DD2434 - Machine Learning, Advanced Course
Assignment 1B

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November 2023

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1 CAVI for Earth quakes

1.1 Question 1.1

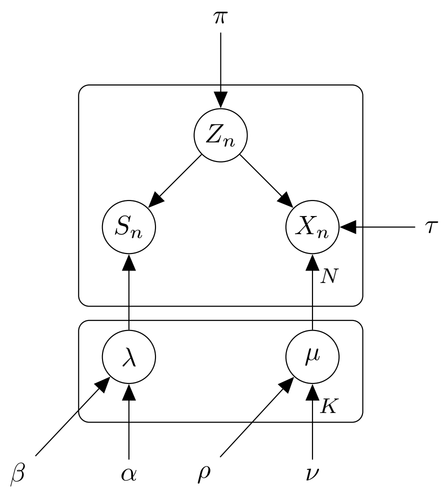

Figure 1: Directed Graphical Model for the Earthquake problem

1.2 Question 1.2

Let us take the Alternative 1 in 2D. Here, we know these distributions:

- $p(Z_n|\pi) = Categorical(\pi)$
- $p(S_n|Z_n = k, \lambda_k) = Poisson(\lambda_k)$
- $p(X_n|Z_n = k, \mu_k, \tau) = Normal(\mu_k, \tau \cdot I)$
- $p(\mu_k|\nu, \rho) = Normal(\nu, \rho \cdot I)$
- $p(\lambda_k|\alpha, \beta) = Gamma(\alpha, \beta)$

Where, ρ and τ define precision and not standard variation. Then we have:

$$\begin{aligned}
\log p(X, S, Z, \lambda, \mu|\pi, \tau, \alpha, \beta, \nu, \rho) &= \log p(X|S, Z, \lambda, \mu, \pi, \tau, \alpha, \beta, \nu, \rho) \\
&\quad + \log p(S, Z, \lambda, \mu|\pi, \alpha, \beta, \nu, \rho) \\
&= \log p(X|Z, \mu, \tau) + \log p(S|Z, \lambda, \mu, \pi, \alpha, \beta, \nu, \rho) \\
&\quad + \log p(Z, \lambda, \mu|\pi\alpha, \beta, \nu, \rho) \\
&= \log p(X|Z, \mu, \tau) + \log p(S|Z, \lambda) + \log p(Z|\pi) \\
&\quad + \log p(\lambda, \mu|\alpha, \beta, \nu, \rho) \\
\log p(X, S, Z, \lambda, \mu|\pi, \tau, \alpha, \beta, \nu, \rho) &= \log p(X|Z, \mu, \tau) + \log p(S|Z, \lambda) + \log p(Z|\pi) \\
&\quad + \log p(\mu|\nu, \rho) + \log p(\lambda|\alpha, \beta)
\end{aligned} \tag{1}$$

Where:

$$
\begin{aligned}
\log p(X|Z,\mu,\tau) &= \sum_{n=1}^{N}\sum_{k=1}^{K}\log p(X_n|Z_n=k,\mu_k,\tau) \\
\log p(S|Z,\lambda) &= \sum_{n=1}^{N}\sum_{k=1}^{K}\log p(S_n|Z_n=k,\lambda_k) \\
\log p(Z|\pi) &= \sum_{n=1}^{N}\log p(Z_n|\pi) \\
\log p(\mu|\nu,\rho) &= \sum_{k=1}^{K}\log p(\mu_k|\nu,\rho) \\
\log p(\lambda|\alpha,\beta) &= \sum_{k=1}^{K}\log p(\lambda_k|\alpha,\beta)
\end{aligned} \tag{2}
$$

1.3 Question 1.3

Here, the mean field approximation is not an approximation but an equality because Z, μ, λ are independent. Therefore we have:

$$
\begin{aligned}
\log q^*(Z_n) &\stackrel{+}{=} \mathbb{E}_{\mu,\lambda}[\log p(X_n, S_n, Z_n, \lambda, \mu|\pi, \tau, \alpha, \beta, \nu, \rho)] \\
&\stackrel{+}{=} \mathbb{E}_{\mu,\lambda}[\log p(X_n|Z_n,\mu,\tau) + \log p(S_n|Z_n,\lambda) + \log p(Z_n|\pi)] \\
&= \mathbb{E}_{\mu}\left[\sum_{k=1}^{K}\mathbb{1}_{\{Z_n=k\}}\left(\log\left(\frac{\tau}{2\pi}\right) - \frac{\tau}{2}\left((x_n-\mu_k)^T(x_n-\mu_k)\right)\right)\right] \\
&\qquad + \mathbb{E}_{\lambda}\left[\sum_{k=1}^{K}\mathbb{1}_{\{Z_n=k\}}\left(\log(\pi_k) - \lambda_k + S_n\log(\lambda_k) - \log(S_n!)\right)\right] \\
&\stackrel{+}{=} \sum_{k=1}^{K}\mathbb{1}_{\{Z_n=k\}}\left(\log\left(\frac{\tau}{2\pi}\right) - \frac{\tau}{2}\mathbb{E}_{\mu}\left[(x_n-\mu_k)^T(x_n-\mu_k)\right]\right. \\
&\qquad \left. + \log(\pi_k) + \mathbb{E}_{\lambda}\left[-\lambda_k + S_n\log(\lambda_k)\right] - \log(S_n!)\right)
\end{aligned} \tag{3}
$$

Now, if we take the entire expression that is multiplied by $\mathbb{1}_{\{Z_n=k\}}$ and we call it $u_{n,k}$, we have:

$$
q^*(Z_n) \propto \prod_{k=1}^{K} u_{n,k}^{\mathbb{1}_{\{Z_n=k\}}} \tag{4}
$$

And if we normalize by taking $r_{n,k} = \frac{u_{n,k}}{\sum_{i=1}^{K} u_{n,i}}$ we get:

$$
q^*(Z_n) = \prod_{k=1}^{K} r_{n,k}^{\mathbb{1}_{\{Z_n=k\}}} \tag{5}
$$

Wich means that $q^*(Z_n)$ is a categorical distribution with parameters $r_{n,k}$. There for we have the expectation of Z_n easily because $\mathbb{E}[z_{n,k}] = r_{n,k}$ where $z_{n,k} = \mathbb{1}_{\{S_n=k\}}$. Note that $r_{n,k}$ depends of

the expected value of μ_k, μ_k^2, λ_k and $\log \lambda_k$. We will be able to compute these expected values by finding $q^*(\mu_k)$ and $q^*(\lambda_k)$.
Let us compute $q^*(\mu_k)$:

$$\begin{aligned}
\log q^*(\mu_k) &\stackrel{\pm}{=} \mathbb{E}_{Z,\lambda}[\log p(X, S, Z = k, \lambda_k, \mu_k | \pi, \tau, \alpha, \beta, \nu, \rho)] \\
&\stackrel{\pm}{=} \mathbb{E}_{Z,\lambda}[\log p(X | Z = k, \mu_k, \tau) + \log p(\mu_k | \nu, \rho)] \\
&= \mathbb{E}_{Z,\lambda}\left[\sum_{n=1}^{N} \mathbb{1}_{\{Z_n = k\}} \left(\log\left(\frac{\tau}{2\pi}\right) - \frac{\tau}{2}\left((x_n - \mu_k)^T (x_n - \mu_k)\right)\right)\right] \\
&\qquad + \log\left(\frac{\rho}{2\pi}\right) - \frac{\rho}{2}\left((\mu_k - \nu)^T (\mu_k - \nu)\right) \\
&\stackrel{\pm}{=} \sum_{n=1}^{N} r_{n,k} \left(\log\left(\frac{\tau}{2\pi}\right) - \frac{\tau}{2}\left((x_n - \mu_k)^T (x_n - \mu_k)\right)\right) - \frac{\rho}{2}\left((\mu_k - \nu)^T (\mu_k - \nu)\right) \\
&\stackrel{\pm}{=} \sum_{n=1}^{N} r_{n,k} \left(-\frac{\tau}{2}\left((x_n - \mu_k)^T (x_n - \mu_k)\right)\right) - \frac{\rho}{2}\left((\mu_k - \nu)^T (\mu_k - \nu)\right) \\
&\stackrel{\pm}{=} -\frac{\tau \sum_{n=1}^{N} r_{n,k}}{2}\left(-2\mu_{k,0} x_{n,0} - 2\mu_{k,1} x_{n,1} + \mu_{k,0}^2 + \mu_{k,1}^2\right) \\
&\qquad - \frac{\rho}{2}\left(-2\mu_{k,0}\nu_0 - 2\mu_{k,1}\nu_1 + \mu_{k,0}^2 + \mu_{k,1}^2\right)
\end{aligned} \tag{6}$$

We define $S = \frac{\rho}{\tau \sum_{n=1}^{N} r_{n,k}}$. Then we have:

$$\begin{aligned}
\log q^*(\mu_k) &\stackrel{\pm}{=} -\frac{\tau \sum_{n=1}^{N} r_{n,k}}{2}\Big[(S+N)\mu_{k,0}^2 + (S+N)\mu_{k,1}^2 \\
&\qquad -2\mu_{k,0}(S\nu_0 + \sum_{n=1}^{N} x_{n,0}) - 2\mu_{k,1}(S\nu_1 + \sum_{n=1}^{N} x_{n,1})\Big] \\
&\stackrel{\pm}{=} -\frac{\tau \sum_{n=1}^{N} r_{n,k}}{2(S+N)}\left[\left(\mu_k - \frac{S\nu + \sum_{n=1}^{N} x_n}{S+N}\right)^T \left(\mu_k - \frac{S\nu + \sum_{n=1}^{N} x_n}{S+N}\right)\right]
\end{aligned} \tag{7}$$

Therefore, we have $q^*(\mu_k) = Normal\left(\mu^*, \rho^* \cdot I\right)$. And we can compute the expected value of μ_k and μ_k^2 easily.

$$\begin{aligned}
\mu^* &= \frac{S\nu + \sum_{n=1}^{N} x_n}{S+N} = \frac{\rho\nu + \tau \sum_{n=1}^{N} r_{n,k} x_n}{\rho + N\tau \sum_{n=1}^{N} r_{n,k}} \\
\rho^* &= \frac{\tau \sum_{n=1}^{N} r_{n,k}}{S+N} = \frac{(\tau \sum_{n=1}^{N} r_{n,k})^2}{\rho + N\tau \sum_{n=1}^{N} r_{n,k}}
\end{aligned} \tag{8}$$

And therefore:

$$\begin{aligned}
\mathbb{E}[\mu_k] &= \mu^* \\
\mathbb{E}[\mu_k^2] &= \frac{1}{\rho^*} + \mu^{*T}\mu^*
\end{aligned} \tag{9}$$

Let us compute $q^*(\lambda_k)$:

$$\begin{aligned}
\log q^*(\lambda_k) &\overset{\pm}{=} \mathbb{E}_{Z,\mu}[\log p(X, S, Z = k, \lambda_k, \mu_k | \pi, \tau, \alpha, \beta, \nu, \rho)] \\
&\overset{\pm}{=} \mathbb{E}_{Z,\mu}[\log p(S | Z = k, \lambda_k) + \log p(\lambda_k | \alpha, \beta)] \\
&= \mathbb{E}_Z \left[\sum_{n=1}^{N} \mathbb{1}_{\{Z_n = k\}} \left(-\lambda_k + S_n \log(\lambda_k) - \log(S_n!)\right) \right] \\
&\qquad + \log \left(\frac{\beta^\alpha}{\Gamma(\alpha)} \right) + (\alpha - 1) \log(\lambda_k) - \beta \lambda_k \\
&\overset{\pm}{=} \sum_{n=1}^{N} r_{n,k} \left(-\lambda_k + S_n \log(\lambda_k)\right) + (\alpha - 1) \log(\lambda_k) - \beta \lambda_k \\
&= \left(\alpha + \sum_{n=1}^{N} S_n r_{n,k} - 1 \right) \log(\lambda_k) - \left(\beta + \sum_{n=1}^{N} r_{n,k} \right) \lambda_k
\end{aligned} \tag{10}$$

Therefore, we have $q^*(\lambda_k) = Gamma\left(\alpha + \sum_{n=1}^{N} S_n r_{n,k}, \beta + \sum_{n=1}^{N} r_{n,k}\right)$. And we can compute the expected value of λ_k and $\log \lambda_k$ easily.

$$\begin{aligned}
\mathbb{E}[\lambda_k] &= \frac{\alpha + \sum_{n=1}^{N} S_n r_{n,k}}{\beta + \sum_{n=1}^{N} r_{n,k}} \\
\mathbb{E}[\log \lambda_k] &= \psi(\alpha + \sum_{n=1}^{N} S_n r_{n,k}) - \log(\beta + \sum_{n=1}^{N} r_{n,k})
\end{aligned} \tag{11}$$

2 VAE image generation

Question 5.1

Our objective function is ELBO: $E_{q(z|x)}\big[\log \frac{p(x,z)}{q(z|x)}\big]$
We will show that ELBO can be rewritten as $E_{q(z|x)}\big(\log p(x|z)\big) - D_{KL}\big(q(z|x)||p(z)\big)$ We have:

$$\begin{aligned}
E_{q(z|x)}\big[\log \frac{p(x,z)}{q(z|x)}\big] &= E_{q(z|x)}\big[\log p(x,z) - \log q(z|x)\big] \\
&= E_{q(z|x)}\big[\log p(x|z) + \log p(z) - \log q(z|x)\big] \\
&= E_{q(z|x)}\big[\log p(x|z)\big] + E_{q(z|x)}\big[\log p(z)\big] - E_{q(z|x)}\big[\log q(z|x)\big] \\
&= E_{q(z|x)}\big[\log p(x|z)\big] - E_{q(z|x)}\big[\log \frac{q(z|x)}{p(z)}\big] \\
&= E_{q(z|x)}\big[\log p(x|z)\big] - D_{KL}\big(q(z|x)||p(z)\big)
\end{aligned}$$

Question 5.2

Consider the second term: $-D_{KL}\big(q(z|x)||p(z)\big)$

Question : *Kullback–Leibler divergence can be computed using the closed-form analytic expression when both the variational and the prior distributions are Gaussian. Write down this KL* *divergence in terms of the parameters of the prior and the variational distributions. Your solution should consider a generic case where the latent space is K-dimensional.*

We have:

$$D_{KL}\left(q(z|x)||p(z)\right) = \int q(z|x) \log \frac{q(z|x)}{p(z)}\, dz$$

And we also have:

$$\begin{aligned} q(z|x) = \mathcal{N}(z|\mu(x), \sigma(x)) &= \prod_{i=1}^{K} \left(\frac{1}{\sigma_i \sqrt{2\pi}} \exp\left(-\frac{(z_i - \mu_i)^2}{2\sigma^2} \right) \right) \\ p(z) = \mathcal{N}(z|0, I) = \left(\frac{1}{\sqrt{2\pi}} \right)^K \exp\left(-\frac{1}{2} z^T z \right) &= \prod_{i=1}^{K} \left(\frac{1}{\sqrt{2\pi}} \right) \exp\left(-\frac{1}{2} z_i^2 \right) \end{aligned} \tag{12}$$

Therefore:

$$\begin{aligned} D_{KL}\left(q(z|x)||p(z)\right) &= \int q(z|x) \log \frac{q(z|x)}{p(z)}\, dz \\ &= \int q(z|x) \log \frac{\prod_{i=1}^{K} \left(2\pi\sigma_i^2\right)^{-\frac{1}{2}} \exp\left(-\frac{(z_i-\mu_i)^2}{2\sigma_i^2}\right)}{\prod_{i=1}^{K} (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{z_i^2}{2}\right)}\, dz \\ &= \int q(z|x) \left(\sum_{i=1}^{K} -\log(\sigma_i) - \frac{(z_i - \mu_i)^2}{2\sigma_i^2} + \frac{z_i^2}{2} \right) dz \\ &= \mathbb{E}_{q(z|x)} \left[\sum_{i=1}^{K} -\log(\sigma_i) - \frac{(z_i - \mu_i)^2}{2\sigma_i^2} + \frac{z_i^2}{2} \right] \\ &= \sum_{i=1}^{K} -\log(\sigma_i) - \frac{\mathbb{E}_{q(z|x)}\left[(z_i - \mu_i)^2\right]}{2\sigma_i^2} + \frac{\mathbb{E}_{q(z|x)}\left[z_i^2\right]}{2} \\ &= \sum_{i=1}^{K} -\log(\sigma_i) - \frac{\sigma_i^2}{2\sigma_i^2} + \frac{\mathbb{E}_{q(z|x)}[z_i^2]}{2} \\ &= \sum_{i=1}^{K} -\log(\sigma_i) - \frac{1}{2} + \frac{\sigma_i^2 + \mu_i^2}{2} \\ D_{KL}\left(q(z|x)||p(z)\right) &= \frac{1}{2} \sum_{i=1}^{K} \left(\sigma_i^2 + \mu_i^2 - \log(\sigma_i^2) - 1 \right) \end{aligned} \tag{13}$$

The rest of tht implementation could be find in the appendix A.1.

A Appendix

A.1 VAE image generation

A.1.1 Reparameterization function

```
def reparameterization(self, mean, var):
    # insert your code here
    std = torch.sqrt(var + 1e-10)
    eps = torch.randn_like(std)
    z = mean + std * eps

    return z
```

A.1.2 Loss function

```
def loss_function(x, theta, mean, log_var):  # should return the loss
    function (- ELBO)
    # insert your code here
    # expected log-likelihood
    recon_loss = -torch.sum(x * torch.log(theta + 1e-10) +
                            (1 - x) * torch.log(1 - theta + 1e-10))

    # KL Divergence
    kl_div = -0.5 * torch.sum(1 + log_var - mean.pow(2) - log_var.exp())

    loss = recon_loss + kl_div

    return loss
```

A.1.3 Generation from test data

```
model.eval()
# below we get decoder outputs for test data
with torch.no_grad():
    for batch_idx, (x, _) in enumerate(tqdm(test_loader)):
        x = x.view(batch_size, x_dim)
        # insert your code below to generate theta from x

        # Pass the test images through the encoder and decoder
        mean, log_var = model.Encoder(x)
        # reparameterize to get latent variable
        z = model.reparameterization(mean, torch.exp(log_var))
        # decode the latent variable to get reconstructed image
        theta = model.Decoder(z)
```

A.1.4 Generation from noise

```
with torch.no_grad():
    # insert your code here to create images from noise (it is enough to
        create theta value for each pixel)
    #
    #
    # generated_images = ....  # should be a matrix ( batch_size-by-x_dim )
    generated_images = torch.round(
        model.Decoder(torch.randn(batch_size, latent_dim)))
```