Scheme and Solution

UE18MA251 Linear Algebra

Q.1 a)

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ -2 & 5 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow (2m)$$

$$P_{23}A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -4 \\ 2 & -4 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow (1m)$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \rightarrow (2m) \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow (1m)$$

$$L^T = U \rightarrow (1m)$$

7m

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b)

$$x + y + az = 2b$$
$$x + 3y + (2+2a)z = 7b$$
$$3x + y + (3+3a)z = 11b$$

$$\left(\begin{array}{ccc|c} 1 & 1 & a & 2b \\ 1 & 3 & 2+2a & 7b \\ 3 & 1 & 3+3a & 11b \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & a & 2b \\ 0 & 2 & 2+a & 5b \\ 0 & -2 & 3 & 5b \end{array}\right) \rightarrow (2m)$$

$(1+1)m$

$$\left(\begin{array}{ccc|c} 1 & 1 & a & 2b \\ 0 & 2 & 2+a & 5b \\ 0 & 0 & a+5 & 10b \end{array}\right) \rightarrow (1m)$$

Unique nontrivial solution $a \neq -5$, any $b \rightarrow (1m)$

trivial solution $a \neq -5$, $b = 0 \rightarrow (1m)$

no solution $a = -5$, $b \neq 0 \rightarrow (1m)$

infinity of solutions $a = -5$, $b = 0 \rightarrow (1m)$

7m

2

3) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$ $[A:I] \rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$ — (1M)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right)$$ — (1+1)m

(6M)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right)$$ — (1+1)m

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$ — (1M)

Q.2 a) $u + v + w = 7$
$u + 2v + 2w = 10$
$2u + 3v - 4w = 3$

~~(1+1)m~~

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 2 & 10 \\ 2 & 3 & -4 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & -6 & -11 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -7 & -14 \end{array}\right)$$

(1M)

$-7w = -14$, $w = 2$
$v + w = 3$, $v = 1$ — (1M)
$u + v + w = 7$, $u = 4$

1) $\rho(A) = \rho(A:b) = n$ (unknowns) — (1M)

2) A^{-1} exists ($A_{3\times3}$ matrix has 3 pivots in Echelonform) (1M)

3) $C(A) = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} ; \alpha, \beta, \gamma \in R \right\}$ whole R^3

$b = \begin{pmatrix} 7 \\ 10 \\ \end{pmatrix} \in C(A)$ — (1M)

4) $Ax=0 \Rightarrow Ux=0$
No Free variable $N(A)=\{0\}$ — (1M)

5) $A^T x=0 \Rightarrow x^T A=0$ — (1M)
$N(A^T)=\{0\}$

fM

b) 1) $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \sim \left\{ a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ a,b,c \in R \right\} \longrightarrow$ (1M)

2) $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \sim \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};\ a,b,c \in R \right\}$

3) $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \sim a+b+c+d=0 \sim \begin{pmatrix} -b-c-d \\ b \\ c \\ d \end{pmatrix} \longrightarrow$ (1M)

$\sim \left(b\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix};\ b,c,d \in R \right)$ — (1M)

7M

4) $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \sim a=b=c=d \begin{pmatrix} a \\ a \\ a \\ a \end{pmatrix}$ — (1M)

$\sim \left\{ a\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix};\ a \in R \right\}$

5) $P_3 : \{1, t, t^2, t^3\}$ — (1M)

6) $\begin{pmatrix} a & a \\ -a & 0 \end{pmatrix} \sim \left\{ a\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix};\ a \in R \right\}$ — (1M)

c) $\begin{pmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 5 \\ 0 & 2 & -3 \\ 0 & -8 & +12 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ (4) — (1M)

$C(A) = \left\{ \alpha \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -5 \end{pmatrix} ; \alpha, \beta \in R \right\}$ — (1M)

$C(A^T) = \left\{ \alpha \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix} ; \alpha, \beta \in R \right\}$ — (1M)

6m

$N(A^T) \Rightarrow A^T x = 0 \Rightarrow x^T A = 0$

$\left(\begin{array}{ccc|c} 1 & -1 & 5 & b_1 \\ 0 & 2 & -3 & b_2 - 2b_1 \\ 0 & -8 & 12 & b_3 + 3b_1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 5 & b_1 \\ 0 & 2 & -3 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 + 4(b_2 - 2b_1) \end{array}\right)$

$b_3 + 4b_2 - 5b_1$

$N(A^T) = \left\{ \alpha \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix} ; \alpha \in R \right\}$ — (2M)

$Ax = 0 \Rightarrow Ux = 0$

$\begin{pmatrix} 1 & -1 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$x - y + 5z = 0 \qquad x = -\frac{7}{2} z.$

$2y - 3z = 0 \Rightarrow y = \frac{3}{2} z$

$\therefore (-7, 3, 1) \in N(A)$ — (1M)

Q.3 a) $x - 2y + 3z = 0$

$$\begin{pmatrix} 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \text{ ——— } (1m)$$

$$\begin{pmatrix} 2y-3z \\ y \\ z \end{pmatrix} \sim y \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \text{ ——— } (2m)$$

intersection with xy plane $xy \Rightarrow z = 0$ (1m)

$$\begin{pmatrix} 2y \\ y \\ 0 \end{pmatrix} \sim y \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \text{ ——— } (1m)$$

Basis of vector perpendicular to plane $(1, -2, 3)$ ——— (1m)

6m

b) $y = C + Dt + Ez$

$$\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} \text{ ——— } (1m)$$

$$A^T A \hat{x} = A^T b$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \text{ (1m)}$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \text{ (1m)}$$

$$\left(\begin{array}{ccc|c} 4 & 2 & 2 & 0 \\ 2 & 2 & 1 & -1 \\ 2 & 1 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 4 & 2 & 2 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right) \text{ ——— } (1+1)m$$

8m

$C = -\frac{1}{2}, D = -1, E = 2$ —— $(1+1+1)m$ 6

$y = -\frac{1}{2} - t + 2z$

(c) $P_3 : \{1, t, t^2, t^3\}$ —— (1m)

$P_4 : \{1, t, t^2, t^3, t^4\}$ —— (1m)

6m

$(3t-5)1 = 3t-5 = -5(1)+3(t)+0(t^2)+0(t^3)+0(t^4)$

$(3t-5)t = 3t^2-5t = 0(1)-5(t)+3(t^2)+0(t^3)+0(t^4)$

$(3t-5)t^2 = 3t^3-5t^2 = 0(1)+0(t)-5(t^2)+3(t^3)+0(t^4)$

$(3t-5)t^3 = 3t^4-5t^3 = 0(1)+0(t)+0(t^2)-5(t^3)+3(t^4)$

(1m)

$P_3 \longrightarrow P_4$

$R^4 \xrightarrow[5\times 4]{} R^5$

$$\begin{pmatrix} -5 & 0 & 0 & 0 \\ 3 & -5 & 0 & 0 \\ 0 & 3 & -5 & 0 \\ 0 & 0 & 3 & -5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$ —— (3m)

Q.4 a) $A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & -3 \\ 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & -3 \\ 0 & 0 \end{pmatrix}$ —— (1) m

$q_3 \perp C(A)$ span(q_1, q_2) ——

$q_3 \in N(A^T)$ —— (1m)

$\begin{pmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ —— (1m)

7

$b = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ — (1m)

$A^T A \hat{x} = A^T b$

$A = QR \quad Q = \begin{pmatrix} 1/3 & 1/\sqrt{18} & -2/3 \\ 2/3 & -1/\sqrt{18} & 2/3 \\ -2/3 & 4/\sqrt{18} & 1/3 \end{pmatrix}$ — (1m)

$R = Q^T A = \begin{pmatrix} 3 & -3 \\ 0 & 3 \end{pmatrix}$ — (1m)

$R\hat{x} = Q^T b$

$\begin{pmatrix} 3 & -3 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$ — (1m)

$\hat{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ — (1m)

8M

7b)

$$\begin{array}{ccccc} A & x_0 & = & \lambda_1 x_1 & \\ \end{array}$$

$A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 0.6 \\ 0.2 \\ 1 \end{pmatrix}$ — (1m)

$Ax_1 = \begin{pmatrix} 1 \\ 1 \\ 2.2 \end{pmatrix} = 2.2 \begin{pmatrix} 0.4545 \\ 0.4545 \\ 1 \end{pmatrix}$ — (1m)

$Ax_2 = 2.818 \begin{pmatrix} 0.4839 \\ 0.5484 \\ 1 \end{pmatrix}$ — (1m)

$Ax_3 = 3.1291 \begin{pmatrix} 0.5052 \\ 0.5051 \\ 1 \end{pmatrix}$ — (1m)

$Ax_4 = 3.0205 \begin{pmatrix} 0.5017 \\ 0.4948 \\ 1 \end{pmatrix}$ — (1m)

7M

$$Ax_5 = 2.9861\begin{pmatrix}0.4994\\0.4994\\1\end{pmatrix} \text{ — (1m)}$$

$$Ax_6 = 2.9976\begin{pmatrix}0.4998\\0.5006\\1\end{pmatrix} \text{ — (1M)}$$

Largest eigen value $= 2.998$ — (1M)

c) $a_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $a_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}$ $a_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

5M

$$q_1 = \frac{a_3}{\|a_3\|} = \begin{pmatrix}0\\0\\1\end{pmatrix} \text{ — (1M)}$$

$$q_2 = \frac{a_2 - (q_1^T a_2) q_1}{\| a_2 - (q_1^T a_2) q_1 \|} \text{ — (1M)}$$

$$\begin{pmatrix}0\\1\\1\end{pmatrix} - \left((0\ 0\ 1)\begin{pmatrix}0\\1\\1\end{pmatrix}\right)\begin{pmatrix}0\\0\\1\end{pmatrix} \text{ — (1M)}$$

$$\Rightarrow \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$q_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \text{ — (1M)}$$

$$q_3 = \frac{a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2}{\| a_3 - (q_1^T a) q_1 - (q_2^T a_3) q_2 \|} \text{ — (1M)}$$

$$\Rightarrow \begin{pmatrix}1\\1\\1\end{pmatrix} - \left((0\ 0\ 1)\begin{pmatrix}1\\1\\1\end{pmatrix}\right)\begin{pmatrix}0\\0\\1\end{pmatrix} - \left((0\ 1\ 0)\begin{pmatrix}1\\1\\1\end{pmatrix}\right)\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \text{ — (1M)}$$

Q.5 a)

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$a_1 = |2| = 2$

$a_{11} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$

$a_{21} = \begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 0$ —— (1m)

$\therefore$ Matrix is positive semidefinite —— (1m)

$\because |A| = 0$

Eigen values of $A = 0, 3, 3$ —— (1m)

Matrix is Positive semidefinite —— (1m)

$x^T A x = (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2 \geq 0$

$= 0$ if $x_1 = x_2 = x_3$ —— (2+1)m

8m

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b) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ $\qquad A = U \Sigma V^T$ —— (1m)

$AA^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ —— (1m)

$\lambda_1 = 1, \lambda_2 = 3$ —— (1m)

Eigen values of $A^T A$, $1, 3, 0$ —— (1m)

$U = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ —— (1+1+1) m

$\Sigma = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (2m)

$V = \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$ (1+1+1)m

12m