Non-linear elasticity

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Abstract

This constitutive model encompasses a non-linear, but history independent, relation between the Cauchy stress, $\boldsymbol{\sigma}$, and the linear strain tensor, $\boldsymbol{\varepsilon}$, i.e.:

$$\boldsymbol{\sigma} = f\left(\boldsymbol{\varepsilon}\right)$$

The model is implemented in 3-D, hence it can directly be used for either 3-D or 2-D plane strain problems.

1 Constitutive model

The following strain-energy is defined:

$$U(\boldsymbol{\varepsilon}) = \frac{9}{2} K \varepsilon_\mathrm{m} + \frac{\sigma_0 \, \varepsilon_0}{n+1} \left(\frac{\varepsilon_\mathrm{eq}}{\varepsilon_0} \right)^{n+1} \tag{1}$$

where K is the bulk modulus, ε_0 and σ_0 are a reference strain and stress respectively, and n is an exponent that sets the degree of non-linearity. Finally ε_m and ε_eq are the hydrostatic and equivalent strains (see Appendix B).

This leads to the following stress–strain relation:

$$\boldsymbol{\sigma} = \frac{\partial U}{\partial \boldsymbol{\varepsilon}} = 3K \varepsilon_\mathrm{m} \, \boldsymbol{I} + \frac{2}{3} \frac{\sigma_0}{\varepsilon_0^n} \, \varepsilon_\mathrm{eq}^{n-1} \, \boldsymbol{\varepsilon}_\mathrm{d} \tag{2}$$

see Appendix A for nomenclature.

2 Consistent tangent

The consistent tangent maps a variation in strain, $\delta\boldsymbol{\varepsilon}$, to a variation in stress, $\delta\boldsymbol{\sigma}$, as follows

$$\delta\boldsymbol{\sigma} = \mathbb{C} : \delta\boldsymbol{\varepsilon} \tag{3}$$

The tangent, $\mathbb{C}$, thus corresponds to the derivative of Eq. (2) w.r.t. strain. For this, the chain rule is employed:

$$\mathbb{C} = \frac{\partial}{\partial \boldsymbol{\varepsilon}} \left[\; 3K \varepsilon_\mathrm{m} \, \boldsymbol{I} \; \right] + \frac{\partial}{\partial \boldsymbol{\varepsilon}_\mathrm{d}} \left[\; \frac{2}{3} \frac{\sigma_0}{\varepsilon_0^n} \, \varepsilon_\mathrm{eq}^{n-1} \, \boldsymbol{\varepsilon}_\mathrm{d} \; \right] : \frac{\partial \boldsymbol{\varepsilon}_\mathrm{d}}{\partial \boldsymbol{\varepsilon}} \tag{4}$$

Where:

- the derivative of the volumetric part reads

$$\frac{\partial}{\partial \boldsymbol{\varepsilon}} \left[\; 3K \varepsilon_\mathrm{m} \, \boldsymbol{I} \; \right] = K \boldsymbol{I} \otimes \boldsymbol{I} \tag{5}$$

- the chain rule for the deviatoric part reads

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{\varepsilon}_\mathrm{d}} \left[\; \varepsilon_\mathrm{eq}^{n-1} \, \boldsymbol{\varepsilon}_\mathrm{d} \; \right] &= \frac{\partial \left[\varepsilon_\mathrm{eq}^{n-1} \right]}{\partial \boldsymbol{\varepsilon}_\mathrm{d}} \otimes \boldsymbol{\varepsilon}_\mathrm{d} + \varepsilon_\mathrm{eq}^{n-1} \frac{\partial \boldsymbol{\varepsilon}_\mathrm{d}}{\partial \boldsymbol{\varepsilon}_\mathrm{d}} && (6) \\ &= \tfrac{2}{3} (n-1) \, \varepsilon_\mathrm{eq}^{n-3} \, \boldsymbol{\varepsilon}_\mathrm{d} \otimes \boldsymbol{\varepsilon}_\mathrm{d} + \varepsilon_\mathrm{eq}^{n-1} \, \mathbb{I} && (7) \end{aligned}$$

- and it has been used that

$$\frac{\partial}{\partial \boldsymbol{\varepsilon}_{\mathrm{d}}}\left[\,\varepsilon_{\mathrm{eq}}^{n-1}\,\right] = (n-1)\,\varepsilon_{\mathrm{eq}}^{n-2}\,\frac{2}{3}\frac{\boldsymbol{\varepsilon}_{\mathrm{d}}}{\varepsilon_{\mathrm{eq}}} \tag{8}$$

$$= \tfrac{2}{3}(n-1)\,\varepsilon_{\mathrm{eq}}^{n-3}\,\boldsymbol{\varepsilon}_{\mathrm{d}} \tag{9}$$

Combining the above yields:

$$\mathbb{C} = K\boldsymbol{I}\otimes\boldsymbol{I} + \frac{2}{3}\frac{\sigma_0}{\varepsilon_0^n}\left(\tfrac{2}{3}(n-1)\,\varepsilon_{\mathrm{eq}}^{n-3}\boldsymbol{\varepsilon}_{\mathrm{d}}\otimes\boldsymbol{\varepsilon}_{\mathrm{d}} + \varepsilon_{\mathrm{eq}}^{n-1}\mathbb{I}\right) : \mathbb{I}_{\mathrm{d}} \tag{10}$$

$$= K\boldsymbol{I}\otimes\boldsymbol{I} + \frac{2}{3}\frac{\sigma_0}{\varepsilon_0^n}\left(\tfrac{2}{3}(n-1)\,\varepsilon_{\mathrm{eq}}^{n-3}\boldsymbol{\varepsilon}_{\mathrm{d}}\otimes\boldsymbol{\varepsilon}_{\mathrm{d}} + \varepsilon_{\mathrm{eq}}^{n-1}\,\mathbb{I}_{\mathrm{d}}\right) \tag{11}$$

3 Consistency check

To check if the derived tangent $\mathbb{C}$ a *consistency check* can be performed. A (random) perturbation $\delta\boldsymbol{\varepsilon}$ is applied. The residual is compared to that predicted by the tangent. For the general case of linearisation, the following holds:

$$\boldsymbol{\sigma}\big(\boldsymbol{\varepsilon}_\star + \delta\boldsymbol{\varepsilon}\big) = \boldsymbol{\sigma}\big(\boldsymbol{\varepsilon}_\star\big) + \mathbb{C}\big(\boldsymbol{\varepsilon}_\star\big) : \delta\boldsymbol{\varepsilon} + \mathcal{O}(\delta\boldsymbol{\varepsilon}^2) \tag{12}$$

or

$$\underbrace{\boldsymbol{\sigma}\big(\boldsymbol{\varepsilon}_\star + \delta\boldsymbol{\varepsilon}\big) - \boldsymbol{\sigma}\big(\boldsymbol{\varepsilon}_\star\big)}_{\delta\boldsymbol{\sigma}} - \mathbb{C}\big(\boldsymbol{\varepsilon}_\star\big) : \delta\boldsymbol{\varepsilon} = \mathcal{O}(\delta\boldsymbol{\varepsilon}^2) \tag{13}$$

This allows the introduction of a relative error

$$\eta = \Big|\Big|\delta\boldsymbol{\sigma} - \mathbb{C}(\boldsymbol{\varepsilon}_\star) : \delta\boldsymbol{\varepsilon}\Big|\Big| / \Big|\Big|\delta\boldsymbol{\sigma}\Big|\Big| \tag{14}$$

This *truncation error* thus scales as $\eta \sim ||\delta\boldsymbol{\varepsilon}||^2$ as depicted in Figure 1. As soon as the error becomes sufficiently small the numerical *rounding error* becomes more dominant, the scaling thereof is also included in Figure 1.

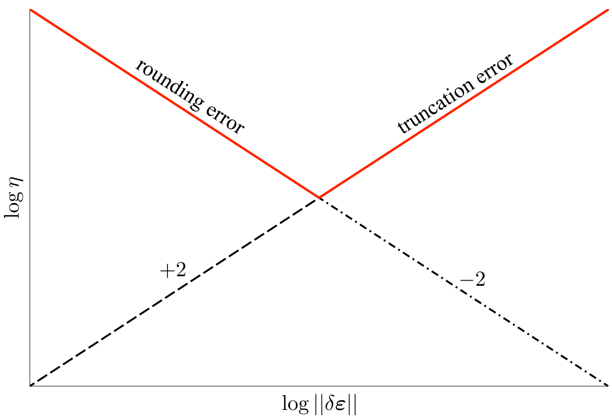

Figure 1. Expected behaviour of the consistency check, see Heath [1, p. 9].

The measurement of η and a function of $||\delta\boldsymbol{\varepsilon}||$, as depicted in Fig. 2, indeed matches the prediction in Fig. 1.

References

[1] M.T. Heath. *Scientific computing.* 2002.

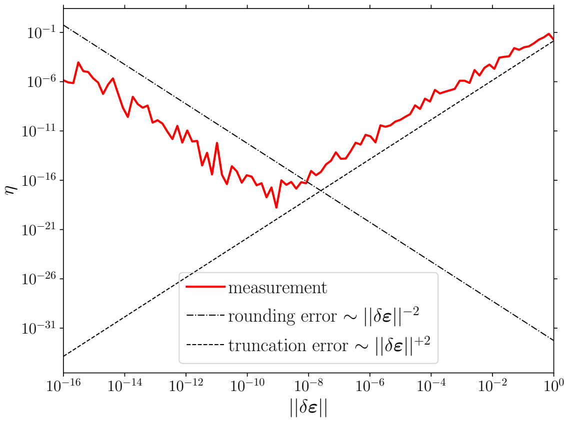

Figure 2. Measured consistency check, cf. Fig. 1.

A Nomenclature

Tensor products

- Dyadic tensor product

$$\mathbb{C} = \boldsymbol{A} \otimes \boldsymbol{B} \tag{15}$$
$$C_{ijkl} = A_{ij}\, B_{kl} \tag{16}$$

- Double tensor contraction

$$C = \boldsymbol{A} : \boldsymbol{B} \tag{17}$$
$$= A_{ij}\, B_{ji} \tag{18}$$

Tensor decomposition

- Deviatoric part $\boldsymbol{A}_{\mathrm{d}}$ of an arbitrary tensor $\boldsymbol{A}$:

$$\mathrm{tr}\,(\boldsymbol{A}_{\mathrm{d}}) \equiv 0 \tag{19}$$

and thus

$$\boldsymbol{A}_{\mathrm{d}} = \boldsymbol{A} - \tfrac{1}{3}\mathrm{tr}\,(\boldsymbol{A}) \tag{20}$$

Fourth order unit tensors

- Unit tensor:

$$\boldsymbol{A} \equiv \mathbb{I} : \boldsymbol{A} \tag{21}$$

and thus

$$\mathbb{I} = \delta_{il}\delta jk \tag{22}$$

- Right-transposition tensor:

$$\boldsymbol{A}^{T} \equiv \mathbb{I}^{RT} : \boldsymbol{A} = \boldsymbol{A} : \mathbb{I}^{RT} \tag{23}$$

and thus

$$\mathbb{I}^{RT} = \delta_{ik}\delta_{jl} \tag{24}$$

- Symmetrisation tensor:

$$\mathrm{sym}\,(\boldsymbol{A}) \equiv \mathbb{I}_\mathrm{s} : \boldsymbol{A} \tag{25}$$

whereby

$$\mathbb{I}_\mathrm{s} = \tfrac{1}{2}\left(\mathbb{I} + \mathbb{I}^{RT}\right) \tag{26}$$

This follows from the following derivation:

$$\begin{aligned}
\mathrm{sym}\,(\boldsymbol{A}) &= \tfrac{1}{2}\left(\boldsymbol{A} + \boldsymbol{A}^T\right) && (27)\\
&= \tfrac{1}{2}\left(\mathbb{I} : \boldsymbol{A} + \mathbb{I}^{RT} : \boldsymbol{A}\right) && (28)\\
&= \tfrac{1}{2}\left(\mathbb{I} + \mathbb{I}^{RT}\right) : \boldsymbol{A} && (29)\\
&= \mathbb{I}_\mathrm{s} : \boldsymbol{A} && (30)
\end{aligned}$$

- Deviatoric and symmetric projection tensor

$$\mathrm{dev}\,(\mathrm{sym}\,(\boldsymbol{A})) \equiv \mathbb{I}_\mathrm{d} : \boldsymbol{A} \tag{31}$$

from which it follows that:

$$\mathbb{I}_\mathrm{d} = \mathbb{I}_\mathrm{s} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I} \tag{32}$$

B Strain measures

- Mean strain

$$\varepsilon_\mathrm{m} = \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\varepsilon}) = \tfrac{1}{3}\,\boldsymbol{\varepsilon} : \boldsymbol{I} \tag{33}$$

- Strain deviator

$$\boldsymbol{\varepsilon}_\mathrm{d} = \boldsymbol{\varepsilon} - \varepsilon_\mathrm{m}\,\boldsymbol{I} = \mathbb{I}_\mathrm{d} : \boldsymbol{\varepsilon} \tag{34}$$

- Equivalent strain

$$\varepsilon_\mathrm{eq} = \sqrt{\tfrac{2}{3}\,\boldsymbol{\varepsilon}_\mathrm{d} : \boldsymbol{\varepsilon}_\mathrm{d}} \tag{35}$$

C Variations

- Strain deviator

$$\delta\boldsymbol{\varepsilon}_\mathrm{d} = \left(\mathbb{I}_\mathrm{s} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I}\right) : \delta\boldsymbol{\varepsilon} = \mathbb{I}_\mathrm{d} : \delta\boldsymbol{\varepsilon} \tag{36}$$

- Mean equivalent strain

$$\delta\varepsilon_\mathrm{m} = \tfrac{1}{3}\boldsymbol{I} : \delta\boldsymbol{\varepsilon} \tag{37}$$

- Von Mises equivalent strain

$$\begin{aligned}
\delta\varepsilon_\mathrm{eq} &= \frac{1}{3}\frac{1}{\varepsilon_\mathrm{eq}}\left(\boldsymbol{\varepsilon}_\mathrm{d} : \delta\boldsymbol{\varepsilon}_\mathrm{d} + \delta\boldsymbol{\varepsilon}_\mathrm{d} : \boldsymbol{\varepsilon}_\mathrm{d}\right) && (38)\\
&= \frac{2}{3}\frac{1}{\varepsilon_\mathrm{eq}}\left(\boldsymbol{\varepsilon}_\mathrm{d} : \delta\boldsymbol{\varepsilon}_\mathrm{d}\right) && (39)\\
&= \frac{2}{3}\frac{\boldsymbol{\varepsilon}_\mathrm{d}}{\varepsilon_\mathrm{eq}} : \delta\boldsymbol{\varepsilon}_\mathrm{d} && (40)
\end{aligned}$$