DD2434 - Machine Learning, Advanced Course Assignment 1B

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1 CAVI for Earth quakes

1.1 Question 1.1

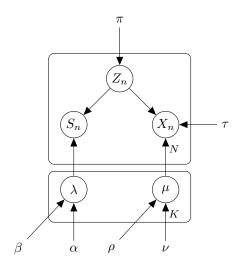


Figure 1: Directed Graphical Model for the Earthquake problem

1.2 Question 1.2

Let us take the Alternative 1 in 2D. Here, we know these distributions:

- $p(Z_n|\pi) = Categorical(\pi)$
- $p(S_n|Z_n = k, \lambda_k) = Poisson(\lambda_k)$
- $p(X_n|Z_n = k, \mu_k, \tau) = Normal(\mu_k, \tau \cdot I)$
- $p(\mu_k|\nu,\rho) = Normal(\nu,\rho\cdot I)$
- $p(\lambda_k | \alpha, \beta) = Gamma(\alpha, \beta)$

Where, ρ and τ define precision and not standard variation. Then we have:

$$\log p(X, S, Z, \lambda, \mu | \pi, \tau, \alpha, \beta, \nu, \rho) = \log p(X | S, Z, \lambda, \mu, \pi, \tau, \alpha, \beta, \nu, \rho)$$

$$+ \log p(S, Z, \lambda, \mu | \pi, \alpha, \beta, \nu, \rho)$$

$$= \log p(X | Z, \mu, \tau) + \log p(S | Z, \lambda, \mu, \pi, \alpha, \beta, \nu, \rho)$$

$$+ \log p(Z, \lambda, \mu | \pi\alpha, \beta, \nu, \rho)$$

$$= \log p(X | Z, \mu, \tau) + \log p(S | Z, \lambda) + \log p(Z | \pi)$$

$$+ \log p(\lambda, \mu | \alpha, \beta, \nu, \rho)$$

$$\log p(X, S, Z, \lambda, \mu | \pi, \tau, \alpha, \beta, \nu, \rho) = \log p(X | Z, \mu, \tau) + \log p(S | Z, \lambda) + \log p(Z | \pi)$$

$$+ \log p(\mu | \nu, \rho) + \log p(\lambda | \alpha, \beta)$$

$$(1)$$

Where:

$$\log p(X|Z, \mu, \tau) = \sum_{n=1}^{N} \sum_{k=1}^{K} \log p(X_n|Z_n = k, \mu_k, \tau)$$

$$\log p(S|Z, \lambda) = \sum_{n=1}^{N} \sum_{k=1}^{K} \log p(S_n|Z_n = k, \lambda_k)$$

$$\log p(Z|\pi) = \sum_{n=1}^{N} \log p(Z_n|\pi)$$

$$\log p(\mu|\nu, \rho) = \sum_{k=1}^{K} \log p(\mu_k|\nu, \rho)$$

$$\log p(\lambda|\alpha, \beta) = \sum_{k=1}^{K} \log p(\lambda_k|\alpha, \beta)$$
(2)

1.3 Question 1.3

Here, the mean field approximation is not an approximation but an equality because Z, μ, λ are independent. Therefore we have:

$$\log q^*(Z_n) \stackrel{\pm}{=} \mathbb{E}_{\mu,\lambda}[\log p(X_n, S_n, Z_n, \lambda, \mu | \pi, \tau, \alpha, \beta, \nu, \rho)]$$

$$\stackrel{\pm}{=} \mathbb{E}_{\mu,\lambda}[\log p(X_n | Z_n, \mu, \tau) + \log p(S_n | Z_n, \lambda) + \log p(Z_n | \pi)]$$

$$= \mathbb{E}_{\mu} \left[\sum_{k=1}^{K} \mathbb{1}_{\{Z_n = k\}} \left(\log \left(\frac{\tau}{2\pi} \right) - \frac{\tau}{2} \left((x_n - \mu_k)^T (x_n - \mu_k) \right) \right) \right]$$

$$+ \mathbb{E}_{\lambda} \left[\sum_{k=1}^{K} \mathbb{1}_{\{Z_n = k\}} \left(\log(\pi_k) - \lambda_k + S_n \log(\lambda_k) - \log(S_n!) \right) \right]$$

$$\stackrel{\pm}{=} \sum_{k=1}^{K} \mathbb{1}_{\{Z_n = k\}} \left(\log \left(\frac{\tau}{2\pi} \right) - \frac{\tau}{2} \mathbb{E}_{\mu} \left[(x_n - \mu_k)^T (x_n - \mu_k) \right] + \log(\pi_k) + \mathbb{E}_{\lambda} \left[-\lambda_k + S_n \log(\lambda_k) \right] - \log(S_n!) \right)$$

$$(3)$$

Now, if we take the entire expression that is multiplied by $\mathbb{1}_{\{Z_n=k\}}$ and we call it $u_{n,k}$, we have:

$$q^*(Z_n) \propto \prod_{k=1}^K u_{n,k}^{\mathbb{1}_{\{Z_n=k\}}}$$
 (4)

And if we normalize by taking $r_{n,k} = \frac{u_{n,k}}{\sum_{i=1}^{K} u_{n,i}}$ we get:

$$q^*(Z_n) = \prod_{k=1}^K r_{n,k}^{\mathbb{1}_{\{Z_n = k\}}}$$
 (5)

Wich means that $q^*(Z_n)$ is a categorical distribution with parameters $r_{n,k}$. There for we have the expectation of Z_n easily because $\mathbb{E}[z_{n,k}] = r_{n,k}$ where $z_{n,k} = \mathbb{1}_{\{S_n = k\}}$. Note that $r_{n,k}$ depends of

the expected value of μ_k , μ_k^2 , λ_k and $\log \lambda_k$. We will be able to compute these expected values by finding $q^*(\mu_k)$ and $q^*(\lambda_k)$. Let us compute $q^*(\mu_k)$:

$$\log q^{*}(\mu_{k}) \stackrel{\pm}{=} \mathbb{E}_{Z,\lambda}[\log p(X, S, Z = k, \lambda_{k}, \mu_{k} | \pi, \tau, \alpha, \beta, \nu, \rho)]$$

$$\stackrel{\pm}{=} \mathbb{E}_{Z,\lambda}[\log p(X | Z = k, \mu_{k}, \tau) + \log p(\mu_{k} | \nu, \rho)]$$

$$= \mathbb{E}_{Z,\lambda} \left[\sum_{n=1}^{N} \mathbb{1}_{\{Z_{n} = k\}} \left(\log \left(\frac{\tau}{2\pi} \right) - \frac{\tau}{2} \left((x_{n} - \mu_{k})^{T} (x_{n} - \mu_{k}) \right) \right) \right]$$

$$+ \log \left(\frac{\rho}{2\pi} \right) - \frac{\rho}{2} \left((\mu_{k} - \nu)^{T} (\mu_{k} - \nu) \right)$$

$$\stackrel{\pm}{=} \sum_{n=1}^{N} r_{n,k} \left(\log \left(\frac{\tau}{2\pi} \right) - \frac{\tau}{2} \left((x_{n} - \mu_{k})^{T} (x_{n} - \mu_{k}) \right) \right) - \frac{\rho}{2} \left((\mu_{k} - \nu)^{T} (\mu_{k} - \nu) \right)$$

$$\stackrel{\pm}{=} \sum_{n=1}^{N} r_{n,k} \left(-\frac{\tau}{2} \left((x_{n} - \mu_{k})^{T} (x_{n} - \mu_{k}) \right) \right) - \frac{\rho}{2} \left((\mu_{k} - \nu)^{T} (\mu_{k} - \nu) \right)$$

$$\stackrel{\pm}{=} -\frac{\tau \sum_{n=1}^{N} r_{n,k}}{2} \left(-2\mu_{k,0} x_{n,0} - 2\mu_{k,1} x_{n,1} + \mu_{k,0}^{2} + \mu_{k,1}^{2} \right)$$

$$- \frac{\rho}{2} \left(-2\mu_{k,0} \nu_{0} - 2\mu_{k,1} \nu_{1} + \mu_{k,0}^{2} + \mu_{k,1}^{2} \right)$$

We define $S = \frac{\rho}{\tau \sum_{n=1}^{N} r_{n,k}}$. Then we have:

$$\log q^*(\mu_k) \stackrel{+}{=} -\frac{\tau \sum_{n=1}^N r_{n,k}}{2} \left[(S+N)\mu_{k,0}^2 + (S+N)\mu_{k,1}^2 -2\mu_{k,0}(S\nu_0 + \sum_{n=1}^N x_{n,0}) - 2\mu_{k,1}(S\nu_1 + \sum_{n=1}^N x_{n,1}) \right]$$

$$\stackrel{+}{=} -\frac{\tau \sum_{n=1}^N r_{n,k}}{2(S+N)} \left[\left(\mu_k - \frac{S\nu + \sum_{n=1}^N x_n}{S+N} \right)^T \left(\mu_k - \frac{S\nu + \sum_{n=1}^N x_n}{S+N} \right) \right]$$
(7)

Therefore, we have $q^*(\mu_k) = Normal(\mu^*, \rho^* \cdot I)$. And we can compute the expected value of μ_k and μ_k^2 easily.

$$\mu^* = \frac{S\nu + \sum_{n=1}^{N} x_n}{S+N} = \frac{\rho\nu + \tau \sum_{n=1}^{N} r_{n,k} x_n}{\rho + N\tau \sum_{n=1}^{N} r_{n,k}}$$

$$\rho^* = \frac{\tau \sum_{n=1}^{N} r_{n,k}}{S+N} = \frac{(\tau \sum_{n=1}^{N} r_{n,k})^2}{\rho + N\tau \sum_{n=1}^{N} r_{n,k}}$$
(8)

And therefore:

$$\mathbb{E}[\mu_k] = \mu^*$$

$$\mathbb{E}[\mu_k^2] = \frac{1}{\rho^*} + \mu^{*T} \mu^*$$
(9)

Let us compute $q^*(\lambda_k)$:

$$\log q^{*}(\lambda_{k}) \stackrel{\pm}{=} \mathbb{E}_{Z,\mu}[\log p(X, S, Z = k, \lambda_{k}, \mu_{k} | \pi, \tau, \alpha, \beta, \nu, \rho)]$$

$$\stackrel{\pm}{=} \mathbb{E}_{Z,\mu}[\log p(S | Z = k, \lambda_{k}) + \log p(\lambda_{k} | \alpha, \beta)]$$

$$= \mathbb{E}_{Z} \left[\sum_{n=1}^{N} \mathbb{1}_{\{Z_{n} = k\}} \left(-\lambda_{k} + S_{n} \log(\lambda_{k}) - \log(S_{n}!) \right) \right]$$

$$+ \log \left(\frac{\beta^{\alpha}}{\Gamma(\alpha)} \right) + (\alpha - 1) \log(\lambda_{k}) - \beta \lambda_{k}$$

$$\stackrel{\pm}{=} \sum_{n=1}^{N} r_{n,k} \left(-\lambda_{k} + S_{n} \log(\lambda_{k}) \right) + (\alpha - 1) \log(\lambda_{k}) - \beta \lambda_{k}$$

$$= \left(\alpha + \sum_{n=1}^{N} S_{n} r_{n,k} - 1 \right) \log(\lambda_{k}) - \left(\beta + \sum_{n=1}^{N} r_{n,k} \right) \lambda_{k}$$

$$(10)$$

Therefore, we have $q^*(\lambda_k) = Gamma\left(\alpha + \sum_{n=1}^N S_n r_{n,k}, \beta + \sum_{n=1}^N r_{n,k}\right)$. And we can compute the expected value of λ_k and $\log \lambda_k$ easily.

$$\mathbb{E}[\lambda_k] = \frac{\alpha + \sum_{n=1}^{N} S_n r_{n,k}}{\beta + \sum_{n=1}^{N} r_{n,k}}$$

$$\mathbb{E}[\log \lambda_k] = \psi(\alpha + \sum_{n=1}^{N} S_n r_{n,k}) - \log(\beta + \sum_{n=1}^{N} r_{n,k})$$
(11)

A Appendix

A.1 Question 1.2

```
import matplotlib.pyplot as plt
import numpy as np
from scipy.stats import gamma, norm
from scipy.special import psi
np.random.seed(14)
def generate_data(mu, tau, N):
    # Insert your code here
    D = np.random.normal(mu, np.sqrt(1/tau), N)
    return D
MU = 1
TAU = 0.5
dataset_1 = generate_data(MU, TAU, 10)
dataset_2 = generate_data(MU, TAU, 100)
dataset_3 = generate_data(MU, TAU, 1000)
# Visulaize the datasets via histograms
# Insert your code here
fig, axs = plt.subplots(1, 3, figsize=(12, 4))
axs[0].hist(dataset_1, bins=20)
axs[1].hist(dataset_2, bins=20)
axs[2].hist(dataset_3, bins=20)
plt.tight_layout()
plt.savefig('12_data.png')
plt.show()
```