FFT

Abstract. beefy floating point units

1. notation

$\mathbb{i}$	$\sqrt{-1}$
$\mathbb{e}$	$2.71\ldots$
π	$3.14\ldots$
$\mathrm{EvalPoly}(a,\vec{x})$	$\sum_{0\le i<n} x_i a^i$, $\vec{x}=(x_0,\ldots,x_{n-1})$
ω_b^a	$\mathbb{e}^{2a\pi\mathbb{i}/b}$ when the base ring is $\mathbb{C}$, Otherwise some compatible b^{th} root of unity.
$\bar{i}^k$	length k bit-reversal of i. Only defined for $0\le i<2^k$

2. fft/ifft

2.1. **convolutions.** For any two vectors $\vec{x}$ and $\vec{y}$ of length n, the *cyclic convolution* $\vec{x}*\vec{y}$ is the length n vector form by the product of the corresponding polynomials modulo ϕ^n-1:

$$\mathrm{EvalPoly}(\phi,\vec{x}*\vec{y})=\mathrm{EvalPoly}(\phi,\vec{x})\cdot\mathrm{EvalPoly}(\phi,\vec{y}) \bmod \phi^n-1.$$

Since the coefficients $\vec{x}*\vec{y}$ appear linearly on the left hand side, they can be computed by an evaluation and interpolation scheme at the n roots of ϕ^n-1. A more general convolution can be defined by

$$\mathrm{EvalPoly}(\phi,\vec{x}*^a\vec{y})=\mathrm{EvalPoly}(\phi,\vec{x})\cdot\mathrm{EvalPoly}(\phi,\vec{y}) \bmod \phi^n-a^n.$$

Replacing ϕ by $a\phi$ has the effect of turning this into an ordinary cyclic convolution after rescaling the input and output by powers of a. Hence, it can be computed the same way. When $a^n=-1$ this is called a *negacyclic convolution*, and when $a^n=\pm\mathbb{i}$ this is called a *right angle convolution*.

2.2. **bit reversals and vector lengths of the form 2^k.** Unless the output of the fft is specifically needed to be in the usual order

$$\left\{\mathrm{EvalPoly}(\omega_{2^k}^i,\vec{x})\right\}_{0\le i<2^k}$$

there is no reason not to give the output in bit-reversed order

$$\left\{\mathrm{EvalPoly}(\omega_{2^k}^{\bar{i}^k},\vec{x})\right\}_{0\le i<2^k}.$$

The reason is that the bit-reversed output is much simpler and faster because it groups similar outputs close together. For example, $\mathrm{EvalPoly}(1,\vec{x})$ and $\mathrm{EvalPoly}(-1,\vec{x})$ are very similar computationally and are right next to each other in the bit-reversed output, but are very far apart in the usual order. Therefore, we will restrict exclusively to bit-reversed outputs.

The usual sequence for calculating a length 16 fft follows the columns below and ends with the fft in the last column.

(1)

x_i	y_i	z_i	w_i	$\mathrm{fft}(\vec{x})$
x_0	x_0+x_8	y_0+y_4	z_0+z_2	w_0+w_1
x_1	x_1+x_9	y_1+y_5	z_1+z_3	(w_0-w_1)
x_2	x_2+x_{10}	y_2+y_6	$(z_0-z_2)\omega_4^0$	w_2+w_3
x_3	x_3+x_{11}	y_3+y_7	$(z_2-z_3)\omega_4^1$	(w_2-w_3)
x_4	x_4+x_{12}	$(y_0-y_4)\omega_8^0$	z_4+z_6	w_4+w_5
x_5	x_5+x_{13}	$(y_1-y_5)\omega_8^1$	z_5+z_7	(w_4-w_5)
x_6	x_6+x_{14}	$(y_2-y_6)\omega_8^2$	$(z_4-z_6)\omega_4^0$	w_6+w_7
x_7	x_7+x_{15}	$(y_3-y_7)\omega_8^3$	$(z_5-z_7)\omega_4^1$	(w_6-w_7)
x_8	$(x_0-x_8)\omega_{16}^0$	y_8+y_{12}	z_8+z_{10}	w_8+w_9
x_9	$(x_1-x_9)\omega_{16}^1$	y_9+y_{13}	z_9+z_{11}	(w_8-w_9)
x_{10}	$(x_2-x_{10})\omega_{16}^2$	$y_{10}+y_{14}$	$(z_8-z_{10})\omega_4^0$	$w_{10}+w_{11}$
x_{11}	$(x_3-x_{11})\omega_{16}^3$	$y_{11}+y_{15}$	$(z_9-z_{11})\omega_4^1$	$(w_{10}-w_{11})$
x_{12}	$(x_4-x_{12})\omega_{16}^4$	$(y_8-y_{12})\omega_8^0$	$z_{12}+z_{14}$	$w_{12}+w_{13}$
x_{13}	$(x_5-x_{13})\omega_{16}^5$	$(y_9-y_{13})\omega_8^1$	$z_{13}+z_{15}$	$(w_{12}-w_{13})$
x_{14}	$(x_6-x_{14})\omega_{16}^6$	$(y_{10}-y_{14})\omega_8^2$	$(z_{12}-z_{14})\omega_4^0$	$w_{14}+w_{15}$
x_{15}	$(x_7-x_{15})\omega_{16}^7$	$(y_{11}-y_{15})\omega_8^3$	$(z_{13}-z_{15})\omega_4^1$	$(w_{14}-w_{15})$

One problem with this approach is that each column accesses many different *twiddle factors* ω_b^a. In the case of a Schönhage–Strassen fft where the base ring is $\mathbb{Z}/(2^m+1)\mathbb{Z}$, this doesn't matter because each twiddle factor is a power of two and implemented via bit shifts. In other cases, these twiddle factors have to be either computed on the fly or precomputed and then retrieved from storage. In the case of precomputation, we have to have in memory the table

(2) $$1,\omega_2^1,1,\omega_4^1,1,\omega_8^1,\omega_8^2,\omega_8^3,1,\omega_{16}^1,\omega_{16}^2,\omega_{16}^3,\omega_{16}^4,\omega_{16}^5,\omega_{16}^6,\omega_{16}^7,1,\dots$$

so that the columns can access this table sequentially. However, such a table is nice because once it is extended to accommodate an fft of a certain length, it can be reused for all ffts of smaller length.

By rearranging the twiddle factors as ($\omega=\omega_{16}$)

(3)

x_i	y_i	z_i	w_i	$\mathrm{fft}(\vec{x})$
x_0	$x_0+\omega^0x_8$	$y_0+\omega^0y_4$	$z_0+\omega^0z_2$	$w_0+\omega^0w_1$
x_1	$x_1+\omega^0x_9$	$y_1+\omega^0y_5$	$z_1+\omega^0z_3$	$w_0+\omega^8w_1$
x_2	$x_2+\omega^0x_{10}$	$y_2+\omega^0y_6$	$z_0+\omega^8z_2$	$w_2+\omega^4w_3$
x_3	$x_3+\omega^0x_{11}$	$y_3+\omega^0y_7$	$z_1+\omega^8z_3$	$w_2+\omega^{12}w_3$
x_4	$x_4+\omega^0x_{12}$	$y_0+\omega^8y_4$	$z_4+\omega^4z_6$	$w_4+\omega^2w_5$
x_5	$x_5+\omega^0x_{13}$	$y_1+\omega^8y_5$	$z_5+\omega^4z_7$	$w_4+\omega^{10}w_5$
x_6	$x_6+\omega^0x_{14}$	$y_2+\omega^8y_6$	$z_4+\omega^{12}z_6$	$w_6+\omega^6w_7$
x_7	$x_7+\omega^0x_{15}$	$y_3+\omega^8y_7$	$z_5+\omega^{12}z_7$	$w_6+\omega^{14}w_7$
x_8	$x_0+\omega^8x_8$	$y_8+\omega^4y_{12}$	$z_8+\omega^2z_{10}$	$w_8+\omega^1w_9$
x_9	$x_1+\omega^8x_9$	$y_9+\omega^4y_{13}$	$z_9+\omega^2z_{11}$	$w_8+\omega^9w_9$
x_{10}	$x_2+\omega^8x_{10}$	$y_{10}+\omega^4y_{14}$	$z_8+\omega^{10}z_{10}$	$w_{10}+\omega^5w_{11}$
x_{11}	$x_3+\omega^8x_{11}$	$y_{11}+\omega^4y_{15}$	$z_9+\omega^{10}z_{11}$	$w_{10}+\omega^{13}w_{11}$
x_{12}	$x_4+\omega^8x_{12}$	$y_8+\omega^{12}y_{12}$	$z_{12}+\omega^6z_{14}$	$w_{12}+\omega^3w_{13}$
x_{13}	$x_5+\omega^8x_{13}$	$y_9+\omega^{12}y_{13}$	$z_{13}+\omega^6z_{15}$	$w_{12}+\omega^{11}w_{13}$
x_{14}	$x_6+\omega^8x_{14}$	$y_{10}+\omega^{12}y_{14}$	$z_{12}+\omega^{14}z_{14}$	$w_{14}+\omega^7w_{15}$
x_{15}	$x_7+\omega^8x_{15}$	$y_{11}+\omega^{12}y_{15}$	$z_{13}+\omega^{14}z_{15}$	$w_{14}+\omega^{15}w_{15}$

there are still the same number of twiddle factor multiplications, but each twiddle factor itself can be reused in each column. Also, as with the previous method, there is a universal (bit-reversed) table

(4) $$1,\quad \omega_2^1,\quad \omega_4^1,\omega_4^3,\quad \omega_8^1,\omega_8^5,\omega_8^3,\omega_8^7,\quad \omega_{16}^1,\omega_{16}^9,\omega_{16}^5,\omega_{16}^{13},\omega_{16}^3,\omega_{16}^{11},\omega_{16}^7,\omega_{16}^{15},\quad \dots$$

TABLE 1. Integer multiplication with complex 53 bit ffts: The two input numbers are viewed as polynomials evaluated at 2^m so that the fft has inputs in the range $[0, 2^m)$. The output coefficients from the ifft are eventually unreliable due to rounding errors: let $f(m)$ denote the maximum bit size of an answer which can be guaranteed correct by this algorithm.

m	upper bound on $f(m)$
22	3008
21	11968
20	30464
19	117632
18	417280
17	1246592
16	1984000

that can be reused. The difference here is that the portion that is used for a specific fft is now half the size (making note of $\omega_4^3 = -\omega_4^1$, $\omega_8^5 = -\omega_8^1$, $\omega_8^7 = -\omega_8^3$, etc.).

Since the output of the fft is given in bit-reversed order, the inverse operation cannot simply replace ω by ω^{-1} and use the same calculation sequence. Thus the ifft has to invert the left-to-right operation by working from the right to the left and inverting each basic operation. This gives slightly different data access patterns but involves essentially the same calculations. For example, the operation $w_8 = z_8 + \omega^2 z_{10}, w_{10} = z_8 - \omega^2 z_{10}$ is inverted by $2z_8 = w_8 + w_{10}, 2z_{10} = \omega^{-2}(w_8 - w_{10})$ and the negative power $\omega^{-\bar{i}^k}$ can be looked up in the table by flipping all bits of i except the highest.

2.3. **fft on sizes other than powers of two.** Suppose the vector to transform has length thrice a power of two: $x_0, x_1, \ldots, x_{3\cdot 2^k-1}$. If, for each $0 \le i < 2^k$, we first perform

$$\begin{aligned} y_i^{(0)} &= x_{i+0\cdot 2^k} + x_{i+1\cdot 2^k} + x_{i+2\cdot 2^k} \\ y_i^{(1)} &= \omega_{3\cdot 2^k}^{i}(x_{i+0\cdot 2^k} + \omega_3^1 x_{i+1\cdot 2^k} + \omega_3^{-1} x_{i+2\cdot 2^k}) \\ y_i^{(2)} &= \omega_{3\cdot 2^k}^{-i}(x_{i+0\cdot 2^k} + \omega_3^{-1} x_{i+1\cdot 2^k} + \omega_3^{1} x_{i+2\cdot 2^k}), \end{aligned}$$

then the three ffts of the length 2^k sequences $y^{(0)}, y^{(1)}, y^{(2)}$ will give the fft of the original sequence. However, while the transforms of $y^{(0)}$ and $y^{(1)}$ will have the same ordering, the sequence $y^{(2)}$ will transform to a different ordering due to the negative power. More precisely,

$$\begin{aligned} \mathrm{EvalPoly}(\omega_{3\cdot 2^k}^{3i}, x) &= \mathrm{EvalPoly}(\omega_{2^k}^{i}, y^{(0)}) \\ \mathrm{EvalPoly}(\omega_{3\cdot 2^k}^{3i+1}, x) &= \mathrm{EvalPoly}(\omega_{2^k}^{i}, y^{(1)}) \\ \mathrm{EvalPoly}(\omega_{3\cdot 2^k}^{3i-1}, x) &= \mathrm{EvalPoly}(\omega_{2^k}^{i}, y^{(2)}). \end{aligned}$$

2.4. **considerations for the base ring $\mathbb{C}$.** When arithmetic in the base ring necessarily includes roundoff error, it is important for the fft to be implemented as accurately as possible. Since both (1) and (3) compute the bit-reversed fft, we are free to use either. Using (3) for the forward transform and inverting (1) for the inverse transform gives calculation sequences for both that consist *entirely of fused-multiply-add operations*. The ifft will read often from the table (2), but this is of little consequence since large convolution lengths are not feasible for this base ring (see Table 1).

The case of real input and output data is important and can be optimized. Feeding in purely real data to the fft and getting back real data from the ifft represents only a 50% utilization of the circuits (1) or (3), since the data possesses certain symmetries under complex conjugation in each column of the calculation. First, and most elementary, two real data sets $\vec{a}$ and $\vec{b}$ of the same length can be transformed with one complex transformation of the same length, and this is nothing but the linearity of the fft operation:

$$\begin{aligned}\mathrm{EvalPoly}(\omega, \vec{a} + \mathbb{i}\vec{b}) &= \mathrm{EvalPoly}(\omega, \vec{a}) + \mathbb{i}\,\mathrm{EvalPoly}(\omega, \vec{b}) \\ \overline{\mathrm{EvalPoly}(\omega^{-1}, \vec{a} + \mathbb{i}\vec{b})} &= \mathrm{EvalPoly}(\omega, \vec{a}) - \mathbb{i}\,\mathrm{EvalPoly}(\omega, \vec{b})\end{aligned}$$

This is fine for doubling the processing speed of an even number of real data sets of the same length, but if we would like to process one real data set twice as fast, the following identity can be used.

$$\mathrm{EvalPoly}(\omega, (x_0, \dots, x_{2n-1})) = \mathrm{EvalPoly}(\omega^2, (x_0, x_2, \dots, x_{2n-2})) + \omega\,\mathrm{EvalPoly}(\omega^2, (x_1, x_3, \dots, x_{2n-1})).$$

The cyclic convolution of real coefficients $\{a_i\}$ and $\{b_i\}$ to form product coefficients $\{c_i\}$ can then take the form

$$\begin{matrix} a_0 + \mathbb{i}b_0 \\ a_1 + \mathbb{i}b_1 \\ \dots \\ a_{15} + \mathbb{i}b_{15} \end{matrix} \underset{\text{fft}_{16}}{\Longrightarrow} \begin{matrix} f_0 \\ f_8 \\ f_4 \\ f_{12} \\ \dots \\ f_{14} \\ f_1 \\ \dots \\ f_7 \\ f_{15} \end{matrix} \Longrightarrow \begin{matrix} \hat{a}_0 \cdot \hat{b}_0 = \hat{c}_0 \\ \hat{a}_8 \cdot \hat{b}_8 = \hat{c}_8 \\ \hat{a}_4 \cdot \hat{b}_4 = \hat{c}_4 \\ \dots \\ \hat{a}_{14} \cdot \hat{b}_{14} = \hat{c}_{14} \\ \hat{a}_1 \cdot \hat{b}_1 = \hat{c}_1 \\ \dots \\ \hat{a}_7 \cdot \hat{b}_7 = \hat{c}_7 \\ \hat{a}_{15} \cdot \hat{b}_{15} = \hat{c}_{15} \end{matrix} \Longrightarrow \begin{matrix} \hat{e}_0 + \mathbb{i}\hat{o}_0 \\ \hat{e}_4 + \mathbb{i}\hat{o}_4 \\ \hat{e}_2 + \mathbb{i}\hat{o}_2 \\ \hat{e}_6 + \mathbb{i}\hat{o}_6 \\ \hat{e}_1 + \mathbb{i}\hat{o}_1 \\ \hat{e}_5 + \mathbb{i}\hat{o}_5 \\ \hat{e}_3 + \mathbb{i}\hat{o}_3 \\ \hat{e}_7 + \mathbb{i}\hat{o}_7 \end{matrix} \underset{\text{ifft}_8}{\Longrightarrow} \begin{matrix} c_0 + \mathbb{i}c_1 \\ c_2 + \mathbb{i}c_3 \\ c_4 + \mathbb{i}c_5 \\ c_6 + \mathbb{i}c_7 \\ c_8 + \mathbb{i}c_9 \\ c_{10} + \mathbb{i}c_{11} \\ c_{12} + \mathbb{i}c_{13} \\ c_{14} + \mathbb{i}c_{15} \end{matrix}$$

Denoting by $\bar{i}$ the result of flipping all bits in i but its lowest (set) bit, the $\hat{a}_i$ and $\hat{b}_i$ are recovered from the f_i, and the e_i and the o_i are recovered from the $\hat{c}_i$ by

$$\begin{aligned} f_i &= \hat{a}_i + \mathbb{i}\hat{b}_i, \quad \hat{c}_{i+0} = \hat{e}_i + \omega^i \hat{o}_i \\ \overline{f_{\bar{i}}} &= \hat{a}_i - \mathbb{i}\hat{b}_i, \quad \hat{c}_{i+8} = \hat{e}_i - \omega^i \hat{o}_i \end{aligned}.$$

The above strategy is well suited for unbalanced multiplications. For balanced multiplications, and certainly for squaring, a right angle convolution is more effective as it outputs the lower and upper coefficients of the product in the real and imaginary components of the convolution without much zero padding of the inputs.

3. Integer multiplication

Figures 1, 2 and 3 show timings for three integer multiplication algorithms:

- variable prime fft: between three and nine 50 bit primes are used along with a truncated recursive matrix Fourier algorithm. Intended for large operands as this is a cache friendly algorithm.
- four prime fft: four fixed 50 bit primes are used to combat the slow nature of a variable number of primes at smaller sizes. Not intended for large operands as no effort has been made to be cache friendly.
- complex fft: 53 bit complex floating point with the real optimizations of Section 2.4. No answers here are provably correct, but coefficient sizes were taken up to about 75% of the observed failure points in Table 1. No truncation is performed at all because the real optimizations complicate matters in the fft and the truncated ifft loses accuracy. The fft sizes instead include small odd multiples (i.e. $1, 3, 5$, see Section 2.3) of powers of two.

For each plotted value of n, the average, maximum, and minimum timings of integer multiplications where the product always has n bits and the smaller operand ranges between $n/2$ and $n/4$ bits are shown with three dots.

FIGURE 1. n bit integer product on desktop Bulldozer

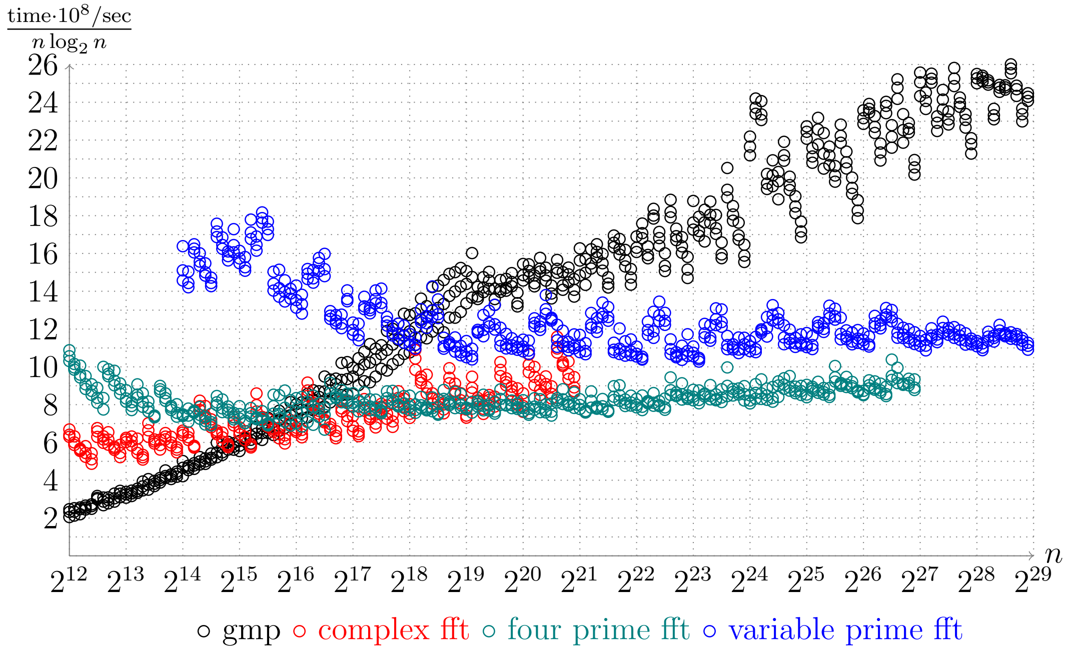
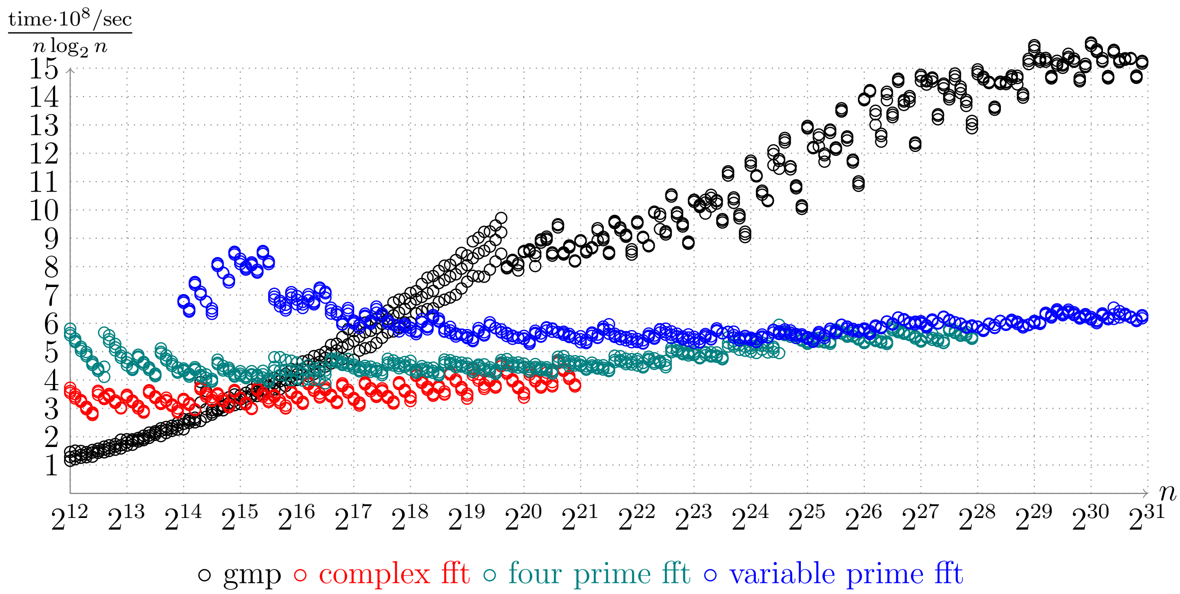


FIGURE 2. n bit integer product on mobile Zen 2



4. TRUNCATION

If the idea of zero padding the real input data to complex data in the previous section sounded bad, then the idea of zero padding the overall input data to the next power-of-two size should sound even worse. For this reason, a *truncated fft* is necessary, which assumes certain portions of the input and output are zero. If the desired output length is denoted by n we expect a runtime proportional to $n \log n$, so the more constant this ratio is the better the truncation is working. In order to graph this ratio, we adopt fftw's notion of a flop: an fft or ifft of length n is equivalent to $5n\log_2(n)$ floating point operations. Figures 4, 5 and 6 show a graph of such a ratio for the three ffts in Section 3 with the input further truncated to length $n/2$.

- The complex fft is not truncated but rather working with lengths of the form 2^k, $3 \cdot 2^k$, or $5 \cdot 2^k$.

FIGURE 3. n bit integer product on desktop Coffee Lake

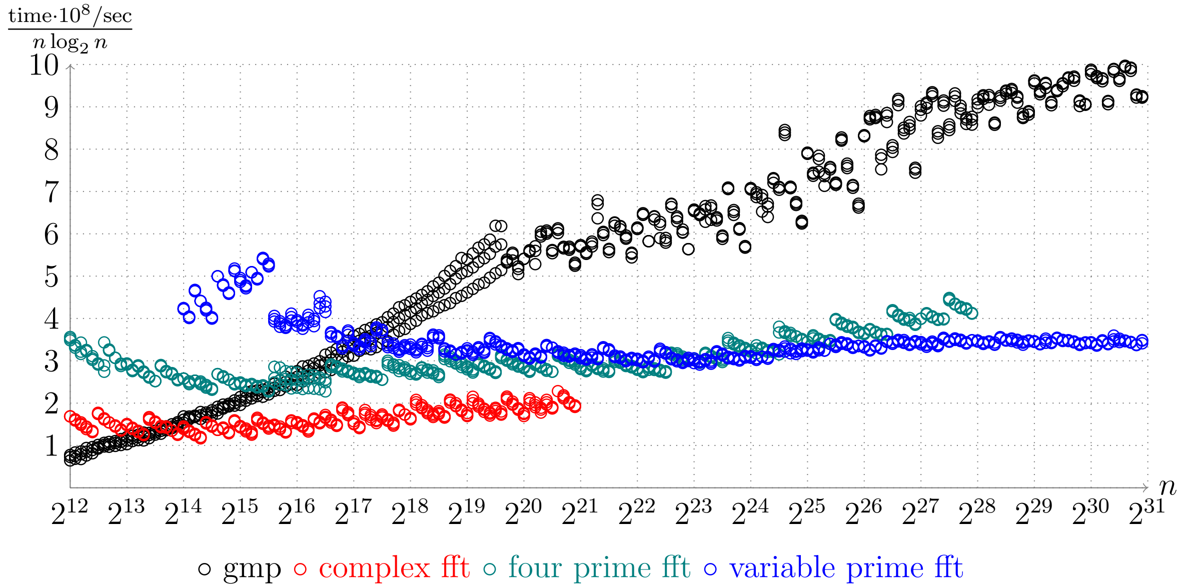
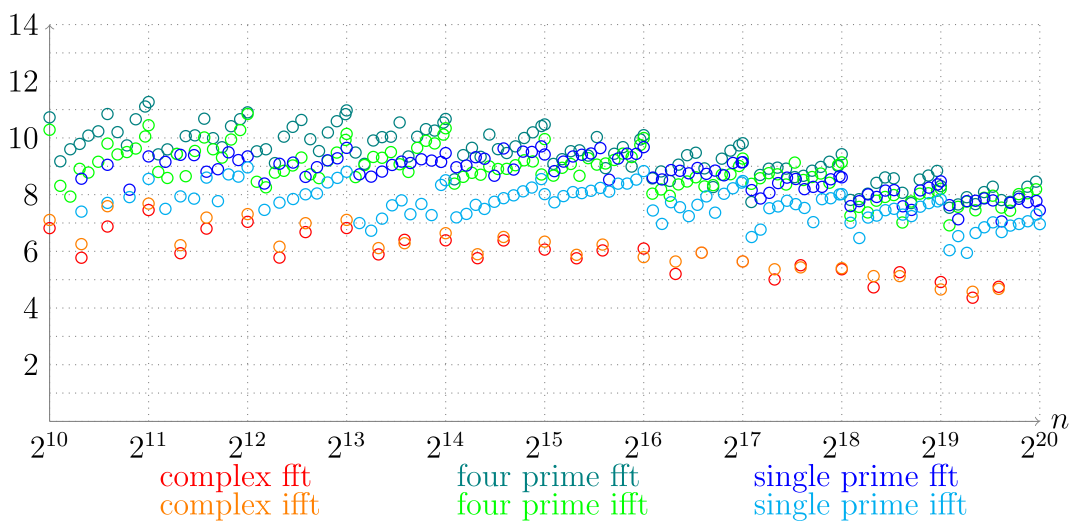


FIGURE 4. Truncation effectiveness in gflops on desktop Bulldozer



- The timings for the four prime fft give the average per-prime timing.
- The single prime fft should be slower than the amortized four prime time for smaller sizes because horizontal action within vector registers is required when processing a single prime.

It seems that the speed of the fft over $\mathbb{C}$ is never more than a factor of two away from the speed over $\mathbb{F}_p$, with the latest processor showing the most impressive performance. What gives the complex fft its special speed in Section 3 is the fact that each data point has a real and imaginary part, so the fft is really processing twice as much data and this is used by the optimizations for real input and output.

Figure 7 shows the corresponding graph for FLINT's Schönhage–Strassen fft, where very large coefficients had to be used to ensure that the graph can continue to $n = 2^{18}$ while keeping the coefficient size constant.

Figure 5. Truncation effectiveness in gflops on mobile Zen 2

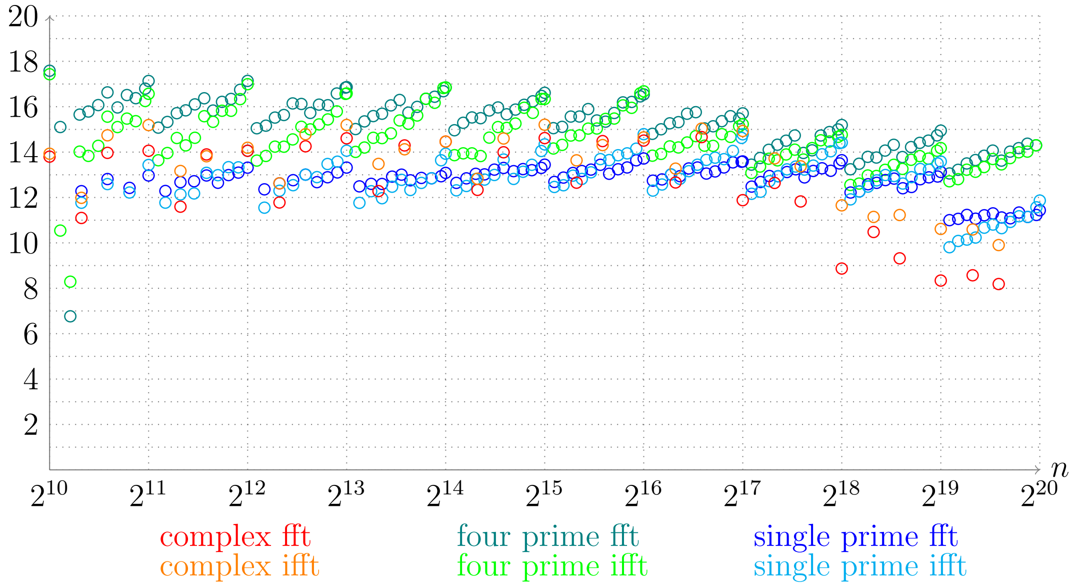


Figure 6. Truncation effectiveness in gflops on desktop Coffee Lake

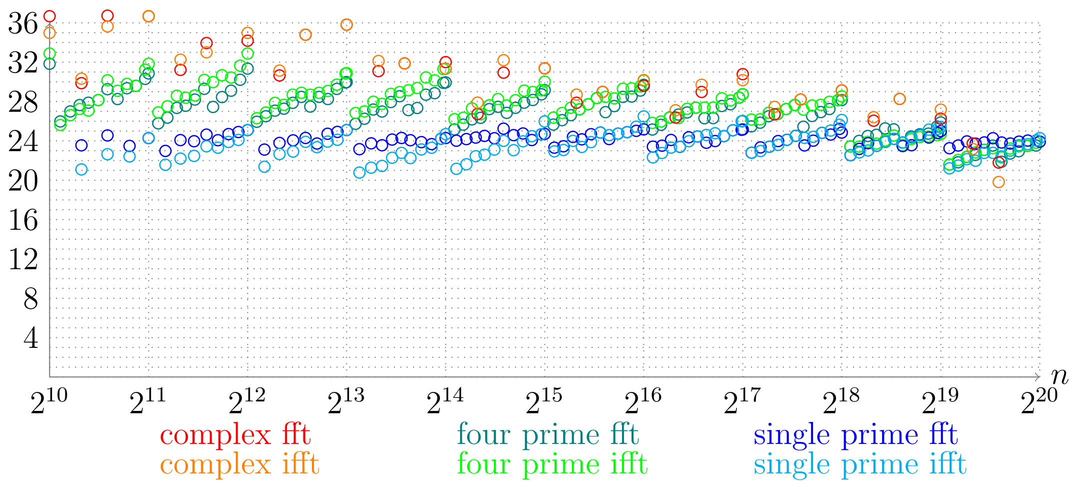


Figure 7. Truncation effectiveness of {i}`fft_mfa_truncate_sqrt2` with 2^{16} bit coefficients