

# Butterworth Filters

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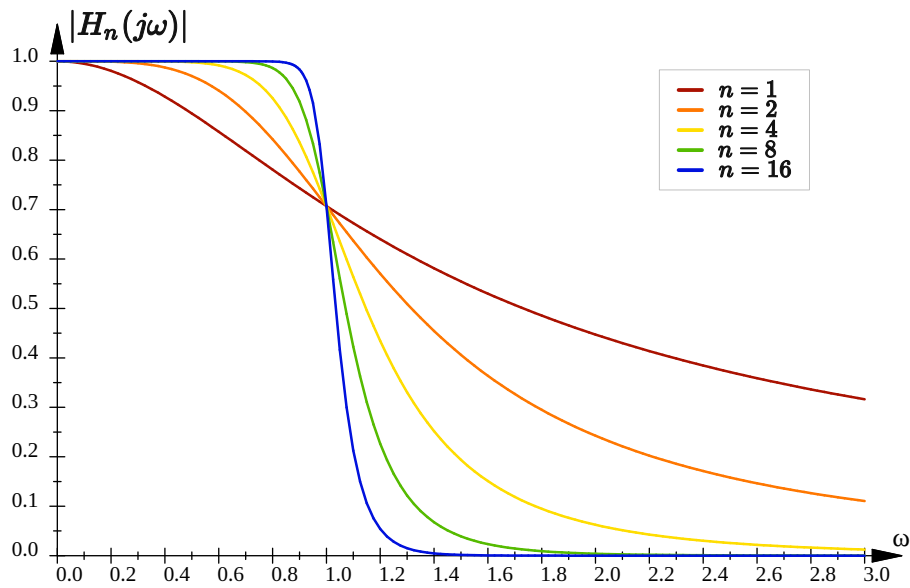
This page will cover the derivation of the transfer functions of low-pass and high-pass Butterworth filters. Butterworth filters are designed to have a very flat frequency response in the passband.

## Definition

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Normalized Butterworth filters are defined in the frequency domain as follows:

$$|H_n(j\omega)| \triangleq \frac{1}{\sqrt{1 + \omega^{2n}}} \quad (1)$$



In order to determine the transfer function, we'll start from the frequency response squared. We'll assume that the transfer function  $H_n(s)$  is a rational function with real coefficients. Therefore,  $\overline{H_n(s)} = H_n(\overline{s})$ .

$$\begin{aligned} |H_n(j\omega)|^2 &= H_n(j\omega) \overline{H_n(j\omega)} \\ &= H_n(j\omega) H_n(\overline{j\omega}) \\ &= H_n(j\omega) H_n(-j\omega) \\ &= \frac{1}{1 + \omega^{2n}} \end{aligned}$$

We're looking for the transfer function  $H_n(s)$ , so we'll use the identity  $s = j\omega \Leftrightarrow \omega = \frac{s}{j}$ .

$$H_n(s)H_n(-s) = \frac{1}{1 + \left(\frac{s}{j}\right)^{2n}}$$

## Poles of $H_n(s)H_n(-s)$

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The poles of this transfer function are given by:

$$\begin{aligned}
 \left(\frac{s}{j}\right)^{2n} &= -1 \\
 \Leftrightarrow s^{2n} &= -1(j)^{2n} \\
 \Leftrightarrow s^{2n} &= -1(-1)^n \\
 \Leftrightarrow s^{2n} &= (-1)^{n+1} \\
 \Leftrightarrow s^{2n} &= e^{j\pi(n+1)}
 \end{aligned}$$

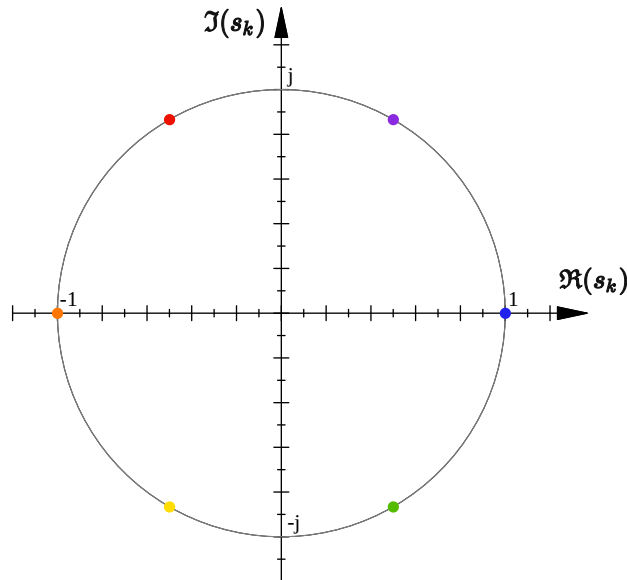
Keep in mind that this is a polynomial of order  $2n$ , so it has  $2n$  complex roots.

$$s_k = e^{j2\pi \frac{2k+n+1}{4n}} \quad k \in \{0, 1, \dots, 2n-1\}$$

For example, for  $n = 3$ , the poles are:

$$\begin{aligned}
 s_0 &= e^{j2\pi \frac{0+3+1}{12}} = e^{j2\pi \frac{2}{6}} \\
 s_1 &= e^{j2\pi \frac{2+3+1}{12}} = e^{j2\pi \frac{3}{6}} \\
 s_2 &= e^{j2\pi \frac{4+3+1}{12}} = e^{j2\pi \frac{4}{6}} \\
 s_3 &= e^{j2\pi \frac{6+3+1}{12}} = e^{j2\pi \frac{5}{6}} \\
 s_4 &= e^{j2\pi \frac{8+3+1}{12}} = e^{j2\pi \frac{6}{6}} \\
 s_5 &= e^{j2\pi \frac{10+3+1}{12}} = e^{j2\pi \frac{1}{6}}
 \end{aligned}$$

These are all points on the unit circle,  $\pi/3 = 60^\circ$  apart.



The poles are stable if they are in the left half plane, if their complex argument is between  $90^\circ$  and  $270^\circ$ :

$$\begin{aligned}
& 2\pi \frac{2k+n+1}{4n} \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \\
\Leftrightarrow & 2k+n+1 \in (n, 3n) \\
\Leftrightarrow & k \in \left( -\frac{1}{2}, n - \frac{1}{2} \right) \\
\Rightarrow & k \in \left( -\frac{1}{2}, n - \frac{1}{2} \right) \cup \{0, 1, \dots, 2n-1\} \\
\Leftrightarrow & k \in \{0, 1, \dots, n-1\} \\
s_{k,stable} &= e^{j2\pi \frac{2k+n+1}{4n}} \quad k \in \{0, 1, \dots, n-1\} \tag{2}
\end{aligned}$$

## Poles of $H_n(s)$

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We want our filter  $H_n(s)$  to be stable, so we pick the poles in the left half plane to be the poles of  $H_n(s)$ . The unstable poles, for  $k \in \{n, n+1, \dots, 2n-1\}$  are the poles of  $H_n(-s)$ . They are the opposites of the poles of  $H_n(s)$ :

$$\begin{aligned}
s_{k,unstable} &= e^{j2\pi \frac{2k+n+1}{4n}} \quad k \in \{n, n+1, \dots, 2n-1\} \\
& \quad l \triangleq k - n \\
&= e^{j2\pi \frac{2(l+n)+n+1}{4n}} \quad l \in \{0, 1, \dots, n-1\} \\
&= e^{j(2\pi \frac{2l+n+1}{4n} + \pi)} \\
&= e^{j\pi} \cdot e^{j2\pi \frac{2l+n+1}{4n}} \\
&= -1 \cdot e^{j2\pi \frac{2l+n+1}{4n}} \\
&= -s_{l,stable}
\end{aligned}$$

## Butterworth Polynomials

We'll define the normalized Butterworth polynomial as follows:

$$B_n(s) \triangleq \prod_{k=0}^{n-1} \left( s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \quad (3)$$

We'll rearrange the product to group each pole with its complex conjugate. Then, using the identity  $e^{j\theta} + e^{-j\theta} = 2 \cos \theta$ , we can further simplify this expression:

Even order  $n$ :

$$\begin{aligned} B_n(s) &= \prod_{k=0}^{n-1} \left( s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \\ &= \prod_{k=0}^{\frac{n}{2}-1} \left( s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \prod_{l=\frac{n}{2}}^{n-1} \left( s - e^{j2\pi \frac{2l+n+1}{4n}} \right) \\ &\quad l = n - k - 1 \\ &= \prod_{k=0}^{\frac{n}{2}-1} \left( s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \left( s - e^{j2\pi \frac{2(n-k-1)+n+1}{4n}} \right) \\ &= \prod_{k=0}^{\frac{n}{2}-1} \left( s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \left( s - \frac{e^{j2\pi \frac{-2k+3n-1}{4n}}}{1} \right) \\ &= \prod_{k=0}^{\frac{n}{2}-1} \left( s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \left( s - \frac{e^{j2\pi \frac{-2k+3n-1}{4n}}}{e^{j2\pi}} \right) \\ &= \prod_{k=0}^{\frac{n}{2}-1} \left( s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \left( s - e^{j2\pi \left( \frac{-2k+3n-1}{4n} - 1 \right)} \right) \\ &= \prod_{k=0}^{\frac{n}{2}-1} \left( s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \left( s - e^{j2\pi \frac{-2k+3n-4n-1}{4n}} \right) \\ &= \prod_{k=0}^{\frac{n}{2}-1} \left( s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \left( s - e^{j2\pi \frac{-2k-n-1}{4n}} \right) \\ &= \prod_{k=0}^{\frac{n}{2}-1} \left( s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \left( s - e^{-j2\pi \frac{2k+n+1}{4n}} \right) \\ &= \prod_{k=0}^{\frac{n}{2}-1} \left( s^2 - s e^{j2\pi \frac{2k+n+1}{4n}} - s e^{-j2\pi \frac{2k+n+1}{4n}} + 1 \right) \\ &= \prod_{k=0}^{\frac{n}{2}-1} \left( s^2 - 2 \cos \left( 2\pi \frac{2k+n+1}{4n} \right) s + 1 \right) \end{aligned}$$

Odd order  $n$ :

In this case,  $n - 1$  is even, and you get a special pole for  $k = \frac{n-1}{2}$ :

$$\begin{aligned} s_{\frac{n-1}{2}} &= e^{j2\pi \frac{2 \frac{n-1}{2} + n + 1}{4n}} \\ &= e^{j2\pi \frac{2n}{4n}} \\ &= e^{j\pi} \\ &= -1 \end{aligned}$$

After isolating this pole, we're left with an even number of complex conjugate poles, just like in the case where  $n$  was even.

In conclusion, the normalized Butterworth polynomial of degree  $n$  is given by:

$$B_n(s) = \begin{cases} \prod_{k=0}^{\frac{n}{2}-1} (s^2 - 2 \cos(2\pi \frac{2k+n+1}{4n}) s + 1) & \text{even } n \\ (s+1) \prod_{k=0}^{\frac{n-1}{2}-1} (s^2 - 2 \cos(2\pi \frac{2k+n+1}{4n}) s + 1) & \text{odd } n \end{cases} \quad (4)$$

## Butterworth Transfer Function $H_n(s)$

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The transfer function  $H_n(s)$  has no zeros, so the numerator is a constant. The poles of  $H_n(s)$  are given by Equation (2), so the denominator is given by Equation (3).

$$H_n(s) = \frac{c}{B_n(s)}$$

We wanted a DC gain of 1 ( $= 0dB$ ) for  $\omega = 0$ :

$$\begin{aligned} |H_n(0j)| &= 1 \\ \Leftrightarrow \left| \frac{c}{B_n(0)} \right| &= 1 \\ \Leftrightarrow \left| \frac{c}{\prod_{k=0}^{n-1} \left( 0 - e^{j2\pi \frac{2k+n+1}{4n}} \right)} \right| &= 1 \\ \Leftrightarrow \frac{|c|}{\prod_{k=0}^{n-1} \left| -e^{j2\pi \frac{2k+n+1}{4n}} \right|} &= 1 \\ \Leftrightarrow \frac{|c|}{1} &= 1 \end{aligned}$$

If we want no phase offset for low frequencies, we can postulate that  $\angle H_n(0j) = 0$ :

$$\begin{aligned} \angle H_n(0j) &= 0 \\ \Leftrightarrow \angle \left( \frac{c}{B_n(0)} \right) &= 0 \\ \Leftrightarrow \angle c - \angle \left( \prod_{k=0}^{\frac{n}{2}-1} \left( 0^2 - 2 \cos \left( 2\pi \frac{2k+n+1}{4n} \right) \cdot 0 + 1 \right) \right) &= 0 \\ \Leftrightarrow \angle c - \angle 1 &= 0 \end{aligned}$$

The derivation is analogous for odd  $n$ .

Therefore,  $c = 1$ , and we've eliminated all unknown parameters from the transfer function:

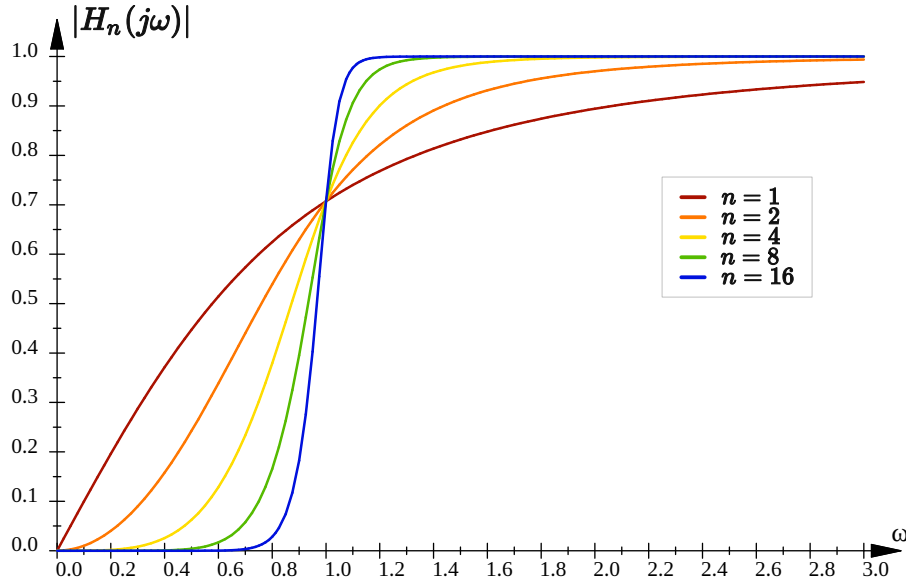
$$H_n(s) = \frac{1}{B_n(s)} \tag{5}$$

## High-Pass Butterworth filters

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Up until now, we only looked at the low-pass Butterworth filter. There's also a high-pass version:

$$|H_{n,hp}(j\omega)| \triangleq \frac{1}{\sqrt{1 + \omega^{-2n}}} \tag{6}$$



We can just multiply the numerator and the denominator by  $\omega^n$  to get a more familiar form:

$$|H_{n,hp}(j\omega)| = \frac{\omega^n}{\sqrt{1 + \omega^{2n}}}$$

As you can see, the poles will be the same as for the low-pass version. On top of that, there now are  $n$  zeros for  $s = 0$ . So the transfer function becomes:

$$H_{n,hp}(s) = \frac{s^n}{B_n(s)} \quad (7)$$

## Non-normalized Butterworth Filters

Up until now, we only looked at normalized Butterworth filters, that have a corner frequency of 1 rad/s. To get a specific corner frequency  $\omega_c$ , we can just scale  $\omega$ , so the definitions become:

$$|H_{n,lp}(j\omega)| \triangleq \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}} \quad (8)$$

$$|H_{n,hp}(j\omega)| \triangleq \frac{1}{\sqrt{1 + \left(\frac{\omega_c}{\omega}\right)^{2n}}} \quad (9)$$

If you start recalculating the transfer functions, you'll quickly realize that this just scales everything by a factor of  $\omega_c$ . The poles no longer lie on the unit circle, but on a circle with radius  $|s_k| = \omega_c$ .

This results in the following transfer functions:

$$H_{n,lp}(s) = \frac{1}{B_n\left(\frac{s}{\omega_c}\right)} \quad (10)$$

$$H_{n,hp}(s) = \frac{s^n}{\omega_c^n B_n\left(\frac{s}{\omega_c}\right)} \quad (11)$$

The gain at the corner frequency can easily be determined from the definitions:

$$\begin{aligned}
 |H_{n,lp}(j\omega_c)| &= |H_{n,hp}(j\omega_c)| = \frac{1}{\sqrt{1 + \left(\frac{\omega_c}{\omega_c}\right)^{2n}}} \\
 &= \frac{1}{\sqrt{2}} \\
 &= \frac{\sqrt{2}}{2} \\
 &\approx 0.707 \\
 20 \log_{10} |H_n(j\omega_c)| &= 20 \log_{10} \left( \frac{\sqrt{2}}{2} \right) \\
 &= 10 \log_{10} \left( \frac{1}{2} \right) \\
 &\approx -3.01 \text{ dB}
 \end{aligned}$$

This is often called the  $-3 \text{ dB}$ -point or the half-power point, because a sinusoidal input signal at that frequency will result in an output signal that has only half of the power of the input signal:  $|H_n(j\omega_c)|^2 = \frac{1}{2}$ .