

DD2434 - Machine Learning, Advanced Course
Assignment 1A

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1 Exponential Family

Question 1.1

$$\begin{aligned}
 p(x|\theta) &= h(x) \exp(\eta(\theta) \cdot T(x) - A(\eta)) \\
 &= h(x) \exp(\eta(\lambda) \cdot T(x) - A(\eta(\lambda))) \\
 &= h(x) \exp(\log \lambda \cdot x - A(\log \lambda)) \\
 &= h(x) \exp(\log \lambda \cdot x - \lambda) \\
 &= h(x) \exp(\log \lambda \cdot x) \exp(-\lambda) \\
 &= e^{-\lambda} \frac{\lambda^x}{x!}
 \end{aligned} \tag{1}$$

We can see that the distribution correspond to a Poisson distribution of parameter λ .

Question 1.2

$$\begin{aligned}
 p(x|\theta) &= h(x) \exp(\eta(\theta) \cdot T(x) - A(\eta)) \\
 &= \exp(\eta([\alpha, \beta]) \cdot [\log x, x] - A(\alpha - 1, -\beta)) \\
 &= \exp([\alpha - 1, -\beta] \cdot [\log x, x] - \log \Gamma(\alpha) + \alpha \log(\beta)) \\
 &= \exp((\alpha - 1) \log x - \beta x - \log \Gamma(\alpha) + \alpha \log(\beta)) \\
 &= \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}
 \end{aligned} \tag{2}$$

We can see that the distribution correspond to a Gamma distribution of parameters α and β .

Question 1.3

$$\begin{aligned}
 p(x|\theta) &= h(x) \exp(\eta(\theta) \cdot T(x) - A(\eta)) \\
 &= \frac{\exp(\eta([\mu, \sigma^2]) \cdot [x, x^2] - A(\eta([\mu, \sigma^2])))}{\sqrt{2\pi}} \\
 &= \frac{\exp([\frac{\mu}{\sigma^2}, -\frac{1}{2\sigma^2}] \cdot [x, x^2] - A([\frac{\mu}{\sigma^2}, -\frac{1}{2\sigma^2}]))}{\sqrt{2\pi}} \\
 &= \frac{\exp(\frac{\mu x}{\sigma^2} - \frac{x^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2} - \log \sigma)}{\sqrt{2\pi}} \\
 &= \frac{\exp(-\frac{(x-\mu)^2}{2\sigma^2})}{\sigma\sqrt{2\pi}}
 \end{aligned} \tag{3}$$

We can see that the distribution correspond to a Normal distribution of parameters μ and σ^2 .

Question 1.4

$$\begin{aligned}
 p(x|\theta) &= h(x) \exp(\eta(\theta) \cdot T(x) - A(\eta)) \\
 &= 2 \exp(\eta(\lambda) \cdot x - A(\eta(\lambda))) \\
 &= 2 \exp(-\lambda x - A(-\lambda)) \\
 &= 2 \exp\left(-\lambda x + \log\left(\frac{\lambda}{2}\right)\right) \\
 &= \lambda e^{-\lambda x}
 \end{aligned} \tag{4}$$

We can see that the distribution correspond to a Exponential distribution of parameter λ .

Question 1.5

$$\begin{aligned}
 p(x|\theta) &= h(x) \exp(\eta(\theta) \cdot T(x) - A(\eta)) \\
 &= \exp(\eta([\psi_1, \psi_2]) \cdot [\log x, \log(1-x)] - A(\eta([\psi_1, \psi_2]))) \\
 &= \exp([\psi_1 - 1, \psi_2 - 1] \cdot [\log x, \log(1-x)] - A([\psi_1 - 1, \psi_2 - 1])) \\
 &= \exp((\psi_1 - 1) \log x + (\psi_2 - 1) \log(1-x) - \log \Gamma(\psi_1) - \log \Gamma(\psi_2) + \log \Gamma(\psi_1 + \psi_2)) \\
 &= \frac{\Gamma(\psi_1 + \psi_2)}{\Gamma(\psi_1)\Gamma(\psi_2)} x^{\psi_1-1} (1-x)^{\psi_2-1}
 \end{aligned} \tag{5}$$

We can see that the distribution correspond to a Beta distribution of parameters ψ_1 and ψ_2 .

2 Dependencies in a Directed Graphical Model

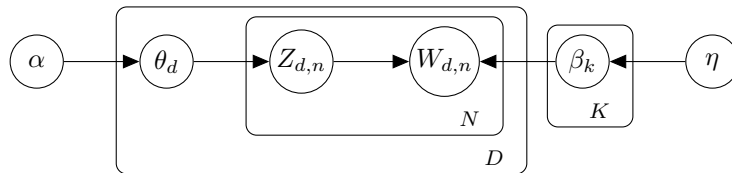
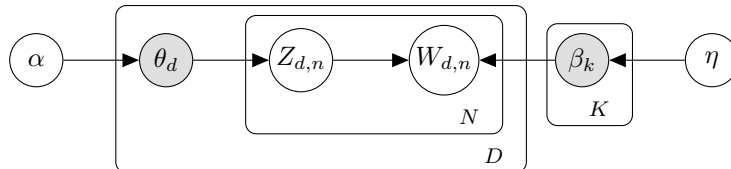


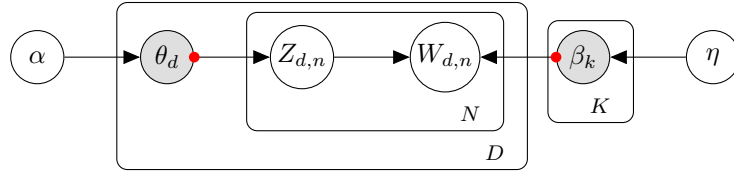
Figure 1: Graphical model of smooth LDA.

Question 2.6

The Bayes net take this form :



Then, if we use the method using the d-separation, we obtain this :



Therefore, we can see that $W_{d,n} \perp W_{d,n+1} | \theta_d, \beta_{1:K}$ is false.

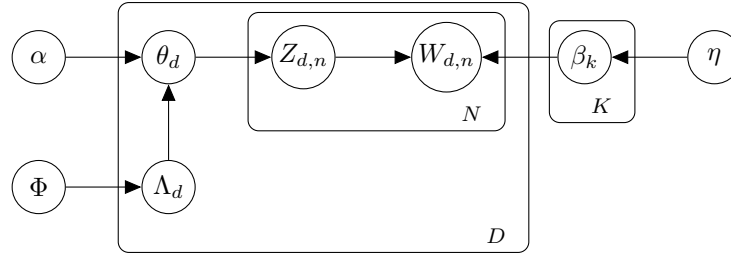


Figure 2: Graphical model of Labeled LDA.

3 CAVI

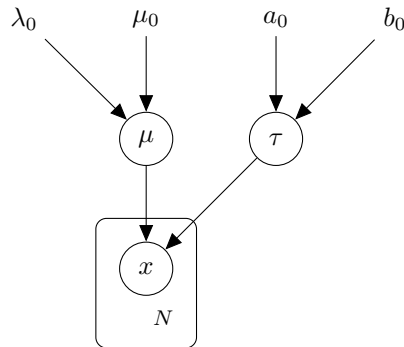


Figure 3: DGM

Question 3.12

In the bishop book, we can see that :

$$p(X|\mu, \tau) = \left(\frac{\tau}{2\pi}\right)^{N/2} \exp \left\{ -\frac{\tau}{2} \sum_{n=1}^N (x_n - \mu)^2 \right\} \quad (6)$$

$$p(\mu|\tau) = \mathcal{N}(\mu|\mu_0, (\lambda_0\tau)^{-1}) \quad (7)$$

$$p(\tau) = \text{Gam}(\tau|a_0, b_0) \quad (8)$$

Then, by using the code in appendix A.1, we obtain :

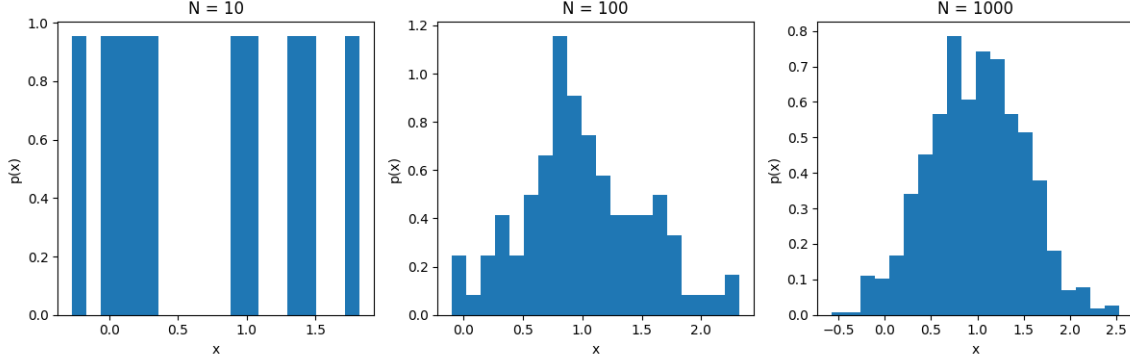


Figure 4: Generated Data.

Question 3.13

Let's find the ML estimates of μ and τ . We know that $\log(q^*(\mu)) = \mathbb{E}_{-\mu}[\log p(X, \mu, \tau)]$. Then, we can write :

$$\begin{aligned}
 \log(q^*(\mu)) &= \mathbb{E}_{-\mu}[\log p(X, \mu, \tau)] \\
 &\stackrel{\pm}{=} \mathbb{E}_{\tau}[\log p(X|\mu, \tau) + \log p(\mu|\tau)] \\
 &= \mathbb{E}_{\tau} \left[\frac{N}{2} \log \left(\frac{\tau}{2\pi} \right) + \frac{\tau}{2} \sum_{n=1}^N (x_n - \mu)^2 + \frac{1}{2} \log \left(\frac{\lambda_0 \tau}{2\pi} \right) + \frac{\lambda_0 \tau}{2} (\mu - \mu_0) \right] \\
 &\stackrel{\pm}{=} \frac{\mathbb{E}_{\tau}[\tau]}{2} \left(\lambda_0 (\mu - \mu_0) + \sum_{n=1}^N (x_n - \mu)^2 \right) \\
 &= -\frac{\mathbb{E}_{\tau}[\tau]}{2} \left(\lambda_0 \mu^2 - 2\lambda_0 \mu \mu_0 + \lambda_0 \mu_0^2 + \sum_{n=1}^N x_n^2 - 2\mu \sum_{n=1}^N x_n + N\mu^2 \right) \\
 &= -\frac{\mathbb{E}_{\tau}[\tau]}{2} \left((\lambda_0 + N)\mu^2 - 2(\lambda_0 \mu_0 + \sum_{n=1}^N x_n)\mu + \lambda_0 \mu_0^2 + \sum_{n=1}^N x_n^2 \right) \\
 &= -\frac{\mathbb{E}_{\tau}[\tau](\lambda_0 + N)}{2} \left(\mu^2 - 2\mu' \frac{\lambda_0 \mu_0 + \sum_{n=1}^N x_n}{\lambda_0 + N} + \frac{\lambda_0 \mu_0^2 + \sum_{n=1}^N x_n^2}{\lambda_0 + N} \right)
 \end{aligned} \tag{9}$$

Therefore we can conclude that $q^*(\mu) = \mathcal{N}(\mu|\mu', \lambda'^{-1})$ with :

$$\mu' = \frac{\lambda_0 \mu_0 + \sum_{n=1}^N x_n}{\lambda_0 + N} \tag{10}$$

$$\lambda' = (\lambda_0 + N)\mathbb{E}[\tau] \tag{11}$$

And for τ we have :

$$\begin{aligned}
 \log(q^*(\tau)) &= \mathbb{E}_{-\tau}[\log p(X, \mu, \tau)] \\
 &\stackrel{\pm}{=} \mathbb{E}_{\mu}[\log p(X|\mu, \tau) + \log p(\mu|\tau)] + \log p(\tau) \\
 &\stackrel{\pm}{=} (a_0 - 1) \log \tau - b_0 \tau + \frac{N}{2} \log \tau - \frac{\tau}{2} \mathbb{E}_{\mu} \left[\sum_{n=1}^N (x_n - \mu)^2 + \lambda_0 (\mu - \mu_0)^2 \right] \\
 &= (a_0 + \frac{N}{2} - 1) \log \tau - \left(b_0 + \frac{1}{2} \mathbb{E}_{\mu} \left[\sum_{n=1}^N (x_n - \mu)^2 + \lambda_0 (\mu - \mu_0)^2 \right] \right) \tau
 \end{aligned} \tag{12}$$

Therefore we can conclude that $q^*(\tau) = \text{Gam}(\tau|a', b')$ with :

$$a' = a_0 + \frac{N}{2} \tag{13}$$

$$\begin{aligned}
 b' &= b_0 + \frac{1}{2} \mathbb{E}_{\mu} \left[\sum_{n=1}^N (x_n - \mu)^2 + \lambda_0 (\mu - \mu_0)^2 \right] \\
 b' &= b_0 + \frac{1}{2} \left(\sum_{n=1}^N x_n^2 + N \mathbb{E}_{\mu}[\mu^2] - 2 \mathbb{E}_{\mu}[\mu] \sum_{n=1}^N x_n + \lambda_0 (\mathbb{E}_{\mu}[\mu^2] + \mu_0^2 - 2 \mu_0 \mathbb{E}_{\mu}[\mu]) \right)
 \end{aligned} \tag{14}$$

Question 3.14

The equation (10.24) in the Bishop is the mean-field approximation which is :

$$q(\mu, \tau) = q(\mu)q(\tau) \tag{15}$$

We have also some formulas to detail :

$$\begin{aligned}
 \mathbb{E}_{\mu}[\mu] &= \mu' \\
 \mathbb{E}_{\mu}[\mu^2] &= \frac{1}{\lambda'} + \mu'^2 \\
 \mathbb{E}_{\tau}[\tau] &= \frac{a'}{b'}
 \end{aligned} \tag{16}$$

The rest of the answer is in the code in appendix A.1.

A Appendix

A.1 Python Code

```
import numpy as np
import scipy.special as sp_spec
import scipy.stats as sp_stats
import numpy.random as np_rand
import matplotlib.pyplot as plt
from sklearn.cluster import KMeans
from sklearn.metrics import adjusted_rand_score
import seaborn as sns

# Question 1.3.12

def generate_data(N: int, mu: float, tau: float) -> np.ndarray:
    return np_rand.normal(mu, tau, N)

def plot_data(X: np.ndarray, ax: plt.Axes) -> None:
    ax.hist(X, bins=20, density=True)
    ax.set_xlabel('x')
    ax.set_ylabel('p(x)')
    ax.set_title(f'N = {len(X)}')

MU = 1
TAU = 0.5
N = [10, 100, 1000]
Xs = [generate_data(n, MU, TAU) for n in N]

fig, axs = plt.subplots(1, 3, figsize=(12, 4))
for i in range(len(Xs)):
    plot_data(Xs[i], axs[i])
plt.tight_layout()
plt.savefig('12_data.png')

# Question 1.3.14

def update_tau(x, mu_prime, lambda_prime, a_0, b_0, mu_0, lambda_0):
    E_mu = mu_prime
    E_mu2 = 1 / lambda_prime + mu_prime ** 2

    a_prime = a_0 + x.shape[0] / 2
    b_prime = b_0 + 0.5 * (np.sum(x ** 2) - 2 * np.sum(x) * E_mu +
                          x.shape[0] * E_mu2 + lambda_0 * (E_mu2 + mu_0 ** 2
                          - 2 * E_mu * mu_0))

    return a_prime, b_prime
```



```
def update_mu(x, a_prime, b_prime, mu_0, lambda_0):  
    E_tau = a_prime / b_prime  
    mu_prime = (lambda_0 * mu_0 + np.sum(x)) / (lambda_0 + x.shape[0])  
    lambda_prime = (lambda_0 + x.shape[0])*E_tau  
  
    return mu_prime, lambda_prime
```