From LBM to Navier-Stokes

Seminar Lattice Boltzmann Methods:
Theory, Implementation and Applications

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This paper outlines the derivation of the Navier-Stokes equations from the Lattice Boltzmann equation by performing the Chapman-Enskog expansion. First, a short introduction on Lattice-Boltzmann methods is given to obtain the necessary basics, then the Chapman-Enskog expansion, i.e. the multiple scale expansion is presented and finally a short discussion on divergence of the obtained equations is given.

1 Introduction

The Navier-Stokes equations are a very common way to describe problems in fluid dynamics. They lead to good solutions in many different fields, for example they are used to model the weather, ocean currents, water flow in a pipe and air flow around a wing. The fluid is described as a continuum in a velocity field or flow field, which is a description of the velocity of the fluid at a given point in space and time. Additionally there are other hydrodynamic quantities like pressure, density and temperature. For incompressible flows the density is a constant.

The Boltzmann equation (or its discrete counterpart) on the other hand describes the fluid in a particle distribution function, which gives the probability to encounter particles near a given position moving with a given velocity at a given time. Thus the fluid is described in the mesoscopic space.

The Navier-Stokes equations can be derived by applying Newton's second law to fluid motion together with some assumptions, but it is of fundamental as well as practical interest to derive the Navier-Stokes equations from the Lattice Boltzmann equation. This derivation shows the necessary steps and assumptions to go from mesoscopic to macroscopic space, shows their relationships and points out the asymptotic equivalence of both schemes. Further, approaches exist in computational simulations to couple the Lattice Boltzmann equation with the Navier Stokes equations.

The derivation of the Navier-Stokes equations from the Lattice Boltzmann equation runs under the name Chapman-Enskog expansion, developed by Chapman and Enskog independently between 1910 and 1920. This expansion can also be used to derive the Euler equations for inviscid fluid and higher order approximations (Burnett and super-Burnett equations.

This paper uses the Einstein summation convention for better readability. According to this convention, when an index variable appears twice in a single term it implies summation of that term over all the values of the index.

2 The Lattice Boltzmann method

2.1 The particle distribution function

An approach to computational fluid dynamics is derived from the Boltzmann equation and is known as the Lattice Boltzmann method. Space is discretized with cubic cells, and the velocity space is broken down to a set of discrete velocities $\mathbf{v_i}$ $(i = 1, \cdots, Q)$, allowing particles to either move to a neighbouring cell or to rest in the current one. The particle distribution functions f_i are defined in the centre of each lattice cell. They can be interpreted in the sense that $f_i(\mathbf{x}, t)$ gives the probability to encounter particles near $\mathbf{x}$ moving with speed $\mathbf{v_i}$ at time t. The particle distribution functions obey the Lattice Boltzmann equation:

$$f_i\left(\mathbf{x} + \mathbf{v_i} \cdot dt, t + dt\right) = f_i\left(\mathbf{x}, t\right) + \Delta_i(f) \tag{1}$$

where the operator $\Delta_i(f)$ models the collisions between molecules. The Lattice Boltzmann equation will be the starting point for the derivation of the Navier-Stokes equations.

Ludwig Boltzmann showed 1972 with the H-theorem, that the particle distributions tend to irreversibly seek towards a local Maxwell-Boltzmann distribution (towards the minimum value of $H = \int f \cdot ln(f) \cdot d\mathbf{v}$). This means that the Maxwell-Boltzmann distribution is the equilibrium particle distribution. In the low Mach number limit discrete transformations of the Maxwell-Boltzmann distribution lead to:

$$f_i^{(eq)}(\rho, \mathbf{u}) = \omega_i \cdot \rho \left(1 + \frac{\mathbf{v_i u}}{c_s^2} + \frac{(\mathbf{v_i u})^2}{2c_s^4} - \frac{\mathbf{u}^2}{2c_s^2}\right) \tag{2}$$

with the macroscopic density ρ and flow velocity $\mathbf{u}$, speed of sound c_s and the lattice weights ω_i, which depend on the set of discrete velocities $\mathbf{v_i}$. The general idea of the Chapman-Enskog expansion is, that the particle distribution is always close to the equilibrium distribution.

2.2 Isotropy

Since isotropy is uniformity in all orientations, a consequence of the discretization of the velocity space is anisotropy. The velocities $\mathbf{v_i}$ and the weights ω_i from the equilibrium distribution $f_i^{(eq)}$ are chosen in a way to be as close as possible to an isotropic description of the system.

Therefore the weights ω_i need to fulfil the following isotropy constraints ($\alpha, \beta, \gamma, \delta \in \{1, \cdots, D\}$):

$$\begin{aligned}
\sum_i \omega_i &= 1 \\
\sum_i \omega_i \cdot v_{i_\alpha} &= 0 \\
\sum_i \omega_i \cdot v_{i_\alpha} \cdot v_{i_\beta} &= c_s^2 \cdot \delta_{\alpha\beta} \\
\sum_i \omega_i \cdot v_{i_\alpha} \cdot v_{i_\beta} \cdot v_{i_\gamma} &= 0 \\
\sum_i \omega_i \cdot v_{i_\alpha} \cdot v_{i_\beta} \cdot v_{i_\gamma} \cdot v_{i_\delta} &= c_s^4(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})
\end{aligned}$$

2.2.1 The D2Q9 discretization

For the D2Q9 (two dimensions and nine velocity vectors) discretization (compare figure 1) the weights ω_i are given by:

$$\omega_i = \begin{cases} 1/36, & |\mathbf{v_i}|_2 = \sqrt{2}dx \\ 4/36, & \textbf{if} \quad |\mathbf{v_i}|_2 = dx \\ 16/36, & |\mathbf{v_i}|_2 = 0 \end{cases} \tag{3}$$

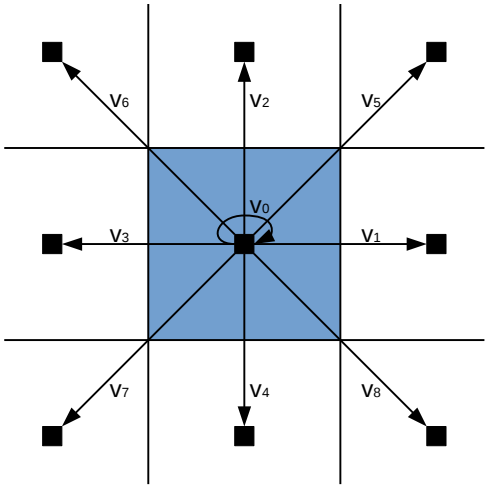

Figure 1: D2Q9 descretization with its velocity vectors $\mathbf{v_i}$

2.3 The BGK collision operator

A collision operator $\Delta_i(f)$ should respect the following constraints:

1. Mass, momentum and energy must be conserved over a collision. This is reflected in the constraint, that there must be exactly five elementary collision invariants $\psi_k(\mathbf{v})$:
   - $\psi_0 = 1$ (proportional to mass)
   - $\psi_{1,2,3} = v_{x,y,z}$ (proportional to momentum)
   - $\psi_4 = \mathbf{v}^2$ (proportional to kinetic energy)

   These collision invariants ψ_k must be conserved by the discrete collision operator:

$$\sum_i \Delta_i(f)\psi_k(\mathbf{v_i}) = 0$$

2. The collision operator must express the tendency to equilibrium distribution (to the discretized Maxwell-Boltzmann distribution in this case). This constraint corresponds to the H-theorem.

Due to the complicated nature of Boltzmann's original collision integral, attempts have been made to find easier collision operators. The most common model which fulfils the above constraints is the Bhatnagar-Gross-Krook (BGK) approximation: [1]

$$\Delta^{BGK}(f) = \frac{1}{\tau}\left(f^{(eq)} - f\right). \tag{4}$$

with the single relaxation time τ. This model simulates collisions via a linear relaxation towards the local equilibrium.

This model will be used later on to derive the Navier-Stokes equation from the Lattice Boltzmann equation. However, analogous derivations are available for other collision operators such as MRT and TRT schemes as well.

2.4 Hydrodynamic quantities and conservation laws

The following macroscopic quantities can be defined via moments of the distribution functions f_i: [11]

$$\textbf{density } \rho(\mathbf{x},t) = \sum_i f_i(\mathbf{x},t) \tag{5}$$

$$\textbf{velocity } \mathbf{u}(\mathbf{x},t) = \frac{1}{\rho}\sum_i \mathbf{v_i} f_i(\mathbf{x},t) \tag{6}$$

The findings from the previous sections[1] imply that the moments (5) and (6) don't change if f is replaced by $f^{(eq)}$.[4]

2.4.1 Momentum flux tensor

The momentum flux tensor $\mathbf{\Pi}$ is given by the complete second moment: ($\circ$ denotes the outer product)

$$\mathbf{\Pi} = \sum_i \mathbf{v_i} \circ \mathbf{v_i} f_i \tag{7}$$

On the other hand the momentum flux tensor on continuum level is defined as:

$$\mathbf{\Pi} = \rho\mathbf{u} \circ \mathbf{u} - \sigma = \rho\mathbf{u} \circ \mathbf{u} + p\mathbf{I} - \sigma' \tag{8}$$

whit the total stress tensor $\sigma = -p\mathbf{I} + \sigma'$, which consists of an isotropic contribution from the thermodynamic pressure $p = \rho \cdot c_s^2$ (ideal gas) and the deviatoric stress tensor σ':

$$\sigma'_{\alpha,\beta} = \nu \cdot \left(\partial_{x_\beta} u_\alpha + \partial_{x_\alpha} u_\beta\right) \tag{9}$$

which describes viscous dissipation.[4]

Further the equilibrium momentum flux tensor $\mathbf{\Pi^{(eq)}}$ is given by the complete second moment tensor of the $f_i^{(eq)}$:

$$\mathbf{\Pi^{(eq)}} = \sum_i \mathbf{v_i v_i} f_i^{(eq)} \tag{10}$$

The second moment tensor is not conserved by the collision operator, thus $\mathbf{\Pi}$ cannot be set equal to $\mathbf{\Pi^{(eq)}}$. In fact, the difference

$$\sigma' = \mathbf{\Pi^{(eq)}} - \mathbf{\Pi} \tag{11}$$

can be identified with the deviatoric stress tensor σ'.

[1] The findings are that the isotropy constraints must be fulfilled, the particle distribution f_i tends to the local equilibrium distribution $f_i^{(eq)}$ and that the collision operator preserves the local density, momentum and energy.

3 Chapman-Enskog expansion

The Chapman-Enskog expansion[3] is a very familiar and widespread method of obtaining information from the Boltzmann equation. It is a multiple scale expansion of the particle distribution function f. It is developed in a power series in the small parameter ϵ. This parameter ϵ is often identified with the Knudsen number Kn:

$$\epsilon = Kn = \frac{\lambda}{L}$$

with the mean free path length λ and the length scale of the macroscopic system L. A vanishing Knusden number describes the continuum limit. In general, the Knudsen number is useful for determining if the continuum mechanics formulation of fluid dynamics can be used: If the Knudsen number is of the order of 1 or larger the system under consideration cannot be described as a continuum. [11] In this case the Chapman-Enskog theory is asymptotically wrong and the Navier-Stokes equations yield a wrong description of the system.

3.1 Multiple scale expansion

This chapter performs the derivation of the Navier-Stokes equations adapting mainly the work of Neumann in [9]. To obtain the Navier-Stokes equations from the Lattice Boltzmann equation the Chapman-Enskog expansion is performed on the particle distribution functions f_i in powers of ϵ and truncated after the term of second order:

$$f_i = f_i^{(0)} + \epsilon^1 f_i^{(1)} + \epsilon^2 f_i^{(2)} + O(\epsilon^3) \tag{12}$$

The idea is to expand the distribution function into a power series near its equilibrium state. Thus the term $f_i^{(0)}$ is identified with the Maxwell-Boltzmann equilibrium distribution $f^{(eq)}$:

$$f_i^{(0)} = f_i^{(eq)}$$

Additionally we have the following two solvability conditions

$$\sum_i f_i^{(k)} = 0 \qquad \textbf{for } k = 1, 2, \cdots \tag{13}$$

$$\sum_i v_{i_\alpha} f_i^{(k)} = 0 \qquad \textbf{for } k = 1, 2, \cdots \tag{14}$$

which ensure, that only the first term $f_i^{(0)}$ contributes to the macroscopic density and momentum. These conditions arise from the fact that replacing the distribution function f_i by the equilibrium distribution $f_i^{(eq)}$ in the hydrodynamic quantities from section 2.4 does not change their value.

3.2 From microscopic to macroscopic scale

The spatial coordinate $\mathbf{x^{(1)}}$ of the macroscopic (hydrodynamic) description can be set in relation to its microscopic counterpart $\mathbf{x}$:

$$\mathbf{x^{(1)}} = \epsilon \cdot \mathbf{x} \tag{15}$$

A similar relation can be formulated for the temporal coordinate t. Since diffusion needs a very short (fast) time scale to consider Brownian motion, whereas advection takes place in a much slower process, a separation of these phenomena needs to be established:

$$t^{(1)} = \epsilon^1 \cdot t \tag{16}$$
$$t^{(2)} = \epsilon^2 \cdot t \tag{17}$$

Thus compressible flows (advective effects) are described in the time scale $t^{(1)}$ and diffusive effects in the time scale $t^{(2)}$.[2]

Finally the macroscopic distribution function $\widetilde{f_i}$ is introduced, redefining the original distribution function f_i using the introduced macroscopic space and time coordinates:

$$f_i(\mathbf{x}, t) = \widetilde{f_i}\left(\mathbf{x^{(1)}}(\mathbf{x}), t^{(1)}(t), t^{(2)}(t)\right) \tag{18}$$

To obtain the new spatial and time derivatives the chain rule applies:

$$\partial_{x_\alpha} = \epsilon \cdot \partial_{x_\alpha^{(1)}} \tag{19}$$
$$\partial_t = \epsilon \cdot \partial_{t^{(1)}} + \epsilon^2 \cdot \partial_{t^{(2)}} \tag{20}$$

These last two equations can be seen as a multiple scale expansion of the space-time derivatives.

This is the main idea of the Chapman-Enskog expansion. It is not only an expansion in the parameter ϵ of the dependent distribution functions f_i, but also of the independent space-time variables x and t. The basic idea of multiple scale analysis is to represent space and time variables in terms of hierarchy of slow/fast scales, such that each variable dominates at its own relevant scale. This not only helps the theoretical analysis, but also leads to greater stability of numerical calculations.[10]

[2] There is another approach in which only the diffusive scaling is considered to obtain the incompressible Navier-Stokes equations directly. This diffusive scaling (developed by Sone) is the natural choice (in a mathematical sense) if the LBE is viewed purely as a numerical method to solve the incompressible Navier-Stokes equation. This approach is explained in [8] but will not be discussed here.

3.3 Left hand side of the LBE

A Taylor expansion at $\mathbf{g} = (\mathbf{x}, t)$ with $\mathbf{h} = (\mathbf{v_i} \cdot dt, dt)$ is performed on the left hand side of the Lattice Boltzmann equation (1):

$$f_i(\mathbf{g} + \mathbf{h}) = f_i(\mathbf{g}) + \langle \nabla f_i(\mathbf{g}), \mathbf{h} \rangle + \frac{1}{2} \langle \mathbf{h}, H_{f_i}(\mathbf{g}) \mathbf{h} \rangle + O(|\mathbf{h}|^3) \tag{21}$$

$$\begin{aligned} f_i(\mathbf{x} + \mathbf{v_i} dt, t + dt) = & f_i(\mathbf{x}, t) \\ & + dt \left(v_{i_\alpha} \partial_{x_\alpha} + \partial_t \right) f_i(\mathbf{x}, t) \\ & + \frac{1}{2} dt^2 \left(v_{i_\alpha} \partial_{x_\alpha} + \partial_t \right)^2 f_i(\mathbf{x}, t) \big) \\ & + O(dt^3) \end{aligned} \tag{22}$$

By introducing the expansion, and the macroscopic relations into the above Taylor expansion we obtain:

$$\begin{aligned} f_i(\mathbf{x} + \mathbf{v_i} dt, t + dt) = & (\widetilde{f}_i^{(0)} + \epsilon \widetilde{f}_i^{(1)} + \epsilon^2 \widetilde{f}_i^{(2)}) \\ & + dt \left(v_{i_\alpha} \epsilon \partial_{x_\alpha^{(1)}} + (\epsilon \partial_{t^{(1)}} + \epsilon^2 \partial_{t^{(2)}}) \right) (\widetilde{f}_i^{(0)} + \epsilon \widetilde{f}_i^{(1)} + \epsilon^2 \widetilde{f}_i^{(2)}) \\ & + \frac{1}{2} dt^2 \left(v_{i_\alpha} \epsilon \partial_{x_\alpha^{(1)}} + (\epsilon \partial_{t^{(1)}} + \epsilon^2 \partial_{t^{(2)}}))^2 (\widetilde{f}_i^{(0)} + \epsilon \widetilde{f}_i^{(1)} + \epsilon^2 \widetilde{f}_i^{(2)}) \right) \\ & + O(dt^3) \end{aligned} \tag{23}$$

or after reordering the terms by the order of ϵ:

$$\begin{aligned} f_i(\mathbf{x} + \mathbf{v_i} dt, t + dt) = & \widetilde{f}_i^{(0)} \\ & + \epsilon \left(\widetilde{f}_i^{(1)} + dt \left(v_{i_\alpha} \partial_{x_\alpha^{(1)}} + \partial_{t^{(1)}} \right) \widetilde{f}^{(0)} \right) \\ & + \epsilon^2 \left(\widetilde{f}_i^{(2)} + dt \left(\partial_{t^{(2)}} \widetilde{f}_i^{(0)} + \left(v_{i_\alpha} \partial_{x_\alpha^{(1)}} + \partial_{t^{(1)}} \right) \left(\widetilde{f}_i^{(1)} + \frac{dt}{2} \left(v_{i_\alpha} \partial_{x_\alpha^{(1)}} + \partial_{t^{(1)}} \right) \widetilde{f}^{(0)} \right) \right) \right) \\ & + O(\epsilon^3) \end{aligned} \tag{24}$$

3.4 Equating coefficients

The next step is to substitute the expanded distribution function and the BGK collision term into the right hand side of the Lattice Boltzmann equation (1):

$$f_i + \Delta_i = \widetilde{f}_i^{(0)} + \epsilon \widetilde{f}_i^{(1)} + \epsilon^2 \widetilde{f}_i^{(2)} - \frac{1}{\tau} \left(\epsilon \widetilde{f}_i^{(1)} + \epsilon^2 \widetilde{f}_i^{(2)} \right) + O(\epsilon^3) \tag{25}$$

Equating coefficients for ϵ^1 of the transformed left hand side (24) and the transformed right hand side (25), brings up the following equation:

$$-\frac{1}{\tau} \widetilde{f}_i^{(1)} = dt \left(v_{i_\alpha} \partial_{x_\alpha^{(1)}} + \partial_{t^{(1)}} \right) \widetilde{f}_i^{(0)} \tag{26}$$

and for ϵ^2:

$$-\frac{1}{\tau}\widetilde{f}_i^{(2)} = dt\left(\partial_{t^{(2)}}\widetilde{f}_i^{(0)} + \left(v_{i_\alpha}\partial_{x_\alpha^{(1)}} + \partial_{t^{(1)}}\right)\left(\widetilde{f}_i^{(1)} + \frac{dt}{2}\left(v_{i_\alpha}\partial_{x_\alpha^{(1)}} + \partial_{t^{(1)}}\right)\widetilde{f}_i^{(0)}\right)\right) \quad (27)$$

This equation can be further simplified with equation (26):

$$-\frac{1}{\tau}\widetilde{f}_i^{(2)} = dt\left(\partial_{t^{(2)}}\widetilde{f}_i^{(0)} + \left(v_{i_\alpha}\partial_{x_\alpha^{(1)}} + \partial_{t^{(1)}}\right)(\widetilde{f}_i^{(1)} - \frac{1}{2\tau}\widetilde{f}_i^{(1)})\right) \quad (28)$$

$$-\frac{1}{\tau}\widetilde{f}_i^{(2)} = dt\left(\partial_{t^{(2)}}\widetilde{f}_i^{(0)} + \frac{1}{\tau}\left(\tau - \frac{1}{2}\right)\left(v_{i_\alpha}\partial_{x_\alpha^{(1)}} + \partial_{t^{(1)}}\right)\widetilde{f}_i^{(1)}\right) \quad (29)$$

3.5 Moment equations

The next step is to calculate the first two moments[3] on equation 26. The zeroth moment is:

$$\sum_i(\partial_{t^{(1)}} + v_{i_\alpha}\partial_{x_\alpha^{(1)}})\widetilde{f}_i^{(0)} = -\frac{1}{dt\cdot\tau}\sum_i \widetilde{f}_i^{(1)} \quad (30)$$

$$\partial_{t^{(1)}}\sum_i \widetilde{f}_i^{(0)} + \partial_{x_\alpha^{(1)}}\sum_i v_{i_\alpha}\widetilde{f}_i^{(0)} = 0 \quad (31)$$

$$\partial_{t^{(1)}}\rho + \partial_{x_\alpha^{(1)}}(\rho\cdot u_\alpha) = 0 \quad (32)$$

The first moment is:

$$\sum_i v_{i_\beta}(\partial_{t^{(1)}} + v_{i_\alpha}\partial_{x_\alpha^{(1)}})\widetilde{f}_i^{(0)} = -\frac{1}{dt\cdot\tau}\sum_i v_{i_\beta}\widetilde{f}_i^{(1)} \quad (33)$$

$$\partial_{t^{(1)}}\sum_i v_{i_\beta}\widetilde{f}_i^{(0)} + \partial_{x_\alpha^{(1)}}\sum_i v_{i_\alpha}v_{i_\beta}\widetilde{f}_i^{(0)} = 0 \quad (34)$$

$$\partial_{t^{(1)}}(\rho\cdot u_\beta) + \partial_{x_\alpha^{(1)}}\Pi_{\alpha,\beta}^{(eq)} = 0 \quad (35)$$

$$\partial_{t^{(1)}}(\rho\cdot u_\beta) + \partial_{x_\beta^{(1)}}p + \partial_{x_\alpha^{(1)}}(\rho\cdot u_\alpha\cdot u_\beta) = 0 \quad (36)$$

The right hand sides vanish because of the solvability conditions (13)-(14) and the integrals are replaced by the macroscopic, hydrodynamic quantities from section 2.4.

The equations (32) and (36) consider only the equilibrium parts of the expansion; no diffusive effects are described. These equations are equivalent to the Euler equations for inviscid fluids.

We perform the same calculations of the two moments on equation 29. The zeroth

[3]The moments are calculated by multiplying the equation by 1 (zeroth moment) and v_{i_β} (first moment) and summating over the velocity space ($i = 1, \cdots, Q$)

moment is:

$$\sum_i \left(\partial_{t^{(2)}} \widetilde{f}_i^{(0)} + \frac{1}{\tau} \left(\tau - \frac{1}{2} \right) \left(v_{i_\alpha} \partial_{x_\alpha^{(1)}} + \partial_{t^{(1)}} \right) \widetilde{f}_i^{(1)} \right) = - \frac{1}{dt \cdot \tau} \sum_i \widetilde{f}_i^{(2)} \tag{37}$$

$$\partial_{t^{(2)}} \sum_i \widetilde{f}_i^{(0)} + \frac{1}{\tau} \left(\tau - \frac{1}{2} \right) \left(\partial_{x_\alpha^{(1)}} \sum_i v_{i_\alpha} \widetilde{f}_i^{(1)} + \partial_{t^{(1)}} \sum_i \widetilde{f}_i^{(1)} \right) = 0 \tag{38}$$

$$\partial_{t^{(2)}} \sum_i \widetilde{f}_i^{(0)} = 0 \tag{39}$$

$$\partial_{t^{(2)}} \rho = 0 \tag{40}$$

The first moment is:

$$\sum_i v_{i_\beta} \left(\partial_{t^{(2)}} \widetilde{f}_i^{(0)} + \frac{1}{\tau} \left(\tau - \frac{1}{2} \right) \left(v_{i_\alpha} \partial_{x_\alpha^{(1)}} + \partial_{t^{(1)}} \right) \widetilde{f}_i^{(1)} \right) = - \frac{1}{dt \cdot \tau} \sum_i v_{i_\beta} \widetilde{f}_i^{(2)} \tag{41}$$

$$\partial_{t^{(2)}} \sum_i v_{i_\beta} \widetilde{f}_i^{(0)} + \frac{1}{\tau} \left(\tau - \frac{1}{2} \right) \left(\partial_{x_\alpha^{(1)}} \sum_i v_{i_\alpha} v_{i_\beta} \widetilde{f}_i^{(1)} + \partial_{t^{(1)}} \sum_i v_{i_\beta} \widetilde{f}_i^{(1)} \right) = 0 \tag{42}$$

$$\partial_{t^{(2)}} (\rho \cdot u_\beta) + \frac{1}{\tau} \left(\tau - \frac{1}{2} \right) \partial_{x_\alpha^{(1)}} \sum_i v_{i_\alpha} v_{i_\beta} \widetilde{f}_i^{(1)} = 0 \tag{43}$$

3.6 Resubstitution

The final step is to combine the above moment equations to obtain the compressible Navier-Stokes equations. By performing $\epsilon \cdot (32) + \epsilon^2 \cdot (40)$ and resubstitution we obtain the continuity equation for compressible flows:

$$\partial_t \rho + \partial_{x_\alpha} (\rho \cdot u_\alpha) = 0 \tag{44}$$

And by performing $\epsilon \cdot (36) + \epsilon^2 \cdot (43)$ and resubstitution we obtain the momentum equation:

$$\partial_t (\rho \cdot u_\beta) + \partial_{x_\alpha} (\rho \cdot u_\alpha \cdot u_\beta) = -\partial_{x_\beta} p + \partial_{x_\alpha} \sigma'_{\alpha,\beta} \tag{45}$$

In the latter equation the following identity was used:[4]

$$\sigma'_{\alpha,\beta} = \nu(\partial_{x_\beta} u_\alpha + \partial_{x_\alpha} u_\beta) = -\frac{1}{\tau} \left(\tau - \frac{1}{2} \right) \sum_i v_{i_\alpha} v_{i_\beta} \epsilon f_i^{(1)} \tag{46}$$

which is valid if the following condition is met by the kinematic viscosity ν and the relaxation time τ:

$$\nu = \rho \cdot c_s^2 \cdot dt \left(\tau - \frac{1}{2} \right) \tag{47}$$

This condition (the viscosity ν must be a positive number) and the fact, that large relaxation time values lead to instability, constraints τ to the interval:

$$\tau \in (0.5, 2.0) \tag{48}$$

[4]The derivation of this identity would exceed the requirements of this paper. For a complete discussion see [5].

4 Range of validity of the Navier-Stokes equations

The Chapman-Enskog expansion introduces an imprecise description of a fluid in terms of macroscopic variables instead of a molecular distribution function (and so the Navier-Stokes equations do). In this chapter the range of validity of the Navier-Stokes equations is discussed.

A simple answer to this question has been given by plausible formal mathematical arguments in [7]: From the Chapman-Enskog expansion results a power series in the parameter ϵ which is truncated at a certain order of ϵ.

At this point it is worth mentioning that this power series is generally not convergent[10], in the sense that higher-order equations arising from third/fourth order truncations (Burnett and super-Burnett equations) are likely to be exposed to numerical instabilities. This is because singular layers may arise from the higher-order derivatives which cannot be captured easily. For a study of these phenomena see [2].

We are not touching any further this issue and concentrate on the second-order Navier-Stokes equations, which seem to be physically sound and robust. The deviations of order ϵ^k from the true distribution, is formally small in ϵ unless the scale of variation (in space or time) is on the order of ϵ. Thus it can be said (by experience with similar mathematical procedures), that the power series is asymptotically valid everywhere except:

1. in a narrow initial layer of order ϵ,
2. in a narrow boundary layer of order ϵ,
3. in any internal shock layers of thickness ϵ.

These regions represent the non-equilibrium flow regimes of a fluid, generated by rapid changes of the system state. This makes clear, that the Chapman-Enskog theory is incomplete. To complete the theory, these layers must be connected to the Chapman-Enskog solution. For further study see [6].

5 Conclusion

The Navier-Stokes equations show good stability and provide a good approximation in real applications. They are used in a lot of different simulations, because they have some big advantages over for example the Lattice Boltzmann methods, mainly adaptive time steps, time implicit schemata and lower memory consumption.

Anyway, this derivation shows, that the Navier-Stokes equations are only a limit case of the Boltzmann equation: when the fluid under consideration can be treated as a continuum. Approaches have been made and research is still done to couple Navier-Stokes and Lattice Boltzmann methods.

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