Consistency of Fanbeam Projections of a Translating Object Along an Arc of a Circle

Abstract

This note aims at extending the results of [2] to the case of a translating object.

1 Introduction

2 Theory

2.1 Problem under consideration

Let us begin with some notations and definitions. We will consider an object in $\mathbb{R}^2$ to be imaged by a fanbeam source that follows an arc of circle with center O and radius R_0 (see Figure 1, left). The object will be identified with its density function $\mathbf{x} \mapsto \mu(\mathbf{x}) \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. The angular velocity of the source will

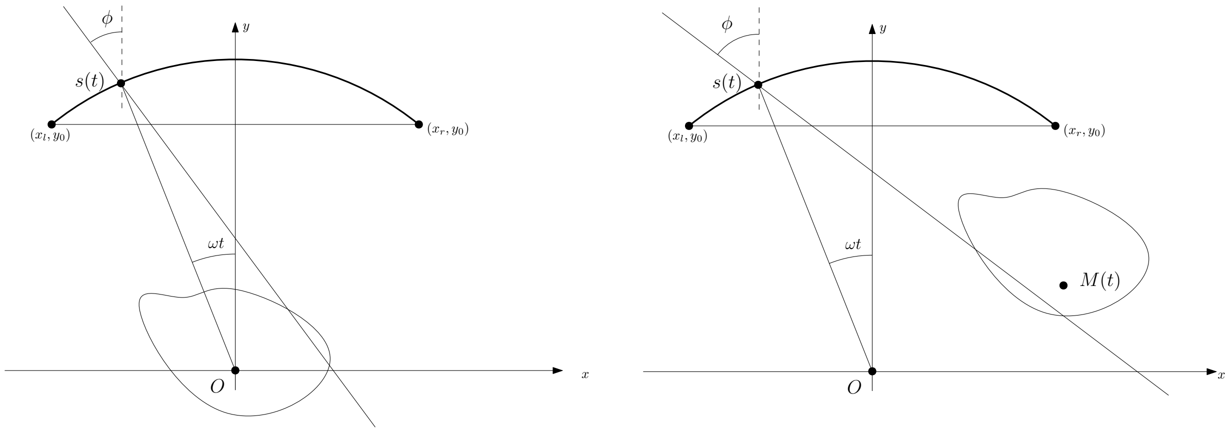

Figure 1: Problem under consideration. The source point $s(t)$ follows the arc of circle depicted in bold. This latter has center O and radius R_0.

be denoted ω, and the time t will range from $-T/2$ to $T/2$, where $T > 0$. Hence, if we denote $s(t)$ the position of the source at time t, one has

$$s(t) = (-R_0 \sin(\omega t), R_0 \cos(\omega t)) \,. \tag{2.1}$$

Furthermore, we will denote $s(T/2) = (x_l, y_l)$ (resp. $s(-T/2) = (x_r, y_r)$) the extreme left (resp. right) position of the source. Since $y_l = y_r = R_0 \cos(\omega T/2)$, we will call y_0 this common value. In the following, we will suppose that $\mathrm{supp}(\mu)$ lies in the half-space $\{y < y_0\}$, and that $y_0 > 0$ (*i.e.* $0 < \omega T < \pi$).

We will suppose that at any time t rays are simultaneously emitted from the source $s(t)$ with angle ϕ ranging from $-\pi/2$ to $\pi/2$. With this setup in mind, we can define the operator giving the acquired data from the object.

Definition 1. *The* fanbeam projection data *of an object with density function μ is a function $(t, \phi) \mapsto T\mu(t, \phi)$ defined by*

$$(T\mu)(t, \phi) = \int_0^{+\infty} \mu\left(s(t) + l\left[\sin\phi, -\cos\phi\right]\right) dl, \tag{2.2}$$

where $t \in [-T/2, T/2]$, $\phi \in [-\pi/2, \pi/2]$ and $s(t)$ is given by (2.1). The operator $\mu \mapsto T\mu$ is called the fanbeam projection operator*.*

Now let us suppose that the object is translating along a line with a constant velocity vector $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ (see Figure 1, right). In other words, if we denote $M_{\mathbf{v}}(t)$ its center of mass at any time t, we have

$$M_{\mathbf{v}}(t) = \left(\left(t + \frac{T}{2} \right) v_1, \left(t + \frac{T}{2} \right) v_2 \right) \tag{2.3}$$

The density function of the object now depends on both the space variable $\mathbf{x} \in \mathbb{R}^2$ and the time t. If we denote it $\mu_{\mathbf{v}}$, we have

$$\mu_{\mathbf{v}}(t, \mathbf{x}) = \mu \left(\mathbf{x} - M_{\mathbf{v}}(t) \right). \tag{2.4}$$

In this regard, the fanbeam projection data will be modified in the following way.

Definition 2. *The* fanbeam projection data of a translating object *with density function μ and translating velocity vector $\mathbf{v}$ is given by*

$$(T_{\mathbf{v}}\mu)(t, \phi) = \left(T\mu_{\mathbf{v}}(t, \cdot) \right)(t, \phi). \tag{2.5}$$

The aim of this note is to derive data consistency conditions (DCCs) from (2.5), in order to retrieve the velocity vector $\mathbf{v}$ from the knowledge of $T_{\mathbf{v}}$.

2.2 Derivation of DCCs

In order to derive DCCs, we will first change our frame of reference, from (O, x, y) to $(M(t), x', y')$, so that the object is at the center and the line between the start point and the end point of the source is still parallel to the x'-axis (see Figure 2, right). In other words, we are performing the following change of variables

$$(x, y) \leftrightarrow (x', y') = \mathcal{R}_\beta \left((x, y) - M_{\mathbf{v}}(t) \right), \tag{2.6}$$

where $\mathcal{R}_\beta$ is the rotation of angle β. This latter is the angle depicted in Figure 2 (left) and is given by

$$\beta = \arctan \left(\frac{Tv_2}{2R_0 \sin(\omega T/2) + Tv_1} \right). \tag{2.7}$$

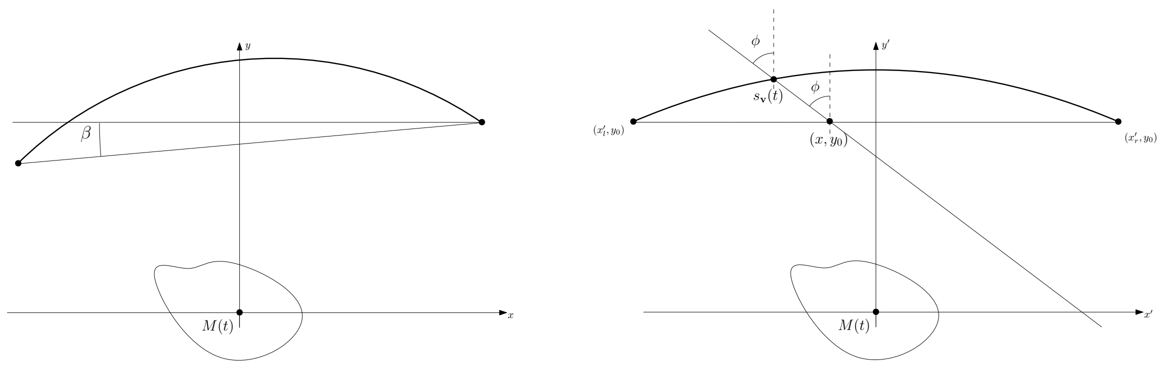

Figure 2: Change of frame: the object is now at center of the coordinates system. Left, only translation of the center of frame; right, after rotation of angle β. Note that the angle ϕ is the same as in Figure 1, right.

In this frame, the coordinates of the source are given by $s_{\mathbf{v}}(t) = \mathcal{R}_\beta \left(s(t) - M_{\mathbf{v}}(t) \right)$. With this in mind, the data is given by the following formula

$$(T_{\mathbf{v}}\mu)(t, \phi) = \int_0^{+\infty} \mu' \left(s_{\mathbf{v}}(t) + l \left[\sin\phi, -\cos\phi \right] \right) dl, \tag{2.8}$$

where $\mu'(\mathbf{x}') = \mu(\mathbf{x})$, *i.e.* the function μ' takes its arguments from the space $(M(t), x', y')$.

In other words, we are now dealing with a fixed object whose density function is given by μ' illuminated by a source following an arc of a cycloid in the frame $(M(t), x', y')$ (see Figure 2, right).

Here, the extreme points $s_{\mathbf{v}}(-T/2)$ and $s_{\mathbf{v}}(T/2)$ have the same y'-coordinate y'_0. We will call x'_l (resp. x'_r) the x'-coordinates of $s_{\mathbf{v}}(T/2)$ (resp. $s_{\mathbf{v}}(-T/2)$). This allows us to define what we call the *virtual fanbeam projection* from a point (x', y'_0).

Definition 3. *For any point x' between x'_l and x'_r, for any angle $\phi \in [-\pi/2, \pi/2]$, the* virtual fanbeam projection *of the object μ is defined by*

$$\left(\tilde{T}\mu\right)(x', \phi) = \int_0^{+\infty} \mu\left((x', y'_0) + l\left[\sin\phi, -\cos\phi\right]\right) dl. \tag{2.9}$$

This is called virtual *since it does not correspond to an actual position of the source.*

In the following lemma, we will make a connection between the virtual fanbeam projection and the fanbeam projection of the translating object.

Lemma 1. *Let us fix a time $t \in [-T/2, T/2]$ and an angle $\phi \in [-\pi/2, \pi/2]$. Let us define*

$$x' = s_{1,\mathbf{v}}(t) + \tan\phi\left(s_{2,\mathbf{v}}(t) - y'_0\right), \tag{2.10}$$

where, for any time t, $(s_{1,\mathbf{v}}(t), s_{2,\mathbf{v}}(t))$ are the coordinates of $s_{\mathbf{v}}(t)$. Then, one has

$$\left(\tilde{T}\mu\right)(x', \phi) = \left(T_{\mathbf{v}}\mu\right)(t, \phi). \tag{2.11}$$

Proof. The idea is that the point (x', y'_0) is the intersection between the line $\{y' = y'_0\}$ and the ray coming from $s_{\mathbf{v}}(t)$ with angle ϕ (see Figure 2). Since $\mathrm{supp}(\mu)$ is under the line $\{y' = y'_0\}$, the integral does not differ if we start from (x', y'_0) or from $s_{\mathbf{v}}(t)$. Indeed, one has

$$\int_0^{+\infty} \mu\left((x', y'_0) + l\left[\sin\phi, -\cos\phi\right]\right) dl = \int_{-\infty}^{+\infty} \mu\left((x', y'_0) + l\left[\sin\phi, -\cos\phi\right]\right) dl. \tag{2.12}$$

Let us now perform the following change of variable

$$l' \leftrightarrow l + \frac{s_{2,\mathbf{v}}(t) - y'_0}{\cos\phi}. \tag{2.13}$$

By definition of x', it gives $x' + l\sin\phi = s_{1,\mathbf{v}}(t) + l'\sin\phi$ and $y'_0 - l\cos\phi = s_{2,\mathbf{v}}(t) - l'\cos\phi$. We then have

$$\int_{-\infty}^{+\infty} \mu\left((x', y'_0) + l\left[\sin\phi, -\cos\phi\right]\right) dl = \int_{-\infty}^{+\infty} \mu\left(s_{\mathbf{v}}(t) + l'\left[\sin\phi, -\cos\phi\right]\right) dl'. \tag{2.14}$$

Restricting the domain of integration from $l' = 0$ to $t = +\infty$ then leads us to the desired result. □

We can now define what are the DCCs of our problem.

Theorem 1. *Let us fix a density function μ. For any integer n, there exist a function $(x', t) \mapsto W_{n,v}(t, x') \in \mathcal{C}^\infty\left([-T/2, T/2] \times [x'_l, x'_r]\right)$ such that*

$$B_n := x' \mapsto \int_{-T/2}^{T/2} \left(T_{\mathbf{v}}\mu\right)(t, \lambda(t))\, W_n(x', t, v) dt \in \mathbb{R}_n[X], \tag{2.15}$$

where $\lambda(t)$ is defined by

$$\lambda_t = \arctan\left(\frac{x' + R_0\sin(\omega t) + \left(t + \frac{T}{2}\right)v}{R_0\cos(\omega t) - y'_0}\right). \tag{2.16}$$

Moreover, it is possible to derive $W_{n,v}(t, x')$ analytically.

Proof. The DCCs derived in [2] heavily rely on a relation between $\tan\phi$ and the angle ωt (see Figure 1). This relation is then differentiated to change variables into the integral in (2.15), which gives the following expression, known to be a polynom according to [1]

$$\int_{-\pi/2}^{\pi/2} \left(\tilde{T}\mu\right)(x', \phi) \frac{\tan^n \phi}{\cos\phi} d\phi. \tag{2.17}$$

We will follow the same path here. First, the formula giving $\tan\phi$ is nearly the same as in [2], *i.e.*

$$\begin{aligned}
\tan\phi &= \frac{x' - s_{\mathbf{v},1}(t)}{s_{\mathbf{v},2}(t) - y_0'} \\
&= \frac{x' + \cos\beta\left(R_0\sin(\omega t) + \left(t + \frac{T}{2}\right)v_1\right) + \sin\beta\left(R_0\cos(\omega t) - \left(t + \frac{T}{2}\right)v_2\right)}{\cos\beta\left(R_0\cos(\omega t) - \left(t + \frac{T}{2}\right)v_2\right) - \sin\beta\left(R_0\sin(\omega t) + \left(t + \frac{T}{2}\right)v_1\right) - y_0'} \\
&:= F(x', t, \mathbf{v}).
\end{aligned}$$

Then, taking its derivative allows us to write the Jacobian for a change of variables from ϕ to t (see Appendix for full detail)

$$\frac{d\phi}{\cos^2\phi} = \frac{\partial F}{\partial t}(x', t, \mathbf{v})dt := J(x', t, \mathbf{v})dt. \tag{2.18}$$

Hence, one can write

$$\begin{aligned}
\frac{\tan^n\phi}{\cos\phi}d\phi &= \tan^n\phi\cos\phi\frac{d\phi}{\cos^2\phi} & (2.19)\\
&= \frac{\left(x' + R_0\sin(\omega t) + \left(t + \frac{T}{2}\right)v\right)^n}{D_{x',t}\left(R_0\cos(\omega t) - y_0'\right)^{n-1}} J(x', t, \mathbf{v})dt & (2.20)\\
&:= W_n(x', t, \mathbf{v})dt, & (2.21)
\end{aligned}$$

where the term $D_{x',t}$ in equation (2.20) refers to the distance between the source point $s_{\mathbf{v}}(t)$ and the virtual point (x', y_0').

With the help of lemma 1, the change of variable from t to ϕ then leads us to

$$B_n(x') = \int_{-\pi/2}^{\pi/2} \left(\tilde{T}\mu\right)(x', \phi) \frac{\tan^n\phi}{\cos\phi} d\phi, \tag{2.22}$$

which is known to be a polynom of degree at most n according to [1] as mentioned above. □

3 Numerical simulations

Let us suppose that we have the projections $g(x', \phi)$. In order to recover v, we can perform the following optimization procedure. Since $B_n(x')$ in equation (2.15) is supposed to be a polynom of order $\leq n$, one can minimize

$$\mathcal{J}(v) = \sum_n \left\|\operatorname{res}\left(B_n(x', v)\right)\right\|^2 \tag{3.1}$$

with respect to v, where res is the residual of the projection onto $\mathbb{R}_n[X]$. The minimization procedure can be done using the gradient of $\mathcal{J}(v)$, given by

$$\nabla\mathcal{J}(v) = 2\sum_n B_n(x', v)\int_{-T/2}^{T/2} g(t, \phi)\frac{\partial W_n}{\partial v}(x', t, v)dt. \tag{3.2}$$

References

[1] Rolf Clackdoyle. Necessary and sufficient consistency conditions for fanbeam projections along a line. *IEEE Transactions on Nuclear Science*, 60(3):1560–1569, 2013.

[2] Rolf Clackdoyle, Michel Defrise, Laurent Desbat, and Johan Nuyts. Consistency of fanbeam projections along an arc of a circle. In *The 13th International Meeting on Fully Three-Dimensional Image Reconstruction in Radiology and Nuclear Medicine*, pages 415–419, 2015.

4 Appendix

Here we will put all the "dirty" computations, so this is draft for the moment...

$$\frac{d\phi}{\cos^2\phi} = \frac{\left(R_0\omega\cos(\omega t) + v\right)\left(R_0\cos(\omega t) - y_0'\right) + R_0\omega\sin(\omega t)\left(x' + R_0\sin(\omega t) + \left(t + \frac{T}{2}\right)v\right)}{\left(R_0\cos(\omega t) - y_0'\right)^2}dt \tag{4.1}$$

$$= \frac{R_0^2\omega - vy_0' + R_0\cos(\omega t)(v - \omega y_0') + R_0\omega\sin(\omega t)(x' + \left(t + \frac{T}{2}\right)v)}{\left(R_0\cos(\omega t) - y_0'\right)^2}dt \tag{4.2}$$

$$\tag{4.3}$$