Consistency of Fanbeam Projections of a Translating Object Along an Arc of a Circle

Abstract

This note aims at extending the results of [2] to the case of a translating object.

1 Introduction

2 Theory

2.1 Problem under consideration

Let us begin with some notations and definitions. We will consider an object in $\mathbb{R}^2$ to be imaged by a fanbeam source that follows an arc of circle with center O and radius R_0 (see Figure 1, left). The object will be identified with its density function $\mathbf{x} \mapsto \mu(\mathbf{x}) \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. The angular velocity of the source will

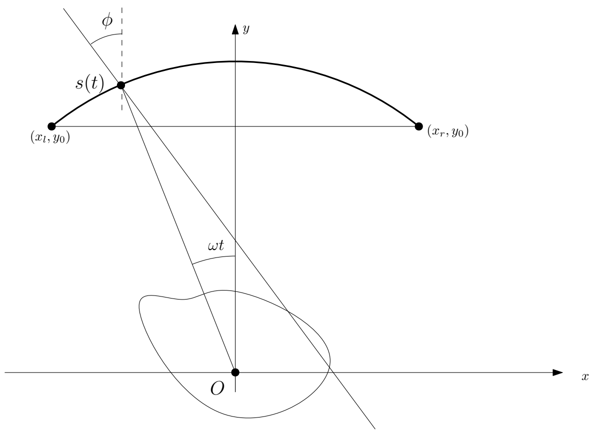

Figure 1: Problem under consideration. The source point $s(t)$ follows the arc of circle depicted in bold. This latter has center O and radius R_0.

be denoted ω, and the time t will range from $-T/2$ to $T/2$, where $T > 0$. Hence, if we denote $s(t)$ the position of the source at time t, one has

$$s(t) = (-R_0 \sin(\omega t), R_0 \cos(\omega t)) . \tag{2.1}$$

Furthermore, we will denote $s(T/2) = (x_l, y_l)$ (resp. $s(-T/2) = (x_r, y_r)$) the extreme left (resp. right) position of the source. Since $y_l = y_r = R_0 \cos(\omega T/2)$, we will call y_0 this common value. In the following, we will suppose that $\mathrm{supp}(\mu)$ lies in the half-space $\{y < y_0\}$, and that $y_0 > 0$ (*i.e.* $0 < \omega T < \pi$).

We will suppose that at any time t rays are simultaneously emitted from the source $s(t)$ with angle ϕ ranging from $-\pi/2$ to $\pi/2$. With this setup in mind, we can define the operator giving the acquired data from the object.

Definition 1. *The* fanbeam projection data *of an object with density function μ is a function $(t, \phi) \mapsto T\mu(t, \phi)$ defined by*

$$(T\mu)(t, \phi) = \int_0^{+\infty} \mu\left(s(t) + l\left[\sin\phi, -\cos\phi\right]\right) dl, \tag{2.2}$$

where $t \in [-T/2, T/2]$, $\phi \in [-\pi/2, \pi/2]$ *and* $s(t)$ *is given by (2.1). The operator* $\mu \mapsto T\mu$ *is called the* fanbeam projection operator*.*

Now let us suppose that the object is translating along the x-axis with a constant velocity $v \in \mathbb{R}$ (see Figure 1, right). In other words, if we denote $M(t)$ its center of mass at any time t, we have

$$M_v(t) = \left(\left(t + \frac{T}{2} \right) v, 0 \right). \tag{2.3}$$

The density function of the object now depends on both the space variable $\mathbf{x} \in \mathbb{R}^2$ and the time t. If we denote it μ_v, we have

$$\mu_v(t, \mathbf{x}) = \mu \left(\mathbf{x} - M_v(t) \right). \tag{2.4}$$

In this regard, the fanbeam projection data will be modified in the following way.

Definition 2. *The* fanbeam projection data of a translating object *with density function* μ *and translating velocity* v *is given by*

$$(T_v\mu)(t, \phi) = \left(T\mu_v(t, \cdot) \right)(t, \phi). \tag{2.5}$$

The aim of this note is to derive data consistency conditions (DCCs) from (2.5), in order to retrieve the velocity v.

2.2 Derivation of DCCs

In order to derive DCCs, we will first change our frame of reference, from (O, x, y) to $(M(t), x, y)$, so that the object is at the center. In this frame, the coordinates of the source are given by $s_v(t) = s(t) - M_v(t)$, so that we have

$$(T_v\mu)(t, \phi) = \int_0^{+\infty} \mu \left(s_v(t) + l \left[\sin \phi, -\cos \phi \right] \right) dl. \tag{2.6}$$

In other words, we are now dealing with a fixed object illuminated by a source following an arc of a cycloid (see Figure 2).

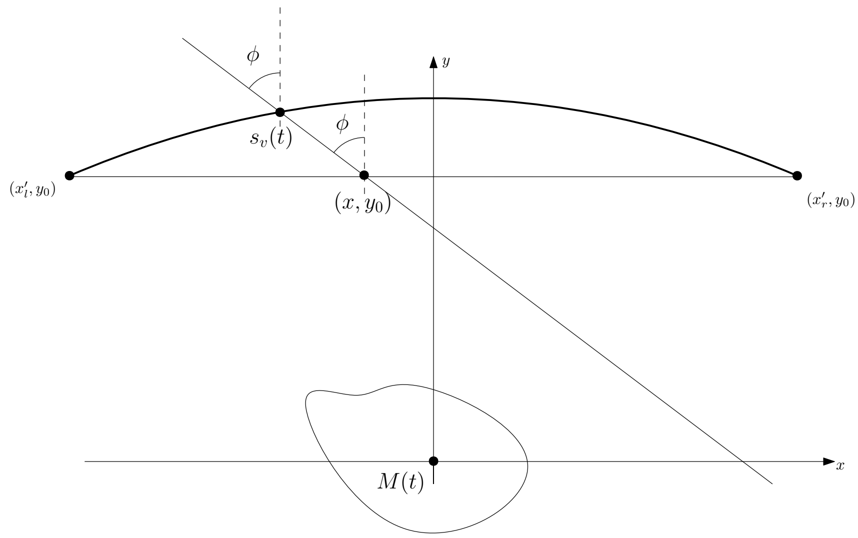

Figure 2: Change of frame: the object is now at center of the coordinates system.

For any time t, let us denote the coordinates of $s_v(t)$ by $(s_{1,v}(t), s_{2,v}(t))$. Here, the extreme points $s_v(-T/2)$ and $s_v(T/2)$ have the same y-coordinate y_0, and $s_{1,v}(-T/2) = x_r$, but $s_{1,v}(T/2)$ differs. We will call it x'_l; note that one has $x'_l = x_l - Tv$. This allows us to define what we call the *virtual fanbeam projection* from a point (x, y_0).

Definition 3. *For any point x between x_l' and x_r, for any angle $\phi \in [-\pi/2, \pi/2]$, the* virtual fanbeam projection *of the object μ is defined by*

$$\left(\tilde{T}\mu\right)(x,\phi) = \int_0^{+\infty} \mu\left((x, y_0) + l\left[\sin\phi, -\cos\phi\right]\right) dl. \tag{2.7}$$

This is called virtual *since it does not correspond to an actual position of the source.*

In the following lemma, we will make a connection between the virtual fanbeam projection and the fanbeam projection of the translating object.

Lemma 1. *Let us fix a time $t \in [-T/2, T/2]$ and an angle $\phi \in [-\pi/2, \pi/2]$. Let us define*

$$x = s_{1,v}(t) + \tan\phi\left(s_{2,v}(t) - y_0\right). \tag{2.8}$$

Then, one has

$$\left(\tilde{T}\mu\right)(x,\phi) = \left(T_v\mu\right)(t,\phi). \tag{2.9}$$

Proof. The idea is that the point (x, y_0) is the intersection between the line $\{y = y_0\}$ and the ray coming from $s_v(t)$ with angle ϕ (see Figure 2). Since $\mathrm{supp}(\mu)$ is under the line $\{y = y_0\}$, the integral does not differ if we start from (x, y_0) or from $s_v(t)$. Indeed, one has

$$\int_0^{+\infty} \mu\left((x, y_0) + l\left[\sin\phi, -\cos\phi\right]\right) dl = \int_{-\infty}^{+\infty} \mu\left((x, y_0) + l\left[\sin\phi, -\cos\phi\right]\right) dl. \tag{2.10}$$

Let us now perform the following change of variable

$$l' \leftrightarrow l + \frac{s_{2,v}(t) - y_0}{\cos\phi}. \tag{2.11}$$

By definition of x, it gives $x + l\sin\phi = s_{1,v}(t) + l'\sin\phi$ and $y_0 - l\cos\phi = s_{2,v}(t) - l'\cos\phi$. We then have

$$\int_{-\infty}^{+\infty} \mu\left((x, y_0) + l\left[\sin\phi, -\cos\phi\right]\right) dl = \int_{-\infty}^{+\infty} \mu\left(s_v(t) + l'\left[\sin\phi, -\cos\phi\right]\right) dl'. \tag{2.12}$$

Restricting the domain of integration from $l' = 0$ to $t = +\infty$ then leads us to the desired result. $\square$

We can now define what are the DCCs of our problem.

Theorem 1. *Let us fix a density function μ. For any integer n, there exist a function $(x,t) \mapsto W_{n,v}(t,x) \in \mathcal{C}^\infty\left([-T/2, T/2] \times [x_l', x_r]\right)$ such that*

$$B_n := x \mapsto \int_{-T/2}^{T/2} \left(T_v\mu\right)(t, \lambda(t))\, W_n(x,t,v) dt \in \mathbb{R}_n[X], \tag{2.13}$$

where $\lambda(t)$ is defined by

$$\lambda_t = \arctan\left(\frac{x + R_0\sin(\omega t) + \left(t + \frac{T}{2}\right)v}{R_0\cos(\omega t) - y_0}\right). \tag{2.14}$$

Moreover, it is possible to derive $W_{n,v}(t,x)$ analytically.

Proof. The DCCs derived in [2] heavily rely on a relation between $\tan\phi$ and the angle ωt (see Figure 1). This relation is then differentiated to change variables into the integral in (2.13), which gives the following expression, known to be a polynom according to [1]

$$\int_{-\pi/2}^{\pi/2} \left(\tilde{T}\mu\right)(x,\phi) \frac{\tan^n\phi}{\cos\phi} d\phi. \tag{2.15}$$

We will follow the same path here. First, the formula giving $\tan\phi$ is nearly the same as in [2], *i.e.*

$$\tan\phi = \frac{x + R_0\sin(\omega t) + \left(t+\frac{T}{2}\right)v}{R_0\cos(\omega t) - y_0}.$$

Then, taking its derivative allows us to write the Jacobian for a change of variables from ϕ to t

$$\begin{aligned}\frac{d\phi}{\cos^2\phi} &= \frac{\left(R_0\omega\cos(\omega t)+v\right)\left(R_0\cos(\omega t)-y_0\right) + R_0\omega\sin(\omega t)\left(x+R_0\sin(\omega t)+\left(t+\frac{T}{2}\right)v\right)}{\left(R_0\cos(\omega t)-y_0\right)^2}dt \\ &= \frac{R_0^2\omega - vy_0 + R_0\cos(\omega t)(v-\omega y_0) + R_0\omega\sin(\omega t)(x+\left(t+\frac{T}{2}\right)v)}{\left(R_0\cos(\omega t)-y_0\right)^2}dt \\ &:= J(x,t,v)dt.\end{aligned}$$

Hence, one can write

$$\begin{aligned}\frac{\tan^n\phi}{\cos\phi}d\phi &= \tan^n\phi\cos\phi\frac{d\phi}{\cos^2\phi} && (2.16)\\ &= \frac{\left(x+R_0\sin(\omega t)+\left(t+\frac{T}{2}\right)v\right)^n}{D_{x,t}\left(R_0\cos(\omega t)-y_0\right)^{n-1}}J(x,t,v)dt && (2.17)\\ &:= W_n(x,t,v)dt, && (2.18)\end{aligned}$$

where the term $D_{x,t}$ in equation (2.17) refers to the distance between the source point $s_v(t)$ and the virtual point (x, y_0).

With the help of lemma 1, the change of variable from t to ϕ then leads us to

$$B_n(x) = \int_{-\pi/2}^{\pi/2}\left(\tilde{T}\mu\right)(x,\phi)\frac{\tan^n\phi}{\cos\phi}d\phi, \tag{2.19}$$

which is known to be a polynom of degree at most n according to [1] as mentioned above. $\square$

3 Numerical simulations

Let us suppose that we have the projections $g(x,\phi)$. In order to recover v, we can perform the following optimization procedure. Since $B_n(x)$ in equation (2.13) is supposed to be a polynom of order $\leq n$, one can minimize

$$\mathcal{J}(v) = \sum_n \|\mathrm{res}\left(B_n(x,v)\right)\|^2 \tag{3.1}$$

with respect to v, where res is the residual of the projection onto $\mathbb{R}_n[X]$. The minimization procedure can be done using the gradient of $\mathcal{J}(v)$, given by

$$\nabla\mathcal{J}(v) = 2\sum_n B_n(x,v)\int_{-T/2}^{T/2} g(t,\phi)\frac{\partial W_n}{\partial v}(x,t,v)dt. \tag{3.2}$$

References

[1] Rolf Clackdoyle. Necessary and sufficient consistency conditions for fanbeam projections along a line. *IEEE Transactions on Nuclear Science*, 60(3):1560–1569, 2013.

[2] Rolf Clackdoyle, Michel Defrise, Laurent Desbat, and Johan Nuyts. Consistency of fanbeam projections along an arc of a circle. In *The 13th International Meeting on Fully Three-Dimensional Image Reconstruction in Radiology and Nuclear Medicine*, pages 415–419, 2015.