CONSISTENCY OF FAN-BEAM PROJECTIONS OF A TRANSLATING OBJECT WITH SOURCES ALONG AN ARC OF A CIRCLE

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TEAM

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- Rolf Clackdoyle
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- Simon Rit

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TIMC

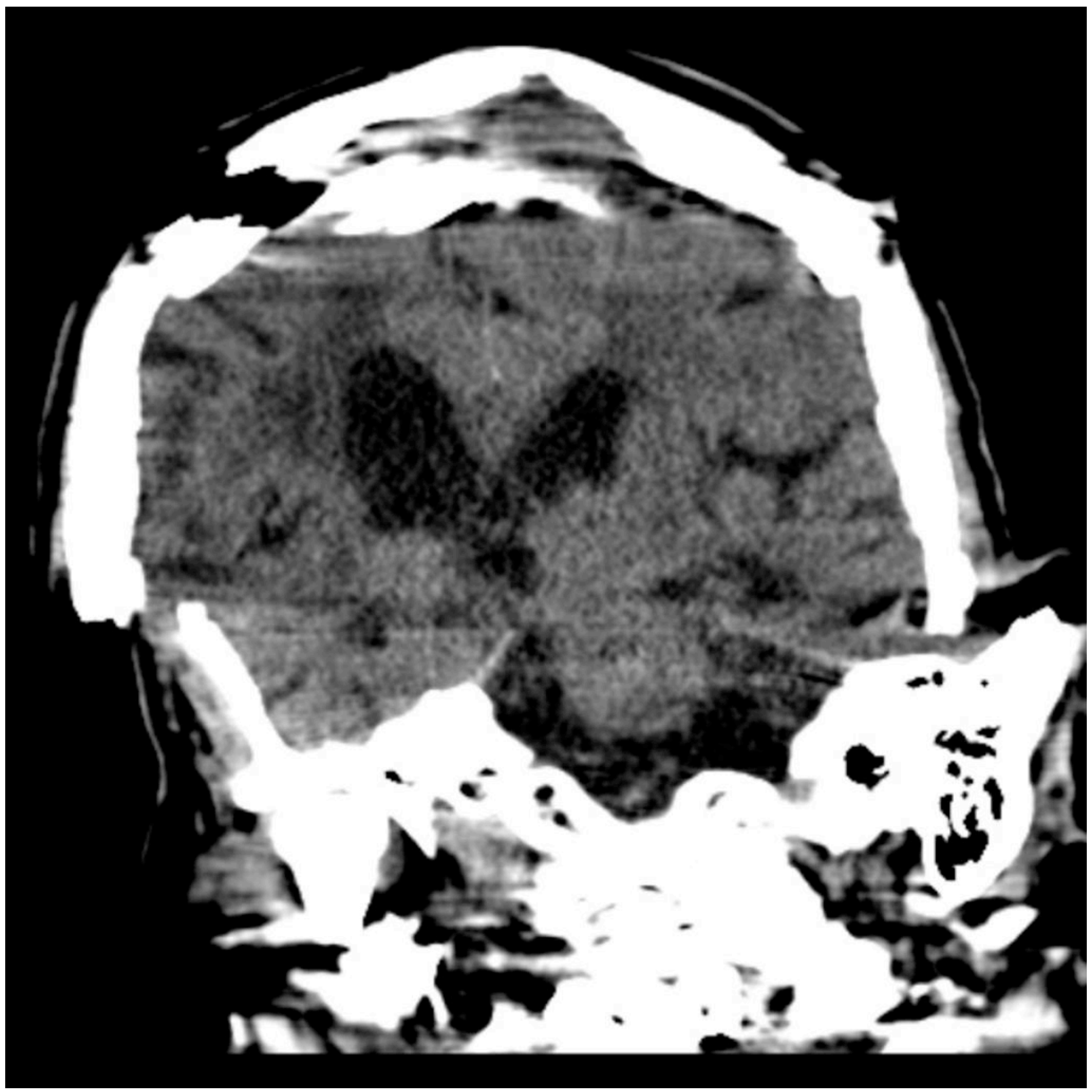
SUMMARY

- Context
- Theory
- Numerical simulations
- Conclusion

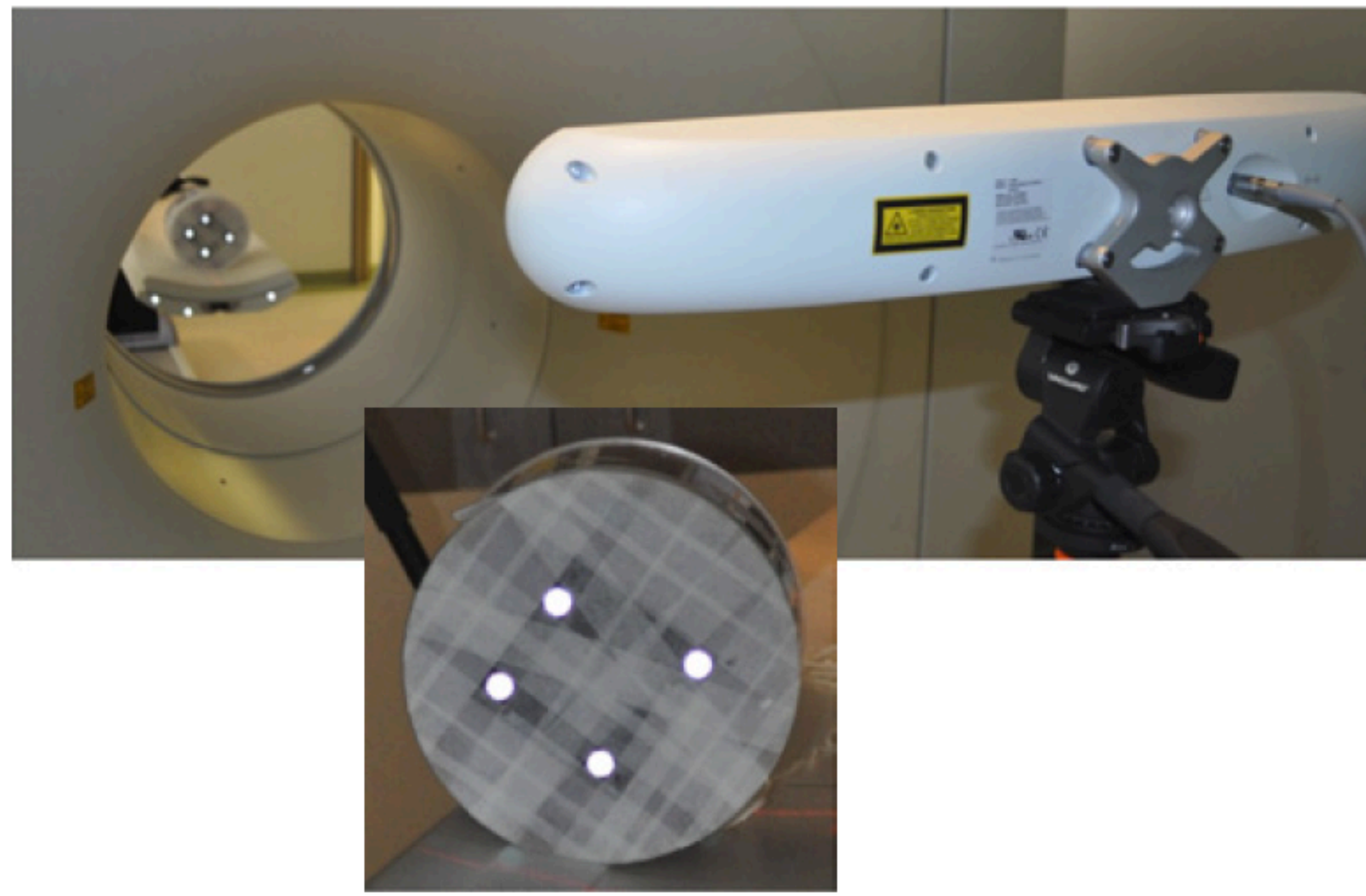
SUMMARY

- **Context**
  - *Medical*
  - *Mathematical*
- Theory
- Numerical simulations
- Conclusion

MEDICAL CONTEXT

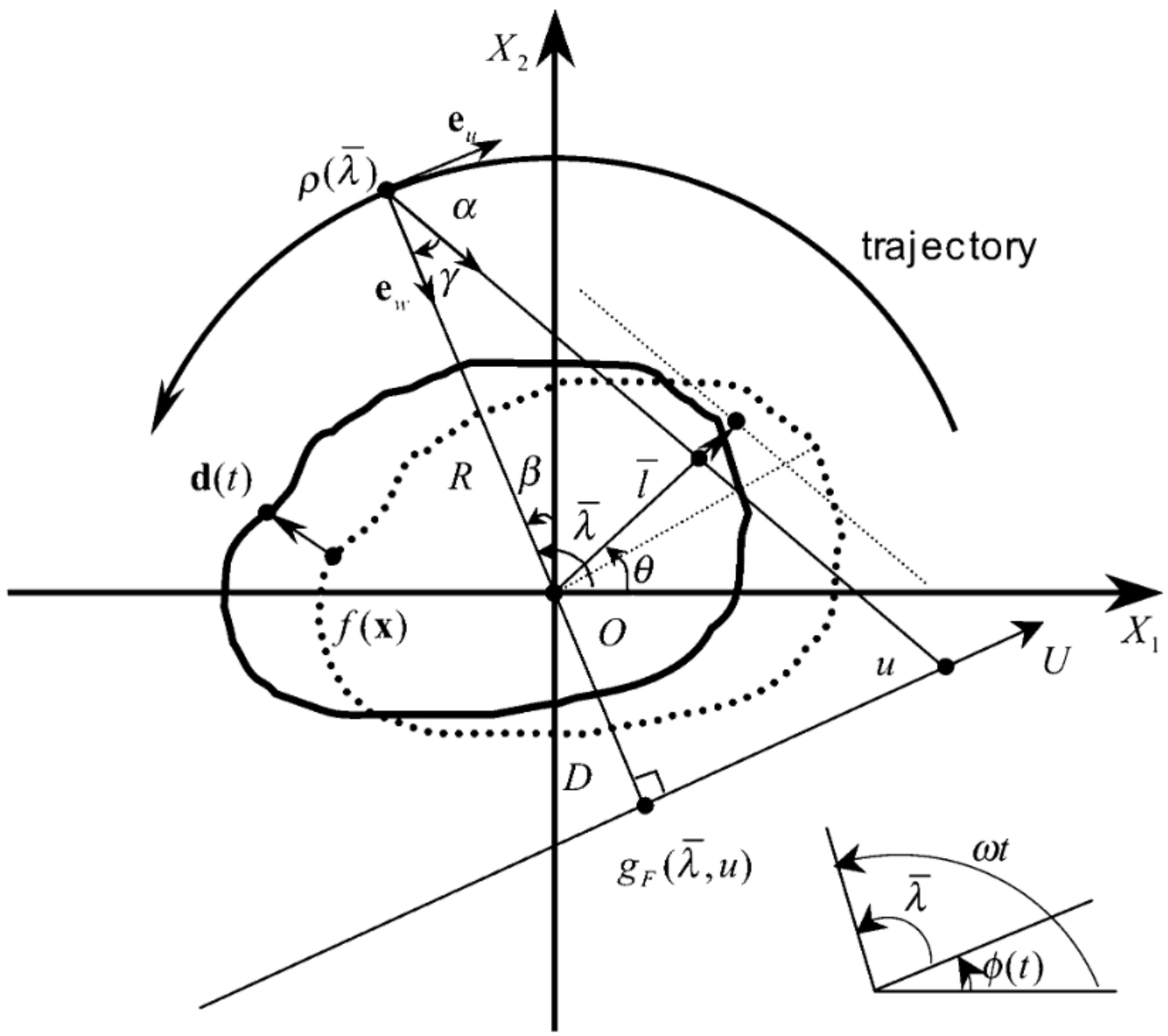


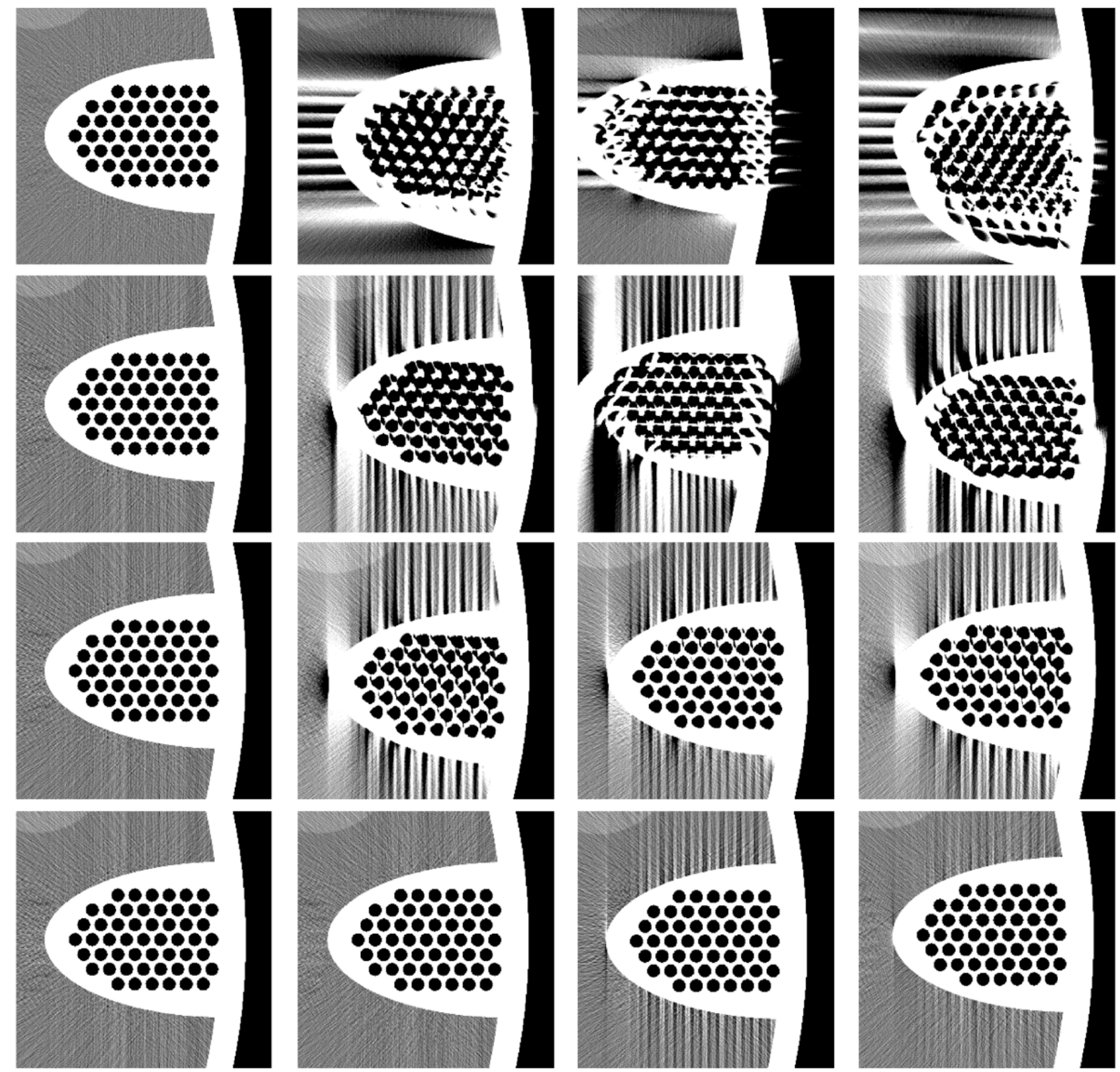
Case courtesy of Dr David Cuete, Radiopaedia.org, rID: 25637



Kim *et al.*, Phys. Med. Biol. 60 (2015)

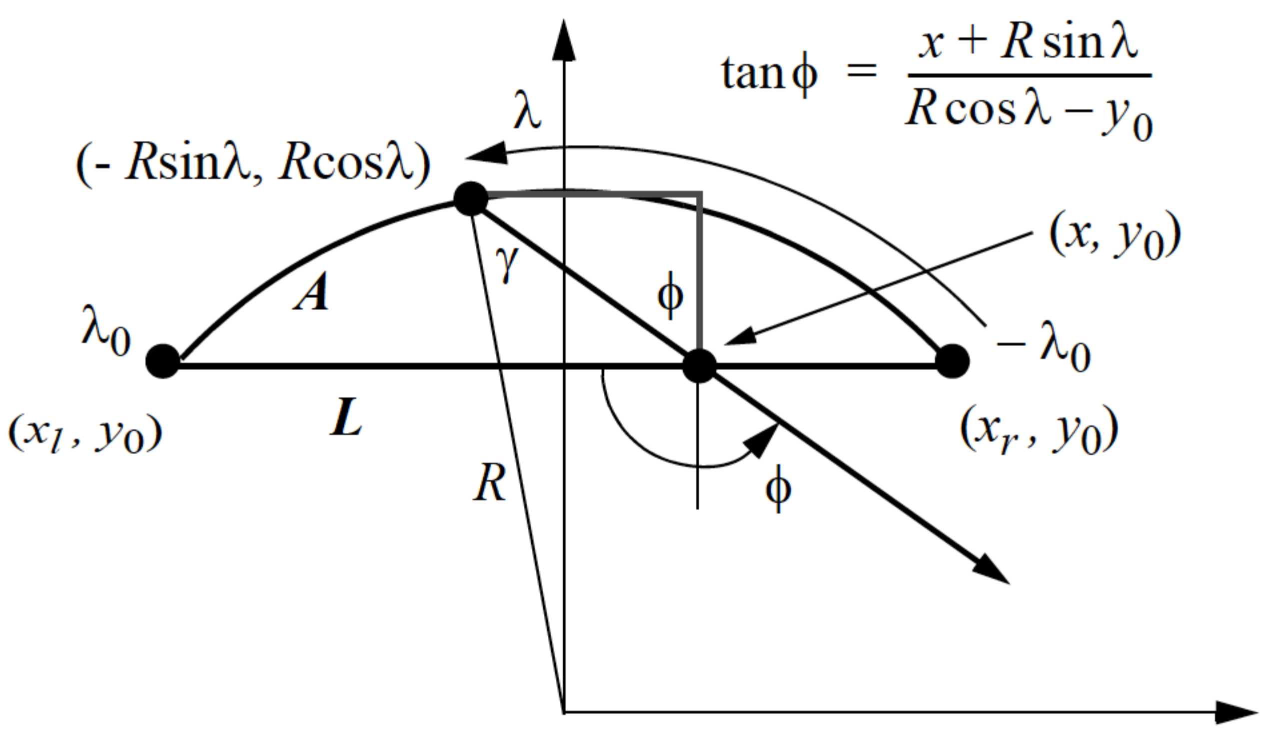
MATHEMATICAL CONTEXT: DATA CONSISTENCY CONDITIONS

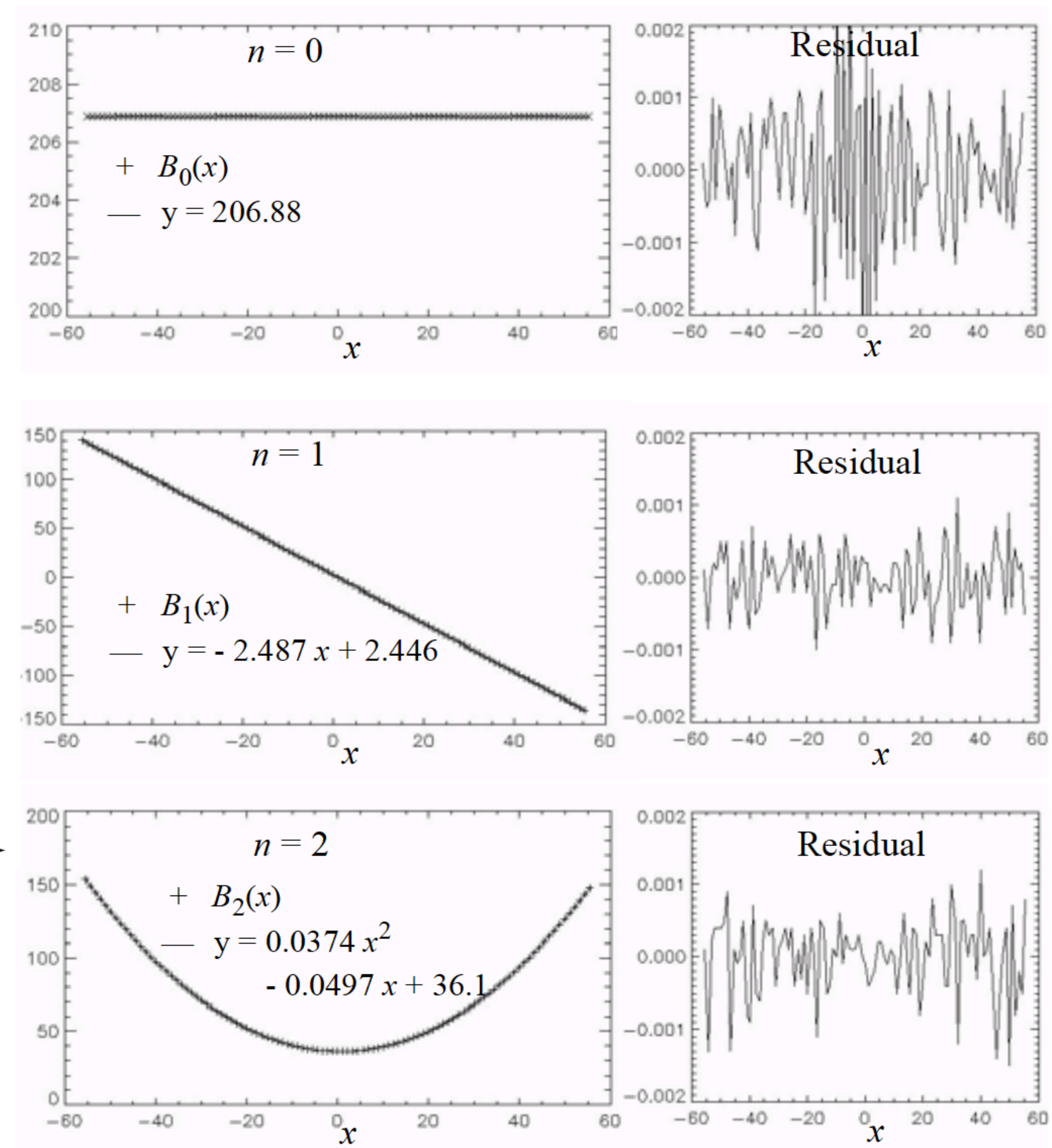




Yu, H., & Wang, G. (2007). *IEEE Trans. on Med. Im.*, 26(2), 249-260

MATHEMATICAL CONTEXT





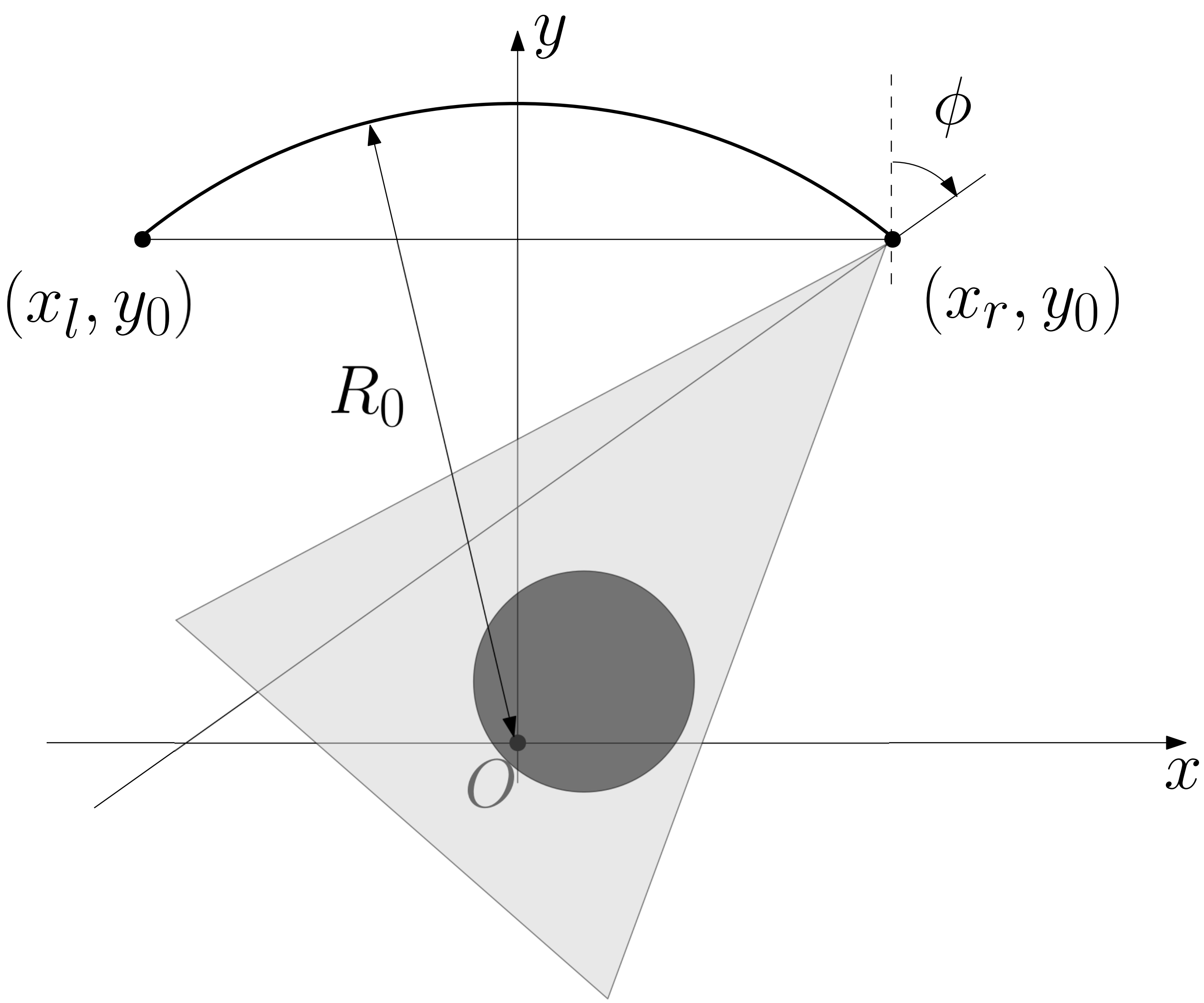
Clackdoyle *et al.*, 13th Fully3D conference

SUMMARY

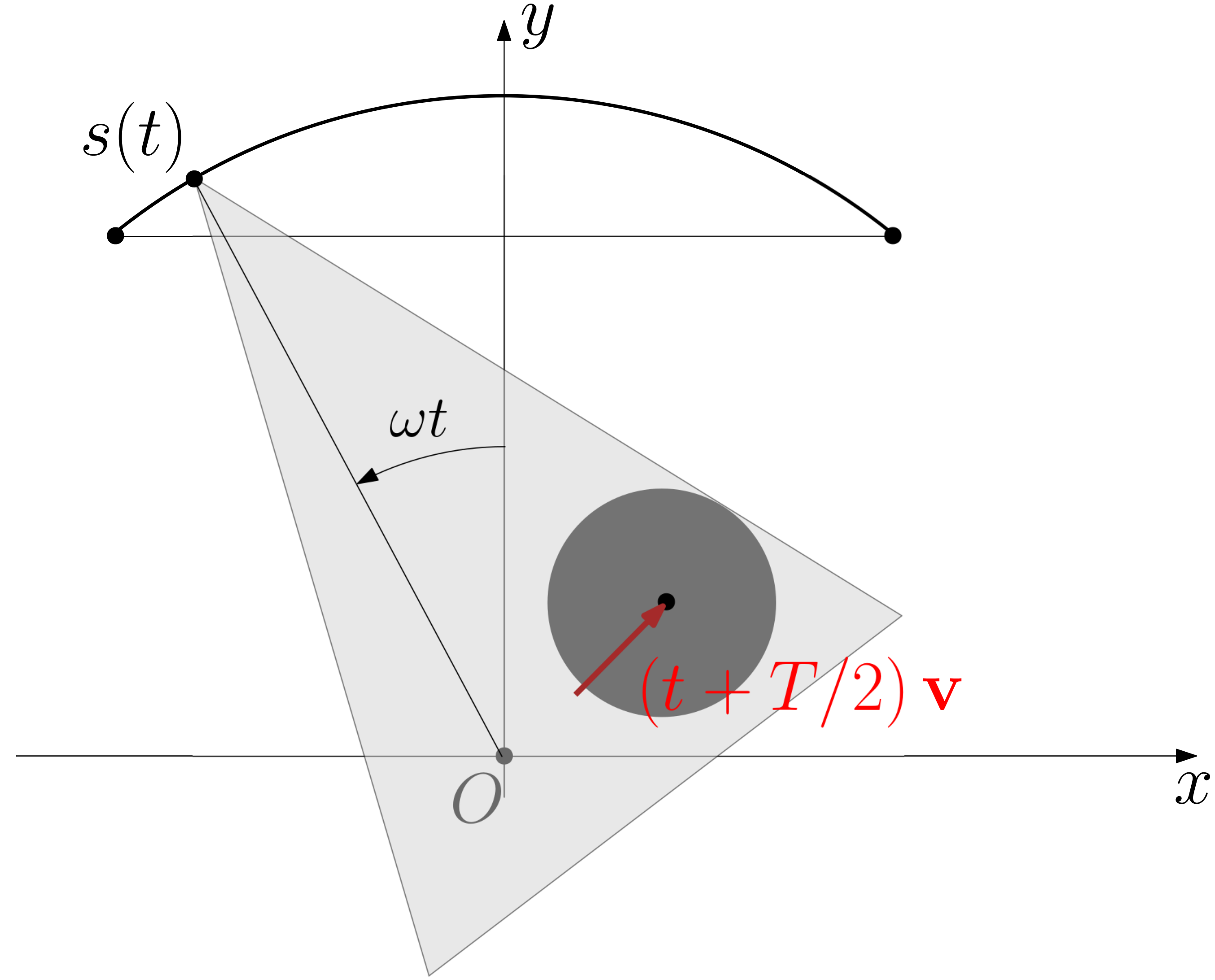
- Context
- **Theory**
  - Setup and notations
  - Derivation of data consistency conditions (DCCs)
- Numerical simulations
- Conclusion

SETUP AND NOTATIONS

$$t = -T/2$$



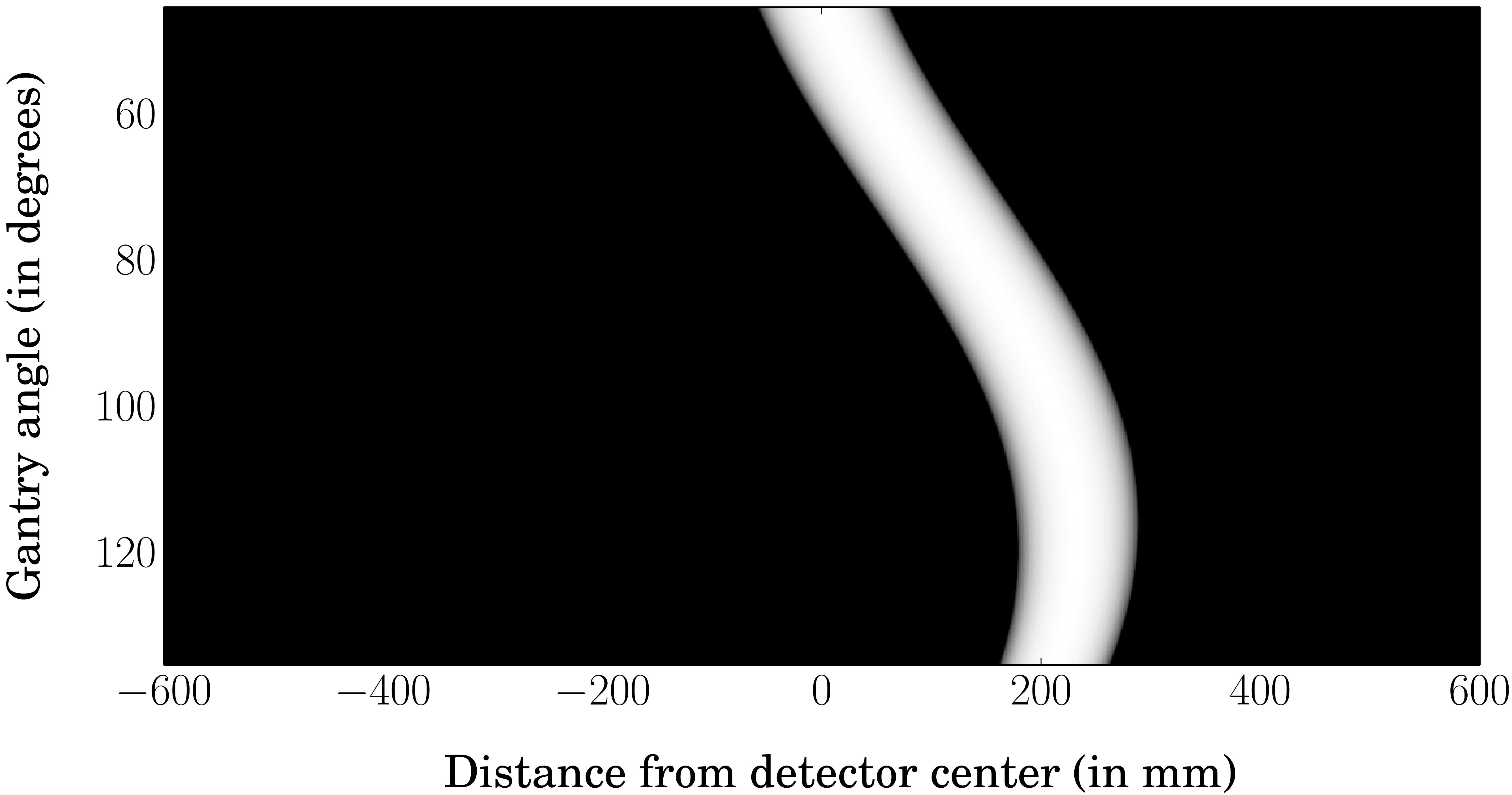
$$-T/2 < t < T/2$$



$$s(t) = R_0\,(-\sin(\omega t), \cos(\omega t))$$

$$\mathbf{v} = (v_1, v_2)$$

EXAMPLE WITH A DISK TRANSLATING ALONG THE X-AXIS



SETUP AND NOTATIONS

Fan-beam projection data of a still object

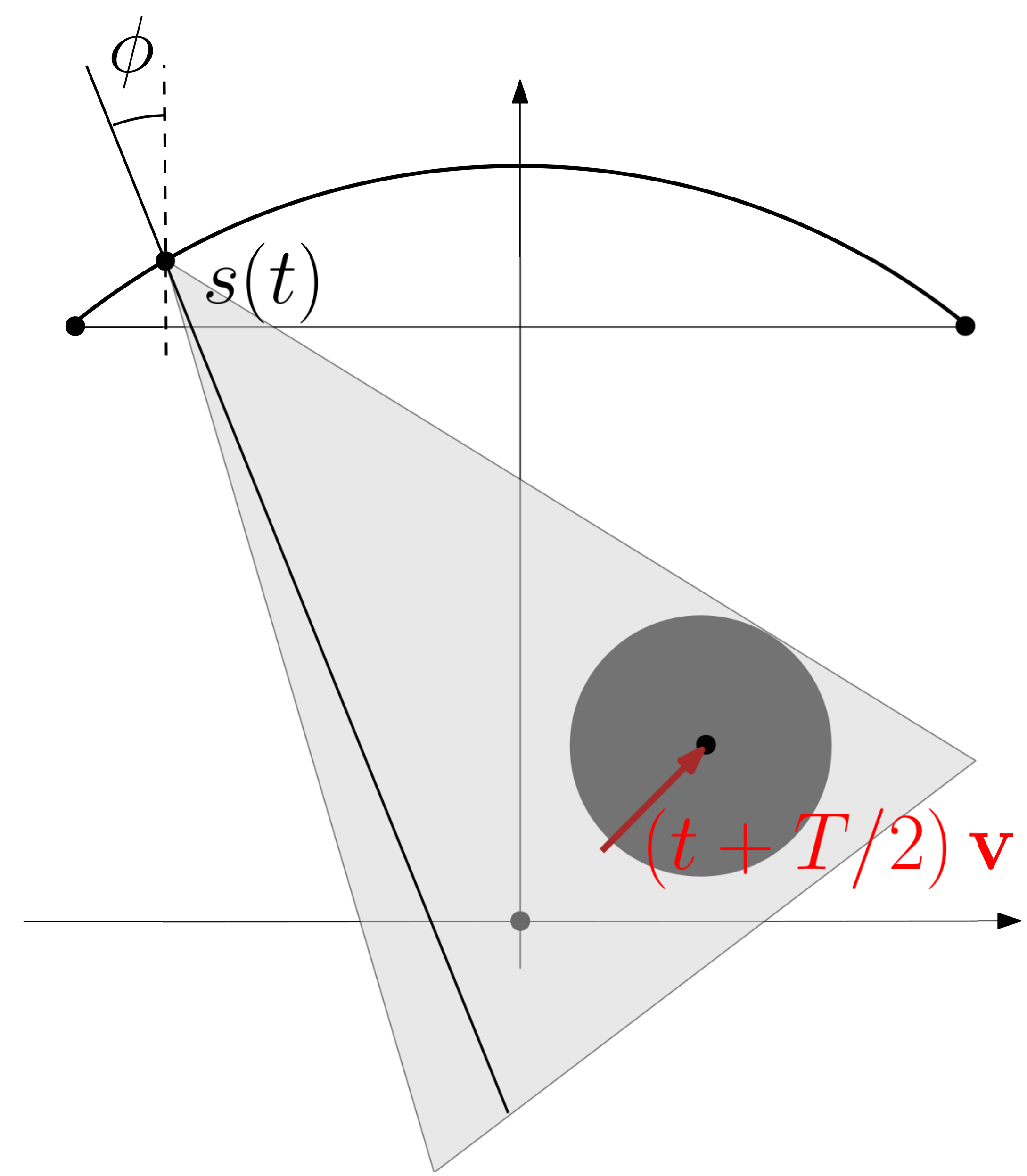
$$(\mathcal{F}\mu)(t,\phi) = \int_0^{+\infty} \mu\left(s(t) + l\left[\sin\phi, -\cos\phi\right]\right) dl$$

Moving object

$$\mu_{\mathbf{v}}(t,\mathbf{x}) = \mu\left(\mathbf{x} - \left(t + \frac{T}{2}\right)\mathbf{v}\right)$$

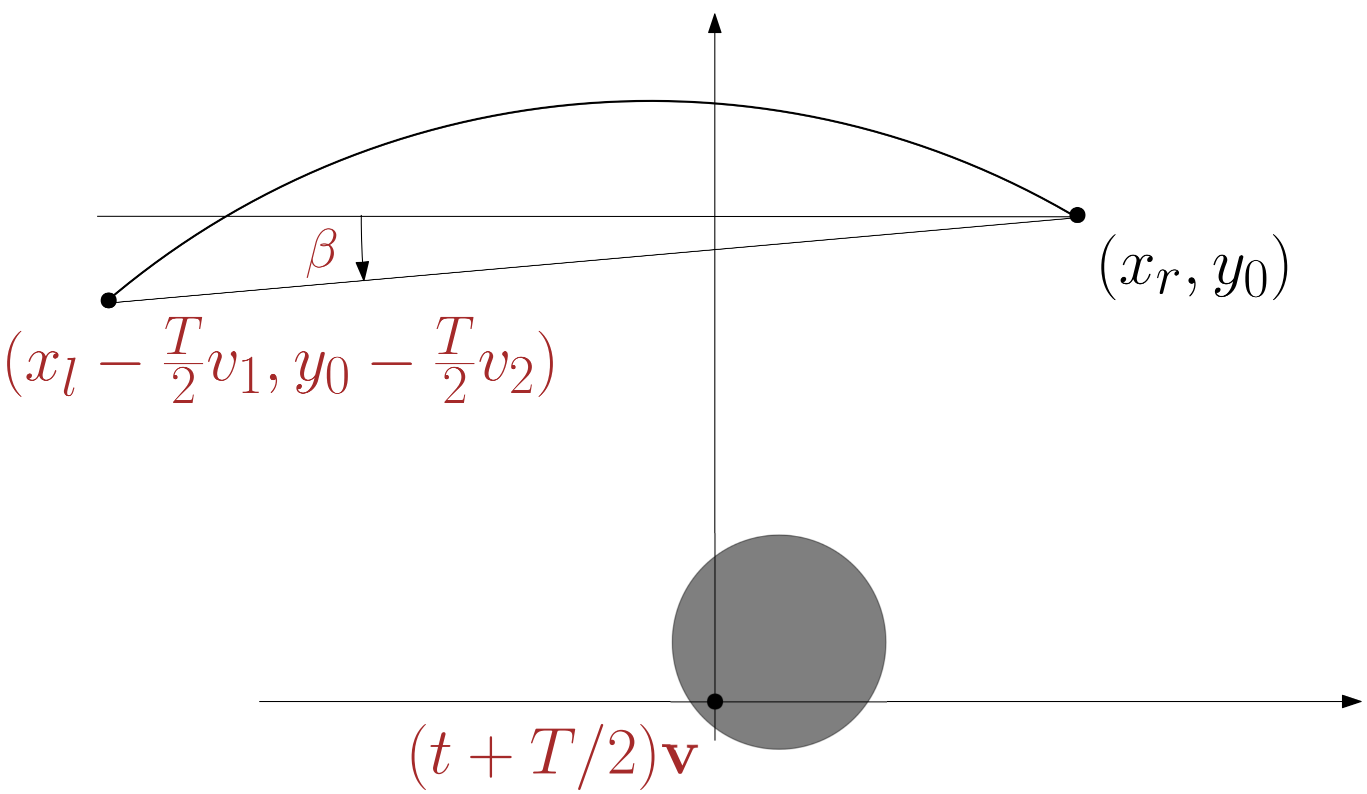
Fan-beam projection data of a moving object

$$(\mathcal{F}_{\mathbf{v}}\mu)(t,\phi) = (\mathcal{F}\mu_{\mathbf{v}})(t,\phi)$$



CHANGE OF REFERENCE FRAME

Translation only

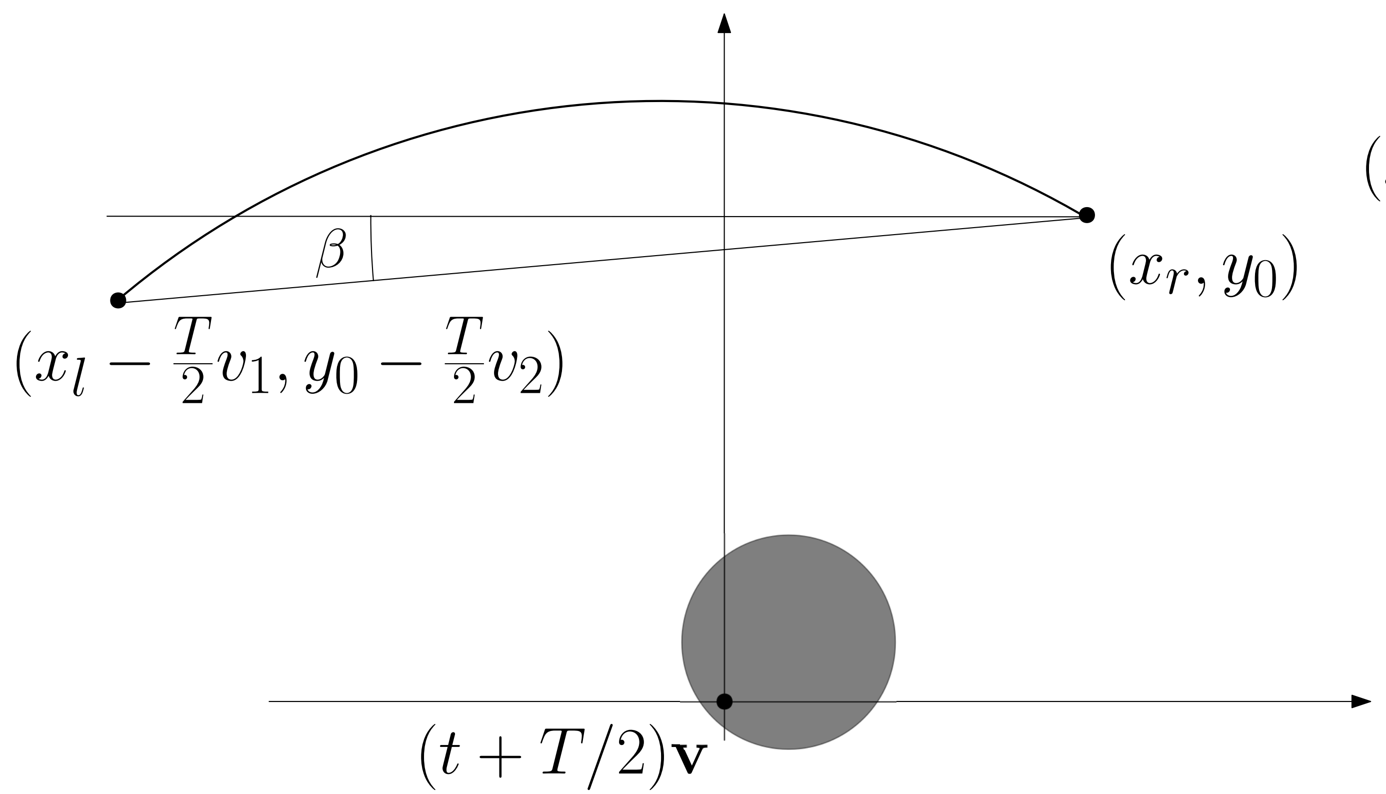


$$\beta = \arctan\left(\frac{Tv_2}{2R_0\sin(\omega T/2) + Tv_1}\right)$$

$\mathcal{R}_\beta, \mathcal{R}_{-\beta}$: rotation matrices

CHANGE OF REFERENCE FRAME

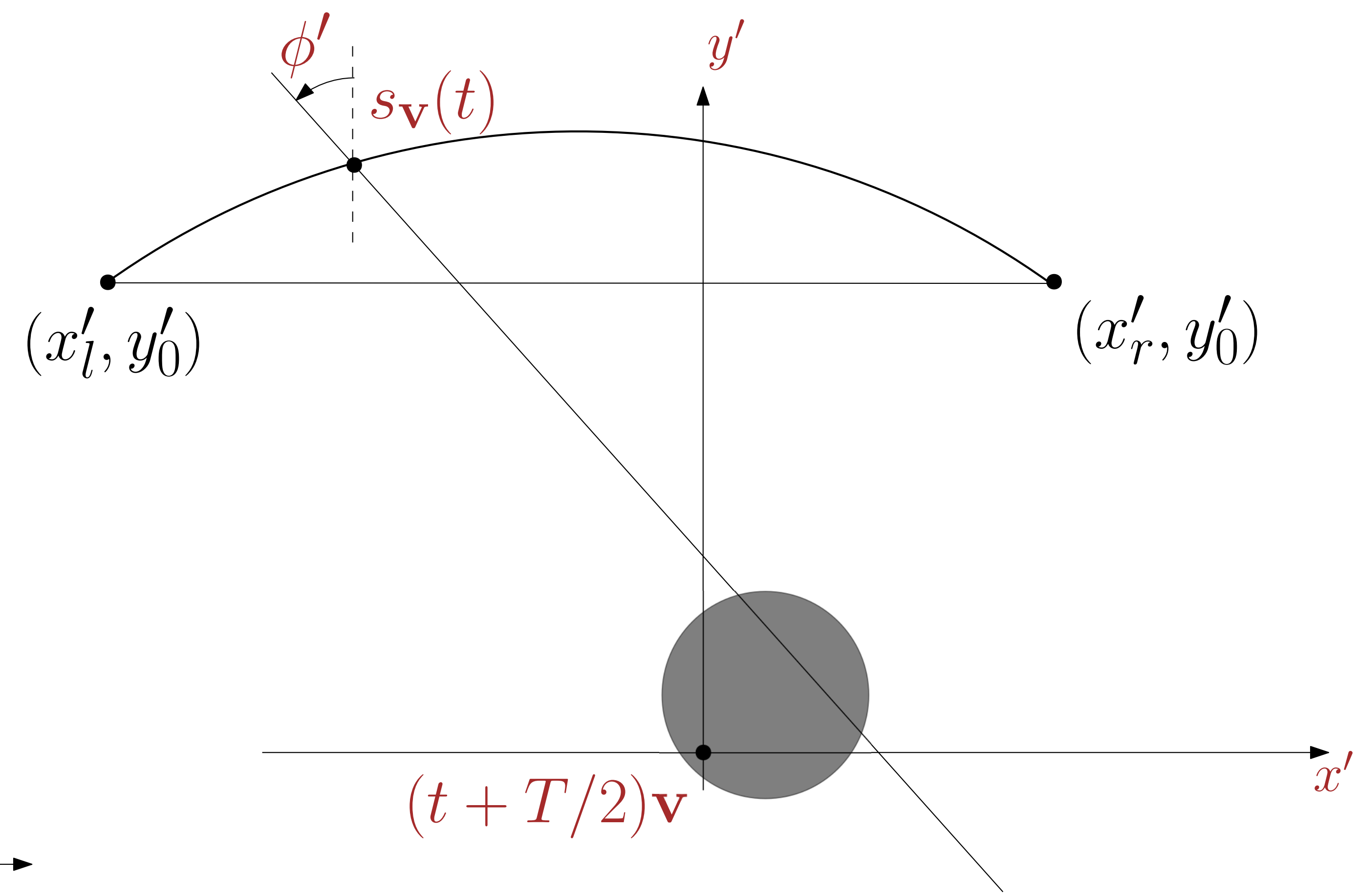
Translation only



$$\beta = \arctan\left(\frac{Tv_2}{2R_0\sin(\omega T/2) + Tv_1}\right)$$

$\mathcal{R}_\beta, \mathcal{R}_{-\beta}$: rotation matrices

Translation and rotation

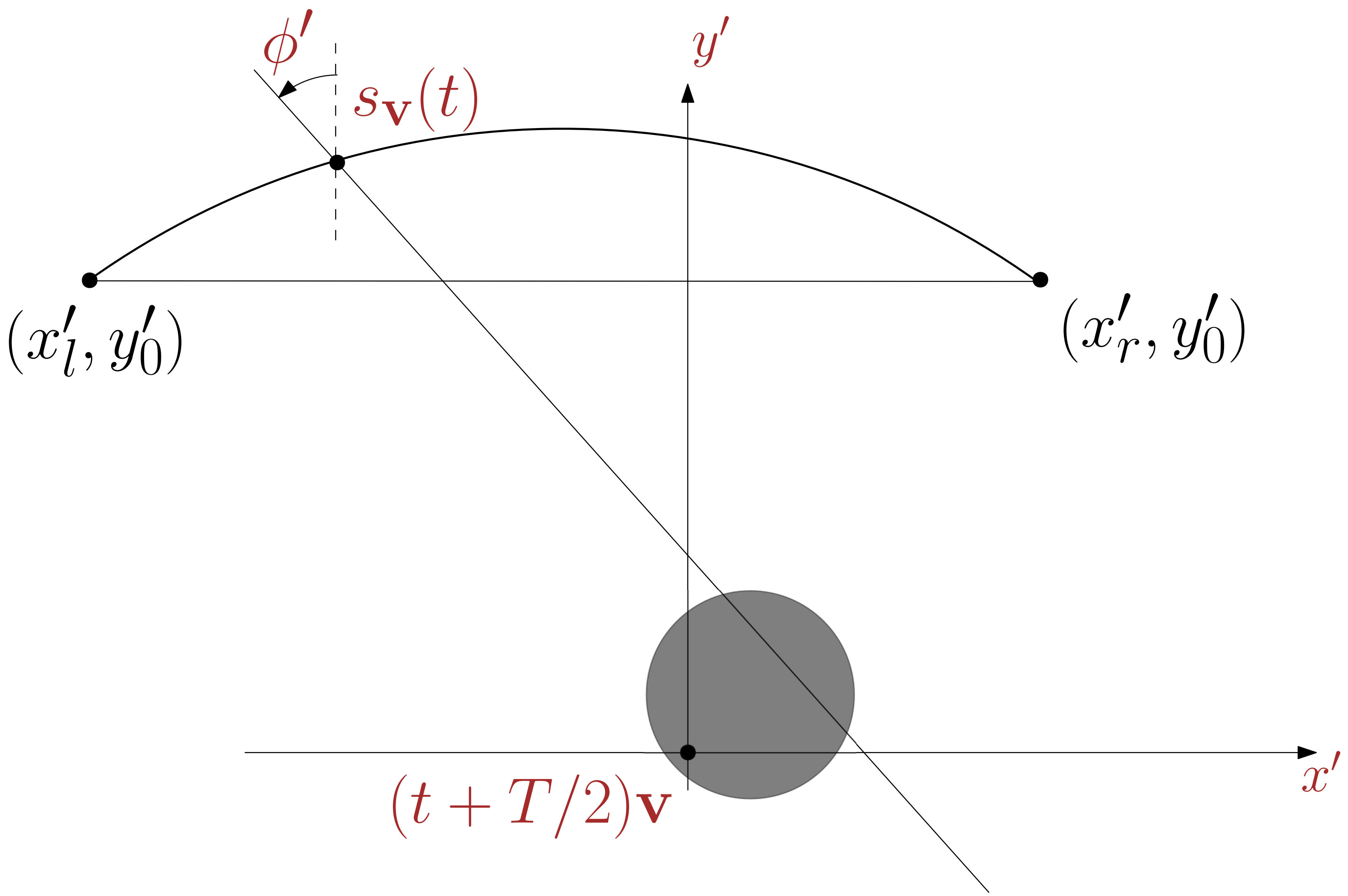


$$(x', y') = \mathcal{R}_{-\beta}\left((x, y) - (t + T/2)\mathbf{v}\right)$$

$$s_{\mathbf{v}}(t) = \mathcal{R}_{-\beta}\left(s(t) - (t + T/2)\mathbf{v}\right)$$

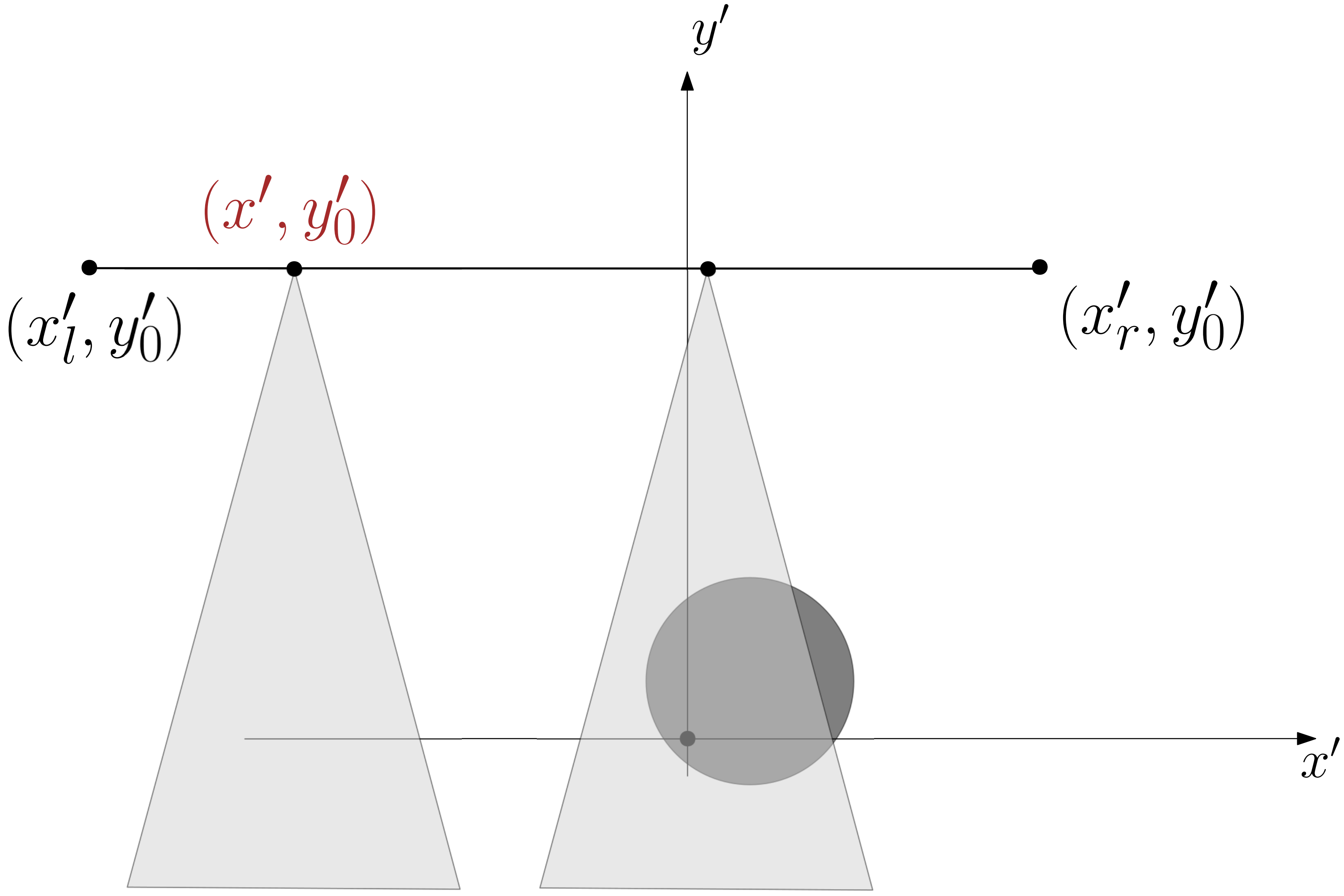
$$\phi' = \phi - \beta$$

CHANGE OF REFERENCE FRAME



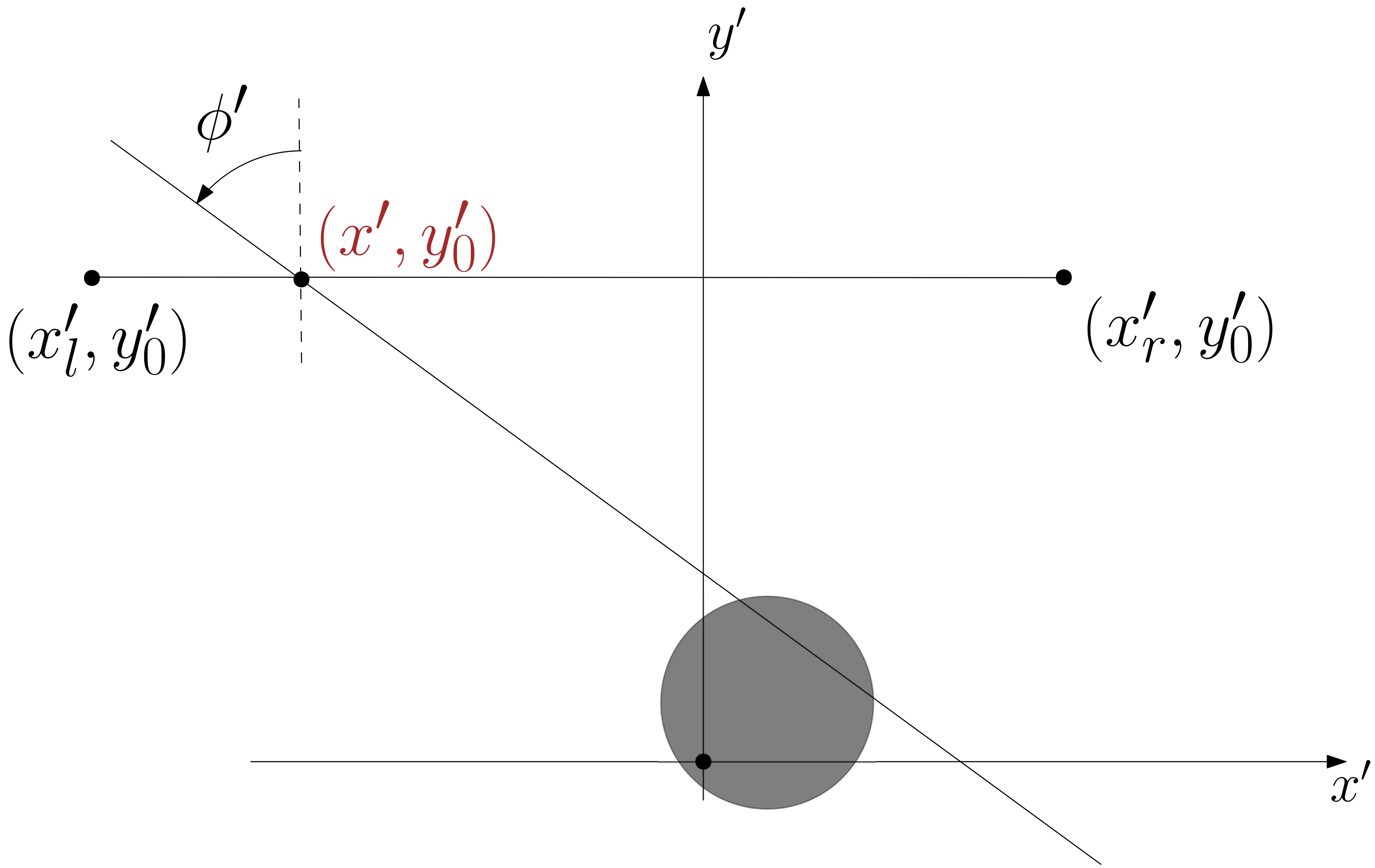
$$(\mathcal{F}_{\mathbf{v}}\mu)(t,\phi) = \int_0^{+\infty} \mu \circ \mathcal{R}_{\beta}\left(s_{\mathbf{v}}(t) + l[\sin(\phi - \beta), -\cos(\phi - \beta)]\right) dl$$

DERIVATION OF DATA CONSISTENCY CONDITIONS



Idea: $x' \mapsto \int_{-\pi/2}^{\pi/2} \tilde{\mathcal{F}}\mu(x', \phi) \frac{\tan^n \phi}{\cos \phi} d\phi \in \mathbb{R}_n[x']$

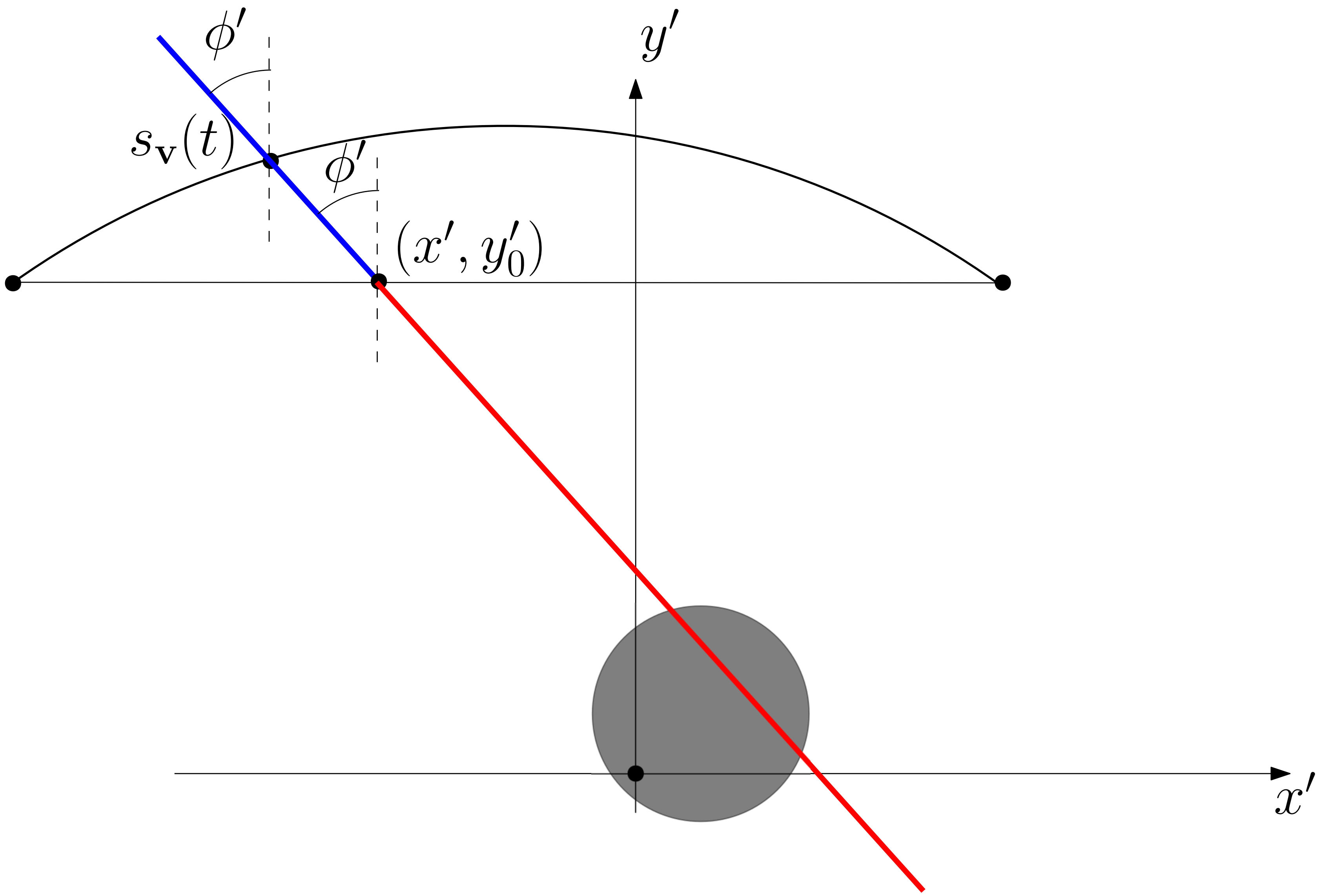
DERIVATION OF DATA CONSISTENCY CONDITIONS



« Virtual » fan-beam projections:

$$(\tilde{\mathcal{F}}\mu)(x', \phi') = \int_0^{+\infty} \mu \circ \mathcal{R}_\beta \left((x', y'_0) + l[\sin\phi', -\cos\phi'] \right) dl$$

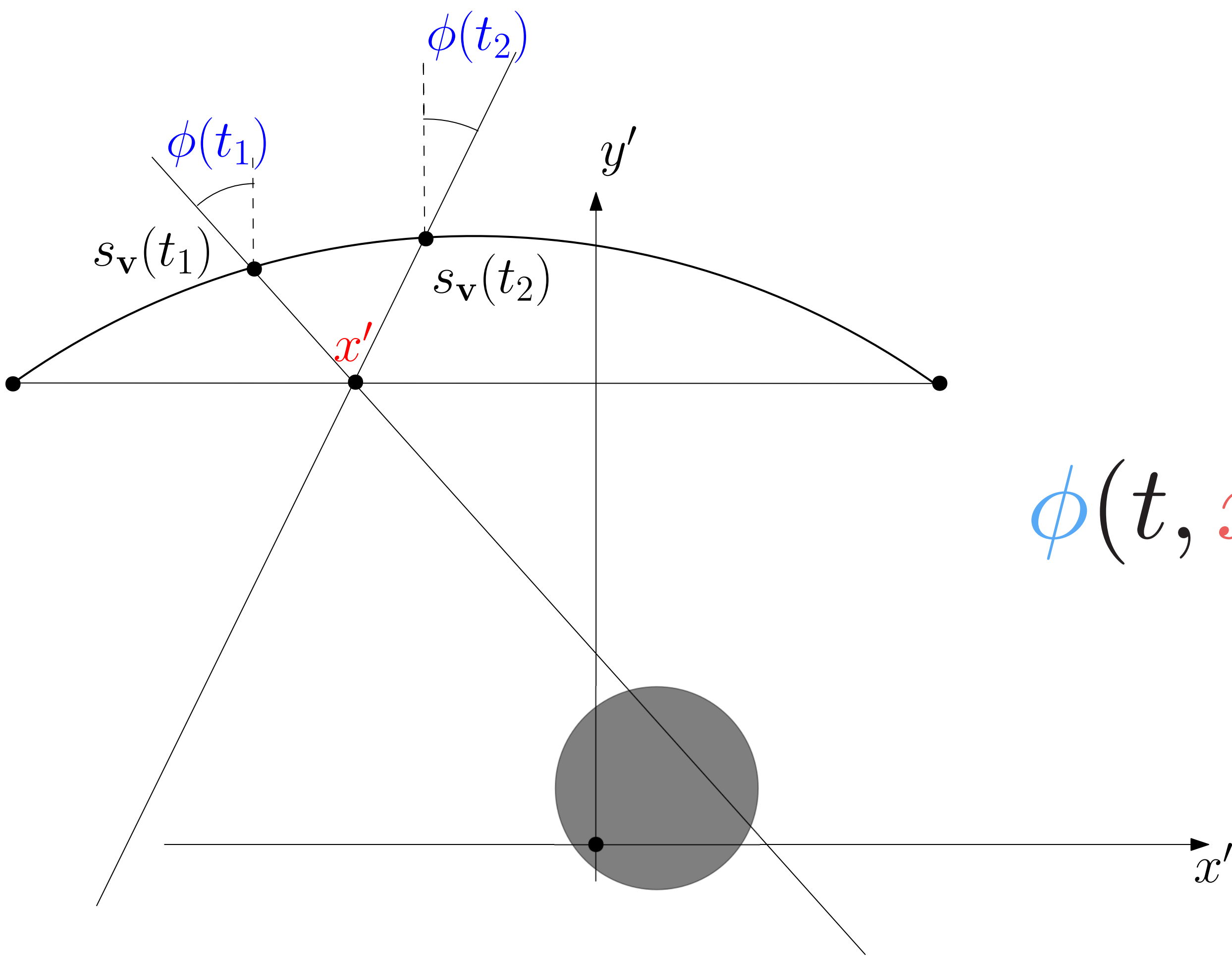
DERIVATION OF DATA CONSISTENCY CONDITIONS



Lemma: $x' = s_{1,\mathbf{v}}(t) + \tan\phi' \left(s_{2,\mathbf{v}}(t) - y_0'\right)$

$$\Longrightarrow\ (\tilde{\mathcal{F}}\mu)(x', \phi') = (\mathcal{F}\mu_{\mathbf{v}})(t, \phi)$$

DERIVATION OF DATA CONSISTENCY CONDITIONS



$$\phi(t, x') = \arctan\left(\frac{x' - s_{1,\mathbf{v}}(t)}{s_{1,\mathbf{v}}(t) - y'_0}\right)$$

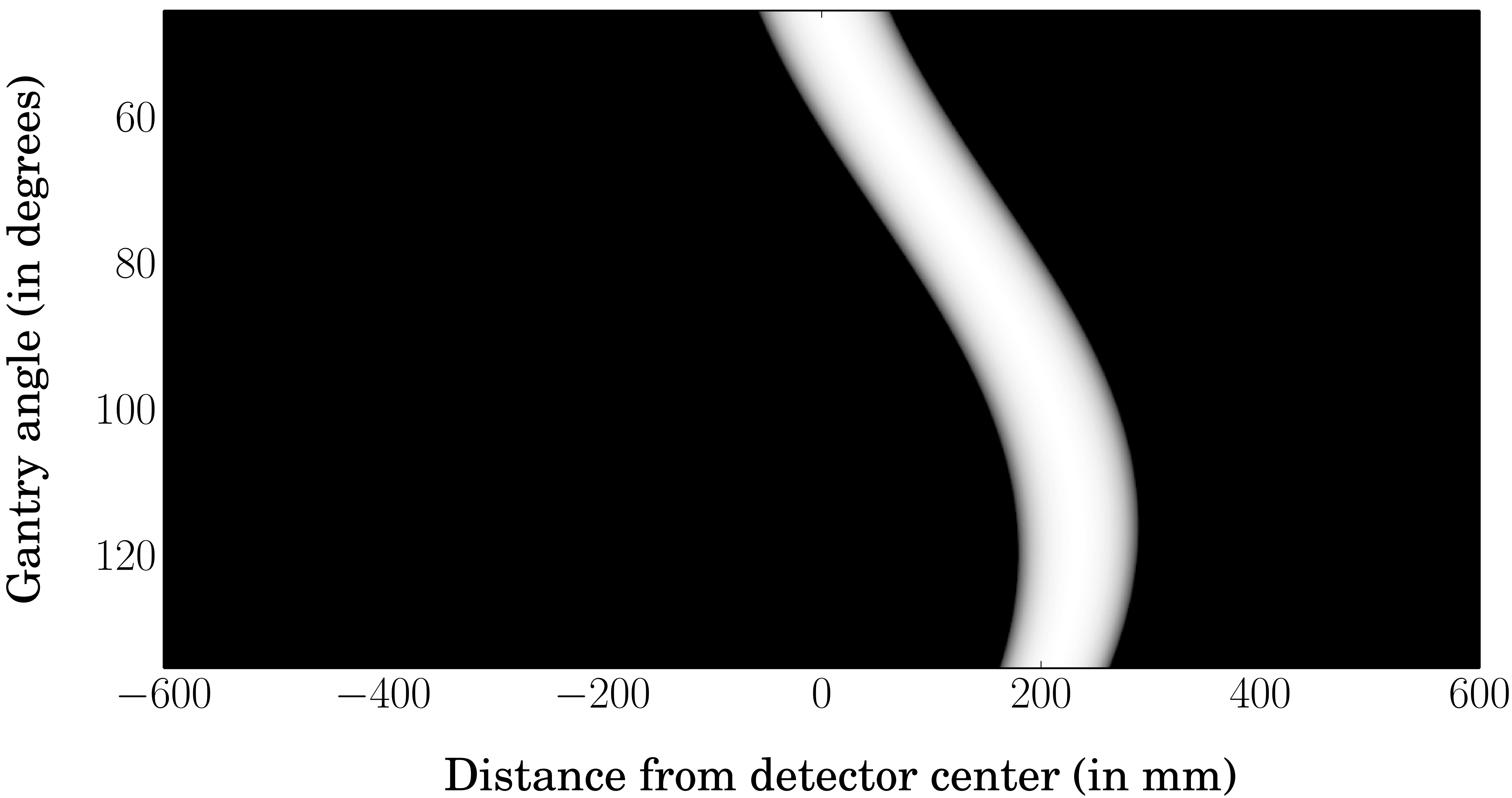
Theorem: the following function is a polynom of order n

$$x' \mapsto \int_{-T/2}^{T/2} \left(\mathcal{F}_{\mathbf{v}}\mu\right)\left(t, \phi(t, x')\right) \tan^n\left[\phi(t, x')\right] \cos\left[\phi(t, x')\right] \frac{\partial \phi}{\partial t} dt$$

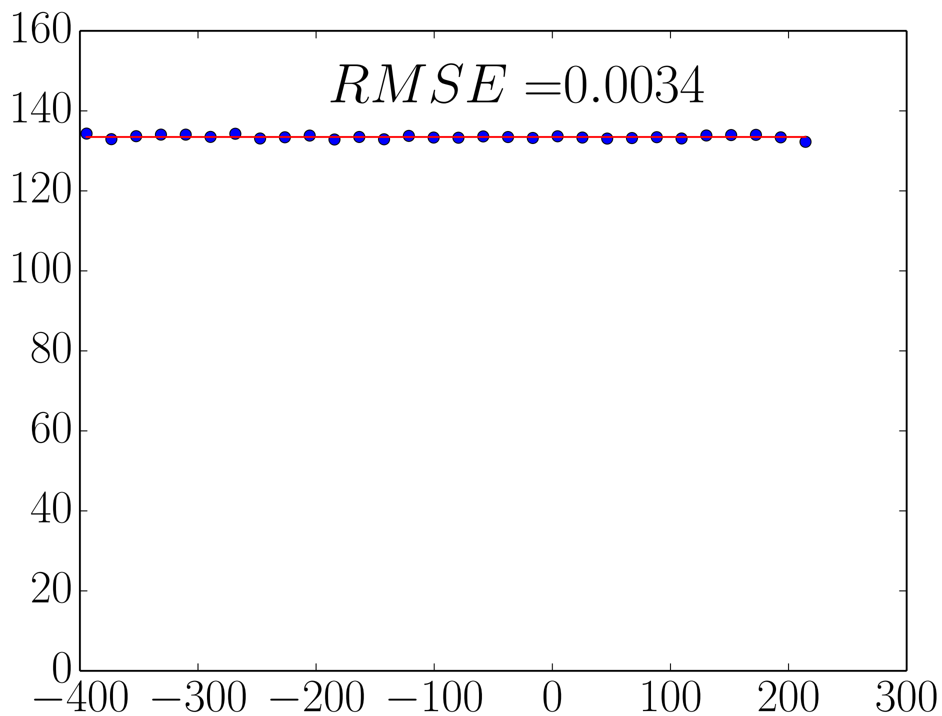
SUMMARY

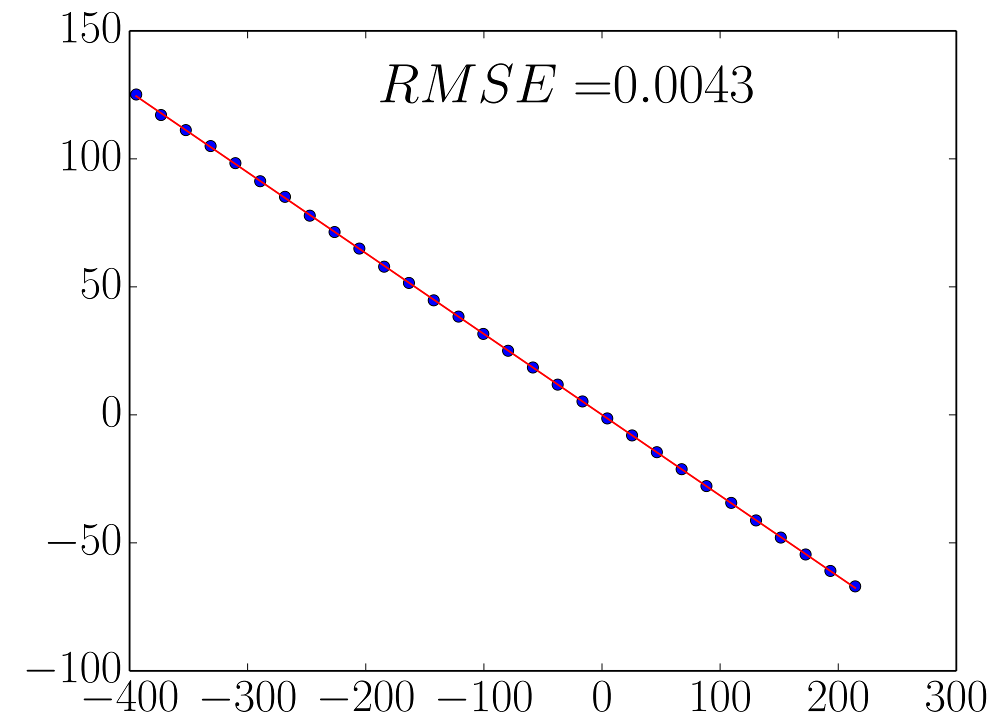
- Context
- Theory
- **Numerical simulations**
  - *Data consistency conditions*
  - *Application: estimation of velocity*
- Conclusion

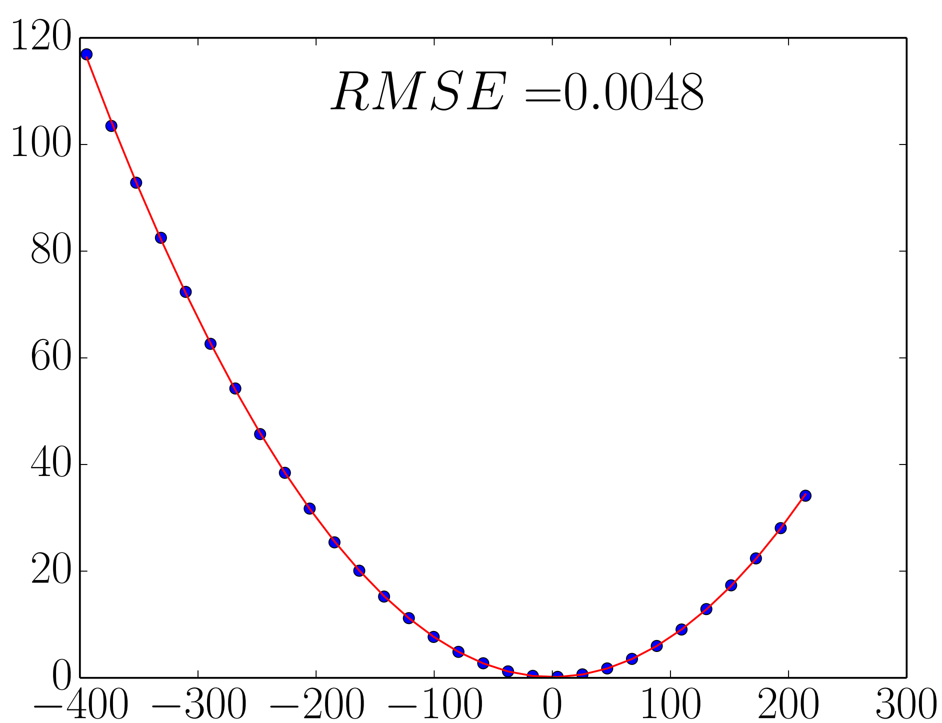
EXAMPLE WITH A DISK TRANSLATING ALONG THE X-AXIS

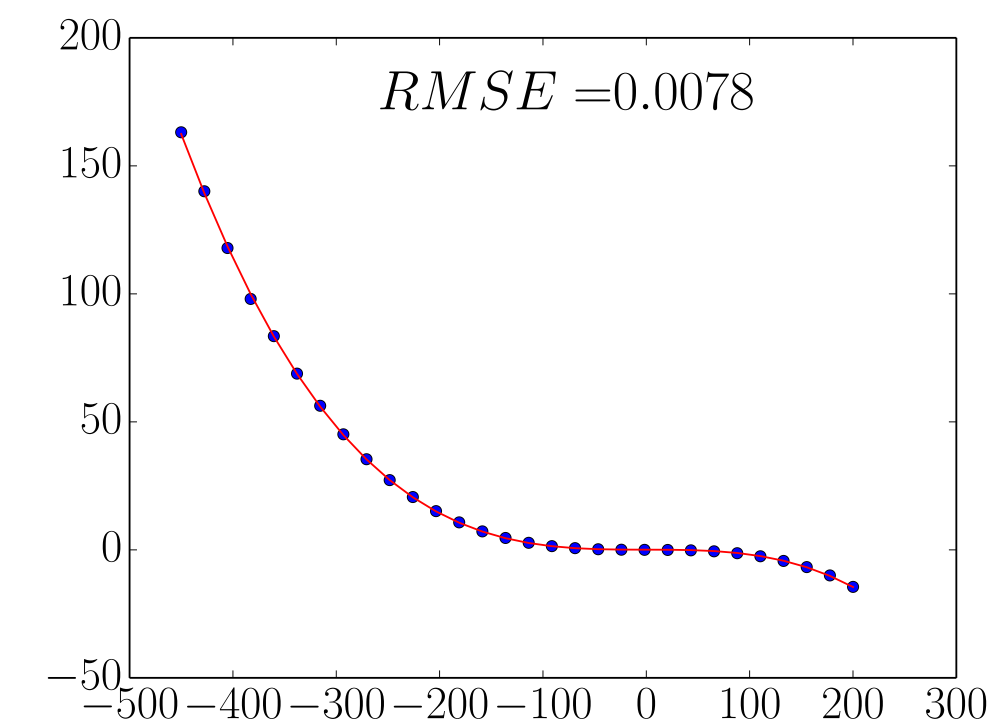


EXAMPLE WITH A DISK TRANSLATING ALONG THE X-AXIS



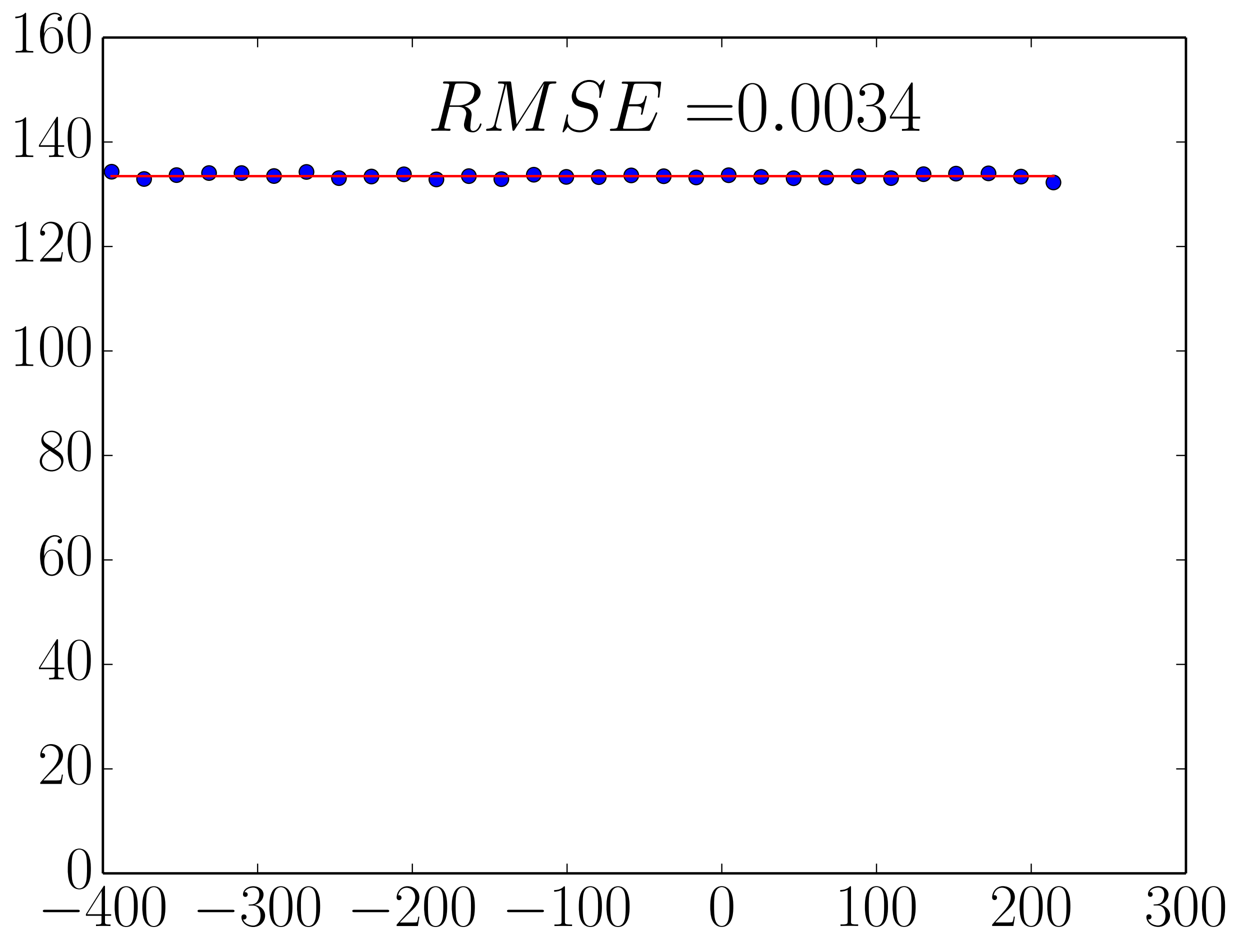




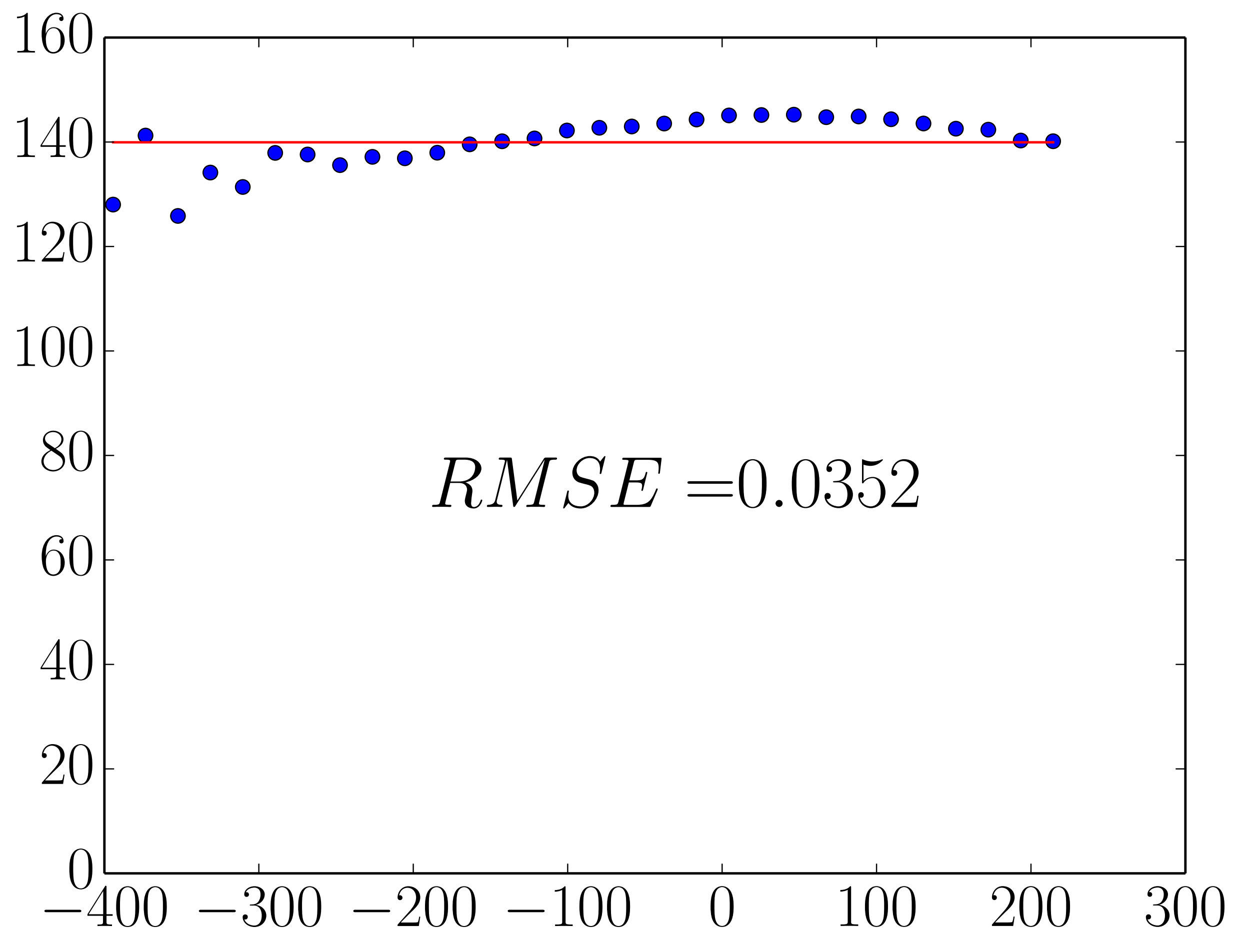


APPLICATION: ESTIMATION OF VELOCITY

True velocity (2 mm/s)



Wrong velocity (0 mm/s)



=> minimize cost function $\mathcal{J}(v) = \mathrm{RMSE}(v)^2$

APPLICATION: ESTIMATION OF VELOCITY

Minimization of cost function $\mathcal{J}(v) = \mathrm{RMSE}(v)^2$

True velocity = 2 mm/s

n	Estimated velocity (in mm/s)	Relative error
0	1.95	2.3 %
1	1.95	2.5 %
2	1.82	8.8 %
3	3.88	94.2 %

SUMMARY

- Context
- Theory
- Numerical simulations
- **Conclusion**
  - *Summary*
  - *Perspectives*

SUMMARY

- Data consistency conditions of **fan-beam** projections of a **translating object**, with sources along an **arc of a circle**
- Idea of the **proof**: change of reference frame, rebinning, change of variables into integration
- **Numerical illustration** with velocity along x-axis
- **Estimation of velocity** with optimization process

PERSPECTIVES

- Numerical simulations with any velocity vector (**not only along x-axis**): *implemented, debugging in progress*
- Improve **optimization process**: *in progress*
- Rigid motion (add **rotation**), **continuous** motion (discretization)
- **Experimentations** with phantom and/or subject