Code journal: Numerical Derivative Visualization

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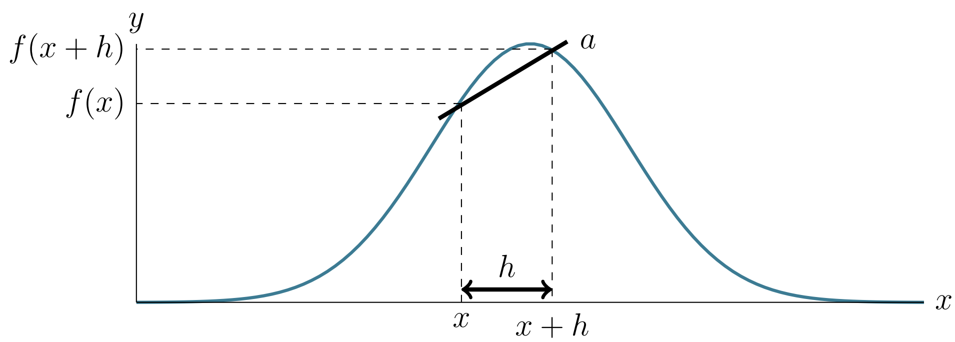
The Derivative of A Function

The derivative of a function $f(x)$, usually called $f'(x)$, is defined as the following limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

Where the term *limit as h approaches 0* means **the value h is getting closest to as h becomes arbitrarily small**, which in this case is 0, even if the expression in question is unsolvable exactly as h = 0. There exists also a geometric definition that states that **a derivative $f'(x)$ is the value of the slope of the tangent line at x**, and this can be represented in a Cartesian plane given an example: say the function $f(x)$ is a curve plotted below, and we grab two arbitrary points in the x axis: x and $x+h$, where h is the **increment** from x to arrive at $x+h$. Now, at these two points, we can evaluate the function $f(x)$ and get, for x a value of $f(x)$, and for $x+h$ a value of $f(x+h)$ in the y axis, which is what we see below.



Line a is a secant line since it crosses the function $f(x)$ twice, and we can calculate the slope of line a as:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{h} \tag{2}$$

In this way, the ratio $\frac{f(x+h)-f(x)}{h}$ where h is an arbitrary increment, would represent the slope of a secant line to $f(x)$. However, if we allow h to get very small and close to 0, the line a becomes a **tangent line of $f(x)$**, and the ratio $\frac{f(x+h)-f(x)}{h}$ where h is very small becomes the **slope of a tangent line**, or the derivative. In other words, the expression in Eq. (1) simply says that when h approaches a value very close to 0, the function $\frac{f(x+h)-f(x)}{h}$ is the **slope of the tangent line of $f(x)$ at point x**. The ratio $\frac{f(x+h)-f(x)}{h}$, **if h was infinitesimal or very small** (for a larger h would mean the slope of a *secant line*), would be the same as calculating a the slope of a line (*tangent* to $f(x)$), commonly known as $\frac{y_2-y_1}{x_2-x_1}$ and expressed below in Eq. (2). Thus, if we take the limit of the ratio $\frac{f(x+h)-f(x)}{h}$ when h is similar to 0, it computes **the slope of the tangent line at every point x of $f(x)$**, known as **the derivative**.

Now, in order to visualize this, we need this ratio in Eq. (2) and a value of h that is considerably small, so that we can evaluate this expression at every value of x and get the slopes of tangent lines of $f(x)$ at every value of x. If we plot all these slopes of the different x values of a definite domain as points $(x, \frac{f(x+h)-f(x)}{h})$, what we are actually plotting is **the derivative of function** $f(x)$. The example function for this plot will be a $sin(x)$ function, with the value of h being $\mathrm{h} = 0.05$, and the domain being $x \in \mathbb{R} | 0 \leq x \leq 6\pi$. In practice, the smallest h is set, the more exact the derivative plot will be, in this case, if h was instead set to $\mathrm{h} = 0.00001$, the plot (black crosses) would get closer to the exact derivative function $f'(x) = cos(x)$.

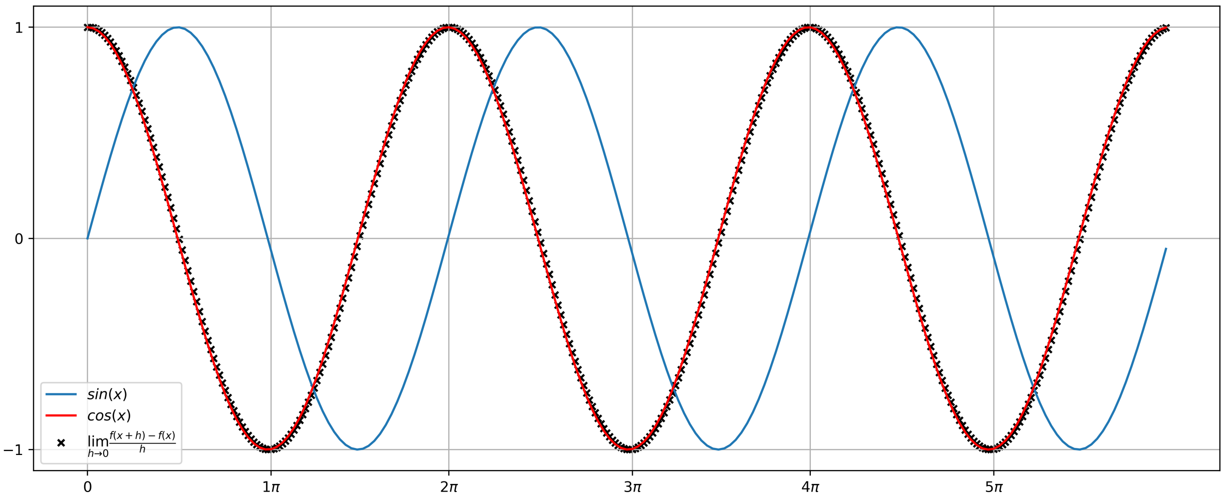


Figure 1: Numerical Derivative Plot (Black), Exact Derivative Plot (Red) of function $f(x)$ (Blue)

Noticeably, what we are plotting are *the values of the slopes at every point x of the domain*, and thus, at the x points where the minimum and maximum of $sin(x)$ are, the derivative or slope is 0, and thus we see the black crosses with height 0 at the corresponding x points of these critical points of $f(x)$. The code for generating such plot is below.

```
# import sin and cos from math, matplotlib.pyplot and numpy
fig = plt.figure(figsize=(12, 5)); ax = fig.add_subplot(1,1,1)
h = 0.05
PI = 3.1416
cycles = 3
xs = [i for i in list(np.arange(0,PI*2 * cycles, 0.1))]
ys = [sin(x) for x in xs]; test = [cos(x) for x in xs]
dy = []; dx = []
for i in list(np.arange(0, PI*2 * cycles, h)):
    x = i - h
    x_plus_h = i
    lim = (sin(x_plus_h) - sin(x)) / h
    dy.append(lim)
    dx.append(i)
x_ticks = [0.0] + [x for x in xs if (x/PI) % 1 < 0.03 and (x/PI) > 1]
ax.plot(xs, ys, label=r"$sin(x)$")
ax.scatter(dx, dy, marker='x', color='black', s=20, label=r"$\lim_{h\to0} \frac
                                        {f(x+h) - f(x)}{h}$")
ax.plot(xs, test, label=r"$cos(x)$", color="red")
ax.set_xticks(x_ticks)
ax.set_yticks([-1, 0, 1])
ax.set_xticklabels(['0'] + [f"{int(t / PI)}" + r"$\pi$" for t in x_ticks[1:]])
```