DATE: PAGE:

Unit - 4

Orthogonalization, Eigenvalues and Eigenvectors

Orthogonal bases

A set of vectors $q_1, q_2 \dots q_n$ is called orthonormal if

$$q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

A matrix that has ~~orthogonal~~ orthonormal columns will be written as Q. For such a matrix, $Q^T Q = I$

i.e. Q^T is a left inverse of Q. In particular, if Q is a square then Q is called an orthogonal matrix.

In this case, $Q^T = Q^{-1}$

Example I: $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ dot product b/w col is 0

unit length of column.

Q is orthogonal.

So that $Q^T Q = I$ $Q^T = Q^{-1}$

Example 2: $Q = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ Q is orthogonal

$Q^T = Q^{-1}$. So that $Q^T Q = I$

Example 3: All permutation matrices are orthogonal

$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is such that $P_{23}^T = P_{23}^{-1}$

So that $P_{23}^T P_{23} = I$

Example 4: $Q = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \\ 0 & 0 \end{bmatrix}$. $\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 2/\sqrt{5} & -1/\sqrt{5} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Q^T is a left inverse of Q.

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* You can never have a Q matrix of size $m \times n$ where $n > m$.

Advantages of Q matrix:

(1) Q preserves norm.

Proof: $\|Qx\|^2 = (Qx)^T (Qx)$

$$= x^T Q^T Q x$$

$$= x^T I x$$

$$= x^T x$$

$$= \|x\|^2.$$

(2) Q preserves angle b/w two vectors

Proof: $(Qx)^T (Qy) = x^T Q^T Q y$

$$= x^T y.$$

$$\therefore \cos\theta = \frac{x^T y}{\|x\| \, \|y\|} = \frac{(Qx)^T (Qy)}{\|Qx\| \, \|Qy\|}$$

(3) If $V_1 V_2 \ldots V_n$ is a basis for a v/s V then any $b \in V$ is a linear combination of the V_is.

$$\therefore b = C_1 V_1 + C_2 V_2 + \ldots C_n V_n$$

If $q_1 q_2 \ldots q_n$ is an orthogonal basis for V. then

$$b = x_1 q_1 + x_2 q_2 + \ldots x_A V_n.$$

$$q_1^T b = x_1$$

$$q_2^T b = x_2$$

$$q_n^T b = x_n$$

$\therefore$ The equation $b = x_1 q_1 + x_2 q_2 + \ldots + x_n q_n$

$$= x_1 \begin{bmatrix} \uparrow \\ q_1 \\ \downarrow \end{bmatrix} + x_2 \begin{bmatrix} \uparrow \\ q_2 \\ \downarrow \end{bmatrix} + x_3 \begin{bmatrix} \uparrow \\ q_n \\ \downarrow \end{bmatrix}$$

can be written as $\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ q_1 & q_2 & q_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b$ $\quad Qx = b$

i.e. $Qx = b$

The solution of the above system of eq^n

$$x = Q^{-1}b = Q^{T}b$$

(4) $b = x_1 q_1 + x_2 q_2 + \dots + x_n q_n$

where any x_i can be solved as $q_i^T b$.

$\therefore b = (q_1^T b)q_1 + (q_2^T b)q_2 + \dots + (q_n^T b)q_n$

$$= \frac{q_1^T b}{q_1^T q_1} q_1 + \frac{q_2^T b}{q_2^T q_2} q_2 + \dots + \frac{q_n^T b}{q_n^T q_n} q_n$$

i.e. b = projection of b onto line through q_1
+ projection of b onto line through q_2
⋮
+ projection of b onto line through q_n

$\therefore$ b = sum of projections of b onto ~~~~ q_i's.

(5) If the columns of a square matrix Q are orthonormal, then its rows are automatically orthonormal.

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

(6) Consider a system of equation, $Qx = b$.
If Q is square, then $x = Q^{-1}b$. or $x = Q^T b$.

If Q is a matrix of size $m \times n$, with $m > n$. Then $Qx = b$ is solvable if and only if b belongs to column space of Q $b \in C(Q)$.
If not, we solve the system by the method of least squares.
The normal equation is: $Q^T Q \hat{x} = Q^T b$ $\therefore \hat{x} = Q^T b$

The point of projection is

$$p = Q\hat{x}$$

$$p = QQ^Tb$$

$QQ^T \neq I$

(Because Q is a tall matrix it will never have a right inverse)

$$P = Q(Q^TQ)^{-1}Q^T$$

$$P = QQ^T$$

Gram- Schimdt Probcess of orthogonalization

To construct an orthonormal set of vectors q_1 q_2 q_3 from a set of linearly independent vectors a, b, c

1) The first vector q_1 can go in any of the 3 directions (a, b, c). Let us choose the direction of q_1 along a.

$$\therefore q_1 = \frac{\vec{a}}{\|a\|}$$

So that $\|q_1\| = 1$

2) The second vector b is independent of a. If it is already orthogonal to a: we can choose q_2 in this direction and write:

$$q_2 = \frac{b}{\|b\|}$$

so that $\|q_2\| = 1$.

If b is not orthogonal to a:

b has components in the direction of q_1 and in $\perp$'r direction.

If we want q_2 to go in this $\perp$'r direction then, we need to subtract the component of b in the direction of q_1 from b.

Consider $B = b - \frac{(q_1^T b) q_1}{\underbrace{(q_1^T q_1)}_{\to 1}}$

Now, $q_2 = \frac{B}{\|B\|}$, so that $\|q_2\| = 1$

3) The third vector c is independent of both a and b. Hence, it is not in the plane of a and b. If it is already orthogonal, then we can choose q_3 in this direction and write $q_3 = \frac{c}{\|c\|}$, so that $\|q_3\| = 1$.

If c is not orthogonal to them: c has components in the direction of q_1, q_2 and in the $\perp$'r direction.

If we want q_3 to go in this $\perp$'r direction, then we need to subtract the components of c in q_1 and q_2 directions from c.

Consider $C = c - \frac{(q_1^T c) q_1}{\underbrace{q_1^T q_1}_{\to 1}} - \frac{(q_2^T c) q_2}{\underbrace{q_2^T q_2}_{\to 1}}$

Now, $q_3 = \frac{C}{\|C\|}$, so that $\|q_3\| = 1$

Proof:

$$q_1^T B = q_1^T (b - (q_1^T b) q_1)$$
$$= q_1^T b - (q_1^T b) q_1^T q_1$$
$$= q_1^T b - q_1^T b$$
$$= 0$$

This implies that they are orthogonal. (B is orthogonal to q_1)

$$q_1^T c = q_1^T\left(c - (q_1^T c)q_1 - (q_1^T c)q_2\right)$$
$$= q_1^T c - (q_1^T c)(q_1^T q_1) - 0$$
$$= q_1^T c - q_1^T c$$
$$= 0$$

This implies that q_1 is orthogonal to c.

Similarly, q_2 is orthogonal to c.

$A = QR$ Factorisation

Given a set of linearly independent vectors a, b, c we construct a set of orthonormal vectors q_1, q_2, q_3 using the GS process. If A is a matrix whose columns are a, b, c and Q is the matrix whose col are q_1, q_2, q_3 we now find a relation b/w A & Q. To do this we express a, b, c as linear combination of q_1, q_2, q_3

a = projection of a onto the line through q_1
$= (q_1^T a)\, q_1$

b = sum of projection of b onto lines through q_1 and q_2
$= (q_1^T b)\, q_1 + (q_2^T b)\, q_2$

c = sum of projection of b onto lines through q_1, q_2 and q_3
$= (q_1^T c)\, q_1 + (q_2^T c)\, q_2 + (q_3^T c)\, q_3$.

$$\therefore A = \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix}$$

$$\boxed{A = QR}$$

A and Q will be equivalent (If A is □ Q is also □.
If A is ▯ Q is also ▯)

R is always square matrix. (and upper triangular in nature)

When the system $Ax = b$ is inconsistent we solve it by the method of least squares. The normal equations are $A^T A \hat{x} = A^T b$

DATE: PAGE:

$$(QR)^T QR\hat{x} = (QR)^T b$$

$$R^T Q^T Q R\hat{x} = R^T Q^T b$$

$$R^T R\hat{x} = R^T Q^T b$$

$\boxed{R\hat{x} = Q^T b}$ this is of the form: $\boxed{Ux = c}$

The above system can be solved by back substitution, since R is upper triangular

Q1. Find a 3rd column so that the matrix $Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & x \\ 1/\sqrt{3} & 2/\sqrt{14} & y \\ 1/\sqrt{3} & -3/\sqrt{14} & z \end{bmatrix}$ is orthogonal

$$1/\sqrt{3}\,x + 1/\sqrt{3}\,y + 1/\sqrt{3}\,z = 0$$

$$x^2 + y^2 + z^2 = 1$$

$x+y+z=0$ $\quad 1/\sqrt{14}\,x + 2/\sqrt{14}\,y + -3/\sqrt{14}\,z = 0.$

$x+2y-3z=0$ $\quad x+y+1=0$

$$x+2y-3=0$$

$$x+y=-1$$

$$x+2y=3$$

$y=4$ $\quad x=-5$

$x, y, z = (-5, 4, 1)$

$\dfrac{-5}{\sqrt{42}}, \dfrac{4}{\sqrt{42}}, \dfrac{1}{\sqrt{42}}$

$$(x, y, z) = \left(\frac{-5}{\sqrt{42}}, \frac{4}{\sqrt{42}}, \frac{1}{\sqrt{42}}\right) \text{ or } \left(\frac{5}{\sqrt{42}}, \frac{-4}{\sqrt{42}}, \frac{-1}{\sqrt{42}}\right)$$

Q2. $Q = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}_{3\times 2}$, $x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}_{2\times 1}$ and $y = \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix}$

verify that : (i) $Q^T Q = I$

(ii) $\|Qx\| = \|x\|$ and $\|Qy\| = \|y\|$

(iii) $(Qx)^T Qy = x^T y$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\|x\|$ $\sqrt{11}$ $\qquad\qquad$ $\|y\|$ $\sqrt{54}$

$$\|Qx\| \quad \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & 2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix} = \sqrt{54}$$

$$\|Qy\| : \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \sqrt{11}$$

$$(Qx)^T (Qy) = \underset{1\times 3}{\begin{bmatrix} 3 & -1 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix}} = 3+7+2 = 12.$$

$$x^T y = \begin{bmatrix} \sqrt{2} & 3 \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix} = -6+18 = 12.$$

Hence verified.

Q3. If W is a subspace spanned by the orthogonal vectors $(2, 5, -1)$ and $(-2, 1, 1)$. Find the point in W that is closest to $(1, 2, 3)$.

$$W = \begin{bmatrix} 2/\sqrt{30} & -2/\sqrt{6} \\ 5/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{30} & 1/\sqrt{6} \end{bmatrix} \qquad p = A\hat{x}$$

$$\hat{x} = \frac{Q^T b}{Q^T Q}$$

$$\hat{x} = \begin{bmatrix} 2/\sqrt{30} & 5/\sqrt{30} & -1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9/\sqrt{30} \\ 3/\sqrt{6} \end{bmatrix} = \hat{x}$$

$$p = \begin{bmatrix} 2/\sqrt{30} & -2/\sqrt{6} \\ 5/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{30} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 9/\sqrt{30} \\ 3/\sqrt{6} \end{bmatrix} = \begin{bmatrix} -12/30 \\ 1 \\ 6/30 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} = (-2/5, 2, 1/5)$$

$$p = \frac{a_1^T b}{a_1^T a_1} a_1 + \frac{a_2^T b}{a_2^T a_2} a_2$$

$$p = [2 \quad 5 \quad -1] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{9}{30} a_1 +$$

[2

$$\frac{3}{10} (2, 5, -1) + \frac{1}{2} (-2, 1, 1)$$

$$[-2, 1, 1] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{3}{6} a_2$$

$$\left(\frac{3}{5} - 1\right), \left(\frac{3}{2} + \frac{1}{2}\right), \left(\frac{-3}{10} + \frac{1}{2}\right)$$

$$\left(-\frac{2}{5}, 2, \frac{1}{5}\right)$$

Gram Schmidt Orthogonalization :

→ Given : A set of linearly independent vectors say a, b, c.

(note : necessary condition to apply Gram Schmidt orthogonalization is to have set of independent vectors)

→ Aim : To Find a set of orthonormal vectors say q_1, q_2, q_3

(note : number of orthonormal vectors obtained are equal to number of independent vectors given).

→ Procedure :-

(note : Always construct orthonormal vectors corresponding to the independent vectors in the given order.)

→ Objective :-

"Every vector in a given vector space say R^n, can always be expressed in terms of the linear combination of its one-Dimensional projection onto the orthonormal bases vectors".

e.g. vector $b \in R^n$

$$b = x_1 q_1 + x_2 q_2 + \dots + x_n q_n$$

where $x_i \in R$ q_i - orthonormal bases set s.t. $b = \frac{(q_1^T b) q_1}{q_1^T q_1} + \frac{(q_2^T b) q_2}{q_2^T q_2} + \cdots + \frac{(q_n^T b) q_n}{q_n^T q_n}$

$\because$ $q_i^T q_j = 1$ whenever $i = j$

$\therefore$ $b = (q_1^T b) q_1 + (q_2^T b) q_2 + \cdots + (q_n^T b) q_n$

Gram Schmidt Process to solve Problems.

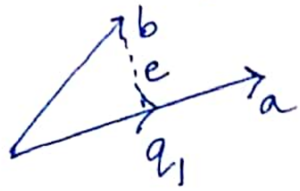
Step I: Find q_1 (consider only vector a)

$\boxed{q_1 = \frac{a}{\|a\|}}$

(both 'q_1' and 'a' are in same direction).

Step II: Find q_2 (consider vectors b and q_1)

project vector 'b' onto 'q_1'



s.t. error vector $\perp q_1$

$e \perp q_1$

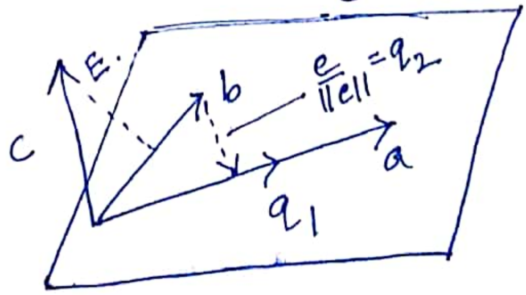
$e = b - p = b - \hat{x} q_1 = b - (q_1^T b) q_1$

$\boxed{e = b - (q_1^T b) q_1}$

normalise vector 'e'

$\boxed{q_2 = \frac{e}{\|e\|}}$

Step III: Find q_3 (consider vectors c and q_1 and q_2)



project 'c' onto the plane spanned by q_1 and q_2

$$\because \quad c = (q_1^T c) q_1 + (q_2^T c) q_2$$

$$E \perp q_1 \text{ and } q_2$$

$$E = c - p.$$

$$E = c - (\hat{x} q_1 + \hat{y} q_2)$$

$$\hat{x} = \frac{(q_1^T c)}{q_1^T q_1} \qquad \hat{y} = \frac{(q_2^T c)}{q_2^T q_2}$$

$$E = c - ((q_1^T c) q_1 + (q_2^T c) q_2)$$

$$\boxed{E = c - (q_1^T c) q_1 - (q_2^T c) q_2}$$

Normalise vector 'E'

$$\boxed{q_3 = \frac{E}{\|E\|}}$$

Step IV: Vectors q_1, q_2, q_3 are the required orthonormal vectors w.r.t to the given independent vectors a, b, c.

Problems on Gram Schmidt Process.

Use GS process to find a set of orthonormal vectors q_1, q_2, q_3 from the given independent vectors a, b, c

$$a = (1,1,1),\ b = (0,1,1),\ c = (0,0,1)$$

Step I : $q_1 = \frac{a}{\|a\|}$

$$\|a\| = \sqrt{(1)^2+(1)^2+(1)^2} = \sqrt{3}$$

$$q_1 = \frac{1}{\sqrt{3}}(1,1,1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\boxed{q_1 = \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}}$$

Step II : $e = b - (q_1^T b) q_1$

$$e = \begin{pmatrix}0\\1\\1\end{pmatrix} - \left\{\left(\frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\right)\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$$

$$e = \begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{2}{\sqrt{3}}\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$$

$$e = \begin{pmatrix}-2/3\\1/3\\1/3\end{pmatrix}$$

$$q_2 = \frac{e}{\|e\|} = \begin{pmatrix}-2/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\end{pmatrix}$$

$$\because \|e\| = \sqrt{(-2/3)^2+(1/3)^2+(1/3)^2} = \frac{\sqrt{6}}{3}$$

$$\boxed{q_2 = \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}}$$

<u>Step III</u>: $E = c - (q_1^T c)q_1 - (q_2^T c)q_2$

$$(q_1^T c)q_1 = \left\{\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$$

$$(q_1^T c)q_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} = \begin{pmatrix}1/3\\1/3\\1/3\end{pmatrix}$$

$$(q_2^T c)q_2 = \left\{\begin{pmatrix}\frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\begin{pmatrix}-2/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\end{pmatrix}$$

$$(q_2^T c)q_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}-2/\sqrt{6}\\1/\sqrt{6}\\1/\sqrt{6}\end{pmatrix} = \begin{pmatrix}-1/3\\1/6\\1/6\end{pmatrix}$$

$$E = \begin{pmatrix}0\\0\\1\end{pmatrix} - \left\{\begin{pmatrix}1/3\\1/3\\1/3\end{pmatrix} + \begin{pmatrix}-1/3\\1/6\\1/6\end{pmatrix}\right\}$$

$$E = \begin{pmatrix}0\\0\\1\end{pmatrix} - \begin{pmatrix}0\\1/2\\1/2\end{pmatrix}$$

$$E = \begin{pmatrix}0\\-1/2\\-1/2\end{pmatrix}$$

$$\|E\| = \sqrt{0^2 + (-1/2)^2 + (-1/2)^2} = 1/\sqrt{2}$$

$$q_3 = \frac{E}{\|E\|} = \begin{pmatrix}0\\-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

$$\boxed{q_3 = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$$

Given vectors :-

$a = (1,1,1)$, $b = (0,1,1)$, $c = (0,0,1)$

Apply G S Process

Obtained Orthonormal vectors :-

$q_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $q_2 = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$q_3 = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Problems to Solve :-

1. Apply Gram Schmidt process to
$a = (1,0,1)$, $b = (1,0,0)$, $c = (2,1,0)$

solution :-

$q_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $q_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$, $q_3 = (0,1,0)$

2. Find orthonormal vectors by G.S. from
$a = (1,-1,0,0)$
$b = (0,1,-1,0)$
$c = (0,0,1,-1)$

Need for Gram Schmidt Process.

Factorize $A = QR$.

$Q \rightarrow$ orthogonal Matrix
(square matrix with orthonormal vectors)
$R \rightarrow$ upper triangular matrix.

Given :- a set of independent vectors
say $a, b, c,$ Find q_1, q_2, q_3 by G.S.

then

$\therefore$ $a = (q_1^T a) q_1$

$b = (q_1^T b) q_1 + (q_2^T b) q_2$

$c = (q_1^T c) q_1 + (q_2^T c) q_2 + (q_3^T c) q_3$

Matrix equivalent.

$$\underset{A}{\begin{pmatrix} | & | & | \\ a & b & c \\ | & | & | \end{pmatrix}} = \underset{Q}{\begin{pmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{pmatrix}} \underset{R}{\begin{pmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{pmatrix}}$$

Problems :-

1. Apply G.S. to
$a = (1,1,1)$, $b = (0,1,1)$, $c = (0,0,1)$
and obtain QR factorization.

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$$A = QR$$

$$R = Q^T A$$

$$R = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$R = \begin{pmatrix} \sqrt{3} & 2/\sqrt{3} & 2/\sqrt{3} \\ 0 & 2/\sqrt{6} & 2/\sqrt{6} \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$R = \begin{pmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{pmatrix}$$

Live Session notes :- 16/3/20

Eigen Values and Eigenvectors :- Page no. 1.

Topics :-

1) What are eigen values and vectors?
2) Geometrical meaning.
3) Why "eigen" ..
4) Properties
5) Largest eigen value.
6) Applications — Diagonalization. ($A = S \Delta S^{-1}$) — powers and Products.

Later, $A = U \Sigma V^T \rightarrow$ SVD. (Unit - 5)

$$A_{n \times n} x_{n \times 1} = b_{n \times 1}$$

$$= \lambda x$$

$\Rightarrow$ $Ax = \lambda x$, λ = eigenvalue

x = eigen vector.

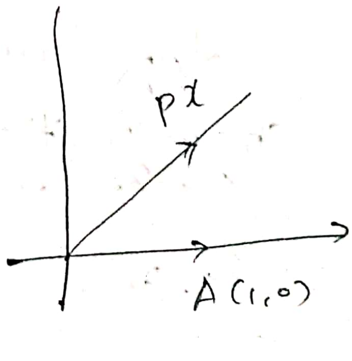
$$Ax - \lambda x = 0$$

$$(A - \lambda I) x = 0$$

If $\lambda = 0 \Rightarrow b = 0$

$\lambda = 1 \Rightarrow b = x$.

λ = scale factor.



eigen - Specific, characteristic.

If $\lambda_i > 0 \Rightarrow |A| > 0$.

If $\lambda_i < 0 \Rightarrow |A| < 0$.

$$Ax = \lambda X$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

$$(A - \lambda I)x = 0. \Rightarrow |A - \lambda I| = 0.$$

$$\Rightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0.$$

$$\Rightarrow (-1)^n \lambda^n + C_1 \lambda^{n-1} + C_2 \lambda^{n-2} + \cdots + C_n = 0. \quad (1)$$

LHS is called characteristic polynomial.

Solving (1), we get $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called characteristic roots or latent roots or eigenvalues.

$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \sum a_{ii}$ = Trace of the matrix.

$\lambda_1 \cdot \lambda_2 \cdots \lambda_n = |A|$.

Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Find the eigen values and eigenvectors of A

Solution:

The characteristic eqn of A is

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0.$

$\Rightarrow (8-\lambda)\left[(7-\lambda)(3-\lambda) - 16\right] + 6\left[-6(3-\lambda) + 8\right] + 2\left[24 - 2(7-\lambda)\right] = 0.$

$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0.$

$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0.$

$\Rightarrow \lambda = 0, 3, 15.$

Sum of the eigen values $= 0 + 3 + 15 = 18$

Sum of the diagonal elements $= 8 + 7 + 3 = 18.$

Case(i) $\lambda = 0.$

$Ax = \lambda x \Rightarrow Ax = 0.$

$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_1, \lambda_2 \ldots \lambda_n$

Using the method of cross-multiplication, Consider first two rows (independent)

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x}{24-14} = \frac{-y}{-32+12} = \frac{z}{56-36}$$

$$\frac{x}{10} = \frac{-y}{-20} = \frac{z}{20}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

Eigen vector corresponding to $\lambda = 0$ is $K(1, 2, 2)'$, where $K \neq 0$.

III^{ly} For $\lambda = 3$, $Ax = 3x$

$$\Rightarrow (A - 3I)(x) = 0$$

$$\Rightarrow \begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving, we get the eigen vector is $K(2, 1, -2)$.

Also for $\lambda = 15$, the eigen vector is $K(2, -2, 1)$.

Note :

For a given eigen value, there are infinite eigen vectors.

Problem : 2 (Repeated eigen values)

Find the eigen values and eigen vectors of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

Solution :

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0.$$

Solving, we get $\lambda = 1, 1, 5$ (eigen values)

$(\because 5+1+1 = 2+3+2)$.

For $\lambda = 1$, $(A - \lambda I)X = 0$

$$\Rightarrow (A - I)X = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} = 0.$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

The eigenvectors are,

$K(-2,1,0)$; $K(-1,0,1)$.

For $\lambda = 5$, $K(1,1,1)$. where $K \neq 0$.

Live Class : 17/3/2020. at 8.10 - 9.40.

Topic : Properties of Eigen values & The Largest Eigen vectors & Values.

Properties :-

1] eigen values are unique.

2] For a λ, eigen vectors are not unique.

3] A & A^T have the same eigenvalues & eigenvectors.

4] Eigen values of a diagonal matrix are diagonal entries.

5] Products of the diagonal elements = determinant of the matrix.

6] Sum of the eigenvalues = Trace of the matrix = Sum of the elements of the principal diagonal.

7] If λ is an eigen value of A, then $\frac{1}{\lambda}$ is the eigenvalue of A^{-1}.

8] If $\lambda_1, \lambda_2, \ldots, \lambda_m$ are eigen values of A, then $\lambda_1^n, \lambda_2^n, \ldots, \lambda_m^n$ are eigenvalues of A^n where $n \in I^+$.

9] If A is Singular matrix, then zero's are the eigen values.

Problems:

Find the eigen values and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Verify that trace equals to the sum of the eigenvalues and the determinant equals to their product. If we shift A to $A - 7I$, then what are the eigenvalues ~~are~~ of $A - 7I$, and how are they related to those of A.

Ans:-

The eigen values are given by

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda) + 2 = 0$$

$$4 - \lambda - 4\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\left[\lambda^2 - (\text{sum of the diagonal elements})\lambda + |A| = 0 \right]$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 2, \lambda = 3.$$

The eigenvalues are $\lambda = 2, \lambda = 3$.

The eigen vectors are given by

$$AX = \lambda X$$

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

When $\lambda = 2$,

$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$R_2 \to R_2 + 2R_1$

$$\begin{array}{cc} PC & FV \end{array}$$
$$\begin{bmatrix} \boxed{-1} & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x - y = 0 \Rightarrow x = -y$$

y is free variable. Let $y = 1$,

then $x = -1$.

$\therefore$ The eigen vector is $X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ or $K\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

When $\lambda = 3$,

$$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$-2x - y = 0$$

y is free variable. Let $y = 1$.

$$X_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \text{ or } K\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}.$$

$$\left.\begin{array}{l} \text{Trace} = 1+4 = 5 \\ \text{Sum of the eigenvalues} = 2+3 = 5 \end{array}\right\} \text{equal.}$$

$$\left.\begin{array}{l} \text{Product of the eigenvalues} = 2\times 3 = 6. \\ \text{Det. } |A| = \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = 4+2 = 6. \end{array}\right\} \text{equal.}$$

Construct $A - \lambda I$.

$$A - \lambda I = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix}$$

$\underline{\lambda = 7}$

$$A - 7I = \begin{pmatrix} 1-7 & -1 \\ 2 & 4-7 \end{pmatrix}$$

$$A - 7I = \begin{pmatrix} -6 & -1 \\ 2 & -3 \end{pmatrix} = B \text{ (say)}.$$

The eigen values are given by

$$|B - \lambda I| = 0.$$

$$\begin{vmatrix} -6-\lambda & -1 \\ 2 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - (-9)\lambda + 20 = 0$$

$$\lambda^2 + 9\lambda + 20 = 0$$

$$(\lambda+4)(\lambda+5) = 0$$

$$\lambda = -4, -5$$

The eigenvectors are given by

$(B-\lambda I)X = 0.$

$$\begin{pmatrix} -6-\lambda & -1 \\ 2 & -3-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

When $\lambda = -4$, $\begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow R_2 + R_1 = R_2$

$$\begin{pmatrix} -2 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-2x - y = 0$ Let $y = 1$

$-2x = 1 \Rightarrow x = -1/2$

$X_1 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ or $K\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}.$

When $\lambda = -5$, $\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-x - y = 0 \Rightarrow x = -y.$

Let $y = 1$, $x = -1.$

$\therefore X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ or $K\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

Eigen values of A are : 2, 3

Eigenvalues of $B = A - 7I$ are : $-4, -5$.

~~&~~ $-4 = 2 -$

Problems :-

Find the eigenvalues of the matrixes A, A^2, A^{-1} and $A + 4I$, given $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

Ans :-

The eigenvalues of A are

$\lambda = 1, \lambda = 3$.

The eigenvalues of A^2 are : $1^2, 3^2 = 1, 9$.

" " " A^{-1} : $\frac{1}{1}, \frac{1}{3}$

" " " $A + 4I$: $1+4, 3+4 = 5, 7$.

Problem :

Write the 3 different 2×2 matrices whose eigenvalues are 4 and 5.

Ans :- Given, eigenvalues are 4, 5.

$\therefore |A| = 20$.

1) $A = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$ (diagonal matrix)

2) $A = \begin{bmatrix} 4 & 10 \\ 0 & 5 \end{bmatrix}$ (upper Δ matrix)

3) $A = \begin{bmatrix} 4 & 0 \\ -1 & 5 \end{bmatrix}$ (lower Δ matrix).

$\times$

Rayleigh's Power Method

1] What 2] Why 3] Procedure. 4] Adv. & Disadv.
5] Applications.

Finding largest eigen value and corresponding eigen vector.

→ It is an iterative process. Writing an algorithms is easy!

Procedure :-

1] Given :- A, initial approximation of eigen vector.

2] $Ax = \lambda x$

3] Apply initial eigen vector as x_0.
$Ax_0 = \lambda_1 x_1$, $Ax_1 = \lambda_2 x_2$, ... $Ax_{n-1} = \lambda_n x_n$

4] Stop the process when two successive iterations are same.

Problem :-
Find the largest eigenvalue and eigenvector corresponding to the matrix $A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}$.

Ans :-
Since initial eigen vector is not given, assume it one of the form $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Let $x^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

First Iteration:-

$$AX^{(0)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix} = \lambda_1 x^{(1)}$$

Second Iteration:-

$$AX^{(1)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -4 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \lambda_2 x^{(2)}$$

Third Iteration:-

$$AX^{(2)} = \begin{bmatrix} 4 & +1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 5.2 \\ -5.2 \end{bmatrix} = 5.6 \begin{bmatrix} 1 \\ 0.9285 \\ -0.9285 \end{bmatrix}$$

$$= 5.6 \begin{bmatrix} 1 \\ 0.929 \\ -0.929 \end{bmatrix} = \lambda_3 x^{(3)}$$

Fourth Iteration:-

$$AX^{(3)} = \begin{bmatrix} 5.857 \\ 5.714 \\ -5.714 \end{bmatrix} = 5.857 \begin{bmatrix} 1 \\ 0.976 \\ -0.976 \end{bmatrix} = \lambda_4 X^{(4)}$$

Fifth Iteration:-

$$AX^{(4)} = \begin{bmatrix} 5.9512 \\ 5.9024 \\ +5.9024 \end{bmatrix} = 5.9512 \begin{bmatrix} 1 \\ 0.9908 \\ -0.9908 \end{bmatrix} = \lambda_5 X^{(5)}$$

Sixth iteration:-

$$AX^{(5)} = \begin{bmatrix} 5.9816 \\ 5.9632 \\ -5.9632 \end{bmatrix} = 5.9816 \begin{bmatrix} 1 \\ 0.9969 \\ -0.9969 \end{bmatrix} = \lambda_6 X^{(6)}$$

Seventh Iteration:-

$$AX_6 = \begin{pmatrix} 5.9938 \\ 5.9876 \\ -5.9876 \end{pmatrix} = 5.9938 \begin{pmatrix} 1 \\ 0.999 \\ -0.999 \end{pmatrix} = \lambda_7 X^{(7)}$$

Stop the process

The Largest eigen value $= 5.994 \cong 6.$

" " " vector $\approx \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$

* Google, twitter uses this method.

* Smallest eigen value of A = Largest eigen value of A^{-1}.

18/03/2020

Diagonalization

It is the process of finding an Invertible matrix S and diagonal matrix Λ, such that the given matrix A satisfies

$$A = S\Lambda S^{-1}$$

Examples :- (i) All diagonal matrix are diagonalizable Eg:- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(ii) All Symmetric Matrix are diagonalizable Eg:- $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Non Examples :- (i) Shear matrix $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ $(a \neq 0)$

$\left(k\begin{pmatrix}1\\0\end{pmatrix}\right.$ are Eigen vectors)

(ii) A Jordan block $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$

$\left(k\begin{pmatrix}1\\0\\0\end{pmatrix}\right.$ are Eigen vectors)

(iii) A rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\theta \neq 0, \pi$

(No Eigen Vectors)

$A = S\Lambda S^{-1}$ → Λ: Eigen value matrix.

S: Eigen matrix → Vector

Diagonalizability of A $\Leftrightarrow$ Existence of a basis of Eigen vectors

$A = S\Lambda S^{-1}$

S: Column form a basis of Eigen spaces

NOTE : If A is $n \times n$ matrix has n distinct Eigen values, then A must be diagonalizable.

The condition is not necessary for e.g. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\lambda = 1, 1, 2$

Problems:-

① Show that $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is diagonalizable.

Soln:- For this matrix $\lambda_1 = 0$, $\lambda_2 = 3$, $\lambda_3 = 15$ (Eigen values)

$V_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ $V_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ $V_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ (Eigen vectors)

Set $S = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$ $\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}$

Scaling the column of S, we get the orthogonal matrix P,

$A = P \Lambda P^T$.

$\Rightarrow$ A is diagonalizable.

② Show that $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is not diagonalizable.

Since it is an upper $\triangle$lar matrix $\lambda = 1, 1, 3$ are the Eigen values

$\lambda = 1$, $(A - \lambda I)X = 0$

$\Rightarrow \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Thus $(x_1, 0, 0)$ are the Eigen vectors

Thus there are not Enough ~~vectors~~ Eigen vectors (we need two independent Eigen vectors)

The number of independent Eigen vectors is no more than the multiplicity of the respective Eigen value.

$\Rightarrow$ It is not diagonilizable.

(3) Show that $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ is diagonalizable.

Since it is upper $\triangle$ matrix $\lambda = 1, 1, 3$ are the Eigen values.

$\lambda = 1$ $\quad (A - \underset{\substack{\downarrow \\ \lambda = 1}}{\lambda I})x = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} x = 0$ $\quad$ Eigen vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 3$ $\quad (A - 3I)x = 0 \Rightarrow \begin{pmatrix} -2 & 0 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} x = 0$ $\quad$ Eigen vector $\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$

$S = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$\therefore$ A is diagonalizable.

We can compute powers of a diagonalizable matrix A as follows,

$A = S\Lambda S^{-1}$ then $A^n = (S\Lambda S^{-1})^n$

$= S\Lambda \cancel{S^{-1}} \cancel{S}\Lambda \cancel{S^{-1}} \cancel{S}\Lambda \cancel{S^{-1}} \underbrace{\ldots\ldots S\Lambda S^{-1}}_{(n \text{ times})}$

$= S\Lambda^n S^{-1}$

Examples :-

(1) let $A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$, Find A^4.

Solution :- Here $\lambda = 2, 2, 4$ are the Eigen values and

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are the Eigen Vectors.

$$S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Matrix A Satisfy $A = S \Lambda S^{-1}$

$$\therefore A^4 = S \Lambda^4 S^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 256 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 136 & -120 & 120 \\ 0 & 16 & 0 \\ 120 & -120 & 136 \end{pmatrix}$$

If A and B are diagonalizable matrices Can we compute the product the same way?

Not unless A and B have the same Eigen vectors.

Cayley Hamilton theorem :-

Every Square matrix A, satisfies its own characteristic Equation.

Eg:- If A has characteristic polynomial

$$\lambda^2 - 3\lambda + 2 = 0$$

then $A^2 - 3A + 2I = [0] \longrightarrow$ zero matrix

Thus $A - 3A + 2A^{-1} = [0]$ (Multiply by A^{-1})

Then $A^{-1} = \dfrac{3I - A}{2}$

If $A = S\Lambda S^{-1}$, $B = S\Gamma S^{-1}$ then $AB = BA = S\Lambda\Gamma S^{-1}$

Theorem :- $AB = BA$ for diagonalizable A and B iff A & B have the same Eigen vectors.