

Consistency of Fanbeam Projections of a Translating Object Along an Arc of a Circle

Abstract

This note aims at extending the results of [1] to the case of a translating object.

1 Introduction

2 Theory

2.1 Problem under consideration

Let us begin with some notations and definitions. We will consider an object in \mathbb{R}^2 to be imaged by a fanbeam source that follows an arc of circle with center O and radius R_0 (see Figure 1, left). The object will be identified with its density function $\mathbf{x} \mapsto \mu(\mathbf{x}) \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. The angular velocity of the source will

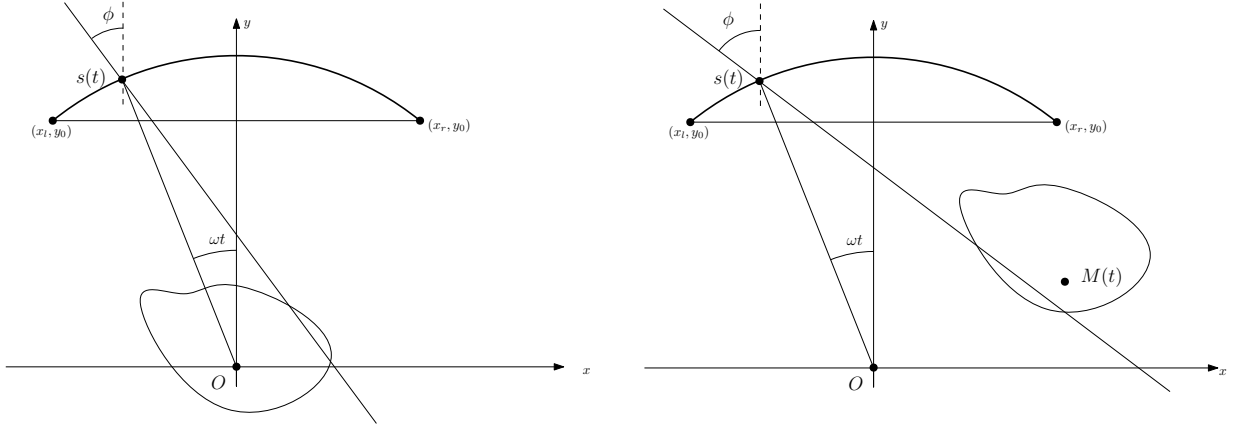


Figure 1: Problem under consideration. The source point $s(t)$ follows the arc of circle depicted in bold. This latter has center O and radius R_0 .

be denoted ω , and the time t will range from $-T/2$ to $T/2$, where $T > 0$. Hence, if we denote $s(t)$ the position of the source at time t , one has

$$s(t) = (-R_0 \sin(\omega t), R_0 \cos(\omega t)). \quad (2.1)$$

Furthermore, we will denote $s(T/2) = (x_l, y_l)$ (resp. $s(-T/2) = (x_r, y_r)$) the extreme left (resp. right) position of the source. Since $y_l = y_r = R_0 \cos(\omega T/2)$, we will call y_0 this common value. In the following, we will suppose that $\text{supp}(\mu)$ lies in the half-space $\{y < y_0\}$, and that $y_0 > 0$ (i.e. $0 < \omega T < \pi$).

We will suppose that at any time t rays are simultaneously emitted from the source $s(t)$ with angle ϕ ranging from $-\pi/2$ to $\pi/2$. With this setup in mind, we can define the operator giving the acquired data from the object.

Definition 1. The fanbeam projection data of an object with density function μ is a function $(t, \phi) \mapsto T\mu(t, \phi)$ defined by

$$(T\mu)(t, \phi) = \int_0^{+\infty} \mu(s(t) + l[\sin \phi, -\cos \phi]) dl, \quad (2.2)$$

where $t \in [-T/2, T/2]$, $\phi \in [-\pi/2, \pi/2]$ and $s(t)$ is given by (2.1). The operator $\mu \mapsto T\mu$ is called the fanbeam projection operator.

Now let us suppose that the object is translating along a line with a constant velocity vector $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ (see Figure 1, right). In other words, if we denote $M_{\mathbf{v}}(t)$ its center of mass at any time t , we have

$$M_{\mathbf{v}}(t) = \left(\left(t + \frac{T}{2} \right) v_1, \left(t + \frac{T}{2} \right) v_2 \right) \quad (2.3)$$

The density function of the object now depends on both the space variable $\mathbf{x} \in \mathbb{R}^2$ and the time t . If we denote it $\mu_{\mathbf{v}}$, we have

$$\mu_{\mathbf{v}}(t, \mathbf{x}) = \mu(\mathbf{x} - M_{\mathbf{v}}(t)). \quad (2.4)$$

In this regard, the fanbeam projection data will be modified in the following way.

Definition 2. The fanbeam projection data of a translating object with density function μ and translating velocity vector \mathbf{v} is given by

$$(T_{\mathbf{v}}\mu)(t, \phi) = (T\mu_{\mathbf{v}}(t, \cdot))(t, \phi). \quad (2.5)$$

The aim of this note is to derive data consistency conditions (DCCs) from (2.5), in order to retrieve the velocity vector \mathbf{v} from the knowledge of $T_{\mathbf{v}}$.

2.2 Derivation of DCCs

In order to derive DCCs, we will first change our frame of reference, from (O, x, y) to $(M(t), x', y')$, so that the object is at the center and the line between the start point and the end point of the source is still parallel to the x' -axis (see Figure 2, right). In other words, we are performing the following change of variables

$$(x, y) \leftrightarrow (x', y') = \mathcal{R}_{\beta}((x, y) - M_{\mathbf{v}}(t)), \quad (2.6)$$

where \mathcal{R}_{β} is the rotation of angle β . This latter is the angle depicted in Figure 2 (left) and is given by

$$\beta = \arctan \left(\frac{Tv_2}{2R_0 \sin(\omega T/2) + Tv_1} \right). \quad (2.7)$$

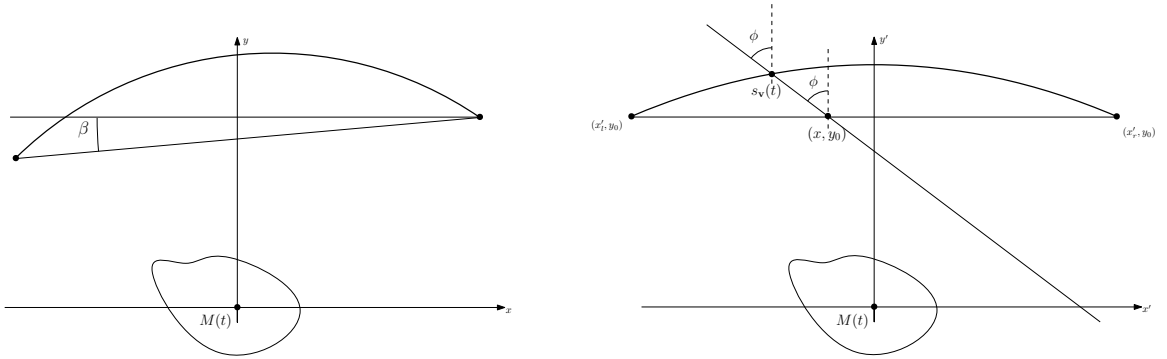


Figure 2: Change of frame: the object is now at center of the coordinates system. Left, only translation of the center of frame; right, after rotation of angle β . Note that the angle ϕ is the same as in Figure 1, right.

In this frame, the coordinates of the source are given by $s_{\mathbf{v}}(t) = \mathcal{R}_{\beta}(s(t) - M_{\mathbf{v}}(t))$. With this in mind, the data is given by the following formula

$$(T_{\mathbf{v}}\mu)(t, \phi) = \int_0^{+\infty} \mu'(s_{\mathbf{v}}(t) + l[\sin \phi, -\cos \phi]) dl, \quad (2.8)$$

where $\mu'(\mathbf{x}') = \mu(\mathbf{x})$, i.e. the function μ' takes its arguments from the space $(M(t), x', y')$.

In other words, we are now dealing with a fixed object whose density function is given by μ' illuminated by a source following an arc of a cycloid in the frame $(M(t), x', y')$ (see Figure 2, right).

Here, the extreme points $s_{\mathbf{v}}(-T/2)$ and $s_{\mathbf{v}}(T/2)$ have the same y' -coordinate y'_0 . We will call x'_l (resp. x'_r) the x' -coordinates of $s_{\mathbf{v}}(T/2)$ (resp. $s_{\mathbf{v}}(-T/2)$). This allows us to define what we call the *virtual fanbeam projection* from a point (x', y'_0) .

Definition 3. For any point x' between x'_l and x'_r , for any angle $\phi \in [-\pi/2, \pi/2]$, the virtual fanbeam projection of the object μ is defined by

$$\left(\tilde{T}\mu\right)(x', \phi) = \int_0^{+\infty} \mu((x', y'_0) + l[\sin \phi, -\cos \phi]) dl. \quad (2.9)$$

This is called *virtual* since it does not correspond to an actual position of the source.

In the following lemma, we will make a connection between the virtual fanbeam projection and the fanbeam projection of the translating object.

Lemma 1. Let us fix a time $t \in [-T/2, T/2]$ and an angle $\phi \in [-\pi/2, \pi/2]$. Let us define

$$x' = s_{1,\mathbf{v}}(t) + \tan \phi (s_{2,\mathbf{v}}(t) - y'_0), \quad (2.10)$$

where, for any time t , $(s_{1,\mathbf{v}}(t), s_{2,\mathbf{v}}(t))$ are the coordinates of $s_{\mathbf{v}}(t)$. Then, one has

$$\left(\tilde{T}\mu\right)(x', \phi) = (T_{\mathbf{v}}\mu)(t, \phi). \quad (2.11)$$

We can now define what are the DCCs of our problem.

Theorem 1. Let us fix a density function μ . For any integer n , there exist a function $(x', t) \mapsto W_{n,v}(t, x') \in C^\infty([-T/2, T/2] \times [x'_l, x'_r])$ such that

$$B_n := x' \mapsto \int_{-T/2}^{T/2} (T_{\mathbf{v}}\mu)(t, \lambda(t)) W_n(x', t, v) dt \in \mathbb{R}_n[X], \quad (2.12)$$

where $\lambda(t)$ is defined by

$$\lambda_t = \arctan \left(\frac{x' + R_0 \sin(\omega t) + (t + \frac{T}{2})v}{R_0 \cos(\omega t) - y'_0} \right). \quad (2.13)$$

Moreover, it is possible to derive $W_{n,v}(t, x')$ analytically.

3 Numerical simulations

3.1 Principles

Let us suppose that we have the projections $(T_{\mathbf{v}}\mu)(t, \phi)$. In order to recover v , we can perform the following optimization procedure. Since $B_n(x')$ in equation (2.12) is supposed to be a polynomial of order $\leq n$, one can minimize

$$\mathcal{J}(v) = \sum_n \|\text{res}(B_n(\cdot, v))\|^2 \quad (3.1)$$

with respect to v , where res is the residual of the projection onto $\mathbb{R}_n[X]$. This will give us the velocity v using only the knowledge of the data $(T_{\mathbf{v}}\mu)(t, \phi)$.

3.2 Application

Here, we will consider an object translating along the x -axis, i.e. $v_2 = 0$. The object under consideration is an ellipse of uniform density, whose axis are 30 and 15 millimeters respectively. The source is rotating around the object with radius $R_0 = 600$ mm, with angle velocity $\omega = 1 \text{ rad} \cdot \text{s}^{-1}$. With a velocity of the ellipse given by $v_1 = 0.3 \text{ mm} \cdot \text{s}^{-1}$, we obtain the sinogram depicted in Figure 3.

With this configuration in mind, the functions $x' \mapsto B_n(x')$ for $n = 0, 1, 2, 3$ can be visualized in Figure 4.

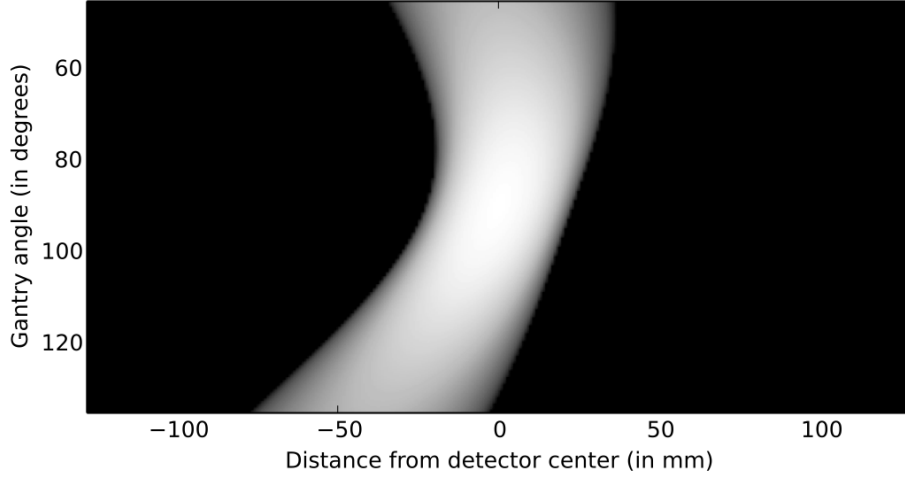


Figure 3: Sinogram of a moving ellipse, with semi-axis $a = 30$ and $b = 15$, translating along the x -axis with velocity $v_1 = 0.3$. It is illuminated by a source at distance $R_0 = 600$, rotating with angle velocity $\omega = 1$.

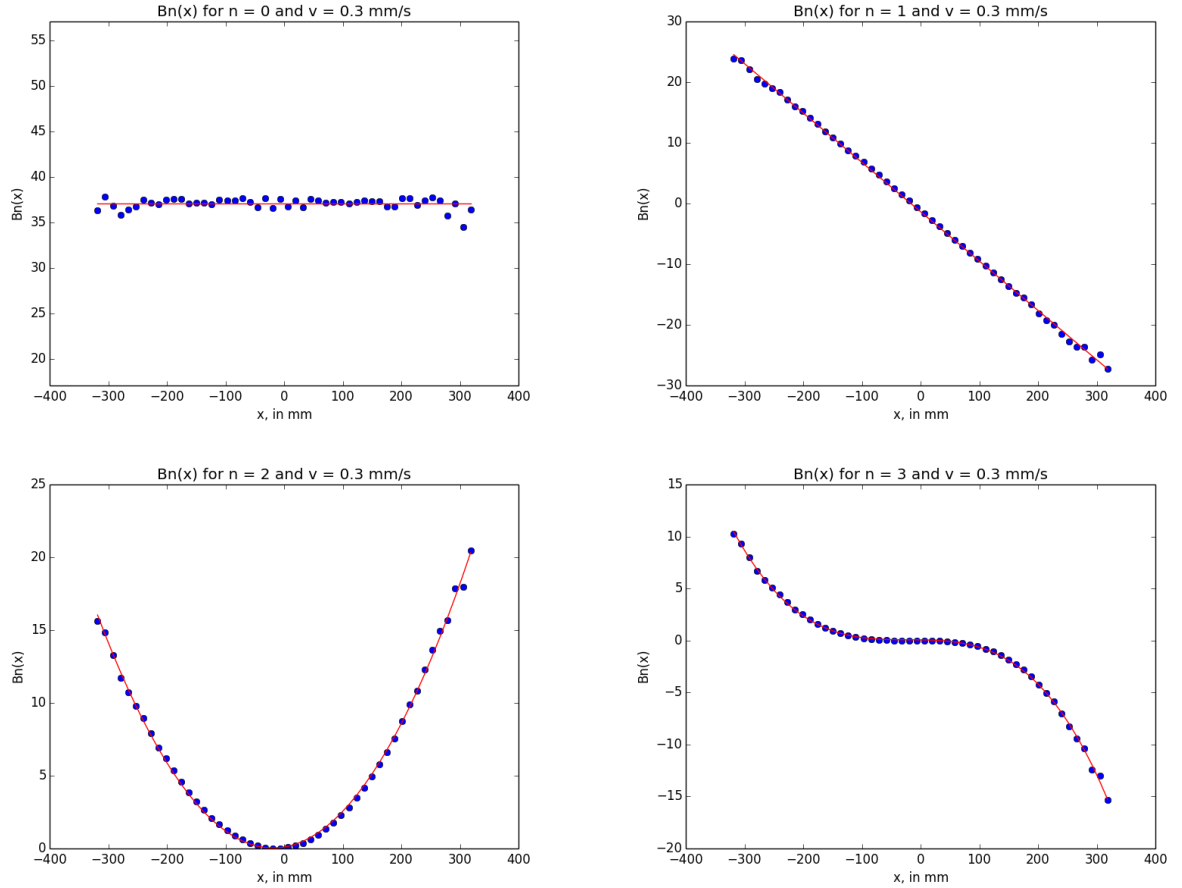


Figure 4: From left to right, and top to bottom, functions $x' \mapsto B_n(x')$ for $n = 0, 1, 2, 3$ respectively.

References

- [1] Rolf Clackdoyle, Michel Defrise, Laurent Desbat, and Johan Nuyts. Consistency of fanbeam projections along an arc of a circle. In *The 13th International Meeting on Fully Three-Dimensional Image*

Reconstruction in Radiology and Nuclear Medicine, pages 415–419, 2015.