Convolutions

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Convolutions

The convolution of two signals f and g is defined as:

$$\begin{aligned} * :\ & (\mathbb{Z} \to \mathbb{R}) \times (\mathbb{Z} \to \mathbb{R}) \to (\mathbb{Z} \to \mathbb{R}) : \\ & f[n], g[n] \mapsto (f * g)[n] \triangleq \sum_{k=-\infty}^{+\infty} f[k]\ g[n-k] \end{aligned} \tag{1}$$

If no ambiguity exists, a shorthand can be used:

$$f * g \triangleq \sum_{k=-\infty}^{+\infty} f[k]\ g[n-k]$$

Properties

The convolution operator can be seen as a product of discrete functions, and it has many properties usually associated with multiplication.

Commutativity: $f * g = g * f$

Associativity: $(f * g) * h = f * (g * h)$

Distributivity: $(f + g) * h = f * h + g * h$

Identity

Convolution with the Kronecker delta function results in the original signal, thanks to the sifting property of the delta function:

$$f * \delta = f = \delta * f$$

Unilateral signals

If the first signal is unilateral (i.e. $\forall n < 0 : f[n] = 0$), the lower bound of the summation becomes zero instead of minus infinity:

$$f * g = \sum_{k=0}^{+\infty} f[k]\ g[n-k]$$

Signals as a sum of delta functions

Any discrete signal can be written as an infinite sum of scaled and shifted Kronecker delta functions.

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\ \delta[n-k]$$

You can easily see that all terms where $n \neq k$ are zero, because the Kronecker delta is zero in that case. Only the term for $n = k$ is non-zero, in which case the Kronecker delta is one, so the result is just $x[k]$. This is an application the sifting property of the delta function, covered in the previous page.

This once again shows that the Kronecker delta is the identity signal with respect to convolution operator, $x[n] = (x * \delta)[n]$.

DTLTI systems as convolutions with the impulse response

You can express the output of any discrete-time linear time-invariant system T with any input $x[n]$ as the convolution of the input with the impulse response of the system, $h[n] \triangleq T(\delta[n])$:

$$T(x[n]) = (x * h)[n] \tag{2}$$

Proof

The proof itself is very simple: We just decompose the input as a sum of delta functions, as described in a previous section, and then we use the linearity and time-invariance to bring the T operator inside of the summation.

$$
\begin{aligned}
y[n] &= T(x[n]) \\
&= T\left(\sum_{k=0}^{\infty} x[k]\ \delta[n-k]\right) \\
&= \sum_{k=0}^{\infty} T\Big(x[k]\ \delta[n-k]\Big) \\
&= \sum_{k=0}^{\infty} x[k]\ T\Big(\delta[n-k]\Big) \\
&= \sum_{k=0}^{\infty} x[k]\ h[n-k] \\
&\triangleq (x * h)[n]
\end{aligned}
$$

Because of the linearity of the system, T can be brought inside of the summation, and since $x[k]$ is a constant factor independent of the time step n, it can be moved outside of the T operator. T applied to the Kronecker delta is (by definition) the impulse response of T, $h[n]$. In this case, it is shifted by k time steps, which is allowed because of the time-invariance of T. $\square$

An important consequence is that every DTLTI transformation can be uniquely represented by its impulse response, in other words, there is a one-to-one correspondence between the definition of transformation T and its impulse response $h[n]$.