On a Simple and Efficient Approach to Probability Distribution Function Aggregation

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***Abstract*—In group decision making, it is inevitable that the individual decision maker's subjectivity is involved, which causes difficulty in reaching a group decision. One of the difficulties is to aggregate a small set of expert opinions with the individual subjectivity or uncertainty modeled with probability theory. This difficult problem is called probability distribution function aggregation (DFA). This paper presents a simple and efficient approach to the DFA problem. The main idea of the proposed approach is that the DFA problem is modeled as a nonlinear function of a set of probability distribution functions, and then a linear feedback iteration scheme is proposed to solve the nonlinear function, leading to a group judgment or decision. Illustration of this new approach is given by a well-known DFA example which was solved with the Delphi method. The DFA problem is a part of the group decision problem. Therefore, the proposed algorithm is also useful to the decision making problem in general. Another contribution of the this paper is the proposed notation of systematically representing the imprecise group decision problem with the classification of imprecise information into three classes, namely incomplete information, vague information, and uncertain information. The particular DFA problem dealt with in this paper is then characterized with this general notation.**

***Index Terms*—Aggregation, group judgment, linear feedback iteration, probability distribution.**

I. Introduction

A HUMAN and another human communicate on things with language. The emergence of the modern computer leads to two phenomena: 1) computer processing of data which further carry the semantics of information (what it is) and knowledge (how to do it and why it is) and 2) human and computer communication on things with data languages [36]. In order to be consistent with the cognition model of a large population of readers perhaps, the word "data" is replaced by the word "information" in the following discussion in this paper with the understanding that there are three categories of information, namely: 1) what it is; 2) how to do it; and 3) why it is. Quite naturally, an issue arises, that is, information precision (including information accuracy) in these communications [17]. Imprecision in information may be in the form of: 1) incompleteness; 2) uncertainty; and 3) ambiguity [40]. An example of imprecision 1) is that among a group of decision makers or experts one or more experts may be absent (so are their opinions), and this may cause some defect decision out of the group. An example of imprecision 2) is that the expert is not confident with his opinion or unsure about the chance of occurring with a particular event. An example of imprecision 3) is that the expert may have difficulty to express his opinion in a crisp number say "temperature of weather today is 30 °C" but may express his opinion with the word or human language (in English) like "today is very hot." It is noted that the three types of imprecision in information expression or exhibition are mutually exclusive.

Computer processing of the information that includes all three types of imprecision is a challenge. Processing of the imprecision 1) of information is out of the scope of this paper. The imprecision 2) of information can be straightforwardly modeled with the mathematical framework of probability theory, and the imprecision 3) of information can be straightforwardly modeled with the mathematical framework of fuzzy logics theory. It is noted that by straightforwardly in the context of this paper it is meant that a particular theory (e.g., fuzzy logics theory) is developed for a particular type of problem (e.g., ambiguity). Just as implied by Zadeh [18] that fuzzy logics theory is essentially a tool for computer processing of words ("word computing"). It is the opinion of the authors of this paper that imprecision 2) is better handled with probability theory.

A generic conceptual model of group decision making (GDM) can be described in the following, adapted from [4] and [7]. First, there is a thing (X) on which a group of experts must make a decision, and the presentation of the group's decision is denoted by x. Second, experts may have different opinions denoted by $x1, x2, \ldots, xn$ (where n is the total number of experts); so x represents a degree of agreement (DOA) among the n experts. In literature, a full agreement is called consensus [4], [7], but in practice the term consensus is used

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with the understanding of a partial agreement, so is this paper. There are two general problems with GDM. First is how to increase the DOA among the experts through a process called consensus reaching process [4], [7]. Second is how to aggregate to the "best" x from $\{x1, x2, \ldots, xn\}$. By best in the context of GDM, it is meant the highest DOA among $\{x1, x2, \ldots, xn\}$. Let the measure of the DOA among the expert opinions called DOA. It may be clear that the solution to the second problem should also be useful to the solution to the first problem, as the solution to the second problem may provide a tool (i.e., the measure of the DOA) to the consensus reaching process. With this tool (DOA), a consensus reaching process only needs to focus on the presentation of the x with its DOA to influence over the experts in changing their opinion and eventually to come up with the x which has the highest DOA. This paper focuses on the second problem.

It is noted that in both first and second problems in GDM, as mentioned above, there is an issue regarding the importance of experts; that is, it is possible that one expert's opinion may be more important than another experts'. The different importance of experts' opinions implies that the relationship between x and $\{x1, x2, \ldots, xn\}$ is not as simple as $x = (x1 + x2 + \ldots + xn)/n$; in general x is a complex function (denoted as f) of $x1, x2, \ldots, xn$. The f may be nonlinear; for instance, the ordered weighted average (OWA) of $x1, x2, \ldots, xn$ is a nonlinear function with the importance of each expert represented by a weight (denoted as w) [29].

There are a number of situations in GDM in terms of how the imprecision in the expert opinion in introduced. Several example situations can be illustrated.

Situation 1: 1) x_i is a crisp number (e.g., the number of bombs is 100 [24]); 2) confidence to cast x_i is not full, expressed by a crisp probability number (e.g., 95%) or the chance of occurring of an x_i is not fully sure, expressed by a crisp probability number (e.g., 50%); 3) importance of an expert is expressed by a crisp number with the domain of (0, 1); and 4) confidence to cast the importance of an expert is not full, expressed by a crisp probability number (e.g., 95%).

Situation 2: 1) x_i is expressed in a vague manner (e.g., word or a range of crisp numbers); 2) confidence to cast x_i is full or the chance of occurring of an x_i is fully sure; 3) importance of an expert is expressed in a vague manner (e.g., word or a range of crisp numbers); and 4) confidence to cast the importance of an expert is full.

Situation 3: 1) x_i is expressed in a vague manner (e.g., word or a range of crisp numbers); 2) confidence to cast x_i is not full, expressed by a crisp probability number (e.g., 95%) or the chance of occurring of an x_i is not fully sure, expressed by a crisp probability number (e.g., 50%); 3) importance of an expert is expressed in a vague manner (e.g., word or a range of crisp numbers); and 4) confidence to cast the importance of an expert is full.

In fact, we can define a notation that has four fields to facilitate the definition of all the situations. The first field is regarding the form of expression of x_i; the second field is regarding the confidence on x_i; the third field is regarding the form of w_i; the fourth field is regarding the confidence on w_i. Let S denote situation. Let X denote the variable of the first field, CX denote the variable of the second field, W denote the variable of the third field, and finally CW denote the variable of the fourth field. So, $S = [X|CX|W|CW]$, where the domain of X is ("crisp" and "vague"), the domain of CX is (a probability number α within (0, 1], the domain of W is (crisp and vague), the domain of CW is (a probability number β within (0, 1]. With this notation, situation 1 ($S1$) is expressed as: $S1 = [\text{crisp}, \alpha, \text{crisp}, \beta]$. On a general remark, the aforementioned situations are useful in practice, and algorithms or methods to solve the second problem of GDM on these situations need to be developed.

In the literature, many studies were performed on the second problem of GDM on the situation of [crisp, α, crisp, 1] and indeed, the contribution of this paper is on this problem. For simplicity in the following discussion, this problem is also called distribution function aggregation (DFA). Furthermore, many studies have been performed on the second problem of GDM on the situation of [vague, 1, vague, 1], (see [8], [9], [33], [34]) based on fuzzy theory. However, it is our view that fuzzy logics theory may not be best suitable to solving the DFA problem, albeit not impossible. On the other hand, the method for solving the DFA problem may be useful to solving the second problem on the situation of [vague, α, vague, 1] (for example) after defuzzification of the word expression of x_i, details of which are out of the scope of this paper.

The objective of this paper is to develop a new method for solving the DFA problem. This paper is organized as follows. In Section II, a literature review on the method to solve the DFA problem is presented, which not only gives a further background for the proposed method but also helps to elaborate on the novelty of the proposed method. Section III presents the proposed method in detail. Section IV gives an example to show how the method works. Section V is a discussion of some implication from the proposed method. Section VI is the conclusion.

II. Related Work for DFA

The main challenge of the DFA problem is how to determine the importance of the expert, namely the importance weight in a simplistic sense to state the challenge. In a most general sense, however, the importance of the expert is in fact encoded in the function of x with respect to $\{x1, x2, \ldots, xn\}$. There are many studies related to how to determine the weight (based on the simplistic sense), e.g., the weight assigned by the moderator, the weight based on the ranking of the expert, etc. All of these are quite subjective [17].

Abandoning the simplistic sense, a linear programming approach is proposed for the DFA problem recently in [19]. The idea is to make the consensus as a quadratic function of individual probability distribution functions. Then the DFA problem becomes a multiobjective optimization problem. Some other approaches are based on the maximum entropy of a group decision or consensus probability distribution (see the work in [35]).

In our previous work [39], a new approach to the DFA problem was proposed. The approach in [39] is as follows.

First, a consensus from a group of experts or group judgment or group decision is considered as a function of probability density functions (PDFs) of all experts, namely

$$f(x) = \emptyset(f_1(x), f_2(x), \ldots, f_m(x)) \tag{1}$$

where m is the total number of experts or PDFs, $f_i(x)$ is the PDF of the ith expert, and x is the target event. Second, (1) is further approximated to

$$f(x) = \sum_{i=1}^{m} w_i f_i(x) \tag{2}$$

where w_i is the weight associated with the ith expert ($0 \leq w_i \leq 1$ and $\sum w_i = 1$). The w_i is determined by the following procedure.

Step 1: Construct the overlapping area of $f_i(x)$ and $f(x)$, denoted as A_i.

Step 2: Maximize A_{TOT} (which is the sum of A_i) under the constraint of $\sum w_i = 1$, with w_i being the variable of this optimization problem (solved by a simulated annealing algorithm).

The philosophy behind the above approach is as this. The overlapping area A_i represents how similar the opinion of the ith expert with the consensus or group opinion. The total overlapping area A_{TOT} represents the consensus of all the experts, and maximization of A_{TOT} is thus meant to achieve the best consensus. However, our previous approach requires employing optimization techniques (simulated annealing in this case), which is computationally overhead and complicated. This paper proposes a new approach which advances our previous approach in [39], particularly by devising a linear feedback iteration scheme to solve the nonlinear function, i.e., (1).

It is noted that the idea of feedback was also proposed in [3]. However, there are the following differences between their work and ours (both previous one and present one). First, the feedback in their work refers to the opportunity that the experts can revise their opinions on the target during the iteration process. This in fact falls into the first problem of GDM as defined before in this paper. Second, they did have a group decision which is a weighted average, which is computed by the so-called qualifier in the OWA approach [29]. Note that the OWA is primarily based on the simplistic sense of the importance of the expert but it does capture a certain aspect of the nonlinearity of the function (f).

III. Proposed Algorithm—Linear Feedback Iteration

The general idea of the proposed approach in this paper is that (1) is viewed as a nonlinear function of $f_i(x)$, say $\emptyset$. If $\emptyset$ is found, $f(x)$ is then determined. So the key is to find $\emptyset$. At this point, a linear feedback iteration process is employed to find $\emptyset$. First, (1) is linearized to (2) with a set of coefficients or weights w_i. Second, the determination of the weights follows a procedure in which the $f(x)$ obtained from (2) at each generation of iteration will be used to update the weights for the next generation. This iteration goes on until the following condition is satisfied:

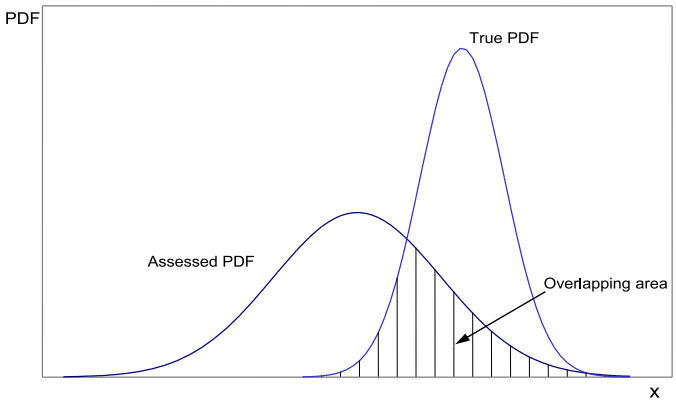

Fig. 1. Overlapping area between assessed PDF and true DF.

$$\sqrt{\sum \left(w_i^{(n-1)} - w_i^{(n)} \right)^2} < \varepsilon \tag{3}$$

or

$$\left| 1 - \int \min\left\{ f^{(n)}(x), f^{(n-1)}(x) \right\} dx \right| < \varepsilon \tag{4}$$

where ε is a small number. It is noted that the integral in (4) represents the overlapping area between $f^{(n)}(x)$ and $f^{(n-1)}(x)$ (see Fig. 1).

The updating equation for w_i is constructed as follows.

Step 1: Construct the overlapping area of $f_i(x)$ and $f(x)$, denoted as A_i. The idea behind this step was explained before.

Step 2: Construct the updating equation as follows:

$$w_i = \frac{A_i}{A_{\mathrm{TOT}}} \tag{5}$$

where A_{TOT} is the sum of A_i. The idea underlying (5) is that the coefficient in (2) represents the weight of the expert coincidentally, and therefore, the updating of the weight may be determined semantically (the concept of weight). A particular knowledge would be: if an expert has his opinion closer to the consensus opinion, then a higher weight is given to that expert. Equation (5) is thus the result of applying this knowledge.

A judicious determination of the initial weights should speed up the convergence. Here, a method is proposed to assign initial weights. Suppose m_i and d_i are measures of the magnitude and dispersion of $f_i(x)$, respectively. Rearrange m_i and d_i, $i = 1, \ldots, m$, in an ascending order, respectively. Denote that the rank numbers of the measures of magnitude and dispersion are $r_m^{(i)}$ and $r_d^{(i)} \in (1, 2, \ldots, m)$, respectively. Note that a general philosophy behind finding a consensus from a group of experts is to aggregate $f_i(x)$ ($i = 1, \ldots, m$) into $f(x)$ that has the following features in the shape of its function: 1) small dispersion and 2) high average magnitude. Along this line of thinking, assign a higher score to an expert if the magnitude and dispersion of his or her PDF are closer to a kind of

"middle point," i.e., $(m+1)/2$ among that group of experts. Hence, the initial score (s_i) of each expert is calculated by

$$s_m^{(i)} = \left(r_m^{(i)} - 1 + a\right)\left(m - r_m^{(i)} + a\right)$$
$$s_d^{(i)} = \left(r_d^{(i)} - 1 + a\right)\left(m - r_d^{(i)} + a\right),\ a \in (0,1) \quad (6)$$
$$s_i = \sqrt{\left[s_m^{(i)}\right]^2 + \left[s_d^{(i)}\right]^2}. \quad (7)$$

In the above equation, $s_m^i(s_d^i)$ stands for the score of the ith expert assigned in terms of the magnitude (dispersion) of the PDF of the ith expert. Because the dispersion and magnitude are somewhat on an orthogonal relation, take the score s^i as the square root of the s_m^i square and s_d^i square, i.e., (7). Finally, the initial weight of expert i ($w_i^{(1)}$) is calculated by the following equation:

$$w_i^{(1)} = s_i / \sum s_i \quad (8)$$

where the superscript "(1)" stands for initial iteration. In (6), the parameter a is intended to adjust the closeness to the middle point. The determination of parameter a is empirical. Through an experimental procedure, the following procedure is proposed to determine a.

Step 1: Set parameter b in (9) as 10 (some other values are possible).

Step 2: Determine parameter a using

$$\max\left(w_1^{(1)}, \ldots, w_m^{(1)}\right) \Big/ \min\left(w_1^{(1)}, \ldots, w_m^{(1)}\right) \approx b. \quad (9)$$

The parameter b is chosen empirically. Through some trial and error, b is recommended to be about 10.

In summary, our algorithm has the following steps.

Step 1: Elicit individual distribution $f_i(x)$.

Step 2: Calculate the initial weights $\{w_i^{(1)}\}$.

Step 3: Aggregate the distributions (calculate A_i and w_i), which thus completes one generation of iteration.

Step 4: Check if the stopping condition is satisfied. If yes, terminate the aggregation process; otherwise, go back to step 3 (executes the next generation of iteration).

It is noted that the proposed procedure is similar to the iterative learning control in control engineering [23]. In there, a control law is established through several generations of repeating the actions of systems. For instance, suppose that a task for a robot is to follow a trajectory. The corresponding iterative learning control law for the robot is established by having the robot to do the task several times. Indeed, the summation of information of each generation of iterations leads to a nonlinear relation with independent variables (errors in the control problem) or functions (individual PDFs in our case). This note actually also implies that our algorithm in this paper is significantly different from our previous approach [39]. The nature of our previous approach is to find the optimal weight for the weighted average, i.e., (2).

IV. Illustration

The example is taken from [24], where the well-known DELPHI method was applied to the selection of an optimal U.S. industrial target system and to the estimation of the numbers of A-bombs required to reduce the munitions output by a prescribed amount. Reproduce the initial estimates given by seven experts in Table I. After a series of feedbacks and re-estimations, the corrected final estimates are [24]

$$\text{smallest}(X_{0.1}) = 167, \text{median}(X_{0.5}) = 276$$
$$\text{largest}(X_{0.9}) = 360.$$

TABLE I
Estimates of the Experts (From [24])

	$x_{0.1}$	$x_{0.5}$	$x_{0.9}$
Expert 1	75	125	200
Expert 2	25	50	150
Expert 3	100	150	175
Expert 4	250	300	800
Expert 5	70	200	500
Expert 6	-	1000	-
Expert 7	2500	5000	10000

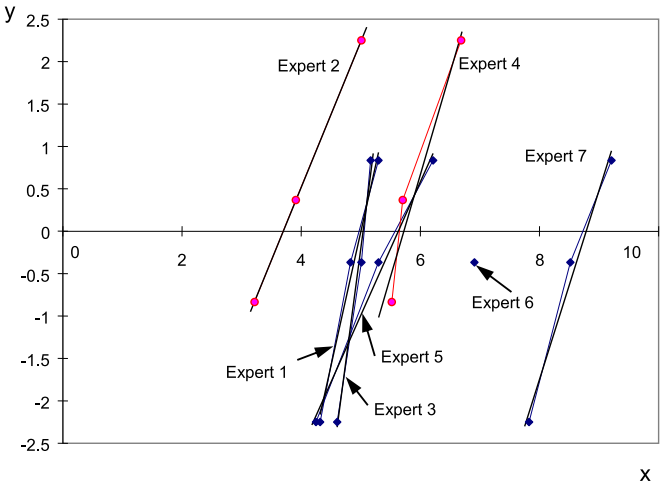

Fig. 2. Fitting of individual assessments.

In this example, the event under consideration is a discrete variable ($X = 1, 2, \ldots$). The discrete variable problem is first converted to a continuous variable problem by

$$P(x = i) = \int_{i-0.5}^{i+0.5} f(x)dx. \quad (10)$$

In the remainder of this paper, the problem is considered as a continuous variable problem. Furthermore, expert 6 is removed for further reconsideration because of missing data (see Table I). The proposed algorithm is run in the following.

A. Step 1: Elicitation of Expert Distributions

Consider the Weibull model to get the distribution for each expert because the Weibull and inverse Weibull distributions with the domain $(0, \infty)$ have the advantage of flexibility in fitting data, and their model parameters can be determined by regression. The cumulative distribution function (CDF) of Weibull distribution is given by

$$F(x) = 1 - \exp\left(-(x/\eta)^{\beta}\right). \quad (11)$$

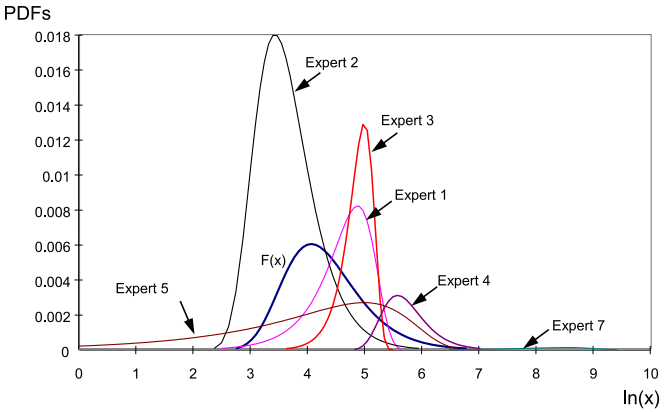

Fig. 3. Aggregated and individual distributions.

TABLE II
TYPES OF THE MODEL AND THE MODEL PARAMETERS OF EXPERT DISTRIBUTIONS

	Model	β	η
Expert 1	Weibull	3.1525	148.89
Expert 2	Inverse Weibull	1.7208	40.52
Expert 3	Weibull	5.3357	154.2
Expert 4	Inverse Weibull	2.4079	305.87
Expert 5	Weibull	1.5742	279.03
Expert 7	Weibull	2.2249	6530.87

TABLE III
INITIAL WEIGHTS OF THE EXPERT

	r_m	r_d	w_i
Expert 1	2	2.5	0.2088
Expert 2	1	2.5	0.1364
Expert 3	3	1	0.1604
Expert 4	5	5	0.1737
Expert 5	4	4	0.2919
Expert 7	6	6	0.0288

The CDF of the inverse Weibull distribution is given by

$$F(x) = \exp\left(-(x/\eta)^{-\beta}\right) \tag{12}$$

where the parameters $\eta(>0)$ and $\beta(>0)$ are called scale and shape parameters, respectively.

Under the following transformations

$$x = \ln(x), y = \ln(-\ln(1 - F(x))) \tag{13}$$

and the transformations

$$x = \ln(x), y = -\ln(-\ln(F(x))). \tag{14}$$

Equations (11) and (12) become a straight line on the x-y plane

$$y = \beta(x - \ln(\eta)). \tag{15}$$

Using the data listed in Table I, both model types (i.e., the Weibull or inverse Weibull) and their parameters for all expert distributions are given in Table II. Fig. 2 plots the fitted models which show the appropriateness of using the Weibull model to fit for this example. Fig. 3 further plots the PDFs of experts using either the Weibull model or the inverse Weibull model indicated in Table II. The aggregated PDF on Fig. 3 will be discussed later.

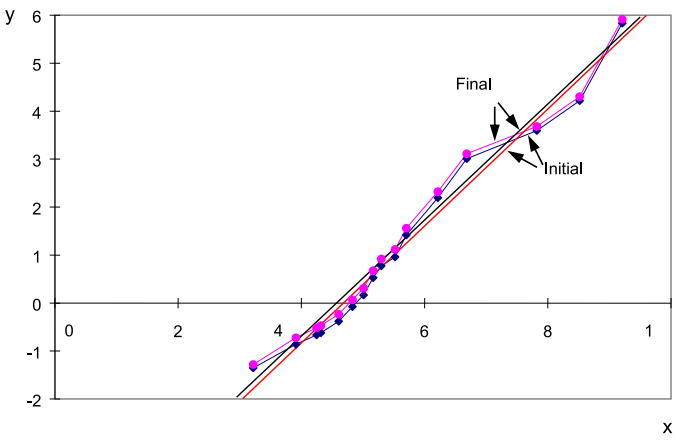

Fig. 4. Inverse Weibull plots.

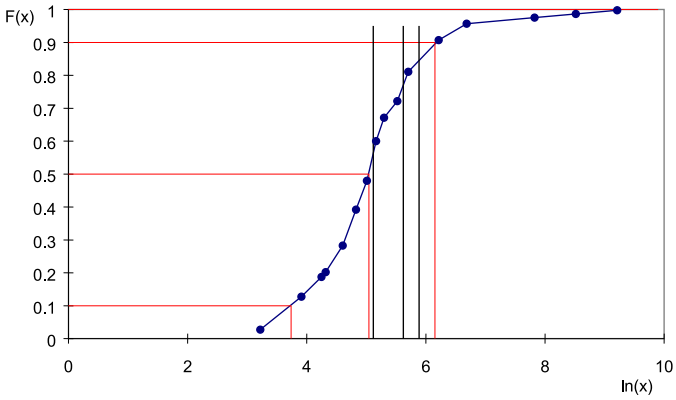

Fig. 5. Aggregated distribution and corrected final estimates.

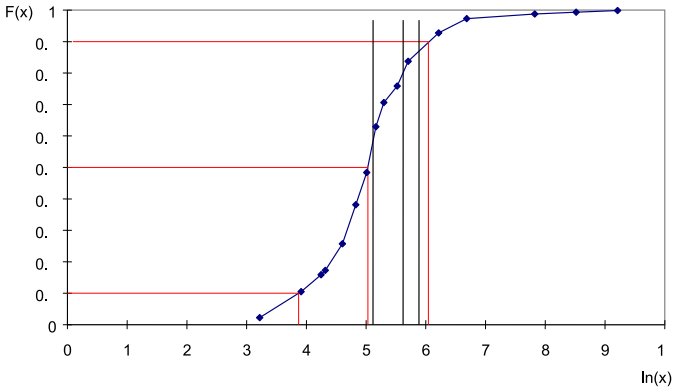

Fig. 6. Distribution aggregated by the Histogram method.

B. Step 2: Determination of the Initial Weights

Use the estimate $x_{0.5}$ as a measure of the magnitude (i.e., r_m) and the difference $(x_{0.9} - x_{0.1})$ as a measure of the dispersion (i.e., r_d). Further choose b in (9) by 10. The corresponding rank numbers in terms of the magnitude and dispersion, respectively, are given in Table III. Note that according to the dispersion measure, experts 1 and 2 are ranked the same in the rank range of 2 and 3. Therefore, assign the same ranking number to them by taking an average of ranks 2 and 3; i.e., their ranking is calculated as 2.5 [i.e., $(2 + 3)/2$]. From (9), we

TABLE IV
SAMPLE DISTRIBUTION OF $F(x)$

x	Expert 1	Expert 2	Expert 3	Expert 4	Expert 5	Expert 7	F(x)
25	0.0036	0.1	0.0000	0.0000	0.0222	0.0000	0.0209
50	0.0316	0.5	0.0025	0.0000	0.0646	0.0000	0.0940
70	0.0885	0.6768	0.0147	0.0000	0.1	0.0000	0.1423
75	0.1	0.7071	0.0211	0.0000	0.1187	0.0000	0.1554
100	0.2481	0.8095	0.1	0.0000	0.1803	0.0000	0.2309
125	0.5	0.8660	0.2784	0.0002	0.2461	0.0002	0.3390
150	0.6407	0.9	0.5	0.0038	0.3137	0.0002	0.4290
175	0.8107	0.9225	0.9	0.0216	0.3811	0.0003	0.5545
200	0.9	0.9379	0.9818	0.0619	0.5	0.0004	0.6301
250	0.9940	0.9573	1	0.1	0.5688	0.0007	0.6820
300	0.9999	0.9686	1	0.5	0.6740	0.0011	0.7849
500	1	0.9868	1	0.7362	0.9	0.0033	0.8945
800	1	0.9941	1	0.9	0.9947	0.0093	0.9518
2500	1	0.9992	1	0.9937	1	0.1	0.9729
5000	1	0.9997	1	0.9988	1	0.5	0.9854
10000	1	0.9999	1	0.9998	1	0.9	0.9971

TABLE V
ITERATIVE PROCESS AND RELEVANT VALUES

	$w_i^{(1)}$	$A_i^{(2)}$	$w_i^{(2)}$	$A_i^{(3)}$	$w_i^{(3)}$	$A_i^{(4)}$	$w_i^{(4)}$
Expert 1	0.2088	0.5827	0.2175	0.5715	0.2162	0.5687	0.2160
Expert 2	0.1364	0.5027	0.1876	0.5369	0.2031	0.5471	0.2078
Expert 3	0.1604	0.4028	0.1504	0.3924	0.1484	0.3899	0.1481
Expert 4	0.1737	0.4191	0.1565	0.3950	0.1494	0.3874	0.1471
Expert 5	0.2919	0.6974	0.2603	0.6769	0.2560	0.6705	0.2546
Expert 7	0.0288	0.0742	0.0277	0.0711	0.0269	0.0697	0.0265
$\sum A_i$		2.6789		2.6438		2.6332	
β	1.2190	1.2070		1.2077		1.2094	
η	108.04	99.73		97.33		96.57	
Equation (8)		0.064		0.018		0.005	
Equation (9)		0.036		0.011		0.004	

TABLE VI
EXPERTS' RANKS

	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$	$r^{(4)}$
Expert 1	2	2	2	2
Expert 2	5	3	3	3
Expert 3	4	5	5	4
Expert 4	3	4	4	5
Expert 5	1	1	1	1
Expert 7	6	6	6	6

TABLE VII
ITERATION PROCESS AND RELEVANT VALUES

	$w_i^{(1)}$	$w_i^{(2)}$	$w_i^{(3)}$	$w_i^{(4)}$	$w_i^{(5)}$
Expert 1	0.2160	0.2271	0.2384	0.2431	0.2451
Expert 2	0.2078	0.1717	0.1624	0.1603	0.1600
Expert 3	0.1481	0.1840	0.1954	0.2000	0.2018
Expert 4	0.1471	0.1307	0.1253	0.1222	0.1206
Expert 5	0.2546	0.2682	0.2638	0.2611	0.2597
Expert 7	0.0265	0.0182	0.0147	0.0133	0.0128
$\sum A_i$		2.7534	2.7858	2.7949	2.7977
Equation (8)		0.0569	0.0201	0.0080	0.0036

get the parameter $a = 0.5$. The initial weights are calculated from (8) and are given in the last column of Table III.

C. Step 3: Aggregation of Expert Distributions

Pick up values of the variable x from $\{x\} = \{x_{0.1}^{(i)}, x_{0.5}^{(i)}, x_{0.9}^{(i)}\}$, $i = 1, 2, \ldots, m$, where m is the number of experts (6 in this case) and list them in the first column of Table IV. The aggregated distribution $F(x)$ can be obtained, and its corresponding value for each x is listed in the last column of Table IV. Note that in aggregating $F(x)$, when a particular value x_j is given by a certain expert, take it for $F_i(x_j)$; otherwise we obtain $F_i(x_j)$ from the model (the Weibull or inverse Weibull).

Furthermore, find that the inverse Weibull model best fits the aggregated distribution $F(x)$ with the model parameters $\eta = 108.04$ and $\beta = 1.2190$, respectively. In the following, use the analytical method to calculate the overlapping area A_i for the iteration process. Take $\varepsilon = 0.01$ in the stopping criterion given by (3) or (4). The result of the iteration process is put in Table V. It can be seen from Table V that the iteration process stops after the third round iteration. The final aggregated distribution is fitted by an inverse Weibull model with parameters $\eta = 96.57$ and $\beta = 1.2094$. Fig. 4 shows the initial and final aggregated distributions. Note that when calculate fractiles, use the interpolation instead of using the fitted model.

TABLE VIII
COMPARISON OF RELEVANT VALUES FOR TWO METHODS

	Method I		Method II		Initial Weights	
	w_i	r_i	w_i	r_i	w_i	r_i
Expert 1	0.2160	2	0.2451	2	0.2088	2
Expert 2	0.2078	3	0.1600	4	0.1364	5
Expert 3	0.1481	4	0.2018	3	0.1604	4
Expert 4	0.1471	5	0.1206	5	0.1737	3
Expert 5	0.2546	1	0.2597	1	0.2919	1
Expert 7	0.0265	6	0.0128	6	0.0288	6
$x_{0.1}$	42		48			
$x_{0.5}$	155		153			
$x_{0.9}$	470		384			
Reference [24]	Smallest =167		Median =276		Largest =360	

The three fractiles obtained by the three-point interpolation scheme are: $x_{0.1} = 42$, $x_{0.5} = 155$, and $x_{0.9} = 470$, respectively. The final aggregated PDF is plotted in Fig. 3 together with the PDFs of individual experts.

Fig. 5 shows the estimates obtained with our approach and the corrected final estimates drawn from [24] along with the aggregated $F(x)$. In this figure, the red lines are the results obtained using our method (in particular the line on the right corresponds to $x_{0.9}$, the line in the middle to $x_{0.5}$, and the line on the left to $x_{0.1}$); whereas the black lines are the corrected results presented in [24]. It can be seen from this figure that our results reasonably agree with the corrected final estimates in terms of the order of magnitude; yet the discrepancy shown in Fig. 5 will be discussed later in this paper. Further note that the estimated weights (Table I) are close to the initial weights (Table II), which provides some justification of our method to determine the initial weights using (7) through (8). Furthermore, the ranking of experts is determined according to their weights. The best and worst experts are given "1" and "6," respectively. Table VI shows the result of the ranking where expert 5 is the best and expert 7 is the worst.

The numerical method, i.e., (A2), is also used to calculate A_i. At this point, we use the weights listed in the last column of Table V as the initial weights in the new iteration process. The iteration results are listed in Table VII. From the table it can be seen that the weights change compared with the result using the analytical method; in particular, the ranks of experts 2 and 3 have exchanged. Fig. 6 shows the estimates obtained with our approach and the corrected final ones drawn from [24] along with the aggregated $F(x)$. The three fractiles are: $x_{0.1} = 48, x_{0.5} = 153$, and $x_{0.9} = 384$, respectively. Note that the results obtained with the analytical method are closer to the corrected final ones drawn from [24] than those obtained by the numerical method.

The comparative results with the two methods for calculating the overlapping area A_i are listed in Table VIII. It can be seen from the table that the ranks for experts 1, 5, and 7 obtained with the both methods are the same. We further define the following measure for accuracy of methods, in which we take the results in [24] as a reference:

$$c = \sqrt{\left[(x_{0.1} - 167)^2 + (x_{0.5} - 276)^2 + (x_{0.9} - 360)^2\right]/3}. \tag{16}$$

The measures for methods I and II, respectively, $c = 119$ and $c = 100$. This implies that the method II is superior to the method I. Further note that both methods appear to have a similar convergence rate in the iteration process. On a final note, both methods are easy to use such that one can use those spreadsheet programs (e.g., Excel).

V. DISCUSSION

The discrepancy between the result found with our approach and that drawn from [24] may be attributed to a fact that their "correct" result may not necessarily be reliable. In their work [24], they did not present how their correct results were obtained, and they did note that their process was carried out in a view of certain median considerations and should be given a firmer theoretical foundation. Inaccuracy with their correct result may also be speculated as there was no information in their paper about the smallest and largest estimates in terms of the probability or fractile. On the other hand, our approach is effective given the fact that there is a large discrepancy among the experts at their initial estimates, i.e., the ratio of the largest value (10 000) over the smallest value (25) being 400 (see Table I).

Another note lies in the proof of the convergence of the proposed algorithm. At this point, no analytical proof of the convergence of the solution generated with the proposed algorithm has been achieved. A simulated proof is given in this paper (see Appendix B). From the simulation, one can clearly see the convergence of the proposed algorithm. In fact, the proposed algorithm is very much close to the iterative learning control algorithm in the system control [32]. The fact that the iteration learning control algorithm is convergent may also give some confidence to the proposed algorithm in this paper for GDM.

VI. Conclusion

This paper presented a new approach for aggregation of individual experts' judgments which are represented by a probability distribution function. A linear feedback iteration scheme was designed to find the aggregated result. The linearization of the aggregated function happens to be a weighted average of individual member distribution functions. This means that the coefficient with f_i in the linearized function f corresponds to the weight of f_i (i.e., the importance of the ith expert). As such, an updating scheme for the coefficients of f was designed based on the definition of the importance of the ith expert, particularly the similarity of the ith expert's opinion to the group opinion with the overlapping between the area of f_i and the area of f.

The proposed approach was applied to a case in literature with the well-known approach called DELPHI. The approach has been shown to be very efficient; particularly for the example discussed in this paper, the convergence was reached by 3–4 iterations. A finding was also made in this paper, that is, the best aggregated distribution function should be in a narrow and centralized shape. This finding was further employed to devise the scheme for determining the initial weights in the proposed algorithm.

Appendix A
Calculation of A_i

In order to make the above procedure work, it is needed to compute A_i. Two methods are proposed for this purpose.

A. Method I—Analytical Method

If $f(x)$ and $f_i(x)$ are known, the overlapping area A_i can be calculated by

$$A_i = \int \min\{f(x), f_i(x)\}dx. \tag{A1}$$

There are a couple of ways to approximate (A1). One way is to find the intersections between $f(x)$ and $f_i(x)$, then to find $F(x)$ and $F_i(x)$. Note that w_i is initially known, and thus $F(x)$ can be found with (2). However, it may be difficult to find the intersections between $f(x)$ and $f_i(x)$ analytically, because $F(x)$ can be in a complicated form. Another way is to use a simple distribution $F^*(x)$ to approximate $F(x)$; while it should be easy to find the intersections between $F^*(x)$ and $F_i(x)$. $F^*(x)$ can be obtained by fitting a sample taken from $F(x)$. Specifically, let $\{x_j\} = x_1, x_2, \ldots, x_n$ be a set of points which depend on a particular application problem under study. Then, we can find $\{F(x_i)\} = \{F(x_1), F(x_2), \ldots, F(x_n)\}$. $F^*(x)$ can be obtained by fitting the $\{F(x_i)\}$.

B. Method II—Numerical Method

Let $\{x_j\} = x_1, x_2, \ldots, x_n$ be a set of points, $p_j = F(x_j) - F(x_{j-1}), F(x_0) = 0$, and $p_{ij} = F_i(x_j) - F_i(x_{j-1})$ $(i = 1, 2, \ldots, m$, where m is the number of experts). Then $\{p_j\}$ and $\{p_{ij}\}$ are histograms of the aggregated and the ith expert's distributions, respectively. An approximation of (A2) is given as follows:

$$A_i = \sum_{j=1}^{n} \min\left[p_j, p_{ij}\right]. \tag{A2}$$

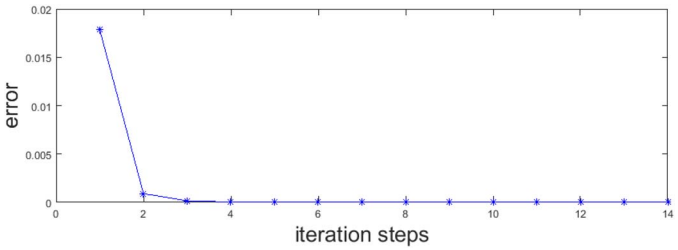

Fig. 7. Errors in the iteration procedure (when the PDFs of all experts are normal distribution).

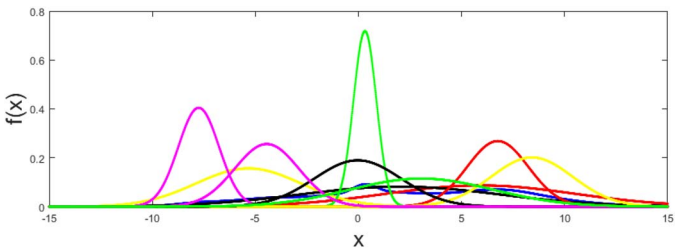

Fig. 8. Curves of ten experts PDF and the aggregated PDF (when the PDFs of all experts are normal distribution).

Appendix B
Simulated Proof of the Convergence of the Proposed Algorithm

A simulated proof of the convergence of the proposed algorithms is conducted. Let us consider that the PDFs of all experts are normal distribution $[f_i(x) = (1/\sqrt{2\pi}\sigma_i)\exp(-((x-\mu_i)^2/2\sigma_i^2))$, where μ_i is expectation and σ_i is standard deviation]. The instances are created by randomly generating the parameters in the PDF. The stopping rule is that either the algorithm has reached 100 iterations or the error $\|\boldsymbol{w}^j - \boldsymbol{w}^{j-1}\| = \sum_{i=1}^{10}(w_i^j - w_i^{j-1})^2$ is less than 10^{-10}. The algorithm was implemented in the MATLAB environment. The parameters of the PDFs of the experts are generated in particular based on the following assumptions.

1) The expectation of expert i follows the uniform distribution in the interval $[-10, 10]$.
2) The standard deviation of expert i follows the uniform distribution in the interval $[0, 5]$.
3) There are ten experts.

Fig. 7 shows the result of convergence for one instance of the ten experts; particularly, the convergence is achieved after 14 steps of iteration. Fig. 8 shows the curves of the PDFs of ten experts and the aggregated PDF (which is in blue color).

Furthermore, we consider that there are 100 instances of the ten experts. Fig. 9 shows the result of the algorithm; particularly, the average steps of iteration are 16.6 and the averaged running time is no more than 14 s.

Let us further consider that the PDFs of the experts are Weibull distribution $[f_i(x) = (k_i/\lambda_i)(x/\lambda_i)^{k_i-1}\exp(-(x/\lambda_i)^{k_i})$, where $k_i > 0$ is shape parameter and λ_i is scale parameter].

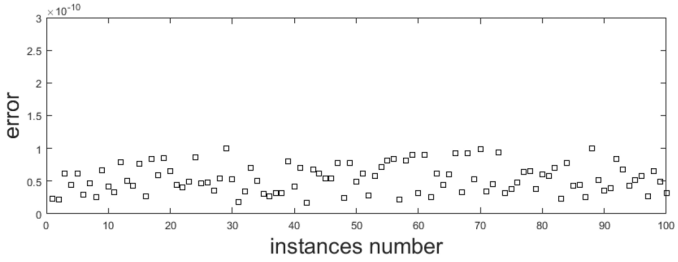

Fig. 9. Converged errors of 100 instances (when the PDFs of all experts are normal distribution).

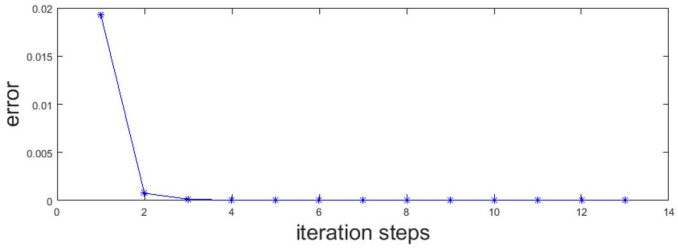

Fig. 10. Errors in the iteration procedure (when the PDFs of the experts are Weibull distribution).

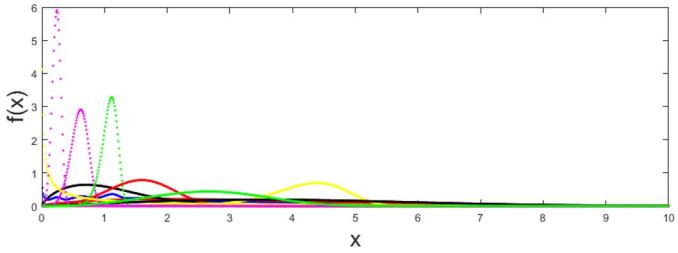

Fig. 11. Curves of ten experts PDF and the aggregated PDF (when the PDFs of the experts are Weibull distribution).

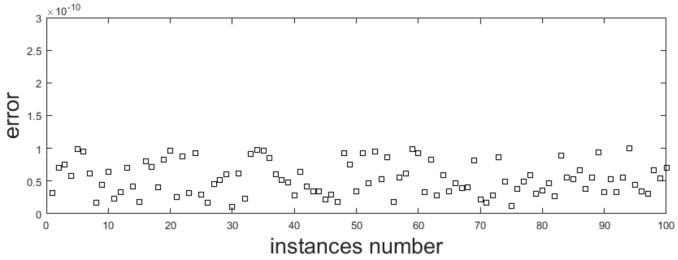

Fig. 12. Converged errors of 100 instances (when the PDFs of the experts are Weibull distribution).

The same stopping rule as for the normal distribution is taken for the Weibull distribution. The parameters of the PDFs of the experts are generated in particular based on the following assumptions.

1) The shape parameter of expert i follows the uniform distribution in the interval [0, 10].
2) The scale parameter of expert i follows the uniform distribution in the interval [0, 5].
3) There are ten experts.

Fig. 10 shows the result of convergence for one instance of the ten experts; particularly, the convergence is achieved after 60 steps of iteration. Fig. 11 shows the curves of PDFs of ten experts and the aggregated PDF (which is in blue color). Further, we consider that there are 100 instances of the ten experts. Fig. 12 shows the result of the algorithm; particularly, the average number of steps of iteration is 14 and the average running time is no more than 15 s.

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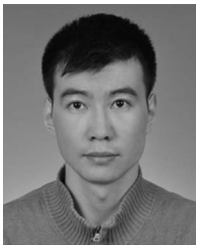

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