$x \vee \neg x$

$\neg\neg x \quad \neg x$

$\forall n > 0$

Claim: $\neg\left(\forall n \in \mathbb{Z}^+ \left(n^2+1 \geq 2^n\right)\right)$

Proof: $\equiv \exists n \in \mathbb{Z}^+ \left(n^2+1 < 2^n\right)$

counterexample:

$n = 5$ $\square$

let $n<0$ arbitrary

We know $2^n < 1$

We know $n^2 > 0$

so $n^2 + 1 \geq 1$

$\therefore 2^n \leq n^2 + 1$

QED

Claim: $\forall n \left(2 \mid n \rightarrow 2 \mid n^2 \right)$

Proof: let $n \in \mathbb{R}$ arbitrary

Suppose $2 \mid n$

By def. $n = 2 \cdot k$ where $k \in \mathbb{Z}$

Thus $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$

Since $\mathbb{Z}$ is closed under mult, $2k^2 \in \mathbb{Z}$

and $2 \mid n^2$ $\square$

Claim: $\forall n \in \mathbb{Z}\ (2 \mid n^2 \rightarrow 2 \mid n)$

Proof: ~~Let $n \in \mathbb{Z}$ arbitrary~~

~~$\exists k \in \mathbb{Z}$ s.t. $n^2 = 2k$~~ BAD IDEA

Instead: prove contrapositive

$\forall n \in \mathbb{Z}\ (n \text{ odd} \rightarrow n^2 \text{ odd})$

(See prev.) $\square$

$\forall n,a,b \in \mathbb{Z} \, (n \nmid ab \rightarrow n \nmid a \wedge n \nmid b)$

let $n,a,b \in \mathbb{Z}$ arbitrary

Contra: $(n \mid a \vee n \mid b \rightarrow n \mid ab)$

Case 1: $n \mid a$

$\exists k \in \mathbb{Z}$ st $n = ka$

$\forall n, a, b \in \mathbb{Z} \, (n \nmid ab \rightarrow n \nmid a \wedge n \nmid b)$

Let $n, a, b \in \mathbb{Z}$ arbitrary

Contra: $(n \mid a \vee n \mid b \rightarrow n \mid ab)$

Case 1: $n \mid a$

$\exists k \in \mathbb{Z}$ st $a = nk$

$ab = n\underbrace{kb}_{\in \mathbb{Z}}$

So $n \mid ab$

$\forall n, a, b \in \mathbb{Z} \, (n \nmid ab \rightarrow n \nmid a \land n \nmid b)$

let $n, a, b \in \mathbb{Z}$ arbitrary

Contra: $(n \mid a \lor n \mid b \rightarrow n \mid ab)$

Case 1: $n \mid a$

$\exists k \in \mathbb{Z}$ st $a = nk$

$ab = n\underbrace{kb}_{\in \mathbb{Z}}$

so $n \mid ab$