DD2434 - Machine Learning, Advanced Course
Assignment 1A

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Contents

1   Exponential Family

Question 1.1

$$
\begin{aligned}
p(x|\theta) &= h(x)\exp(\eta(\theta)\cdot T(x) - A(\eta)) \\
&= h(x)\exp(\eta(\lambda)\cdot T(x) - A(\eta(\lambda))) \\
&= h(x)\exp(\log\lambda \cdot x - A(\log\lambda)) \\
&= h(x)\exp(\log\lambda \cdot x - \lambda) \\
&= h(x)\exp(\log\lambda \cdot x)\exp(-\lambda) \\
&= e^{-\lambda}\frac{\lambda^x}{x!}
\end{aligned}
\tag{1}
$$

We can see that the distribution correspond to a <u>Poisson distribution of parameter λ</u>.

Question 1.2

$$
\begin{aligned}
p(x|\theta) &= h(x)\exp(\eta(\theta)\cdot T(x) - A(\eta)) \\
&= \exp(\eta([\alpha,\beta])\cdot[\log x, x] - A(\alpha - 1, -\beta)) \\
&= \exp([\alpha - 1, -\beta]\cdot[\log x, x] - \log\Gamma(\alpha) + \alpha\log(\beta)) \\
&= \exp((\alpha - 1)\log x - \beta x - \log\Gamma(\alpha) + \alpha\log(\beta)) \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}
\end{aligned}
\tag{2}
$$

We can see that the distribution correspond to a <u>Gamma distribution of parameters α and β</u>.

Question 1.3

$$
\begin{aligned}
p(x|\theta) &= h(x)\exp(\eta(\theta)\cdot T(x) - A(\eta)) \\
&= \frac{\exp(\eta([\mu,\sigma^2])\cdot[x,x^2] - A(\eta([\mu,\sigma^2])))}{\sqrt{2\pi}} \\
&= \frac{\exp([\frac{\mu}{\sigma^2}, -\frac{1}{2\sigma^2}]\cdot[x,x^2] - A([\frac{\mu}{\sigma^2}, -\frac{1}{2\sigma^2}]))}{\sqrt{2\pi}} \\
&= \frac{\exp(\frac{\mu x}{\sigma^2} - \frac{x^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2} - \log\sigma)}{\sqrt{2\pi}} \\
&= \frac{\exp(-\frac{(x-\mu)^2}{2\sigma^2})}{\sigma\sqrt{2\pi}}
\end{aligned}
\tag{3}
$$

We can see that the distribution correspond to a <u>Normal distribution of parameters μ and σ^2</u>.

Question 1.4

$$
\begin{aligned}
p(x|\theta) &= h(x)\exp(\eta(\theta)\cdot T(x) - A(\eta)) \\
&= 2\exp(\eta(\lambda)\cdot x - A(\eta(\lambda))) \\
&= 2\exp(-\lambda x - A(-\lambda)) \\
&= 2\exp\left(-\lambda x + \log\left(\frac{\lambda}{2}\right)\right) \\
&= \lambda e^{-\lambda x}
\end{aligned}
\tag{4}
$$

We can see that the distribution correspond to a <u>Exponential distribution of parameter λ</u>.

Question 1.5

$$
\begin{aligned}
p(x|\theta) &= h(x)\exp(\eta(\theta)\cdot T(x) - A(\eta)) \\
&= \exp(\eta([\psi_1,\psi_2])\cdot[\log x, \log(1-x)] - A(\eta([\psi_1,\psi_2]))) \\
&= \exp([\psi_1-1,\psi_2-1]\cdot[\log x, \log(1-x)] - A([\psi_1-1,\psi_2-1])) \\
&= \exp((\psi_1-1)\log x + (\psi_2-1)\log(1-x) - \log\Gamma(\psi_1) - \log\Gamma(\psi_2) + \log\Gamma(\psi_1+\psi_2)) \\
&= \frac{\Gamma(\psi_1+\psi_2)}{\Gamma(\psi_1)\Gamma(\psi_2)}x^{\psi_1-1}(1-x)^{\psi_2-1}
\end{aligned}
\tag{5}
$$

We can see that the distribution correspond to a <u>Beta distribution of parameters ψ_1 and ψ_2</u>.
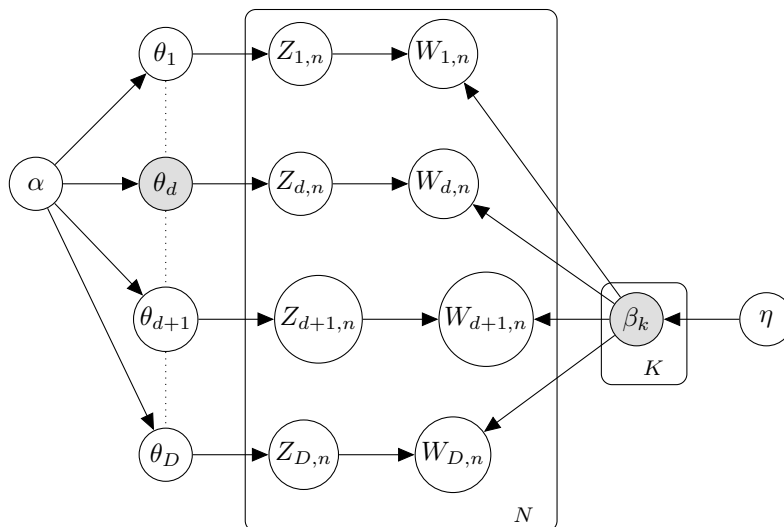
2 Dependencies in a Directed Graphical Model
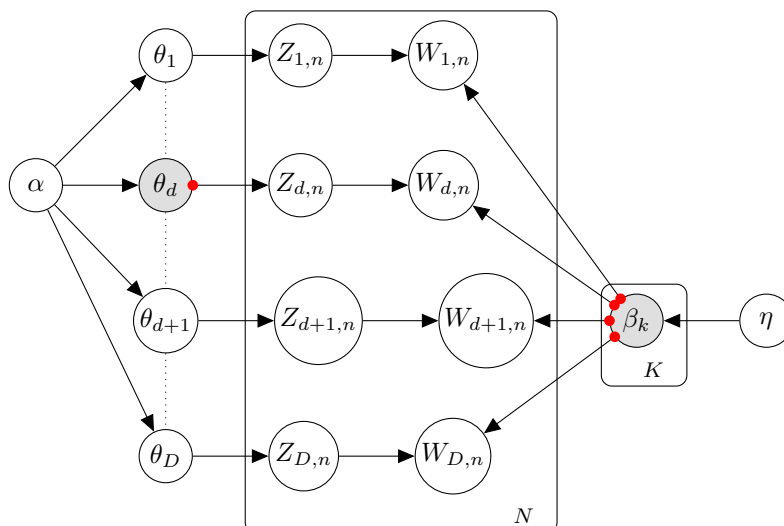
Question 2.6

Figure 1: Graphical model of smooth LDA.

The Bayes net take this form :

Then, if we use the method using the d-separation, we obtain this :

Therefore, we can see that $W_{d,n} \perp W_{d,n+1}|\theta_d, \beta_{1:K}$ is <u>true</u>.

Question 2.7

The Bayes net take this form (with d-separation marks) :

Therefore, we can see that $\theta_d \perp \theta_{d+1} | Z_{d,1:N}$ is <u>false</u>.

Question 2.8

The Bayes net take this form (with d-separation marks) :

Therefore, we can see that $\theta_d \perp \theta_{d+1} | \alpha, Z_{1:D,1:N}$ is <u>true</u>.

Question 2.9

Figure 2: Graphical model of Labeled LDA.

The Bayes net take this form (with d-separation marks) :

Therefore, we can see that $W_{d,n} \perp W_{d,n+1} | \Lambda_d, \beta_{1:K}$ is <u>false</u>.

Question 2.10

The Bayes net take this form (with d-separation marks) :

Therefore, we can see that $\theta_d \perp \theta_{d+1} | Z_{d,1:N}, Z_{d+1,1:N}$ is <u>false</u>.

Question 2.11

The Bayes net take this form (with d-separation marks) :

Therefore, we can see that $\Lambda_d \perp \Lambda_{d+1} | \Phi, Z_{1:D,1:N}$ is <u>false</u>.
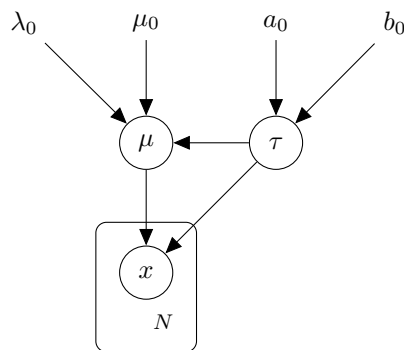
3 CAVI

Figure 3: DGM

Question 3.12

In the bishop book, we can see that :

$$p(X|\mu,\tau) = \left(\frac{\tau}{2\pi}\right)^{N/2} \exp\left\{-\frac{\tau}{2}\sum_{n=1}^{N}(x_n - \mu)^2\right\} \tag{6}$$

$$p(\mu|\tau) = \mathcal{N}(\mu|\mu_0, (\lambda_0\tau)^{-1}) \tag{7}$$

$$p(\tau) = \text{Gam}(\tau|a_0, b_0) \tag{8}$$
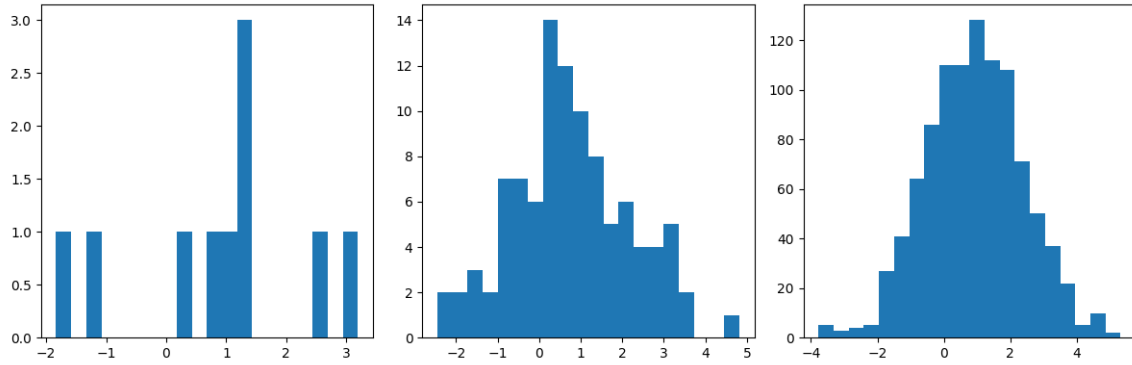
Then, by using the code in appendix A.1, we obtain :

Figure 4: Generated Data

Question 3.13

Let's find the ML estimates of μ and τ. We know that $\log(q^*(\mu)) = \mathbb{E}_{-\mu}[\log p(X, \mu, \tau)]$. Then, we can write :

$$
\begin{aligned}
\log(q^*(\mu)) &= \mathbb{E}_{-\mu}[\log p(X, \mu, \tau)] \\
&\overset{+}{=} \mathbb{E}_{\tau}[\log p(X|\mu,\tau) + \log p(\mu|\tau)] \\
&= \mathbb{E}_{\tau}\left[\frac{N}{2}\log\left(\frac{\tau}{2\pi}\right) - \frac{\tau}{2}\sum_{n=1}^{N}(x_n - \mu)^2 + \frac{1}{2}\log\left(\frac{\lambda_0\tau}{2\pi}\right) - \frac{\lambda_0\tau}{2}(\mu - \mu_0)^2\right] \\
&\overset{+}{=} -\frac{\mathbb{E}_{\tau}[\tau]}{2}\left(\lambda_0(\mu - \mu_0)^2 + \sum_{n=1}^{N}(x_n - \mu)^2\right) \\
&= -\frac{\mathbb{E}_{\tau}[\tau]}{2}\left(\lambda_0\mu^2 - 2\lambda_0\mu\mu_0 + \lambda_0\mu_0^2 + \sum_{n=1}^{N}x_n^2 - 2\mu\sum_{n=1}^{N}x_n + N\mu^2\right) \\
&= -\frac{\mathbb{E}_{\tau}[\tau]}{2}\left((\lambda_0 + N)\mu^2 - 2(\lambda_0\mu_0 + \sum_{n=1}^{N}x_n)\mu + \lambda_0\mu_0^2 + \sum_{n=1}^{N}x_n^2\right) \\
&\overset{+}{=} -\frac{\mathbb{E}_{\tau}[\tau](\lambda_0 + N)}{2}\left(\mu^2 - 2\mu\frac{\lambda_0\mu_0 + \sum_{n=1}^{N}x_n}{\lambda_0 + N}\right)
\end{aligned}
\tag{9}
$$

Therefore we can conclude that $q^*(\mu) = \mathcal{N}(\mu|\mu_N, \lambda_N^{-1})$ with :

$$\mu_N = \frac{\lambda_0 \mu_0 + \sum_{n=1}^{N} x_n}{\lambda_0 + N} \tag{10}$$

$$\lambda_N = (\lambda_0 + N)\mathbb{E}[\tau] \tag{11}$$

And for τ we have :

$$
\begin{aligned}
\log(q^*(\tau)) &= \mathbb{E}_{-\tau}[\log p(X, \mu, \tau)] \\
&\overset{+}{=} \mathbb{E}_\mu[\log p(X|\mu, \tau) + \log p(\mu|\tau)] + \log p(\tau) \\
&\overset{+}{=} (a_0 - 1)\log \tau - b_0 \tau + \frac{N+1}{2}\log \tau - \frac{\tau}{2}\mathbb{E}_\mu\left[\sum_{n=1}^{N}(x_n - \mu)^2 + \lambda_0(\mu - \mu_0)^2\right] \\
&= \left(a_0 + \frac{N+1}{2} - 1\right)\log \tau - \left(b_0 + \frac{1}{2}\mathbb{E}_\mu\left[\sum_{n=1}^{N}(x_n - \mu)^2 + \lambda_0(\mu - \mu_0)^2\right]\right)\tau
\end{aligned}
\tag{12}
$$

Therefore we can conclude that $q^*(\tau) = \text{Gam}(\tau|a_N, b_N)$ with :

$$a_N = a_0 + \frac{N+1}{2} \tag{13}$$

$$
\begin{aligned}
b_N &= b_0 + \frac{1}{2}\mathbb{E}_\mu\left[\sum_{n=1}^{N}(x_n - \mu)^2 + \lambda_0(\mu - \mu_0)^2\right] \\
b_N &= b_0 + \frac{1}{2}\left(\sum_{n=1}^{N} x_n^2 + N\mathbb{E}_\mu[\mu^2] - 2\mathbb{E}_\mu[\mu]\sum_{n=1}^{N} x_n + \lambda_0\left(\mathbb{E}_\mu[\mu^2] + \mu_0^2 - 2\mu_0\mathbb{E}_\mu[\mu]\right)\right)
\end{aligned}
\tag{14}
$$

With :

$$
\begin{aligned}
\mathbb{E}_{q(\mu)}[\mu] &= \mu_N \\
\mathbb{E}_{q(\mu)}[\mu^2] &= \frac{1}{\lambda_N} + \mu_N^2 \\
\mathbb{E}_{q(\tau)}[\tau] &= \frac{a_N}{b_N}
\end{aligned}
\tag{15}
$$

If we take non-informative priors then $a_0 = b_0 = \mu_0 = \lambda_0 = 0$, then we have :

$$
\begin{aligned}
\mu_N &= \overline{x} \\
\lambda_N &= N\mathbb{E}[\tau] \\
a_N &= \frac{N+1}{2} \\
b_N &= \frac{1}{2}\mathbb{E}_\mu\left[\sum_{n=1}^{N}(x_n - \mu)^2\right]
\end{aligned}
\tag{16}
$$

And by using $\mathbb{E}[\tau] = \frac{a_N}{b_N}$ we obtain :

$$
\begin{aligned}
\frac{1}{\mathbb{E}[\tau]} &= \frac{b_N}{a_N} \\
\frac{1}{\mathbb{E}[\tau]} &= \frac{2}{2(N+1)} \mathbb{E}_\mu \left[\sum_{n=1}^{N} (x_n - \mu)^2 \right] \\
\frac{1}{\mathbb{E}[\tau]} &= \frac{N}{N+1} \left(\overline{x^2} - 2\overline{x}\mathbb{E}[\mu] + \mathbb{E}[\mu^2] \right)
\end{aligned}
\tag{17}
$$

And, with the fact that $\mathbb{E}[\mu] = \mu_N$ and $\mathbb{E}[\mu^2] = \frac{1}{\lambda_N} + \mu_N^2$, we obtain :

$$
\begin{aligned}
\mathbb{E}[\mu] &= \overline{x} \\
\mathbb{E}[\mu^2] &= \frac{1}{N\mathbb{E}[\tau]} + \overline{x}^2
\end{aligned}
\tag{18}
$$

And therefore:

$$
\begin{aligned}
\frac{1}{\mathbb{E}[\tau]} = \frac{N}{N+1} \left(\overline{x^2} - 2\overline{x}^2 + \frac{1}{N\mathbb{E}[\tau]} + \overline{x}^2 \right) &\Leftrightarrow \frac{1}{\mathbb{E}[\tau]} - \frac{1}{(N+1)\mathbb{E}[\tau]} = \frac{N}{N+1} \left(\overline{x^2} - \overline{x}^2 \right) \\
&\Leftrightarrow \frac{N+1-1}{(N+1)\mathbb{E}[\tau]} = \frac{N}{N+1} \left(\overline{x^2} - \overline{x}^2 \right) \\
&\Leftrightarrow \frac{1}{\mathbb{E}[\tau]} = \left(\overline{x^2} - \overline{x}^2 \right) \\
&\Leftrightarrow \frac{1}{\mathbb{E}[\tau]} = \frac{1}{N} \sum_{n=1}^{N} (x_n - \overline{x})^2
\end{aligned}
\tag{19}
$$

Which define the ML estimates. The implementation is in the code in appendix A.1.

Question 3.14

The posterior is defined as $p(\mu, \tau | x)$. Then, we can write :

$$
\begin{aligned}
p(\mu, \tau | x) &= \frac{p(x|\mu,\tau)p(\mu,\tau)}{p(x)} \\
&\propto p(x|\mu,\tau)p(\mu,\tau)
\end{aligned}
\tag{20}
$$

Where $x|\mu, \tau \sim \mathcal{N}(\mu|\mu, \tau^{-1})$ and $\mu, \tau \sim NormalGamma(\mu_0, \lambda_0, a_0, b_0)$. Therefore, as we saw in the question 1.3 in the Module 1 exercise, we have $\mu, \tau | x \sim NormalGamma(\mu', \lambda', a', b')$, where :

$$
\begin{aligned}
\mu' &= \frac{N\overline{x} + \mu_0\lambda_0}{N + \lambda_0} \\
\lambda' &= N + \lambda_0 \\
a' &= a_0 + \frac{N-1}{2} \\
b' &= b_0 + \frac{1}{2} \left(\sum_{n=1}^{N} x_n^2 + \lambda_0\mu_0^2 - \frac{(N\overline{x} + \mu_0\lambda_0)^2}{N + \lambda_0} \right)
\end{aligned}
\tag{21}
$$

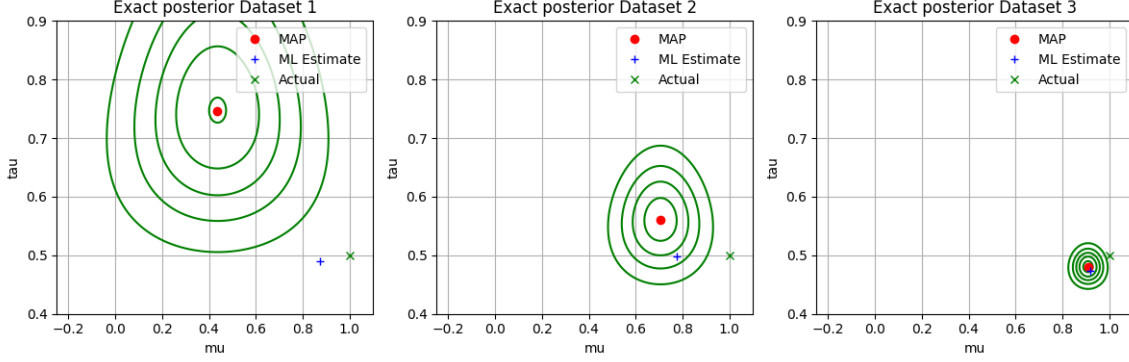Therefore, if we plot the contour for each datasets we obtain :

Figure 5: Contours of exact posteriors by datasets

The rest of the answer is in the code in appendix A.1.

Question 3.15

The equation (10.24) in the Bishop is the mean-field approximation which is :

$$q(\mu, \tau) = q(\mu)q(\tau) \tag{22}$$

This time, we take the result of the question 3.13 without setting the priors to 0. Then, we have :

$$
\begin{aligned}
q(\mu) &= \mathcal{N}(\mu | \mu_N, \lambda_N^{-1}) \\
q(\tau) &= \mathrm{Gam}(\tau | a_N, b_N)
\end{aligned}
\tag{23}
$$

with updates equations in the cavi algorithm described by :

$$
\begin{aligned}
\mu_N &= \frac{\lambda_0 \mu_0 + N \overline{x}}{\lambda_0 + N} \\
\lambda_N &= (\lambda_0 + N)\mathbb{E}[\tau] \\
a_N &= a_0 + \frac{N+1}{2} \\
b_N &= b_0 + \frac{1}{2}\left(\sum_{n=1}^{N} x_n^2 + N\mathbb{E}_\mu[\mu^2] - 2\mathbb{E}_\mu[\mu] \sum_{n=1}^{N} x_n + \lambda_0 \left(\mathbb{E}_\mu[\mu^2] + \mu_0^2 - 2\mu_0 \mathbb{E}_\mu[\mu] \right) \right)
\end{aligned}
\tag{24}
$$

and the expectations are the ones described in the equation (15).
Now, we need to find the ELBO formula :

$$
\begin{aligned}
\mathcal{L}(q) &= \mathbb{E}_{q(\mu), q(\tau)}[\log p(X, \mu, \tau)] - \mathbb{E}_{q(\mu), q(\tau)}[\log q(\mu, \tau)] \\
&= \mathbb{E}_{q(\mu), q(\tau)}[\log p(X | \mu, \tau) + \log p(\mu, \tau)] - \mathbb{E}_{q(\mu), q(\tau)}[\log q(\mu) + \log q(\tau)] \\
&= \mathbb{E}_{q(\mu), q(\tau)}[\log p(X | \mu, \tau)] + \mathbb{E}_{q(\mu), q(\tau)}[\log p(\mu, \tau)] - \mathbb{E}_{q(\mu)}[\log q(\mu)] - \mathbb{E}_{q(\tau)}[\log q(\tau)] \\
&= \mathbb{E}_{q(\mu), q(\tau)}[\log p(X | \mu, \tau)] + \mathbb{E}_{q(\mu), q(\tau)}[\log p(\mu, \tau)] + \mathbb{H}_q[\mu] + \mathbb{H}_q[\tau]
\end{aligned}
\tag{25}
$$

If we compute the first term we have:

$$
\begin{aligned}
\mathbb{E}_{q(\mu),q(\tau)}[\log p(X|\mu,\tau)] &= \mathbb{E}_{q(\mu),q(\tau)}\left[\frac{N}{2}\log\left(\frac{\tau}{2\pi}\right) - \frac{\tau}{2}\sum_{n=1}^{N}(x_n - \mu)^2\right] \\
&= \frac{N}{2}\left(\mathbb{E}_{q(\tau)}[\log\tau] - \log(2\pi)\right) \\
&\quad - \frac{\mathbb{E}_{q(\tau)}[\tau]}{2}\left(\sum_{n=1}^{N}x_n^2 - 2\mathbb{E}_{q(\mu)}[\mu]N\overline{x_n} + N\mathbb{E}_{q(\mu)}[\mu^2]\right) \\
&\overset{\pm}{=} \frac{N}{2}\mathbb{E}_{q(\tau)}[\log\tau] - \frac{\mathbb{E}_{q(\tau)}[\tau]}{2}\left(\sum_{n=1}^{N}x_n^2 - 2\mathbb{E}_{q(\mu)}[\mu]N\overline{x_n} + N\mathbb{E}_{q(\mu)}[\mu^2]\right)
\end{aligned}
\tag{26}
$$

And the second one is:

$$
\begin{aligned}
\mathbb{E}_{q(\mu),q(\tau)}[\log p(\mu,\tau)] &= \mathbb{E}_{q(\mu),q(\tau)}\left[\log\left(\frac{b_0^{a_0}\sqrt{\lambda_0}}{\Gamma(a_0)\sqrt{2\pi}}\right) + (a_0 - \frac{1}{2})\log\tau - b_0\tau - \frac{\lambda_0\tau(\mu - \mu_0)^2}{2}\right] \\
&= \log\left(\frac{b_0^{a_0}\sqrt{\lambda_0}}{\Gamma(a_0)\sqrt{2\pi}}\right) + (a_0 - \frac{1}{2})\mathbb{E}_{q(\tau)}[\log\tau] - b_0\mathbb{E}_{q(\tau)}[\tau] \\
&\quad - \frac{\lambda_0\mathbb{E}_{q(\tau)}[\tau]}{2}\left(\mathbb{E}_{q(\mu)}[\mu^2] - 2\mu_0\mathbb{E}_{q(\mu)}[\mu] + \mu_0^2\right) \\
&\overset{\pm}{=} (a_0 - \frac{1}{2})\mathbb{E}_{q(\tau)}[\log\tau] - b_0\mathbb{E}_{q(\tau)}[\tau] - \frac{\lambda_0\mathbb{E}_{q(\tau)}[\tau]}{2}\left(\mathbb{E}_{q(\mu)}[\mu^2] - 2\mu_0\mathbb{E}_{q(\mu)}[\mu] + \mu_0^2\right)
\end{aligned}
\tag{27}
$$

And we can compute all of that because entropies are known and we have the following expectations:

$$
\begin{aligned}
\mathbb{E}_{q(\mu)}[\mu] &= \mu_N \\
\mathbb{E}_{q(\mu)}[\mu^2] &= \frac{1}{\lambda_N} + \mu_N^2 \\
\mathbb{E}_{q(\tau)}[\tau] &= \frac{a_N}{b_N} \\
\mathbb{E}_{q(\tau)}[\log\tau] &= \psi(a_N) - \log b_N
\end{aligned}
\tag{28}
$$

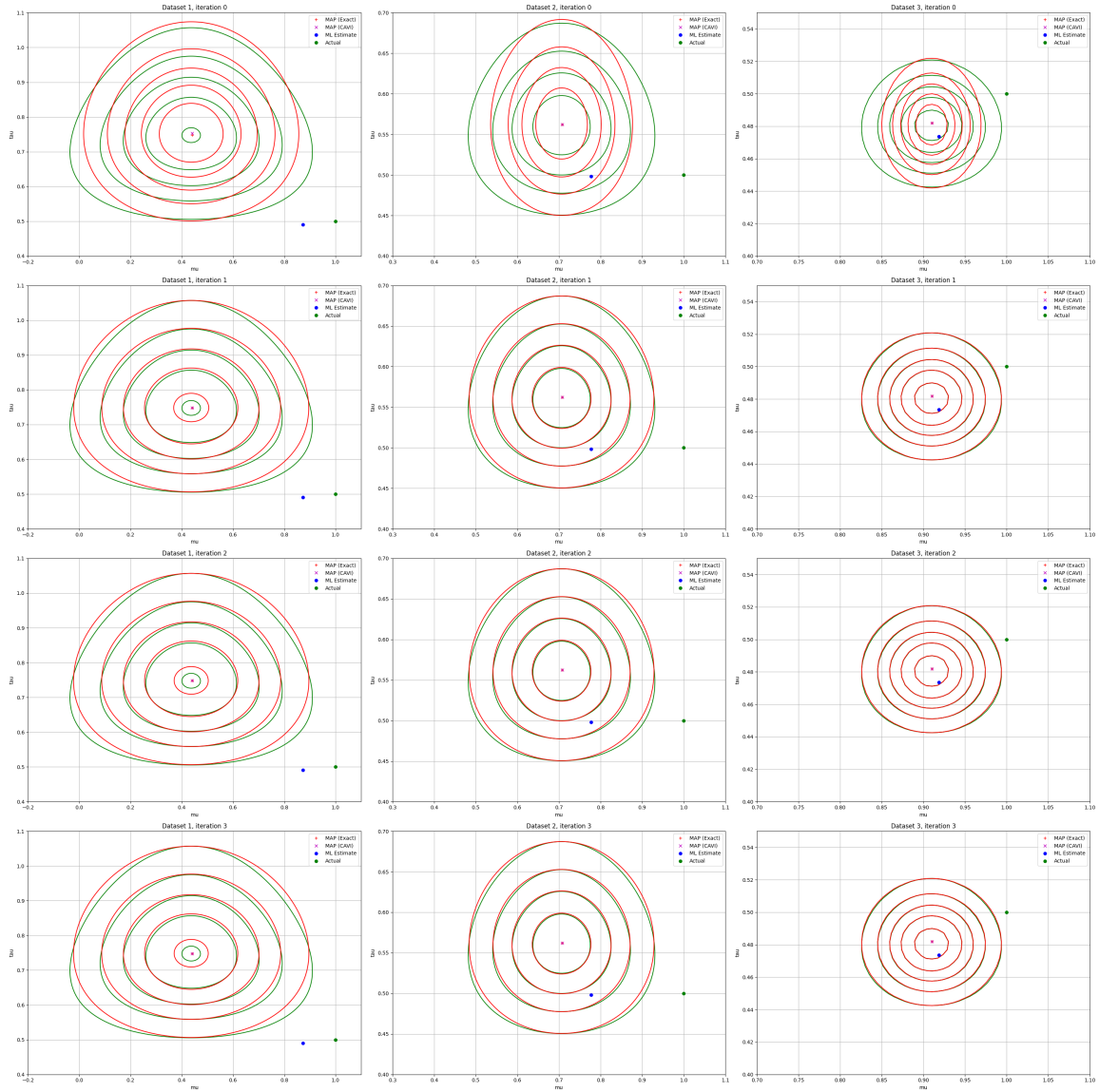Then we obtain this result :

Figure 6: Contours of the approximations by VI and the exact posterior by datasets, by iterations
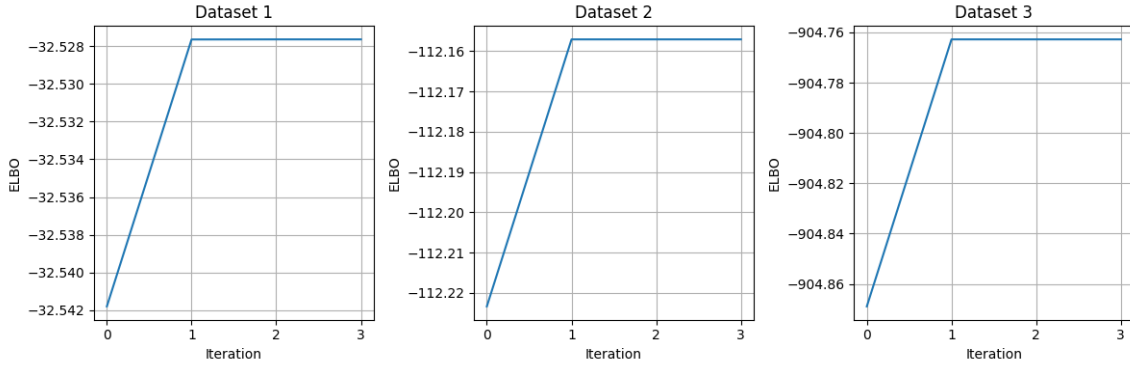
And we obtain an elbo plot :

Figure 7: ELBO plot by datasets

The code is in appendix A.1.

4   SVI - LDA

Question 4.16

According to the Hoffman paper, the local hidden variables are defined by the model where the distribution of each observation x_n only depends on its corresponding local variable z_n and the global variables $\beta_{1:K}$. Therefore, we can write:

$$p(x, z, \beta | alpha) = p(\beta | \alpha) \prod_{n=1}^{N} p(x_n, z_n | \beta) \tag{29}$$

Because:

$$p(x_n, z_n | x_{-n}, z_{-n}, \beta, \alpha) = p(x_n, z_n | \beta, \alpha) \tag{30}$$

Question 4.17

In this figure the global hidden variables are the topics $\beta_{1:K}$ and the local hidden variables are the topic proportions θ_d and the topic assignments $z_{d,1:N}$.

Question 4.18

The ELBO formula is:

$$\mathcal{L}(q) = \mathbb{E}_{q(\theta, z, \beta)}[\log p(w, \theta, z, \beta)] - \mathbb{E}_{q(\theta, z, \beta)}[\log q(\theta, z, \beta)] \tag{31}$$

And here we recall that we have:

$$p(w, z, \theta, \beta) = \prod_{k=1}^{K} p(\beta_k) \prod_{d=1}^{D} p(\theta_d) \prod_{d=1}^{D} \prod_{n=1}^{N} p(z_{dn} | \theta_d) p(w_{dn} | \beta_{z_{dn}}) \tag{32}$$

Therefore we have:

$$\mathbb{E}_{q(\theta,z,\beta)}[\log p(w,\theta,z,\beta)]$$

$$= \mathbb{E}_{q(\theta,z,\beta)}\left[\sum_{k=1}^{K}\log p(\beta_k) + \sum_{d=1}^{D}\log p(\theta_d) + \sum_{d=1}^{D}\sum_{n=1}^{N}\log p(z_{dn}|\theta_d)p(w_{dn}|\beta_{z_{dn}})\right]$$

$$- \mathbb{E}_{q(\theta,z,\beta)}\left[\sum_{k=1}^{K}\log q(\beta_k) + \sum_{d=1}^{D}\log q(\theta_d) + \sum_{d=1}^{D}\sum_{n=1}^{N}\log q(z_{dn}|\theta_d) + \log q(w_{dn}|\beta_{z_{dn}})\right]$$

$$= \sum_{k=1}^{K}\mathbb{E}_{q(\beta_k)}[\log p(\beta_k)] + \sum_{d=1}^{D}\mathbb{E}_{q(\theta_d)}[\log p(\theta_d)] + \sum_{d=1}^{D}\sum_{n=1}^{N}\mathbb{E}_{q(z_{dn}),q(\theta_d)}[\log p(z_{dn}|\theta_d)]$$

$$+ \sum_{d=1}^{D}\sum_{n=1}^{N}\mathbb{E}_{q(z_{dn}),q(\beta_{z_{dn}})}[\log p(w_{dn}|\beta_{z_{dn}})]$$

$$+ \sum_{k=1}^{K}\mathbb{H}_q[\beta_k] + \sum_{d=1}^{D}\mathbb{H}_q[\theta_d] + \sum_{d=1}^{D}\sum_{n=1}^{N}\mathbb{H}_q[z_{dn}|\theta_d] + \mathbb{H}_q[w_{dn}|\beta_{z_{dn}}]$$

$$(33)$$

Using Hoffman updates equations, we can write:

$$\lambda_k = \eta + \sum_{d=1}^{D}\sum_{n=1}^{N}z_{dn}^k w_{dn}$$

$$\gamma_d = \alpha + \sum_{n=1}^{N}z_{dn}$$

$$\phi_{dn} = \log \beta_{k,w_{dn}} + \log \theta_{dk}$$

$$(34)$$

And by using the expectations given in the Hoffman paper, we can write:

$$\sum_{k=1}^{K}\mathbb{E}_{q(\beta_k)}[\log p(\beta_k)] = \sum_{k=1}^{K}\sum_{\nu=1}^{W}(\eta - 1)\mathbb{E}_{q(\beta_{k\nu})}[\log \beta_{k\nu}]$$

$$= \sum_{k=1}^{K}\sum_{\nu=1}^{W}(\eta - 1)\left(\Psi(\lambda_{k\nu}) - \Psi\left(\sum_{y=1}^{W}\lambda_{ky}\right)\right)$$

$$(35)$$

$$\sum_{d=1}^{D}\mathbb{E}_{q(\theta_d)}[\log p(\theta_d)] = \sum_{d=1}^{D}\sum_{k=1}^{K}(\alpha - 1)\mathbb{E}_{q(\theta_{dk})}[\log \theta_{dk}]$$

$$= \sum_{d=1}^{D}\sum_{k=1}^{K}(\alpha - 1)\left(\Psi(\gamma_{dk}) - \Psi\left(\sum_{v=1}^{K}\gamma_{dv}\right)\right)$$

$$(36)$$

$$\sum_{d=1}^{D}\sum_{n=1}^{N}\mathbb{E}_{q(z_{dn}),q(\theta_d)}[\log p(z_{dn}|\theta_d)] = \sum_{d=1}^{D}\sum_{n=1}^{N}\sum_{k=1}^{K}\mathbb{E}_{q(z_{dn})}[z_{dn}^k]\mathbb{E}_{q(\theta_{dk})}[\log \theta_{dk}]$$

$$= \sum_{d=1}^{D}\sum_{n=1}^{N}\sum_{k=1}^{K}\phi_{dn}^k\left(\Psi(\gamma_{dk}) - \Psi\left(\sum_{v=1}^{K}\gamma_{dv}\right)\right)$$

$$(37)$$

$$\sum_{d=1}^{D}\sum_{n=1}^{N}\mathbb{E}_{q(z_{dn}),q(\beta_{z_{dn}})}[\log p(w_{dn}|\beta_{z_{dn}})] = \sum_{d=1}^{D}\sum_{n=1}^{N}\sum_{k=1}^{K}\sum_{\nu=1}^{W}w_{dn}^{k}\mathbb{E}_{q(z_{dn})}[z_{dn}^{k}]\mathbb{E}_{q(\beta_{k\nu})}[\log \beta_{k\nu}]$$

$$= \sum_{d=1}^{D}\sum_{n=1}^{N}\sum_{k=1}^{K}\sum_{\nu=1}^{W}w_{dn}^{k}\phi_{dn}^{k}\left(\Psi(\lambda_{k\nu}) - \Psi\left(\sum_{y=1}^{W}\lambda_{ky}\right)\right) \quad (38)$$

And the entropies can be found on Wikipedia.

Question 4.19

The code is in appendix A.2.

5 BBVI

Question 5.20

We have the simple model:

$$X|\theta \sim \mathcal{N}(\theta, \sigma^2)$$
$$\theta \sim Gamma(\alpha, \beta) \quad (39)$$

With α, β and σ^2 known. Now, we will derive the gradient estimate w.r.t. ν without Rao-Blackwellization using one sample $z \sim q_\nu(\theta)$, $q_\nu(\theta) = LogNormal(\nu, \epsilon^2)$. We recall the formula:

$$\nabla_\lambda \mathcal{L} \approx \frac{1}{S}\sum_{s=1}^{S}\nabla_\lambda \log q(z_s|\lambda)\left(\log p(x, z_s) - \log q(z_s|\lambda)\right) \quad (40)$$

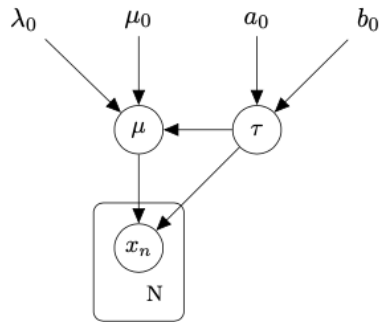Where $z_s \sim q(z|\lambda)$. Therefore, here we have:

$$\nabla_\nu \mathcal{L} \approx \nabla_\nu \log q(z|\nu)\left(\log p(x, z) - \log q(z|\nu)\right)$$

$$\approx \nabla_\nu \log\left(\frac{\exp\left(-\frac{(\log\theta-\nu)^2}{2\epsilon^2}\right)}{\theta\epsilon\sqrt{2\pi}}\right)\left(\log\left(\frac{\exp\left(-\frac{(x-\theta)^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}}\right)\right.$$

$$+ \log\left(\frac{\beta^\alpha\theta^{\alpha-1}e^{-\beta\theta}}{\Gamma(\alpha)}\right) - \log\left(\frac{\exp\left(-\frac{(\log\theta-\nu)^2}{2\epsilon^2}\right)}{\theta\epsilon\sqrt{2\pi}}\right)\right)$$

$$\approx \nabla_\nu\left(-\frac{(\log\theta-\nu)^2}{2\epsilon^2}\right)\left(-\frac{(x-\theta)^2}{2\sigma^2} - \log\left(\sigma\sqrt{2\pi}\right)\right. \quad (41)$$

$$\left. -\beta\theta + (\alpha-1)\log\theta + \log\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right) + \frac{(\log\theta-\nu)^2}{2\epsilon^2} + \log\left(\theta\epsilon\sqrt{2\pi}\right)\right)$$

$$\approx \frac{\log\theta-\nu}{\epsilon^2}\left(\frac{(\sigma(\log\theta-\nu))^2 - (\epsilon(x-\theta))^2}{2\sigma^2\epsilon^2} + \log\left(\frac{\epsilon}{\sigma}\right)\right.$$

$$\left. -\beta\theta + \alpha\log\theta + \log\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)\right)$$

A   Appendix

A.1   CAVI

Assignment 1.3 - CAVI

Consider the model defined by Equation (10.21)-(10-23) in Bishop, for which DGM is presented below:

Question 1.3.12:

Implement a function that generates data points for the given model.

```python
import numpy as np
from scipy.stats import gamma, norm
from scipy.special import psi
from scipy.special import gamma as gamma_func
np.random.seed(14)

def generate_data(mu, tau, N):
    # Insert your code here
    D = np.random.normal(mu, np.sqrt(1/tau), N)

    return D
```

Set $\mu = 1$, $\tau = 0.5$ and generate datasets with size N=10,100,1000. Plot the histogram for each of 3 datasets you generated.

```python
MU = 1
TAU = 0.5

dataset_1 = generate_data(MU, TAU, 10)
dataset_2 = generate_data(MU, TAU, 100)
dataset_3 = generate_data(MU, TAU, 1000)

# Visulaize the datasets via histograms
# Insert your code here
import matplotlib.pyplot as plt
fig, axs = plt.subplots(1, 3, figsize=(12, 4))
axs[0].hist(dataset_1, bins=20)
axs[1].hist(dataset_2, bins=20)
axs[2].hist(dataset_3, bins=20)
plt.tight_layout()
plt.savefig('../images/12_data.png')
plt.show()
```
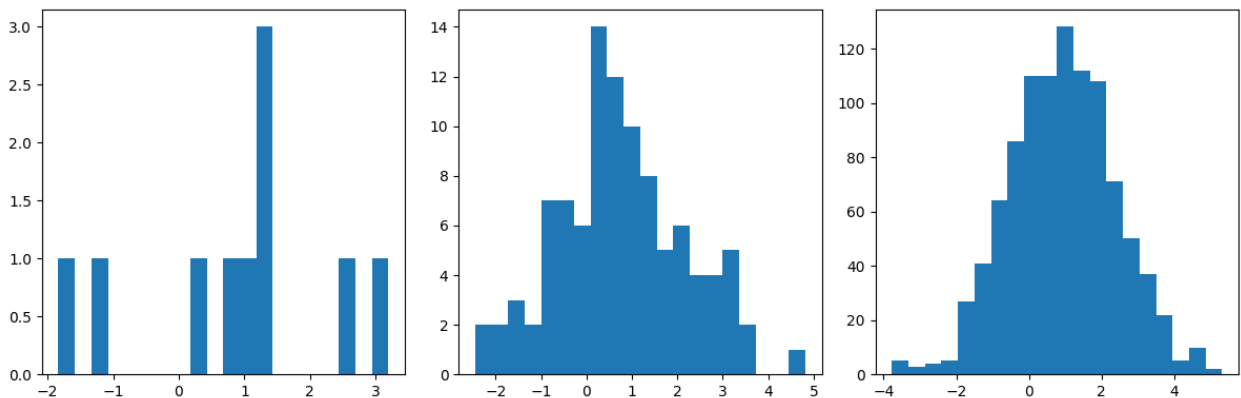

Question 1.3.13:

Find ML estimates of the variables μ and τ

```python
def ML_est(data):
    # insert your code
    N = len(data)
```

```
    x_mean = np.mean(data)
    x_var = np.var(data)

    tau_ml = 1 / x_var
    mu_ml = x_mean

    return mu_ml, tau_ml
```

Question 1.3.14:

What is the exact posterior? First derive it in closed form, and then implement a function that computes it for the given parameters:

```
In [ ]: def compute_exact_posterior(D, a_0, b_0, mu_0, lambda_0):
            # your implementation
            x_mean = np.mean(D)
            N = len(D)

            mu_prime = (lambda_0 * mu_0 + N * x_mean) / (lambda_0 + N)
            lambda_prime = lambda_0 + N
            a_prime = a_0 + (N-1)/ 2
            b_prime = b_0 + 0.5 * (np.sum(D**2) +
                                    lambda_0 * mu_0**2 - lambda_prime * mu_prime**2)

            exact_post_distribution = (a_prime, b_prime, mu_prime, lambda_prime)

            return exact_post_distribution
```

```
In [ ]: # prior parameters
        mu_0 = 0
        lambda_0 = 10
        a_0 = 20
        b_0 = 20
```

```
In [ ]: mus = np.linspace(-0.25, 1.1, 200)
        taus = np.linspace(0.4, 0.9, 200)

        fig, axs = plt.subplots(1, 3, figsize=(12, 4))
        for i, dataset in enumerate([dataset_1, dataset_2, dataset_3]):
          mu_ml, tau_ml = ML_est(dataset)

          a_T, b_T, mu_T, lambda_T = compute_exact_posterior(
              dataset, a_0, b_0, mu_0, lambda_0)

          Z_exact = np.zeros((len(mus), len(taus)))
          pTau = gamma(a=a_T, loc=0, scale=1/b_T)
          for j, tau in enumerate(taus):
              pMu = norm(loc=mu_T, scale=1/np.sqrt(lambda_T*tau))
              Z_exact[:, j] = pMu.pdf(mus) * pTau.pdf(tau)
          # Finding the maximum of the exact posterior
          mu_max_exact = mus[np.argmax(np.max(Z_exact, axis=1))]
          tau_max_exact = taus[np.argmax(np.max(Z_exact, axis=0))]
          # Plotting the results
          axs[i].contour(*np.meshgrid(mus, taus), Z_exact.T,
                          levels=5, colors=['green'])
          axs[i].plot(mu_max_exact, tau_max_exact, 'ro', label='MAP')
          axs[i].plot(mu_ml, tau_ml, 'b+', label='ML Estimate')
          axs[i].plot(MU, TAU, 'gx', label='Actual')
          axs[i].legend()
          axs[i].grid()
          axs[i].set_xlabel('mu')
          axs[i].set_ylabel('tau')
          axs[i].set_title('Exact posterior Dataset {}'.format(i+1))
        plt.tight_layout()
        plt.savefig('../images/14_contours.png')
        plt.show()
```
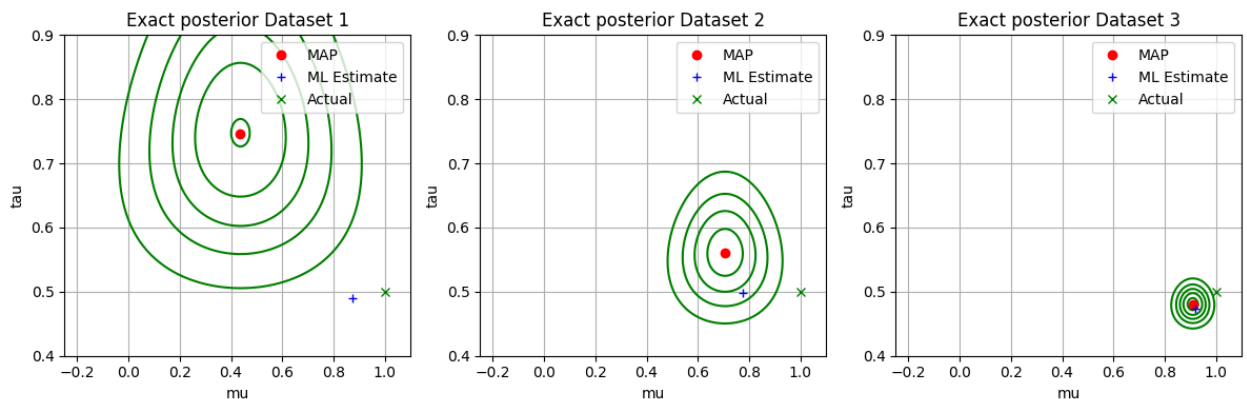
Question 1.3.15:

You will implement the VI algorithm for the variational distribution in Equation (10.24) in Bishop. Start with introducing the prior parameters:

Continue with a helper function that computes ELBO:

```
In [ ]: def compute_E_tau(a_N, b_N):
            E_tau = a_N / b_N

            return E_tau

        def compute_E_mu_2(mu_N, lambda_N):
            E_mu_2 = mu_N**2 + 1/lambda_N

            return E_mu_2

        def compute_E_log_tau(a_N, b_N):
            E_log_tau = psi(a_N) - np.log(b_N)

            return E_log_tau
```

```
In [ ]: def compute_elbo(D, a_0, b_0, mu_0, lambda_0, a_N, b_N, mu_N, lambda_N):
            N = len(D)
            x_mean = np.mean(D)
            x_2_sum = np.sum(D**2)
            E_tau = compute_E_tau(a_N, b_N)
            E_mu_2 = compute_E_mu_2(mu_N, lambda_N)
            E_log_tau = compute_E_log_tau(a_N, b_N)

            # compute the elbo
            # E[log p(D|mu, tau)]
            E_log_p_D = N/2 * E_log_tau - 0.5*E_tau * (x_2_sum - 2*N*x_mean*mu_N + N*E_mu_2)

            # E[log p(mu, tau)]
            E_log_p_mu_tau = (a_0-0.5)*E_log_tau - b_0*E_tau - 0.5*lambda_0*E_tau*(E_mu_2 + mu_0**2 - 2*mu_0*mu_N)

            # Entropy of mu
            entropy_mu = norm.entropy(loc=mu_N, scale=1/np.sqrt(lambda_N))
            # Entropy of tau
            entropy_tau = gamma.entropy(a=a_N, scale=1/b_N)

            elbo = E_log_p_D + E_log_p_mu_tau + entropy_mu + entropy_tau

            return elbo
```

Now, implement the CAVI algorithm:

```
In [ ]: def CAVI(D, a_0, b_0, mu_0, lambda_0, iter=5):
            # make an initial guess for the expected value of tau
            E_tau = 1

            N = len(D)
            x_mean = np.mean(D)
            x_2_sum = np.sum(D**2)

            # Constants
            a_N = a_0 + (N+1) / 2
            mu_N = (lambda_0 * mu_0 + N * x_mean) / (lambda_0 + N)
            E_mu = mu_N

            # Variables
            b_Ns = []
            lambda_Ns = []

            # ELBO
            elbos = []

            # CAVI iterations ...
            for i in range(iter):
                # update the values for the variational parameters
                lambda_N = (lambda_0 + N) * E_tau

                E_mu_2 = compute_E_mu_2(mu_N, lambda_N)
                b_N = b_0 + 0.5 * (x_2_sum + N*E_mu_2 - 2*N*E_mu*x_mean + lambda_0*(E_mu_2 - 2*E_mu*mu_0 + mu_0**2))

                E_tau = compute_E_tau(a_N, b_N)

                b_Ns.append(b_N)
                lambda_Ns.append(lambda_N)
                # save ELBO for each iteration, plot them afterwards to show convergence
                elbos.append(compute_elbo(D, a_0, b_0, mu_0, lambda_0, a_N , b_N, mu_N, lambda_N))

            return a_N, b_N, mu_N, lambda_N, elbos, b_Ns, lambda_Ns
```

Run the VI algorithm on the datasets. Compare the inferred variational distribution with the exact posterior and the ML estimate. Visualize the results and discuss your findings.

```python
In [ ]: def compute_z_exact(mus, taus, a_, b_, mu_, lambda_):
            z = np.zeros((len(mus), len(taus)))
            pTau = gamma(a=a_, loc=0, scale=1/b_)
            for j, tau in enumerate(taus):
                pMu = norm(loc=mu_, scale=1/np.sqrt(lambda_*tau))
                z[:, j] = pMu.pdf(mus) * pTau.pdf(tau)

            return z

        def compute_z_cavi(mus, taus, a_, b_, mu_, lambda_):
            pTau = gamma(a=a_, loc=0, scale=1/b_)
            pMu = norm(loc=mu_, scale=1/np.sqrt(lambda_))
            z = np.outer(pMu.pdf(mus), pTau.pdf(taus))
            return z

In [ ]: iter = 4 # number of iterations for CAVI
        mus = np.linspace(-0.2, 1.1, 200)
        taus = np.linspace(0.1, 1.1, 200)

        xlims = [[-0.2, 1.1], [0.3, 1.1], [0.7, 1.1]]
        ylims = [[0.4, 1.1], [0.4, 0.7], [0.4, 0.55]]

        elbos_list = []

        fig, axs = plt.subplots(iter, 3, figsize=(30, 30))
        for i, dataset in enumerate([dataset_1, dataset_2, dataset_3]):
          mu_ml, tau_ml = ML_est(dataset)
          a_N, b_N, mu_N, lambda_N, elbos, b_Ns, lambda_Ns = CAVI(dataset, a_0, b_0, mu_0, lambda_0, iter=iter)
          a_T, b_T, mu_T, lambda_T = compute_exact_posterior(
                dataset, a_0, b_0, mu_0, lambda_0)

          elbos_list.append(elbos)

          for j in range(iter):
            Z_exact = compute_z_exact(mus, taus, a_T, b_T, mu_T, lambda_T)
            Z_cavi = compute_z_cavi(mus, taus, a_N, b_Ns[j], mu_N, lambda_Ns[j])
            # Finding the maximum of the exact posterior
            mu_max_exact = mus[np.argmax(np.max(Z_exact, axis=1))]
            tau_max_exact = taus[np.argmax(np.max(Z_exact, axis=0))]
            # Finding the maximum of the CAVI approximation
            mu_max_cavi = mus[np.argmax(np.max(Z_cavi, axis=1))]
            tau_max_cavi = taus[np.argmax(np.max(Z_cavi, axis=0))]
            # Plotting the results
            axs[j, i].contour(*np.meshgrid(mus, taus), Z_exact.T,
                        levels=5, colors=['green'])
            axs[j, i].contour(*np.meshgrid(mus, taus), Z_cavi.T,
                        levels=5, colors=['red'])
            axs[j, i].plot(mu_max_exact, tau_max_exact, 'r+', label='MAP (Exact)')
            axs[j, i].plot(mu_max_cavi, tau_max_cavi, 'mx', label='MAP (CAVI)')
            axs[j, i].plot(mu_ml, tau_ml, 'bo', label='ML Estimate')
            axs[j, i].plot(MU, TAU, 'go', label='Actual')
            axs[j, i].legend()
            axs[j, i].grid()
            axs[j, i].set_xlabel('mu')
            axs[j, i].set_ylabel('tau')
            axs[j, i].set_title(f'Dataset {i+1}, iteration {j}')
            axs[j, i].set_xlim(xlims[i])
            axs[j, i].set_ylim(ylims[i])
        plt.tight_layout()
        plt.savefig('../images/15_contours.png')
        plt.show()

        # Plot ELBOs
        fig, axs = plt.subplots(1, 3, figsize=(12, 4))
        for i in range(3):
          axs[i].plot(elbos_list[i])
          axs[i].set_xlabel('Iteration')
          axs[i].set_ylabel('ELBO')
          axs[i].set_title(f'Dataset {i+1}')
          axs[i].grid()
        plt.tight_layout()
        plt.savefig('../images/15_elbo.png')
        plt.show()
```
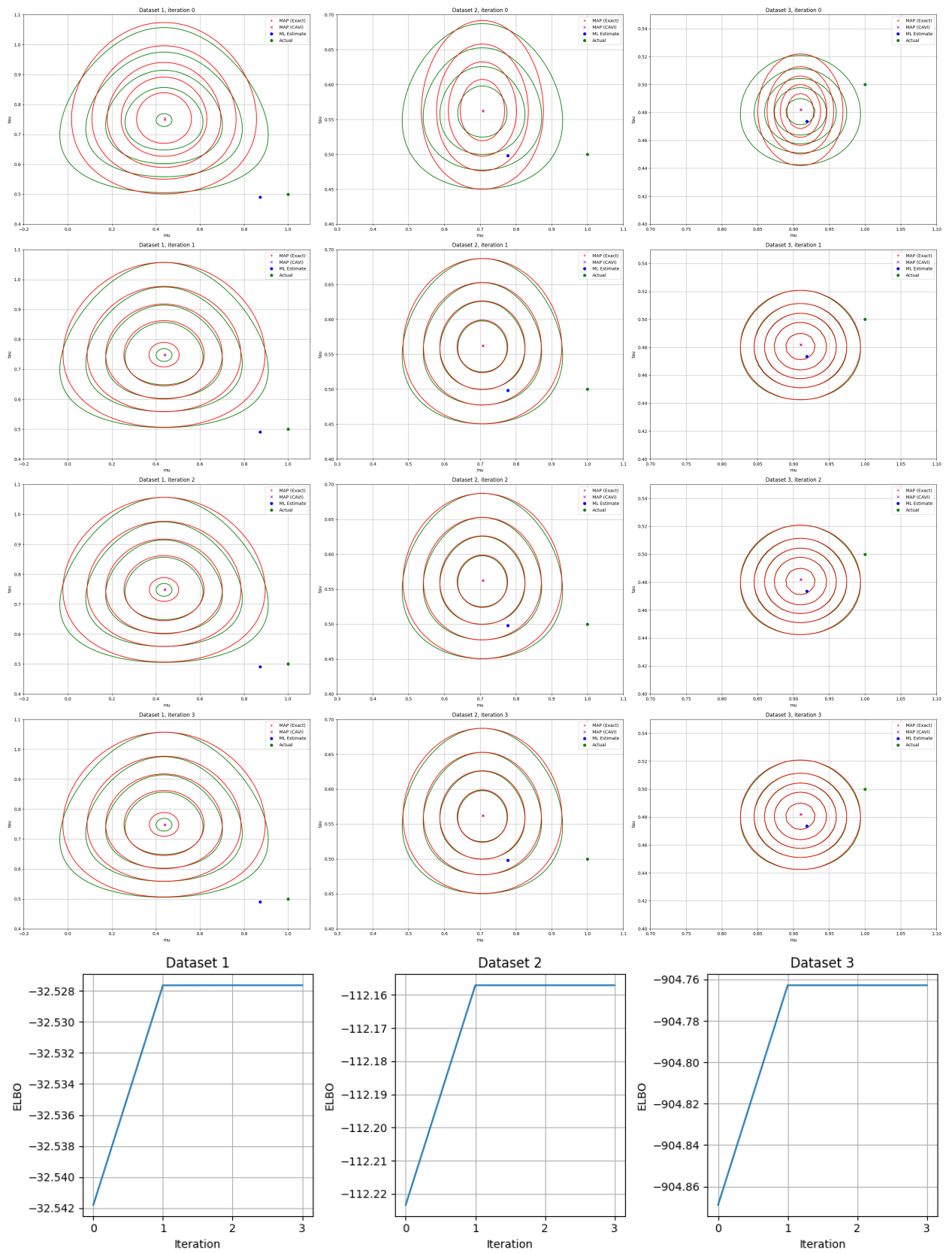
A.2 SVI