Numerical differentiation using imaginary number

1 Imaginary numbers

Imaginary number $$x_{Real} + h_{Imaginary}$$	
Algebra $$h^n = ?$$	

Complex number $$x_{Real} + hi$$	
Basic algebra $$i^2 = -1$$ $$(x+hi)^2 = x^2 - h^2 + 2xhi$$	

Dual number $$x_{Real} + h\varepsilon$$	
Basic algebra $$\varepsilon^2 = 0 \quad \varepsilon^n = \varepsilon^{n-2} \cdot \varepsilon^2 = 0$$ $$(x+h\varepsilon)^2 = x^2 + 2xh\varepsilon$$ $$(h\varepsilon)^2 = h^2 \cdot 0 = 0$$	

Hyper-Dual number (Another imaginary number ε_2 is newly introduced) $$x_{Real} + h_{Imaginary} = x_{Real} + h\varepsilon_1 + h\varepsilon_2$$ $h\varepsilon_1 + h\varepsilon_2$: Imaginary number part	
Basic algebra $$\varepsilon_1^2 = 0 \quad \varepsilon_2^2 = 0 \quad \varepsilon_1\varepsilon_2 \neq 0$$	
$$(x + h\varepsilon_1 + h\varepsilon_2)^2 = x^2 + (h\varepsilon_1)^2 + (h\varepsilon_2)^2 + 2[x \cdot h\varepsilon_1 + x \cdot h\varepsilon_2 + h\varepsilon_1 \cdot h\varepsilon_2]$$	
$$\therefore (x + h\varepsilon_1 + h\varepsilon_2)^2 = x^2 + 2[x \cdot h\varepsilon_1 + x \cdot h\varepsilon_2 + h\varepsilon_1 \cdot h\varepsilon_2]$$	

2 Single-variated numerical differentiation

2.1 Taylor series

Definition
$$F(x+\Delta x) = F(x) + \frac{1}{1!}\frac{dF(x)}{dx}\Delta x + \frac{1}{2!}\frac{d^2F(x)}{dx^2}(\Delta x)^2 + \cdots$$
In numerical analysis,
$$F(x+\Delta x) = F(x) + \frac{1}{1!}\frac{dF(x)}{dx}\Delta x + \frac{1}{2!}\frac{d^2F(x)}{dx^2}(\Delta x)^2 + O(h^3)$$
Truncation (numerical) error
$$O(h^3 = (\Delta x)^3) = \frac{1}{3!}\frac{d^3F(x)}{dx^3}(\Delta x)^3 + \frac{1}{4!}\frac{d^4F(x)}{dx^4}(\Delta x)^4 + \cdots$$
→ Which means the order of magnitude of the error is about $(\Delta x)^3$.
Example
$$F(x+\Delta x) = F(x) + \frac{1}{1!}\frac{dF(x)}{dx}\Delta x + O(h^2)$$
$$F(x+\Delta x) = F(x) + O(h)$$

2.2 Real number

Two different taylor series
$$F(x+\Delta x) = F(x) + \frac{dF(x)}{dx}\Delta x + O(h^2)$$
$$F(x-\Delta x) = F(x) + \frac{dF(x)}{dx}(-\Delta x) + O(h^2)$$
1st order derivative (Forward difference)
$$\frac{dF(x)}{dx} = \frac{F(x+\Delta x) - F(x)}{\Delta x} - \frac{O(h^2)}{\Delta x} \approx \frac{F(x+\Delta x) - F(x)}{\Delta x}$$
1st order derivative (Mid-point rule: Central difference)
$$\begin{aligned} &F(x+\Delta x) - F(x-\Delta x) \\ &\quad = [F(x) - F(x)] + \left[\frac{dF(x)}{dx}\Delta x + \frac{dF(x)}{dx}\Delta x\right] + \frac{1}{2}\left[\frac{dF(x)}{dx}(\Delta x)^2 + \frac{dF(x)}{dx}(\Delta x)^2\right] + O(h^3) \end{aligned}$$
$$\frac{dF(x)}{dx} = \frac{F(x+\Delta x) - F(x-\Delta x)}{2\Delta x} - \frac{O(h^3)}{\Delta x} \approx \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

- Numerical differentiation using real number is also known as “Finite difference method”.
- This is why the numberical (Truncation) error is accommodated in the numerical differentiation

2.3 Dual number

Taylor series w.r.t. Dual number
$$F(x+h\varepsilon) = F(x) + \frac{dF(x)}{dx}h\varepsilon + \frac{1}{2!}\frac{d^2F(x)}{dx^2}(h\varepsilon)^2 + \cdots$$
$$F(x+h\varepsilon) = F(x) + \frac{dF(x)}{dx}h\varepsilon$$

- Taylor series expanded in the basis of Dual number can be obtained by simply replacing Δx with hε.
- Then, Taylor series results in the function value with taking the Dual number as an argument.

1st order derivative
$\frac{dF(x)}{dx}(\mathrm{Img}) = \frac{F(x+h\varepsilon)-F(x)}{h} = 0 + \frac{dF(x)}{dx}(h\varepsilon) \neq \text{Real number}$
$\frac{dF(x)}{dx}(\mathrm{Real}) = \mathrm{IMG}_{\varepsilon}\left(\frac{F(x+h\varepsilon)-F(x)}{h\varepsilon}\right)$

- In virtue of the Dual number characteristic ($\varepsilon^2 = 0$), the numerical error completely vanishes.
- The differentiation value is acquired in a form of the imaginary number. Thereby, we need an additional operation: $\mathrm{IMG}_{\varepsilon}(x + h\varepsilon) = h$.
- BUT, higher order of derivative is not obtainable; only 1st order derivative is available because higher-order terms disappear ($\varepsilon^{n\geq 2} = 0$).
- This is why we need "Hyper-Dual number". (This is introduced by Dr. Fike in his thesis.)

2.4 Hyper-Dual number

Taylor series w.r.t. Hyper-dual number
$$F(x+h\varepsilon_1+h\varepsilon_2)=F(x)+\frac{dF(x)}{dx}(h\varepsilon_1+h\varepsilon_2)+\frac{1}{2!}\frac{d^2F(x)}{dx^2}(h\varepsilon_1+h\varepsilon_2)^2$$
$$\therefore\ F(x+h\varepsilon_1+h\varepsilon_2)=F(x)+\frac{dF(x)}{dx}h\varepsilon_1+\frac{dF(x)}{dx}h\varepsilon_2+\frac{d^2F(x)}{dx^2}(h)^2\varepsilon_1\varepsilon_2$$

- "Hyper"-dual number stands for introducing a secondary imaginary number ($h\varepsilon_2$) to the dual number.
- For expansion, replace Δx in normal Taylor series with $h\varepsilon_1+h\varepsilon_2$: $\Delta x\rightarrow(h\varepsilon_1+h\varepsilon_2)$
- Resultantly, we can expand the Taylor series up to the second-order derivative, which means we might add third or more imaginary number for the higher-order expansion. (This is not necessary for plasticity😊)
- The first order of derivative is the same as the way of Dual number simply defining ($h\varepsilon_2=0$).

2nd order derivative
$$F(x+h\varepsilon_1+h\varepsilon_2)=F(x)+\frac{dF(x)}{dx}h\varepsilon_1+\frac{dF(x)}{dx}h\varepsilon_2+\frac{d^2F(x)}{dx^2}(h)^2\varepsilon_1\varepsilon_2$$
$$F(x-h\varepsilon_1-h\varepsilon_2)=F(x)-\frac{dF(x)}{dx}h\varepsilon_1-\frac{dF(x)}{dx}h\varepsilon_2+\frac{d^2F(x)}{dx^2}(h)^2\varepsilon_1\varepsilon_2$$
$$\frac{d^2F(x)}{dx^2}(Img)=\frac{F(x+h\varepsilon_1+h\varepsilon_2)+F(x-h\varepsilon_1-h\varepsilon_2)-2F(x)}{(h)^2\varepsilon_1\varepsilon_2}$$
$$\frac{d^2F(x)}{dx^2}(Real)=IMG_{\varepsilon_1\varepsilon_2}\left(\frac{F(x+h\varepsilon_1+h\varepsilon_2)+F(x-h\varepsilon_1-h\varepsilon_2)-2F(x)}{(h)^2\varepsilon_1\varepsilon_2}\right)$$

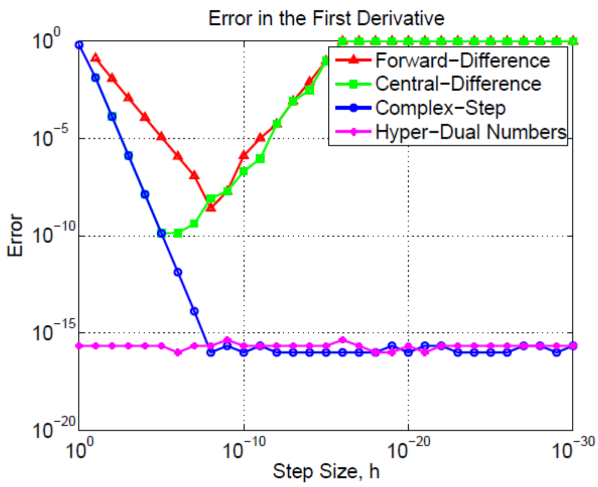

Figure 1.1: The accuracy of several first-derivative calculation methods, presented as a function of step size for the function $f(x)=\frac{e^x}{\sqrt{\sin^3 x+\cos^3 x}}$.

3 Multi-variated numerical differentiation

3.1 Multi-variated Taylor seires

Single variation

$$f(x_1+\Delta x_1, x_2) = f(x_1, x_2) + \Delta x_1 \frac{\partial f(x_1, x_2)}{\partial x_1} + \frac{\Delta x_1^2}{2}\frac{\partial^2 f(x_1, x_2)}{\partial x_1\,\partial x_1} + O(h^3)$$

$$f(x_1-\Delta x_1, x_2) = f(x_1, x_2) - \Delta x_1 \frac{\partial f(x_1, x_2)}{\partial x_1} + \frac{\Delta x_1^2}{2}\frac{\partial^2 f(x_1, x_2)}{\partial x_1\,\partial x_1} + O(h^3)$$

Multiple variation

$$\begin{aligned} f(x_1+\Delta x_1, x_2+\Delta x_2) &= f(x_1, x_2) + \Delta x_1 \frac{\partial f(x_1, x_2)}{\partial x_1} + \Delta x_2 \frac{\partial f(x_1, x_2)}{\partial x_2} \\ &+ \frac{1}{2}\left\{\Delta x_1^2 \frac{\partial^2 f(x_1, x_2)}{\partial x_1\,\partial x_1} + \Delta x_2^2 \frac{\partial^2 f(x_1, x_2)}{\partial x_2\,\partial x_2} + 2\Delta x_1 \Delta x_2 \frac{\partial^2 f(x_1, x_2)}{\partial x_1\,\partial x_2}\right\} + O(h^3) \end{aligned}$$

$$\begin{aligned} f(x_1-\Delta x_1, x_2-\Delta x_1) &= f(x_1, x_2) - \Delta x_1 \frac{\partial f(x_1, x_2)}{\partial x_1} - \Delta x_2 \frac{\partial f(x_1, x_2)}{\partial x_2} \\ &+ \frac{1}{2}\left\{\Delta x_1^2 \frac{\partial^2 f(x_1, x_2)}{\partial x_1\,\partial x_1} + \Delta x_2^2 \frac{\partial^2 f(x_1, x_2)}{\partial x_2\,\partial x_2} + 2\Delta x_1 \Delta x_2 \frac{\partial^2 f(x_1, x_2)}{\partial x_1\,\partial x_2}\right\} + O(h^3) \end{aligned}$$

$$\begin{aligned} f(x_1+\Delta x_1, x_2-\Delta x_2) &= f(x_1, x_2) + \Delta x_1 \frac{\partial f(x_1, x_2)}{\partial x_1} - \Delta x_2 \frac{\partial f(x_1, x_2)}{\partial x_2} \\ &+ \frac{1}{2}\left\{\Delta x_1^2 \frac{\partial^2 f(x_1, x_2)}{\partial x_1\,\partial x_1} + \Delta x_2^2 \frac{\partial^2 f(x_1, x_2)}{\partial x_2\,\partial x_2} - 2\Delta x_1 \Delta x_2 \frac{\partial^2 f(x_1, x_2)}{\partial x_1\,\partial x_2}\right\} + O(h^3) \end{aligned}$$

$$\begin{aligned} f(x_1-\Delta x_1, x_2+\Delta x_2) &= f(x_1, x_2) - \Delta x_1 \frac{\partial f(x_1, x_2)}{\partial x_1} + \Delta x_2 \frac{\partial f(x_1, x_2)}{\partial x_2} \\ &+ \frac{1}{2}\left\{\Delta x_1^2 \frac{\partial^2 f(x_1, x_2)}{\partial x_1\,\partial x_1} + \Delta x_2^2 \frac{\partial^2 f(x_1, x_2)}{\partial x_2\,\partial x_2} - 2\Delta x_1 \Delta x_2 \frac{\partial^2 f(x_1, x_2)}{\partial x_1\,\partial x_2}\right\} + O(h^3) \end{aligned}$$

1st order differentiation

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{f(x_1+\Delta x_1, x_2) - f(x_1-\Delta x_1, x_2)}{2\Delta x_1}$$

2nd order differentiation

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1\,\partial x_1} = \frac{f(x_1+\Delta x_1, x_2) + f(x_1-\Delta x_1, x_2) - 2f(x_1, x_2)}{\Delta x_1^2}$$

$$\begin{aligned} \frac{\partial^2 f(x_1, x_2)}{\partial x_1\,\partial x_2} &= \frac{f(x_1+\Delta x_1, x_2+\Delta x_2) + f(x_1-\Delta x_1, x_2-\Delta x_1) - 2f(x_1, x_2)}{2\Delta x_1 \Delta x_2} \\ &- \frac{1}{2\Delta x_1 \Delta x_2}\left\{\Delta x_1^2 \frac{\partial^2 f(x_1, x_2)}{\partial x_1\,\partial x_1} + \Delta x_2^2 \frac{\partial^2 f(x_1, x_2)}{\partial x_2\,\partial x_2}\right\} \end{aligned}$$

$$\begin{aligned} &\frac{\partial^2 f(x_1, x_2)}{\partial x_1\,\partial x_2} \\ &= \frac{f(x_1+\Delta x_1, x_2+\Delta x_2) + f(x_1-\Delta x_1, x_2-\Delta x_1) - f(x_1+\Delta x_1, x_2-\Delta x_2) - f(x_1-\Delta x_1, x_2+\Delta x_1)}{\Delta x_1 \Delta x_2} \end{aligned}$$

3.2 Real number

Single variation
$$f(x_1+h,x_2)=f(x_1,x_2)+h\frac{\partial f(x_1,x_2)}{\partial x_1}+\frac{h^2}{2}\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1}+O(h^3)$$
$$f(x_1-h,x_2)=f(x_1,x_2)-h\frac{\partial f(x_1,x_2)}{\partial x_1}+\frac{h^2}{2}\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1}+O(h^3)$$
Multiple variation
$$\begin{aligned}&f(x_1+h,x_2+h)\\&\quad=f(x_1,x_2)+h\left\{\frac{\partial f(x_1,x_2)}{\partial x_1}+\frac{\partial f(x_1,x_2)}{\partial x_2}\right\}\\&\quad+\frac{h^2}{2}\left\{\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1}+\frac{\partial^2 f(x_1,x_2)}{\partial x_2\,\partial x_2}+2\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_2}\right\}+O(h^3)\end{aligned}$$
$$\begin{aligned}&f(x_1-h,x_2-h)\\&\quad=f(x_1,x_2)-h\left\{\frac{\partial f(x_1,x_2)}{\partial x_1}+\frac{\partial f(x_1,x_2)}{\partial x_2}\right\}\\&\quad+\frac{h^2}{2}\left\{\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1}+\frac{\partial^2 f(x_1,x_2)}{\partial x_2\,\partial x_2}+2\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_2}\right\}+O(h^3)\end{aligned}$$
1st order differentiation (Best: $h=10^{-6}$)
$$\frac{\partial f(x_1,x_2)}{\partial x_1}=\frac{f(x_1+h,x_2)-f(x_1-h,x_2)}{2h}$$
2nd order differentiation (Best: $h=10^{-3}$)
$$\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1}=\frac{f(x_1+h,x_2)+f(x_1-h,x_2)-2f(x_1,x_2)}{h^2}$$
$$\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_2}=\frac{f(x_1+h,x_2+h)+f(x_1-h,x_2-h)-2f(x_1,x_2)}{2h^2}-\frac{1}{2}\left\{\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1}+\frac{\partial^2 f(x_1,x_2)}{\partial x_2\,\partial x_2}\right\}$$

3.3 Complex number

Single variation
$$f(x_1+hi,x_2)=f(x_1,x_2)+hi\frac{\partial f(x_1,x_2)}{\partial x_1}-\frac{h^2}{2}\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1}+O((hi)^3)$$
$$f(x_1-hi,x_2)=f(x_1,x_2)-hi\frac{\partial f(x_1,x_2)}{\partial x_1}-\frac{h^2}{2}\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1}+O((hi)^3)$$
Multiple variation
$$\begin{aligned}&f(x_1+hi,x_2+hi)\\&=f(x_1,x_2)+hi\left\{\frac{\partial f(x_1,x_2)}{\partial x_1}+\frac{\partial f(x_1,x_2)}{\partial x_2}\right\}\\&-\frac{h^2}{2}\left\{\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1}+\frac{\partial^2 f(x_1,x_2)}{\partial x_2\,\partial x_2}+2\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_2}\right\}+O(h^3)\end{aligned}$$
$$\begin{aligned}&f(x_1-hi,x_2-hi)\\&=f(x_1,x_2)-hi\left\{\frac{\partial f(x_1,x_2)}{\partial x_1}+\frac{\partial f(x_1,x_2)}{\partial x_2}\right\}\\&-\frac{h^2}{2}\left\{\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1}+\frac{\partial^2 f(x_1,x_2)}{\partial x_2\,\partial x_2}+2\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_2}\right\}+O(h^3)\end{aligned}$$
1st order differentiation
$$\frac{\partial f(x_1,x_2)}{\partial x_1}=IMG\left\{\frac{f(x_1+h,x_2)-f(x_1-h,x_2)}{2hi}\right\}$$
2nd order differentiation
$$\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1}=REAL\left\{\frac{f(x_1+h,x_2)+f(x_1-h,x_2)-2f(x_1,x_2)}{-h^2}\right\}$$
$$\begin{aligned}\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_2}&=REAL\left\{\frac{f(x_1+hi,x_2+hi)+f(x_1-hi,x_2-hi)-2f(x_1,x_2)}{-h^2}\right\}\\&-\frac{1}{2}\left\{\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1}+\frac{\partial^2 f(x_1,x_2)}{\partial x_2\,\partial x_2}\right\}\end{aligned}$$

3.4 Generalized complex number

Generalized complex number
$\Delta x_\theta = h(\cos\theta + i\sin\theta)$
Single variation
$f(x_1+\Delta x_\theta, x_2) = f(x_1,x_2) + \Delta x_\theta \frac{\partial f(x_1,x_2)}{\partial x_1} + \frac{(\Delta x_\theta)^2}{2}\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1} + O((hi)^3)$
$f(x_1-\Delta x_\theta, x_2) = f(x_1,x_2) - \Delta x_\theta \frac{\partial f(x_1,x_2)}{\partial x_1} + \frac{(\Delta x_\theta)^2}{2}\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1} + O((hi)^3)$
Multiple variation
$f(x_1+\Delta x_\theta, x_2+\Delta x_\theta) = f(x_1,x_2) + \Delta x_\theta\left\{\frac{\partial f(x_1,x_2)}{\partial x_1} + \frac{\partial f(x_1,x_2)}{\partial x_2}\right\} + \frac{(\Delta x_\theta)^2}{2}\left\{\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1} + \frac{\partial^2 f(x_1,x_2)}{\partial x_2\,\partial x_2} + 2\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_2}\right\} + O(h^3)$
$f(x_1-\Delta x_\theta, x_2-\Delta x_\theta) = f(x_1,x_2) - \Delta x_\theta\left\{\frac{\partial f(x_1,x_2)}{\partial x_1} + \frac{\partial f(x_1,x_2)}{\partial x_2}\right\} + \frac{(\Delta x_\theta)^2}{2}\left\{\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1} + \frac{\partial^2 f(x_1,x_2)}{\partial x_2\,\partial x_2} + 2\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_2}\right\} + O(h^3)$
1st order differentiation
$\frac{\partial f(x_1,x_2)}{\partial x_1} = \mathrm{IMG}\left\{\frac{f(x_1+\Delta x_\theta, x_2) - f(x_1-\Delta x_\theta, x_2)}{2\Delta x_\theta}\right\}$
2nd order differentiation
$\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1} = \mathrm{REAL}\left\{\frac{f(x_1+\Delta x_\theta, x_2) + f(x_1-\Delta x_\theta, x_2) - 2f(x_1,x_2)}{(\Delta x_\theta)^2}\right\}$
$\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_2} = \mathrm{REAL}\left\{\frac{f(x_1+\Delta x_\theta, x_2+\Delta x_\theta) + f(x_1-\Delta x_\theta, x_2-\Delta x_\theta) - 2f(x_1,x_2)}{2(\Delta x_\theta)^2}\right\} - \frac{1}{2}\left\{\frac{\partial^2 f(x_1,x_2)}{\partial x_1\,\partial x_1} + \frac{\partial^2 f(x_1,x_2)}{\partial x_2\,\partial x_2}\right\}$

3.5 Hyper-dual number

Single variation (Dual number)
$f(x_1 + h\varepsilon, x_2) = f(x_1, x_2) + h\varepsilon \frac{\partial f(x_1, x_2)}{\partial x_1}$
$f(x_1 - h\varepsilon, x_2) = f(x_1, x_2) - h\varepsilon \frac{\partial f(x_1, x_2)}{\partial x_1}$
Single variation (Hyper-dual number)
$f(x_1 + h\varepsilon_1 + h\varepsilon_2, x_2) = f(x_1, x_2) + (h\varepsilon_1 + h\varepsilon_2) \frac{\partial f(x_1, x_2)}{\partial x_1} + h^2 \varepsilon_1 \varepsilon_2 \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \, \partial x_1}$
$f(x_1 - h\varepsilon_1 - h\varepsilon_2, x_2) = f(x_1, x_2) - (h\varepsilon_1 + h\varepsilon_2) \frac{\partial f(x_1, x_2)}{\partial x_1} + h^2 \varepsilon_1 \varepsilon_2 \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \, \partial x_1}$
Multiple variation
$f(x_1 + h\varepsilon_1 + h\varepsilon_2, x_2 + h\varepsilon_1 + h\varepsilon_2) = f(x_1, x_2) + (h\varepsilon_1 + h\varepsilon_2) \left\{ \frac{\partial f(x_1, x_2)}{\partial x_1} + \frac{\partial f(x_1, x_2)}{\partial x_1} \right\}$ $+ h^2 \varepsilon_1 \varepsilon_2 \left\{ \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \, \partial x_1} + \frac{\partial^2 f(x_1, x_2)}{\partial x_2 \, \partial x_2} + 2 \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \, \partial x_2} \right\}$
$f\big(x_1 - (h\varepsilon_1 + h\varepsilon_2), x_2 - (h\varepsilon_1 + h\varepsilon_2)\big) = f(x_1, x_2) - (h\varepsilon_1 + h\varepsilon_2) \left\{ \frac{\partial f(x_1, x_2)}{\partial x_1} + \frac{\partial f(x_1, x_2)}{\partial x_1} \right\}$ $+ h^2 \varepsilon_1 \varepsilon_2 \left\{ \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \, \partial x_1} + \frac{\partial^2 f(x_1, x_2)}{\partial x_2 \, \partial x_2} + 2 \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \, \partial x_2} \right\}$
1st order differentiation (Dual number)
$\frac{\partial f(x_1, x_2)}{\partial x_1} = \mathrm{IMG}_{\varepsilon} \left\{ \frac{f(x_1 + h\varepsilon, x_2) - f(x_1 - h\varepsilon, x_2)}{2h\varepsilon} \right\}$
2nd order differentiation (Hyper-dual number)
$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \, \partial x_1} = \mathrm{IMG}_{\varepsilon_1 \varepsilon_2} \left\{ \frac{f(x_1 + h\varepsilon_1 + h\varepsilon_2, x_2) + f(x_1 - h\varepsilon_1 - h\varepsilon_2, x_2) - 2f(x_1, x_2)}{2h^2 \varepsilon_1 \varepsilon_2} \right\}$
$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \, \partial x_2} = \mathrm{IMG}_{\varepsilon_1 \varepsilon_2} \left\{ \frac{f(x_1 + h\boldsymbol{\varepsilon}, x_2 + h\boldsymbol{\varepsilon}) + f(x_1 - h\boldsymbol{\varepsilon}, x_2 - h\boldsymbol{\varepsilon}) - 2f(x_1, x_2)}{4h^2 \varepsilon_1 \varepsilon_2} \right\}$ $- \frac{1}{2} \left\{ \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \, \partial x_1} + \frac{\partial^2 f(x_1, x_2)}{\partial x_2 \, \partial x_2} \right\}$