

Consistency of Fanbeam Projections of a Translating Object Along an Arc of a Circle

Abstract

This note aims at extending the results of [2] to the case of a translating object.

1 Introduction

2 Theory

2.1 Problem under consideration

Let us begin with some notations and definitions. We will consider an object in \mathbb{R}^2 to be imaged by a fanbeam source that follows an arc of circle with center O and radius R_0 (see Figure 1, left). The object will be identified with its density function $\mathbf{x} \mapsto \mu(\mathbf{x}) \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. The angular velocity of the source will

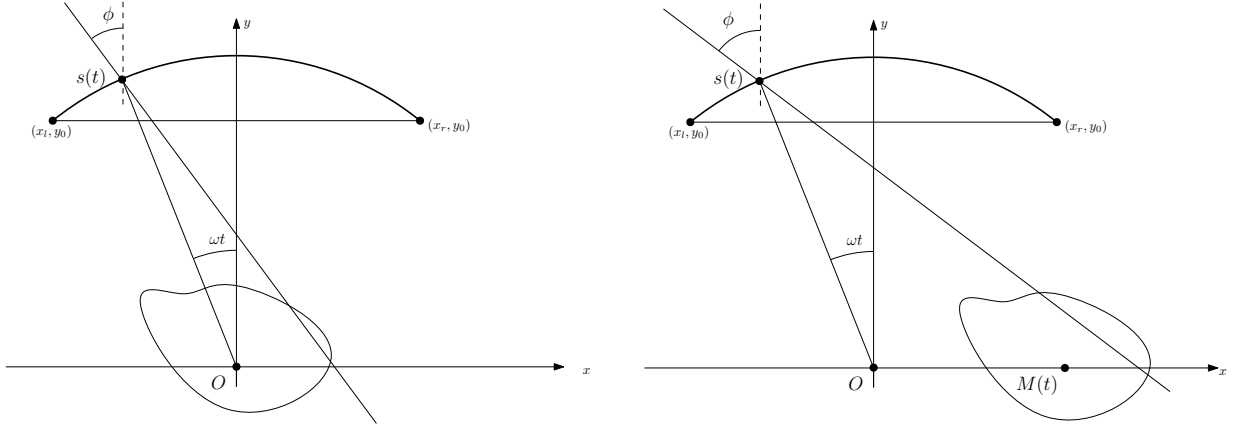


Figure 1: Problem under consideration. The source point $s(t)$ follows the arc of circle depicted in bold. This latter has center O and radius R_0 .

be denoted ω , and the time t will range from $-T/2$ to $T/2$, where $T > 0$. Hence, if we denote $s(t)$ the position of the source at time t , one has

$$s(t) = (-R_0 \sin(\omega t), R_0 \cos(\omega t)). \quad (2.1)$$

Furthermore, we will denote $s(T/2) = (x_l, y_l)$ (resp. $s(-T/2) = (x_r, y_r)$) the extreme left (resp. right) position of the source. Since $y_l = y_r = R_0 \cos(\omega T/2)$, we will call y_0 this common value. In the following, we will suppose that $\text{supp}(\mu)$ lies in the half-space $\{y < y_0\}$.

We will suppose that at any time t rays are simultaneously emitted from the source $s(t)$ with angle ϕ ranging from $-\pi/2$ to $\pi/2$. With this setup in mind, we can define the operator giving the acquired data from the object.

Definition 1. The fanbeam projection data of an object with density function μ is a function $(t, \phi) \mapsto T\mu(t, \phi)$ defined by

$$(T\mu)(t, \phi) = \int_0^{+\infty} \mu(s(t) + l[\sin \phi, -\cos \phi]) dl, \quad (2.2)$$

where $t \in [-T/2, T/2]$, $\phi \in [-\pi/2, \pi/2]$ and $s(t)$ is given by (2.1). The operator $\mu \mapsto T\mu$ is called the fanbeam projection operator.

Now let us suppose that the object is translating along the x -axis with a constant velocity $v \in \mathbb{R}$ (see Figure 1, right). In other words, if we denote $M(t)$ its center of mass at any time t , we have

$$M_v(t) = \left(\left(t + \frac{T}{2} \right) v, 0 \right). \quad (2.3)$$

The density function of the object now depends on both the space variable $\mathbf{x} \in \mathbb{R}^2$ and the time t . If we denote it μ_v , we have

$$\mu_v(t, \mathbf{x}) = \mu(\mathbf{x} - M_v(t)). \quad (2.4)$$

In this regard, the fanbeam projection data will be modified in the following way.

Definition 2. The fanbeam projection data of a translating object with density function μ and translating velocity v is given by

$$(T_v\mu)(t, \phi) = (T\mu_v(t, \cdot))(t, \phi). \quad (2.5)$$

The aim of this note is to derive data consistency conditions (DCCs) from (2.5), in order to retrieve the velocity v .

2.2 Derivation of DCCs

In order to derive DCCs, we will first change our frame of reference, from (O, x, y) to $(M(t), x, y)$, so that the object is at the center. In this frame, the coordinates of the source are given by $s_v(t) = s(t) - M_v(t)$, so that we have

$$(T_v\mu)(t, \phi) = \int_0^{+\infty} \mu(s_v(t) + l[\sin \phi, -\cos \phi]) dl. \quad (2.6)$$

In other words, we are now dealing with a fixed object illuminated by a source following an arc of a cycloid (see Figure 2).

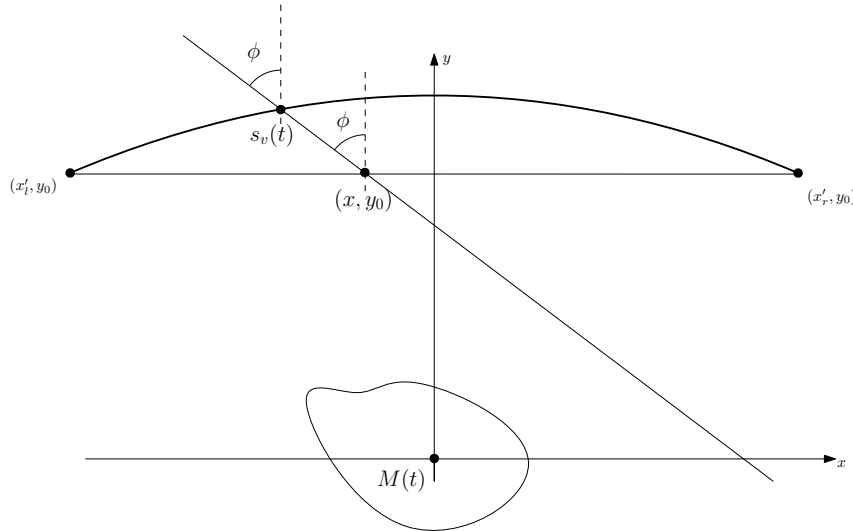


Figure 2: Change of frame: the object is now at center of the coordinates system.

For any time t , let us denote the coordinates of $s_v(t)$ by $(s_{1,v}(t), s_{2,v}(t))$. Here, the extreme points $s_v(-T/2)$ and $s_v(T/2)$ have the same y -coordinate y_0 , and $s_{1,v}(-T/2) = x_r$, but $s_{1,v}(T/2)$ differs. We will call it x'_l ; note that one has $x'_l = x_l - Tv$. This allows us to define what we call the *virtual fanbeam projection* from a point (x, y_0) .

Definition 3. For any point x between x'_l and x_r , for any angle $\phi \in [-\pi/2, \pi/2]$, the virtual fanbeam projection of the object μ is defined by

$$\left(\tilde{T}\mu\right)(x, \phi) = \int_0^{+\infty} \mu((x, y_0) + l[\sin \phi, -\cos \phi]) dl. \quad (2.7)$$

This is called virtual since it does not correspond to an actual position of the source.

In the following lemma, we will make a connection between the virtual fanbeam projection and the fanbeam projection of the translating object.

Lemma 1. Let us fix a time $t \in [-T/2, T/2]$ and an angle $\phi \in [-\pi/2, \pi/2]$. Let us define

$$x = s_{1,v}(t) + \tan \phi (s_{2,v}(t) - y_0). \quad (2.8)$$

Then, one has

$$\left(\tilde{T}\mu\right)(x, \phi) = (T_v\mu)(t, \phi). \quad (2.9)$$

Proof. The idea is that the point (x, y_0) is the intersection between the line $\{y = y_0\}$ and the ray coming from $s_v(t)$ with angle ϕ (see Figure 2). Since $\text{supp}(\mu)$ is under the line $\{y = y_0\}$, the integral does not differ if we start from (x, y_0) or from $s_v(t)$. Indeed, one has

$$\int_0^{+\infty} \mu((x, y_0) + l[\sin \phi, -\cos \phi]) dl = \int_{-\infty}^{+\infty} \mu((x, y_0) + l[\sin \phi, -\cos \phi]) dl. \quad (2.10)$$

Let us now perform the following change of variable

$$l' \leftrightarrow l + \frac{s_{2,v}(t) - y_0}{\cos \phi}. \quad (2.11)$$

By definition of x , it gives $x + l \sin \phi = s_{1,v}(t) + l' \sin \phi$ and $y_0 - l \cos \phi = s_{2,v}(t) - l' \cos \phi$. We then have

$$\int_{-\infty}^{+\infty} \mu((x, y_0) + l[\sin \phi, -\cos \phi]) dl = \int_{-\infty}^{+\infty} \mu(s_v(t) + l'[\sin \phi, -\cos \phi]) dl'. \quad (2.12)$$

Restricting the domain of integration from $l' = 0$ to $t = +\infty$ then leads us to the desired result. \square

We can now define what are the DCCs of our problem.

Theorem 1. Let us fix a density function μ . For any integer n , there exist a function $(x, t) \mapsto W_{n,v}(t, x) \in \mathcal{C}^\infty([-T/2, T/2] \times [x'_l, x_r])$ such that

$$B_n := x \mapsto \int_{-T/2}^{T/2} (T_v\mu)(t, \lambda(t)) W_n(x, t, v) dt \in \mathbb{R}_n[X], \quad (2.13)$$

where $\lambda(t)$ is defined by

$$\lambda_t = \arctan \left(\frac{x + R_0 \sin(\omega t) + (t + \frac{T}{2})v}{R_0 \cos(\omega t) - y_0} \right). \quad (2.14)$$

Moreover, it is possible to derive $W_{n,v}(t, x)$ analytically.

Proof. The DCCs derived in [2] heavily rely on a relation between $\tan \phi$ and the angle ωt (see Figure 1). This relation is then differentiated to change variables into the integral in (2.13), which gives the following expression, known to be a polynom according to [1]

$$\int_{-\pi/2}^{\pi/2} \left(\tilde{T}\mu\right)(x, \phi) \frac{\tan^n \phi}{\cos \phi} d\phi. \quad (2.15)$$

We will follow the same path here. First, the formula giving $\tan \phi$ is nearly the same as in [2], *i.e.*

$$\tan \phi = \frac{x + R_0 \sin(\omega t) + \left(t + \frac{T}{2}\right) v}{R_0 \cos(\omega t) - y_0}.$$

Then, taking its derivative allows us to write the Jacobian for a change of variables from ϕ to t

$$\begin{aligned} \frac{d\phi}{\cos^2 \phi} &= \frac{(R_0 \omega \cos(\omega t) + v)(R_0 \cos(\omega t) - y_0) + R_0 \omega \sin(\omega t) \left(x + R_0 \sin(\omega t) + \left(t + \frac{T}{2}\right) v\right)}{(R_0 \cos(\omega t) - y_0)^2} dt \\ &= \frac{R_0^2 \omega - v y_0 + R_0 \cos(\omega t)(v - \omega y_0) + R_0 \omega \sin(\omega t)(x + \left(t + \frac{T}{2}\right) v)}{(R_0 \cos(\omega t) - y_0)^2} dt \\ &:= J(x, t, v) dt. \end{aligned}$$

Hence, one can write

$$\frac{\tan^n \phi}{\cos \phi} d\phi = \tan^n \phi \cos \phi \frac{d\phi}{\cos^2 \phi} \quad (2.16)$$

$$= \frac{\left(x + R_0 \sin(\omega t) + \left(t + \frac{T}{2}\right) v\right)^n}{D_{x,t} (R_0 \cos(\omega t) - y_0)^{n-1}} J(x, t, v) dt \quad (2.17)$$

$$:= W_n(x, t, v) dt, \quad (2.18)$$

where the term $D_{x,t}$ in equation (2.17) refers to the distance between the source point $s_v(t)$ and the virtual point (x, y_0) .

With the help of lemma 1, the change of variable from t to ϕ then leads us to

$$B_n(x) = \int_{-\pi/2}^{\pi/2} \left(\tilde{T}\mu\right)(x, \phi) \frac{\tan^n \phi}{\cos \phi} d\phi, \quad (2.19)$$

which is known to be a polynomial of degree at most n according to [1] as mentioned above. \square

3 Numerical simulations

Let us suppose that we have the projections $g(x, \phi)$. In order to recover v , we can perform the following optimization procedure. Since $B_n(x)$ in equation (??) is supposed to be a polynomial of order $\leq n$, one can minimize

$$\mathcal{J}(v) = \sum_n \|\text{res}(B_n(x, v))\|^2 \quad (3.1)$$

with respect to v , where res is the residual of the projection onto $\mathbb{R}_n[X]$. The minimization procedure can be done using the gradient of $\mathcal{J}(v)$, given by

$$\nabla \mathcal{J}(v) = 2 \sum_n B_n(x, v) \int_{-T/2}^{T/2} g(t, \phi) \frac{\partial W_n}{\partial v}(x, t, v) dt. \quad (3.2)$$

References

- [1] Rolf Clackdoyle. Necessary and sufficient consistency conditions for fanbeam projections along a line. *IEEE Transactions on Nuclear Science*, 60(3):1560–1569, 2013.
- [2] Rolf Clackdoyle, Michel Defrise, Laurent Desbat, and Johan Nuyts. Consistency of fanbeam projections along an arc of a circle. In *The 13th International Meeting on Fully Three-Dimensional Image Reconstruction in Radiology and Nuclear Medicine*, pages 415–419, 2015.