# Plane Fitting using PCA

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Given a set of 3D points $\{p_i\}$, we want to find out a plane (i.e., unit normal $n$ and center $q$) that describes this set of points as

$$n^\intercal(p_i - q) = 0, \forall i \tag{1}$$

Of course real world data points usually won't strictly satisfy equation (1). Thus we define function

$$\mathrm{dist}(p_i; n, q) \triangleq n^\intercal(p_i - q) \tag{2}$$

which represents the *signed* point-plane distance between the plane $(n, q)$ and data point $p_i$ (recall that $n$ is a unit normal vector).

Now the problem of plane fitting can be formulated as a least square problem with the following cost function

$$\begin{aligned} \mathrm{cost}(n, q) &\triangleq \sum_i \mathrm{dist}^2(p_i; n, q) \\ &= \sum_i (n^\intercal(p_i - q))^2 \\ &= n^\intercal[\cdots, p_i - q, \cdots][\cdots, p_i^\intercal - q^\intercal, \cdots]^\intercal n \\ &= n^\intercal A(q) A(q)^\intercal n \end{aligned} \tag{3}$$

where $A(q) \triangleq [\cdots, p_i - q, \cdots]$.

If $n$ is fixed to some value, to minimize the cost function (3), we need to zero the following partial derivative (check matrix calculus)

$$\mathbf{0} = \frac{\partial \mathrm{cost}(n, q)}{\partial q} \equiv \sum_i (2nn^\intercal q - 2nn^\intercal p_i). \tag{4}$$

Solving equation (4) leads to the optimal plane center $q^*$ as (no matter what $n$ is)

$$q^* = \frac{1}{|\{p_i\}|} \sum_i p_i. \tag{5}$$

Now to solve the optimal plane normal $n$, we can plug equation (5) back to the cost function (3) as

$$\mathrm{cost}(n; q^*) \triangleq n^\intercal A(q^*) A(q^*)^\intercal n = n^\intercal B(q^*) n \tag{6}$$

where $q^*$ becomes a fixed parameter of this new cost function, and $B(q^*) \triangleq A(q^*)A(q^*)^\intercal$. Recall that $A(q)$ is a $3 \times |\{p_i\}|$ matrix, thus $B(q)$ should be a $3 \times 3$ *positive-definite* matrix. Now the optimal plane normal $n^*$ needs to be solved by

$$\begin{aligned} n^* = \arg\min_{n} \quad & n^\intercal B(q^*) n \\ \text{s.t.} \quad & n^\intercal n = 1 \end{aligned} \tag{7}$$

which reminds us a classic problem that can be solved by the singular value decomposition $A(q) \equiv U(q)S(q)V(q)^\intercal$, leading to (omitting $(q)$ in $B$ and other relevant matrices for clarity)

$$\begin{aligned} n^\intercal B n =& n^\intercal U S V^\intercal V S^\intercal U^\intercal n \\ =& n^\intercal U S S^\intercal U^\intercal n. \end{aligned} \tag{8}$$

Since $S = [\mathrm{diag}(\sigma_1, \sigma_2, \sigma_3), \mathbf{0}, \cdots, \mathbf{0}]$.

If we give each data point a corresponding weight $w_i$, then the distance equation $(2)$ can be changed to a weighted version as

$$\mathrm{wdist}(p_i, w_i; n, q) \triangleq w_i n^\intercal (p_i - q) \tag{9}$$

TO BE CONTINUED...

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