Consistency of Fanbeam Projections Along an Arc of a Cycloid

Abstract

This note aims at extending the results of [1] to the case of an arc of cycloid, instead of an arc of circle.

1 Introduction

Let us begin with some notations. The reader shall refer to Figure 1, which is analogous to [1, Figure 2]. The object to be imaged in given by its density function $\mu(x)$ and is illuminated by a source that follows an arc of cycloid. In other words, the fanbeam projection data $g(t,\gamma)$ is given by

$$g(t,\gamma) = \int_0^{+\infty} \mu\left(s(t) + l\left[\sin(\gamma+\lambda_t), -\cos(\gamma+\lambda_t)\right]\right) dl \tag{1.1}$$

The cycloid is parametrized by t in the following way

$$s(t) = \left(-R_0\sin(\omega t) - \left(t + \frac{T}{2}\right)v, R_0\cos(\omega t)\right),$$

where R_0, ω and v are fixed constants.

We will denote R_t the distance between the center O and the source point $s(t)$, and λ_t the angle formed with the y-axis. Namely, one has

$$R_t = \sqrt{\left(R_0\sin(\omega t) + tv\right)^2 + R_0^2\cos^2(\omega t)}, \tag{1.2}$$

$$\lambda_t = \arctan\left(\frac{R_0\sin(\omega t) + tv}{R_0\cos(\omega t)}\right). \tag{1.3}$$

2 Derivation of DCCs

The DCCs derived in [1] heavily rely on a relation between $\tan\phi$ and the angle λ_t, which is then differentiated to change variables into the following integral

$$B_n(x) = \int_{-\pi/2}^{\pi/2} g_v(x,\phi)\frac{\tan^n\phi}{\cos\phi}d\phi, \tag{2.1}$$

where $g_v(x,\phi)$ is the virtual fanbeam projection whose virtual source is located at (x, y_0). In other words,

$$g_v(x,\phi) = \int_0^{\infty} \mu\left((x,y_0) + l(\sin\phi, -\cos\phi)\right) dl. \tag{2.2}$$

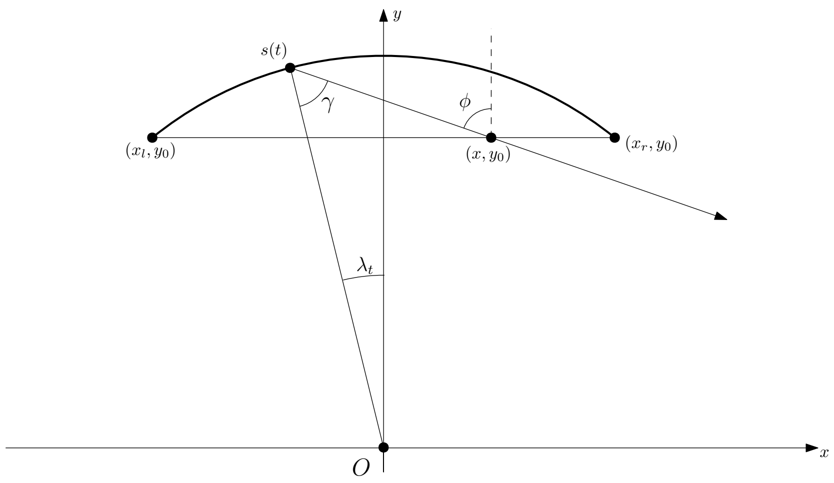

Figure 1: Problem under consideration. The source point $s(t)$ follows the cycloid depicted in bold, and the virtual point has coordinates (x, y_0).

We will follow the same path in this note. First, the formula giving $\tan\phi$ is nearly the same as in [1], *i.e.*

$$\tan\phi = \frac{x + R_0\sin(\omega t) + tv}{R_0\cos(\omega t) - y_0}$$

Then, taking its derivative allows us to write the Jacobian for a change of variables from ϕ to t

$$\begin{aligned}
\frac{d\phi}{\cos^2\phi} &= \frac{(R_0\omega\cos(\omega t) + v)\,(R_0\cos(\omega t) - y_0) + R_0\omega\sin(\omega t)\,(x + R_0\sin(\omega t) + tv)}{(R_0\cos(\omega t) - y_0)^2}dt \\
&= \frac{R_0^2\omega - vy_0 + R_0\cos(\omega t)(v - \omega y_0) + R_0\omega\sin(\omega t)(x + tv)}{(R_0\cos(\omega t) - y_0)^2}dt \\
&:= J(x,t)dt.
\end{aligned}$$

Hence, one can write

$$\begin{aligned}
\frac{\tan^n\phi}{\cos\phi}d\phi &= \tan^n\phi\cos\phi\frac{d\phi}{\cos^2\phi} && (2.3) \\
&= \frac{(x + R_0\sin(\omega t) + tv)^n}{D_{x,t}\,(R_0\cos(\omega t) - y_0)^{n-1}}J(x,t)dt && (2.4) \\
&:= W_n(x,t)dt, && (2.5)
\end{aligned}$$

where the term $D_{x,t}$ in equation (2.4) refers to the distance between the source point $s(t)$ and the virtual point (x, y_0). In any case, equation (2.1) can be re-written as

$$B_n(x) = \int_{-T/2}^{T/2} g(t,\gamma)W_n(x,t)dt, \tag{2.6}$$

where T is defined such that $s(-T/2) = (x_r, y_0)$ and $s(T/2) = (x_l, y_0)$. Note that the angle γ depends on x and t in the following way

$$\cos\gamma = \frac{(R_0\sin(\omega t) + tv)\,(x + R_0\sin(\omega t) + tv) - R_0\cos(\omega t)\,(y_0 - R_0\cos(\omega t))}{R_t D_{x,t}}. \tag{2.7}$$

3 Numerical simulations

To be continued.

References

[1] Rolf Clackdoyle, Michel Defrise, Laurent Desbat, and Johan Nuyts. Consistency of fanbeam projections along an arc of a circle. In *The 13th International Meeting on Fully Three-Dimensional Image Reconstruction in Radiology and Nuclear Medicine*, pages 415–419, 2015.