Energy barrier

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Protocol

The protocol is as follows.

1. An element is selected for triggering (in the example below chosen in the center of the system). Its location is denoted by $\vec{r}'$.

2. A perturbation around a stress- and strain-free configuration is considered. To this end, the selected element (only) is subjected to an eigen stress $\boldsymbol{\sigma}'$. The corresponding equilibrium configuration then constitutes to the perturbation that will be used. It is characterised by the stress field $\delta\vec{u}(\vec{r})$, and corresponding stress $\delta\boldsymbol{\sigma}(\vec{r})$ and strain $\delta\boldsymbol{\varepsilon}(\vec{r})$ fields.

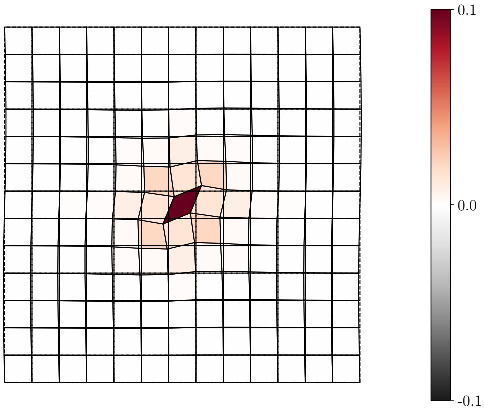
3. Two types of perturbations are considered:

   - Simple shear: $\boldsymbol{\sigma}' = \boldsymbol{\sigma}'_s = \vec{e}_x\vec{e}_y + \vec{e}_x\vec{e}_y$. Gives: $\delta\vec{u}_s(\vec{r})$, $\delta\boldsymbol{\sigma}_s(\vec{r})$, and $\delta\boldsymbol{\varepsilon}_s(\vec{r})$.
   - Pure shear: $\boldsymbol{\sigma}' = \boldsymbol{\sigma}'_p = \vec{e}_x\vec{e}_x - \vec{e}_y\vec{e}_y$. Gives: $\delta\vec{u}_p(\vec{r})$, $\delta\boldsymbol{\sigma}_p(\vec{r})$, and $\delta\boldsymbol{\varepsilon}_p(\vec{r})$.

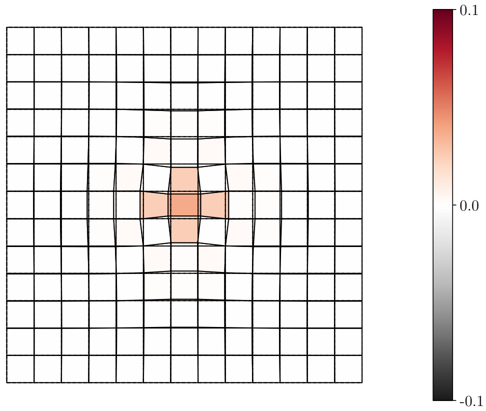
   For the triggered element the strain (and) stress are empirically of the following structure:

   - Simple shear perturbation: $\delta\boldsymbol{\varepsilon}_s(\vec{r}') = \delta\gamma(\vec{e}_x\vec{e}_y + \vec{e}_x\vec{e}_y)$
   - Pure shear perturbation: $\delta\boldsymbol{\varepsilon}_p(\vec{r}') = \delta\mathcal{E}(\vec{e}_x\vec{e}_x - \vec{e}_y\vec{e}_y)$

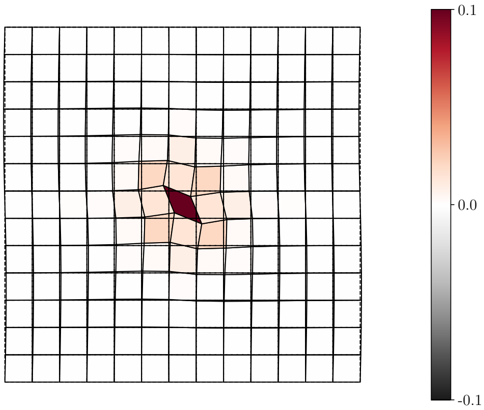
4. A perturbation $\Delta\vec{u}(\vec{r}) = s\delta\vec{u}_s(\vec{r}) + p\delta\vec{u}_p(\vec{r})$ is then applied such that the yield surface is reached in the triggered element in such a way that the change in potential energy introduced by the perturbation is minimal.



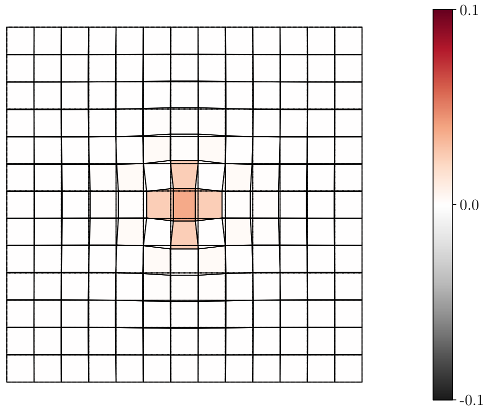
(a) Simple shear: $\delta\vec{u}_s(\vec{r})$



(b) Pure shear: $\delta\vec{u}_p(\vec{r})$



(c) Simple shear: $-\delta\vec{u}_s(\vec{r})$



(d) Pure shear: $-\delta\vec{u}_p(\vec{r})$

Figure 1. Perturbation modes. The shown colour is the energy change resulting from the perturbation.

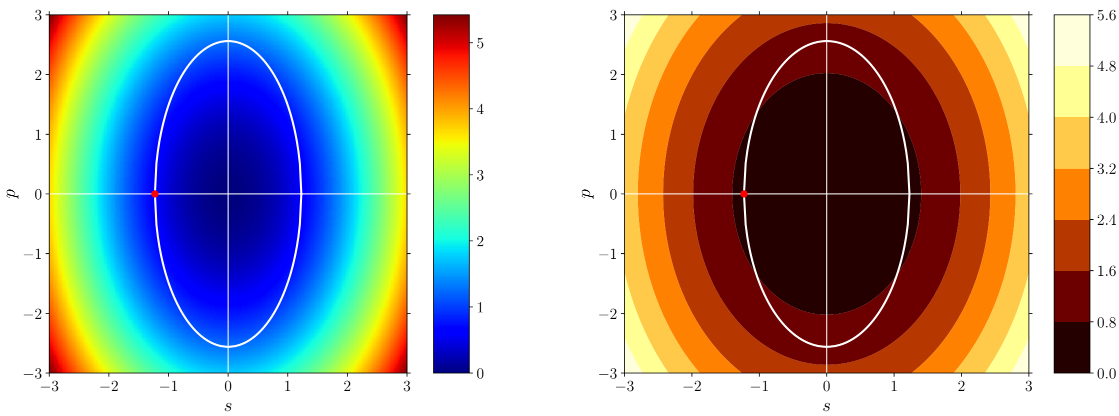
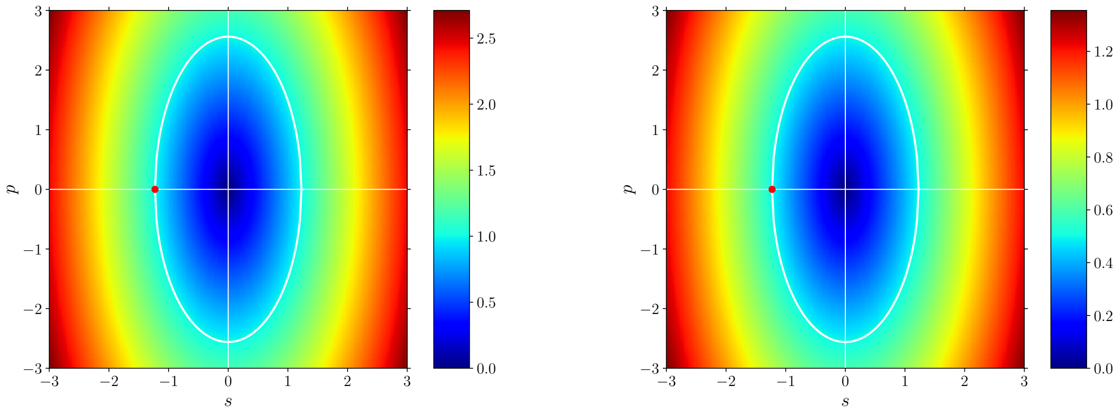


Figure 2. Change of internal energy, ΔE, for a perturbation: $\Delta\vec{u}(\vec{r}) = s\delta\vec{u}_s(\vec{r}) + p\delta\vec{u}_p(\vec{r})$. A contour plot is also shown.



(a) Equivalent stress.

(b) Equivalent strain.

Figure 3. Resulting (a) equivalent stress and (b) equivalent strain for a perturbation: $\Delta\vec{u}(\vec{r}) = s\delta\vec{u}_s(\vec{r}) + p\delta\vec{u}_p(\vec{r})$.

Exploring the yield surface

Yield surface
Initially the strain deviator in the triggered element reads

$$\boldsymbol{\varepsilon}_\mathrm{d}(\vec{r}') = \begin{bmatrix} \mathcal{E} & \gamma \\ \gamma & -\mathcal{E} \end{bmatrix} \tag{1}$$

After triggering the strain deviator is

$$\boldsymbol{\varepsilon}^*_\mathrm{d}(\vec{r}') = \begin{bmatrix} \mathcal{E} + p\delta\mathcal{E} & \gamma + s\delta\gamma \\ \gamma + s\delta\gamma & -\mathcal{E} - p\delta\mathcal{E} \end{bmatrix} \tag{2}$$

To reach the yield surface one thus needs to solve

$$(\mathcal{E} + p\delta\mathcal{E})^2 + (\gamma + s\delta\gamma)^2 = \varepsilon_y^2 \tag{3}$$

for (s, p) (with ε_y the relevant yield strain).

Change of energy
The energy in the system reads

$$E = \frac{1}{2} \int_\Omega \boldsymbol{\sigma}(\vec{r}) : \boldsymbol{\varepsilon}(\vec{r}) \; \mathrm{d}\Omega \tag{4}$$

After triggering:

$$E^* = \frac{1}{2} \int_\Omega (\boldsymbol{\sigma}(\vec{r}) + \Delta\boldsymbol{\sigma}(\vec{r})) : (\boldsymbol{\varepsilon}(\vec{r}) + \Delta\boldsymbol{\varepsilon}(\vec{r})) \; \mathrm{d}\Omega \tag{5}$$

where $\Delta\boldsymbol{\sigma}(\vec{r}) = s\delta\boldsymbol{\sigma}_s(\vec{r}) + p\delta\boldsymbol{\sigma}_p(\vec{r})$ and $\Delta\boldsymbol{\varepsilon}(\vec{r}) = s\delta\boldsymbol{\varepsilon}_s(\vec{r}) + p\delta\boldsymbol{\varepsilon}_p(\vec{r})$. It is straightforward to show that the change of energy

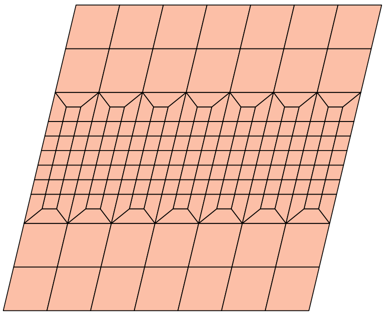
$$\Delta E = E^* - E = \int_\Omega \big(\boldsymbol{\sigma}(\vec{r}) + \tfrac{1}{2}\Delta\boldsymbol{\sigma}(\vec{r}) \big) : \Delta\boldsymbol{\varepsilon}(\vec{r}) \; \mathrm{d}\Omega \tag{6}$$

whereby in practice integration is performed numerically, e.g.

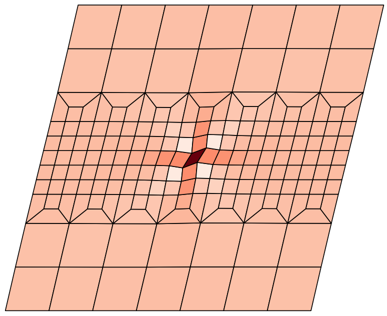
$$\Delta E = E^* - E = \sum_q \delta\Omega_q \; \big(\boldsymbol{\sigma}_q + \tfrac{1}{2}\Delta\boldsymbol{\sigma}_q \big) : \Delta\boldsymbol{\varepsilon}_q \tag{7}$$

Example

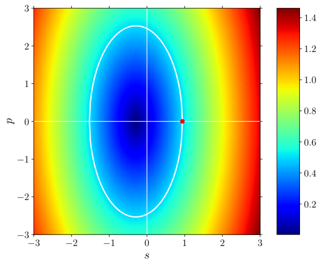
Two examples are included: A homogeneous medium that is subjected to shear in Fig. 4, and the same problem additionally subjected to a vertical perturbation of the top and bottom boundaries Fig. 5.



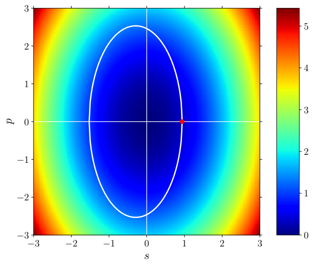
(a) Initial configuration.



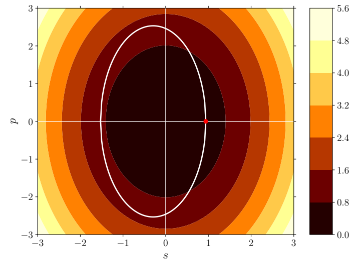
(b) After perturbation.



(c) ε

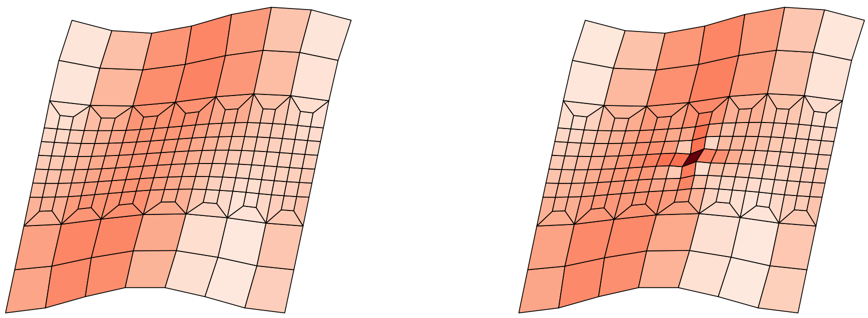


(d) ΔE

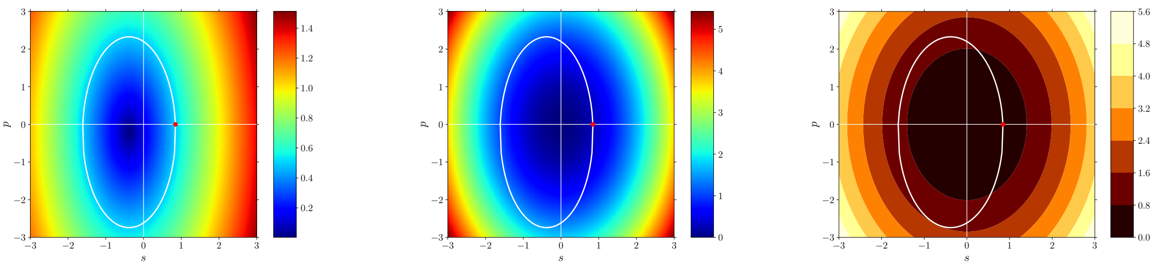


(e) ΔE

Figure 4. (a–b) Starting and perturbed configuration for a homogeneous sheared system. (c–e) Phase diagram of a perturbation $\Delta\vec{u}(\vec{r}) = s\delta\vec{u}_s(\vec{r}) + p\delta\vec{u}_p(\vec{r})$ of the configuration in (a). The perturbation is applied based on (s, p) that lie on the yield surface and that minimise the increase in potential energy, as shown using a red dot.



(a) Initial configuration. **(b)** After perturbation.



(c) ε **(d)** ΔE **(e)** ΔE

Figure 5. (a–b) Starting and perturbed configuration for a homogeneous sheared system. (c–e) Phase diagram of a perturbation $\Delta\vec{u}(\vec{r}) = s\delta\vec{u}_s(\vec{r}) + p\delta\vec{u}_p(\vec{r})$ of the configuration in (a). The perturbation is applied based on (s, p) that lie on the yield surface and that minimise the increase in potential energy, as shown using a red dot.