

# Convolutions

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## Convolutions

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The convolution of two signals  $f$  and  $g$  is defined as:

$$\begin{aligned} * : (\mathbb{Z} \rightarrow \mathbb{R}) \times (\mathbb{Z} \rightarrow \mathbb{R}) &\rightarrow (\mathbb{Z} \rightarrow \mathbb{R}) : \\ f[n], g[n] &\mapsto (f * g)[n] \triangleq \sum_{k=-\infty}^{+\infty} f[k] g[n - k] \end{aligned} \quad (1)$$

If no ambiguity exists, a shorthand can be used:

$$f * g \triangleq \sum_{k=-\infty}^{+\infty} f[k] g[n - k]$$

### Properties

The convolution operator can be seen as a product of discrete functions, and it has many properties usually associated with multiplication.

Commutativity:  $f * g = g * f$

Associativity:  $(f * g) * h = f * (g * h)$

Distributivity:  $(f + g) * h = f * h + g * h$

### Identity

Convolution with the Kronecker delta function results in the original signal, thanks to the sifting property of the delta function:

$$f * \delta = f = \delta * f$$

### Unilateral signals

If the first signal is unilateral (i.e.  $\forall n < 0 : f[n] = 0$ ), the lower bound of the summation becomes zero instead of minus infinity:

$$f * g = \sum_{k=0}^{+\infty} f[k] g[n - k]$$

## Signals as a sum of delta functions

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Any discrete signal can be written as an infinite sum of scaled and shifted Kronecker delta functions.

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n - k]$$

You can easily see that all terms where  $n \neq k$  are zero, because the Kronecker delta is zero in that case. Only the term for  $n = k$  is non-zero, in which case the Kronecker delta is one, so the result is just  $x[k]$ . This is an application of the sifting property of the delta function, covered in the [previous page](#).

This once again shows that the Kronecker delta is the identity signal with respect to convolution operator,  $x[n] = (x * \delta)[n]$ .

## DTLTI systems as convolutions with the impulse response

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You can express the output of any discrete-time linear time-invariant system  $T$  with any input  $x[n]$  as the convolution of the input with the impulse response of the system,  $h[n] \triangleq T(\delta[n])$ :

$$T(x[n]) = (x * h)[n] \quad (2)$$

### Proof

The proof itself is very simple: We just decompose the input as a sum of delta functions, as described in a previous section, and then we use the linearity and time-invariance to bring the  $T$  operator inside of the summation.

$$\begin{aligned}
 y[n] &= T(x[n]) \\
 &= T\left(\sum_{k=0}^{\infty} x[k] \delta[n-k]\right) \\
 &= \sum_{k=0}^{\infty} T(x[k] \delta[n-k]) \\
 &= \sum_{k=0}^{\infty} x[k] T(\delta[n-k]) \\
 &= \sum_{k=0}^{\infty} x[k] h[n-k] \\
 &\triangleq (x * h)[n]
 \end{aligned}$$

Because of the linearity of the system,  $T$  can be brought inside of the summation, and since  $x[k]$  is a constant factor independent of the time step  $n$ , it can be moved outside of the  $T$  operator.  $T$  applied to the Kronecker delta is (by definition) the impulse response of  $T$ ,  $h[n]$ . In this case, it is shifted by  $k$  time steps, which is allowed because of the time-invariance of  $T$ .  $\square$

An important consequence is that every DTLTI transformation can be uniquely represented by its impulse response, in other words, there is a one-to-one correspondence between the definition of transformation  $T$  and its impulse response  $h[n]$ .