Rigid Body Transformations

Two distinct positions and orientations of the same rigid body

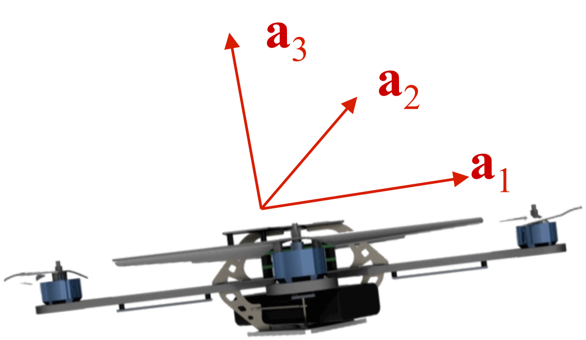
Reference Frames

We associate with any position and orientation a *reference frame*

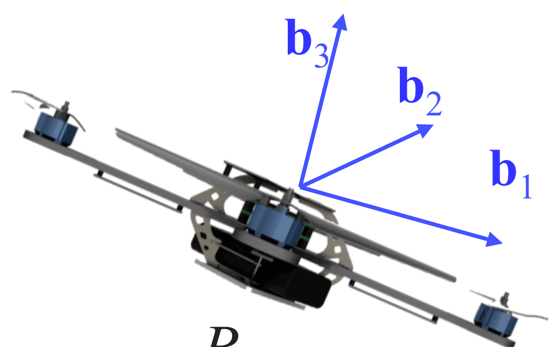
In reference frame $\{A\}$, we can find three linearly independent vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ that are basis vectors.

We can write any vector as a linear combination of the basis vectors in either frame.

$$\mathbf{v} = v_1\,\mathbf{a}_1 + v_2\,\mathbf{a}_2 + v_3\mathbf{a}_3$$



A



B

Notation

Vectors

- $\mathbf{x}, \mathbf{y}, \mathbf{a}, \ldots$
- ${}^{A}\mathbf{x}$
- $u, v, p, q, \ldots$

Potential for Confusion!

Reference Frames

- $A, B, C, \ldots$
- $a, b, c, \ldots$

Matrices

- $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$

Transformations

- ${}^{A}\mathbf{A}_{B}\ {}^{A}\mathbf{R}_{B}\ {}^{A}\boldsymbol{\xi}_{B}$
- $\mathbf{A}_{ab}\ \mathbf{R}_{ab}$
- $g_{ab},\ h_{ab},\ \ldots$

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Rigid Body Displacement

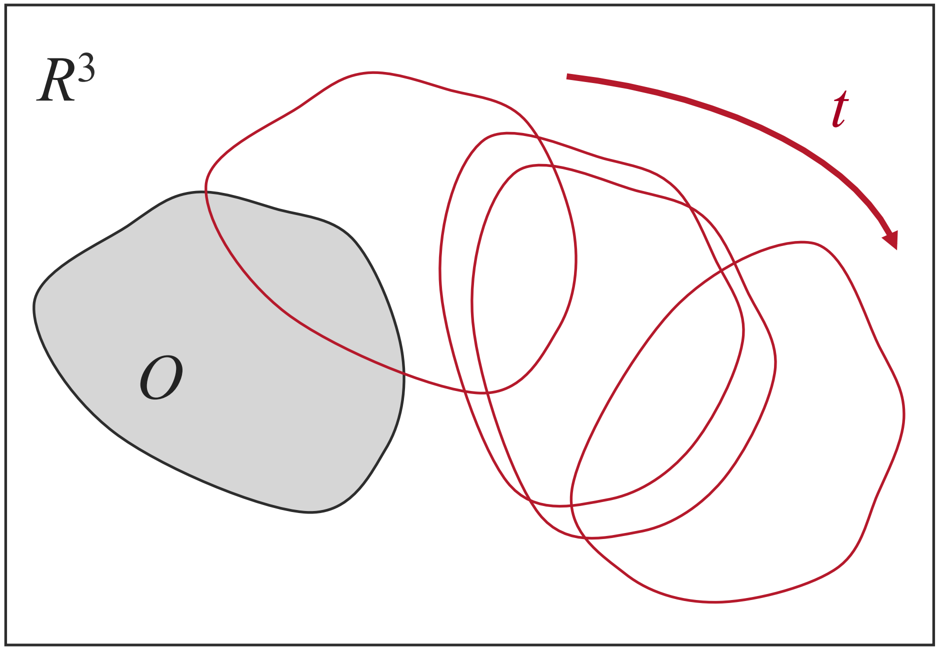
Object

$$O \subset R^3$$

Rigid Body Displacement

Map

$$g : O \rightarrow R^3$$

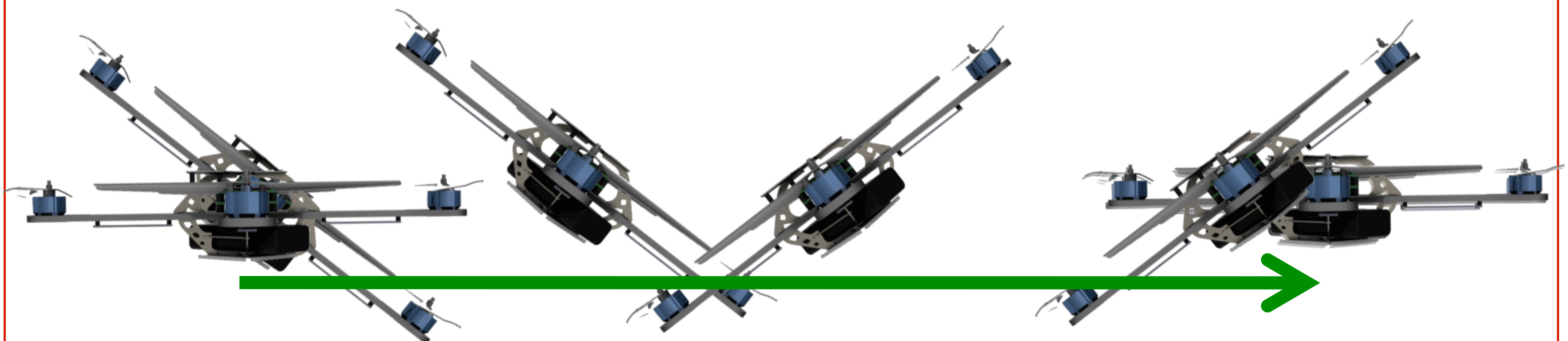


Rigid Body Motion

Continuous family of maps

$$g(t) : O \rightarrow R^3$$

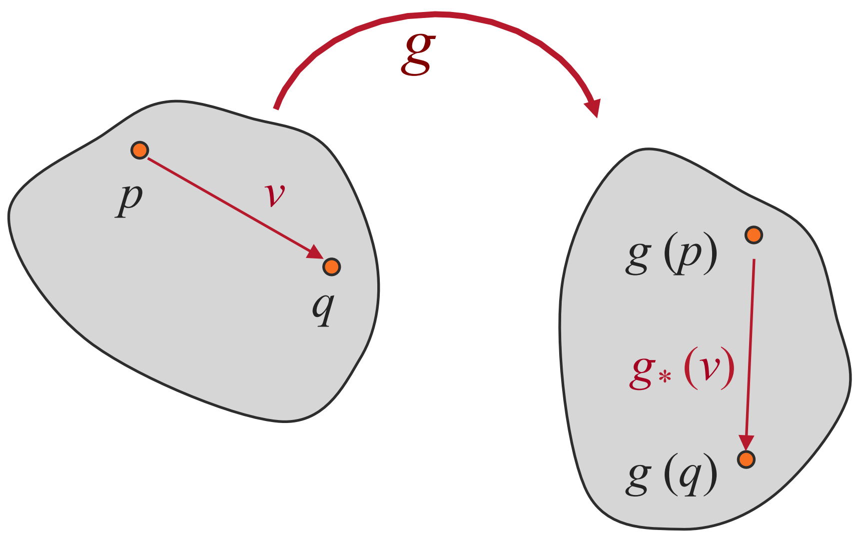
Example



Rigid Body Displacement

A displacement is a transformations of points

- Transformation (g) of points induces an action (g_*) on vectors



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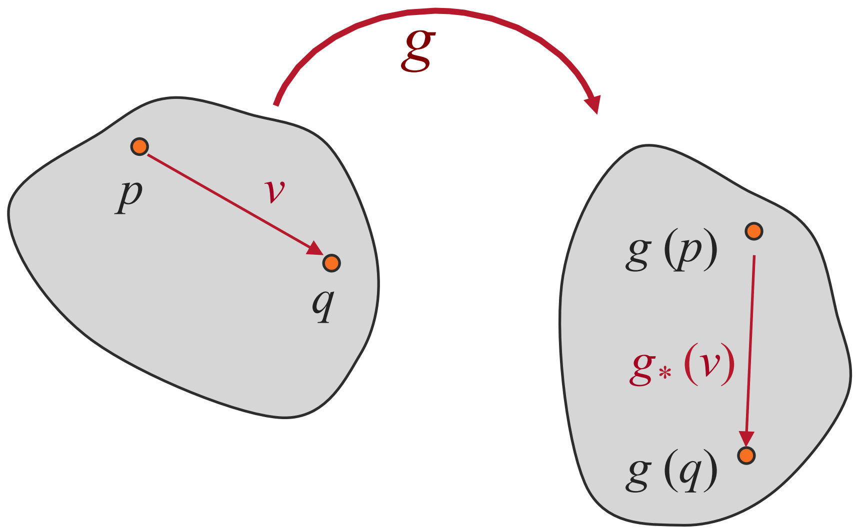
What makes *g* a *rigid body displacement?*



$$\|g(p) - g(q)\| = \|p - q\|$$

1. Lengths are preserved

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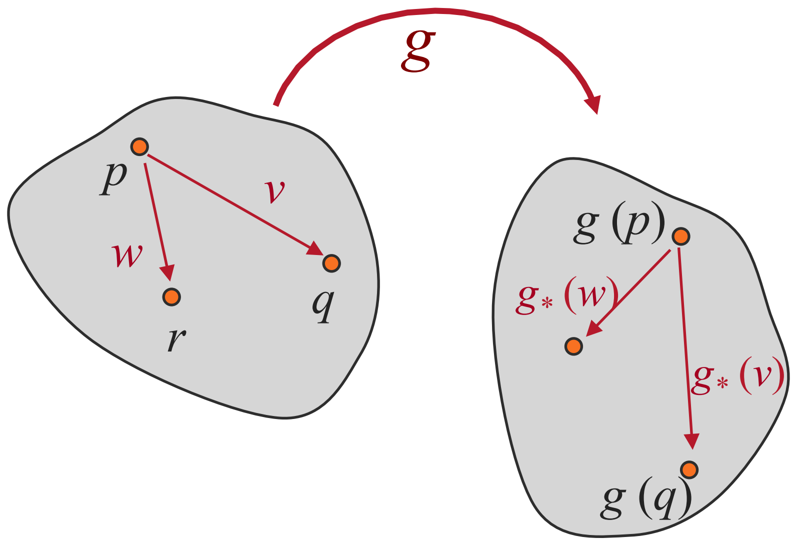
What makes g a *rigid body displacement*?



$$g_\star(v) \times g_\star(w) = g_\star(v \times w)$$

2. Cross products are preserved

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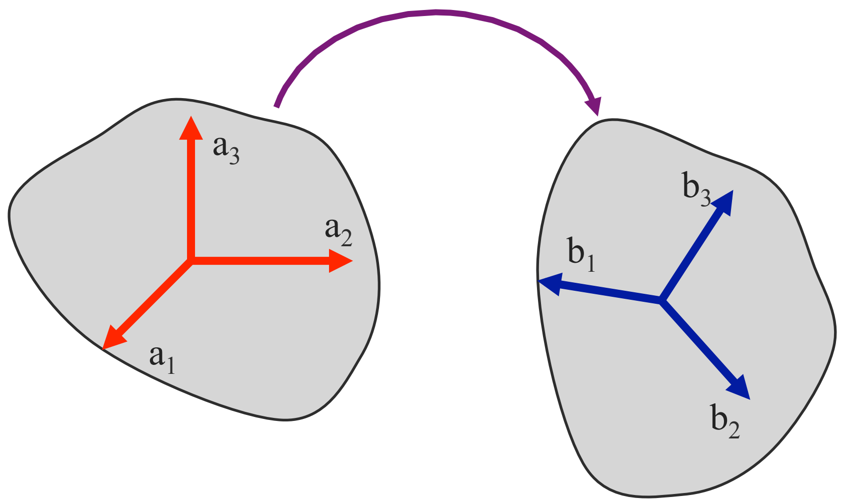
g is a *rigid body* displacement



mutually orthogonal unit vectors get mapped to mutually orthogonal unit vectors

You should be able to prove

- orthogonal vectors are mapped to orthogonal vectors

- g_* preserves inner products

$$g_\star(v) \cdot g_\star(w) = g_\star(v \cdot w)$$

Summary

Rigid body displacements are transformations (maps) that satisfy two important properties

1. The map preserves lengths

2. Cross products are preserved by the induced map

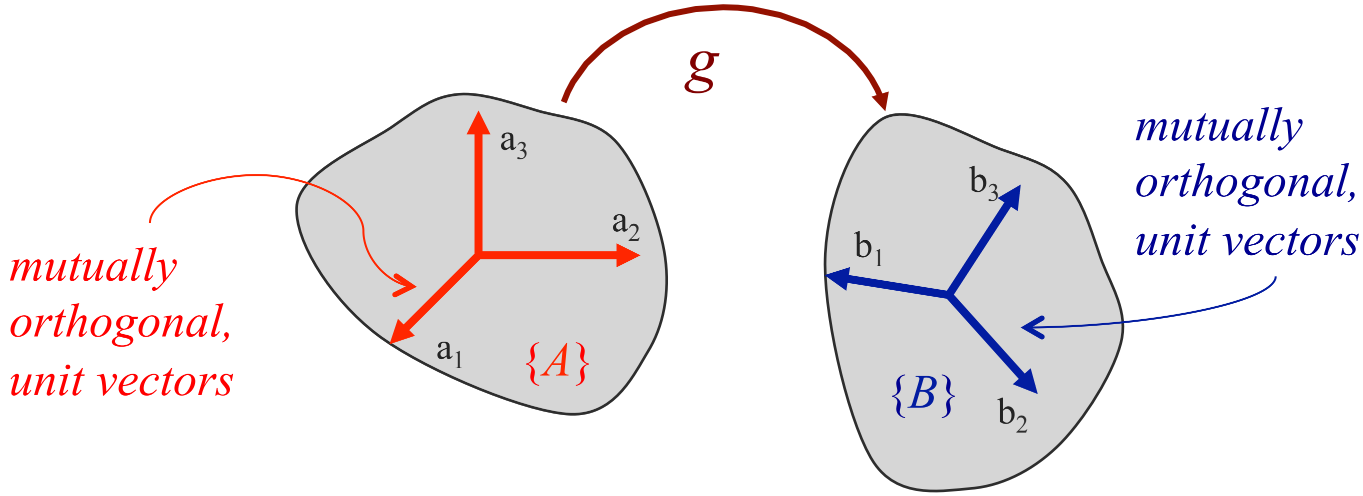
Note

Rigid body displacements and rigid body transformations are used interchangeably

1. Transformations generally used to describe relationship between reference frames attached to different rigid bodies.

2. Displacements describe relationships between two positions and orientation of a frame attached to a displaced rigid body

g is a *rigid body* displacement



$$\mathbf{b}_1 = R_{11}\mathbf{a}_1 + R_{12}\mathbf{a}_2 + R_{13}\mathbf{a}_3$$
$$\mathbf{b}_2 = R_{21}\mathbf{a}_1 + R_{22}\mathbf{a}_2 + R_{23}\mathbf{a}_3$$
$$\mathbf{b}_3 = R_{31}\mathbf{a}_1 + R_{32}\mathbf{a}_2 + R_{33}\mathbf{a}_3$$

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

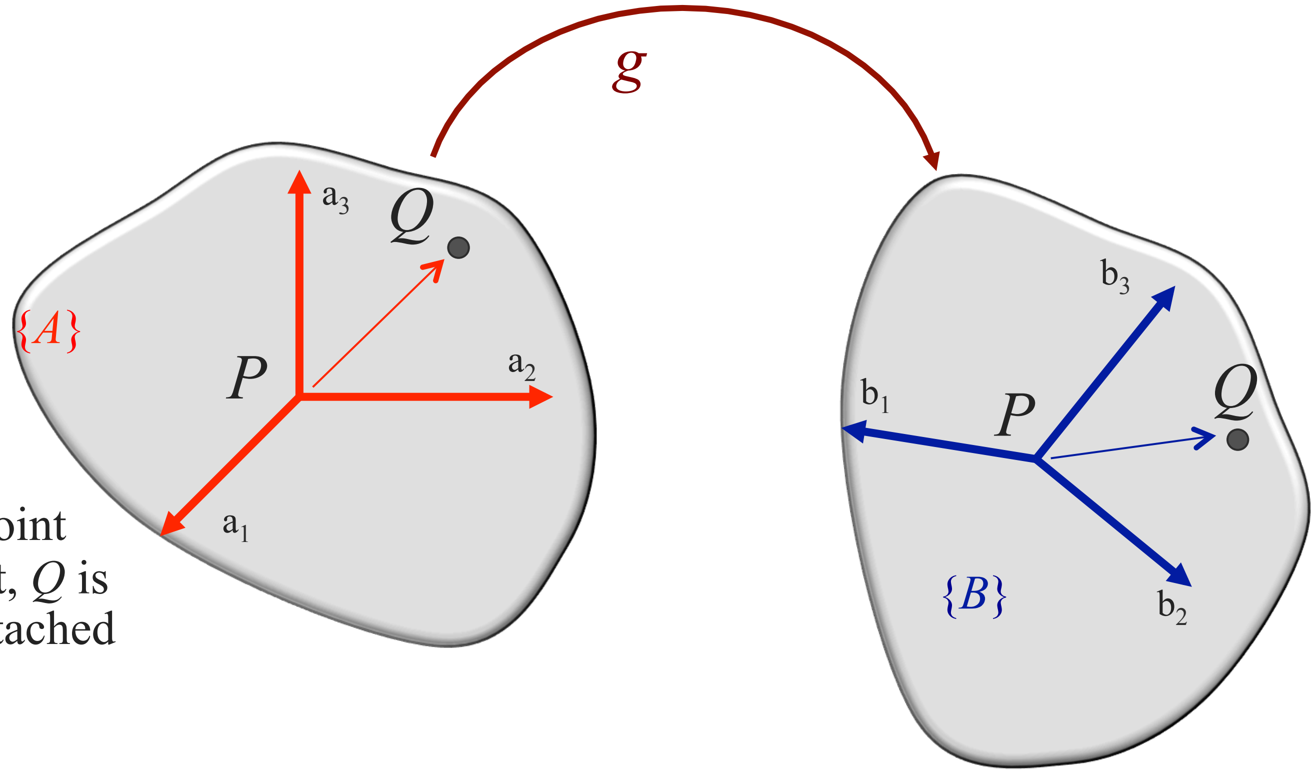
rotation matrix

Penn Engineering

Properties of a Rotation Matrix

- Orthogonal
  - Matrix times its transpose equals the identity
- Special orthogonal
  - Determinant is +1
- Closed under multiplication
  - The product of any two rotation matrices is another rotation matrix
- The inverse of a rotation matrix is also a rotation matrix

g is a *rigid body* displacement



P is a reference point fixed to the object, Q is a generic point attached to the object.

$$\overrightarrow{PQ} = q_1 \mathbf{a}_1 + q_2 \mathbf{a}_2 + q_3 \mathbf{a}_3$$

$$\overrightarrow{PQ} = q_1 \mathbf{b}_1 + q_2 \mathbf{b}_2 + q_3 \mathbf{b}_3$$

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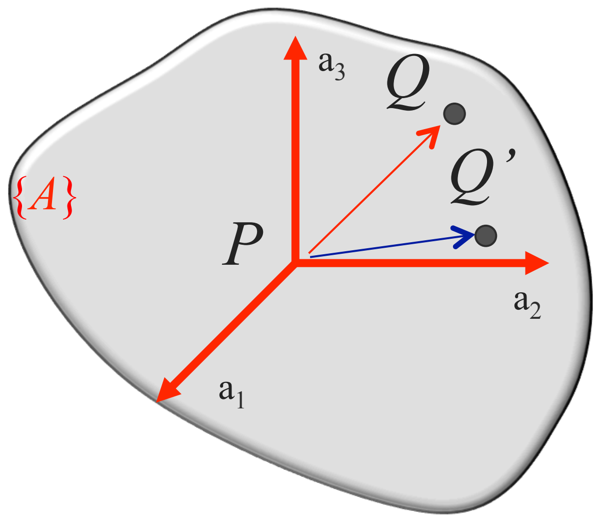
g is a *rigid body* displacement



$$\overrightarrow{PQ} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$\overrightarrow{PQ'} = q_1'\mathbf{a}_1 + q_2'\mathbf{a}_2 + q_3'\mathbf{a}_3$$

$$\begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \text{ or } \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix}$$

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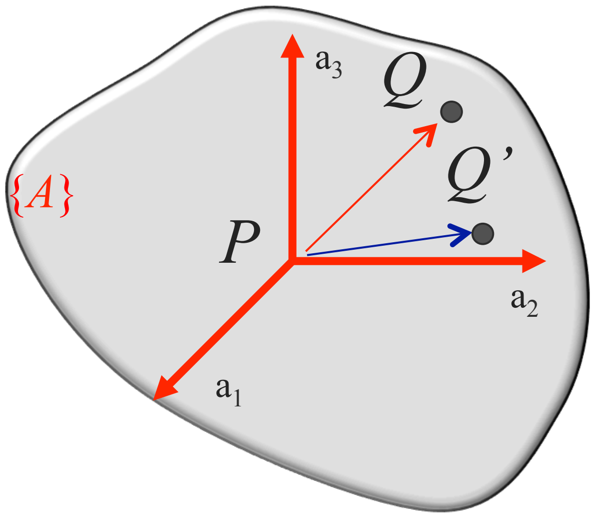
g is a *rigid body* displacement



$$\overrightarrow{PQ} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$\overrightarrow{PQ'} = q'_1\mathbf{a}_1 + q'_2\mathbf{a}_2 + q'_3\mathbf{a}_3$$

Verify $$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} q'_1 \\ q'_2 \\ q'_3 \end{bmatrix}$$

rotation matrix

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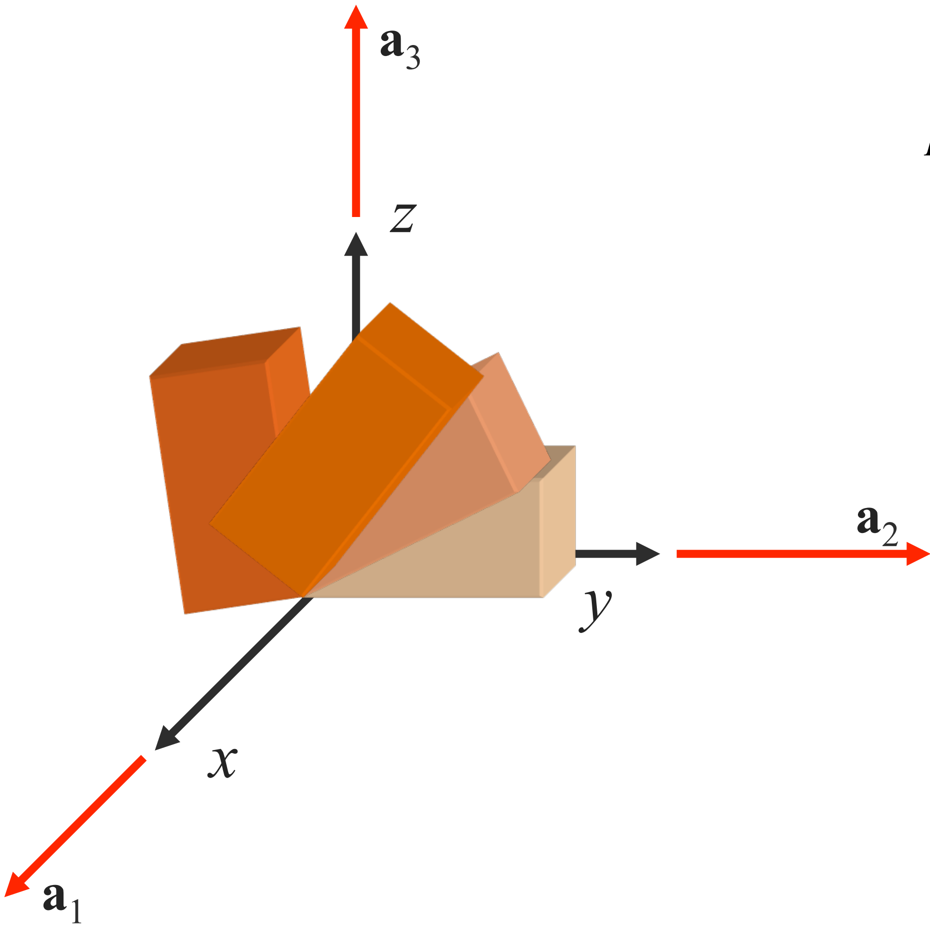
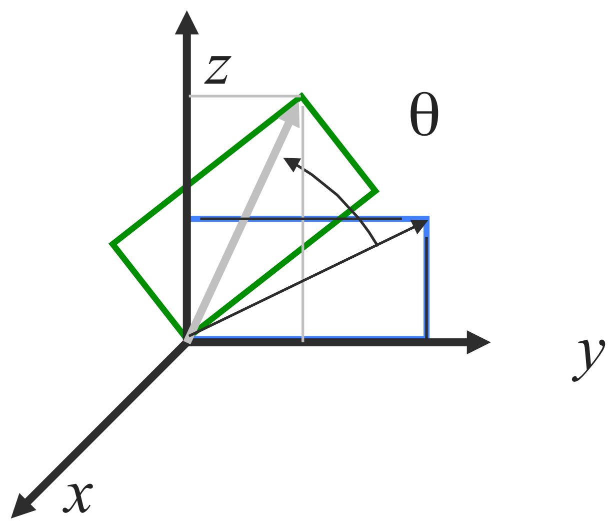
Example: Rotation

- Rotation about the *x*-axis through θ



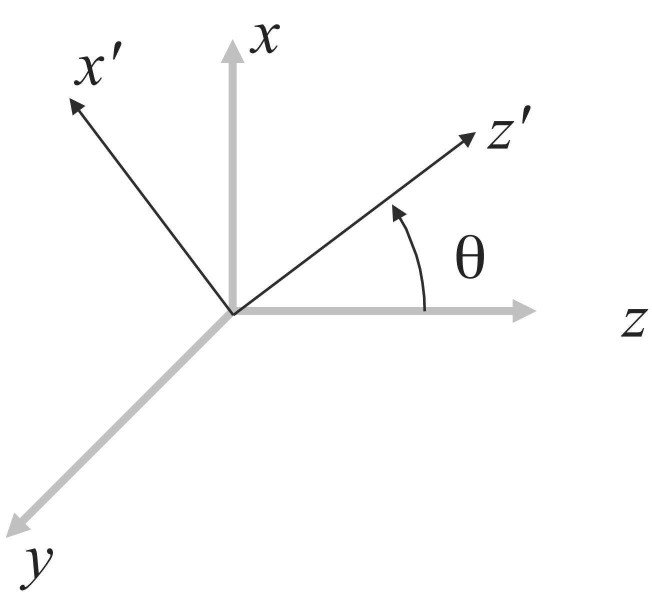
$$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



Example: Rotation

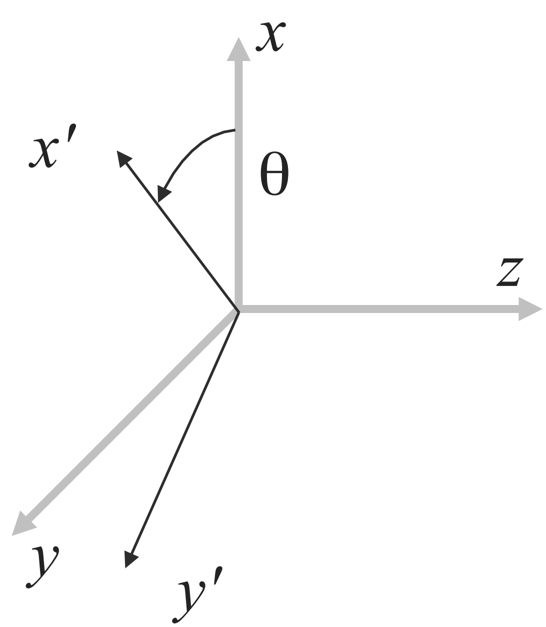
Rotation about the *y*-axis through θ

$$Rot(y,\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$$



Rotation about the *z*-axis through θ

$$Rot(z,\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$



Rotations

Special Orthogonal Matrices

$$\left\{R \in \mathbb{R}^{3\times 3} \mid R^T R = RR^T = I,\ \det R = 1\right\}$$

Special Orthogonal group in 3 dimensions

- The group of rotations is called *SO*(3)
- Coordinates for *SO*(3)

  1 Rotation matrices

  2 Euler angles

  3 Axis angle parameterization

  4 Exponential coordinates

  5 Quaternions

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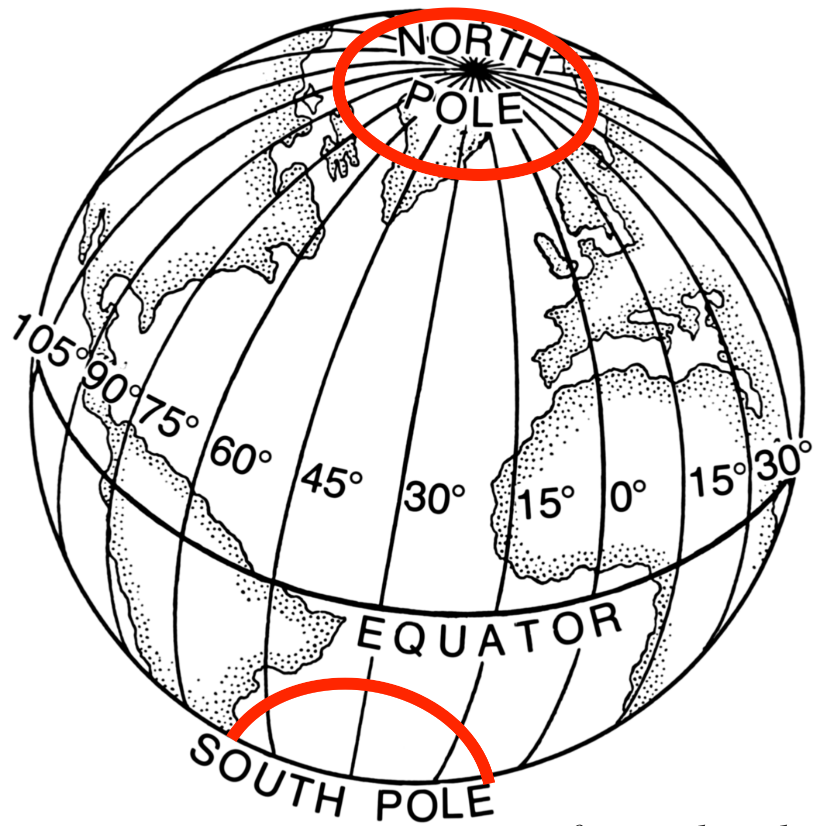
Coordinates for a Sphere

- Parameterize using a set of local coordinate charts (latitude and longitude)

- We want a collection of charts to describe the Earth's surface



Images from wikipedia

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What is the minimum number of charts you need to cover the Earth's surface?

Images from wikipedia

What is the minimum number of charts you need to cover *SO*(3)?

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = RR^T = I,\ \det R = 1\}$$

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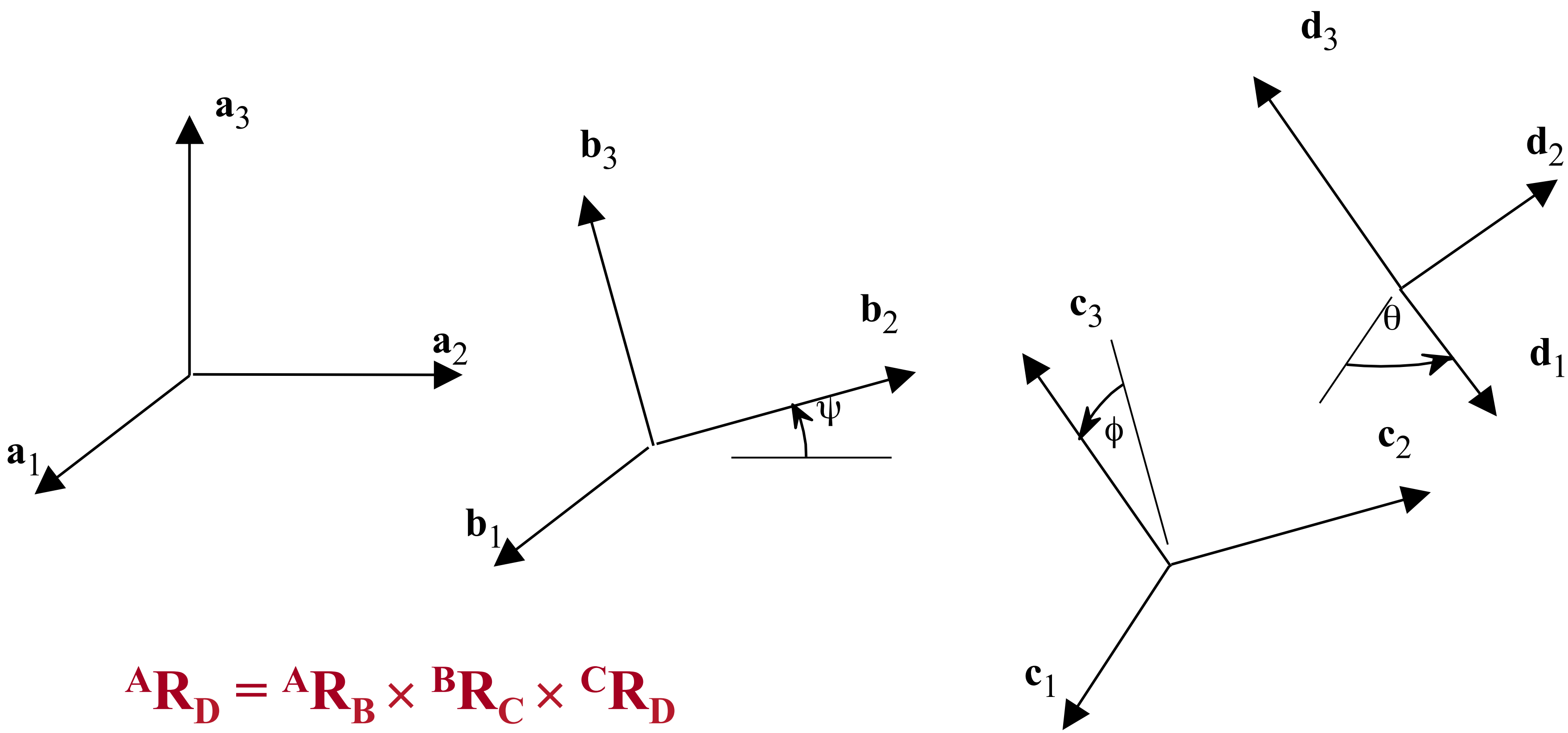
Euler Angles

Composition of Three Rotations



$${}^{A}\mathbf{R}_{D} = {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C} \times {}^{C}\mathbf{R}_{D}$$

$${}^{A}\mathbf{R}_{D} = \mathrm{Rot}(x, \psi) \times \mathrm{Rot}(y, \phi) \times \mathrm{Rot}(z, \theta)$$

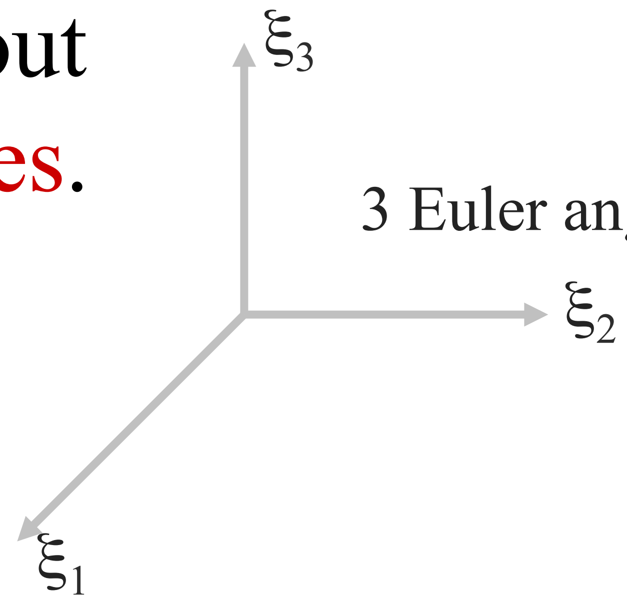
roll *pitch* *yaw*

Euler Angles

Image from wikipedia

Any rotation can be described by three successive rotations about linearly independent axes.

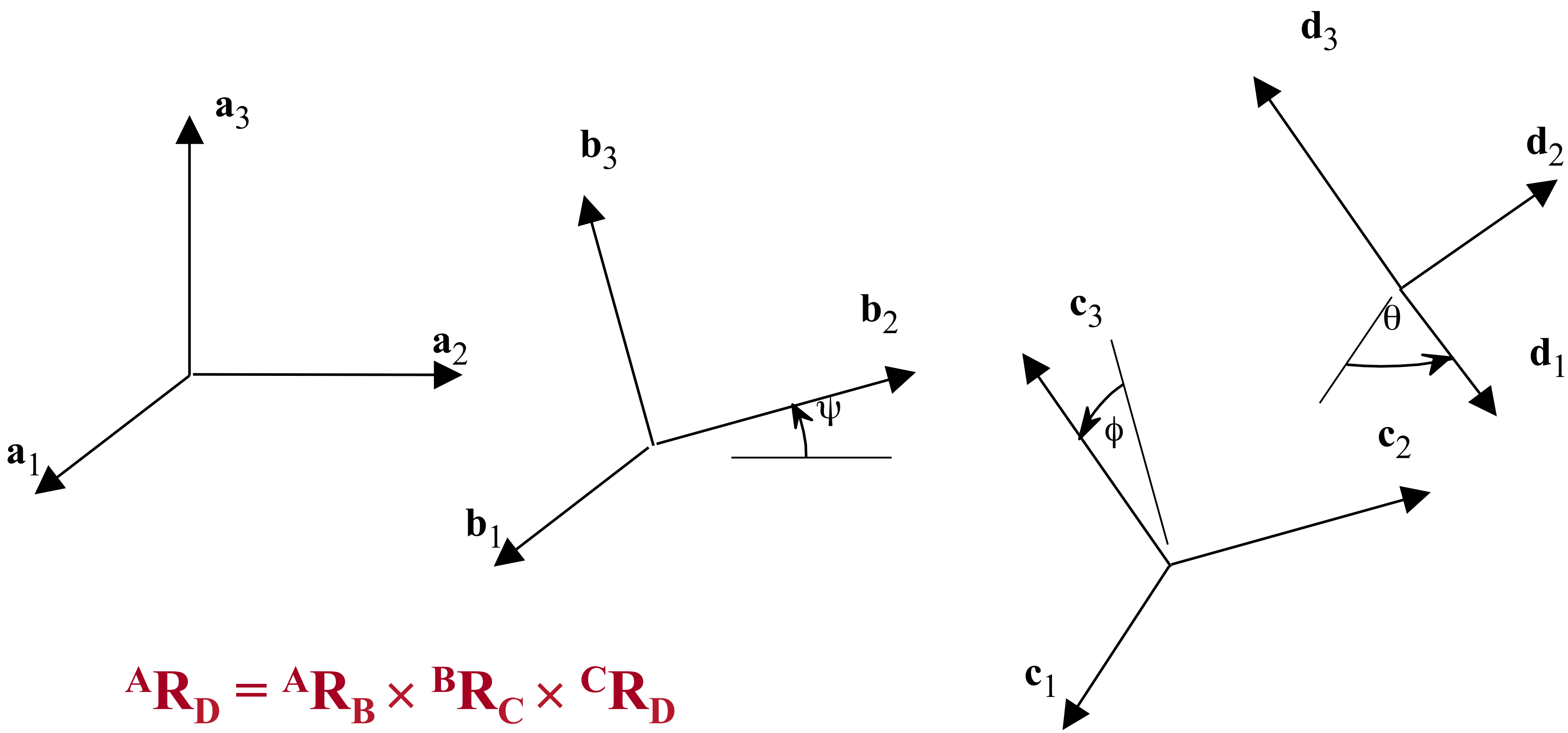
$$\begin{pmatrix} 3 \times 3 \text{ rotation matrix} \end{pmatrix}$$



3 Euler angles

Almost 1-1 transformation

X-Y-Z Euler Angles



$$^{A}\mathbf{R}_{D} = {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C} \times {}^{C}\mathbf{R}_{D}$$

$$^{A}\mathbf{R}_{D} = \mathrm{Rot}(x, \psi) \times \mathrm{Rot}(y, \phi) \times \mathrm{Rot}(z, \theta)$$

roll *pitch* *yaw*

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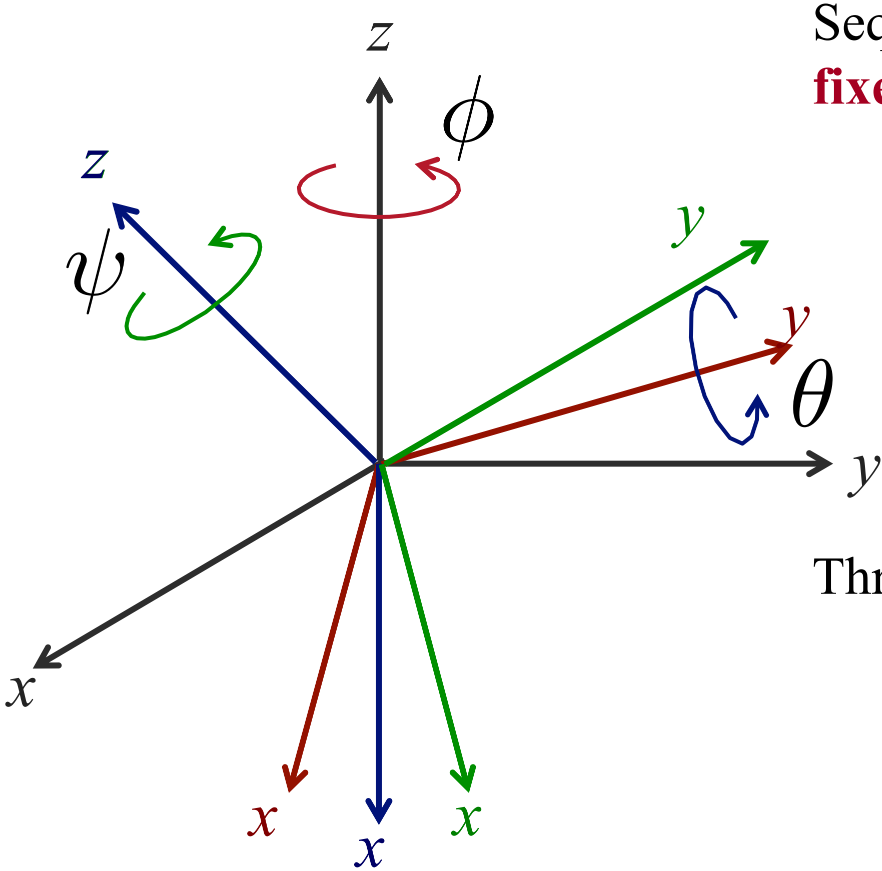
Z-Y-Z Euler Angles



Sequence of three rotations about **body-fixed** axes

- Rot(z, ϕ)
- Rot(y, θ)
- Rot(z, ψ)

Are these linearly independent?

Three Euler Angles

- ϕ, θ, and ψ
- Parameterize rotations

Note

- $\theta = 0$ is a special (singular) case

$$\mathbf{R} = \mathrm{Rot}(z, \phi) \times \mathrm{Rot}(y, \theta) \times \mathrm{Rot}(z, \psi)$$

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Determination of Euler Angles

$$\mathbf{R} = \mathrm{Rot}(z, \phi) \times \mathrm{Rot}(y, \theta) \times \mathrm{Rot}(z, \psi)$$

$$R = \begin{bmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi & \boxed{\cos\phi\sin\theta} \\ \sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \boxed{\cos\theta} \end{bmatrix}$$

$R_{31} = -\sin\theta\cos\psi$

$R_{32} = \sin\theta\sin\psi$

$$\begin{bmatrix} R_{11} & R_{12} & \boxed{R_{13}} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & \boxed{R_{33}} \end{bmatrix}$$

$R_{33} = \cos\theta$

$R_{13} = \sin\theta\cos\phi$

$R_{23} = \sin\theta\sin\phi$

known rotation matrix

Determination of Euler Angles

$$R = \begin{bmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta \\ \sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix}$$

If $|R_{33}| < 1$,

$$\theta = \sigma \arccos(R_{33}), \quad \sigma = \pm 1$$

$$\psi = a\tan 2\left(\frac{R_{32}}{\sin\theta}, \frac{-R_{31}}{\sin\theta}\right)$$

$$\phi = a\tan 2\left(\frac{R_{23}}{\sin\theta}, \frac{R_{13}}{\sin\theta}\right)$$

Two sets of Euler angles for every **R** for almost all **R**'s!

If $R_{33} = 1$,

$$R = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\psi & 0 \\ \cos\phi\sin\psi + \sin\phi\cos\psi & -\sin\phi\sin\psi + \cos\phi\cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$f(\phi + \psi)$

If $R_{33} = -1$,

$$R = \begin{bmatrix} -\cos\phi\cos\psi - \sin\phi\sin\psi & \cos\phi\sin\psi - \sin\phi\cos\psi & 0 \\ \cos\phi\sin\psi - \sin\phi\cos\psi & \sin\phi\sin\psi + \cos\phi\cos\psi & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$f(\phi + \psi)$

Infinite set of Euler Angles!

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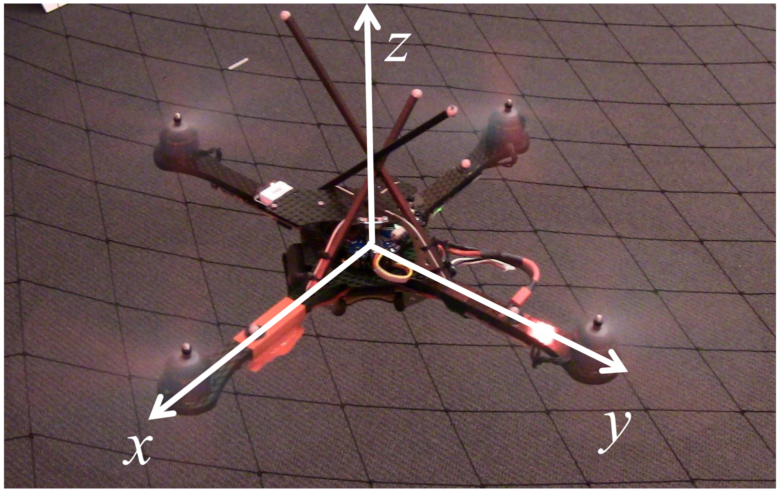
Z-X-Y Euler Angles

Sequence of three rotations about **body-fixed** axes

- Rot(z, ψ)
- Rot(x, ϕ)
- Rot(y, θ)



Verify

$$R = \begin{bmatrix} c\psi c\theta - s\phi s\psi s\theta & -c\phi s\psi & c\psi s\theta + c\theta s\phi s\psi \\ c\theta s\psi + c\psi s\phi s\theta & c\phi c\psi & s\psi s\theta - c\theta s\phi c\psi \\ -c\phi s\theta & s\phi & c\phi c\theta \end{bmatrix}$$

N. Michael, D. Mellinger, Q. Lindsey, V. Kumar, *The GRASP Multiple Micro-UAV Testbed*, IEEE Robotics & Automation Magazine, vol.17, no.3, pp.56-65, Sept. 2010

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What is the minimum number of sets of Euler angles you need to cover $SO(3)$?

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = RR^T = I\}$$

Axis/Angle Representation

Special Orthogonal Matrices

$$\{R \in \mathbb{R}^{3\times 3} \mid R^T R = R R^T = I,\ \det R = 1\}$$

Special Orthogonal group in 3 dimensions

- Coordinates for *SO*(3)
  1 Rotation matrices
  2 Euler angles
  3 Axis angle parameterization
  4 Exponential coordinates
  5 Quaternions

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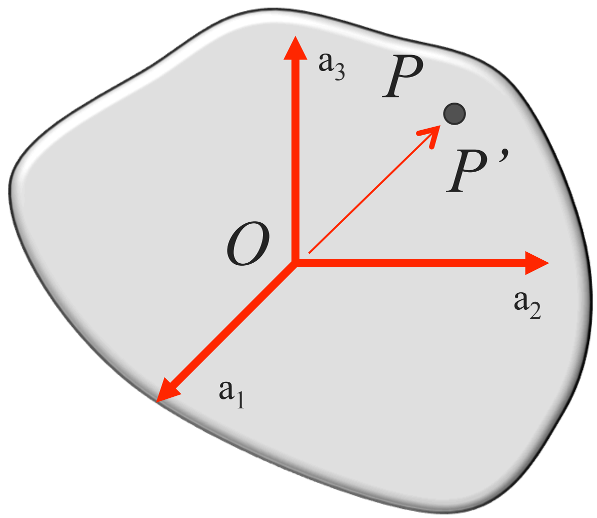
Euler's Theorem

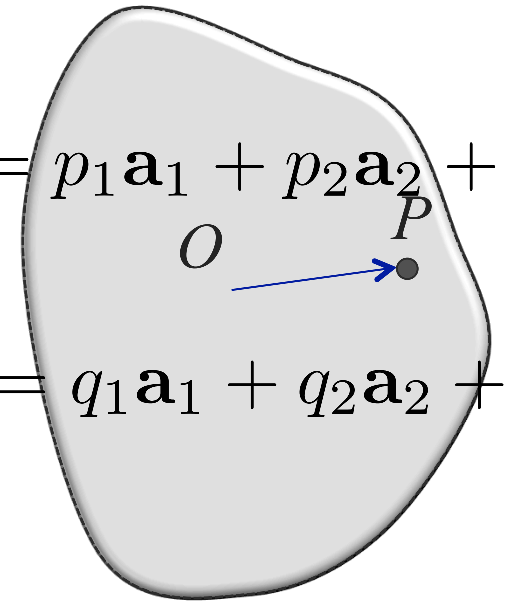
Rotations

Any displacement of a rigid body such that a point on the rigid body, say O, remains fixed, is equivalent to a rotation about a fixed axis through the point O.

Rotation with O fixed



$$\overrightarrow{OP} = p_1\mathbf{a}_1 + p_2\mathbf{a}_2 + p_3\mathbf{a}_3$$



$$\overrightarrow{OP'} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$\mathbf{q} \quad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \quad \mathbf{p}$$

$$\mathbf{q} = R\mathbf{p}$$

Proof of Euler's Theorem

$$\mathbf{q} = R\mathbf{p}$$

Is there a point **p** that maps onto itself?

If there were such a point **p** ...

$$\mathbf{p} = R\mathbf{p}$$

Solve eigenvalue problem

$$R\mathbf{p} = \lambda\mathbf{p}$$

Verify $\lambda=1$ is an eigenvalue for any R

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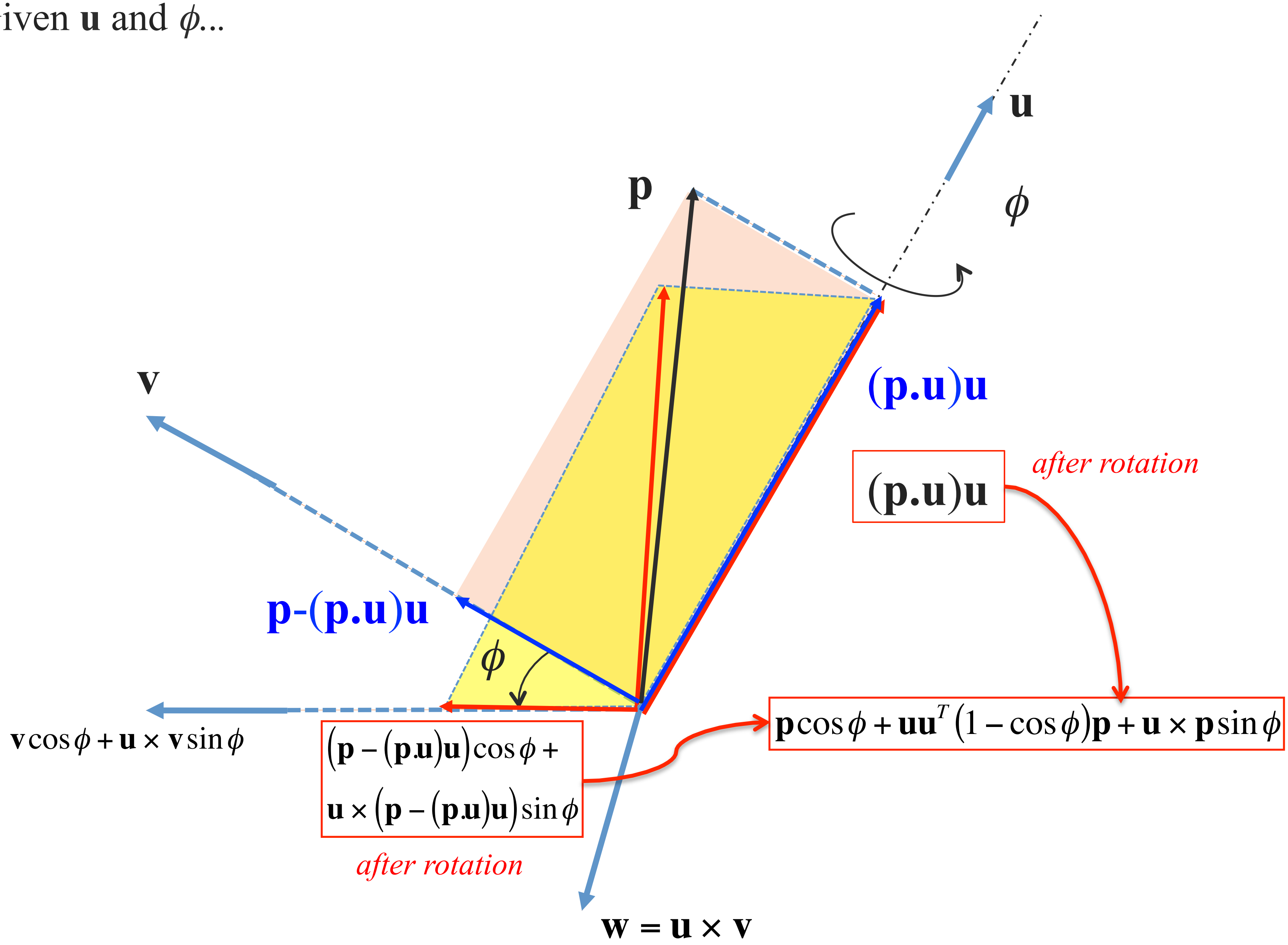
How does one find the rotation matrix for a general axis and angle of rotation?

Note we already know the answer if the axis of rotation is one of the coordinate axes.

Given **u** and ϕ...



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1-1 correspondence between any 3×1 vector and a 3×3 skew symmetric matrix

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ -a_2 b_1 + a_1 b_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

a

linear operator

For any vector **b**

$$\mathbf{a} \times \mathbf{b} = \mathbf{A}_{3x3}\, \mathbf{b}$$

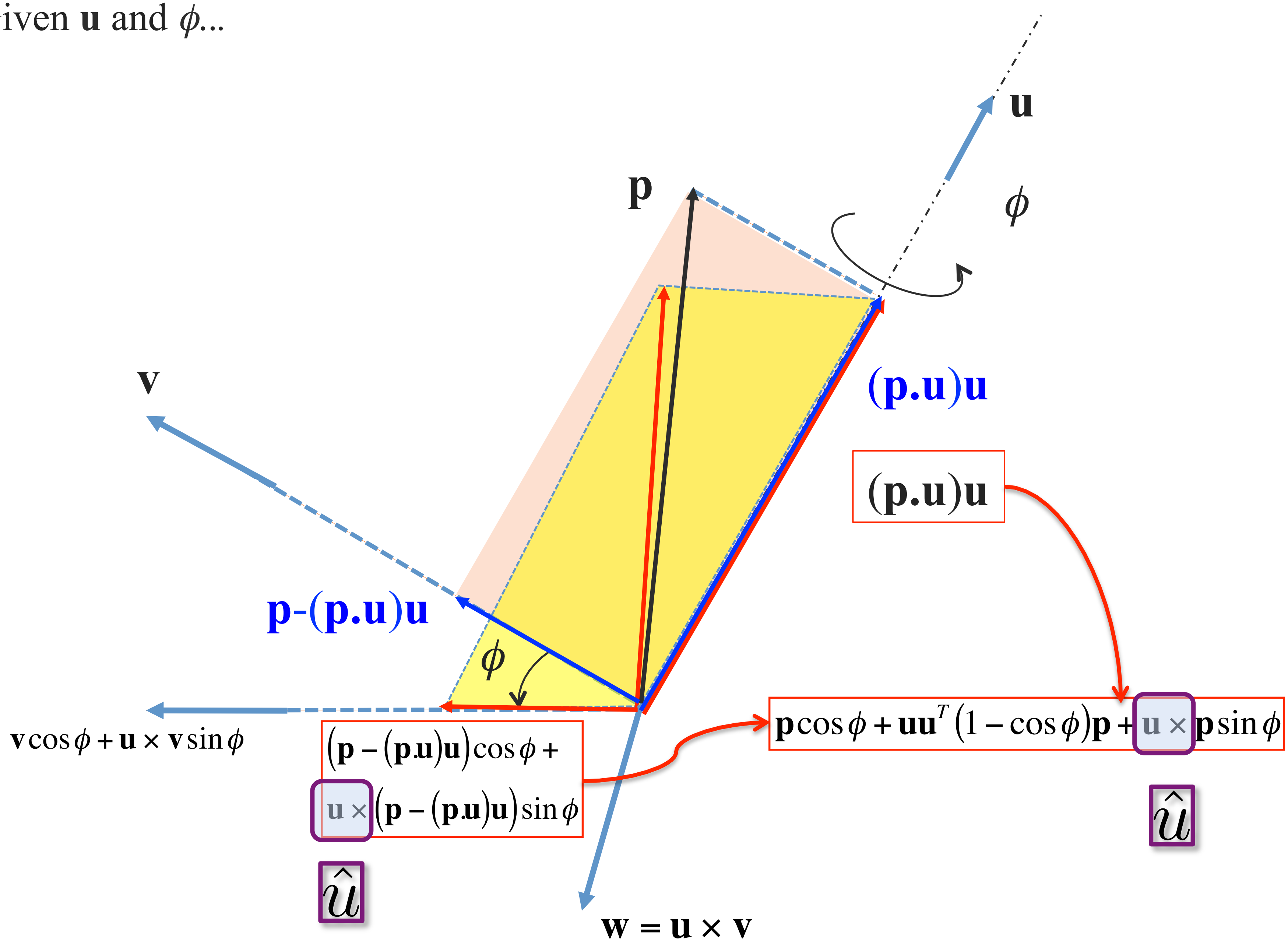
Notation

$\mathbf{A}$

$\mathbf{a}$^

$\hat{\mathbf{a}}$

Given **u** and ϕ...



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Axis/Angle to Rotation Matrix

Rotation of a generic vector p about u through ϕ

$$Rp = p\cos\phi + uu^T(1-\cos\phi)p + \hat{u}p\sin\phi$$

Axis of rotation	u
Rotation angle	ϕ

Rodrigues' formula

$$Rot(u,\phi) = I\cos\phi + uu^T(1-\cos\phi) + \hat{u}\sin\phi$$

Image from wikipedia

1. Set u to be a unit vector along x (or y or z). Verify result is the same as $Rot(x, \phi)$.

2. Is the (axis, angle) to rotation matrix map *onto*? 1-1?

$Rot(u,\phi)$ and $Rot(-u, 2\pi-\phi)$?

restrict ϕ to the interval $[0,\pi]$?

Euler's theorem

Penn Engineering

Axis/Angle to Rotation Matrix

Rotation of a generic vector p about u through ϕ

$$Rp = p\cos\phi + uu^T(1-\cos\phi)p + \hat{u}p\sin\phi$$

Axis of rotation	u
Rotation angle	ϕ

Rodrigues' formula

$$Rot(u,\phi) = I\cos\phi + uu^T(1-\cos\phi) + \hat{u}\sin\phi$$

Lets extract the axis and the angle from the rotation matrix, R

Verify

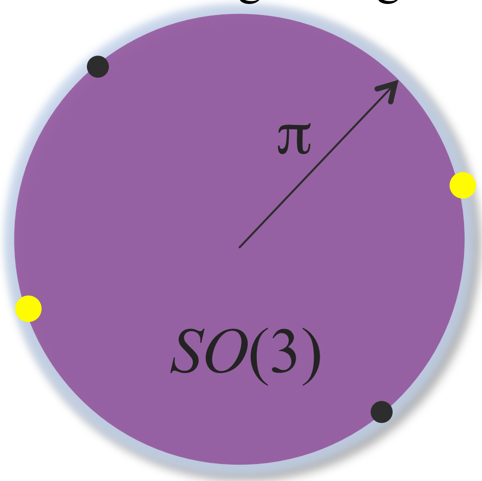
$$\cos\phi = \frac{\tau - 1}{2} \qquad \hat{u} = \frac{1}{2\sin\phi}(R - R^T)$$ (u, without solving for eigenvector)

1. (axis, angle) to rotation matrix map is many to 1

2. restricting angle to the interval [0,π] makes it 1-1 except for

$\tau = 3 \Rightarrow \phi = 0 \Rightarrow$ *no unique axis*

$\tau = -1 \Rightarrow \phi = \pi \Rightarrow$ *u* or -*u*



Rotations and Angular Velocities

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Time Derivatives of Rotations

Rotation matrix

$$R(t)$$

Orthogonality

$$R^T(t)R(t) = I \qquad \xrightarrow{\frac{d}{dt}(\,.\,)} \qquad \dot{R}^T R + R^T \dot{R} = 0$$

$$R(t)R^T(t) = I \qquad\qquad R\dot{R}^T + \dot{R}R^T = 0$$

$R^T\dot{R}$ and $\dot{R}R^T$ are skew symmetric

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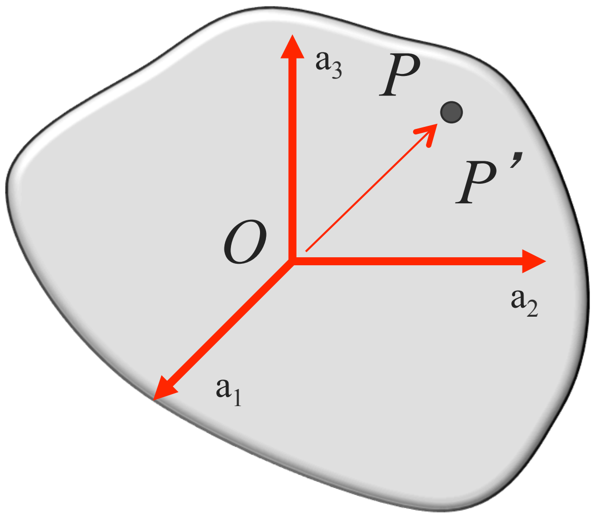
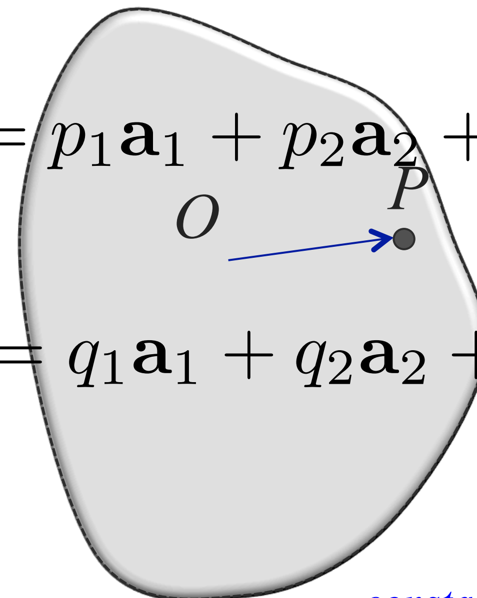
Rotation with O fixed





$$\overrightarrow{OP} = p_1\mathbf{a}_1 + p_2\mathbf{a}_2 + p_3\mathbf{a}_3$$

$$\overrightarrow{OP'} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

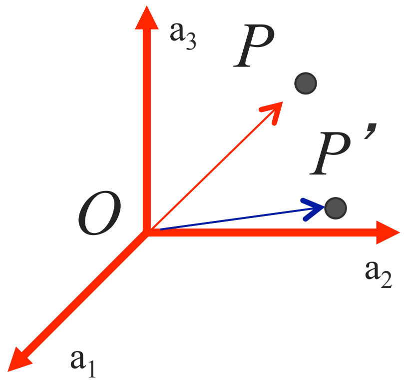
$$\underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}}_{q} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}}_{p}$$

constant coordinates of P in body-fixed frame

$$\boxed{q(t) = R(t)p}$$

changing coordinates of P as the rigid body rotates

Rotation with O fixed



$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$\boxed{q(t) = R(t)p}$$

$$\dot{q} = \dot{R}p$$

velocity in inertial frame

position in body-fixed frame

$$R^T\dot{q} = \boxed{R^T\dot{R}}p$$

$\hat{\omega}^b$

velocity in body-fixed frame

encodes angular velocity in body-fixed frame

$$\dot{q} = \boxed{\dot{R}R^T}q$$

velocity in inertial frame

encodes angular velocity in inertial frame

$\hat{\omega}^s$

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Exercise

What is the angular velocity for a rotation about the *z* axis?

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \dot{R} = \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\theta}$$

Angular velocity for a rotation about the *z*-axis

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T \dot{R} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}$$

$$= \dot{R} R^T = \dot{\theta} \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta} = \boxed{\widehat{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \dot{\theta}}$$

Two Rotations

$$R = R_z(\theta) R_x(\phi)$$

$$\begin{aligned}\hat{\omega}^b = R^T \dot{R} &= (R_z R_x)^T (\dot{R}_z R_x + R_z \dot{R}_x) \\ &= R_x^T R_z^T \dot{R}_z R_x + R_x^T \dot{R}_x\end{aligned}$$

$$\begin{aligned}\hat{\omega}^s = \dot{R} R^T &= (\dot{R}_z R_x + R_z \dot{R}_x)(R_z R_x)^T \\ &= \dot{R}_z R_z^T + R_z \dot{R}_x R_x^T R_z^T\end{aligned}$$

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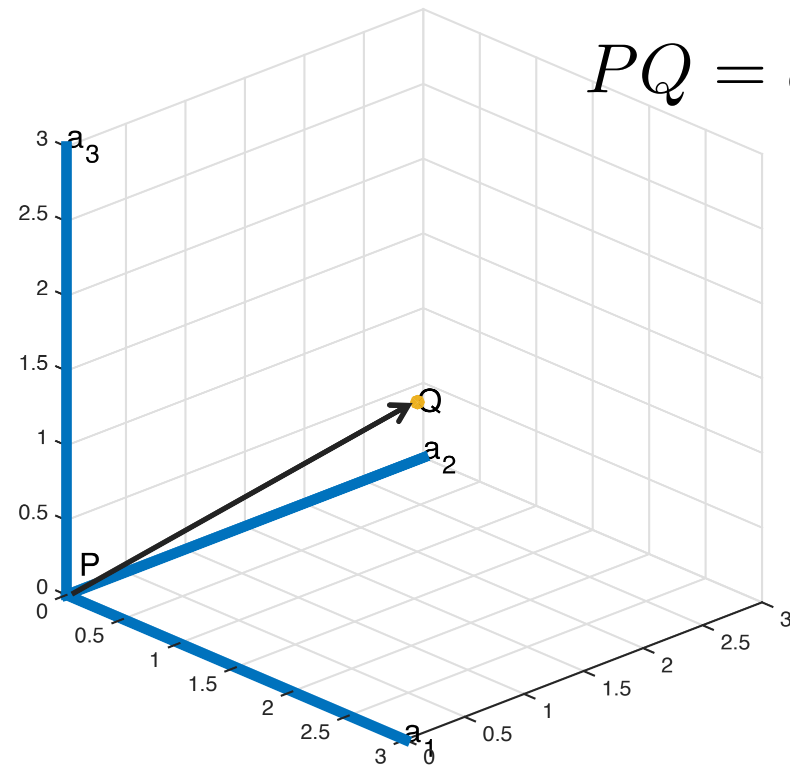
Rigid-Body Displacements

Rigid-Body Displacement

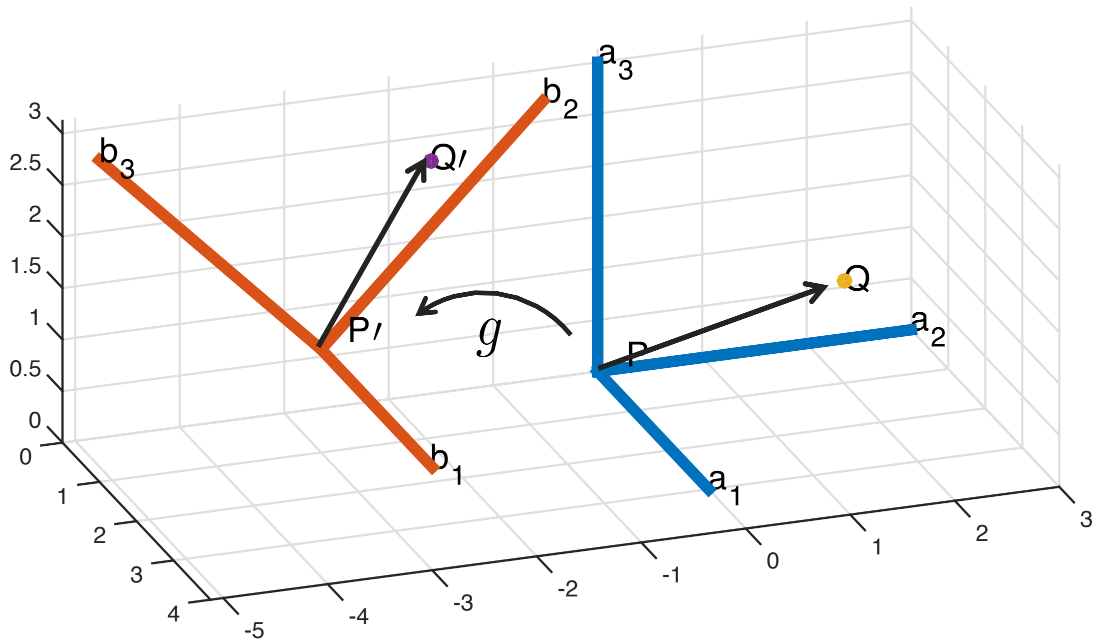
Consider Frame A and vector PQ.

$$PQ = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$



Rigid-Body Displacement

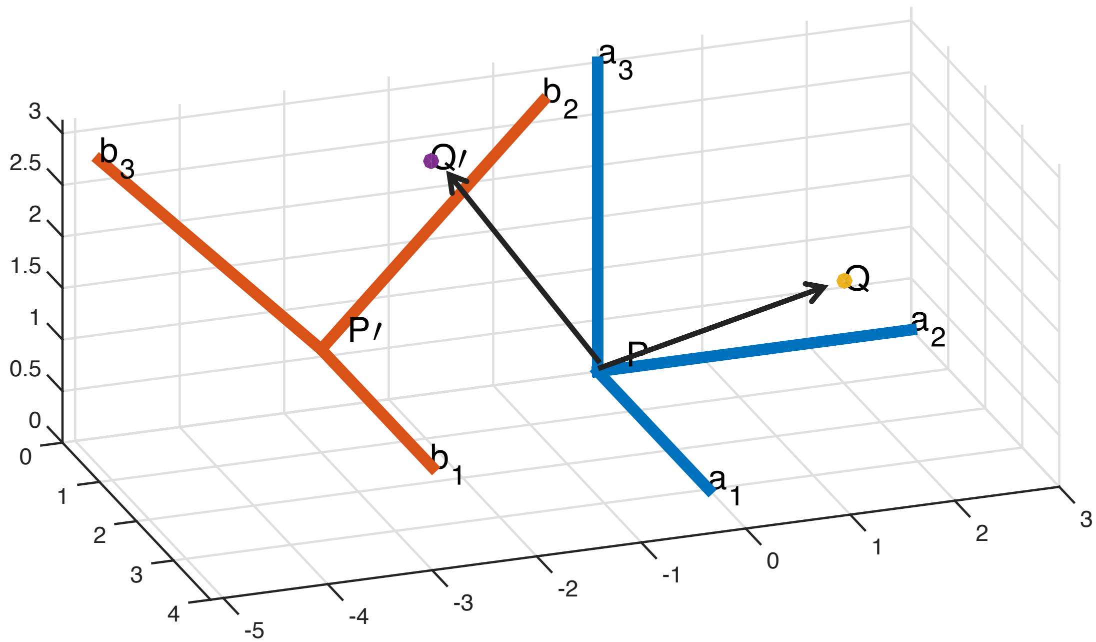
Let Frame B be Frame A, after rigid-body displacement g.



$$PQ = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$
$$P'Q' = q_1\mathbf{b}_1 + q_2\mathbf{b}_2 + q_3\mathbf{b}_3$$

Rigid-Body Displacement

Let Frame B be Frame A, after rigid-body displacement g.



$$PQ = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$PQ' = q_1'\mathbf{a}_1 + q_2'\mathbf{a}_2 + q_3'\mathbf{a}_3$$

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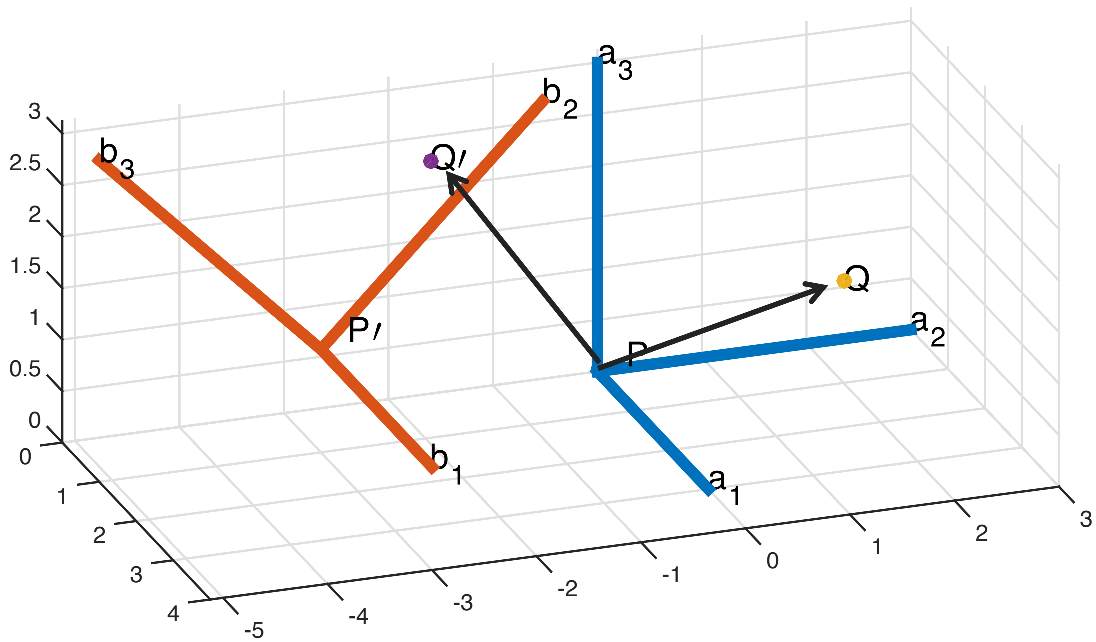
Rigid-Body Displacement

Let Frame B be Frame A, after rigid-body displacement g.



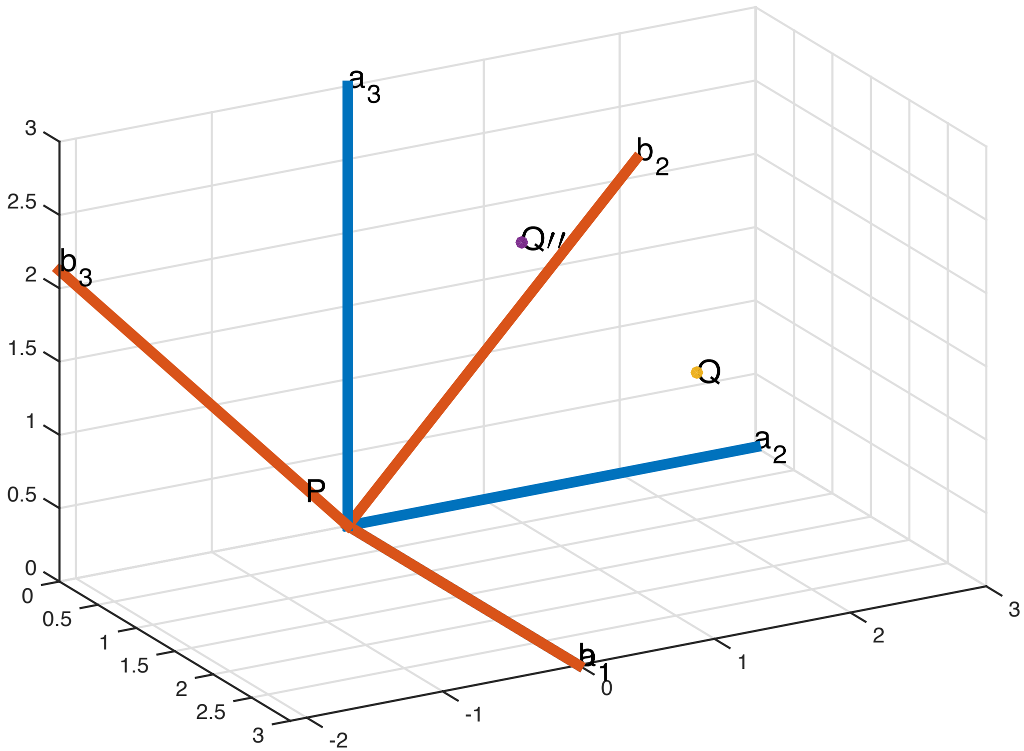
$$PQ = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$PQ' = q_1'\mathbf{a}_1 + q_2'\mathbf{a}_2 + q_3'\mathbf{a}_3$$

$$\boxed{\begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} = R \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \mathbf{d}}$$

Rotation

Translate frame B so reference frames share an origin.



$$PQ = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$PQ'' = q_1\mathbf{b}_1 + q_2\mathbf{b}_2 + q_3\mathbf{b}_3$$

$$= q_1''\mathbf{a}_1 + q_2''\mathbf{a}_2 + q_3''\mathbf{a}_3$$

$$\begin{bmatrix} q_1'' \\ q_2'' \\ q_3'' \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

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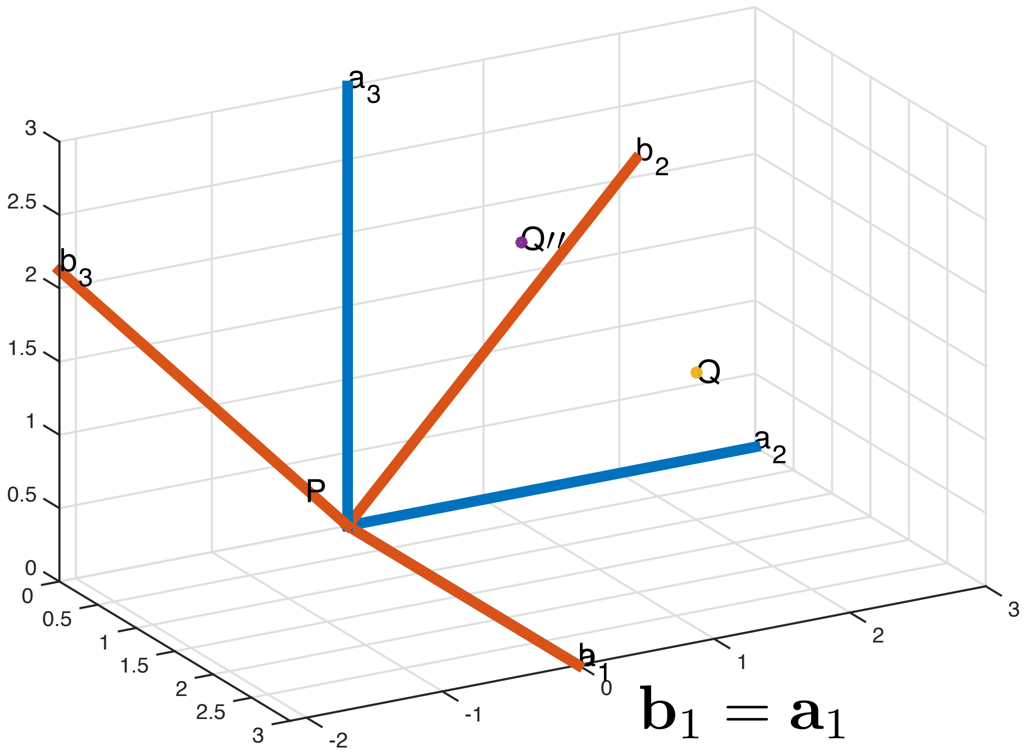
Rotation

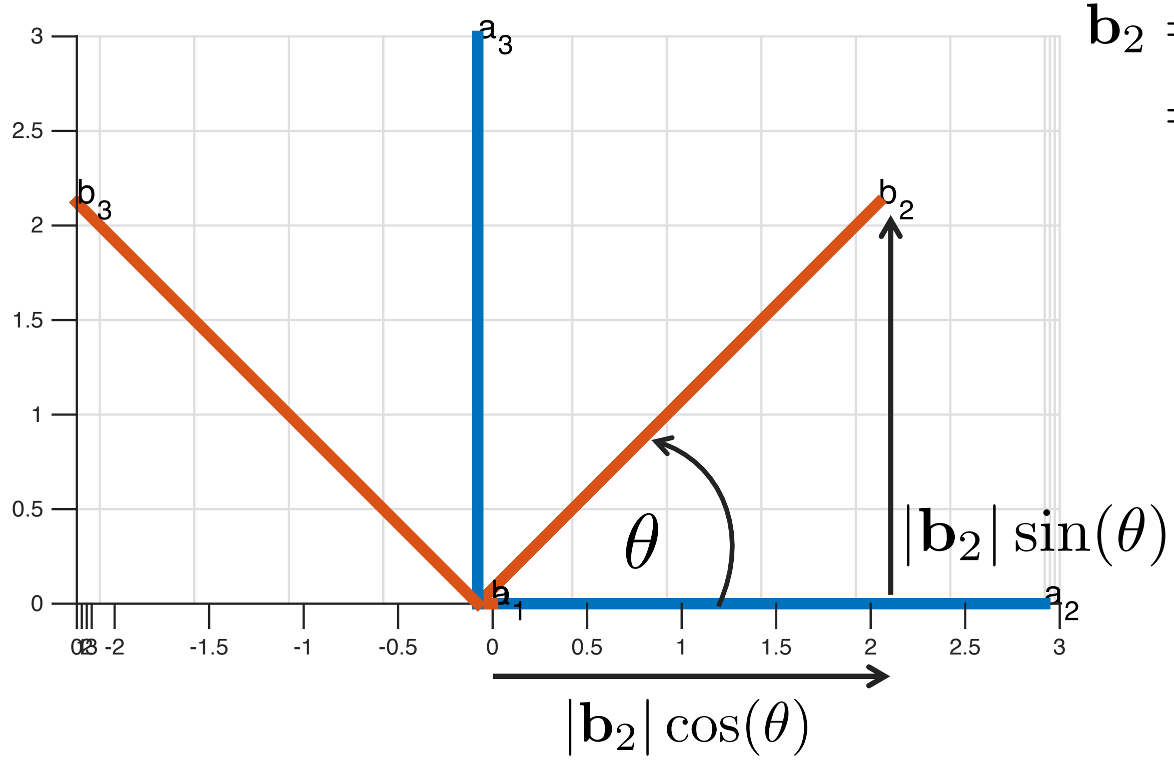
Translate Frame B so reference frames share an origin.



$$\mathbf{b}_1 = \mathbf{a}_1$$

Rotation

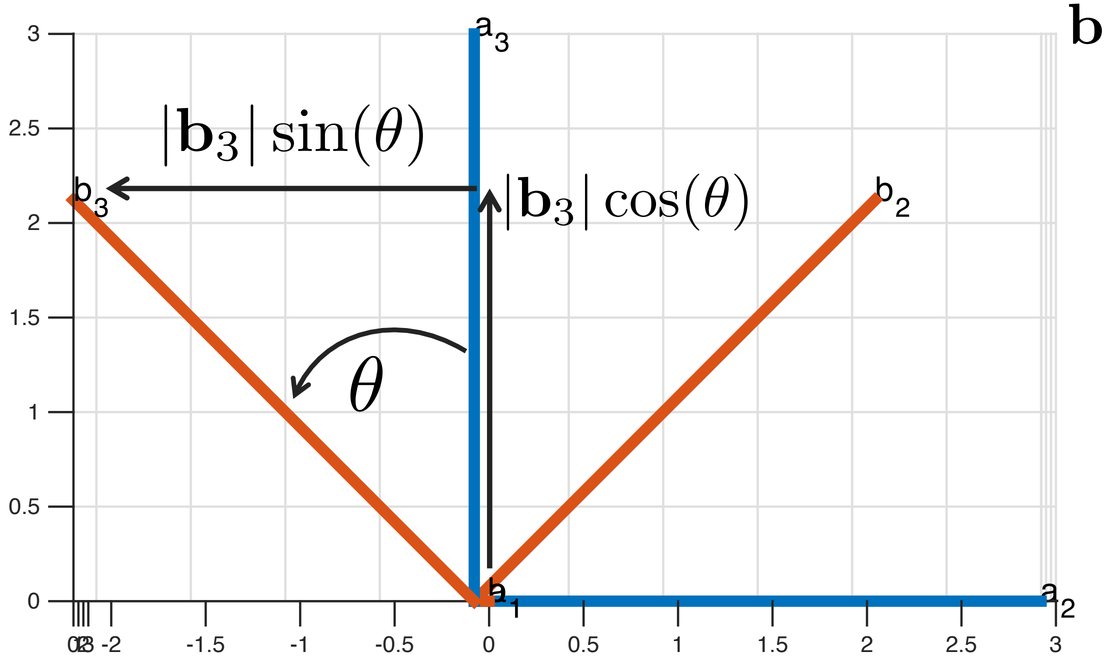
Translate Frame B so reference frames share an origin.



$$\begin{aligned}\mathbf{b}_2 &= |\mathbf{b}_2|\cos(\theta)\mathbf{a}_2 + |\mathbf{b}_2|\sin(\theta)\mathbf{a}_3 \\ &= \cos(\theta)\mathbf{a}_2 + \sin(\theta)\mathbf{a}_3\end{aligned}$$

Rotation

Frame B so the two frames share an origin.



$$\mathbf{b}_3 = -|\mathbf{b}_3|\sin(\theta)\mathbf{a}_2 + |\mathbf{b}_3|\cos(\theta)\mathbf{a}_3$$
$$= -\sin(\theta)\mathbf{a}_2 + \cos(\theta)\mathbf{a}_3$$

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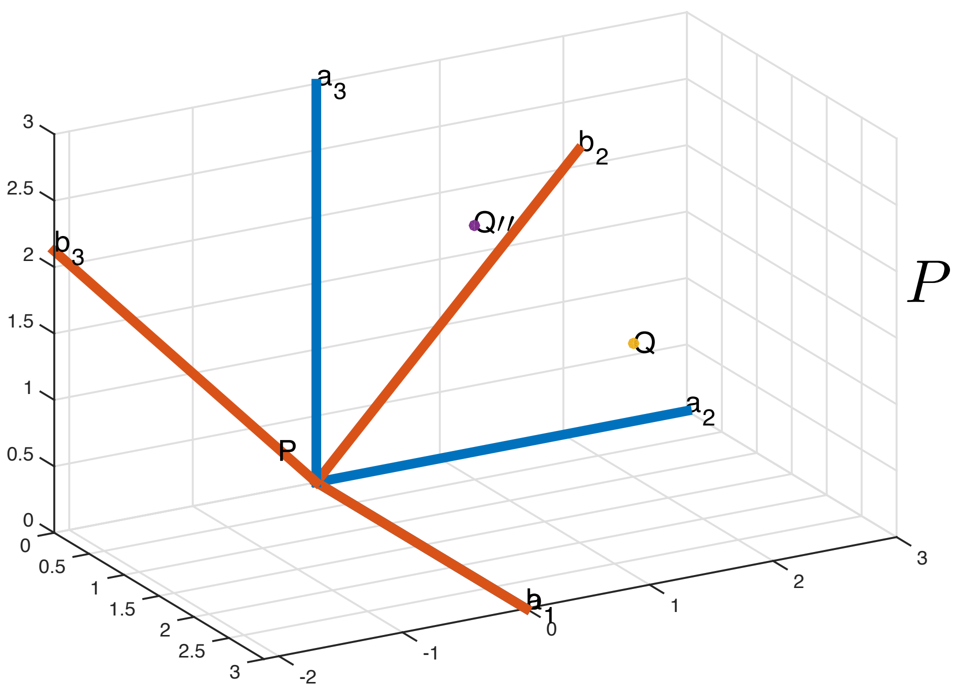
Rotation

Translate frame B so reference frames share an origin.



$$PQ'' = q_1\mathbf{b}_1 + q_2\mathbf{b}_2 + q_3\mathbf{b}_3$$
$$= q_1''\mathbf{a}_1 + q_2''\mathbf{a}_2 + q_3''\mathbf{a}_3$$

$$PQ'' = q_1(\mathbf{a}_1) + q_2(\cos(\theta)\mathbf{a}_2 + \sin(\theta)\mathbf{a}_3) + q_3(-\sin(\theta)\mathbf{a}_2 + \cos(\theta)\mathbf{a}_3)$$

$$= q_1\mathbf{a}_1 + (q_2\cos(\theta) - q_3\sin(\theta))\mathbf{a}_2 + (q_2\sin(\theta) + q_3\cos(\theta))\mathbf{a}_3$$

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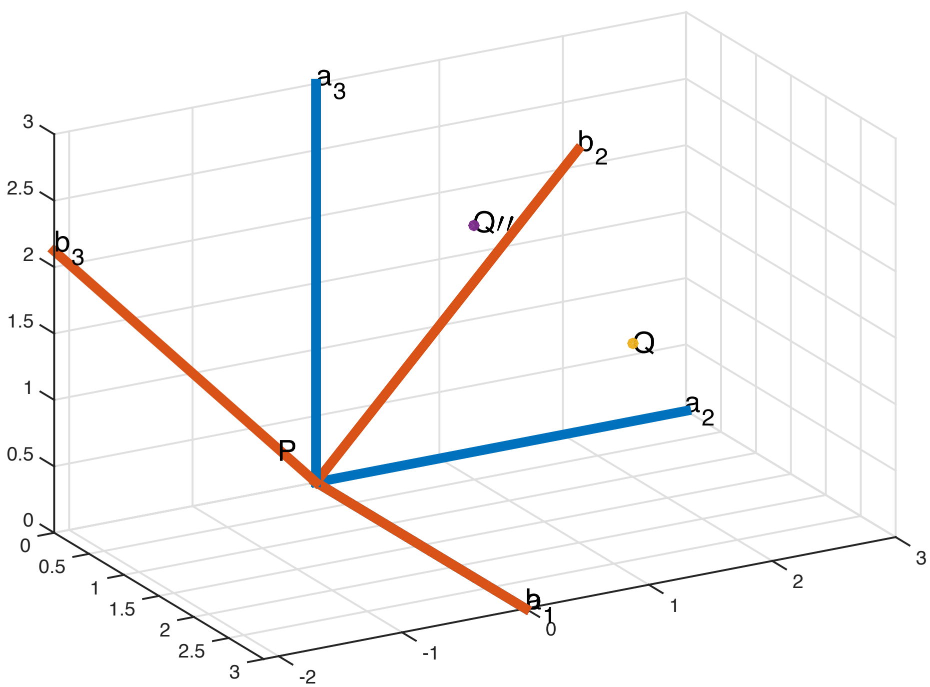
Rotation

Translate frame B so reference frames share an origin.



$$PQ'' = q_1''\mathbf{a}_1 + q_2''\mathbf{a}_2 + q_3''\mathbf{a}_3$$

$$PQ'' = q_1\mathbf{a}_1 + (q_2\cos(\theta) - q_3\sin(\theta))\mathbf{a}_2 + (q_2\sin(\theta) + q_3\cos(\theta))\mathbf{a}_3$$

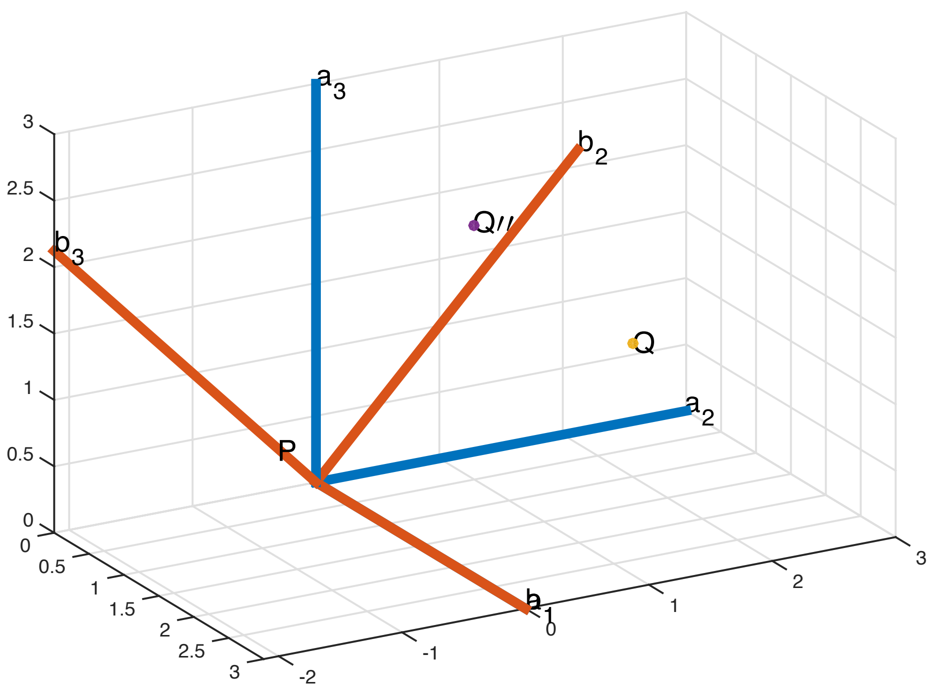
$$q_1'' = q_1$$

$$q_2'' = q_2\cos(\theta) - q_3\sin(\theta)$$

$$q_3'' = q_2\sin(\theta) + q_3\cos(\theta)$$

Rotation

Translate frame B so reference frames share an origin.



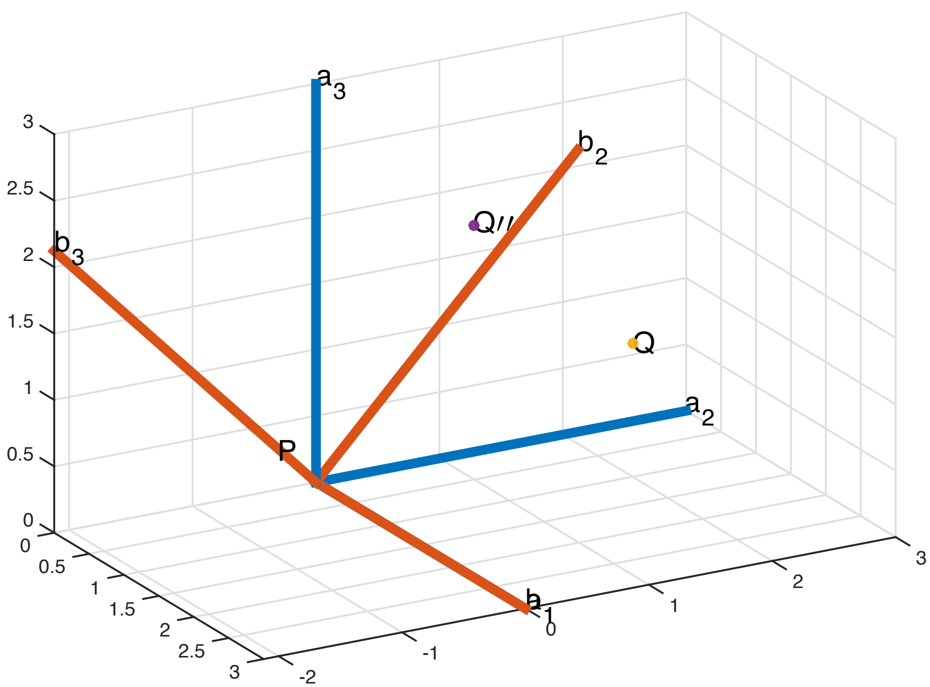
$$PQ'' = q_1(\mathbf{a}_1) + q_2(\cos(\theta)\mathbf{a}_2 + \sin(\theta)\mathbf{a}_3) + q_3(-\sin(\theta)\mathbf{a}_2 + \cos(\theta)\mathbf{a}_3)$$

$$PQ'' = q_1''\mathbf{a}_1 + q_2''\mathbf{a}_2 + q_3''\mathbf{a}_3$$

$$\begin{bmatrix} q_1'' \\ q_2'' \\ q_3'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Rotation

Translate frame B so reference frames share an origin.



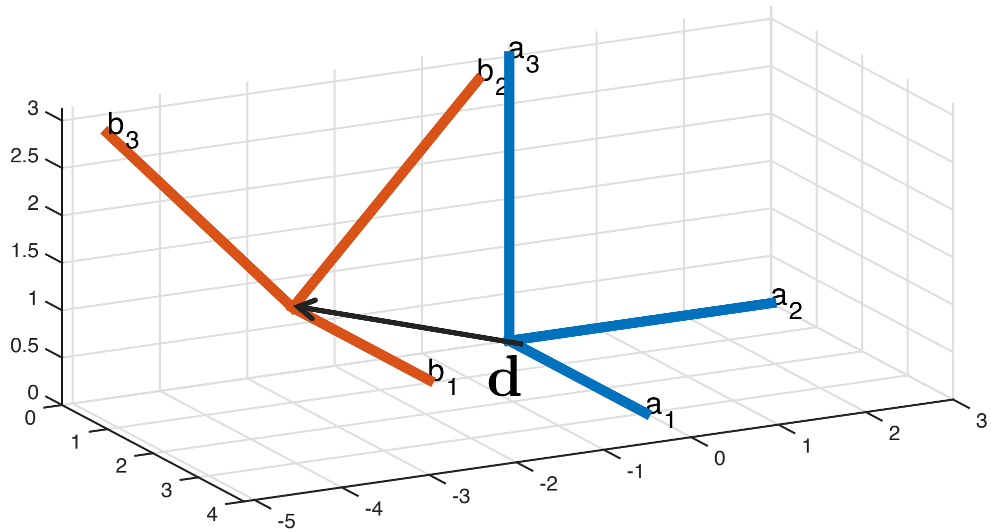
$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} = Rot(x, \theta)$$

$$\theta = \frac{\pi}{4} \rightarrow$$

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Translation

Let **d** be the vector from the origin of Frame A to the origin of Frame B, expressed in terms of Frame A.

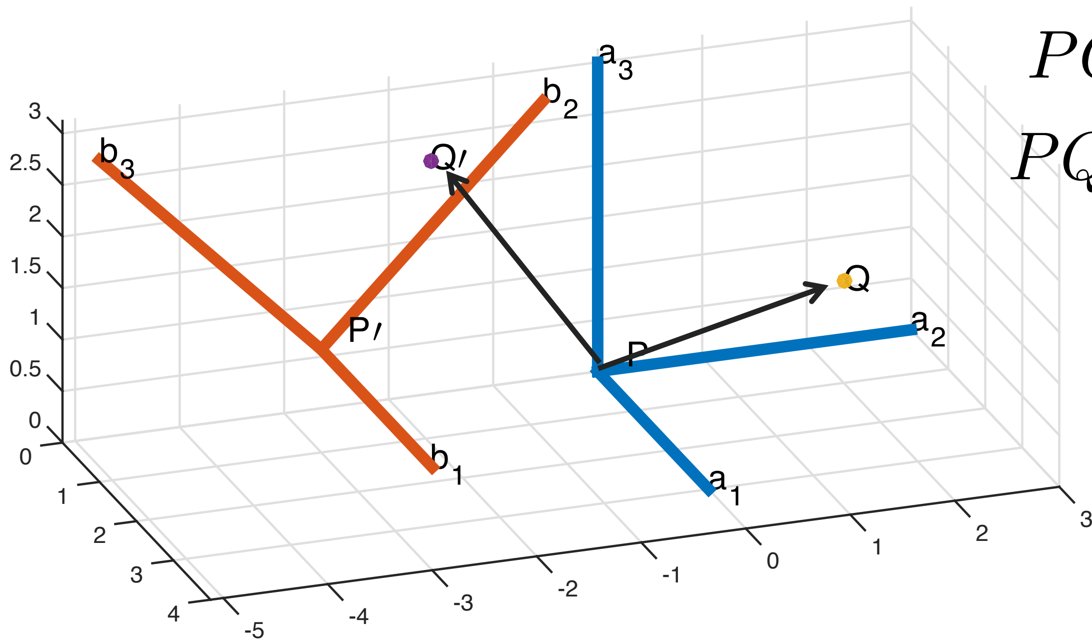


$$\mathbf{d} = 1\mathbf{a}_1 - 3\mathbf{a}_2 + 1\mathbf{a}_3$$

We can characterize a rigid-body displacement with a rotation matrix and translation vector.

Rigid-Body Displacement

Let Frame B be Frame A, after rigid-body displacement g.



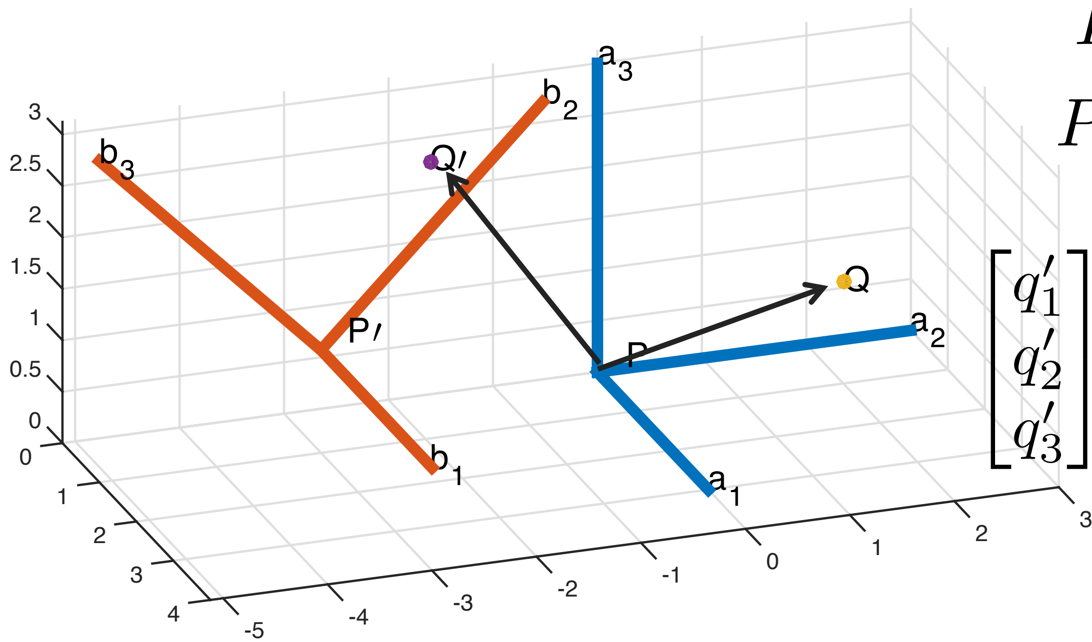
$$PQ = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$PQ' = q_1'\mathbf{a}_1 + q_2'\mathbf{a}_2 + q_3'\mathbf{a}_3$$

$$\begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} = R \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \mathbf{d}$$

Rigid-Body Displacement

Let Frame B be Frame A, after rigid-body displacement g.



$$PQ = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$PQ' = q_1'\mathbf{a}_1 + q_2'\mathbf{a}_2 + q_3'\mathbf{a}_3$$

$$\begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2.29 \\ 3.12 \end{bmatrix}$$

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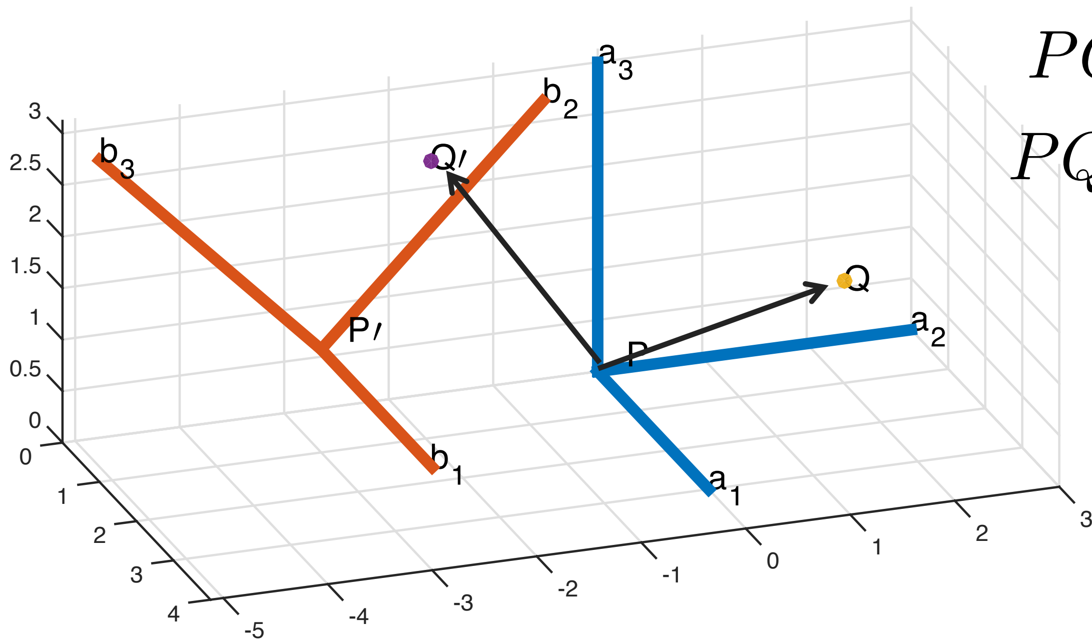
Rigid-Body Displacement

Let Frame B be Frame A, after rigid-body displacement g.



$$PQ = q_1 \mathbf{a}_1 + q_2 \mathbf{a}_2 + q_3 \mathbf{a}_3$$

$$PQ' = q_1' \mathbf{a}_1 + q_2' \mathbf{a}_2 + q_3' \mathbf{a}_3$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = R^T \left(\begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} - \mathbf{d} \right)$$

Properties of Functions

Function

A *function* is a relation that assigns each element in a set of inputs X, called the *domain*, to exactly one element in a set of outputs Y, called the *codomain* (or *range*).

$$f : X \to Y$$

Function

$$f : X \to Y$$

One-to-one (injective): for all a, b in X, if $f(a) = f(b)$, then $a = b$

No two inputs from the domain will map to the same output in the codomain.

Onto (surjective): for all y in Y, there is an x in X such that $f(x) = y$

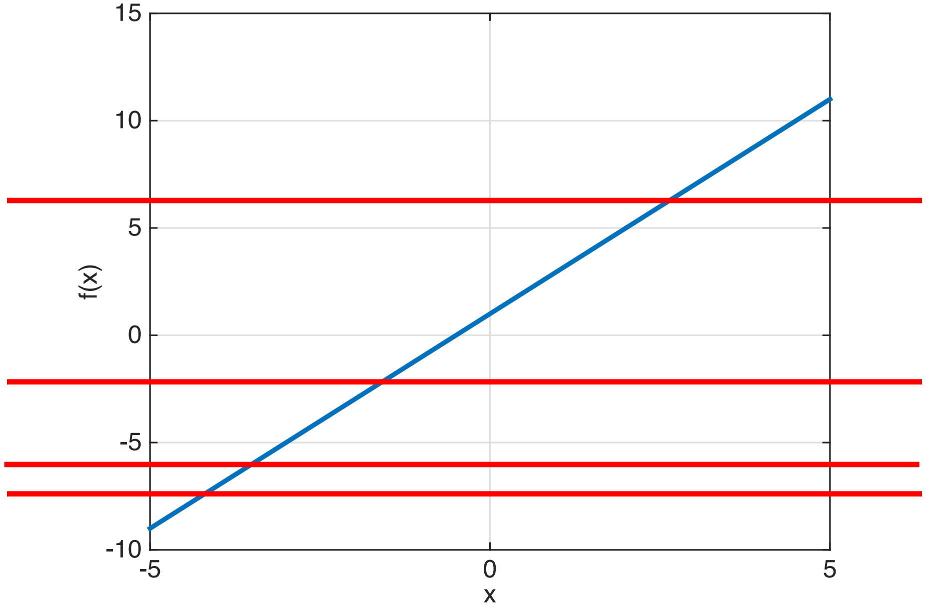
Every output in the codomain has an input in the domain that maps to it.

Example 1: One-to-one Functions

Consider:

$$f : R \to R \quad \text{such that} \quad f(x) = 2x + 1$$

This function **is** one-to-one.

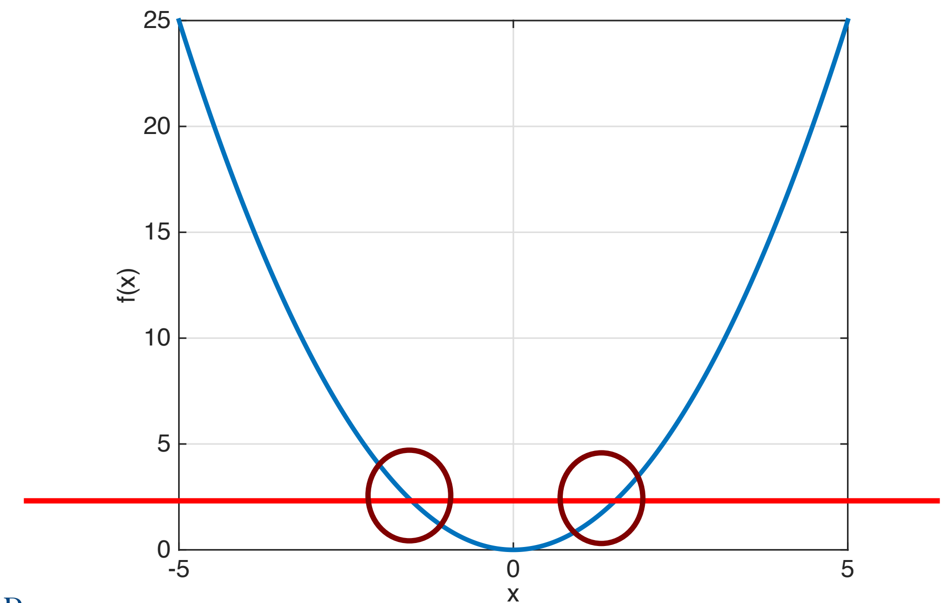


Example 2: One-to-one Functions

Consider:

$$f : R \to R \qquad \text{such that} \qquad f(x) = x^2$$

This function **is not** one-to-one.



$$f(1) = 1$$

$$f(-1) = 1$$

$$f(x) = f(-x)$$

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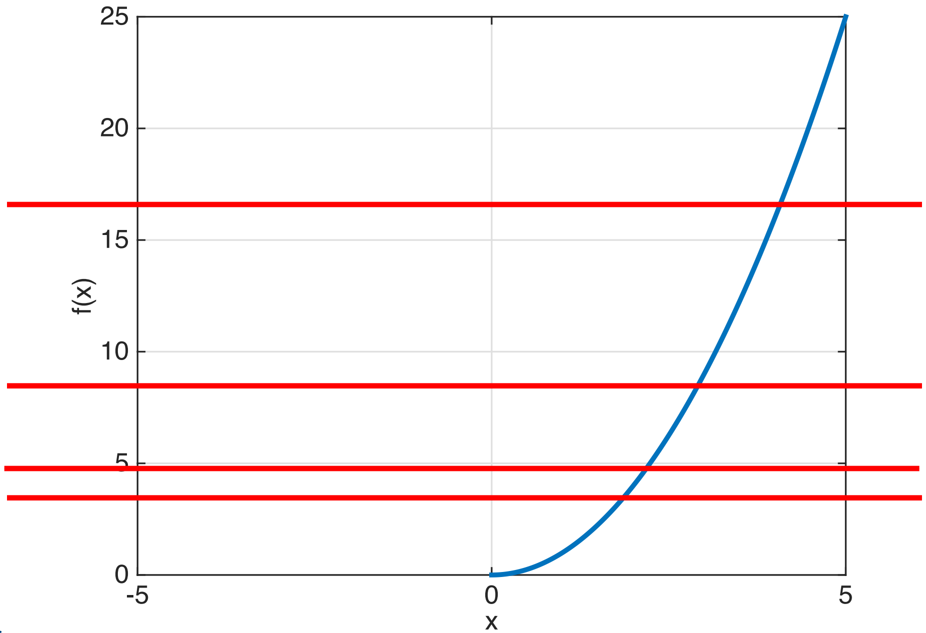
Example 2: One-to-one Functions

Consider:

$$f : \boxed{[0, \infty)} \to R \quad \text{such that} \quad f(x) = x^2$$

This function **is** one-to-one.



We have removed the "redundant" values of x from the domain.

Example 3: Onto Functions

Consider:

$f : R \to R$ such that $f(x) = e^x$

This function **is not** onto.

For any $y \leq 0$, there is no x such that $e^x = y$.

Example 3: Onto Functions

Consider:

$$f : R \rightarrow \boxed{(0, \infty)} \quad \text{such that} \quad f(x) = e^x$$

This function **is** onto.

The specified codomain no longer includes the values $y \leq 0$.

Inverse Tangent with atan2

atan (arctangent) Function

Recall:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{y}{x}$$

The function $\theta = \tan^{-1}(\frac{y}{x})$ returns the angle θ for which $\tan(\theta) = \frac{y}{x}$.

$$\tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}} \longrightarrow \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$$

$$\text{atan}(\frac{y}{x}) = \tan^{-1}(\frac{y}{x})$$

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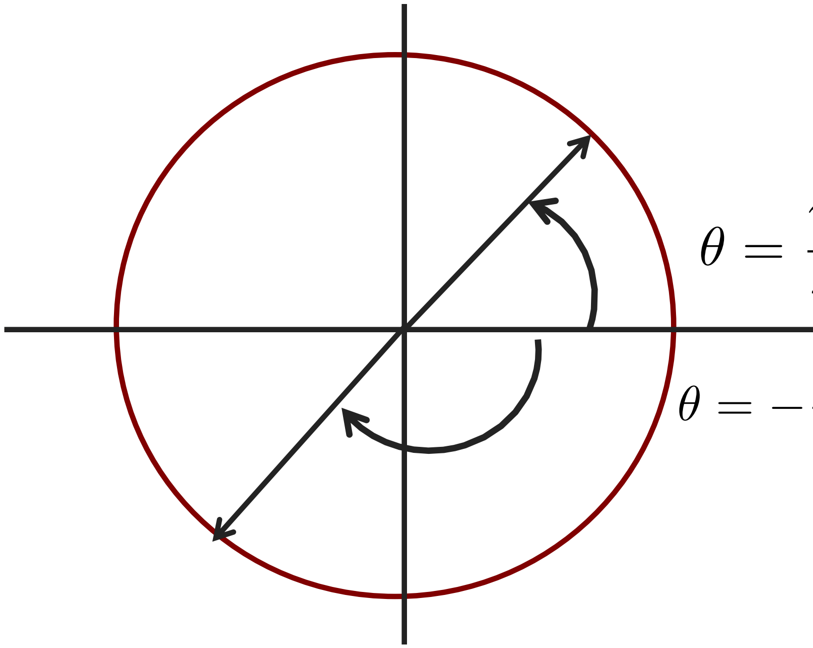
atan (arctangent) Function



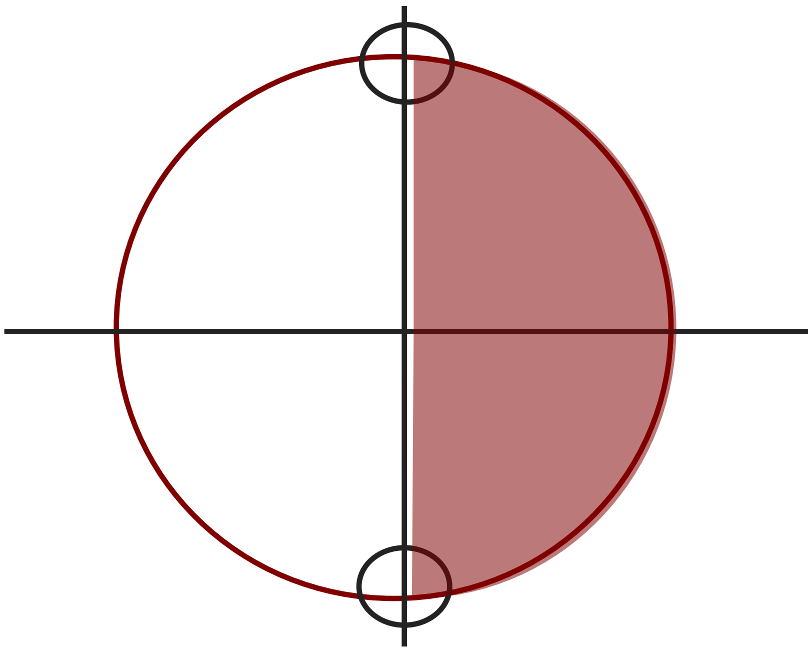
$$\theta = \frac{\pi}{4} \rightarrow \begin{aligned} \sin(\theta) &= \frac{\sqrt{2}}{2} \\ \cos(\theta) &= \frac{\sqrt{2}}{2} \\ \tan(\theta) &= 1 \end{aligned}$$

$$\theta = -\frac{3\pi}{4} \rightarrow \begin{aligned} \sin(\theta) &= -\frac{\sqrt{2}}{2} \\ \cos(\theta) &= -\frac{\sqrt{2}}{2} \\ \tan(\theta) &= 1 \end{aligned}$$

$$\tan^{-1}(\frac{1}{1}) = \tan^{-1}(\frac{-1}{-1})$$

The atan function cannot distinguish between opposite points on the unit circle.

atan (arctangent) Function



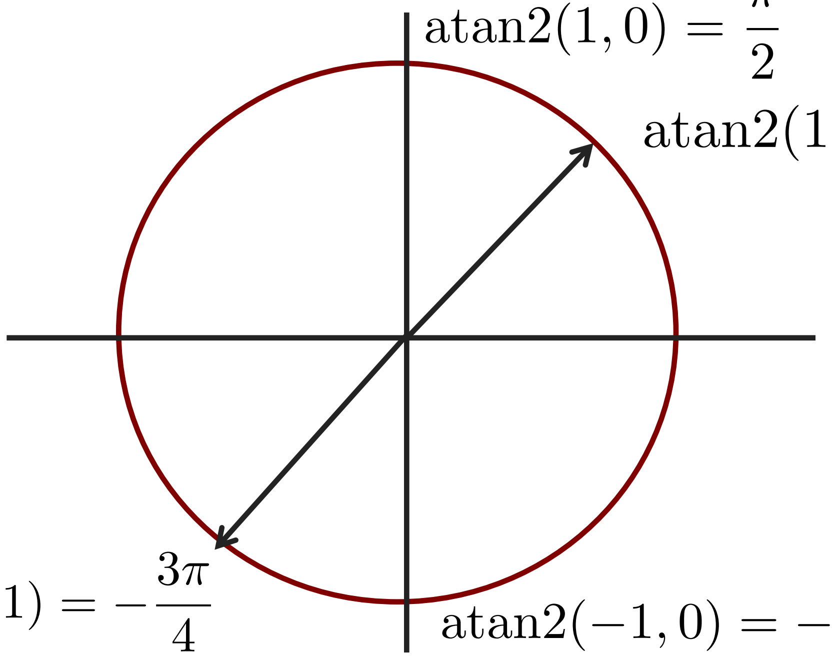
$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{y}{x} = \frac{\pm 1}{0} = \text{undefined}$$

The atan function fails when $\theta = \pm\frac{\pi}{2}$.

Returns values in range $(-\frac{\pi}{2}, \frac{\pi}{2})$

atan2

$\operatorname{atan2}(y, x)$ is an implementation of the atan function that takes into account ratio and the signs of y and x.



Returns values in range $(-\pi, \pi]$

Eigenvalues and Eigenvectors of Matrices

Determinant

A determinant is a scalar property of square matrices, denoted $\det(A)$ or $|A|$.

- Think of rows of an n x n matrix as n vectors in R^n.
- The determinant represents the "space contained" by these vectors.

In this course, we will be working with 2x2 or 3x3 matrices.

Determinant (2x2 Matrix)

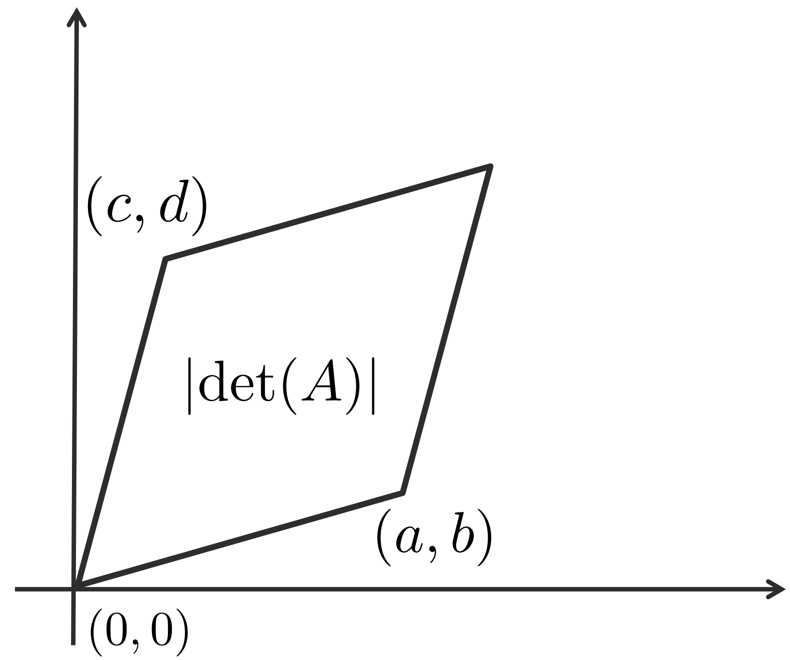
Consider:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Area of parallelogram
defined by the rows.



Example: Determinant (2x2 Matrix)

Consider:

$$\begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$$

Determinant:

$$\begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} = (1)(1) - (3)(4) = -11$$

Determinant (3x3 Matrix)

Consider:

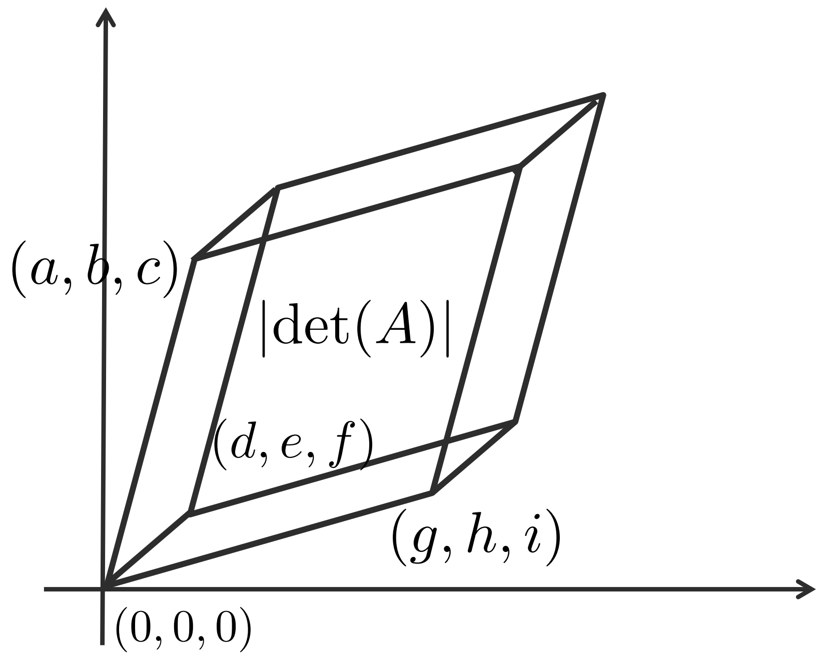
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Determinant:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh$$

$$- ceg - bdi - afh$$

Volume of parallelepiped defined by the rows.



Example: Determinant (3x3 Matrix)

Consider:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 3 & -1 \end{bmatrix}$$

Determinant:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 3 & -1 \end{vmatrix}$$

$$\begin{aligned} &= (1)(-4)(-1) + (2)(1)(0) + (3)(0)(3) \\ &\quad - (3)(-4)(0) - (2)(0)(-1) - (1)(1)(3) \\ &= 1 \end{aligned}$$

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Eigenvalues and Eigenvectors

A matrix is a transformation.

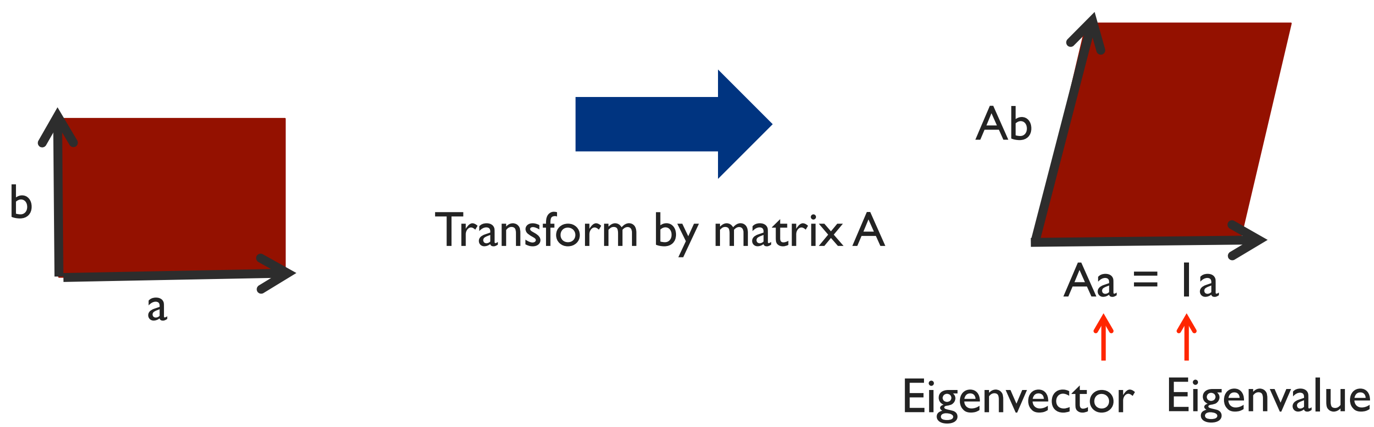
$$y = Ax$$

Eigenvalues and Eigenvectors

Eigenvectors are vectors associated by a square matrix that do not change in direction when multiplied by the matrix.

Eigenvalues are scalar values representing how much each eigenvector changes in length.

$$A\mathbf{v} = \lambda\mathbf{v}$$



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Finding Eigenvalues

1. Calculate:

$$\det(A - \lambda \mathbf{I})$$

2. Find solutions to:

$$\det(A - \lambda \mathbf{I}) = 0$$

There will be n eigenvalues for an n x n matrix, but not all of them have to be distinct or real values.

Example: Finding Eigenvalues

Consider:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

1. Calculate $\det(A - \lambda \mathbf{I})$

$$\begin{aligned} \det(A - \lambda \mathbf{I}) &= \det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) \\ &= (2-\lambda)^2 - 1 \\ &= \lambda^2 - 4\lambda + 3 \end{aligned}$$

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Example: Finding Eigenvalues

Consider:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

2. Find solutions to $\det(A - \lambda \mathbf{I}) = 0$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda_1 = 1, \lambda_2 = 3$$

2 eigenvalues for a 2 x 2 matrix

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Finding Eigenvectors

1. For each eigenvalue, solve the equation:

$$A\mathbf{v} = \lambda\mathbf{v}$$

or:

$$(A - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0}$$

Notice that if $\mathbf{v}$ is an eigenvector, then $\alpha\mathbf{v}$ is also an eigenvector, where α is any scalar.

Thus, we typically think about **linearly independent** eigenvectors.

Eigenvectors

n vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ are **linearly independent (LI)** if the only solution to the equation:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_n\mathbf{v}_n = 0$$

is $a_1 = a_2 = ... = a_n = 0$.

There will be at least one LI eigenvector for each eigenvalue.

If eigenvalues are repeated, there might be multiple LI eigenvectors for that eigenvalue.

Example: Finding Eigenvectors

Consider:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

1. For $\lambda_1 = 1$:

$$\begin{aligned} &(A - \lambda_1 \mathbf{I})\,\mathbf{v}_1 \\ &= \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathbf{v}_1 \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{v}_1 \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

$\Rightarrow$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_1 = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Quaternions

Quaternion Definition

Quaternion:

$$q = (q_0, q_1, q_2, q_3)$$

This can be interpreted as a constant + vector:

$$q = (q_0, \mathbf{q})$$

Operations with Quaternions

Quaternion addition/subtraction:

$$p \pm q = (p_0 \pm q_0, \mathbf{p} \pm \mathbf{q})$$

Quaternion multiplication:

$$pq = (p_0 q_0 - \mathbf{p}^T \mathbf{q}, p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q})$$

Quaternion inverse:

$$q^{-1} = (q_0, -\mathbf{q})$$

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Axis-Angle Representation to Quaternion

Quaternions can be used to represent rigid-body rotations.

Recall the axis-angle representation of rotations:

Angle of rotation: ϕ

Axis of rotation: $\mathbf{u}$

The equivalent quaternion is:

$$q = (\cos(\frac{\phi}{2}), u_1 \sin(\frac{\phi}{2}), u_2 \sin(\frac{\phi}{2}), u_3 \sin(\frac{\phi}{2}))$$

Quaternions to Axis-Angle Representation

Given a quaternion:

$$q = (q_0, q_1, q_2, q_3)$$

The equivalent axis-angle representation is:

Angle of rotation: $2\cos^{-1}(q_0)$

Axis of rotation: $\mathbf{u}_2 = \begin{bmatrix} \frac{q_1}{\sqrt{1-q_0^2}} \\ \frac{q_2}{\sqrt{1-q_0^2}} \\ \frac{q_3}{\sqrt{1-q_0^2}} \end{bmatrix}$

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Vector Rotation with Quaternions

To rotate a vector **p** in R^3 by the quaternion q:

1. Define quaternion:

$$p = (0, \mathbf{p})$$

2. The result after rotation is:

$$p' = qpq^{-1} = (0, \mathbf{p}')$$

We can easily compose two rotations:

$$q = q_2 q_1$$

Properties of Quaternions

- $q = (q_0, q_1, q_2, q_3)$ **and** $-q = (-q_0, -q_1, -q_2, -q_3)$ represent the same rotation.

- Compact representation of rotations, with only 4 parameters.
- No singularities
- Quaternion product is more numerically stable than matrix multiplication.

Matrix Derivative

Matrix Derivative

Recall the following expressions from lecture:

$$\dot{R}^T R + R^T \dot{R} = 0$$

$$R\dot{R}^T + \dot{R}R^T = 0$$

What is $\dot{R}$?

Matrix Derivative

R is a matrix where each component is a function of time.

$$R = \begin{bmatrix} R_{11}(t) & R_{12}(t) & R_{13}(t) \\ R_{21}(t) & R_{22}(t) & R_{32}(t) \\ R_{31}(t) & R_{32}(t) & R_{33}(t) \end{bmatrix}$$

$\dot{R}$ is a matrix whose components are the time derivatives of the components of R.

$$\dot{R} = \begin{bmatrix} \frac{dR_{11}(t)}{dt} & \frac{dR_{12}(t)}{dt} & \frac{dR_{13}(t)}{dt} \\ \frac{dR_{21}(t)}{dt} & \frac{dR_{22}(t)}{dt} & \frac{dR_{23}(t)}{dt} \\ \frac{dR_{31}(t)}{dt} & \frac{dR_{32}(t)}{dt} & \frac{dR_{33}(t)}{dt} \end{bmatrix}$$

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Matrix Derivative Properties

Properties for scalar function derivatives apply to matrix derivatives as well:

$$\frac{d}{dt}(A \pm B) = \dot{A} \pm \dot{B}$$

$$\frac{d}{dt}(AB) = \dot{A}B + A\dot{B}$$

$$\frac{d}{dt}(A(\theta(t))) = \frac{dA}{d\theta}\dot{\theta}$$

Example 1: Matrix Derivative

Consider:

$$R = \begin{bmatrix} 2t & t^2 & e^t \\ \sin(t) & \cos(t) & \tan(t) \\ 5 & ln(t) & 0 \end{bmatrix}$$

The time derivative is:

$$\dot{R} = \begin{bmatrix} 2 & 2t & e^t \\ \cos(t) & -\sin(t) & \sec^2(t) \\ 0 & \frac{1}{t} & 0 \end{bmatrix}$$

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Example 2: Matrix Derivative

Consider: $\theta(t)$ Use chain rule!

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = Rot(z, \theta)$$

$$\dot{R} = \begin{bmatrix} -\dot{\theta}\sin(\theta) & -\dot{\theta}\cos(\theta) & 0 \\ \dot{\theta}\cos(\theta) & -\dot{\theta}\sin(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}$$

Skew-Symmetric Matrices and the Hat Operator

Matrix Transpose

Every matrix has a *transpose*, denoted A^T.

- Let A be a n x m matrix and A_{ij} be the element in the i^{th} row and j^{th} column of A.
- The transpose is defined by $A_{ij}^T = A_{ji}$, that is, the rows and columns of A are "flipped".

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Example 1: Matrix Transpose

Consider:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

The transpose is:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Example 2: Matrix Transpose

Consider:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

2x3 matrix

The transpose is:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

3x2 matrix

Example 3: Matrix Transpose

Consider:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

The transpose is:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \longleftarrow (A^T)^T = A$$

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Example 4: Matrix Transpose

Consider:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

The transpose is:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Matrix Symmetry

A matrix is *symmetric* if:

$$A^T = A$$

A matrix is *skew-symmetric* if:

$$A^T = -A$$

3x3 Skew-Symmetric Matrices

A matrix is *skew-symmetric* if:

$$A^T = -A$$

Consider a 3x3 matrix:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

This matrix is skew-symmetric if:

$$A^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = - \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = -A$$

3x3 Skew-Symmetric Matrices

Matching components gives the constraints:

$$\left.\begin{aligned} A_{11} &= -A_{11} \\ A_{22} &= -A_{22} \\ A_{33} &= -A_{33} \end{aligned}\right\} \text{Must be 0}$$

$$\begin{aligned} A_{21} &= -A_{12} \\ A_{13} &= -A_{31} \\ A_{23} &= -A_{32} \end{aligned}$$

$$\Rightarrow \quad A = \begin{bmatrix} 0 & A_{12} & A_{13} \\ A_{21} & 0 & A_{23} \\ A_{31} & A_{32} & 0 \end{bmatrix}$$

3x3 Skew-Symmetric Matrices

Matching components gives the constraints:

$$A_{11} = -A_{11}$$

$$A_{22} = -A_{22}$$

$$A_{33} = -A_{33}$$

$$\left.\begin{aligned} A_{21} &= -A_{12} \\ A_{13} &= -A_{31} \\ A_{23} &= -A_{32} \end{aligned}\right\}$$

$$\Rightarrow \quad A = \begin{bmatrix} 0 & -A_{21} & A_{13} \\ A_{21} & 0 & -A_{32} \\ -A_{13} & A_{32} & 0 \end{bmatrix}$$

A 3x3 skew-symmetric matrix only has 3 independent parameters!

3x3 Skew-Symmetric Matrices

We can concisely represent a skew-symmetric matrix as a 3x1 vector:

$$A = \begin{bmatrix} 0 & -A_{21} & A_{13} \\ A_{21} & 0 & -A_{32} \\ -A_{13} & A_{32} & 0 \end{bmatrix} \Rightarrow a = \begin{bmatrix} A_{32} \\ A_{13} \\ A_{21} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

We use the *hat operator* to switch between these two representations.

$$\hat{a} = \widehat{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Example: 3x3 Skew-Symmetric Matrices

Consider:

$$\omega = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The corresponding skew-symmetric matrix is:

$$\hat{\omega} = \begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$$

Vector Cross Product

The hat operator is also used to denote the cross product between two vectors.

$$\mathbf{u} \times \mathbf{v} = \hat{\mathbf{u}}\mathbf{v}$$

$$= \underbrace{\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}}_{[u\times]} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

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Representation of Angular Velocities

Recall we defined the angular velocity vectors:

$$\hat{\omega}^b = R^T \dot{R}$$

$$\hat{\omega}^s = \dot{R} R^T$$

$R^T \dot{R}$ and $\dot{R} R^T$ are skew-symmetric.

We are guaranteed to find vectors ω^b, ω^s that satisfy the given definitions of angular velocity.

Dynamics of a Quadrotor

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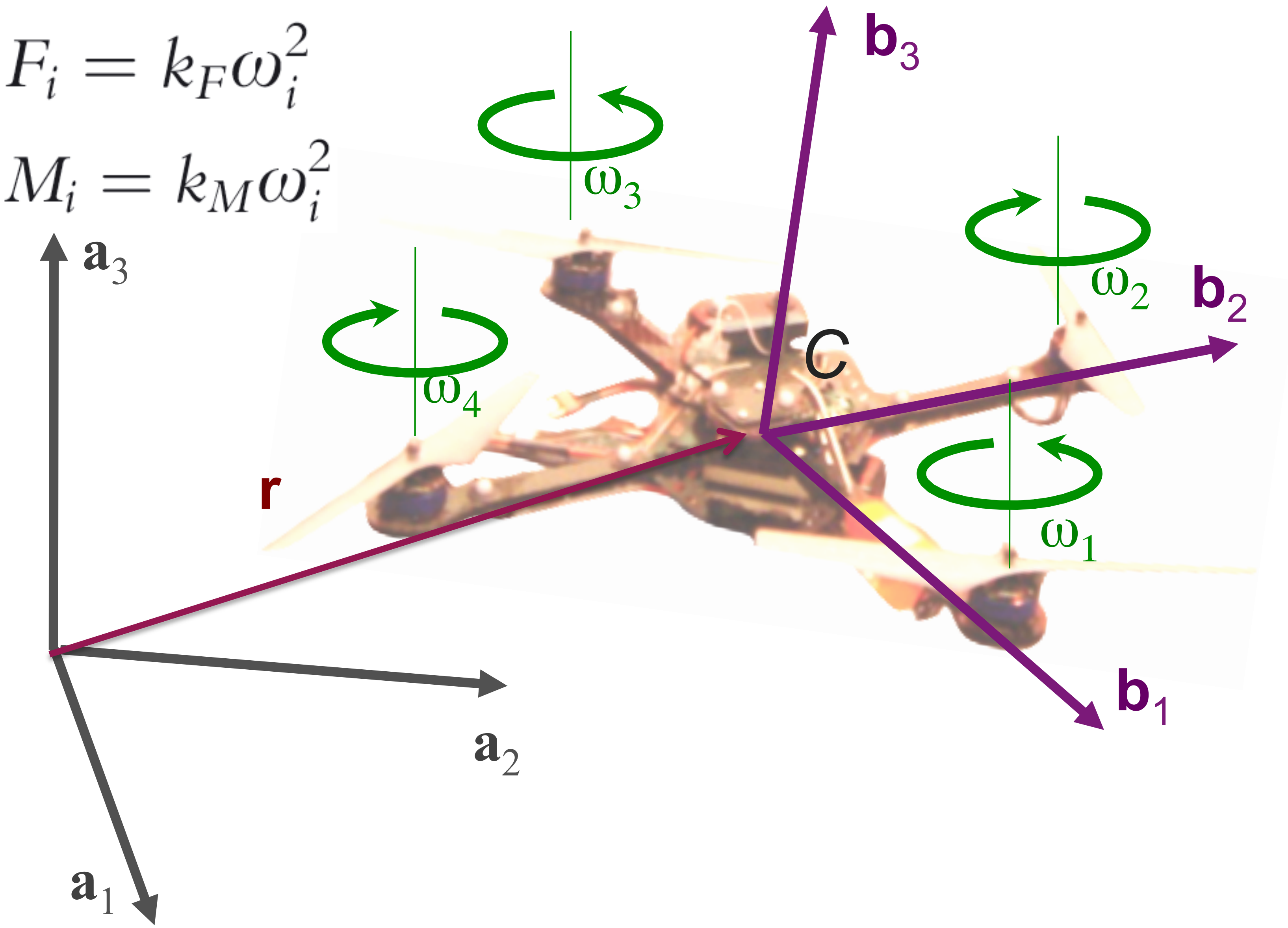
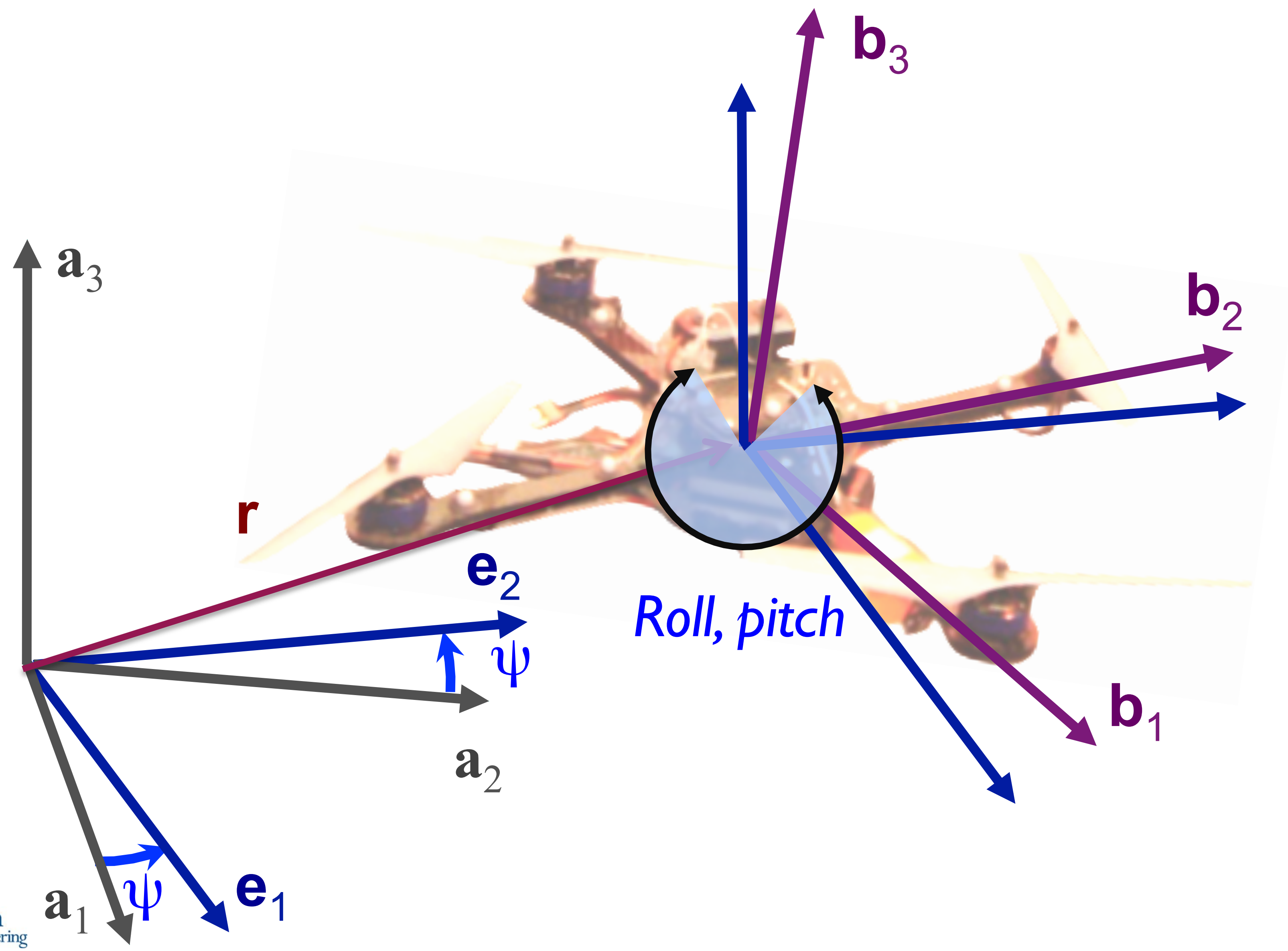
$$F_i = k_F \omega_i^2$$

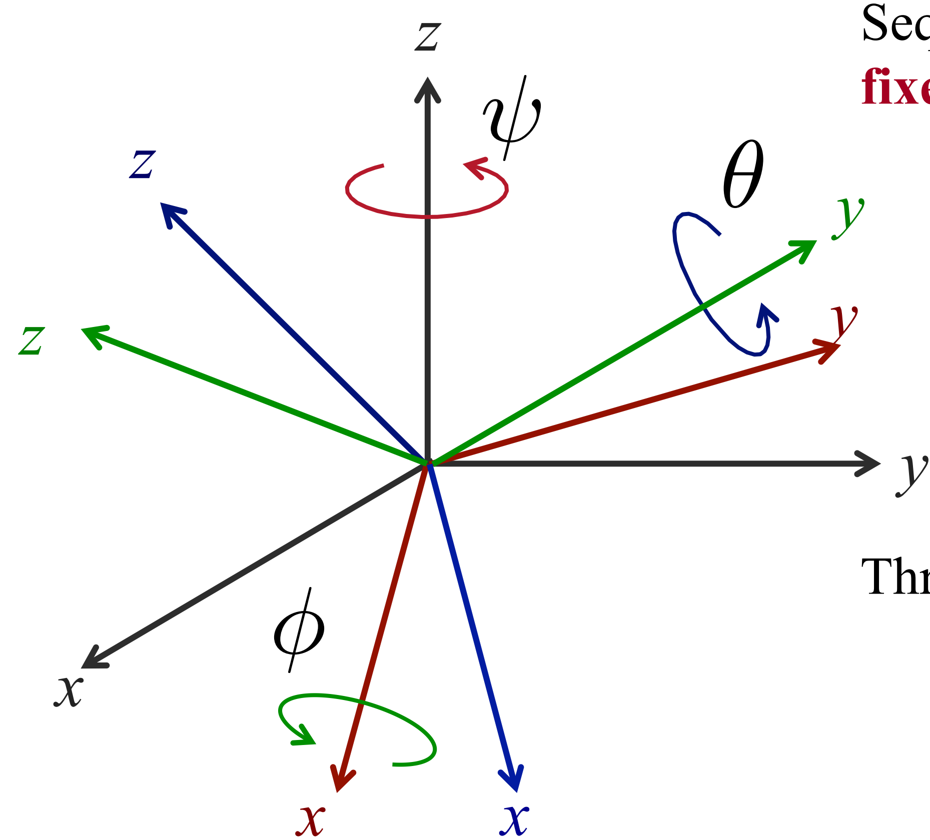
$$M_i = k_M \omega_i^2$$



Euler Angles



Z-X-Y Euler Angles



Sequence of three rotations about **body-fixed** axes

- Rot(z, ψ)
- Rot(x, ϕ)
- Rot(y, θ)

Three Euler Angles

- ϕ, θ, and ψ
- Parameterize rotations

Note there are singularities!

$$\mathbf{R} = \mathrm{Rot}(z, \psi) \times \mathrm{Rot}(x, \phi) \times \mathrm{Rot}(y, \theta)$$

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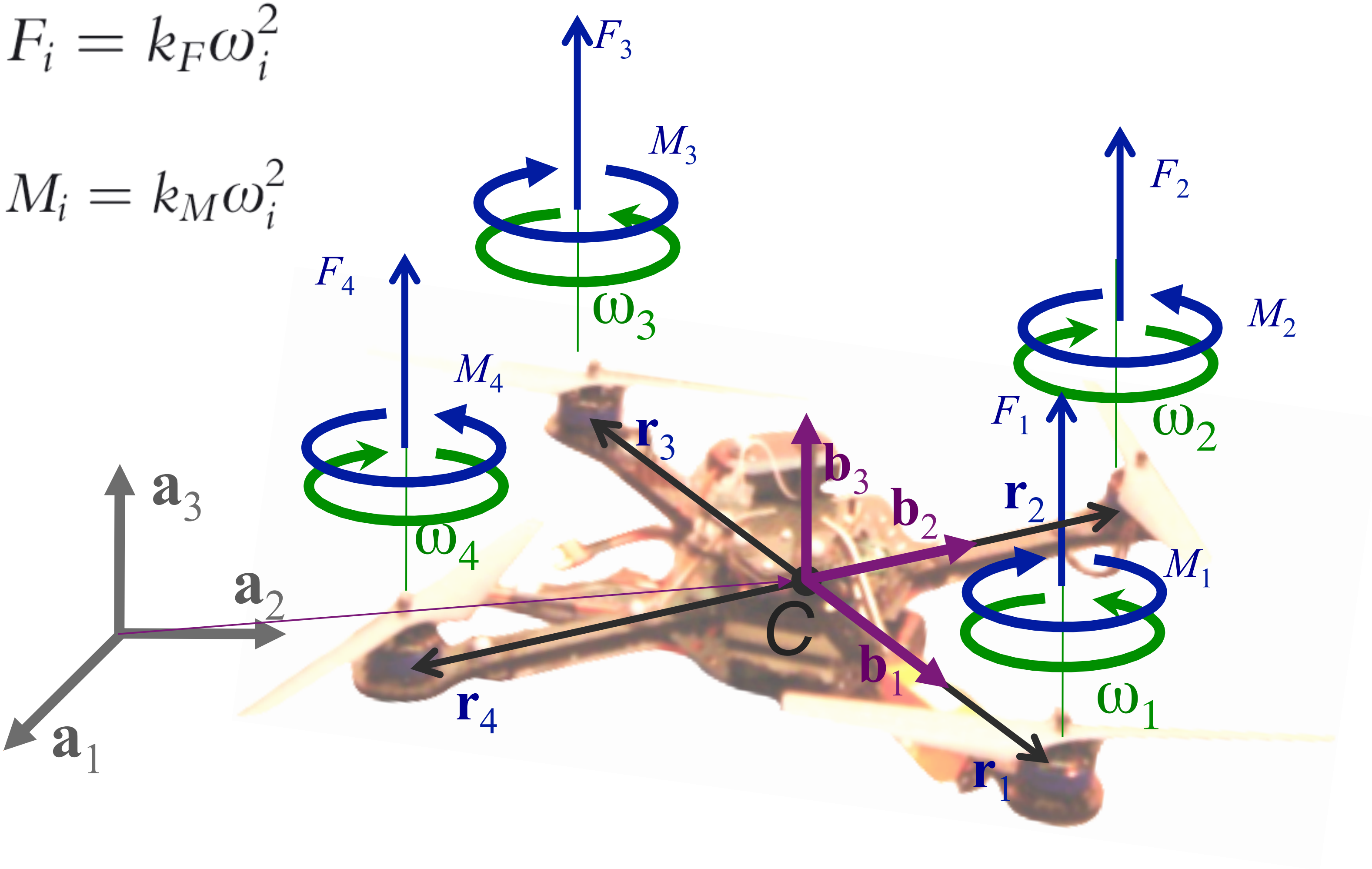
External Forces and Moments

$$F_i = k_F \omega_i^2$$

$$M_i = k_M \omega_i^2$$



$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 - mg\mathbf{a}_3$$

$$\mathbf{M} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \mathbf{r}_4 \times \mathbf{F}_4$$
$$+\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_4$$

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Newton-Euler Equations

System of Particles

Rigid Body

Newton-Euler Equations

System of Particles

Rigid Body

Newton-Euler Equations

Newton's Equations of Motion for a Single Particle of mass *m*

$$\mathbf{F} = m\mathbf{a}$$

Newton-Euler Equations

System of Particles

Rigid Body

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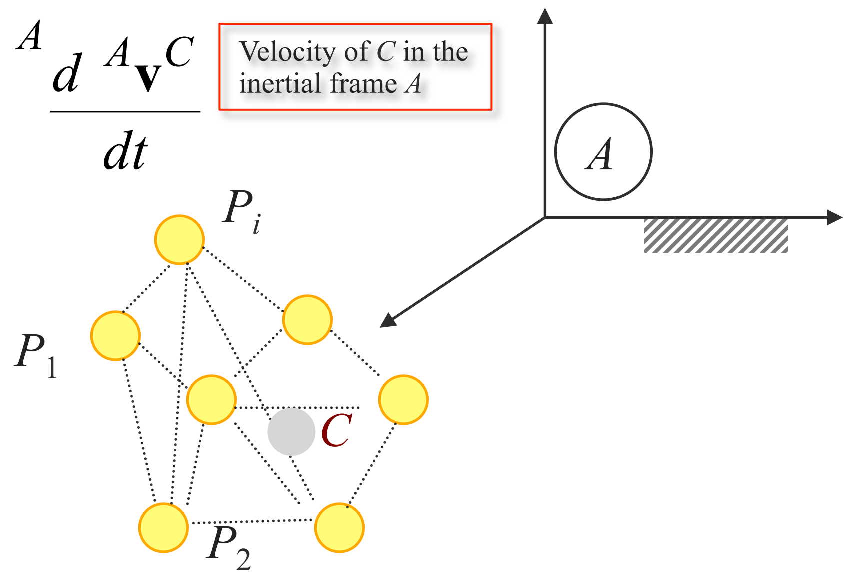
Newton's Second Law for a System of Particles

The center of mass for a system of particles, *S*, accelerates in an inertial frame (*A*) as if it were a single particle with mass *m* (equal to the total mass of the system) acted upon by a force equal to the net external force.

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i = m \frac{{}^{A}d\ {}^{A}\mathbf{v}^{C}}{dt}$$

Velocity of C in the inertial frame A



Center of mass

$$\mathbf{r}_c = \frac{1}{m} \sum_{i=1,N} m_i \mathbf{p}_i$$

Rate of Change of Linear Momentum

Derivative in the inertial frame A

$$\mathbf{F} = \frac{{}^{A}d\,\mathbf{L}}{dt}$$

Linear momentum of the system of particles in the inertial frame A

Also true for a rigid body

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Rotational equations of motion for a rigid body

The rate of change of angular momentum of the rigid body *B* relative to *C* in *A* is equal to the resultant moment of all external forces acting on the body relative to *C*

Angular momentum of the rigid body *B* with the origin *C* in the inertial frame *A*

Derivative in the inertial frame *A*

Net moment from all external forces and torques about the reference C

$$\frac{{}^{A}d\,{}^{A}\mathbf{H}_{C}^{S}}{dt} = \mathbf{M}_{C}^{S}$$

angular velocity of B in A

Angular momentum

$$\boxed{{}^{A}\mathbf{H}_{C}^{S} = \mathbf{I}_{C} \cdot {}^{A}\omega^{B}}$$

inertia tensor with C as the origin

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Principal Axes and Principal Moments

Principal axis of inertia

u is a unit vector along a principal axis if **I . u** is parallel to **u**

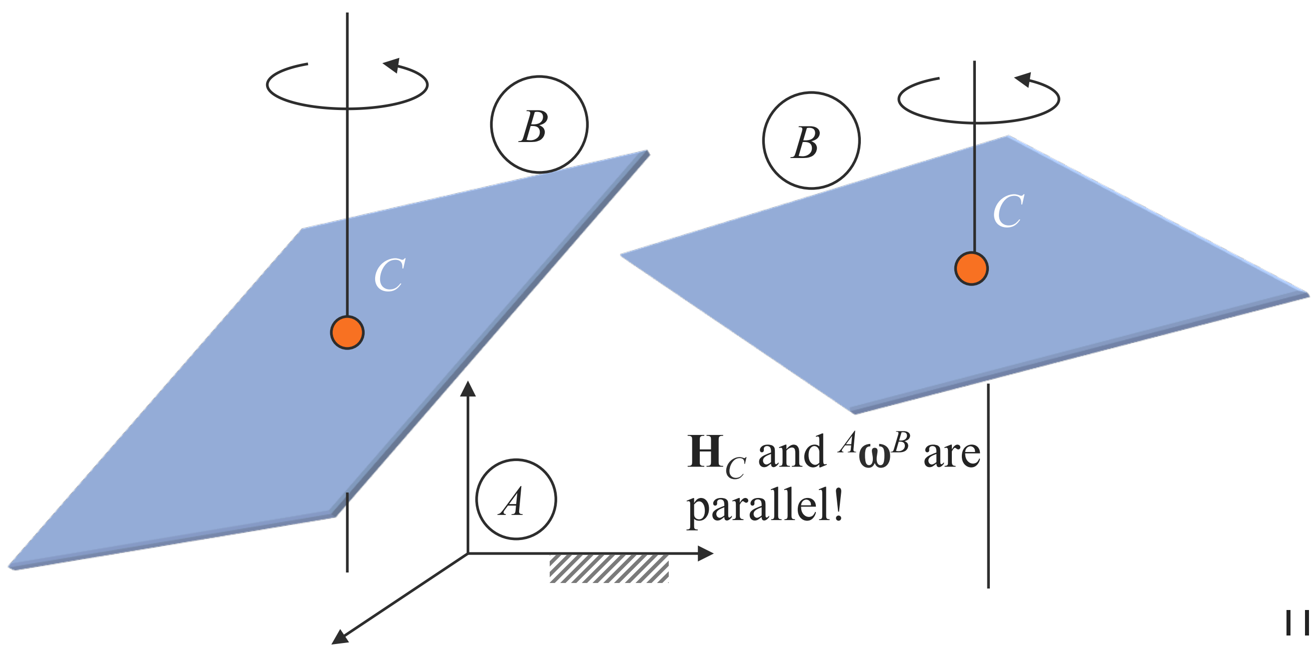
There are 3 independent principal axes!

Principal moment of inertia

The moment of inertia with respect to a principal axis, **u . I . u**, is called a principal moment of inertia.

Physical interpretation

$\mathbf{H}_C$ and ${}^A\boldsymbol{\omega}^B$ are not parallel!



Euler's Equations

1

$$\frac{{}^{A}d\mathbf{H}_C}{dt} = \mathbf{M}_C$$

$$\begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} \overset{p}{\omega_1} \\ \overset{q}{\omega_2} \\ \overset{r}{\omega_3} \end{bmatrix} = \begin{bmatrix} M_{C,1} \\ M_{C,2} \\ M_{C,3} \end{bmatrix}$$

2

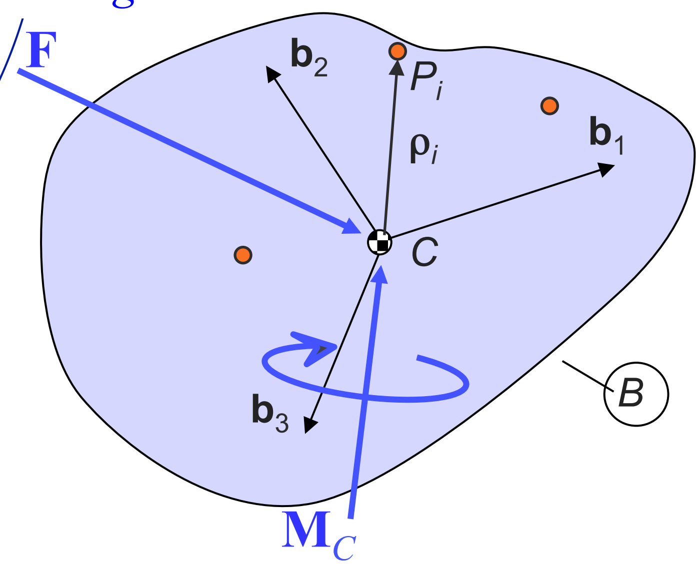
differentiating

Let $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, be along principal axes and

$${}^{A}\boldsymbol{\omega}^{B} = \omega_1\, \mathbf{b}_1 + \omega_2\, \mathbf{b}_2 + \omega_3 \mathbf{b}_3$$

$$\frac{{}^{B}d\mathbf{H}_C}{dt} + {}^{A}\omega^{B} \times \mathbf{H}_C = \mathbf{M}_C$$

$$\frac{{}^{B}d\mathbf{H}_C}{dt} = I_{11}\dot{\omega}_1\mathbf{b}_1 + I_{22}\dot{\omega}_2\mathbf{b}_2 + I_{33}\dot{\omega}_3\mathbf{b}_3$$

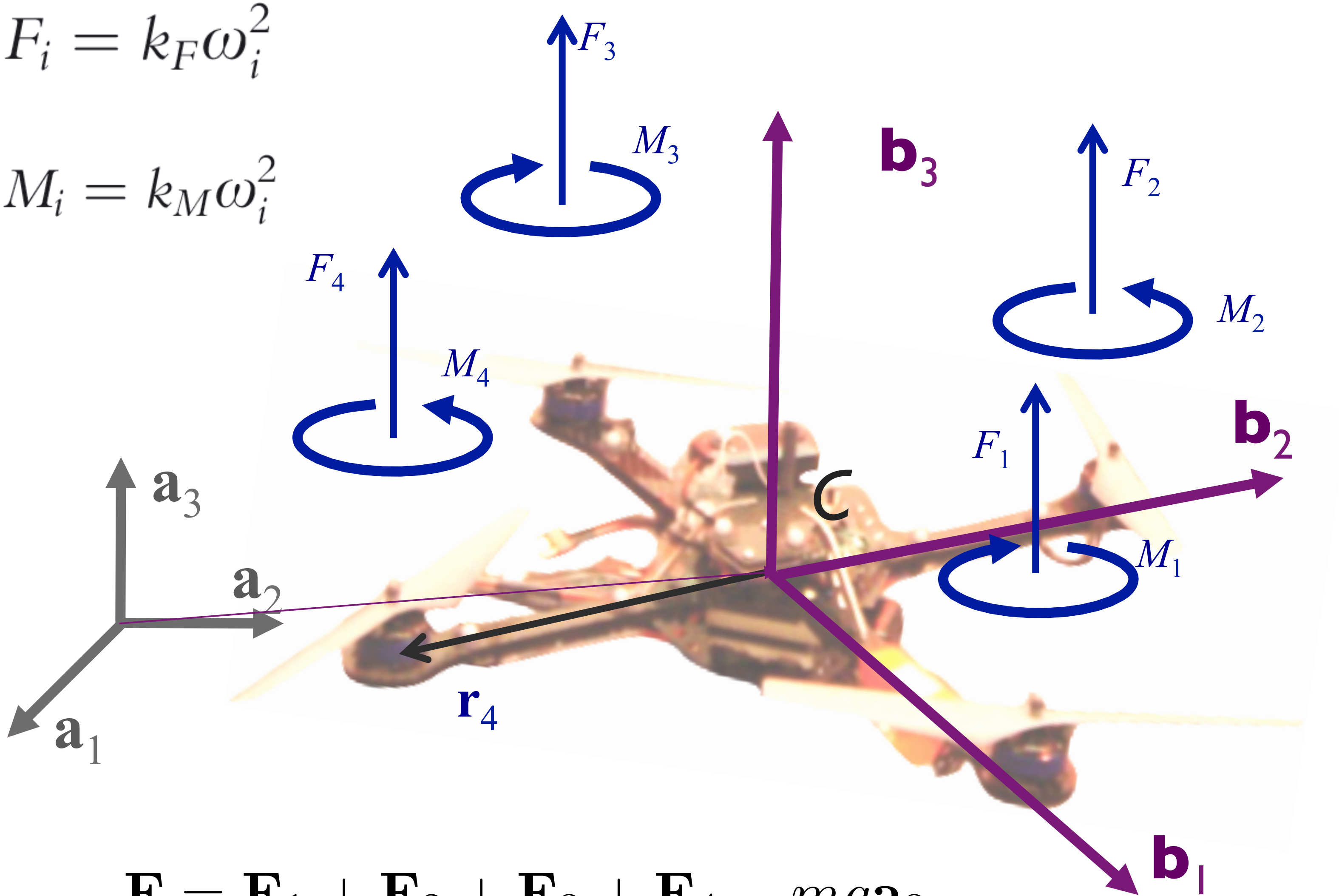


net external moment

Quadrotor Equations of Motion

$$F_i = k_F \omega_i^2$$

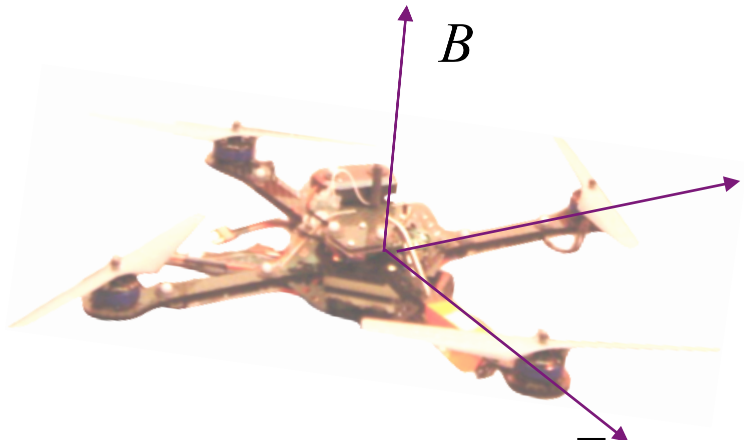
$$M_i = k_M \omega_i^2$$



$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 - mg\mathbf{a}_3$$

$$\mathbf{M} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \mathbf{r}_4 \times \mathbf{F}_4 + \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_4$$

Newton-Euler Equations



$$ {}^{A}\boldsymbol{\omega}^{B} = p\,\mathbf{b}_1 + q\,\mathbf{b}_2 + r\,\mathbf{b}_3 $$

Rotation of thrust vector from *B* to *A*

$$ m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ \underbrace{F_1 + F_2 + F_3 + F_4}_{u_1} \end{bmatrix} $$

Components in the inertial frame along $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$

$$ I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \underbrace{\begin{bmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix}}_{\mathbf{u}_2} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix} $$

Components in the body frame along $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$, the principal axes

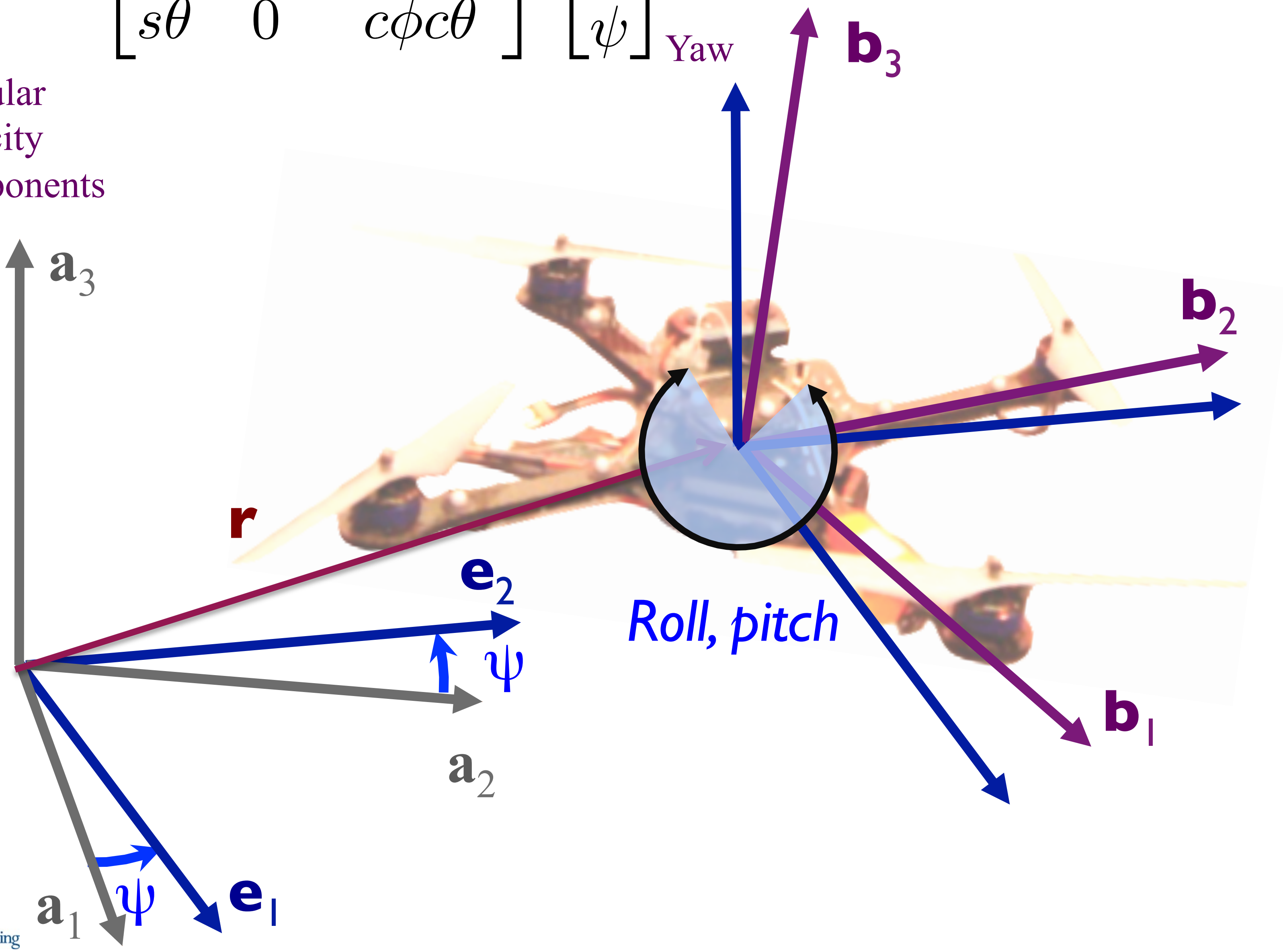
How do we estimate all the parameters in this model?

$$m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix}$$

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} c\theta & 0 & -c\phi s\theta \\ 0 & 1 & s\phi \\ s\theta & 0 & c\phi c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \begin{matrix} \text{Roll} \\ \text{Pitch} \\ \text{Yaw} \end{matrix}$$

Angular velocity components in B



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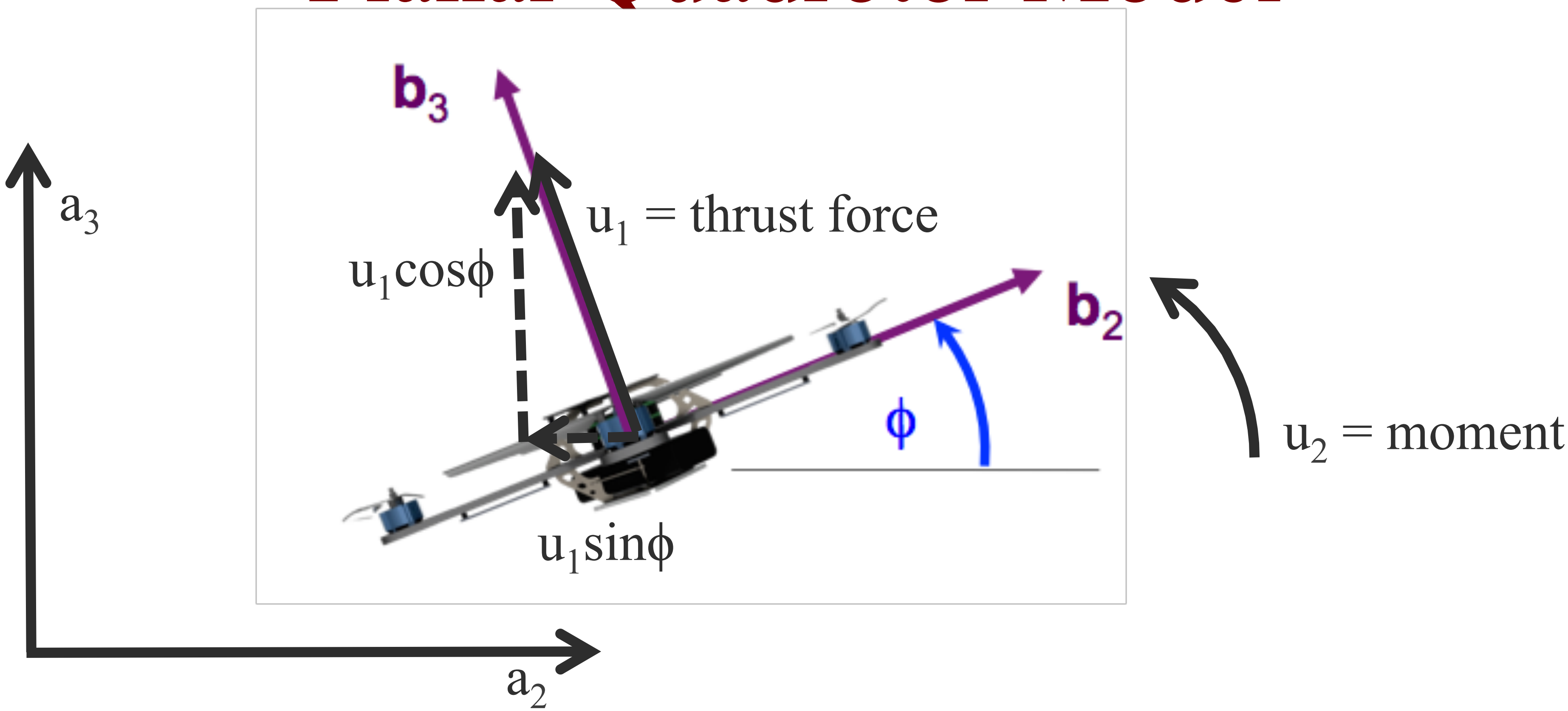
Planar Quadrotor Model



$$\begin{bmatrix} \ddot{y} \\ \ddot{z} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{m}\sin\phi & 0 \\ \frac{1}{m}\cos\phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

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State Space for Quadrotors

State Vector

- q describes the configuration (position) of the system
- x describes the state of the system

$$q = \begin{bmatrix} x \\ y \\ z \\ \varphi \\ \theta \\ \psi \end{bmatrix}, \quad x = \begin{bmatrix} q \\ -- \\ \dot{q} \end{bmatrix}$$

Planar Quadrotor

$$q = \begin{bmatrix} y \\ z \\ \varphi \end{bmatrix}, \quad x = \begin{bmatrix} q \\ -- \\ \dot{q} \end{bmatrix}$$

Equilibrium at Hover

- q_e describes the equilibrium configuration of the system
- x_e describes the equilibrium state of the system

$$q_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 0 \\ 0 \\ \psi_0 \end{bmatrix}, \quad x_e = \begin{bmatrix} q_e \\ -- \\ 0 \end{bmatrix}$$

$$q_e = \begin{bmatrix} y_0 \\ z_0 \\ 0 \end{bmatrix}, \quad x_e = \begin{bmatrix} q_e \\ -- \\ 0 \end{bmatrix}$$

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Dynamical Systems in State-Space Form

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Dynamical Systems

Systems where the effects of actions do not occur immediately.

Evolution of the system's states is governed by a set of ordinary differential equations.

Ordinary differential equations are often rearranged into *state-space form*.

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

Matrices

State-Space Form

Given an ordinary differential equation:

1. Identify the order, n, of the system
2. Define the states $x_1 = y(t), x_2 = \dot{y}(t), \ldots, x_n = y^{(n-1)}(t)$

   (n-1)st derivative

3. Create the *state vector* $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^T = \begin{bmatrix} y & \dot{y} & \ldots & y^{(n-1)} \end{bmatrix}^T$
4. Write the coupled first-order differential equations:

$$\frac{d}{dt}x_1 = \frac{d}{dt}y = \dot{y} = x_2$$

$$\frac{d}{dt}x_2 = \frac{d}{dt}\dot{y} = \ddot{y} = x_3$$

$$\ldots$$

$$\frac{d}{dt}x_n = \frac{d}{dt}y^{(n-1)} = g(y, \dot{y}, \ldots, y^{(n-1)}, \mathbf{u}) = g(x_1, x_2, \ldots, x_n, \mathbf{u})$$

From governing ODEs

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State-Space Form

5. Write system of first-order differential equations as matrix:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \dots \\ g(x_1, x_2, \dots, x_n, \mathbf{u}) \end{bmatrix}$$

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

Example 1: Mass-Spring System

$$m\ddot{y}(t) + ky(t) = u(t)$$

1. Identify n = 2
2. Define states $x_1 = y, x_2 = \dot{y}$
3. Create the state vector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T = \begin{bmatrix} y & \dot{y} \end{bmatrix}^T$
4. Write the coupled first-order differential equations:

$$\frac{d}{dt}x_1 = \frac{d}{dt}y = \dot{y} = x_2$$

$$\frac{d}{dt}x_2 = \frac{d}{dt}\dot{y} = \ddot{y} = \frac{u(t) - ky(t)}{m} = \frac{u(t) - kx_1}{m}$$

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Example 1: Mass-Spring System

$$m\ddot{y}(t) + ky(t) = u(t)$$

5. Write system of first-order differential equations as matrix:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{u(t) - kx_1}{m} \end{bmatrix}$$

This system is actually *linear*:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

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Example 2: Planar Quadrotor Model

$$m\ddot{y} = -\sin(\phi)u_1$$

$$m\ddot{z} = \cos(\phi)u_1 - mg$$

$$I_{xx}\ddot{\phi} = u_2$$

1. Identify n = 2
2. Define states $x_1 = y, x_2 = z, x_3 = \phi, x_4 = \dot{y}, x_5 = \dot{z}, x_6 = \dot{\phi}$
3. Define the state vector

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T = \begin{bmatrix} y & z & \phi & \dot{y} & \dot{z} & \dot{\phi} \end{bmatrix}^T$$

Example 2: Planar Quadrotor Model

$$m\ddot{y} = -\sin(\phi)u_1$$
$$m\ddot{z} = \cos(\phi)u_1 - mg$$
$$I_{xx}\ddot{\phi} = u_2$$

4. Define the system of first-order differential equations:

$$\frac{d}{dt}x_1 = \frac{d}{dt}y = \dot{y} = x_4$$

$$\frac{d}{dt}x_2 = \frac{d}{dt}z = \dot{z} = x_5$$

$$\frac{d}{dt}x_3 = \frac{d}{dt}\phi = \dot{\phi} = x_6$$

$$\frac{d}{dt}x_4 = \frac{d}{dt}\dot{y} = \ddot{y} = \frac{-\sin(\phi)u_1}{m} = \frac{-\sin(x_3)u_1}{m}$$

$$\frac{d}{dt}x_5 = \frac{d}{dt}\dot{z} = \ddot{z} = \frac{\cos(\phi)u_1}{m} - g = \frac{\cos(x_3)u_1}{m} - g$$

$$\frac{d}{dt}x_6 = \frac{d}{dt}\dot{\phi} = \ddot{\phi} = \frac{u_2}{I_{xx}}$$

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Example 2: Planar Quadrotor Model

$$m\ddot{y} = -\sin(\phi)u_1$$
$$m\ddot{z} = \cos(\phi)u_1 - mg$$
$$I_{xx}\ddot{\phi} = u_2$$

5. Write system of first-order differential equations as matrix:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ \frac{-\sin(x_3)u_1}{m} \\ \frac{\cos(x_3)u_1}{m} - g \\ \frac{u_2}{I_{xx}} \end{bmatrix}$$

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