T)$$

$$\therefore A^T(b - A\hat{x}) = 0$$

$$A^Tb - \hat{x}A^TA = 0 \Rightarrow A^TA\hat{x} = A^Tb$$ [Normal eqn]

$$\Rightarrow \hat{x} = \frac{A^Tb}{A^TA} = (A^TA)^{-1}(A^Tb)$$

is least squares solⁿ

Thus, the projection is $p = A(A^TA)^{-1}A^Tb$

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Projection matrix

We have $p = A\hat{x} = A(A^TA)^{-1}A^Tb$

$= [A(A^TA)^{-1}A^T]b$

$p = Pb$

where $P = A(A^TA)^{-1}A^T$ is the projection matrix

* When b is in the $C(A)$, then $p = b$

i.e., $p = A\hat{x} = A(A^TA)^{-1}A^Tb$

Since $b \in C(A)$, then $Ax = b$

$$p = A(A^TA)^{-1}A^T(Ax)$$
$$= A(A^TA)^{-1}A^TAx$$
$$= A(A^{-1}(A^{T^{-1}}A^T))Ax$$
$$= A(A^{+}A)x = Ax$$
$$p = b \quad [A \text{ is invertible}]$$

* When b is in the $N(A^T)$ then $b = 0$

$$p = A(A^TA)^{-1}A^Tb$$

Since $b \in N(A^T) \Rightarrow A^Tb = 0$

$$p = A(A^TA)^{-1}0$$
$$p = 0$$

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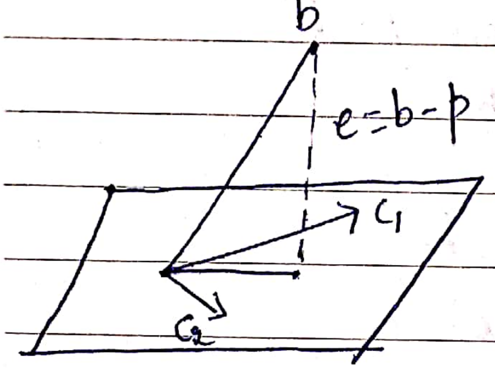
* If A has independent columns, then A^TA is square, symmetric and invertible

* Projection of b onto the column space of matrix of order 3X2

Consider $Ax=b$

$A_{3\times 2}\, x_{2\times 1} = b_{3\times 1}$ and Rank 2



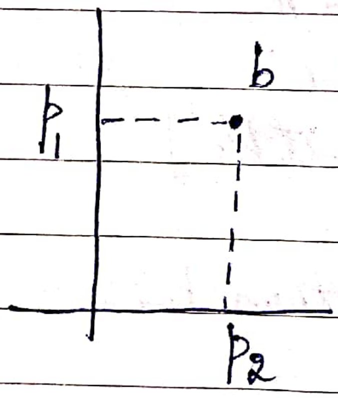
Col 1 ⊥ e & Col 2 ⊥ e

$Ae=0$

$A(b-p)=0$

$\Rightarrow$ every column of A is $\perp^r$ to e

*



$b = p_1 + p_2$

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Problems

1. Let S be the 2D subspace of $\mathbb{R}^3$ spanned by $v_1:(1,2,1)$ and $v_2:(1,-1,1)$. Write the vector $v:(-2,2,2)$ as the sum of a vector in S and a vector orthogonal to S.

Sol^n By data, $S = \text{Span}\{v_1, v_2\} = C(A)$

consider $A^T x = 0$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

The special solution is $(-1,0,1)$

i.e., $a = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \in S^{\perp} = N(A^T)$

Now, projection of v on the $N(A^T)$
i.e., projec^n of v on the line

$$p_2 = \hat{x} a = \left(\frac{a^T v}{a^T a} \right) a = \left(\frac{\begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}}{\begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \right) \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

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$$p_1 + p_2 = b$$

$$p_2 = b - p_1 = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

2. Write out $E^2 = \|Ax - b\|^2$ and set to zero its derivatives wrt u and v if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $x = \begin{bmatrix} u \\ v \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$. Compare the resulting equation with $A^TA\hat{x} = A^Tb$, confirming that calculus as well as geometry gives the normal equation. Find the solution $\hat{x}$ and the projection $p = A\hat{x}$. Why is $b = p$?

Solⁿ Given $E^2 = \|Ax - b\|^2$ (By calculus)

$$Ax = b$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ u+v \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$E^2 = \|Ax - b\|^2 = \left\| \begin{matrix} u-1 \\ v-3 \\ u+v-4 \end{matrix} \right\|^2$$

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$$L^2 = c_1^2 + c_2^2 + c_3^2$$

$$L^2 = (u-1)^2 + (v-3)^2 + (u+v-4)^2$$

$$\frac{\partial L^2}{\partial u} = 2(u-1) + 2(u+v-4) = 0$$

$$\Rightarrow u-1+u+v-4=0 \Rightarrow 2u+v=5$$

$$\frac{\partial L^2}{\partial v} = 2(v-3) + 2(u+v-4) = 0$$

$$\Rightarrow u+2v=7$$

By geometry, consider the normal eq^n

$$A^TA\hat{x} = A^Tb$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\Rightarrow 2u+v=5 \text{ and } u+2v=7$$

The sol^n of the above eq^{n}s are : $u=1, v=3$

$$\hat{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

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The projection is $p = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$p = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = b.$$

$\therefore$ This 'b' is in the $C(A)$ and so $p = b$.

3. Find the projection of b onto the $C(A)$, $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$. Split b into $p + q$ with p in the $C(A)$ and q is $\perp^r$ to that space. Which of four subspace contain q?

Sol^n: Consider the normal equation,

$$A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix} \hat{x} = \begin{bmatrix} -11 \\ 27 \end{bmatrix}$$

$R_2 \to R_2 + \frac{8}{6} R_1$

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$$\begin{bmatrix} 6 & -8 \\ 0 & 22/3 \end{bmatrix} \hat{x} = \begin{bmatrix} -11 \\ 27 \end{bmatrix}$$

$$\Rightarrow x = 203/66 \ , \ y = 81/22$$

$$\hat{x} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Thus, the projection is

$$p = A\hat{x} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

We have

$$p + q = b$$

$$q = b - p = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

p is in the $C(A)$ and q is in the $N(A^T)$.

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4. Find the matrix which projects every vector in $\mathbb{R}^2$ onto the line through $(1,2)$. Also find the matrix which projects onto the orthogonal complements on this line. Hence obtain two rel^{n}s b/w these two matrices and explain why there are such relations.

(i) $P = \dfrac{aa^T}{a^Ta} = \dfrac{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}{5} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}$

(ii) Let $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $u = \begin{bmatrix} x \\ y \end{bmatrix}$

$u^T v = 0$

$\Rightarrow x + 2y = 0$

i.e., $\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\underset{\text{(Choose)}}{y = 1} \Rightarrow x = -2 \quad \therefore u = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a basis for the orthogonal complement of a given line.

$Q = \dfrac{uu^T}{u^Tu} = \dfrac{\begin{bmatrix} -4 & -2 \\ -2 & 1 \end{bmatrix}}{5}$

$P + Q = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix} + \begin{bmatrix} -4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

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$$P \cdot Q = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix} \begin{bmatrix} -4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix} = 0$$

If $\begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}$ projects every vector in $\mathbb{R}^2$ onto V then find the projection matrix onto $V^\perp$.

$$Q = I - P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$$

5. Solve $Ax = b$ by least squares, where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Also find p and verify that the error vector e is $\perp^r$ to $C(A)$.

Soln

$$Ax = b$$

$$A^TA\hat{x} = A^Tb$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A:b = \left[\begin{array}{cc:c} 2 & 1 & 1 \\ 1 & 2 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - \frac{1}{2}R_1} \sim \left[\begin{array}{cc:c} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \end{array}\right]$$

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$\Rightarrow x = y = 1/3$

$\therefore \hat{x} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$

$p = A\hat{x} = \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix}$; $e = p - b = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}$

$$e^T A = \begin{bmatrix} 2/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Thus $e^T A = 0 \Rightarrow e \perp C(A)$

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Best fitting line $b = C + Dt$ by least squares method.

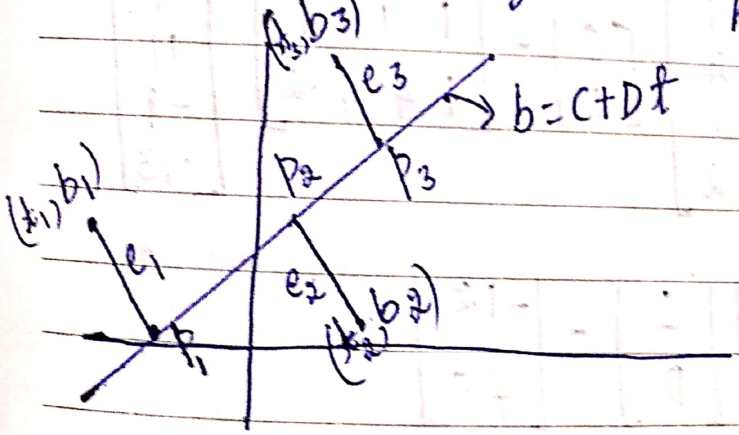
Consider $b = C + Dt$ be the best fitting curve. It passes through $(t_1, b_1), (t_2, b_2)$ and (t_3, b_3).

$$C + Dt_1 = b_1$$
$$C + Dt_2 = b_2$$
$$C + Dt_3 = b_3$$

i.e., $Ax = b$

where $A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{bmatrix}$, $x = \begin{bmatrix} C \\ D \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

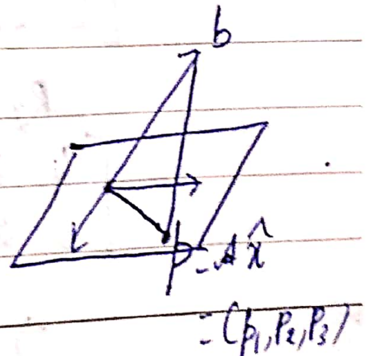
$\hat{x} = (C, D)$ is the best approximate sol^n that minimizes the squared errors



To find $\hat{x}$, we use

$$\boxed{A^T A \hat{x} = A^T b}$$

Note: The st line approximation matches the projection p of b.



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Problems

1. Find the equation of the line that runs through four points $(1,-1)$, $(4,11)$, $(-1,-9)$ and $(-2,-13)$

Sol^n Let $b = C + Dt$ be the best fitting line

Now, $C + D(1) = -1$

$C + D(4) = 11$

$C + D(-1) = -9$

$C + D(-2) = -13$

i.e., $Ax = b$

$$\begin{bmatrix} 1 & 1 \\ 1 & 4 \\ 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \\ -9 \\ -13 \end{bmatrix}$$

Consider $A^T A \hat{x} = A^T b$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 4 \\ 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & -1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 11 \\ -9 \\ -13 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 22 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} -12 \\ 78 \end{bmatrix}$$

$R_2 \to R_2 - \frac{1}{2} R_1$

$$\begin{bmatrix} 4 & 2 \\ 0 & 21 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} -12 \\ 84 \end{bmatrix}$$

$\Rightarrow C = -5,\ D = 4 \Rightarrow$ the best fitting line is $b = 4t - 5$

Date ....

2. The different denominations in rupees (X) and their average life in years (Y) are given below. Using the method least squares fit a straight line in the form $Y = a + bX$. And hence find the average life of a Rs 10 note.

Sol^n

X	1	2	3
y	1	2	3

Let $Y = a + bx$ be the best fitting curve

Now, $1 = a + b(1)$

$2 = a + b(2)$

$3 = a + b(3)$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Consider $(A^T A)\hat{x} = A^T b$

Solving above eq^n, we get $a = 0$, $b = 1$

$\therefore$ the best fitting line is $Y = X$

The average life of Rs 10 is given as

$$\underline{\underline{Y = 10 \text{ years}}}$$

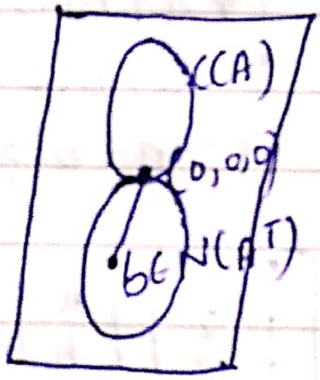
Date ....

1. If V is the subspace spanned by $(1,1,0,1)$ and $(0,0,1,0)$ find a basis for the orthogonal complement $V^\perp$. Find also the vector in V closest to the vector $b=(0,1,0,-1)$ in $V^\perp$.

Sol^n basis for $V^\perp = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Since $b \in V^\perp$

$\Rightarrow$ projection of b on V is zero vector



2. If S is the subspace spanned by $(0,1,-1)$, $(1,0,-2)$, obtain the matrix which projects every vector in $\mathbb{R}^3$ onto $S^\perp$. Hence or otherwise obtain the matrix that projects onto S.