DD2434 - Machine Learning, Advanced Course
Assignment 1A

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Contents

1 Exponential Family

Question 1.1

$$\begin{aligned}
p(x|\theta) &= h(x)\exp(\eta(\theta)\cdot T(x) - A(\eta)) \\
&= h(x)\exp(\eta(\lambda)\cdot T(x) - A(\eta(\lambda))) \\
&= h(x)\exp(\log\lambda\cdot x - A(\log\lambda)) \\
&= h(x)\exp(\log\lambda\cdot x - \lambda) \\
&= h(x)\exp(\log\lambda\cdot x)\exp(-\lambda) \\
&= e^{-\lambda}\frac{\lambda^x}{x!}
\end{aligned} \tag{1}$$

We can see that the distribution correspond to a Poisson distribution of parameter λ.

Question 1.2

$$\begin{aligned}
p(x|\theta) &= h(x)\exp(\eta(\theta)\cdot T(x) - A(\eta)) \\
&= \exp(\eta([\alpha,\beta])\cdot[\log x, x] - A(\alpha-1, -\beta)) \\
&= \exp([\alpha-1, -\beta]\cdot[\log x, x] - \log\Gamma(\alpha) + \alpha\log(\beta)) \\
&= \exp((\alpha-1)\log x - \beta x - \log\Gamma(\alpha) + \alpha\log(\beta)) \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}
\end{aligned} \tag{2}$$

We can see that the distribution correspond to a Gamma distribution of parameters α and β.

Question 1.3

$$\begin{aligned}
p(x|\theta) &= h(x)\exp(\eta(\theta)\cdot T(x) - A(\eta)) \\
&= \frac{\exp(\eta([\mu,\sigma^2])\cdot[x, x^2] - A(\eta([\mu,\sigma^2])))}{\sqrt{2\pi}} \\
&= \frac{\exp([\frac{\mu}{\sigma^2}, -\frac{1}{2\sigma^2}]\cdot[x, x^2] - A([\frac{\mu}{\sigma^2}, -\frac{1}{2\sigma^2}]))}{\sqrt{2\pi}} \\
&= \frac{\exp(\frac{\mu x}{\sigma^2} - \frac{x^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2} - \log\sigma)}{\sqrt{2\pi}} \\
&= \frac{\exp(-\frac{(x-\mu)^2}{2\sigma^2})}{\sigma\sqrt{2\pi}}
\end{aligned} \tag{3}$$

We can see that the distribution correspond to a Normal distribution of parameters μ and σ^2.

Question 1.4

$$
\begin{aligned}
p(x|\theta) &= h(x)\exp(\eta(\theta)\cdot T(x) - A(\eta)) \\
&= 2\exp(\eta(\lambda)\cdot x - A(\eta(\lambda))) \\
&= 2\exp(-\lambda x - A(-\lambda)) \\
&= 2\exp\left(-\lambda x + \log\left(\frac{\lambda}{2}\right)\right) \\
&= \lambda e^{-\lambda x}
\end{aligned} \tag{4}
$$

We can see that the distribution correspond to a Exponential distribution of parameter λ.

Question 1.5

$$
\begin{aligned}
p(x|\theta) &= h(x)\exp(\eta(\theta)\cdot T(x) - A(\eta)) \\
&= \exp(\eta([\psi_1,\psi_2])\cdot[\log x, \log(1-x)] - A(\eta([\psi_1,\psi_2]))) \\
&= \exp([\psi_1 - 1, \psi_2 - 1]\cdot[\log x, \log(1-x)] - A([\psi_1 - 1, \psi_2 - 1])) \\
&= \exp((\psi_1 - 1)\log x + (\psi_2 - 1)\log(1-x) - \log\Gamma(\psi_1) - \log\Gamma(\psi_2) + \log\Gamma(\psi_1+\psi_2)) \\
&= \frac{\Gamma(\psi_1+\psi_2)}{\Gamma(\psi_1)\Gamma(\psi_2)} x^{\psi_1 - 1}(1-x)^{\psi_2 - 1}
\end{aligned} \tag{5}
$$

We can see that the distribution correspond to a Beta distribution of parameters ψ_1 and ψ_2.

2 Dependencies in a Directed Graphical Model

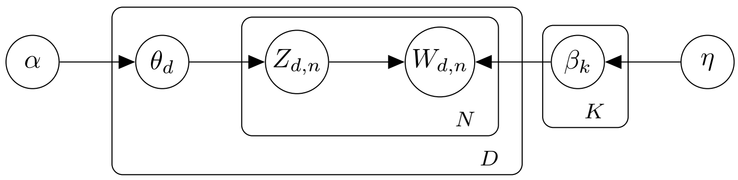
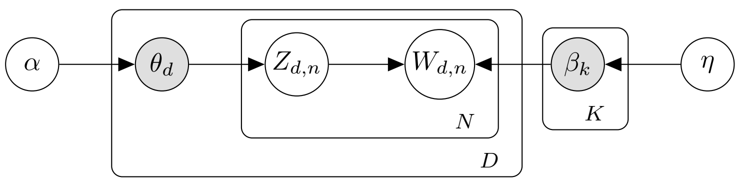


Figure 1: Graphical model of smooth LDA.

Question 2.6

The Bayes net take this form :



Then, if we use the method using the d-separation, we obtain this :

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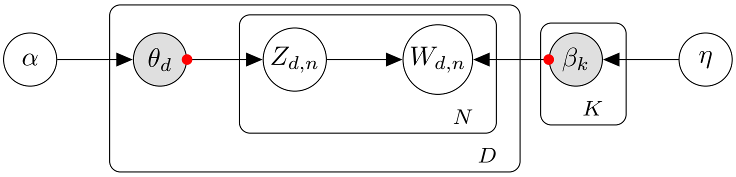

Therefore, we can see that $W_{d,n} \perp W_{d,n+1} | \theta_d, \beta_{1:K}$ is <u>false</u>.

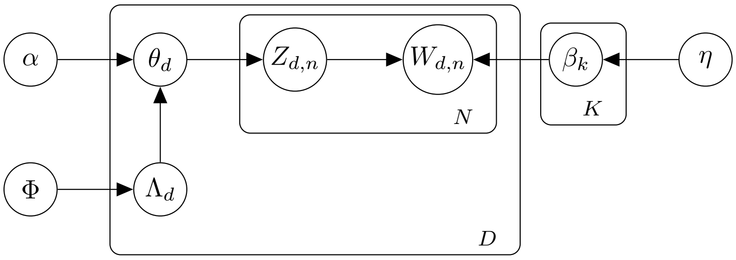

Figure 2: Graphical model of Labeled LDA.

3 CAVI

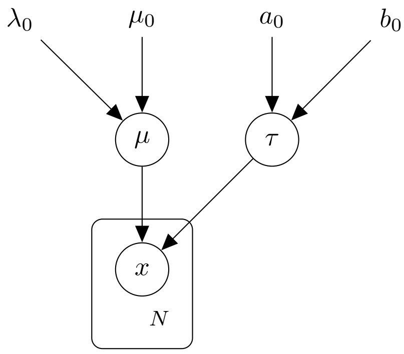

Figure 3: DGM

Question 3.12

In the bishop book, we can see that :

$$p(X|\mu, \tau) = \left(\frac{\tau}{2\pi}\right)^{N/2} \exp\left\{-\frac{\tau}{2}\sum_{n=1}^{N}(x_n - \mu)^2\right\} \tag{6}$$

$$p(\mu|\tau) = \mathcal{N}(\mu|\mu_0, (\lambda_0\tau)^{-1}) \tag{7}$$

$$p(\tau) = \text{Gam}(\tau|a_0, b_0) \tag{8}$$

Then, by using the code in appendix A.1, we obtain :

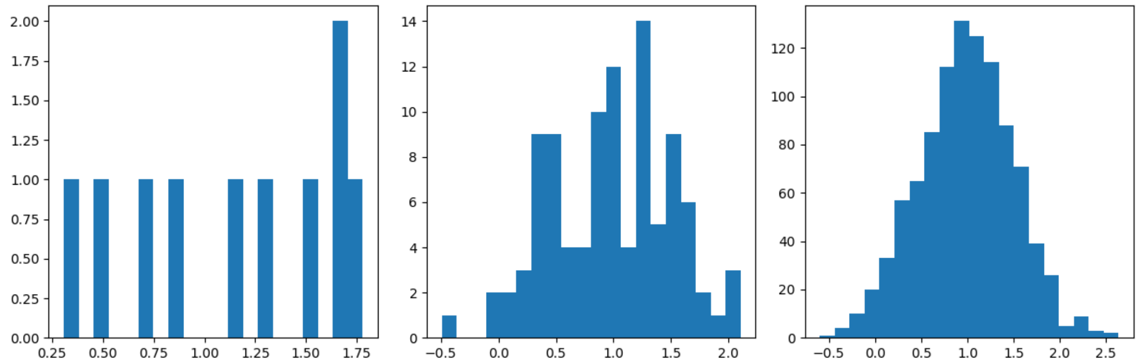

Figure 4: Generated Data.

Question 3.13

Let's find the ML estimates of μ and τ. We know that $\log(q^*(\mu)) = \mathbb{E}_{-\mu}[\log p(X, \mu, \tau)]$. Then, we can write :

$$\begin{aligned}
\log(q^*(\mu)) &= \mathbb{E}_{-\mu}[\log p(X, \mu, \tau)] \\
&\stackrel{\pm}{=} \mathbb{E}_{\tau}[\log p(X|\mu, \tau) + \log p(\mu|\tau)] \\
&= \mathbb{E}_{\tau}\left[\frac{N}{2}\log\left(\frac{\tau}{2\pi}\right) + \frac{\tau}{2}\sum_{n=1}^{N}(x_n - \mu)^2 + \frac{1}{2}\log\left(\frac{\lambda_0\tau}{2\pi}\right) + \frac{\lambda_0\tau}{2}(\mu - \mu_0)\right] \\
&\stackrel{\pm}{=} \frac{\mathbb{E}_{\tau}[\tau]}{2}\left(\lambda_0(\mu - \mu_0) + \sum_{n=1}^{N}(x_n - \mu)^2\right) \\
&= -\frac{\mathbb{E}_{\tau}[\tau]}{2}\left(\lambda_0\mu^2 - 2\lambda_0\mu\mu_0 + \lambda_0\mu_0^2 + \sum_{n=1}^{N}x_n^2 - 2\mu\sum_{n=1}^{N}x_n + N\mu^2\right) \\
&= -\frac{\mathbb{E}_{\tau}[\tau]}{2}\left((\lambda_0 + N)\mu^2 - 2(\lambda_0\mu_0 + \sum_{n=1}^{N}x_n)\mu + \lambda_0\mu_0^2 + \sum_{n=1}^{N}x_n^2\right) \\
&= -\frac{\mathbb{E}_{\tau}[\tau](\lambda_0 + N)}{2}\left(\mu^2 - 2\mu\frac{\lambda_0\mu_0 + \sum_{n=1}^{N}x_n}{\lambda_0 + N} + \frac{\lambda_0\mu_0^2 + \sum_{n=1}^{N}x_n^2}{\lambda_0 + N}\right)
\end{aligned} \tag{9}$$

Therefore we can conclude that $q^*(\mu) = \mathcal{N}(\mu|\mu_N, \lambda_N^{-1})$ with :

$$\mu_N = \frac{\lambda_0\mu_0 + \sum_{n=1}^{N}x_n}{\lambda_0 + N} \tag{10}$$

$$\lambda_N = (\lambda_0 + N)\mathbb{E}[\tau] \tag{11}$$

And for τ we have :

$$\begin{aligned}
\log(q^*(\tau)) &= \mathbb{E}_{-\tau}[\log p(X, \mu, \tau)] \\
&\stackrel{\pm}{=} \mathbb{E}_\mu[\log p(X|\mu, \tau) + \log p(\mu|\tau)] + \log p(\tau) \\
&\stackrel{\pm}{=} (a_0 - 1)\log\tau - b_0\tau + \frac{N}{2}\log\tau - \frac{\tau}{2}\mathbb{E}_\mu\left[\sum_{n=1}^{N}(x_n-\mu)^2 + \lambda_0(\mu-\mu_0)^2\right] \\
&= (a_0 + \frac{N}{2} - 1)\log\tau - \left(b_0 + \frac{1}{2}\mathbb{E}_\mu\left[\sum_{n=1}^{N}(x_n-\mu)^2 + \lambda_0(\mu-\mu_0)^2\right]\right)\tau
\end{aligned} \tag{12}$$

Therefore we can conclude that $q^*(\tau) = \text{Gam}(\tau|a_N, b_N)$ with :

$$a_N = a_0 + \frac{N}{2} \tag{13}$$

$$\begin{aligned}
b_N &= b_0 + \frac{1}{2}\mathbb{E}_\mu\left[\sum_{n=1}^{N}(x_n-\mu)^2 + \lambda_0(\mu-\mu_0)^2\right] \\
b_N &= b_0 + \frac{1}{2}\left(\sum_{n=1}^{N}x_n^2 + N\mathbb{E}_\mu[\mu^2] - 2\mathbb{E}_\mu[\mu]\sum_{n=1}^{N}x_n + \lambda_0\left(\mathbb{E}_\mu[\mu^2] + \mu_0^2 - 2\mu_0\mathbb{E}_\mu[\mu]\right)\right)
\end{aligned} \tag{14}$$

With :

$$\begin{aligned}
\mathbb{E}_{q(\mu)}[\mu] &= \mu_N \\
\mathbb{E}_{q(\mu)}[\mu^2] &= \frac{1}{\lambda_N} + \mu_N^2 \\
\mathbb{E}_{q(\tau)}[\tau] &= \frac{a_N}{b_N}
\end{aligned} \tag{15}$$

If we take non-informative priors then $a_0 = b_0 = \mu_0 = \lambda_0 = 0$, then we have :

$$\begin{aligned}
\mu_N &= \overline{x} \\
\lambda_N &= N\mathbb{E}[\tau] \\
a_N &= \frac{N}{2} \\
b_N &= \frac{1}{2}\mathbb{E}_\mu\left[\sum_{n=1}^{N}(x_n-\mu)^2\right]
\end{aligned} \tag{16}$$

And by using $\mathbb{E}[\tau] = \frac{a_N}{b_N}$ we obtain :

$$\begin{aligned}
\frac{1}{\mathbb{E}[\tau]} &= \frac{b_N}{a_N} \\
\frac{1}{\mathbb{E}[\tau]} &= \frac{2}{2N}\mathbb{E}_\mu\left[\sum_{n=1}^{N}(x_n-\mu)^2\right] \\
\frac{1}{\mathbb{E}[\tau]} &= \overline{x^2} - 2\overline{x}\mathbb{E}[\mu] + \mathbb{E}[\mu^2]
\end{aligned} \tag{17}$$

And, with the fact that $\mathbb{E}[\mu] = \mu_N$ and $\mathbb{E}[\mu^2] = \frac{1}{\lambda_N} + \mu_N^2$, we obtain :

$$\begin{aligned} \mathbb{E}[\mu] &= \overline{x} \\ \mathbb{E}[\mu^2] &= \frac{1}{N\mathbb{E}[\tau]} + \overline{x}^2 \end{aligned} \tag{18}$$

And therefore:

$$\frac{1}{\mathbb{E}[\tau]} = \frac{N}{N-1}(\overline{x^2} - \overline{x}^2) = \frac{N}{N-1}\sum_{n=1}^{N}(x_n - \overline{x})^2 \tag{19}$$

Wich define the ML estimates. The implementation is in the code in appendix A.2.

Question 3.14

The posterior is defined as $p(\mu, \tau|x)$. Then, we can write :

$$\begin{aligned} p(\mu, \tau|x) &= \frac{p(x|\mu, \tau)p(\mu, \tau)}{p(x)} \\ &\propto p(x|\mu, \tau)p(\mu, \tau) \end{aligned} \tag{20}$$

Where $x|\mu, \tau \sim \mathcal{N}(\mu|\mu, \tau^{-1})$ and $\mu, \tau \sim NormalGamma(\mu_0, \lambda_0, a_0, b_0)$. Therefore, as we saw in the question 1.3 in the Module 1 exercise, we have $\mu, \tau|x \sim NormalGamma(\mu', \lambda', a', b')$, where :

$$\begin{aligned} \mu' &= \frac{N\overline{x} + \mu_0\lambda_0}{N + \lambda_0} \\ \lambda' &= N + \lambda_0 \\ a' &= a_0 + \frac{N}{2} \\ b' &= b_0 + \frac{1}{2}\left(\sum_{n=1}^{N} x_n^2 + \lambda_0\mu_0^2 - \frac{(N\overline{x} + \mu_0\lambda_0)^2}{N + \lambda_0}\right) \end{aligned} \tag{21}$$

The rest of the answer is in the code in appendix A.3.

Question 3.15

The equation (10.24) in the Bishop is the mean-field approximation which is :

$$q(\mu, \tau) = q(\mu)q(\tau) \tag{22}$$

Now, we need to find the ELBO formula :

$$\begin{aligned} \mathcal{L}(q) &= \mathbb{E}_q[\log p(X, \mu, \tau)] - \mathbb{E}_q[\log q(\mu, \tau)] \\ &= \mathbb{E}_q[\log p(X|\mu, \tau) + \log p(\mu, \tau)] - \mathbb{E}_q[\log q(\mu) + \log q(\tau)] \\ &= \mathbb{E}_q[\log p(X|\mu, \tau)] + \mathbb{E}_q[\log p(\mu, \tau)] - \mathbb{E}_q[\log q(\mu)] - \mathbb{E}_q[\log q(\tau)] \\ &= \mathbb{E}_q[\log p(X|\mu, \tau)] + \mathbb{E}_q[\log p(\mu, \tau)] + \mathbb{H}_q[\mu] + \mathbb{H}_q[\tau] \end{aligned} \tag{23}$$

A Appendix

A.1 Question 3.12

```
import matplotlib.pyplot as plt
import numpy as np


def generate_data(mu, tau, N):
    # Insert your code here
    D = np.random.normal(mu, tau, N)

    return D


mu = 1
tau = 0.5

dataset_1 = generate_data(mu, tau, 10)
dataset_2 = generate_data(mu, tau, 100)
dataset_3 = generate_data(mu, tau, 1000)

# Visulaize the datasets via histograms
# Insert your code here
fig, axs = plt.subplots(1, 3, figsize=(12, 4))
axs[0].hist(dataset_1, bins=20)
axs[1].hist(dataset_2, bins=20)
axs[2].hist(dataset_3, bins=20)
plt.tight_layout()
plt.savefig('12_data.png')
plt.show()
```

A.2 Question 3.13

```
def ML_est(data):
    # insert your code
    N = len(data)
    x_mean = np.mean(data)
    x_var = np.var(data, ddof=1)

    tau_ml = 1 / (N * x_var)
    mu_ml = x_mean

    return mu_ml, tau_ml
```

A.3 Question 3.14

```
def compute_exact_posterior(D, a_0, b_0, mu_0, lambda_0):
    # your implementation
```

```
    x_mean = np.mean(D)
    N = len(D)

    mu_prime = (lambda_0 * mu_0 + N * x_mean) / (lambda_0 + N)
    lambda_prime = lambda_0 + N
    a_prime = a_0 + N / 2
    b_prime = b_0 + 1 / 2 * (np.sum((D - x_mean)**2) +
                             lambda_0 * mu_0**2 - lambda_prime * mu_prime
                                 **2)

    exact_post_distribution = (mu_prime, lambda_prime, a_prime, b_prime)

    return exact_post_distribution
```

A.4 Question 3.15

```
from scipy.stats import gamma, norm

# prior parameters
mu_0 = 0
lambda_0 = 1
a_0 = 1
b_0 = 1


def compute_elbo(D, a_0, b_0, mu_0, lambda_0, a_N, b_N, mu_N, lambda_N):
    # given the prior and posterior parameters together with the data,
    # compute ELBO here
    N = len(D)

    return elbo


def CAVI(D, a_0, b_0, mu_0, lambda_0):
    # make an initial guess for the expected value of tau
    initial_guess_exp_tau = 1

    N = len(D)
    x_mean = np.mean(D)
    x_2_sum = np.sum(D**2)

    # Constants
    a_N = a_0 + N / 2
    mu_N = (lambda_0 * mu_0 + N * x_mean) / (lambda_0 + N)
    E_mu = mu_N

    # Variational parameters
    b_N = b_0
    lambda_N = lambda_0

    # ELBO
```

```
    elbos = []

    # CAVI iterations ...
    for i in range(100):
        # update the values for the variational parameters
        E_tau = a_N / b_N
        E_mu_2 = 1 / lambda_N + mu_N**2

        lambda_N = (lambda_0 + N) * E_tau
        b_N = b_0 + 1 / 2 * (x_2_sum + N*E_mu_2 - 2*N*E_mu *
                             x_mean + lambda_0*(E_mu_2 - 2*E_mu*mu_0 + mu_0
                                 **2))
        # save ELBO for each iteration, plot them afterwards to show
            convergence
        elbos.append(compute_elbo(D, a_0, b_0, mu_0,
                         lambda_0, a_N, b_N, mu_N, lambda_N))

    return a_N, b_N, mu_N, lambda_N, elbos
```