

# Code journal: Numerical Derivative Visualization

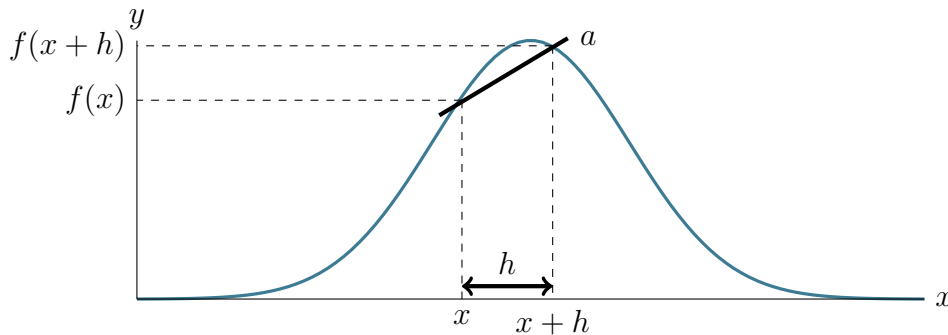
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## The Derivative of A Function

The derivative of a function  $f(x)$ , usually called  $f'(x)$ , is defined as the following limit:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (1)$$

Where the term *limit as  $h$  approaches 0* means  **$h$  becomes arbitrarily small or very close to 0**, even if the expression in question is unsolvable exactly as  $h = 0$ . There exists also a geometric definition that states that **a derivative  $f'(x)$  is the value of the slope of the tangent line at  $x$** , and this can be represented in a Cartesian plane given an example: say the function  $f(x)$  is a curve plotted below, and we grab two arbitrary points in the  $x$  axis:  $x$  and  $x+h$ , where  $h$  is the **increment** from  $x$  to arrive at  $x+h$ . Now, at these two points, we can evaluate the function  $f(x)$  and get, for  $x$  a value of  $f(x)$ , and for  $x+h$  a value of  $f(x+h)$  in the  $y$  axis, which is what we see below.



Line  $a$  is a secant line since it crosses the function  $f(x)$  twice, and we can calculate the slope of line  $a$  as:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{x+h - x} = \frac{f(x+h) - f(x)}{h} \quad (2)$$

In this way, the ratio  $\frac{f(x+h)-f(x)}{h}$  where  $h$  is an arbitrary increment, would represent the slope of a secant line to  $f(x)$ . However, if we allow  $h$  to get very small and close to 0, the line  $a$  becomes a **tangent line of  $f(x)$** , and the ratio  $\frac{f(x+h)-f(x)}{h}$  where  $h$  is very small becomes the **slope of a tangent line**, or the derivative. In other words, the expression in Eq. (1) simply says that when  $h$  approaches a value very close to 0, the function  $\frac{f(x+h)-f(x)}{h}$  is the **slope of the tangent line of  $f(x)$  at point  $x$** . The ratio  $\frac{f(x+h)-f(x)}{h}$ , if  **$h$  was infinitesimal or very small** (for a larger  $h$  would mean the slope of a *secant line*), would be the same as calculating a the slope of a line (*tangent* to  $f(x)$ ), commonly known as  $\frac{y_2-y_1}{x_2-x_1}$  and expressed below in Eq. (2). Thus, if we take the limit of the ratio  $\frac{f(x+h)-f(x)}{h}$  when  $h$  is similar to 0, it computes **the slope of the tangent line at every point  $x$  of  $f(x)$** , known as **the derivative**.

Now, in order to visualize this, we need this ratio in Eq. (2) and a value of  $h$  that is considerably small, so that we can evaluate this expression at every value of  $x$  and get the slopes of tangent

lines of  $f(x)$  at every value of  $x$ . If we plot all these slopes of the different  $x$  values of a definite domain as points  $(x, \frac{f(x+h)-f(x)}{h})$ , what we are actually plotting is **the derivative of function  $f(x)$** . The example function for this plot will be a  $\sin(x)$  function, with the value of  $h$  being  $h = 0.05$ , and the domain being  $x \in \mathbb{R} | 0 \leq x \leq 6\pi$ . In practice, the smallest  $h$  is set, the more exact the derivative plot will be, in this case, if  $h$  was instead set to  $h = 0.00001$ , the plot (black crosses) would get closer to the exact derivative function  $f'(x) = \cos(x)$ .

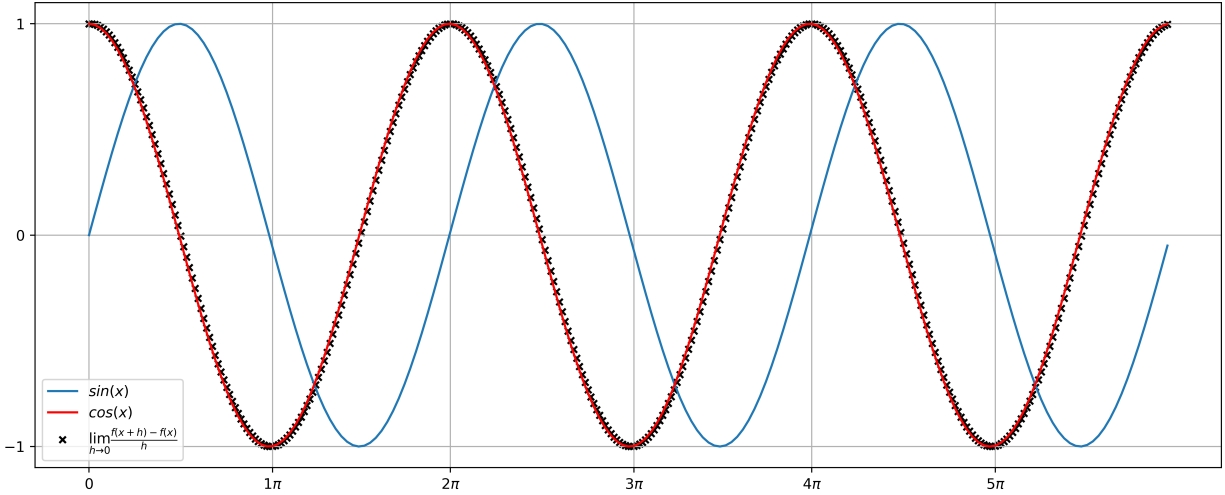


Figure 1: Numerical Derivative Plot (Black), Exact Derivative Plot (Red) of function  $f(x)$  (Blue)

Noticeably, what we are plotting are *the values of the slopes at every point  $x$  of the domain*, and thus, at the  $x$  points where the minimum and maximum of  $\sin(x)$  are, the derivative or slope is 0, and thus we see the black crosses with height 0 at the corresponding  $x$  points of these critical points of  $f(x)$ . The code for generating such plot is below.

```
# import sin and cos from math, matplotlib.pyplot and numpy
fig = plt.figure(figsize=(12, 5)); ax = fig.add_subplot(1,1,1)
h = 0.05
PI = 3.1416
cycles = 3
xs = [i for i in list(np.arange(0,PI*2 * cycles, 0.1))]
ys = [sin(x) for x in xs]; test = [cos(x) for x in xs]
dy = []; dx = []
for i in list(np.arange(0, PI*2 * cycles, h)):
    x = i - h
    x_plus_h = i
    lim = (sin(x_plus_h) - sin(x)) / h
    dy.append(lim)
    dx.append(i)
x_ticks = [0.0] + [x for x in xs if (x/PI) % 1 < 0.03 and (x/PI) > 1]
ax.plot(xs, ys, label=r"$sin(x)$")
ax.scatter(dx, dy, marker='x', color='black', s=20, label=r"$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$")
ax.plot(xs, test, label=r"$cos(x)$", color="red")
ax.set_xticks(x_ticks)
ax.set_yticks([-1, 0, 1])
ax.set_xticklabels(['0'] + [f"{int(t / PI)}" + r"$\pi$" for t in x_ticks[1:]])
```