# Consistency of Fanbeam Projections of a Translating Object Along an Arc of a Circle

Thomas Boulier, Rolf Clackdoyle, Jérôme Lesaint, Laurent Desbat

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## I. Introduction

In the field of CT-imaging, data consistency conditions (DCCs) use the redundancy of the data in order to establish relations that must be fulfilled between projections. These conditions are called *full* when they are not only necessary but also sufficient. One of the best known DCCs are the Helgason-Ludwig (H-L) conditions [3], [4], which are full for parallel projections. Full DCCs are also known for the case of fan-beam projections with source position taken along a straight line [1].

In [2], the fan-beam DCCs were modified to handle the case of a source moving along an arc of a circle. Briefly, all rays passing through the same point along a "virtual" line between the two extreme positions of the source are gathered to form a fan-beam projection. The DCCs for fanbeam projections along a line from [1] could then be applied; numerical examples showed a situation where an artificial detector attenuation was added, and the DCCs were invoked to recover the unknown attenuation coefficient. The case of a moving object has been tackled in [8], [7], where the authors used ~~parallel~~-beam H-L conditions ~~in~~ the case of an object undergoing rigid body motion while ~~illuminated~~ by a fan-beam source along a ~~line.~~ Those conditions allowed the authors to recover the parameters of ~~the movement,~~ in order to suppress the artifacts caused by such movement. However, using the parallel-beam H-L conditions means that the fan-beam projections had to cover all rays in the plane so that a fan-to-parallel rebinning could be achieved. The rebinning was achieved mathematically; however, the requirement of at least a standard fan-beam shortscan of 180° plus fan-angle ~~is~~ required. In our work, we are concerned with the opposite extreme of a short arc of fan-beam measurements.

In this article, we will ~~be interested in these two situations. We~~ will suppose that an object is translating while ~~illuminated~~ by a fan-beam source which is moving along an arc of a circle.

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The authors would like to thank Simon Rit for his help using RTK.
This work is partially supported by the Agence Nationale de la Recherche (France), project “DROITE”, number ANR-12-BS01-0018.

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### A. Problem under consideration

Let us begin with some notation and definitions. We will consider an object in $\mathbb{R}^2$ to be imaged in a fanbeam geometry with sources on an arc of circle with center $O$ and radius $R_0$ (see Figure 1, top). The object is identified with its density function $\mathbf{x} \mapsto \mu(\mathbf{x}) \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. The angular velocity of the

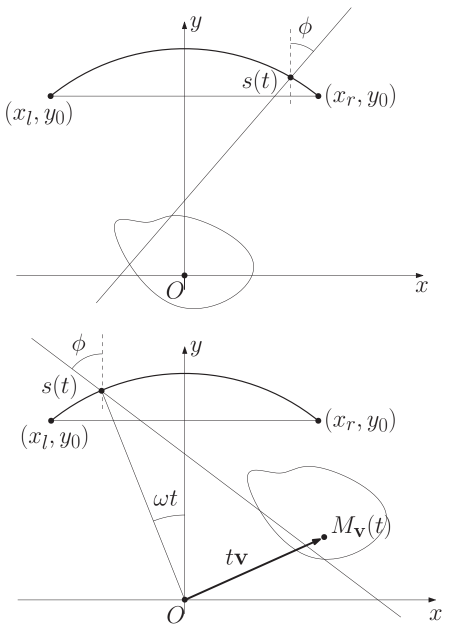


Fig. 1. Problem under consideration. The source point $s(t)$ follows the arc of circle depicted in bold. The circle has center $O$ and radius $R_0$. The object is continually undergoing a translation during the movement of the source.

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$$s(t) = (-R_0 \sin(\omega t), R_0 \cos(\omega t)) \,. \tag{1}$$

Furthermore, we will let $s(T/2) = (x_l, y_l)$ (resp. $s(-T/2) = (x_r, y_r)$) denote the extreme left (resp. right) position of the source. Since $y_l = y_r = R_0 \cos(\omega T/2)$, we will call $y_0$ ~~its common value~~. In the following, we will ~~suppose~~ that supp$(\mu)$

# Résumé des commentaires sur DCC_translation_CT_meeting_2018_Boulier.RC.pdf

## Page : 1

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"fanbeam" in title. "fan beam" in abstract. "fan-beam" in paragraph 2 of main text.

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parallel

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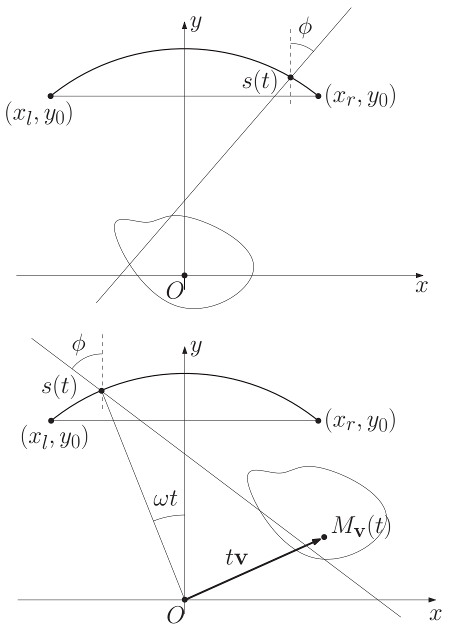


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lies in the half-space $\{y < y_0\}$, and that $y_0 > 0$ (~~i.e.~~ $< \omega T < \pi$).

We will suppose that at any time $t$, rays are simultaneously emitted from the source $s(t)$ with angle $\phi$ ranging from $-\pi/2$ to $\pi/2$. With this setup in mind, we can define the operator modeling the acquired data from the object.

**Definition 1.** *The* fanbeam projection data *of an object with density function $\mu$ is a function $(t, \phi) \mapsto \mathcal{F}\mu(t, \phi)$ defined by*

$$(\mathcal{F}\mu)(t, \phi) = \int_0^{+\infty} \mu\left(s(t) + l\left[\sin\phi, -\cos\phi\right]\right) dl, \quad (2)$$

*where $t \in [-T/2, T/2]$, $\phi \in [-\pi/2, \pi/2]$ and $s(t)$ is given by (1). The operator $\mu \mapsto \mathcal{F}\mu$ is called the* fanbeam projection operator.

Now let us suppose that the object is translating along a line with a constant velocity vector $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ (see Figure 1, bottom). In other words, if we let $M_{\mathbf{v}}(t)$ denote its center of mass at any time $t$, we have

$$M_{\mathbf{v}}(t) = (t + T/2)\,\mathbf{v} \quad (3)$$

The density function of the object now depends on both the spatial variable $\mathbf{x} \in \mathbb{R}^2$ and the time $t$. If we denote the ~~time varying~~ object by $\mu_{\mathbf{v}}$, we have

$$\mu_{\mathbf{v}}(t, \mathbf{x}) = \mu\left(\mathbf{x} - M_{\mathbf{v}}(t)\right). \quad (4)$$

In this regard, the fanbeam projection data will be modified in the following way.

**Definition 2.** *The* fanbeam projection data of a translating object *with density function $\mu$ and velocity vector $\mathbf{v}$ is given by*

$$(\mathcal{F}_{\mathbf{v}}\mu)(t, \phi) = (\mathcal{F}\mu_{\mathbf{v}})(t, \phi). \quad (5)$$

The aim of this note is to derive data consistency conditions (DCCs) from (5), in order to retrieve the velocity vector $\mathbf{v}$ from the knowledge of a single element of the range of $\mathcal{F}_{\mathbf{v}}$.

### B. Derivation of DCCs

In order to derive DCCs, we will first change our reference frame, from $(O, x, y)$ to $(M(t), x', y')$, so that the origin is the center of mass of the object at any time $t$ and the line between the start point and the end point of the source is still parallel to the $x'$-axis (see Figure 2). In other words, we are performing the following change of variables

$$(x, y) \leftrightarrow (x', y') = \mathcal{R}_\beta\left((x, y) - M_{\mathbf{v}}(t)\right), \quad (6)$$

where $\mathcal{R}_\beta$ rotation by $\beta$. The angle ~~beta~~ is depicted in Figure 2 (top) and is given by

$$\beta = \arctan\left(\frac{T v_2}{2R_0 \sin(\omega T/2) + T v_1}\right). \quad (7)$$

Note that in this equation, the denominator can be equal to zero. ~~Such~~ situation ~~occurs~~ in particular when $v_1 < 0$. We studied those particular cases, both in terms of physical meaning and numerical implications. For the sake of simplicity, we will assume ~~this abstract~~ that $v_1$ is not too negative (~~e.g.~~ $v_1 > -2R_0/T$) to ensure that the denominator is non-zero.

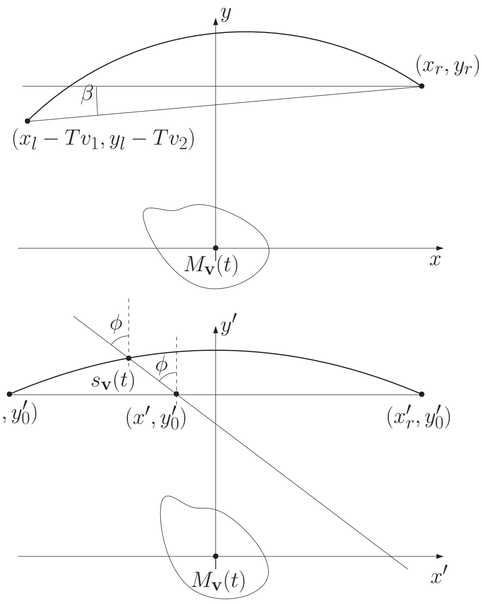


Fig. 2. Change of frame: the object is now at center of the ~~coordinates~~ system. Top: only translation of the center of frame; bottom: after rotation of angle $\beta$. Note that the angle $\phi$ is the same as in Figure 1, bottom. The virtual source position $(x', y'_0)$ is defined in terms of $s_{\mathbf{v}}(t)$ and $\phi$; see equation (10).

In this frame, the coordinates of the source position is given by $s_{\mathbf{v}}(t) = \mathcal{R}_\beta\left(s(t) - M_{\mathbf{v}}(t)\right)$. With this in mind, the data are given by the following formula

$$(\mathcal{F}_{\mathbf{v}}\mu)(t, \phi) = \int_0^{+\infty} \mu \circ \mathcal{R}_\beta\left(s_{\mathbf{v}}(t) + l\left[\sin\phi, -\cos\phi\right]\right) dl. \quad (8)$$

In other words, we are now dealing with a fixed object whose density function is given by $\mu \circ \mathcal{R}_\beta$, ~~illuminated~~ a source following an arc of a cycloid in the frame $(M(t), x', y')$ (see Figure 2, bottom).

Here, the extreme points $s_{\mathbf{v}}(-T/2)$ and $s_{\mathbf{v}}(T/2)$ have the same $y'$-coordinate, $y'_0$. We will call $x'_l$ (resp. $x'_r$) the $x'$-coordinates of $s_{\mathbf{v}}(T/2)$ (resp. $s_{\mathbf{v}}(-T/2)$).

Now we define what we call the *virtual fanbeam projection* from a point $(x', y'_0)$.

**Definition 3.** *For any point $x'$ between $x'_l$ and $x'_r$, and for any angle $\phi \in [-\pi/2, \pi/2]$, the* virtual fanbeam projection *of the object $\mu$ is defined by*

$$\left(\tilde{\mathcal{F}}\mu\right)(x', \phi) = \int_0^{+\infty} \mu\left((x', y'_0) + l\left[\sin\phi, -\cos\phi\right]\right) dl. \quad (9)$$

*This is called* virtual *since it does not correspond to an actual position of the source.*

In the following lemma, we will make the connection between the virtual fanbeam projection and the fanbeam projection of the translating object.

**Lemma 1.** *Let us fix a time $t \in [-T/2, T/2]$ and an angle $\phi \in [-\pi/2, \pi/2]$. Let us define*

$$x' = s_{1,\mathbf{v}}(t) + \tan\phi\left(s_{2,\mathbf{v}}(t) - y'_0\right), \quad (10)$$

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Thomas... please check equation (8). It doesn't seem quite right (e.g. the source s(t) seems to be rotated twice). Also, it would be much nicer if it were to clearly follow from equations (5), (4), (2). Laurent suggests inserting R_beta R_{-beta} in front of mu. It also looks like phi should be phi+beta (or phi-beta) in the right hand side.

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Thomas... make sure you have the proof (for general beta) that equations (8) and (9) are equal to each other. [It doesn't seem likely right now, because (8) doesn't seem right.]

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