FAIR CAUSAL INFERENCE FOR FUNCTIONAL DATA

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Who Am I?

- Tim
- PhD Candidate in Economics, University of Bonn
- Focus:
  - Econometrics & Statistics
  - Causal Inference
  - Programming

Motivation

- Foot striking patterns:
  - forefoot vs heel

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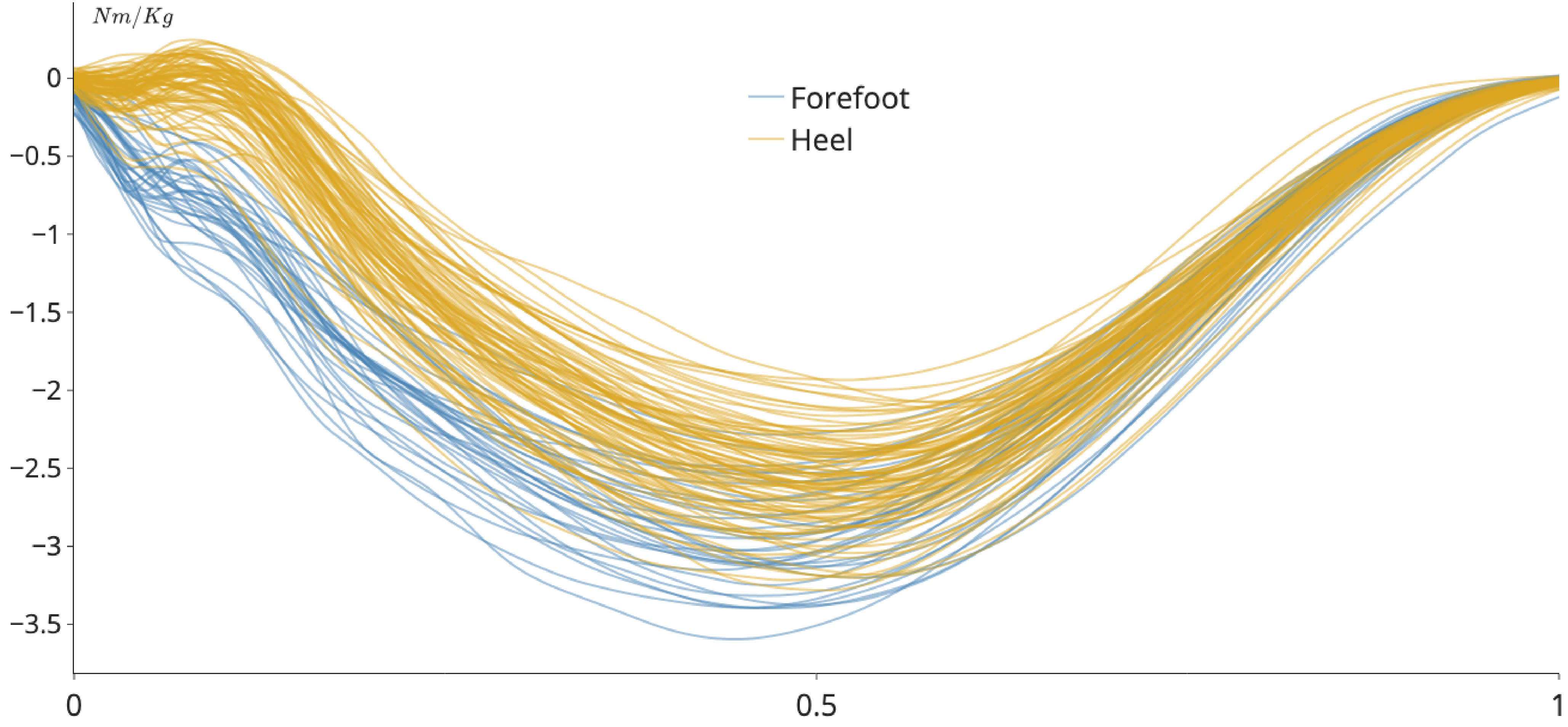
  - forefoot vs heel
- Consider one metric: Force on ankle joints
- ***What's the (causal) effect of forefoot running on ankle joint loading?***

Data



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Outcomes	Controls	Treatment
$Y_i \in C^1[0,1]$	$X_i \in \mathbb{R}^p$	$W_i \in \{0,1\}$

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  - SUTVA: $Y_i = Y_i(W_i)$

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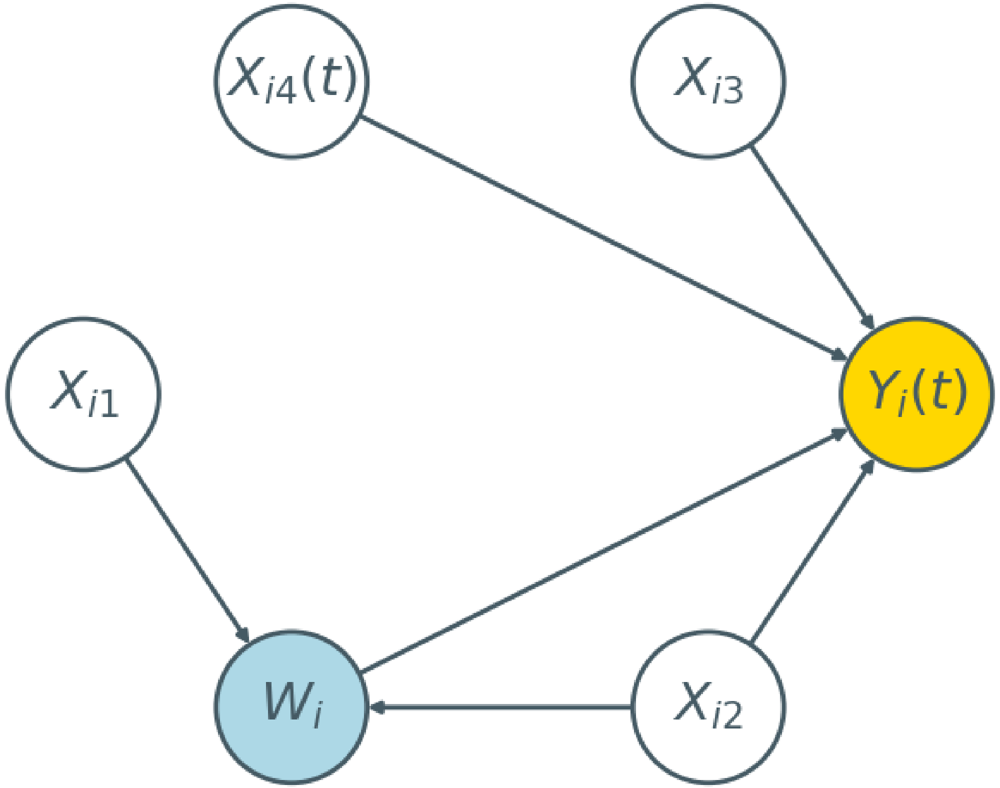
  - $\eta < \mathbb{P}[W_i = 1 | X_i] < 1 - \eta$

Plan

1. **Find relevant control variables**
   - Utilize causal graphs from *causal inference* literature
2. **Choose a suitable estimator**
   - Utilize methods from *econometrics* literature
3. **Construct confidence bands**
   - Utilize results from *functional data* literature

Find relevant control variables

Directed Acyclical Graph



- For $t \in [0, 1]$
- Structure may change with t
- Set of variables used for prediction of outcome and treatment may differ

Choose a suitable estimator

Augmented Inverse Propensity Score Weighting

$$\hat{\tau}(t) = \frac{1}{n}\sum_{i=1}^{n} \hat{\mathbb{E}}[Y_i(t)|X_i, W_i = 1] - \hat{\mathbb{E}}[Y_i(t)|X_i, W_i = 0]$$

$$+ \frac{1}{n}\sum_{i=1}^{n} W_i \frac{Y_i(t) - \hat{\mathbb{E}}[Y_i(t)|X_i, W_i = 1]}{\hat{\mathbb{P}}[W_i = 1|X_i]} - (1 - W_i)\frac{Y_i(t) - \hat{\mathbb{E}}[Y_i(t)|X_i, W_i = 0]}{1 - \hat{\mathbb{P}}[W_i = 1|X_i]}$$

- (Non-)parametric estimators of nuisance functions:
  - $\hat{\mathbb{E}}[Y_i(t)|X_i, W_i = w]$
  - $\hat{\mathbb{P}}[W_i = 1|X_i]$

Properties and Requirements

Properties of $\hat{\tau}(t)$:

- Consistent for $\tau(t)$
- Doubly robust
- Semiparametric efficient

Requirements:

- Cross-fitting
- Nuisance functions are estimated at $o_P(n^{-1/4})$ rates

Construct Confidence Bands

Simultaneous Confidence Bands

- **To Show:**
  - Asymptotically Gaussian estimator of τ
  - Uniformly consistent estimator of its covariance kernel c (and its 1st and 2nd partial derivatives)
- Liebl and Reimherr (2022):
  - **Get:** Simultaneous and fair confidence bands
  - *Fairness:* Control balance of false-positive rate over $[0, 1]$

Theorem

Under *regularity conditions* on the continuity and differentiability of functions and distributions of the functional errors

$$\sqrt{n}(\hat{\tau} - \tau) \xrightarrow{d} \mathcal{GP}(0, c).$$

And, we can construct an estimator of c and its partial derivatives that is uniformly consistent.

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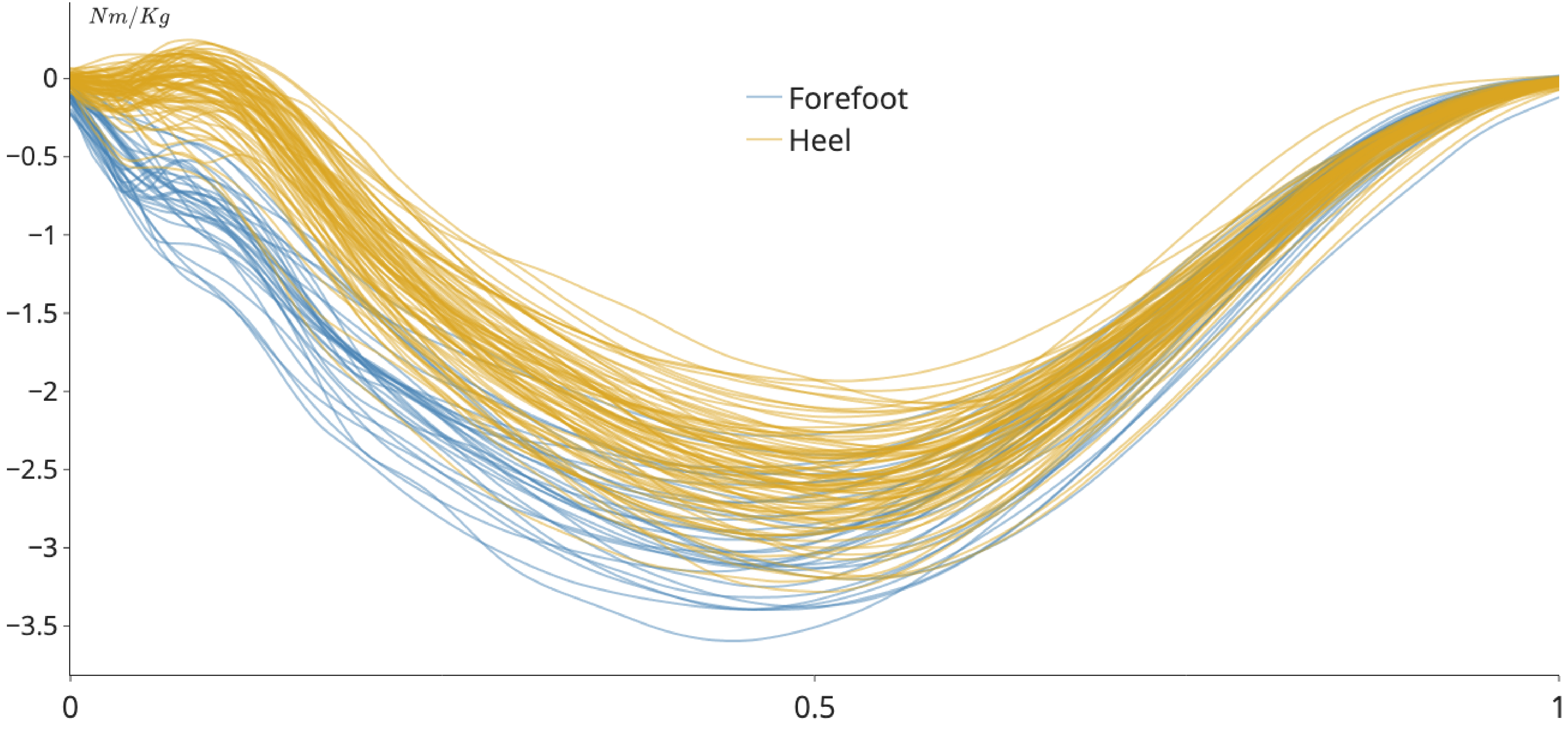
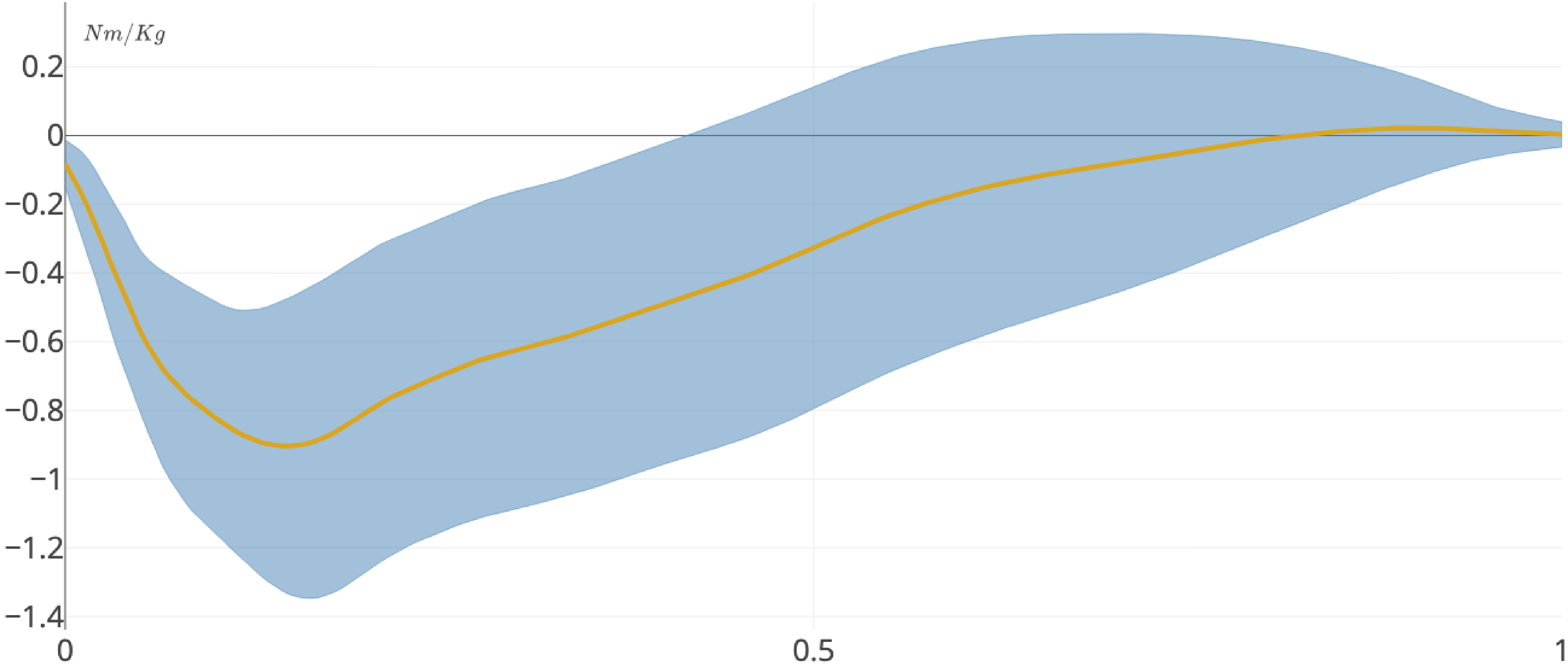
- Construct sample analogue estimator $\hat{c}$ of c
- Show that $\hat{c}$ and its derivatives converge uniformly

Application



Result



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