

DD2434 - Machine Learning, Advanced Course  
Assignment 1B

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# 1 CAVI for Earth quakes

## 1.1 Question 1.1

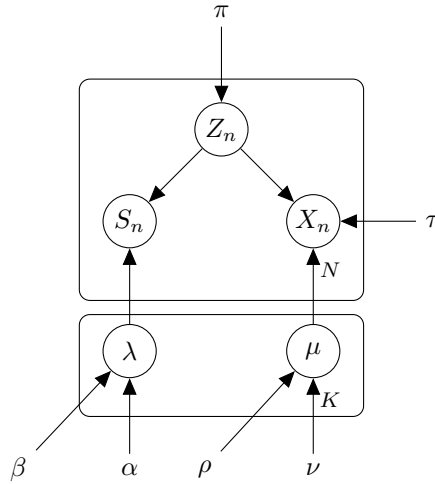


Figure 1: Directed Graphical Model for the Earthquake problem

## 1.2 Question 1.2

Let us take the Alternative 1 in 2D. Here, we know these distributions:

- $p(Z_n|\pi) = \text{Categorical}(\pi)$
- $p(S_n|Z_n = k, \lambda_k) = \text{Poisson}(\lambda_k)$
- $p(X_n|Z_n = k, \mu_k, \tau) = \text{Normal}(\mu_k, \tau \cdot I)$
- $p(\mu_k|\nu, \rho) = \text{Normal}(\nu, \rho \cdot I)$
- $p(\lambda_k|\alpha, \beta) = \text{Gamma}(\alpha, \beta)$

Where,  $\rho$  and  $\tau$  define precision and not standard variation. Then we have:

$$\begin{aligned}
 \log p(X, S, Z, \lambda, \mu|\pi, \tau, \alpha, \beta, \nu, \rho) &= \log p(X|S, Z, \lambda, \mu, \pi, \tau, \alpha, \beta, \nu, \rho) \\
 &\quad + \log p(S, Z, \lambda, \mu|\pi, \alpha, \beta, \nu, \rho) \\
 &= \log p(X|Z, \mu, \tau) + \log p(S|Z, \lambda, \mu, \pi, \alpha, \beta, \nu, \rho) \\
 &\quad + \log p(Z, \lambda, \mu|\pi, \alpha, \beta, \nu, \rho) \\
 &= \log p(X|Z, \mu, \tau) + \log p(S|Z, \lambda) + \log p(Z|\pi) \\
 &\quad + \log p(\lambda, \mu|\alpha, \beta, \nu, \rho) \\
 \log p(X, S, Z, \lambda, \mu|\pi, \tau, \alpha, \beta, \nu, \rho) &= \log p(X|Z, \mu, \tau) + \log p(S|Z, \lambda) + \log p(Z|\pi) \\
 &\quad + \log p(\mu|\nu, \rho) + \log p(\lambda|\alpha, \beta)
 \end{aligned} \tag{1}$$

Where:

$$\begin{aligned}
 \log p(X|Z, \mu, \tau) &= \sum_{n=1}^N \sum_{k=1}^K \log p(X_n|Z_n = k, \mu_k, \tau) \\
 \log p(S|Z, \lambda) &= \sum_{n=1}^N \sum_{k=1}^K \log p(S_n|Z_n = k, \lambda_k) \\
 \log p(Z|\pi) &= \sum_{n=1}^N \log p(Z_n|\pi) \\
 \log p(\mu|\nu, \rho) &= \sum_{k=1}^K \log p(\mu_k|\nu, \rho) \\
 \log p(\lambda|\alpha, \beta) &= \sum_{k=1}^K \log p(\lambda_k|\alpha, \beta)
 \end{aligned} \tag{2}$$

### 1.3 Question 1.3

Here, the mean field approximation is not an approximation but an equality because  $Z, \mu, \lambda$  are independent. Therefore we have:

$$\begin{aligned}
 \log q^*(Z_n) &\stackrel{\pm}{=} \mathbb{E}_{\mu, \lambda} [\log p(X_n, S_n, Z_n, \lambda, \mu|\pi, \tau, \alpha, \beta, \nu, \rho)] \\
 &\stackrel{\pm}{=} \mathbb{E}_{\mu, \lambda} [\log p(X_n|Z_n, \mu, \tau) + \log p(S_n|Z_n, \lambda) + \log p(Z_n|\pi)] \\
 &= \mathbb{E}_{\mu} \left[ \sum_{k=1}^K \mathbb{1}_{\{Z_n=k\}} \left( \log \left( \frac{\tau}{2\pi} \right) - \frac{\tau}{2} ((x_n - \mu_k)^T (x_n - \mu_k)) \right) \right] \\
 &\quad + \mathbb{E}_{\lambda} \left[ \sum_{k=1}^K \mathbb{1}_{\{Z_n=k\}} (\log(\pi_k) - \lambda_k + S_n \log(\lambda_k) - \log(S_n!)) \right] \\
 &\stackrel{\pm}{=} \sum_{k=1}^K \mathbb{1}_{\{Z_n=k\}} \left( \log \left( \frac{\tau}{2\pi} \right) - \frac{\tau}{2} \mathbb{E}_{\mu} [(x_n - \mu_k)^T (x_n - \mu_k)] \right. \\
 &\quad \left. + \log(\pi_k) + \mathbb{E}_{\lambda} [-\lambda_k + S_n \log(\lambda_k)] - \log(S_n!) \right)
 \end{aligned} \tag{3}$$

Now, if we take the entire expression that is multiplied by  $\mathbb{1}_{\{Z_n=k\}}$  and we call it  $u_{n,k}$ , we have:

$$q^*(Z_n) \propto \prod_{k=1}^K u_{n,k}^{\mathbb{1}_{\{Z_n=k\}}} \tag{4}$$

And if we normalize by taking  $r_{n,k} = \frac{u_{n,k}}{\sum_{i=1}^K u_{n,i}}$  we get:

$$q^*(Z_n) = \prod_{k=1}^K r_{n,k}^{\mathbb{1}_{\{Z_n=k\}}} \tag{5}$$

Wich means that  $q^*(Z_n)$  is a categorical distribution with parameters  $r_{n,k}$ . There for we have the expectation of  $Z_n$  easily because  $\mathbb{E}[z_{n,k}] = r_{n,k}$  where  $z_{n,k} = \mathbb{1}_{\{S_n=k\}}$ . Note that  $r_{n,k}$  depends of

the expected value of  $\mu_k$ ,  $\mu_k^2$ ,  $\lambda_k$  and  $\log \lambda_k$ . We will be able to compute these expected values by finding  $q^*(\mu_k)$  and  $q^*(\lambda_k)$ .

Let us compute  $q^*(\mu_k)$ :

$$\begin{aligned}
 \log q^*(\mu_k) &\stackrel{\pm}{=} \mathbb{E}_{Z,\lambda}[\log p(X, S, Z = k, \lambda_k, \mu_k | \pi, \tau, \alpha, \beta, \nu, \rho)] \\
 &\stackrel{\pm}{=} \mathbb{E}_{Z,\lambda}[\log p(X | Z = k, \mu_k, \tau) + \log p(\mu_k | \nu, \rho)] \\
 &= \mathbb{E}_{Z,\lambda} \left[ \sum_{n=1}^N \mathbb{1}_{\{Z_n=k\}} \left( \log \left( \frac{\tau}{2\pi} \right) - \frac{\tau}{2} ((x_n - \mu_k)^T (x_n - \mu_k)) \right) \right] \\
 &\quad + \log \left( \frac{\rho}{2\pi} \right) - \frac{\rho}{2} ((\mu_k - \nu)^T (\mu_k - \nu)) \\
 &\stackrel{\pm}{=} \sum_{n=1}^N r_{n,k} \left( \log \left( \frac{\tau}{2\pi} \right) - \frac{\tau}{2} ((x_n - \mu_k)^T (x_n - \mu_k)) \right) - \frac{\rho}{2} ((\mu_k - \nu)^T (\mu_k - \nu)) \quad (6) \\
 &\stackrel{\pm}{=} \sum_{n=1}^N r_{n,k} \left( -\frac{\tau}{2} ((x_n - \mu_k)^T (x_n - \mu_k)) \right) - \frac{\rho}{2} ((\mu_k - \nu)^T (\mu_k - \nu)) \\
 &\stackrel{\pm}{=} -\frac{\tau \sum_{n=1}^N r_{n,k}}{2} (-2\mu_{k,0}x_{n,0} - 2\mu_{k,1}x_{n,1} + \mu_{k,0}^2 + \mu_{k,1}^2) \\
 &\quad - \frac{\rho}{2} (-2\mu_{k,0}\nu_0 - 2\mu_{k,1}\nu_1 + \mu_{k,0}^2 + \mu_{k,1}^2)
 \end{aligned}$$

We define  $S = \frac{\rho}{\tau \sum_{n=1}^N r_{n,k}}$ . Then we have:

$$\begin{aligned}
 \log q^*(\mu_k) &\stackrel{\pm}{=} -\frac{\tau \sum_{n=1}^N r_{n,k}}{2} \left[ (S + N)\mu_{k,0}^2 + (S + N)\mu_{k,1}^2 \right. \\
 &\quad \left. - 2\mu_{k,0}(S\nu_0 + \sum_{n=1}^N x_{n,0}) - 2\mu_{k,1}(S\nu_1 + \sum_{n=1}^N x_{n,1}) \right] \quad (7) \\
 &\stackrel{\pm}{=} -\frac{\tau \sum_{n=1}^N r_{n,k}}{2(S + N)} \left[ \left( \mu_k - \frac{S\nu + \sum_{n=1}^N x_n}{S + N} \right)^T \left( \mu_k - \frac{S\nu + \sum_{n=1}^N x_n}{S + N} \right) \right]
 \end{aligned}$$

Therefore, we have  $q^*(\mu_k) = \text{Normal}(\mu^*, \rho^* \cdot I)$ . And we can compute the expected value of  $\mu_k$  and  $\mu_k^2$  easily.

$$\begin{aligned}
 \mu^* &= \frac{S\nu + \sum_{n=1}^N x_n}{S + N} = \frac{\rho\nu + \tau \sum_{n=1}^N r_{n,k}x_n}{\rho + N\tau \sum_{n=1}^N r_{n,k}} \\
 \rho^* &= \frac{\tau \sum_{n=1}^N r_{n,k}}{S + N} = \frac{(\tau \sum_{n=1}^N r_{n,k})^2}{\rho + N\tau \sum_{n=1}^N r_{n,k}} \quad (8)
 \end{aligned}$$

And therefore:

$$\begin{aligned}
 \mathbb{E}[\mu_k] &= \mu^* \\
 \mathbb{E}[\mu_k^2] &= \frac{1}{\rho^*} + \mu^{*T} \mu^* \quad (9)
 \end{aligned}$$

Let us compute  $q^*(\lambda_k)$ :

$$\begin{aligned}
 \log q^*(\lambda_k) &\stackrel{\pm}{=} \mathbb{E}_{Z,\mu}[\log p(X, S, Z = k, \lambda_k, \mu_k | \pi, \tau, \alpha, \beta, \nu, \rho)] \\
 &\stackrel{\pm}{=} \mathbb{E}_{Z,\mu}[\log p(S | Z = k, \lambda_k) + \log p(\lambda_k | \alpha, \beta)] \\
 &= \mathbb{E}_Z \left[ \sum_{n=1}^N \mathbb{1}_{\{Z_n=k\}} (-\lambda_k + S_n \log(\lambda_k) - \log(S_n!)) \right] \\
 &\quad + \log \left( \frac{\beta^\alpha}{\Gamma(\alpha)} \right) + (\alpha - 1) \log(\lambda_k) - \beta \lambda_k \\
 &\stackrel{\pm}{=} \sum_{n=1}^N r_{n,k} (-\lambda_k + S_n \log(\lambda_k)) + (\alpha - 1) \log(\lambda_k) - \beta \lambda_k \\
 &= \left( \alpha + \sum_{n=1}^N S_n r_{n,k} - 1 \right) \log(\lambda_k) - \left( \beta + \sum_{n=1}^N r_{n,k} \right) \lambda_k
 \end{aligned} \tag{10}$$

Therefore, we have  $q^*(\lambda_k) = \text{Gamma} \left( \alpha + \sum_{n=1}^N S_n r_{n,k}, \beta + \sum_{n=1}^N r_{n,k} \right)$ . And we can compute the expected value of  $\lambda_k$  and  $\log \lambda_k$  easily.

$$\begin{aligned}
 \mathbb{E}[\lambda_k] &= \frac{\alpha + \sum_{n=1}^N S_n r_{n,k}}{\beta + \sum_{n=1}^N r_{n,k}} \\
 \mathbb{E}[\log \lambda_k] &= \psi \left( \alpha + \sum_{n=1}^N S_n r_{n,k} \right) - \log \left( \beta + \sum_{n=1}^N r_{n,k} \right)
 \end{aligned} \tag{11}$$

## 2 VAE image generation

### Question 5.1

Our objective function is ELBO:  $E_{q(z|x)} \left[ \log \frac{p(x,z)}{q(z|x)} \right]$

We will show that ELBO can be rewritten as  $E_{q(z|x)} (\log p(x|z)) - D_{KL}(q(z|x) || p(z))$ . We have:

$$\begin{aligned}
 E_{q(z|x)} \left[ \log \frac{p(x,z)}{q(z|x)} \right] &= E_{q(z|x)} [\log p(x,z) - \log q(z|x)] \\
 &= E_{q(z|x)} [\log p(x|z) + \log p(z) - \log q(z|x)] \\
 &= E_{q(z|x)} [\log p(x|z)] + E_{q(z|x)} [\log p(z)] - E_{q(z|x)} [\log q(z|x)] \\
 &= E_{q(z|x)} [\log p(x|z)] - E_{q(z|x)} \left[ \log \frac{q(z|x)}{p(z)} \right] \\
 &= E_{q(z|x)} [\log p(x|z)] - D_{KL}(q(z|x) || p(z))
 \end{aligned}$$

### Question 5.2

Consider the second term:  $-D_{KL}(q(z|x) || p(z))$

**Question :** Kullback-Leibler divergence can be computed using the closed-form analytic expression when both the variational and the prior distributions are Gaussian. Write down this KL

divergence in terms of the parameters of the prior and the variational distributions. Your solution should consider a generic case where the latent space is  $K$ -dimensional.

We have:

$$D_{KL}(q(z|x)||p(z)) = \int q(z|x) \log \frac{q(z|x)}{p(z)} dz$$

And we also have:

$$\begin{aligned} q(z|x) &= \mathcal{N}(z|\mu(x), \sigma(x)) = \prod_{i=1}^K \left( \frac{1}{\sigma_i \sqrt{2\pi}} \exp \left( -\frac{(z_i - \mu_i)^2}{2\sigma_i^2} \right) \right) \\ p(z) &= \mathcal{N}(z|0, I) = \left( \frac{1}{\sqrt{2\pi}} \right)^K \exp \left( -\frac{1}{2} z^T z \right) = \prod_{i=1}^K \left( \frac{1}{\sqrt{2\pi}} \right) \exp \left( -\frac{1}{2} z_i^2 \right) \end{aligned} \quad (12)$$

Therefore:

$$\begin{aligned} D_{KL}(q(z|x)||p(z)) &= \int q(z|x) \log \frac{q(z|x)}{p(z)} dz \\ &= \int q(z|x) \log \frac{\prod_{i=1}^K (2\pi\sigma_i^2)^{-\frac{1}{2}} \exp \left( -\frac{(z_i - \mu_i)^2}{2\sigma_i^2} \right)}{\prod_{i=1}^K (2\pi)^{-\frac{1}{2}} \exp \left( -\frac{z_i^2}{2} \right)} dz \\ &= \int q(z|x) \left( \sum_{i=1}^K -\log(\sigma_i) - \frac{(z_i - \mu_i)^2}{2\sigma_i^2} + \frac{z_i^2}{2} \right) dz \\ &= \mathbb{E}_{q(z|x)} \left[ \sum_{i=1}^K -\log(\sigma_i) - \frac{(z_i - \mu_i)^2}{2\sigma_i^2} + \frac{z_i^2}{2} \right] \\ &= \sum_{i=1}^K -\log(\sigma_i) - \frac{\mathbb{E}_{q(z|x)} [(z_i - \mu_i)^2]}{2\sigma_i^2} + \frac{\mathbb{E}_{q(z|x)} [z_i^2]}{2} \\ &= \sum_{i=1}^K -\log(\sigma_i) - \frac{\sigma_i^2}{2\sigma_i^2} + \frac{\mathbb{E}_{q(z|x)} [z_i^2]}{2} \\ &= \sum_{i=1}^K -\log(\sigma_i) - \frac{1}{2} + \frac{\sigma_i^2 + \mu_i^2}{2} \\ D_{KL}(q(z|x)||p(z)) &= \frac{1}{2} \sum_{i=1}^K (\sigma_i^2 + \mu_i^2 - \log(\sigma_i^2) - 1) \end{aligned} \quad (13)$$

The rest of the implementation could be found in the appendix A.1.

## A Appendix

### A.1 VAE image generation

#### A.1.1 Reparameterization function

```
def reparameterization(self, mean, var):  
    # insert your code here  
    std = torch.sqrt(var + 1e-10)  
    eps = torch.randn_like(std)  
    z = mean + std * eps  
  
    return z
```

#### A.1.2 Loss function

```
def loss_function(x, theta, mean, log_var): # should return the loss  
    function (~ ELBO)  
    # insert your code here  
    # expected log-likelihood  
    recon_loss = -torch.sum(x * torch.log(theta + 1e-10) +  
                             (1 - x) * torch.log(1 - theta + 1e-10))  
  
    # KL Divergence  
    kl_div = -0.5 * torch.sum(1 + log_var - mean.pow(2) - log_var.exp())  
  
    loss = recon_loss + kl_div  
  
    return loss
```

#### A.1.3 Generation from test data

```
model.eval()  
# below we get decoder outputs for test data  
with torch.no_grad():  
    for batch_idx, (x, _) in enumerate(tqdm(test_loader)):  
        x = x.view(batch_size, x_dim)  
        # insert your code below to generate theta from x  
  
        # Pass the test images through the encoder and decoder  
        mean, log_var = model.Encoder(x)  
        # reparameterize to get latent variable  
        z = model.reparameterization(mean, torch.exp(log_var))  
        # decode the latent variable to get reconstructed image  
        theta = model.Decoder(z)
```



#### A.1.4 Generation from noise

```
with torch.no_grad():
    # insert your code here to create images from noise (it is enough to
    # create theta value for each pixel)
    #
    #
    # generated_images = .... # should be a matrix ( batch_size-by-x_dim )
    generated_images = torch.round(
        model.Decoder(torch.randn(batch_size, latent_dim)))
```