Consistency of Fanbeam Projections of a Translating Object Along an Arc of a Circle

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***Abstract*—This note aims at extending the results of [1] to the case of a translating object.**

I. Introduction

II. Theory

A. Problem under consideration

Let us begin with some notations and definitions. We will consider an object in $\mathbb{R}^2$ to be imaged by a fanbeam source that follows an arc of circle with center O and radius R_0 (see Figure 1, left). The object will be identified with its density function $\mathbf{x} \mapsto \mu(\mathbf{x}) \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. The angular velocity of the source will be denoted ω, and the time t will range from $-T/2$ to $T/2$, where $T > 0$. Hence, if we denote $s(t)$ the position of the source at time t, one has

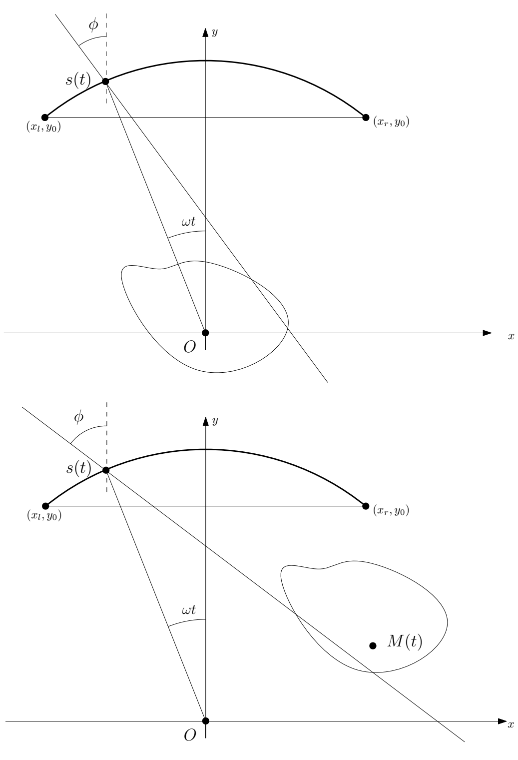

Fig. 1. Problem under consideration. The source point $s(t)$ follows the arc of circle depicted in bold. This latter has center O and radius R_0.

$$s(t) = \left(-R_0 \sin(\omega t), R_0 \cos(\omega t)\right). \qquad \text{(II.1)}$$

Furthermore, we will denote $s(T/2) = (x_l, y_l)$ (resp. $s(-T/2) = (x_r, y_r)$) the extreme left (resp. right) position of the source. Since $y_l = y_r = R_0 \cos(\omega T/2)$, we will call y_0 this common value. In the following, we will suppose that $\mathrm{supp}(\mu)$ lies in the half-space $\{y < y_0\}$, and that $y_0 > 0$ (*i.e.* $0 < \omega T < \pi$).

We will suppose that at any time t rays are simultaneously emitted from the source $s(t)$ with angle ϕ ranging from $-\pi/2$ to $\pi/2$. With this setup in mind, we can define the operator giving the acquired data from the object.

Definition 1. *The* fanbeam projection data *of an object with density function μ is a function $(t, \phi) \mapsto T\mu(t, \phi)$ defined by*

$$(T\mu)(t, \phi) = \int_0^{+\infty} \mu\left(s(t) + l\left[\sin\phi, -\cos\phi\right]\right) dl, \qquad \text{(II.2)}$$

where $t \in [-T/2, T/2]$, $\phi \in [-\pi/2, \pi/2]$ and $s(t)$ is given by (II.1). The operator $\mu \mapsto T\mu$ is called the fanbeam projection operator.

Now let us suppose that the object is translating along a line with a constant velocity vector $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ (see Figure 1, right). In other words, if we denote $M_{\mathbf{v}}(t)$ its center of mass at any time t, we have

$$M_{\mathbf{v}}(t) = \left(\left(t + \frac{T}{2}\right) v_1, \left(t + \frac{T}{2}\right) v_2\right) \qquad \text{(II.3)}$$

The density function of the object now depends on both the space variable $\mathbf{x} \in \mathbb{R}^2$ and the time t. If we denote it $\mu_{\mathbf{v}}$, we have

$$\mu_{\mathbf{v}}(t, \mathbf{x}) = \mu\left(\mathbf{x} - M_{\mathbf{v}}(t)\right). \qquad \text{(II.4)}$$

In this regard, the fanbeam projection data will be modified in the following way.

Definition 2. *The* fanbeam projection data of a translating object *with density function μ and translating velocity vector $\mathbf{v}$ is given by*

$$(T_{\mathbf{v}}\mu)(t, \phi) = \left(T\mu_{\mathbf{v}}(t, \cdot)\right)(t, \phi). \qquad \text{(II.5)}$$

The aim of this note is to derive data consistency conditions (DCCs) from (II.5), in order to retrieve the velocity vector $\mathbf{v}$ from the knowledge of $T_{\mathbf{v}}$.

B. Derivation of DCCs

In order to derive DCCs, we will first change our frame of reference, from (O, x, y) to $(M(t), x', y')$, so that the object is at the center and the line between the start point and the end point of the source is still parallel to the x'-axis (see Figure 2, right). In other words, we are performing the following change of variables

$$(x, y) \leftrightarrow (x', y') = \mathcal{R}_\beta\left((x, y) - M_{\mathbf{v}}(t)\right), \qquad \text{(II.6)}$$

where $\mathcal{R}_\beta$ is the rotation of angle β. This latter is the angle depicted in Figure 2 (left) and is given by

$$\beta = \arctan\left(\frac{T v_2}{2R_0 \sin(\omega T/2) + T v_1}\right). \qquad \text{(II.7)}$$

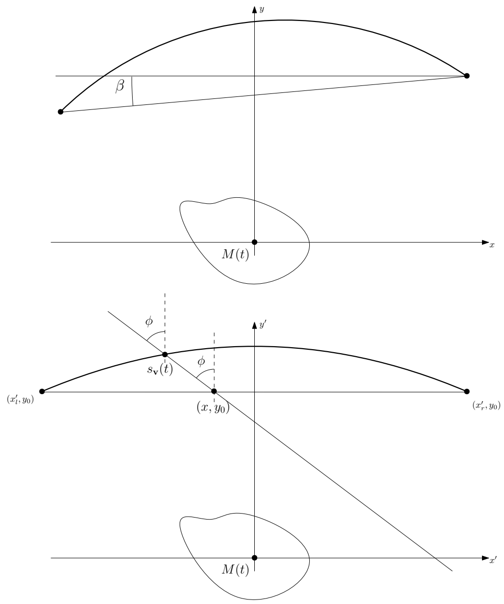

Fig. 2. Change of frame: the object is now at center of the coordinates system. Left, only translation of the center of frame; right, after rotation of angle β. Note that the angle ϕ is the same as in Figure 1, right.

In this frame, the coordinates of the source are given by $s_{\mathbf{v}}(t) = \mathcal{R}_\beta \left(s(t) - M_{\mathbf{v}}(t)\right)$. With this in mind, the data is given by the following formula

$$(T_{\mathbf{v}}\mu)(t, \phi) = \int_0^{+\infty} \mu' \left(s_{\mathbf{v}}(t) + l \left[\sin\phi, -\cos\phi\right]\right) dl, \quad \text{(II.8)}$$

where $\mu'(\mathbf{x}') = \mu(\mathbf{x})$, *i.e.* the function μ' takes its arguments from the space $(M(t), x', y')$.

In other words, we are now dealing with a fixed object whose density function is given by μ' illuminated by a source following an arc of a cycloid in the frame $(M(t), x', y')$ (see Figure 2, right).

Here, the extreme points $s_{\mathbf{v}}(-T/2)$ and $s_{\mathbf{v}}(T/2)$ have the same y'-coordinate y'_0. We will call x'_l (resp. x'_r) the x'-coordinates of $s_{\mathbf{v}}(T/2)$ (resp. $s_{\mathbf{v}}(-T/2)$). This allows us to define what we call the *virtual fanbeam projection* from a point (x', y'_0).

Definition 3. *For any point x' between x'_l and x'_r, for any angle $\phi \in [-\pi/2, \pi/2]$, the* virtual fanbeam projection *of the object μ is defined by*

$$\left(\tilde{T}\mu\right)(x', \phi) = \int_0^{+\infty} \mu\left((x', y'_0) + l\left[\sin\phi, -\cos\phi\right]\right) dl. \quad \text{(II.9)}$$

This is called virtual *since it does not correspond to an actual position of the source.*

In the following lemma, we will make a connection between the virtual fanbeam projection and the fanbeam projection of the translating object.

Lemma 1. *Let us fix a time $t \in [-T/2, T/2]$ and an angle $\phi \in [-\pi/2, \pi/2]$. Let us define*

$$x' = s_{1,\mathbf{v}}(t) + \tan\phi \left(s_{2,\mathbf{v}}(t) - y'_0\right), \quad \text{(II.10)}$$

where, for any time t, $(s_{1,\mathbf{v}}(t), s_{2,\mathbf{v}}(t))$ are the coordinates of $s_{\mathbf{v}}(t)$. Then, one has

$$\left(\tilde{T}\mu\right)(x', \phi) = \left(T_{\mathbf{v}}\mu\right)(t, \phi). \quad \text{(II.11)}$$

We can now define what are the DCCs of our problem.

Theorem 1. *Let us fix a density function μ. For any integer n, there exist a function $(x', t) \mapsto W_{n,v}(t, x') \in \mathcal{C}^\infty\left([-T/2, T/2] \times [x'_l, x'_r]\right)$ such that*

$$B_n := x' \mapsto \int_{-T/2}^{T/2} \left(T_{\mathbf{v}}\mu\right)(t, \lambda(t))\, W_n(x', t, v) dt \in \mathbb{R}_n[X], \quad \text{(II.12)}$$

where $\lambda(t)$ is defined by

$$\lambda_t = \arctan\left(\frac{x' + R_0 \sin(\omega t) + \left(t + \frac{T}{2}\right) v}{R_0 \cos(\omega t) - y'_0}\right). \quad \text{(II.13)}$$

Moreover, it is possible to derive $W_{n,v}(t, x')$ analytically.

III. Numerical simulations

A. Principles

Let us suppose that we have the projections $\left(T_{\mathbf{v}}\mu\right)(t, \phi)$. In order to recover v, we can perform the following optimization procedure. Since $B_n(x')$ in equation (II.12) is supposed to be a polynom of order $\leq n$, one can minimize

$$\mathcal{J}(v) = \sum_n \|\text{res}\left(B_n(\cdot, v)\right)\|^2 \quad \text{(III.1)}$$

with respect to v, where res is the residual of the projection onto $\mathbb{R}_n[X]$. This will give us the velocity v using only the knowledge of the data $\left(T_{\mathbf{v}}\mu\right)(t, \phi)$.

B. Application

Here, we will consider an object translating along the x-axis, *i.e.* $v_2 = 0$. The object under consideration is an ellipse of uniform density, whose axis are 30 and 15 millimeters respectively. The source is rotating around the object with radius $R_0 = 600\,\text{mm}$, with angle velocity $\omega = 1\,\text{rad} \cdot \text{s}^{-1}$. With a velocity of the ellipse given by $v_1 = 0.3\text{mm} \cdot \text{s}^{-1}$, we obtain the sinogram depicted in Figure 3.

With this configuration in mind, the functions $x' \mapsto B_n(x')$ for $n = 0, 1, 2, 3$ can be visualized in Figure 4.

References

[1] Rolf Clackdoyle, Michel Defrise, Laurent Desbat, and Johan Nuyts. Consistency of fanbeam projections along an arc of a circle. In *The 13th International Meeting on Fully Three-Dimensional Image Reconstruction in Radiology and Nuclear Medicine*, pages 415–419, 2015.

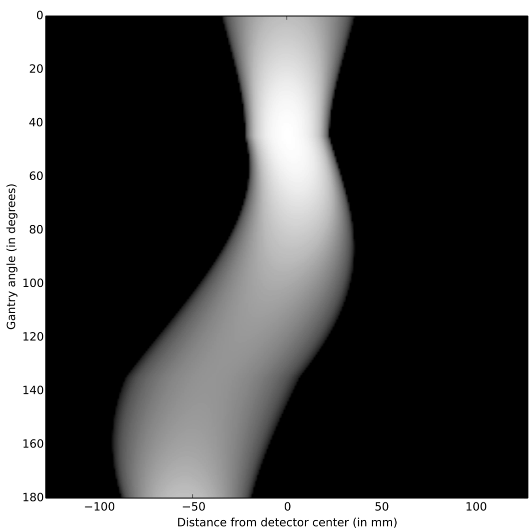

Fig. 3. Sinogram of a moving ellipse, with semi-axis $a = 30$ and $b = 15$, translating along the x-axis with velocity $v_1 = 0.3$. It is illuminated by a source at distance $R_0 = 600$, rotating with angle velocity $\omega = 1$.

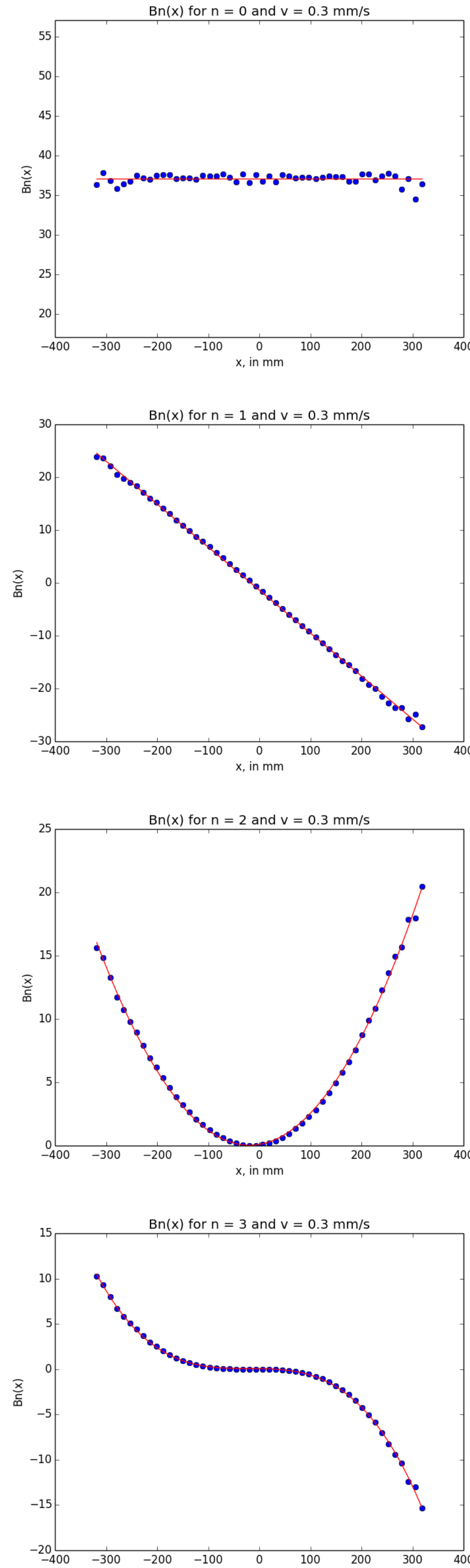

Fig. 4. From left to right, and top to bottom, functions $x' \mapsto B_n(x')$ for $n = 0, 1, 2, 3$ respectively.